SENSITIVITY AND HISTORIC BEHAVIOR FOR CONTINUOUS MAPS ON BAIRE METRIC SPACES

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Abstract. We introduce a notion of sensitivity with respect to a continuous real-valued bounded map which provides a sufficient condition for a continuous transformation, acting on a Baire metric space, to exhibit a Baire generic subset of points with historic behavior (also known as irregular points). The applications of this criterion recover, and extend, several known theorems on the genericity of the irregular set, besides yielding a number of new results, including information on the irregular set of geodesic flows, in both negative and non-positive curvature, and semigroup actions.

1. Introduction

1.1. **Historic behavior.** In what follows, we shall write X to denote a compact metric space and Y will stand for an arbitrary metric space. Given such a space Y and $A \subset Y$, denote by $A' \subset Y$ the set of non-isolated accumulation points in A, that is, $y \in A'$ if and only if y belongs to the closure $\overline{A \setminus \{y\}}$. Let $C(Y, \mathbb{R})$ be the set of real-valued continuous maps on Y and $C^b(Y, \mathbb{R})$ be its subset of bounded elements endowed with the supremum norm $\| \cdot \|_\infty$.

A topological space Z is said to be a Baire space if the intersection of countably many open dense subsets in Z is dense in Z. We say that a set $A \subseteq Z$ is *Baire generic* in a Baire space Z if it contains an intersection of countably many open dense sets of Z (that is, A contains a dense G_δ set).

Given a Baire metric space (Y, d), a continuous map $T : Y \to Y$ and $\varphi \in C^b(Y, \mathbb{R})$, the set of (T, φ)*-irregular points*, or points with historic behavior, is defined by

$$\mathcal{I}(T, \varphi) = \Big\{ y \in Y : \Big(\frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j(y)) \Big)_{n \in \mathbb{N}} \text{ does not converge} \Big\}.$$

Birkhoff's ergodic theorem ensures that, for any Borel T-invariant probability measure μ and every μ-integrable observable $\varphi : Y \to \mathbb{R}$, the sequence of averages $\big(\frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j(y)) \big)_{n \in \mathbb{N}}$ converges at μ-almost every point y in Y. So, the set of (T, φ)-irregular points is negligible with respect to any T-invariant probability measure. In the last decades, though, there has been an intense study concerning the set of points for which Cesàro averages do not converge. Contrary to the previous measure-theoretical description, the set of the irregular points may be Baire generic and, moreover, have full topological pressure, full metric mean dimension or full Hausdorff dimension (see [2, 5, 3, 4, 24, 26, 35]). In [9], the first and the fourth

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named authors obtained a simple and unifying criterion, using first integrals, to guarantee that $\mathcal{I}(T,\varphi)$ is Baire generic in X whenever $T: X \to X$ is a continuous dynamics acting on a compact metric space X. More precisely, given $\varphi \in C(X,\mathbb{R})$, consider the map $L_\varphi : X \to \mathbb{R}$ defined by

$$x \in X \quad \mapsto \quad L_\varphi(x) = \limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ T^j(x). \tag{1.1}$$

This is a first integral with respect to the map T, that is, $L_\varphi \circ T = L_\varphi$. The existence of dense sets of discontinuity points for this first integral turns out to be a sufficient condition for the genericity of the historic behavior.

Theorem 1.1. [9, Theorem A] *Let (X,d) be a compact metric space, $T: X \to X$ be a continuous map and $\varphi : X \to \mathbb{R}$ be a continuous observable. Assume that there exist two dense subsets $A, B \subset X$ such that the restrictions of L_φ to A and to B are constant, though the value at A is different from the one at B. Then $\mathcal{I}(T,\varphi)$ is a Baire generic subset of X.*

The assumptions of the previous theorem are satisfied by a vast class of continuous maps on compact metric spaces, including minimal non-uniquely ergodic homeomorphisms, non-trivial homoclinic classes, continuous maps with the specification property, Viana maps and some partially hyperbolic diffeomorphisms (cf. [9]).

In this work we establish a criterion with a wider scope than the one of Theorem 1.1. It applies to Baire metric spaces and general sequences of bounded continuous real-valued maps, rather than just Cesàro averages, subject to a weaker requirement than the one demanded in the previous theorem. In particular, one obtains new results on the irregular set of several classes of maps and flows, which comprise geodesic flows on certain non-compact Riemannian manifolds, countable Markov shifts and endomorphisms with two physical measures exhibiting intermingled basins of attraction. Regarding semigroup actions, we note that irregular points for group actions with respect to Cesàro averages were first studied in [17]. We will provide additional information on the Baire genericity of irregular sets for averages that take into account the group structure. We refer the reader to Sections 10 and 11 for the precise statements.

In the next subsections we will state our main definitions and results.

1.2. **Sensitivity and genericity of historic behavior.** Given a metric space (Y,d), a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}} \in C^b(Y,\mathbb{R})^{\mathbb{N}}$ and $y \in Y$, let

$$W_\Phi(y) = \big\{\varphi_n(y) : \; n \in \mathbb{N}\big\}'$$

denote the set of accumulation points of the sequence $(\varphi_n(y))_{n\in\mathbb{N}}$. The next notion is inspired by the concept of sensitivity to initial conditions.

Definition 1. *Let (Y,d) be a metric space and $\Phi \in C^b(Y,\mathbb{R})^{\mathbb{N}}$. We say that Y is* Φ-sensitive *(or sensitive with respect to the sequence Φ) if there exist dense subsets A, $B \subset Y$, where B can be equal to A, and $\varepsilon > 0$ such that for any $(a,b) \in A \times B$ one has*

$$\sup_{r\in W_\Phi(a),\; s\in W_\Phi(b)} |r-s| > \varepsilon.$$

In the particular case of a sequence Φ *of Cesàro averages*

$$(\varphi_n)_{n\in\mathbb{N}} = \Big(\frac{1}{n}\sum_{j=0}^{n-1}\varphi\circ T^j\Big)_{n\in\mathbb{N}}$$

associated with a potential $\varphi\in C^b(Y,\mathbb{R})$ *and a continuous map* $T: Y\to Y$*, we say that* Y *is* (T,φ)-sensitive *if the space* Y *is* Φ-sensitive*, and write* W_φ *instead of* W_Φ*.*

We refer the reader to Example 2 for an illustration of this definition. We observe that being (T,φ)-sensitive is a direct consequence of the assumption on L_φ stated in Theorem 1.1, though it may be strictly weaker (cf. Example 4). Our first result concerns Φ-sensitive sequences and strengths Theorem 1.1.

Theorem 1.2. *Let* (Y,d) *be a Baire metric space and* $\Phi=(\varphi_n)_{n\in\mathbb{N}}\in C^b(Y,\mathbb{R})^{\mathbb{N}}$ *be a sequence of continuous bounded maps such that* $\limsup_{n\to+\infty}\|\varphi_n\|_\infty<+\infty$*. If* Y *is* Φ*-sensitive then the set*

$$\mathcal{I}(\Phi)=\Big\{y\in Y:\ \lim_{n\to+\infty}\varphi_n(y)\ \text{ does not exist}\Big\}$$

is a Baire generic subset of Y*. In particular, if* $T: Y\to Y$ *is a continuous map,* φ *belongs to* $C^b(Y,\mathbb{R})$ *and* Y *is* (T,φ)*-sensitive, then* $\mathcal{I}(T,\varphi)$ *is a Baire generic subset of* Y*.*

We emphasize that the previous statement does not require a guiding dynamical system, so it may be applied to general sequences of bounded continuous real-valued maps rather than just Cesàro averages. In particular, we may address the Baire genericity of the irregular set of semigroup actions with respect to averages that take into account the group structure (cf. Section 11 for more details).

1.3. Irregular points for continuous maps on Baire metric spaces with dense orbits.

In this subsection, building over [15, 20], we will discuss the relation between transitivity, existence of dense orbits and the size of the set of irregular points for continuous maps on Baire metric spaces.

Definition 2. Given a continuous map $T: Y\to Y$ on a Baire metric space (Y,d), one says that:

- T is *transitive* if for every non-empty open sets $U, V\subset Y$ there exists $n\in\mathbb{N}$ such that $U\cap T^{-n}(V)\neq\emptyset$.
- T is *strongly transitive* if $\bigcup_{n\geqslant 0}T^n(U)=Y$ for every non-empty open set $U\subset Y$.
- T has a *dense orbit* if there is $y\in Y$ such that $\{T^j(y):\ j\in\mathbb{N}\cup\{0\}\}$ is dense in Y.

It is worthwhile observing that, if the metric space is compact and has no isolated points, then the map is transitive if and only if it has a dense orbit (see [1, Theorem 1.4] and Example 1).

Denote by $\mathrm{Trans}(Y,T)$ the set

$$\{y\in Y:\ \text{ the orbit of } y \text{ by } T \text{ is dense in } Y\}$$

and consider the following notation:

$$\mathcal{H}(Y,T) \coloneqq \{\varphi \in C^b(Y,\mathbb{R})\colon\ \mathcal{I}(T,\varphi) \neq \emptyset\}$$
$$\mathcal{D}(Y,T) \coloneqq \{\varphi \in C^b(Y,\mathbb{R})\colon\ \mathcal{I}(T,\varphi) \text{ is dense in } Y\}$$
$$\mathcal{R}(Y,T) \coloneqq \{\varphi \in C^b(Y,\mathbb{R})\colon\ \mathcal{I}(T,\varphi) \text{ is Baire generic in } Y\}.$$

The set of *completely irregular points* with respect to T is precisely the intersection

$$\bigcap_{\varphi\,\in\,\mathcal{H}(Y,T)} \mathcal{I}(T,\varphi).$$

Clearly $\mathcal{R}(Y,T) \subset \mathcal{D}(Y,T) \subset \mathcal{H}(Y,T)$. Inspired by [20], we aim at finding sufficient conditions on (Y,T) under which the sets $\mathcal{R}(Y,T)$ and $\mathcal{D}(Y,T)$ coincide. We note that the set $\mathcal{H}(Y,T)$ may be uncountable. Moreover, in (cf. [34, Theorem 2.1]), Tian proved that, if Y is compact and T has the almost-product and uniform separation properties, then the set of completely irregular points is either empty or carries full topological entropy. More recently, in [20], Hou, Lin and Tian showed that, for each transitive continuous map on a compact metric space Y, either every point with dense orbit is contained in the basin of attraction of some invariant probability measure μ, defined by

$$B(\mu) \,=\, \Big\{x \in X\colon\ \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j(x)} = \mu \ \text{ (convergence in the weak}^* \text{ topology)}\Big\}$$

(so those points are regular with respect to any continuous potential), or irregular behavior occurs on $\operatorname{Trans}(Y,T)$ and the irregular set is Baire generic for every φ belonging to an open dense subset of $C(Y,\mathbb{R})$ (described in [20, Theorem A]). The next consequence of Theorem 1.2 generalizes this information.

Corollary 1.3. *Let (Y,d) be a Baire metric space and $T : Y \to Y$ be a continuous map such that* $\operatorname{Trans}(Y,T) \neq \emptyset$. *Then:*

(*i*) *When (Y,d) has an isolated point,*

$$\mathcal{D}(Y,T) \neq \emptyset \qquad \Leftrightarrow \qquad \operatorname{Trans}(Y,T) \subseteq \bigcap_{\varphi\,\in\,\mathcal{D}(Y,T)} \mathcal{I}(T,\varphi).$$

Moreover, if Y has an isolated point and $\mathcal{D}(Y,T) \neq \emptyset$, then $\mathcal{R}(Y,T) = \mathcal{D}(Y,T)$.

(*ii*) *If $\mathfrak{F} \subset C^b(Y,\mathbb{R})$ and $\bigcap_{\varphi\,\in\,\mathfrak{F}} \mathcal{I}(T,\varphi)$ is Baire generic in Y, then*

$$\operatorname{Trans}(Y,T) \cap \bigcap_{\varphi\,\in\,\mathfrak{F}} \mathcal{I}(T,\varphi) \neq \emptyset.$$

1.4. **Oscillation of the time averages.** Define, for any $\varphi \in C^b(Y,\mathbb{R})$ and $y \in Y$,

$$\ell_\varphi(y) = \liminf_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j y) \qquad \text{and} \qquad L_\varphi(y) = \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j y)$$

and consider

$$\ell^*_\varphi = \inf_{y\,\in\,\operatorname{Trans}(Y,T)} \ell_\varphi(y) \qquad \text{and} \qquad L^*_\varphi = \sup_{y\,\in\,\operatorname{Trans}(Y,T)} L_\varphi(y). \tag{1.2}$$

For each $\alpha \leqslant \beta$, take the sets

$$I_\varphi[\alpha,\, \beta] \;=\; \Big\{y \in Y\colon\; \ell_\varphi(y) = \alpha \text{ and } \beta = L_\varphi(y)\Big\}$$

and

$$\widehat{I_\varphi[\alpha,\, \beta]} \;=\; \Big\{y \in Y\colon\; \ell_\varphi(y) \leqslant \alpha \text{ and } \beta \leqslant L_\varphi(y)\Big\}.$$

The next result estimates the topological size of the previous level sets for dynamics with dense orbits.

