On projective modules with semilocal endomorphims ring.

C. Lomp, M. Nasrutdinov and I. Sakhaev

October 21, 2010

Abstract

It is proved that a projective left R-module P with semilocal endomorphism ring is finitely generated if and only if $\mathrm{h}dim\,(P) = \mathrm{h}dim\,((P/J(P))$.

1 Preliminaries.

Throughout R denotes an associative ring with unit and all modules will be left unitial R-module. Let $J(M)$ be the *Jacobson radical* of an R-module M. A left R-module is called *semi-local* if $M/J(M)$ is a semi-simple left R-module. For a semisimple module $M = \bigoplus_{iinI} M_i$ where all M_i are simple R-modules, we denote by $\mathrm{l}ength\,(M) = \mathrm{card}(I)$ the length of M.

A submodule N of M is called *small* in M if for every submodule U of M with $N + U = M$ we have $U = M$. We denote a small submodule N of M by $N \ll M$. A module M is said to be *hollow* if $M \neq 0$ and every proper submodule of M is small in M. M is said to have *finite hollow dimension* (or *finite dual Goldie dimension*) if there exists an exact sequence

$$M \xrightarrow{\quad g \quad} \bigoplus_{i=1}^{n} H_i \longrightarrow 0$$

where all the H_i are hollow and the kernel of g is small in M. Then n is an invariant of the module M and called the *hollow dimension* of M and we write $\mathrm{h}dim\,(M) = n$. In [SV, Theorem 1.8] it is shown that in this case M does not allow any epimorphism to a direct sum with more then n summands. Note that for a semisimple module M with finite dual Goldie dimensioin we have $\mathrm{h}dim\,(M) = \mathrm{l}ength\,(M)$. The dual Goldie dimension was investigated in [GP], [HaS], [Lo], [SV], [T76], [T94], [V]. The modules with semilocal endomorphisms ring were investigated in [HeS], [Lo] and [Lo]. In [Lo] C.Lomp has formulated the question: *Is every (self-) projective R-module P with semilocal endomorphisms ring finitely generated ?*

This question is closely related to an old problem of D. Lazard (see [La]): *If P is a projective left R-module with $P/J(P)$ finitely generated then P is finitely generated ?*

H. Zoschinger in [Z81] and I.Sakhaev in [S77], [S85], [S87], [S89], [S91], [S93] have investigated the problem of D. Lazard. A non commutative semilocal ring over which there exists a no finitely generated projective module P such that P/J(P) is finitely generated was constructed by I. Sakhaev and V. Gerasimov in [GS].

2 Proof of the Results.

At first we show how our question is related to the problem of D. Lazard.

Lemma 2.1. *Let P be a projective left R-module with semilocal endomorphism ring. Then the left R-module $P/J(P)$ is finitely generated.*

Proof. By Takeuchi's Theorem (see [Lo, Theorem 2.10] for a short proof) we have $\mathrm{h}dim\,(\mathrm{E}nd\,(P)) = \mathrm{h}dim\,(P) < \infty$ and by [Lo, Theorem 2.7] P is semilocal with $\mathrm{h}dim\,(P) \geq \mathrm{l}ength\,(P/J(P))$. Since every semisimple module with finite length is finitely generated, the result follows. □

Now we will prove the more general confirmation about R-modules M with $\mathrm{h}dim\,(M) < \infty$.

Theorem 2.2. *Let M be a left R-module with $\mathrm{h}dim\,(M)$ finite. Then the following conditions are equivalent:*

(a) M is finitely generated.

(b) $J(M)$ is small in M.

(c) $\mathrm{h}dim\,(M) = \mathrm{h}dim\,(M/J(M))$.

Proof. By [Lo, Theorem 2.7] M is semilocal and $\mathrm{h}dim(M) \geq \mathrm{l}ength(M/J(M))$. Hence $M/J(M)$ is a semisimple module of finite length and hence finitely generated.

By [W, 21.6(4)] a module M is finitely generated if and only if $M/J(M)$ is finitely generated and $J(M) \ll M$. Hence $(a) \Leftrightarrow (b)$ follows from [W, 21.6(4)].

As $\mathrm{h}dim\,(M)$ is finite $(b) \Leftrightarrow (c)$ follows from [V, Theorem 1.20(3)]. □

Corollary 2.3. *A projective left R-module P with semi-local endomorphisms ring is finitely generated if and only if $\mathrm{h}dim\,(P) = \mathrm{h}dim\,(P/J(P))$.*

Proof. By Takeuchi's theorem [Lo, Theorem 3.10] we have $\mathrm{h}dim\,(P) < \infty$. Theorem 2.2 completes the proof. □

Remark 2.4. *In [HeS, Example 10(2)] D. Herbera and A. Shamsuddin gave an example of a cyclic module with semilocal endomorphism ring, but with infinite dual Goldie dimension. Their construction is as follows:*

Take a ring extension $R \subseteq S$ such that R is semilocal, but S is not (for example R a field and S=R[X] the (commutative) polynomial ring with coefficients in R). Consider the (S, R)-bimodule $M := \mathrm{H}om_R\,({}_RS,{}_R\,S/R)$, the sub-bimodule

$N = f \in M \mid f(R) = 0$ *and the ring* $T := \begin{pmatrix} S & M \\ 0 & R \end{pmatrix}$. *Take the right ideal* $I := \begin{pmatrix} 0 & N \\ 0 & R \end{pmatrix}$ *of* T. *Then* $\mathrm{End}_T\,(T/I) \simeq I'/I \simeq R$ *holds as rings, where* $I' = \begin{pmatrix} R & N \\ 0 & R \end{pmatrix}$ *is the idealizer of* I *in* T. *Hence* T/I *is a cyclic* T*-module whose endomorphism ring is isomorphic to the semilocal ring* R. *Since* S *has infinite dual Goldie dimension as right* S*-module,* T/I *has infinite dual Goldie dimension as right* T*-module. To see this let* $U := \begin{pmatrix} 0 & M \\ 0 & R \end{pmatrix}$ *and note that* $S_S = T/U_T$ *is a factor module of* T/I *as* $I \subseteq U$. *Thus* $\infty = \mathrm{h}dim\,(S) = \mathrm{h}dim\,(T/U_T) \leq \mathrm{h}dim\,(T/I)$.

References

[SV] B.Sarath and K.Varadarajan, *Dual Goldie dimension II*, Commun. Algebra 7, 1885-1899 (1979).

[GP] P.Grezeszcuk and E.R.Puczyłowski, *On Goldie and dual Goldie dimension*, J. Pure and Applied Algebra 31, 47-54 (1984)

[HaS] A.Hanna and A.Shamsuddin, *Dual Goldie dimension*, Rendiconti dell' Istitutio di Mathematica dell' Universita di Trieste 24, No.1-2, 25-38 (1992)

[Lo] C.Lomp, *On dual Goldie dimension*, Diplomarbeit, HHU Düsseldorf (1996)

[T76] T.Takeuchi, *On cofinite-dimensional modules*, Hokkaido Math. J. 5, 1-43 (1976)

[T94] T.Takeuchi, *Coranks of a quasi-projective module and its endomorphism ring*, Glasgow Math. J. 36, 381 - 383 (1994)

[V] K.Varadarajan, *Dual Goldie dimension*, Commun. Algebra, 7, 565-610 (1979)

[HeS] D.Herbera and A.Shamsuddin, *Modules with semi-local endomorphism ring*, Proc. American Math. Soc. 123, 3593-3600 (1995)

[Lo] C.Lomp, *On semilocal modules and rings.*, Commun. Algebra 27, No.4, 1921-1935 (1999).

[La] D. Lazard, *Liberte des gros modules projectifs*, J. Algebra 31, 437 - 451 (1974)

[S77] I. Sakhaev, *The finite generation of projective modules.*, Izv. VUZov. Mathematika. 9, 69 - 79 (1977)

[S85] I. Sakhaev, *On the projectivity of finitely generated flat modules over semilocal rings.*, Math. Notes 37 No. 2, 152 - 162 (1985)

[S87] I. Sakhaev, *On the group $K_0(AP^0)$.* Math. Nachrichten 130, 157 - 175 (1987)

[S89] I. Sakhaev, *On SVI-rings and on D. Lazard hyphothesis.* Math. Nachrichten 144, 87 - 98 (1989)

[S91] I.I. Sakhaev, *On lifting finite generacy of a projective module modulo its radical*, Math. Notes 49, 293 - 301 (1991)

[S93] I.I. Sakhaev, *The finite generation of projective modules*, Algebra(Proc. Int. Conf. Krasnoyarsk 1993), edited by Y.L. Ershov et al., 209-216, de Gruyter, Berlin 1996

[Z81] H.Zöschinger, *Projektive Moduln mit endlich erzeugtem Radikalfaktormodul*, Math. Annalen 255, 199 - 206 (1981)

[GS] V.N. Gerasmiov and I.I. Sakhaev, A counterexample to two hypotheses on projective and flat modules, Siberian Math. J. 25, 855 - 859 (1984)

[W] R.Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading (1991)