Theorem 1.4. *Let (Y, d) be a Baire metric space and $T : Y \to Y$ be a continuous map with a dense orbit. Given $\varphi \in C^b(Y, \mathbb{R})$, one has:*

- *(i)* *$I_\varphi[\ell^*_\varphi,\, L^*_\varphi]$ is a Baire generic subset of Y. In particular, $\mathcal{I}(T, \varphi)$ is either Baire generic or meagre in Y.*
- *(ii)* *$\mathcal{I}(T, \varphi)$ is a meagre subset of Y if and only if there exists $C_\varphi \in \mathbb{R}$ such that*

$$I_\varphi[C_\varphi,\, C_\varphi] \;=\; \Big\{y \in Y\colon\; \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j(y)) = C_\varphi\Big\}$$

is a Baire generic set containing $\mathrm{Trans}(Y, T)$.

Consequently, if (Y, d) is a Baire metric space, $T : Y \to Y$ is a continuous map with a dense orbit and $\bigcup_{\varphi \in C^b(Y,T)} \mathcal{I}(T, \varphi)$ is not Baire generic, then $Y \setminus \mathcal{I}(T, \varphi)$ is Baire generic for every $\varphi \in C^b(Y, \mathbb{R})$, and there exists a linear functional $\mathcal{F} : C^b(Y, \mathbb{R}) \to \mathbb{R}$ such that

$$\mathrm{Trans}(Y, T) \subset \Big\{y \in Y : \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j(y)) \;=\; \mathcal{F}(\varphi) \qquad \forall\, \varphi \in C^b(Y, \mathbb{R})\Big\}.$$

Note that, if Y is compact, $\mathcal{F}$ is represented by the space of Borel invariant measures with the weak*-topology. More generally, if Y is a metric space, then the dual of $C^b(Y, \mathbb{R})$ is represented by the regular, bounded, finitely additive set functions with the norm of total variation (cf. [16, Theorem 2, IV.6.2]).

Remark 1.5. Theorem 1.4 implies that, if (Y, d) is a Baire metric space and $T : Y \to Y$ is a continuous minimal map (that is, $\mathrm{Trans}(Y, T) = Y$), then for every $\varphi \in C^b(X, \mathbb{R})$ either $\mathcal{I}(T, \varphi) = \emptyset$ or $\mathcal{I}(T, \varphi)$ is Baire generic.

Remark 1.6. It is unknown whether there exist a Baire metric space (Y, d), a continuous map $T : Y \to Y$ and $\varphi \in C^b(Y, \mathbb{R})$ such that T has a dense orbit and $\mathcal{I}(T, \varphi)$ is a non-empty meagre set.

As a consequence of the proof of Theorem 1.2 and Corollary 1.3 we obtain the next corollary as a counterpart of [20, Lemma 3.1] for continuous maps on Baire metric spaces.

Corollary 1.7. *Let (Y, d) be a Baire metric space and $T : Y \to Y$ be a continuous map with a dense orbit, and take $\varphi \in C^b(Y, \mathbb{R})$. The following conditions are equivalent:*

- *(i)* *Y is (T, φ)-sensitive.*
- *(ii)* *$\mathcal{I}(T, \varphi)$ is Baire generic in X.*
- *(iii)* $\mathrm{Trans}(Y, T) \cap \mathcal{I}(T, \varphi) \neq \emptyset$.

Remark 1.8. Under the assumptions of Corollary 1.7, if Y is (T,φ)-sensitive then one has $\mathcal{R}(Y,T) = \big\{\varphi \in C^b(Y,\mathbb{R})\colon\ \ell^*_\varphi < L^*_\varphi\big\}$. We also note that one has $(i) \Rightarrow (ii)$ in Corollary 1.7 even without assuming the existence of dense orbits.

Remark 1.9. An immediate consequence of the proof of Corollary 1.7 is that, within the setting of continuous maps $T: Y \to Y$ acting on a Baire metric space Y and having a dense orbit, the Definition 1 of (T,φ)-sensitivity is equivalent to the following statement: *There exist a dense set $A \subset Y$ and $\varepsilon > 0$ such that for any $a \in A$ one has*

$$\sup_{r\,\in\, W_\varphi(a),\ s\,\in\, W_\varphi(a)} |r-s| > \varepsilon.$$

Indeed, the latter statement clearly implies Definition 1. Conversely, if X is (T,φ)-sensitive, then we may take a dense orbit $A = \{T^j(x_0)\colon\ j \in \mathbb{N} \cup \{0\}\}$ contained in $\mathcal{I}(T,\varphi)$, whose existence is guaranteed by item (iii) of Corollary 1.7. So, given two distinct accumulation points r_0, s_0 in $W_\varphi(x_0)$ and choosing $\varepsilon > |r_0 - s_0| > 0$, then for every $a, b \in A$ one has $W_\varphi(a) = W_\varphi(b) = W_\varphi(x_0)$ and $\sup_{r_a\,\in\, W_\varphi(a),\ r_b\,\in\, W_\varphi(b)} |r_a - r_b| \geqslant |r_0 - s_0| > \varepsilon$.

The remainder of the paper is organized as follows. In Section 2, we convey the previous results to the particular case of continuous dynamics acting on compact metric spaces. The aforementioned results are then proved in the ensuing sections, where we also discuss their scope and compare them with properties established in other references. In Section 10 we test our assumptions on some examples and in Section 11 we provide some applications, namely within the settings of semigroup actions and geodesic flows on non-compact manifolds.

2. Irregular points for continuous maps on compact metric spaces

Suppose now that (X,d) is a compact metric space. Let $\mathcal{P}(X)$ stand for the set of Borel probability measures on X with the weak*-topology and consider a continuous map $T: X \to X$ with a dense orbit. For every $x \in X$, let δ_x be the Dirac measure supported at x and denote the set of accumulation points in $\mathcal{P}(X)$ of the sequence of empirical measures $\big(\frac{1}{n}\sum_{j=0}^{n-1} \delta_{T^j x}\big)_{n\in\mathbb{N}}$ by $V_T(x)$. Our next result imparts new information about the irregular set without requiring any assumption about the existence of isolated points in X.

The next result is a consequence of [36, Proposition 1 and Proposition 2], [20, Corollary 2.2 and Proposition 3.1] and Theorem 1.4.

Corollary 2.1. *Let (X,d) be a compact metric space and $T: X \to X$ be a continuous map with a dense orbit. Then:*

(a) $X_\Delta := \Big\{x \in X\colon \bigcup_{t\,\in\,\mathrm{Trans}(X,T)} V_T(t) \subseteq V_T(x)\Big\}$ *is Baire generic in X. Moreover, $X_\Delta \subseteq \bigcap_{\varphi\,\in\, C(X,\mathbb{R})} I_\varphi\widehat{[\ell^*_\varphi,\, L^*_\varphi]}$, so the latter set is Baire generic as well.*

(b) $\bigcup_{\varphi\,\in\, C(X,\mathbb{R})} \mathcal{I}(T,\varphi)$ *is either Baire generic or meagre. If it is meagre, there exists a Borel T-invariant measure μ such that*

$$\mathrm{Trans}(X,T) \subset \Big\{x \in X:\ \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j(x)) = \int \varphi\, d\mu \quad \forall\, \varphi \in C(X,\mathbb{R})\Big\}.$$

(c) $\bigcap_{\varphi \in \mathcal{H}(X,T)} \mathcal{I}(T,\varphi)$ *is either Baire generic or meagre. In addition,*

$$\bigcap_{\varphi \in \mathcal{H}(X,T)} \mathcal{I}(T,\varphi) \text{ is Baire generic } \Leftrightarrow \Big\{\varphi \in C(X,\mathbb{R})\colon \mathcal{I}(T,\varphi) \neq \emptyset \text{ and } \ell^*_\varphi = L^*_\varphi\Big\} = \emptyset.$$

We may ask whether the notion of (T,φ)-sensitivity (cf. Definition 1) is somehow related to the classical concepts of sensitivity to initial conditions and expansiveness. Let us recall these two notions.

Definition 3. Let $T : X \to X$ be a continuous map on a compact metric space (X,d). We say that:

- T has *sensitivity to initial conditions* if there exists $\varepsilon > 0$ such that, for every $x \in X$ and any $\delta > 0$, there is $z \in B(x,\delta)$ satisfying
$$\sup_{n \in \mathbb{N}} d(T^n(x), T^n(z)) > \varepsilon.$$
- T is (positively) *expansive* if there exists $\varepsilon > 0$ such that, for any points $x, z \in X$ with $x \neq z$, one has
$$\sup_{n \in \mathbb{N}} d(T^n(x), T^n(z)) > \varepsilon.$$

It is clear that an expansive map has sensitivity to initial conditions. Our next result shows that the condition of (T,φ)-sensitivity often implies sensitivity to initial conditions.

Theorem 2.2. *Let X be a compact metric space, $T : X \to X$ be a continuous map and $\varphi \in C(X,\mathbb{R})$. If X is (T,φ)-sensitive then either T has sensitivity to initial conditions or $\mathcal{I}(T,\varphi)$ has non-empty interior. In particular, if T has a dense set of periodic orbits and X is (T,φ)-sensitive, then T has sensitivity to initial conditions.*

The previous discussion together with well known examples yield the following scheme of connections:

$$\begin{array}{ccc}
\begin{array}{c} T \text{ is strongly transitive} \\ \text{and has dense periodic orbits} \\ \Downarrow \\ T \text{ has dense periodic orbits} \\ \text{and dense pre-orbits} \\ \Downarrow \\ \exists\,\varphi \in C(X,\mathbb{R})\colon\ X \text{ is } (T,\varphi)-\text{sensitive} \\ \text{and the interior of } \mathcal{I}(T,\varphi) = \emptyset \end{array} & \Rightarrow & \begin{array}{c} \text{Expansiveness} \\ \Downarrow \ \not\Uparrow \\ \text{Sensitivity to initial conditions.} \end{array}
\end{array} \tag{2.1}$$

When X is a compact topological manifold there is a link between expansiveness and sensitivity with respect to a well chosen continuous map $\varphi : X \to \mathbb{R}$. Indeed, Coven and Reddy (cf. [14]) proved that, if $T : X \to X$ is a continuous expansive map acting on a compact topological manifold, then there exists a metric $\tilde{d}$ compatible with the topology of (X,d) such that $T : (X,\tilde{d}) \to (X,\tilde{d})$ is a Ruelle-expanding map: there are constants $\lambda > 1$ and $\delta_0 > 0$ such that, for all $x, y, z \in X$, one has

- $\tilde{d}(T(x), T(y)) \geqslant \lambda \tilde{d}(x,y)$ whenever $\tilde{d}(x,y) < \delta_0$;
- $B(x,\delta_0) \cap T^{-1}(\{z\})$ is a singleton whenever $\tilde{d}(T(x), z) < \delta_0$.

In particular, if X is connected then T is topologically mixing. Moreover, as Ruelle-expanding maps admit finite Markov partitions and are semiconjugate to subshifts of finite type, one can choose $\varphi \in C(X, \mathbb{R})$ such that $\mathcal{I}(T, \varphi)$ is a Baire generic subset of X. Moreover, the interior of $\mathcal{I}(T, \varphi)$ is empty by the denseness of the set of periodic points of T. Therefore, summoning Corollary 1.7, we conclude that:

Corollary 2.3. *Let (X, d) be a compact connected topological manifold and $T : X \to X$ be a continuous expansive map. Then there exists $\varphi \in C(X, \mathbb{R})$ such that X is (T, φ)-sensitive.*

It is still an open question whether for each expansive map T on a compact metric space X there exists $\varphi \in C(X, \mathbb{R})$ such that X is (T, φ)-sensitive. Although we have no examples, it is likely to exist continuous maps on compact metric spaces which have sensitivity to initial conditions but for which X is not (T, φ)-sensitive for every $\varphi \in C(X, \mathbb{R})$.

Another consequence of Theorem 1.2 concerns the irregular sets for continuous maps satisfying the strong transitivity condition (see Definition 2).

Corollary 2.4. *Let $T : X \to X$ be a continuous map on a compact metric space X. If T is strongly transitive and $\varphi \in C(X, \mathbb{R})$, then either $\mathcal{I}(T, \varphi) = \emptyset$ or $\mathcal{I}(T, \varphi)$ is a Baire generic subset of X.*

We note that, as strongly transitive homeomorphisms on a compact metric space X are minimal, Corollary 2.4 extends the information in Remark 1.5 to strongly transitive continuous maps on compact metric spaces.

3. Proof of Theorem 1.2

The argument is a direct adaptation of the one used to show [9, Theorem 1]. Suppose that there are dense subsets A, B of Y and $\varepsilon > 0$ such that for any $(a, b) \in A \times B$ there exist $(r_a, r_b) \in \{\varphi_n(a) : n \geq 1\}' \times \{\varphi_n(b) : n \geq 1\}'$ satisfying $|r_a - r_b| > \varepsilon$. Fix $0 < \eta < \frac{\varepsilon}{3}$. Since the maps φ_n are continuous, given an integer $N \in \mathbb{N}$ the set

$$\Lambda_N = \Big\{ y \in Y : |\varphi_n(y) - \varphi_m(y)| \leqslant \eta \quad \forall m, n \geqslant N \Big\}$$

is closed in Y. Moreover:

Lemma 3.1. *Λ_N has empty interior for every $N \in \mathbb{N}$.*

Proof. Assume that there exists $N \in \mathbb{N}$ such that Λ_N has non-empty interior (which we abbreviate into $\operatorname{int}(\Lambda_N) \neq \emptyset$). Hence there exists $a \in A$ such that $a \in \operatorname{int}(\Lambda_N)$. Since φ_N is continuous, there exists $\delta_N > 0$ such that $|\varphi_N(a) - \varphi_N(y)| < \eta$ for every $y \in Y$ satisfying $d(a, y) < \delta_N$. By the denseness of B, one can choose $b \in B$ such that $b \in \operatorname{int}(\Lambda_N)$ and $d(a, b) < \delta_N$. Besides, according to the the definition of Λ_N one has

$$|\varphi_n(a) - \varphi_m(a)| \leqslant \eta \qquad \text{and} \qquad |\varphi_n(b) - \varphi_m(b)| \leqslant \eta \qquad \forall\, m, n \geqslant N.$$

For the previous pair $(a, b) \in A \times B$, choose $(r_a, r_b) \in \{\varphi_n(a) : n \geq 1\}' \times \{\varphi_n(b) : n \geq 1\}'$ satisfying $|r_a - r_b| > \varepsilon$. Fixing $m = N$, taking the limit as n goes to $+\infty$ in the first inequality along a subsequence converging to r_a and taking the limit as n tends to $+\infty$ in the second inequality along a subsequence convergent to r_b, we conclude that

$$|r_a - \varphi_N(a)| \leqslant \eta \qquad \text{and} \qquad |r_b - \varphi_N(b)| \leqslant \eta.$$

Therefore,

$$\varepsilon < |r_a - r_b| \leqslant |\varphi_N(a) - r_a| + |\varphi_N(b) - r_b| + |\varphi_N(a) - \varphi_N(b)| \leqslant 3\eta$$

contradicting the choice of η. Thus Λ_N must have empty interior. □

We can now finish the proof of Theorem 1.2. Using the fact that $\limsup_{n\to+\infty} \|\varphi_n\|_\infty < +\infty$, one deduces that $Y \setminus \mathcal{I}(\Phi) \subset \bigcup_{N=1}^{\infty} \Lambda_N$. Thus, by Lemma 3.1, the set of Φ-regular points (that is, those points for which Φ is a convergent sequence) is contained in a countable union of closed sets with empty interior. This shows that $\mathcal{I}(\Phi)$ is Baire generic, as claimed.

The second statement in the theorem is a direct consequence of the first one. □

Remark 3.2. It is worth mentioning that the proof of Theorem 1.2 also shows that one has

$$\{y \in Y\colon\ \limsup_n \varphi_n(y) - \liminf_n \varphi_n(y) < \eta\} \subseteq \bigcup_{N=1}^{\infty} \Lambda_N$$

and so the set

$$\{y \in Y\colon\ \limsup_n \varphi_n(y) - \liminf_n \varphi_n(y) \geqslant \eta\}$$

is Baire generic in Y.

Remark 3.3. The argument used in the proof of Theorem 1.2 adapts naturally to the context of continuous-time dynamical systems. This fact will be used later, when applying Theorem 1.2 to geodesic flows on non-positive curvature (see Example 6).

4. Proof of Corollary 1.3

We start by showing that the existence of an irregular point with respect to an observable φ whose orbit by T is dense is enough to ensure that Y is (T,φ)-sensitive.

Lemma 4.1. *Let (Y,d) be a Baire metric space, $T : Y \to Y$ be a continuous map such that (Y,T) has a dense orbit, and $\varphi \in C^b(Y,\mathbb{R})$. If $\mathcal{I}(T,\varphi) \cap \mathrm{Trans}(Y,T) \neq \emptyset$ then Y is (T,φ)-sensitive.*

Proof. Suppose that $\varphi \in C^b(Y,\mathbb{R})$ and $\mathcal{I}(T,\varphi)\cap\mathrm{Trans}(Y,T) \neq \emptyset$. Let $y \in Y$ be a point in this intersection. Then there is $\varepsilon > 0$ such that $\varepsilon < \limsup_{n\to+\infty} \varphi_n(y) - \liminf_{n\to+\infty} \varphi_n(y)$. Since φ is a bounded function, the values

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j(z)) \quad \text{and} \quad \limsup_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j(z))$$

are constant for every $z \in \{T^j(y) : j \in \mathbb{N}\cup\{0\}\}$. This invariance, combined with the fact that $\{T^n(y) : n \in \mathbb{N}\cup\{0\}\}$ is a dense subset of Y, implies that Y is (T,φ)-sensitive. □

Let us resume the proof of Corollary 1.3.

(i) Take $y \in \mathrm{Trans}(Y,T)$. Assume that Y has an isolated point and $\mathcal{D}(Y,T) \neq \emptyset$. Then there exists $N \in \mathbb{N}\cup\{0\}$ such that $T^N(y)$ is an isolated point of Y. Take $\psi \in \mathcal{D}(Y,T)$ whose set $\mathcal{I}(T,\psi)$ is dense in Y. Since $\{T^N(y)\}$ is an open subset of Y, one has $\{T^N(y)\}\cap\mathcal{I}(T,\psi) \neq \emptyset$, hence $T^N(y)$ belongs to $\mathcal{I}(T,\psi)$. Therefore, $T^N(y) \in \bigcap_{\psi\in\mathcal{D}(Y,T)} \mathcal{I}(T,\psi)$.

In fact, more is true: $y \in \bigcap_{\psi \in \mathcal{D}(Y,T)} \mathcal{I}(T,\psi)$. Indeed, suppose that there exists $\psi_0 \in D(Y,T)$ such that $y \in Y \setminus \mathcal{I}(T,\psi_0)$. Consider the aforementioned integer $N \in \mathbb{N} \cup \{0\}$ such that $T^N(y) \in \bigcap_{\psi \in \mathcal{D}(Y,T)} \mathcal{I}(T,\psi)$. As $Y \setminus I(T,\psi)$ is T-invariant and $y \in Y \setminus \mathcal{I}(T,\psi_0)$, we have $T^N(y) \in Y \setminus \mathcal{I}(T,\psi_0)$. This contradicts the choice of N. So, we have shown that $\mathrm{Trans}(Y,T) \subseteq \bigcap_{\psi \in \mathcal{D}(Y,T)} \mathcal{I}(T,\psi)$.

We are left to prove that, if (Y,d) has an isolated point and $\mathcal{D}(Y,T) \neq \emptyset$, then $\mathcal{R}(Y,T) = \mathcal{D}(Y,T)$. Suppose that $\mathcal{D}(Y,T) \neq \emptyset$. As we have just proved, $\mathrm{Trans}(Y,T) \subseteq \bigcap_{\psi \in \mathcal{D}(Y,T)} \mathcal{I}(T,\psi)$. This implies that for each $\psi \in \mathcal{D}(Y,T)$ one has $\mathcal{I}(T,\psi) \cap \mathrm{Trans}(Y,T) \neq \emptyset$. By Lemma 4.1 and Theorem 1.2 we conclude that $\mathcal{I}(T,\psi)$ is Baire generic, so $\psi \in \mathcal{R}(Y,T)$.

(ii) Suppose that $\bigcap_{\varphi \in \mathfrak{F}} \mathcal{I}(T,\varphi)$ is Baire generic. By hypothesis, there exists y in Y such that

$$\Omega_y = \overline{\{T^n(y) : n \in \mathbb{N} \cup \{0\}\}} = Y$$

so Y is separable. This implies that, if $\omega(y) = \Omega_y$, then Y does not have isolated points, and so T is transitive. Since Y is a Baire separable metric space, $\mathrm{Trans}(Y,T)$ is Baire generic as well (cf. [1, Proposition 4.7]). Therefore, $\mathrm{Trans}(Y,T) \cap \bigcap_{\varphi \in \mathfrak{F}} \mathcal{I}(T,\varphi)$ is also Baire generic. Thus $\mathrm{Trans}(Y,T) \cap \bigcap_{\varphi \in \mathfrak{F}} \mathcal{I}(T,\varphi)$ is not empty.

Assume now that $\omega(y) \subsetneq \Omega_y$. Then Y has an isolated point. As $\emptyset \neq \mathfrak{F} \subseteq \mathcal{D}(Y,T)$ and Y has an isolated point, by item (i) we know that $\mathrm{Trans}(Y,T) \subseteq \bigcap_{\varphi \in \mathcal{D}(Y,T)} \mathcal{I}(T,\varphi)$. Moreover, as $\mathfrak{F} \subseteq \mathcal{D}(Y,T)$, one has $\bigcap_{\varphi \in \mathcal{D}(Y,T)} \mathcal{I}(T,\varphi) \subseteq \bigcap_{\varphi \in \mathfrak{F}} \mathcal{I}(T,\varphi)$. Consequently,

$$\mathrm{Trans}(Y,T) \subseteq \bigcap_{\varphi \in \mathfrak{F}} \mathcal{I}(T,\varphi).$$

□

5. Proof of Theorem 1.4

We will adapt the proof of [20, Lemma 4.1]. Consider the sets

$$A_{\ell_\varphi^*} = \Big\{y \in Y\colon \liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j y) = \ell_\varphi^*\Big\}$$

$$A_{L_\varphi^*} = \{y \in Y\colon \limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j y) = L_\varphi^*\Big\}$$

and, for every $\alpha > \ell_\varphi^*$,

$$B_\alpha = \bigcap_{N=1}^{+\infty} \bigcup_{n=N}^{+\infty} \Big\{y \in Y\colon \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i y) < \alpha\Big\}.$$

As $\alpha > \ell_\varphi^*$, there exists $y_0 \in \mathrm{Trans}(Y,T)$ such that $y_0 \in B_\alpha$. Consequently, for every $N \in \mathbb{N}$ the set $\bigcup_{n=N}^{+\infty} \Big\{y \in Y\colon \frac{1}{n} \sum_{i=0}^{n-1} \varphi(T^i y) < \alpha\}$ is an open, dense subset of Y. Therefore, B_α is Baire generic in Y for any $\alpha > \ell_\varphi^*$.

Take a convergent sequence $\{\alpha_n\}_{n=1}^{+\infty}$ in $\mathbb{R}$ such that $\lim_{n\to+\infty}\alpha_n = \ell_\varphi^*$ and $\alpha_n > \ell_\varphi^*$ for every $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{+\infty} B_{\alpha_n} \subseteq A_{\ell_\varphi^*}$. This implies that $A_{\ell_\varphi^*}$ is Baire generic in Y. We deduce similarly that $A_{L_\varphi^*}$ is Baire generic in Y. Thus $A_{\ell_\varphi^*} \cap A_{L_\varphi^*} = I_\varphi[\ell_\varphi^*, L_\varphi^*]$ is Baire generic in Y.

We are now going to show that either $I_\varphi[\ell_\varphi^*, L_\varphi^*] \subseteq \mathcal{I}(T,\varphi)$ or $I_\varphi[\ell_\varphi^*, L_\varphi^*] \subseteq Y \setminus \mathcal{I}(T,\varphi)$. For any $y \in I_\varphi[\ell_\varphi^*, L_\varphi^*]$, we have

$$\liminf_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1}\varphi(T^j(y)) \,=\, \ell_\varphi^* \,\leqslant\, L_\varphi^* \,=\, \limsup_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(T^j(y)).$$

Therefore, if $\ell_\varphi^* < L_\varphi^*$, then $I_\varphi[\ell_\varphi^*, L_\varphi^*] \subseteq \mathcal{I}(T,\varphi)$ and $\mathcal{I}(T,\varphi)$ is Baire generic in Y. Otherwise, if $\ell_\varphi^* = L_\varphi^*$, then $I_\varphi[\ell_\varphi^*, L_\varphi^*] \subseteq Y \setminus \mathcal{I}(T,\varphi)$. These inclusions imply that $\mathcal{I}(T,\varphi)$ is either Baire generic or meagre, and characterize the case of a meagre $\mathcal{I}(T,\varphi)$. □

Remark 5.1. According to [30, Proposition 3.11], in the context of Baire ergodic maps T, for any Baire measurable function $\varphi : Y \to \mathbb{R}$ there exist a Baire generic subset $\mathfrak{R}$ of Y and constants c_-^φ and c_+^φ such that

$$\ell_\varphi(y) \,=\, c_-^\varphi \qquad \text{and} \qquad L_\varphi(y) \,=\, c_+^\varphi \qquad \forall\, y \in \mathfrak{R}.$$

6. Proof of Corollary 1.7

$(i) \Rightarrow (ii)$ By Remark 3.2 there exists $\eta > 0$ such that $\mathcal{D} = \{y \in Y : L_\varphi(y) - \ell_\varphi(y) \geq \eta\}$ is a Baire generic, hence dense, subset of Y. Note that, for any z and w in $\mathcal{D}$, one has either $|L_\varphi(z) - \ell_\varphi(w)| > \frac{\eta}{2}$ or $|L_\varphi(w) - \ell_\varphi(z)| > \frac{\eta}{2}$. This implies that Y is (T,φ)-sensitive, and so, by Theorem 1.2, the set $\mathcal{I}(T,\varphi)$ is Baire generic.

$(ii) \Rightarrow (iii)$ Apply item (ii) of Corollary 1.3 with $\mathfrak{F} = \{\varphi\}$.

$(iii) \Rightarrow (i)$ Assume that $\mathrm{Trans}(Y,T) \cap \mathcal{I}(T,\varphi) \neq \emptyset$. Then, there are y_0, $y_1 \in \mathrm{Trans}(Y,T)$ (possibly equal) and convergent subsequences $(\varphi_{n_k}(y_0))_{k\in\mathbb{N}}$ and $(\varphi_{m_k}(y_1))_{k\in\mathbb{N}}$ whose limits are distinct. Let A be the (dense) orbit of y_0, B the (dense) orbit of y_1 and ε equal to half the distance between the limits of the convergent subsequences $(\varphi_{n_k}(y_0))_{k\in\mathbb{N}}$ and $(\varphi_{m_k}(y_1))_{k\in\mathbb{N}}$. This is enough data to confirm that Y is (T,φ)-sensitive. □

7. Proof of Corollary 2.1

(a) Suppose that X is a compact metric space that has a dense orbit. From [36, Proposition 1 and Proposition 2] or [20, Corollary 2.2 and Proposition 3.1], we know that X_Δ is Baire generic in X.

We proceed by showing that $X_\Delta \subseteq \bigcap_{\varphi\in C(X,\mathbb{R})} I_\varphi\widehat{[\ell_\varphi^*, L_\varphi^*]}$. Take x in X_Δ and $\varphi \in C(X,\mathbb{R})$. We claim that $\ell_\varphi(x) \leqslant \ell_\varphi^*$ and $L_\varphi^* \leqslant L_\varphi(x)$. Assume, on the contrary, that $\ell_\varphi(x) > \ell_\varphi^*$. Then there exists a transitive point x_0 such that $\ell_\varphi(x) > \ell_\varphi(x_0)$. Therefore, there is a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\ell_\varphi(x) > \ell_\varphi(x_0) = \lim_{k\to+\infty}\frac{1}{n_k}\sum_{j=0}^{n_k-1}\varphi(T^j(x_0)).$$

Reducing to a subsequence if necessary, we find $\mu \in V_T(x_0)$ such that $\frac{1}{n_k}\sum_{j=0}^{n_k-1}\delta_{T^j(y)}$ converges to μ in the weak*-topology as k goes to $+\infty$, and so $\ell_\varphi(x) > \ell_\varphi(x_0) = \int \varphi\, d\mu$. Since, by hypothesis, μ belongs to $V_T(x)$, there exists an infinite sequence $(n_q)_{q\in\mathbb{N}}$ such that $\frac{1}{n_q}\sum_{j=0}^{n_q-1}\delta_{T^j(x)}$ converges to μ as q goes to $+\infty$. This yields to

$$\int \varphi\, d\mu = \lim_{q\to+\infty}\frac{1}{n_q}\sum_{j=0}^{n_q-1}\varphi(T^j(x)) \geqslant \ell_\varphi(x) > \ell_\varphi(x_0) = \int \varphi\, d\mu$$

which is a contradiction. Therefore, we must have $\ell_\varphi(x) \leqslant \ell^*_\varphi$.

We prove similarly that $L^*_\varphi \leqslant L_\varphi(x)$ for every $x \in X_\Delta$. Thus $x \in I_\varphi\widehat{[\ell^*_\varphi, L^*_\varphi]}$.

(b) We start by establishing the following auxiliary result.

Lemma 7.1. *Let (X,d) be a compact metric space, $T: X\to X$ be a continuous map with a dense orbit. If $\mathcal{R}(X,T)=\emptyset$, then $\bigcap_{\varphi\in C(X,\mathbb{R})} X\setminus\mathcal{I}(T,\varphi)$ is Baire generic in X.*

Proof. Since $\mathcal{R}(X,T)=\emptyset$, from Theorem 1.4 we know that $\mathcal{I}(T,\varphi)$ is meagre for every $\varphi\in C(X,\mathbb{R})$. Hence $X\setminus\mathcal{I}(T,\varphi)$ is Baire generic for all $\varphi\in C(X,\mathbb{R})$. Let S be a countable, dense subset of $C(X,\mathbb{R})$, which exists because X is compact (cf. [22]). Then $\bigcap_{\psi\in S} X\setminus\mathcal{I}(T,\psi)$ is Baire generic in X. We are left to prove that

$$\bigcap_{\psi\in S} X\setminus\mathcal{I}(T,\psi) \subseteq \bigcap_{\varphi\in C(X,\mathbb{R})} X\setminus\mathcal{I}(T,\varphi).$$

Take $x\in\bigcap_{\psi\in S} X\setminus\mathcal{I}(T,\psi)$ and $\varphi\in C(X,\mathbb{R})$; we need to show that x belongs to $X\setminus\mathcal{I}(T,\varphi)$. As S is dense, given $\varepsilon>0$ there exists $\psi\in S$ such that $\|\varphi-\psi\|_\infty<\frac{\varepsilon}{3}$. Since $x\in X\setminus\mathcal{I}(T,\psi)$, there is $N\in\mathbb{N}\cup\{0\}$ such that, for every $n,m\geqslant N$, we have $\|\psi_n(x)-\psi_m(x)\|<\frac{\varepsilon}{3}$. Therefore,

$$\|\varphi_n(x)-\varphi_m(x)\| \leqslant \|\varphi_n(x)-\psi_n(x)\| + \|\psi_n(x)-\psi_m(x)\| + \|\psi_m(x)-\varphi_m(x)\| < \varepsilon$$

so x is in $X\setminus\mathcal{I}(T,\varphi)$. □

Let us go back to the proof of item (b). We begin by showing that $\bigcup_{\varphi\in C(X,\mathbb{R})}\mathcal{I}(T,\varphi)$ is either Baire generic or meagre. Suppose that $\bigcup_{\varphi\in C(X,\mathbb{R})}\mathcal{I}(T,\varphi)$ is not Baire generic, so $\mathcal{R}(X,T)$ is empty. By Lemma 7.1, the set $\bigcap_{\varphi\in C(X,\mathbb{R})} X\setminus\mathcal{I}(T,\varphi)$ is Baire generic in X, and so $\bigcup_{\varphi\in C(X,\mathbb{R})}\mathcal{I}(T,\varphi)$ is a meagre set. Moreover, due to the compactness of X, Riesz representation theorem and Theorem 1.4, there exists a Borel invariant probability measure μ such that

$$\mathrm{Trans}(X,T) \subset \Big\{x\in X:\ \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(T^j(x)) = \int\varphi d\mu \qquad \forall\,\varphi\in C(X,\mathbb{R})\Big\}.$$

Hence, $X_\Delta=\{x\in X:\mu\in V_T(x)\}$ thus, by item (a), X_Δ is Baire generic in X. This implies that the set

$$X_\Delta\cap\bigcap_{\varphi\in C(X,\mathbb{R})} X\setminus\mathcal{I}(T,\varphi) = \Big\{x\in X:\ \lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(T^j(x)) = \int\varphi\, d\mu \quad \forall\,\varphi\in C(X,\mathbb{R})\Big\}$$

is Baire generic in X.

(c) Firstly, we note that, since Corollary 1.7 shows that

$$\mathcal{R}(X,T) \;=\; \{\varphi \in C(X,\mathbb{R})\colon\; \ell^*_\varphi < L^*_\varphi\}$$

then one has

$$\mathcal{H}(X,T) \;=\; \mathcal{R}(X,T) \cup \{\varphi \in C(X,\mathbb{R})\colon\; \mathcal{I}(T,\varphi) \neq \emptyset \;\text{ and }\; \ell^*_\varphi = L^*_\varphi\}.$$

Proposition 7.2. *Let (X,d) be a compact metric space and $T : X \to X$ be a continuous map with a dense orbit. The following statements are equivalent:*

- (i) $\#\Big(\bigcup_{t\,\in\,\mathrm{Trans}(X,T)} V_T(t)\Big) > 1.$
- (ii) $\mathcal{R}(X,T) \neq \emptyset$.
- (iii) $\mathcal{R}(X,T)$ *is open and dense in* $C(X,\mathbb{R})$.
- (iv) $X_\Delta \subseteq \bigcap_{\varphi\,\in\,\mathcal{R}(X,T)} \mathcal{I}(T,\varphi)$.
- (v) $\bigcap_{\varphi\,\in\,\mathcal{R}(X,T)} \mathcal{I}(T,\varphi)$ *is Baire generic.*
- (vi) $\mathrm{Trans}(X,T) \cap \bigcap_{\varphi\,\in\,\mathcal{R}(X,T)} \mathcal{I}(T,\varphi) \neq \emptyset$.
- (vii) $\bigcup_{\varphi\,\in\,C(X,\mathbb{R})} \mathcal{I}(T,\varphi)$ *is Baire generic.*

Proof. We will prove that $(i) \Leftrightarrow (ii)$, $(ii) \Rightarrow \cdots \Rightarrow (vii)$ and $(vii) \Rightarrow (ii)$.

$(i) \Rightarrow (ii)$ Suppose that $\#\left(\cup_{x\,\in\,\mathrm{Trans}(X,T)} V_T(x)\right) > 1$. Then there exist two distinct Borel probability measures μ, ν in $\cup_{x\,\in\,\mathrm{Trans}(X,T)} V_T(x)$ and $\varphi \in C(X,\mathbb{R})$ such that $\int \varphi\, d\mu \neq \int \varphi\, d\nu$. This implies that $\{\mu,\nu\} \subseteq V_T(x)$ for all $x \in X_\Delta$, and so $X_\Delta \subseteq \mathcal{I}(T,\varphi)$. Therefore, $\mathcal{I}(T,\varphi)$ is Baire generic since, by item (a), the set X_Δ is Baire generic.

$(ii) \Rightarrow (i)$ Suppose that that there exists $\varphi \in C(X,\mathbb{R})$ such that $\mathcal{I}(T,\varphi)$ is Baire generic. By Corollary 1.7, one has $\mathrm{Trans}(X,T) \cap \mathcal{I}(T,\varphi) \neq \emptyset$. Given $y \in \mathrm{Trans}(X,T) \cap \mathcal{I}(T,\varphi)$, one has $\ell_\varphi(y) < L_\varphi(y)$, and so $\#V_T(y) > 1$.

$(ii) \Rightarrow (iii)$ For every $x \in X$, denote by $\mathcal{U}_x$ the set $\{\varphi \in C(X,\mathbb{R}) : x \in \mathcal{I}(T,\varphi)\}$. If $\mathcal{U}_x \neq \emptyset$, then $\mathcal{U}_x$ is an open dense subset of $C(X,\mathbb{R})$ (see [20, Lemma 3.2]) and, by Corollary 1.7, $\mathcal{R}(X,T) = \bigcup_{x\,\in\,\mathrm{Trans}(X,T)} \mathcal{U}_x$. Therefore, if $\mathcal{R}(X,T) \neq \emptyset$, then $\mathcal{R}(X,T)$ is open and dense in $C(X,\mathbb{R})$.

$(iii) \Rightarrow (iv)$ Suppose that $\mathcal{R}(X,T)$ is open and dense in $C(X,\mathbb{R})$. In particular, $\mathcal{R}(X,T)$ is not empty. From Corollary 1.7, $\mathcal{R}(X,T) = \{\varphi \in C(X,\mathbb{R})\colon\; \ell^*_\varphi < L^*_\varphi\}$. Using item (a), we know that $X_\Delta \subseteq \bigcap_{\varphi\,\in\,C(X,\mathbb{R})} I_\varphi\widehat{[\ell^*_\varphi, L^*_\varphi]}$. Moreover,

$$X_\Delta \subseteq \bigcap_{\varphi\,\in\,C(X,\mathbb{R})} I_\varphi\widehat{[\ell^*_\varphi, L^*_\varphi]} = \Bigg(\bigcap_{\ell^*_\varphi < L^*_\varphi} I_\varphi\widehat{[\ell^*_\varphi, L^*_\varphi]}\Bigg) \cap \Bigg(\bigcap_{\ell^*_\varphi = L^*_\varphi} I_\varphi\widehat{[\ell^*_\varphi, L^*_\varphi]}\Bigg)$$

and

$$X_\Delta \subseteq \bigcap_{\ell^*_\varphi < L^*_\varphi} I_\varphi\widehat{[\ell^*_\varphi, L^*_\varphi]}.$$

Consequently,

$$X_\Delta \subseteq \bigcap_{\ell^*_\varphi < L^*_\varphi} \mathcal{I}(T,\varphi) = \bigcap_{\varphi \in \mathcal{R}(X,T)} \mathcal{I}(T,\varphi).$$

$(iv) \Rightarrow (v)$ Suppose that $X_\Delta \subseteq \bigcap_{\varphi \in \mathcal{R}(X,T)} \mathcal{I}(T,\varphi)$. By item (a), X_Δ is Baire generic; hence $\bigcap_{\varphi \in \mathcal{R}(X,T)} \mathcal{I}(T,\varphi)$ is Baire generic in X.

$(v) \Rightarrow (vi)$ Suppose that $\bigcap_{\varphi \in \mathcal{R}(X,T)} \mathcal{I}(T,\varphi)$ is Baire generic. By item (ii) of Corollary 1.3, we know that $\mathrm{Trans}(X,T) \cap \bigcap_{\varphi \in \mathcal{R}(X,T)} \mathcal{I}(T,\varphi) \neq \emptyset$.

$(vi) \Rightarrow (vii)$ This is clear from Corollary 1.7.

$(vii) \Rightarrow (ii)$ Suppose that $\bigcup_{\varphi \in C(X,\mathbb{R})} \mathcal{I}(T,\varphi)$ is Baire generic. If $\mathcal{R}(X,T) = \emptyset$, using Lemma 7.1 one deduces that $\bigcap_{\varphi \in C(X,\mathbb{R})} X \setminus \mathcal{I}(T,\varphi)$ is Baire generic. Then

$$\emptyset = \Big(\bigcup_{\varphi \in C(X,\mathbb{R})} \mathcal{I}(T,\varphi) \Big) \cap \Big(\bigcap_{\varphi \in C(X,\mathbb{R})} X \setminus \mathcal{I}(T,\varphi) \Big)$$

is Baire generic as well, so it is not empty. This contradiction ensures that $\mathcal{R}(X,T) \neq \emptyset$.

□

We now resume the proof of item (c). Suppose that $\mathcal{H}(X,T)$ is not empty. If $\{\varphi \in C(X,\mathbb{R}) : \mathcal{I}(T,\varphi) \neq \emptyset \text{ and } \ell^*_\varphi = L^*_\varphi\}$ is empty, then $\mathcal{H}(X,T) = \mathcal{R}(X,T)$ is not empty. Therefore, by Proposition 7.2, we conclude that $\bigcap_{\varphi \in \mathcal{H}(X,T)} \mathcal{I}(T,\varphi)$ is Baire generic.

Assume, otherwise, that $\{\varphi \in C(X,\mathbb{R}) : \mathcal{I}(T,\varphi) \neq \emptyset \text{ and } \ell^*_\varphi = L^*_\varphi\}$ is not empty. Then there exists $\varphi \in C(X,\mathbb{R})$ such that $\mathcal{I}(T,\varphi) \neq \emptyset$ and $\ell^*_\varphi = L^*_\varphi$.

Claim: *The set $\mathcal{I}(T,\varphi)$ is meagre.*

In fact, suppose that $\mathcal{I}(T,\varphi) \neq \emptyset$ and $\ell^*_\varphi = L^*_\varphi$. Denote by γ this common value. Then by item (i) of Theorem 1.4, the set $I_\varphi[\gamma,\gamma]$ is Baire generic in Y. Moreover, we have $\mathrm{Trans}(X,T) \subseteq I_\varphi[\gamma,\gamma]$. Therefore, by item (ii) of Theorem 1.4, $\mathcal{I}(T,\varphi)$ is meagre. □

As $\bigcap_{\varphi \in \mathcal{H}(X,T)} \mathcal{I}(T,\varphi) \subseteq \mathcal{I}(T,\varphi)$, we conclude from the previous Claim that the set $\bigcap_{\varphi \in \mathcal{H}(X,T)} \mathcal{I}(T,\varphi)$ is meagre as well. So, we have shown that $\bigcap_{\varphi \in \mathcal{H}(X,T)} \mathcal{I}(T,\varphi)$ is either Baire generic or meagre.

Actually, we have proved more: the set $\bigcap_{\varphi \in \mathcal{H}(X,T)} \mathcal{I}(T,\varphi)$ is Baire generic if and only if $\{\varphi \in C(X,\mathbb{R}) \colon \mathcal{I}(T,\varphi) \neq \emptyset \text{ and } \ell^*_\varphi = L^*_\varphi\}$ is empty. □

As a consequence of the Corollary 2.1 we get the following dichotomy between uniquely ergodicity and Baire genericity of historic behavior.

Scholium 7.3. *If (X,d) is a compact metric space and $T : X \to X$ is a continuous minimal map, then either T is uniquely ergodic or the set $\bigcap_{\varphi \in \mathcal{H}(X,T)} \mathcal{I}(T,\varphi)$ is Baire generic.*

Proof. Let $T : X \to X$ be a continuous minimal map which is not uniquely ergodic. Suppose, by contradiction, that there is φ in $C(X,\mathbb{R})$ such that $\mathcal{I}(T,\varphi) \neq \emptyset$ and $\ell^*_\varphi = L^*_\varphi$. Then there exists $x_0 \in X$ such that $\ell_\varphi(x_0) < L_\varphi(x_0)$. Since T is minimal, one has $\mathrm{Trans}(X,T) = X$; in particular, the orbit of x_0 is dense in X. Consequently,

$$\ell^*_\varphi \leqslant \ell_\varphi(x_0) < L_\varphi(x_0) \leqslant L^*_\varphi.$$

But this contradicts the assumption $\ell^*_\varphi = L^*_\varphi$. Thus

$$\{\varphi \in C(X,\mathbb{R})\colon \mathcal{I}(T,\varphi) \neq \emptyset \text{ and } \ell^*_\varphi = L^*_\varphi\} = \emptyset$$

and therefore, by item (c) of Corollary 2.1, the set $\bigcap_{\varphi\in\mathcal{H}(X,T)} \mathcal{I}(T,\varphi)$ is Baire generic. $\square$

8. Proof of Theorem 2.2

Let (X,d) be a compact metric space and $\varphi \in C(X,\mathbb{R})$ such that X is (T,φ)-sensitive. Recall that this means that there exist dense sets $A, B \subset X$ and $\varepsilon > 0$ such that, for any pair $(a,b) \in A\times B$ there is $(r_a, r_b) \in \{\varphi_n(a) : n \in \mathbb{N}\}' \times \{\varphi_n(b) : n \in \mathbb{N}\}'$ satisfying $|r_a - r_b| > \varepsilon$. Using the uniform continuity of φ, one can choose $\delta > 0$ such that $|\varphi(z) - \varphi(w)| < \frac{\varepsilon}{2}$ for every $z, w \in X$ with $d(z,w) < \delta$.

Assume, by contradiction, that $\mathcal{I}(T,\varphi)$ has empty interior and T has no sensitivity to initial conditions. The latter condition implies that for each $\theta > 0$ there exist $x_\theta \in X$ and an open neighborhood U_{x_θ} of x_θ such that $d(T^n(x_\theta), T^n(x)) \leqslant \theta$ for every $x \in U_{x_\theta}$ and $n \in \mathbb{N}\cup\{0\}$. Choose $\theta = \frac{\delta}{3}$ and take x_θ and U_{x_θ} as above.

As $\mathcal{I}(T,\varphi)$ has empty interior, the set $X \setminus \mathcal{I}(T,\varphi)$ is dense in X. Therefore, there are $a \in A\cap U_{x_\theta}$, $b \in B \cap U_{x_\theta}$ and a φ-regular point $c \in U_{x_\theta}$ satisfying

$$|\varphi(T^n(a)) - \varphi(T^n(c))| < \tfrac{\varepsilon}{2} \quad \text{and} \quad |\varphi(T^m(b)) - \varphi(T^m(c))| < \tfrac{\varepsilon}{2} \qquad \forall\, n,m \in \mathbb{N}\cup\{0\}.$$

Consequently, for every $n, m \in \mathbb{N}\cup\{0\}$, one has

$$\Big|\tfrac{1}{n}\sum_{j=0}^{n-1}\varphi(T^j(a)) - \tfrac{1}{n}\sum_{j=0}^{n-1}\varphi(T^j(c))\Big| < \tfrac{\varepsilon}{2} \quad \text{and} \quad \Big|\tfrac{1}{m}\sum_{j=0}^{m-1}\varphi(T^j(b)) - \tfrac{1}{m}\sum_{j=0}^{m-1}\varphi(T^j(c))\Big| < \tfrac{\varepsilon}{2}.$$

Taking the limit as n goes to $+\infty$ in the first inequality along a subsequence $(n_k)_{k\in\mathbb{N}}$ converging to r_a and taking the limit as m tends to $+\infty$ in the second inequality along a subsequence $(m_k)_{k\in\mathbb{N}}$ convergent to r_b, and using that c is a φ-regular point, we obtain

$$\Big|r_a - \lim_{k\to+\infty}\frac{1}{n_k}\sum_{j=0}^{n_k-1}\varphi(T^j(c))\Big| \leqslant \frac{\varepsilon}{2} \quad \text{and} \quad \Big|r_b - \lim_{k\to\infty}\frac{1}{m_k}\sum_{j=0}^{m_k-1}\varphi(T^j(c))\Big| \leqslant \frac{\varepsilon}{2}$$

and so $|r_a - r_b| \leqslant \varepsilon$. We have reached a contradiction, thus proving that either T has sensitivity on initial conditions or the irregular set $\mathcal{I}(T,\varphi)$ has non-empty interior, as claimed.

Assume now that X is (T,φ)-sensitive and T has a dense set of periodic orbits. As the periodic points of T are φ-regular and form a dense subset of X, the set $\mathcal{I}(T,\varphi)$ must have empty interior. Thus, by the previous statement, T has sensitivity on initial conditions. $\square$

9. Proof of Corollary 2.4

Let $T : X \to X$ be a strongly transitive continuous endomorphism of a compact metric space X. Given $\varphi \in C(X,\mathbb{R})$ satisfying $\mathcal{I}(T,\varphi) \neq \emptyset$, let us show that $\mathcal{I}(T,\varphi)$ is a Baire generic subset of X. Fix $x_0 \in \mathcal{I}(T,\varphi)$ and let $\varepsilon = L_\varphi(x_0) - \ell_\varphi(x_0) > 0$. The strong transitivity assumption ensures that, for every non-empty open subset U of X, there is $N \in \mathbb{N} \cup \{0\}$ such that $x_0 \in T^N(U)$. Thus, the pre-orbit $\mathcal{O}^-_T(x_0) = \big\{x \in X\colon\, T^n(x) = x_0 \text{ for some } n \in \mathbb{N}\big\}$ of x_0 is dense in X. Moreover,

$$\sup_{r,\, s \,\in\, W_\varphi(x)} |r - s| > \varepsilon \qquad \forall\, x \in \mathcal{O}^-_T(x_0).$$

This proves that X is (T,φ)-sensitive and so, by Theorem 1.2, the set $\mathcal{I}(T,\varphi)$ is Baire generic in X. □

10. Examples

In this section we discuss the hypothesis and derive some consequences of the main results. The first example helps to clarify the requirements in Theorem 1.2 and Corollary 1.3, besides calling our attention to the difference between transitivity and the existence of a dense orbit.

Example 1. Consider the space $X = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ endowed with the Euclidean metric. Let $T : X \to X$ be the continuous map given by $T(0) = 0$ and $T(1/n) = 1/(n+1)$ for every $n \in \mathbb{N}$. Notice that X has isolated points and T has a dense orbit, though $\text{Trans}(X,T) = \{1\}$. However, T is not transitive. Moreover, $\mathcal{I}(T,\varphi) = \emptyset$ for every $\varphi \in C(X,\mathbb{R})$.

The second example illustrates Definition 1.

Example 2. Consider the shift space $X = \{0,1\}^{\mathbb{N}}$ endowed with the metric defined by

$$d((a_n)_{n\,\in\,\mathbb{N}},\, (b_n)_{n\,\in\,\mathbb{N}}) \;=\; \sum_{n=1}^{+\infty} \frac{|a_n - b_n|}{2^n}$$

and take the shift map $\sigma : X \to X$ given by $\sigma(\,(a_n)_{n\,\in\mathbb{N}}\,) = (a_{n+1})_{n\,\in\mathbb{N}}$. Then the sets

$$\begin{aligned} A &= \big\{(a_n)_{n\,\in\,\mathbb{N}}\,|\;\exists\, N \in\, \mathbb{N}\colon\; a_n = 0 \quad \forall\, n \geqslant N\big\} \\ B &= \big\{(a_n)_{n\,\in\,\mathbb{N}}\,|\;\exists\, N \in\, \mathbb{N}\colon\; a_n = 1 \quad \forall\, n \geqslant N\big\} \end{aligned}$$

are the stable sets of the fixed points $\overline{0}$ and $\overline{1}$ of σ, and are dense subsets of X. Besides, if $\varphi \in C(X,\mathbb{R})$, then for every $a \in A$ and $b \in B$ one has

$$\lim_{n\,\to\,+\infty} \varphi(\sigma^n(a)) = \varphi(\overline{0}) \qquad \text{and} \qquad \lim_{n\,\to\,+\infty} \varphi(\sigma^n(a)) = \varphi(\overline{1}).$$

Therefore,

$$\lim_{n\,\to\,+\infty} \frac{1}{n} \sum_{j=1}^{n-1} \varphi(\sigma^j(a)) = \varphi(\overline{0}) \qquad \text{and} \qquad \lim_{n\,\to\,+\infty} \frac{1}{n} \sum_{j=1}^{n-1} \varphi(\sigma^j(b)) = \varphi(\overline{1}).$$

And so, if $\varphi(\overline{0}) \neq \varphi(\overline{1})$ and one chooses $\varepsilon = |\varphi(\overline{0}) - \varphi(\overline{1})|/2$, then we conclude that for every $(a,b) \in A \times B$ there are $r_a \in W_\varphi(a)$ and $r_b \in W_\varphi(b)$ such that $|r_a - r_b| > \varepsilon$. Consequently, X is (σ,φ)-sensitive with respect to any $\varphi \in C(X,\mathbb{R})$ satisfying $\varphi(\overline{0}) \neq \varphi(\overline{1})$. So, by Theorem 1.2, for every such maps φ the set $\mathcal{I}(T,\varphi)$ is Baire generic in X.

Remark 10.1. We note that a similar reasoning shows that, if $T : Y \to Y$ is a continuous map acting on a Baire metric space such that T has two periodic points with dense pre-orbits, then there exists $\varphi \in C^b(Y,\mathbb{R})$ whose set $\mathcal{I}(T,\varphi)$ is Baire generic in Y.

In the following example we will address the irregular set in the context of countable Markov shifts. These symbolic systems appear naturally as models for non-uniformly hyperbolic dynamical systems on compact manifolds (see [29] and references therein), hyperbolic systems with singularities [10], including Sinai dispersing billiards, and certain classes of piecewise monotone interval maps [19], which encompass the piecewise expanding Lorenz interval maps, just to mention a few.

Example 3. Let $\mathcal{A}$ be a countable set, $\mathbb{A} = (a_{i,j})_{i,j\,\in\mathcal{A}}$ be a matrix of zeroes and ones and $\Sigma_{\mathbb{A}} \subset \mathcal{A}^{\mathbb{N}}$ be the subset

$$\Sigma_{\mathbb{A}} = \big\{(x_n)_{n\,\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}} \colon\; a_{x_n, x_{n+1}} = 1 \quad \forall\, n \in \mathbb{N}\big\}.$$

Endow $\Sigma_{\mathbb{A}}$ with the metric

$$d((x_n)_{n\,\in\mathbb{N}},\,(y_n)_{n\,\in\mathbb{N}}) = \begin{cases} 2^{-\min\{k\,\in\mathbb{N}\colon\, x_k \neq y_k\}} & \text{if } \{k\,\in\mathbb{N}\colon\; x_k \neq y_k\} \neq \emptyset \\ 0 & \text{otherwise.}\end{cases}$$

We note that $a_{i,j} = 0$ for all but finitely many values of $(i,j) \in \mathcal{A}\times\mathcal{A}$ if and only if $\Sigma_{\mathbb{A}}$ is a compact metric space. Besides, the metric space $(\Sigma_{\mathbb{A}}, d)$ has a countable basis for the topology, generated by the countably many cylinders, and it is invariant by the shift map $\sigma\colon \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$.

If $\sigma_{|\Sigma_{\mathbb{A}}}$ has the periodic specification property (see examples in [32]), then the set of periodic points of $\sigma_{|\Sigma_{\mathbb{A}}}$ is dense in $\Sigma_{\mathbb{A}}$ and all the points in $\Sigma_{\mathbb{A}}$ have dense pre-orbits. Therefore, there exists $\varphi \in C^b(\Sigma_{\mathbb{A}}, \mathbb{R})$ such that $\Sigma_{\mathbb{A}}$ is (σ,φ)-sensitive, whose $\mathcal{I}(\sigma,\varphi)$ is Baire generic by Theorem 1.2. Indeed, for each $\varphi \in C^b(\Sigma_{\mathbb{A}}, \mathbb{R})$,

(a) either there exists a constant $c_\varphi \in \mathbb{R}$ such that, for every periodic point p,

$$\frac{1}{\pi(p)} \sum_{j=0}^{\pi(p)-1} \varphi(\sigma^j(p)) = c_\varphi$$

where $\pi(p) \in \mathbb{N}$ denotes the minimal period of p;

(b) or there are two periodic points $p,\, q \in \Sigma_{\mathbb{A}}$ such that

$$\frac{1}{\pi(p)} \sum_{j=0}^{\pi(p)-1} \varphi(\sigma^j(p)) \neq \frac{1}{\pi(q)} \sum_{j=0}^{\pi(q)-1} \varphi(\sigma^j(q))$$

and so $\Sigma_{\mathbb{A}}$ is (σ,φ)-sensitive, since the pre-orbits of p and q are dense and therefore provide two sets A and B complying with the Definition 1 (see Remark 10.1).

The next example shows that the existence of a discontinuous first integral L_φ with two dense level sets for a continuous map $T : Y \to Y$ acting on a metric space Y may be indeed stronger than requiring (T,φ)-sensitivity.

Example 4. Let $(\Psi_t)_{t\,\in\mathbb{R}}$ be the Bowen's example, that is, a smooth Morse-Smale flow on $\mathbb{S}^2$ with hyperbolic singularities $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, S, N\}$ and displaying an attracting union of four separatrices, as illustrated in Figure 1. More precisely, there exist four separatrices $\gamma_1, \gamma_2, \gamma_3$ and γ_4 associated to hyperbolic singularities σ_1, σ_2 of saddle type, while all the other singularities are repellers. Let $\varphi : \mathbb{S}^2 \to \mathbb{R}$ be a continuous observable satisfying $\varphi(x) \in [0,1]$ for every $x \in \mathbb{S}^2$, $\varphi(\sigma_1) = 1$ and $\varphi(\sigma_2) = 0$. Consider the time-one map $T : Y \to Y$ of the flow

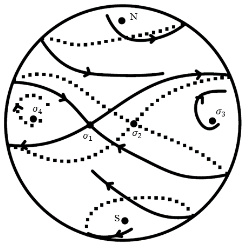

FIGURE 1. Flow with historic behavior.

$(\Psi_t)_{t\in\mathbb{R}}$. As the orbit by T of every point in $Y = \mathbb{S}^2 \setminus \big(\bigcup_{i=1}^4 \gamma_i\big)$ accumulate on the closure of the union of the separatrices, it is immediate that the first integral L_φ, defined by (1.1), is everywhere constant in Y: $L_\varphi(y) = 1$ for every $y \in Y$. However, as $W_\varphi(y) = [0,1]$ for every $y \in Y$, the space Y is (T,φ)-sensitive.

Our final example in this section concerns a skew-product admitting two invariant probability measures whose basins of attraction have positive Lebesgue measure and are dense.

Example 5. Consider the annulus $\mathbb{A} = \mathbb{S}^1 \times [0,1]$ and the map $T : \mathbb{A} \to \mathbb{A}$ given by

$$T(x,t) \;=\; \Big(3x \,(\text{mod } 1),\, t + \frac{t(1-t)}{32}\cos(2\pi x)\Big) \qquad \forall\,(x,t) \in \mathbb{S}^1 \times [0,1].$$

In [21], Kan proved that T admits two physical measures (that is, T-invariant probability measures whose basins of attraction have positive Lebesgue measure), namely $\mu_0 = \text{Leb}_{\mathbb{S}^1} \times \delta_0$ and $\mu_1 = \text{Leb}_{\mathbb{S}^1} \times \delta_1$, whose basins of attraction $B(\mu_0)$ and $B(\mu_1)$ are intermingled, that is, for every non-empty open set $\mathcal{U} \subset \mathbb{A}$

$$\text{Leb}_{\mathbb{A}}\big(\mathcal{U} \cap B(\mu_0)\big) > 0 \quad \text{ and } \quad \text{Leb}_{\mathbb{A}}\big(\mathcal{U} \cap B(\mu_1)\big) > 0$$

where

$$B(\mu) \;=\; \Big\{x \in \mathbb{A}\colon\; \Big(\frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j(y))\Big)_{n\in\mathbb{N}} \;\text{ converges to } \int \varphi\, d\mu, \quad \forall\,\varphi \in C(\mathbb{A},\mathbb{R})\Big\}.$$

Later, Bonatti, Díaz and Viana introduced in [6] the concept of Kan-like map and proved that any such map robustly admits two physical measures. More recently, Gan and Shi [18] showed that, in the space of C^2 diffeomorphisms of $\mathbb{A}$ preserving the boundary, every C^2 Kan-like map T_0 admits a C^2-open neighborhood $\mathscr{V}$ such that the following holds: for each $T \in \mathscr{V}$ and every non-empty open set $\mathcal{U} \subset \mathbb{A}$,

$$\text{the interior of } \mathbb{A} \subset \bigcup_{n\geqslant 0} T^n(\mathcal{U}).$$

Using Theorem 1.2 we conclude that, if T is a Kan-like map and $\varphi \in C(\mathbb{A},\mathbb{R})$ satisfies the inequality $\int \varphi\, d\mu_0 \neq \int \varphi\, d\mu_1$, then $\mathcal{I}(T,\varphi)$ is a Baire generic subset of $\mathbb{A}$. More generally, the argument that established Corollary 2.4 also ensures that, for any $\varphi \in C(\mathbb{A},\mathbb{R})$, one has

(a) either $\mathcal{I}(T,\varphi) \cap \text{interior of } \mathbb{A} = \emptyset$

(b) or $\mathcal{I}(T,\varphi)$ is a Baire generic subset of $\mathbb{A}$.

In particular, when $\mathcal{I}(T,\varphi)$ is dense then it is Baire generic; thus $\mathcal{D}(\mathbb{A},T)=\mathcal{R}(\mathbb{A},T)$.

11. Applications

As it will become clear in the remainder of this section, Theorem 1.2 has a wide range of applications according to the class of sequences $\Phi=(\varphi_n)_{n\geq 1}$ of observables one considers. Let us provide two such applications, one with a geometric motivation and another in the context of semigroup actions.

Application 1. The irregular sets of uniformly hyperbolic maps and flows on compact Riemannian manifolds have been extensively studied. One of the reasons for this success is that these dynamical systems can be modeled by symbolic dynamical systems which satisfy the so-called specification property. Irregular sets for continuous maps acting on compact metric spaces and satisfying the specification property have been studied in [23]. Many difficulties arise, though, if one drops the compactness assumption. An important example of a hyperbolic dynamical system with non-compact phase space is given by the geodesic flow on a complete connected negatively curved manifold. The next example applies Theorem 1.2 precisely to this setting.

Example 6. Let (M,g) be a connected, complete Riemannian manifold. We will discuss the Baire genericity of points with historic behavior in the following two cases (we refer the reader to [12] for precise definitions and more information):

- (I) (M,g) is negatively curved and the non-wandering set of the geodesic flow $(\Psi_t^g)_{t\,\in\,\mathbb{R}}$ contains more than two hyperbolic periodic orbits.
- (II) (M,g) has non-positive curvature, its universal curvature has no flat strips and the geodesic flow $(\Psi_t^g)_{t\,\in\,\mathbb{R}}$ has at least three periodic orbits.

Regarding (I), by [12, Theorem 1.1] it is known that the space $\mathcal{E}$ of Borel ergodic probability measures fully supported on the non-wandering set Ω (that is, every point in Ω belongs to their support) is a G_δ dense subset of all Borel probability measures on T^1M which are invariant by the geodesic flow. In particular, as $\#\mathcal{E}\geqslant 2$ due to the assumption that the geodesic flow has at least two distinct periodic orbits, one can choose a continuous observable $\varphi: T^1M\to\mathbb{R}$ such that

$$\inf_{\mu\,\in\,\mathcal{E}}\int\varphi\,d\mu<\sup_{\mu\,\in\,\mathcal{E}}\int\varphi\,d\mu.$$

As the ergodic basins of attraction of the probability measures in $\mathcal{E}$ are dense in Ω, one concludes that

$$L_\varphi(\cdot)=\limsup_{t\,\to\,+\infty}\,\frac{1}{t}\int_0^t\varphi(\Psi_s^g(\cdot))\,ds$$

is a first integral for the geodesic flow $(\Psi_t^g)_{t\,\in\,\mathbb{R}}$. Moreover, there are subsets $A,B\subset T^1M$ which are dense in Ω and whose L_φ value is constant, though the value in A is different from the one in B. The existence of A and B means that Ω is (T,Φ)-sensitive, where $T=\Psi_1^g$ is the time-1 map of the geodesic flow and Φ is defined by $\Phi=\int_0^1(\varphi\circ\Psi_s^g)\,ds$. Then Theorem 1.2 ensures that $\mathcal{I}(T,\Phi)$ is Baire generic. Consequently,

$$\mathcal{I}((\Psi_t^g)_{t\,\in\,\mathbb{R}},\,\varphi)=\Big\{y\in\Omega:\lim_{t\,\to\,+\infty}\,\frac{1}{t}\int_0^t\varphi(\Psi_s^g(y))\,ds\quad\text{does not exist}\Big\}$$

is a Baire generic subset of Ω as well.

The previous argument can be easily adapted to the case (II) to yield the conclusion that $I((\Psi^g_t)_{t\in\mathbb{R}},\varphi)$ is Baire generic since, according to [13, Theorem 1.1], the space $\mathcal{E}$ of Borel ergodic probability measures with full support on the non-wandering set Ω also form a G_δ dense set.

Application 2. Recall that a locally compact group G is *amenable* if for every compact set $K \subset G$ and $\delta > 0$ there is a compact set $F \subset G$, called (K,δ)-invariant, such that $m(F\Delta KF) < \delta\, m(F)$, where m denotes the counting measure on G if G is discrete, and stands for the Haar measure in G otherwise. We refer the reader to [28] for alternative formulations of this concept. A sequence $(F_n)_n$ of compact subsets of G is a *Følner sequence* if, for every compact $K \subset G$ and every $\delta > 0$, the set F_n is (K,δ)-invariant for every sufficiently large $n \in \mathbb{N}$ (whose estimate depends on K). A Følner sequence $(F_n)_n$ is *tempered* if there exists $C > 0$ such that

$$m\Big(\bigcup_{1\leqslant k<n} F_k^{-1}\,F_n\Big) \leqslant C\,m\big(\,F_n\,\big) \qquad \forall\, n\in\mathbb{N}.$$

It is known that every Følner sequence has a tempered subsequence and that every amenable group has a tempered Følner sequence (cf. [25, Proposition 1.4]). Furthermore, if G is an amenable group acting on a probability space (X,μ) by measure preserving maps and $(F_n)_n$ is a tempered Følner sequence, then for every $\varphi \in L^1(\mu)$ the limit

$$\lim_{n\to+\infty} \frac{1}{m(F_n)} \int_{F_n} \varphi(g(x))\,dm(g)$$

exists for μ-almost every $x \in X$; if, in addition, the G-action is ergodic, the previous limit is μ-almost everywhere constant and coincides with $\int \varphi\,d\mu$ (cf. [25, Theorem 1.2]).

Let (X,d) be a compact metric space and G be a group. We say that a Borel probability measure μ on X is G-*invariant* (or invariant by the action $\Gamma : G\times X \to X$ of G on X) if $\mu(g^{-1}(A)) = \mu(A)$ for every measurable set A and every $g \in G$. We denote the space of G-invariant probability measures by $\mathcal{M}_G(X)$, whose subset of ergodic elements is $\mathcal{E}_G(X)$. A group action of G on X is said to be *uniquely ergodic* if it admits a unique G-invariant ergodic probability measure (a property equivalent to the existence of a unique G-invariant probability measure if G is a countable amenable group, due to the ergodic decomposition; see [27]). Given $\mu \in \mathcal{M}_G(X)$, the basin of attraction of μ is defined by

$$B(\mu) = \Big\{x \in X\colon\ \frac{1}{|F_n|}\sum_{g\,\in\,F_n} \delta_{g(x)} \to \mu\ \ \text{(convergence in the weak}^*\text{ topology)}\Big\}$$

where $\delta_{g(x)}$ stands for the Dirac probability measure supported on the point $g(x)$.

Consider $\varphi \in C(X,\mathbb{R})$ and the sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ of continuous bounded maps defined by

$$\varphi_n(x) = \frac{1}{m(F_n)}\int_{F_n} \varphi(g(x))\,dm(g). \tag{11.1}$$

Clearly, $\|\varphi_n\|_\infty \leqslant \|\varphi\|_\infty$ for every $n \in \mathbb{N}$. Moreover, if we assume that there are fully supported G-invariant ergodic Borel probability measures $\mu_1 \neq \mu_2$, then there is $\varphi \in C(X,\mathbb{R})$ such that X is Φ-sensitive, since the basins of attraction of μ_1 and μ_2 are disjoint and both dense in X. Thus, under these assumptions, Theorem 1.2 ensures that, for the map φ, the irregular set

$$\Big\{x \in X\colon \lim_{n\to+\infty} \frac{1}{m(F_n)} \int_{F_n} \varphi(g(x))\, dm(g) \ \text{ does not exist}\Big\}$$

is a Baire generic subset of X.

In what follows, we will introduce a requirement weaker than the previous one, which is satisfied by countable amenable group actions with the specification property and also ensures that the irregular set of some potential is Baire generic. Afterwards, we will check it on an example (cf. Example 7).

Definition 4. Following [11, 31], we say that a continuous group action $\Gamma : G \times X \to X$ has the *specification property* if for every $\varepsilon > 0$ there exists a finite set $K_\varepsilon \subset G$ (depending on ε) such that: for any finite sample of points $x_0, x_1, x_2, \ldots, x_\kappa$ in X and any collection of finite subsets $\hat{F}_0, \hat{F}_1, \hat{F}_2, \ldots, \hat{F}_\kappa$ of G satisfying the condition

$$K_\varepsilon \hat{F}_i \cap \hat{F}_j = \emptyset \qquad \text{for every distinct } \ 0 \leqslant i, j \leqslant \kappa \tag{11.2}$$

there exists a point $x \in X$ such that

$$d(g(x),\, g(x_i)) < \varepsilon \qquad \text{for every } \ g \in \bigcup_{0 \leqslant j \leqslant \kappa} \hat{F}_j. \tag{11.3}$$

In rough terms, the previous property asserts that any finite collection of pieces of orbits can be shadowed by a true orbit provided that there is no overlap of the (translated) group elements that parameterize the orbits. We note that, if G is generated by a single map g, then Definition 4 coincides with the classical notion of specification for g (cf. [33]).

It is known that, if X is a compact metric space and $T\colon X \to X$ is a continuous map with the specification property, then the basin of attraction of any T-invariant ergodic probability measure is dense in X (see [33, Proof of Theorem 4]). To extend this information to countable amenable group actions with the specification property (and the counting measure m, which we denote by $|\cdot|$), consider the *generalized basin of attraction* of any G-invariant ergodic probability measure μ, defined by

$$C(\mu) = \Big\{x \in X\colon \ \mu \in V(x)\Big\}$$

where $V(x)$ denotes the set of accumulation points, in the weak*-topology, of the sequence

$$\Big(\frac{1}{|F_n|} \sum_{g \in F_n} \delta_{g(x)}\Big)_{n \in \mathbb{N}}.$$

Lemma 11.1. *Let G be a countable amenable group, $(F_n)_{n\in\mathbb{N}}$ be a tempered Følner sequence, (X, d) be a compact metric space and $\Gamma\colon G \times X \to X$ be a continuous group action satisfying the specification property. If μ is a G-invariant ergodic probability measure on X then $C(\mu)$ is a dense subset of X.*

Proof. As X is compact, the space $C(X, \mathbb{R})$ is separable. Given a dense sequence $(\varphi_\ell)_{\ell\in\mathbb{N}}$ in $C(X, \mathbb{R})$ and $\ell \in \mathbb{N}$, by the Ergodic theorem for countable amenable group actions [25] there is a full μ-measure subset $X_\ell \subset X$ such that, for every $x \in X_\ell$,

$$\lim_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \varphi_\ell(g(x)) = \int \varphi_\ell \, d\mu.$$

Therefore,

$$\bigcap_{\ell \in \mathbb{N}} X_\ell \subseteq B(\mu) \subseteq C(\mu)$$

and so

$$\mu(B(\mu)) = 1 = \mu(C(\mu)).$$

Moreover, given $z \in \bigcap_{\ell \in \mathbb{N}} X_\ell$, a continuous map $\varphi \in C(X, \mathbb{R})$ and a compact set $F \subset G$, as $\lim_{n \to +\infty} |F_n| = +\infty$ one also has

$$\lim_{n \to +\infty} \frac{1}{|F_n \setminus F|} \sum_{g \in F_n \setminus F} \varphi(g(z)) = \int \varphi \, d\mu.$$

We are left to show that $C(\mu)$ is dense in X. The following argument is inspired by [33, Theorem 4].

We start by noticing that, given $\varepsilon > 0$, let $K_\varepsilon \subset G$ be given by the specification property. Then, for every $k \in \mathbb{N}$, one can choose a positive integer n_k such that, as k goes to $+\infty$ one obtains

$$\frac{K_{\delta/2^k}}{|F_{n_k}|} \to 0 \qquad \text{and} \qquad \frac{|F_{n_{k-1}}|}{|F_{n_k}|} \to 0.$$

Recall that, by the compactness of X, given a continuous map $\varphi \in C(X, \mathbb{R})$, its modulus of uniform continuity, defined by

$$\zeta_\varepsilon(\varphi) = \sup \big\{ |\varphi(u) - \varphi(v)| \colon \, v \in B(u, \varepsilon), \; u \in X \big\}$$

where $B(u, \varepsilon)$ stands for the ball in X centered at u with radius ε, converges to zero as ε goes to 0^+.

Fix $x \in X$ and $\delta > 0$. We claim that $C(\mu)$ intersects the closed ball $\overline{B(x, \delta)}$ of radius δ around x. The idea to prove this assertion is to shadow pieces of orbits of increasing size in $C(\mu)$ which start at the ball $B(x, \delta)$. More precisely, take $z \in \bigcap_{\ell \in \mathbb{N}} X_\ell$ and consider $x_0 = x$, $x_1 = z$ and the finite sets (which satisfy (11.2))

$$\begin{aligned} \hat{F}_0 &= \{id\} \\ \hat{F}_1 &= \big(K_{\delta/2}^{-1} [F_{n_1} \setminus \hat{F}_0]\big) \setminus K_{\delta/2} \end{aligned}$$

where $K_\alpha^{-1} = \{g^{-1} \colon g \in K_\alpha\}$ for every $\alpha > 0$. By the specification property there is $y_1 \in B(x, \delta/2)$ such that $d(g(y_1), g(z)) < \varepsilon$ for every $g \in \hat{F}_1$. In particular, given a continuous map $\varphi \in C(X, \mathbb{R})$ one has

$$\Big| \frac{1}{|\hat{F}_1|} \sum_{g \in \hat{F}_1} \varphi(g(y_1)) - \frac{1}{|\hat{F}_1|} \sum_{g \in \hat{F}_1} \varphi(g(z)) \Big| < \zeta_{\frac{\delta}{2}}(\varphi).$$

Consider now $x_2 = y_1$ and $x_3 = z$ and the finite sets (which satisfy (11.2))

$$\begin{aligned} \hat{F}_2 &= \hat{F}_0 \cup \hat{F}_1 \\ \hat{F}_3 &= (K_{\delta/2^2}^{-1} [F_{n_2} \setminus \hat{F}_2]) \setminus (K_{\delta/2^2} \hat{F}_2) \end{aligned}$$

Hence, setting $\hat{F}_4 = \hat{F}_2 \cup \hat{F}_3$ and using the specification property once more, one obtains a point $y_2 \in B(y_1, \delta/2^2)$ such that $d(g(y_2), g(z)) < \varepsilon$ for every $g \in \hat{F}_3$, from which it is immediate that for every continuous map $\varphi \in C(X, \mathbb{R})$ one has

$$\Big|\frac{1}{|\hat{F}_3|} \sum_{g \in \hat{F}_3} \varphi(g(y_1)) - \frac{1}{|\hat{F}_3|} \sum_{g \in \hat{F}_3} \varphi(g(z))\Big| < \zeta_{\frac{\delta}{2^2}}(\varphi).$$

Proceeding recursively, given $y_j \in B(y_{j-1}, \delta/2^j)$ and the finite set $\hat{F}_j \subset G$ containing $\{id\}$, we take

$$\hat{F}_{j+1} = (K^{-1}_{\delta/2^j}\,[F_{n_{j+1}} \setminus \hat{F}_j]) \setminus (K_{\delta/2^j} \hat{F}_j) \tag{11.4}$$

and, by the specification property, we find $y_{j+1} \in B(y_j, \delta/2^{j+1})$ satisfying

$$d(g(y_{j+1}), g(y_j)) < \varepsilon \quad \text{for every } g \in \hat{F}_j$$

$$d(g(y_{j+1}), g(z)) < \varepsilon \quad \text{for every } g \in \hat{F}_{j+1}$$

and, for every continuous map $\varphi \in C(X, \mathbb{R})$,

$$\Big|\frac{1}{|\hat{F}_{j+1}|} \sum_{g \in \hat{F}_{j+1}} \varphi(g(y_{j+1})) - \frac{1}{|\hat{F}_{j+1}|} \sum_{g \in \hat{F}_{j+1}} \varphi(g(z))\Big| < \zeta_{\frac{\delta}{2^j}}(\varphi).$$

Thus, by construction, $(y_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $\overline{B(x, \delta)}$, hence convergent to some point $y_\infty \in \overline{B(x, \delta)}$. Moreover, the choice of the sets $\hat{F}_k$ ensures that

$$\lim_{k \to +\infty} \frac{|\hat{F}_k \,\Delta\, F_{n_k}|}{|F_{n_k}|} = 0.$$

In addition, from the initial selection of z and the previous estimates we conclude that, for every continuous map $\varphi \in C(X, \mathbb{R})$, the subsequence of averages given by

$$\Big(\frac{1}{|\hat{F}_n|} \sum_{g \in \hat{F}_n} \varphi(g(y_\infty))\Big)_{n \in \mathbb{N}}$$

converges to $\int \varphi \, d\mu$. Thus y_∞ belongs to $C(\mu)$. □

The following dichotomy is a direct consequence of Lemma 11.1 and Theorem 1.2.

Corollary 11.2. *Let G be a countable amenable group, $(F_n)_{n \in \mathbb{N}}$ be a tempered Følner sequence, X be a compact metric space and $\Gamma \colon G \times X \to X$ be a continuous group action satisfying the specification property. Then, for every $\varphi \in C(X, \mathbb{R})$, either*

$$\int \varphi \, d\mu_1 = \int \varphi \, d\mu_2 \qquad \forall\, \mu_1,\, \mu_2 \in \mathcal{M}_G(X)$$

or

$$\Big\{x \in X \colon \lim_{n \to +\infty} \frac{1}{|F_n|} \sum_{g \in F_n} \varphi(g(x)) \quad \textit{does not exist}\Big\}$$

is a Baire generic subset of X.

Proof. Fix $\varphi \in C(X,\mathbb{R})$ and suppose that there are two probability measures μ_1, $\mu_2 \in \mathcal{M}_G(X)$, which we may assume to be ergodic (using the ergodic decomposition [27]), such that

$$\int \varphi \, d\mu_1 \neq \int \varphi \, d\mu_2.$$

Their generalized basins of attraction $C(\mu_1)$ and $C(\mu_2)$ are both dense in X by Lemma 11.1. Define $\varepsilon = \frac{1}{2} \,|\int \varphi \, d\mu_1 - \int \varphi \, d\mu_2| > 0$. Using the sets $A = C(\mu_1)$, $B = C(\mu_2)$ and ε we confirm that X is Φ-sensitive, where $\Phi = \big(\varphi_n\big)_{n\in\mathbb{N}}$ and

$$\varphi_n(x) = \frac{1}{|F_n|} \sum_{g \,\in\, F_n} \varphi(g(x)).$$

Consequently, by Theorem 1.2, the irregular set

$$\Big\{x \in X\colon \lim_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\,\in\, F_n} \varphi(g(x)) \text{ does not exist}\Big\}$$

is a Baire generic in X. □

We observe that Corollary 11.2 has an immediate consequence regarding the empirical measures distributed along elements of a tempered Følner sequence: under the assumptions of this corollary, one has that either the amenable group action is uniquely ergodic or the set

$$\Big\{x \in X\colon \Big(\frac{1}{|F_n|} \sum_{g\,\in\, F_n} \delta_{g(x)}\Big)_{n\,\in\,\mathbb{N}} \text{ does not converge in the weak}^* \text{ topology}\Big\}$$

is Baire generic in X. This extends Furstenberg's theorem (see [27, Theorem 3.2.7]), according to which an amenable group action by homeomorphisms on a compact metric space is uniquely ergodic if and only if there is a constant c such that the sequence of averages (11.1) of every continuous function converges to c.

Example 7. Consider the 2-torus $\mathbb{T}^2$, with a Riemannian metric d, and the linear Anosov diffeomorphisms g_1 and g_2 of $\mathbb{T}^2$ induced by the matrices $A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. As $A_1 = A_2^2$, the maps g_1 and g_2 commute and induce a $\mathbb{Z}^2$-action on the 2-torus given by

$$\begin{array}{rccc} \Gamma\colon & \mathbb{Z}^2 \times \mathbb{T}^2 & \to & \mathbb{T}^2 \\ & \big((m,n),\, x\big) & \mapsto & \big(g_1^m \circ g_2^n\big)(x). \end{array}$$

Moreover, this action has the specification property. Let us see why.

Given $\varepsilon > 0$, let $k_\varepsilon \in \mathbb{N}$ be provided by the specification property for the Anosov diffeomorphism g_2 and $K_\varepsilon = [-k_\varepsilon,\, k_\varepsilon]^2 \subset \mathbb{Z}^2$. For any finite collection of points $x_0, x_1, x_2, \dots, x_\kappa$ in $\mathbb{T}^2$ and any choice of finite subsets $\hat{F}_0, \hat{F}_1, \hat{F}_2, \dots, \hat{F}_\kappa$ of $\mathbb{Z}^2$ satisfying the condition

$$K_\varepsilon \hat{F}_i \cap \hat{F}_j = \emptyset \qquad \text{for every distinct } \; 0 \leqslant i,j \leqslant \kappa \tag{11.5}$$

we claim that there exists a point $x \in \mathbb{T}^2$ such that

$$d(\Gamma_{(m,n)}(x), \Gamma_{(m,n)}(x_i)) < \varepsilon \qquad \text{for every } \; (m,n) \in \bigcup_{0\leqslant j \leqslant \kappa} \hat{F}_j.$$

Indeed, consider the map $\Theta : \mathbb{Z}^2 \to \mathbb{Z}$ given by $\Theta((m,n)) = 2m+n$ and notice that

$$\Gamma((m,n),\cdot) = g_2^{\Theta((m,n))}(\cdot) \qquad \forall\, m,n \in \mathbb{Z}. \tag{11.6}$$

The choice of the sets K_ε together with assumption (11.5) imply that $\Theta(K_\varepsilon \hat{F}_i) \cap \Theta(\hat{F}_j) = \emptyset$ for every $i \neq j$. Consequently,

$$\inf\left\{|u-v|\colon u \in \Theta(\hat{F}_i),\ v \in \Theta(\hat{F}_j)\right\} \geqslant k_\varepsilon.$$

To find $x \in \mathbb{T}^2$ as claimed, we are left to apply the specification property of g_2 (valid since g_2 is an Anosov diffeomorphism) and the equality (11.6).

Consider now a tempered Følner sequence $(F_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}^2$; for instance, the one defined by $F_n = [-n,n]^2 \subset \mathbb{Z}^2$. If P is the common fixed point by both g_1 and g_2, then the probability measure $\mu_1 = \delta_P$ belongs to $\mathcal{M}_G(X)$, and the same happens with the Lebesgue measure (say μ_2) on $\mathbb{T}^2$. Thus, given $\varphi \in C(\mathbb{T}^2,\mathbb{R}) \setminus \{0\}$ such that $\varphi \geqslant 0$ and $\varphi(P) = 0$, then $\int \varphi\, d\mu_1 = 0 \neq \int \varphi\, d\mu_2$, and therefore Corollary 11.2 asserts that

$$\left\{x \in \mathbb{T}^2\colon\ \lim_{n\to+\infty} \frac{1}{|F_n|} \sum_{g\in F_n} \varphi(g(x)) \ \text{ does not exist}\right\}$$

is a Baire generic subset of $\mathbb{T}^2$.

Application 3. Let G be a free semigroup, finitely generated by a finite set of self-maps $G_1 = \{Id, g_1, \cdots, g_p\}$ of a probability measure space $(X,\mathfrak{B},\mu)$. Assume that μ is invariant by $g_i : X \to X$ for every $1 \leqslant i \leqslant p$. The choice of G_1 endows G with a norm $|\cdot|$ defined as follows: given $g \in G$, then $|g|$ stands for the length of the shortest word over the alphabet G_1 representing g. Denote by G_k the set $\{g \in G\colon\ |g| = k\}$.

Now take $\varphi \in L^\infty(X,\mu)$ and consider the sequence of its spherical averages

$$k \in \mathbb{N} \quad \mapsto \quad s_k(\varphi) \;=\; \frac{1}{\#G_k} \sum_{g\in G_k} \varphi \circ g$$

where $\#$ stands for the cardinal of a finite set (if $G_k = \emptyset$, we set $s_k(\varphi) = 0$). Next consider the Cesàro averages of the previous spherical averages, that is,

$$n \in \mathbb{N} \quad \mapsto \quad \Phi_n \;=\; \frac{1}{n}\sum_{k=0}^{n-1} s_k(\varphi). \tag{11.7}$$

The main results of [7, 8] establish the pointwise convergence of the sequence $\left(\Phi_n\right)_{n\in\mathbb{N}}$ at μ-almost every point $x \in X$.

As a consequence of Theorem 1.2, if there exists a dense set of points $x \in X$ such that $W_\Phi(x)$ is not a singleton, then the set of Φ-irregular points is Baire generic in X. Let us check this information through an example.

Example 8. Consider the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C}\colon\ |z| = 1\}$ and the self-maps of $\mathbb{S}^1$ given by $g_1(z) = z^4$ and $g_2(z) = z^7$. These transformations commute and have two fixed points in common, whose pre-orbits by g_1 and g_2 are dense in $\mathbb{S}^1$. Take the free semigroup G generated by $G_1 = \{Id, g_1, g_2\}$, and let φ be in $C(\mathbb{S}^1,\mathbb{R})$.

Regarding the averages (11.7) of φ, in this case they are given by

$$\Phi_n = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(\varphi\circ g_1^j\circ g_2^{k-j}).$$

Let $z_0\in\mathbb{S}^1$ be a common fixed point for g_1 and g_2. The sequence $\big(\Phi_n(z_0)\big)_{n\in\mathbb{N}}$ converges to $\varphi(z_0)$ as n goes to $+\infty$. We claim that, for every x in the pre-orbit $\mathcal{O}^-(z_0)$ of z_0 by the semigroup action (made up by the pre-images of z_0 by all the elements of the semigroup G), the sequence $\big(\Phi_n(x)\big)_{n\in\mathbb{N}}$ converges to $\varphi(z_0)$ as well. Let us show this claim.

Given $x\in\mathcal{O}^-(z_0)$, there exists $g=g_{i_n}\circ\cdots\circ g_{i_2}\circ g_{i_1}\in G$, where $i_j\in\{1,2\}$, such that $g(x)=z_0$. Yet, as g_1 and g_2 commute, one can simply write $g=g_1^a\circ g_2^b$ for some non-negative integers a and b. Assume that $a,b\geqslant 1$ (the remaining cases are identical). If $k\geqslant a+b$, the sum

$$\frac{1}{2^k}\sum_{j=0}^{k}\binom{k}{j}(\varphi\circ g_1^j\circ g_2^{k-j})(x) \tag{11.8}$$

may be rewritten as

$$\frac{1}{2^k}\left[\sum_{j=0}^{a-1}\binom{k}{j}(\varphi\circ g_1^j\circ g_2^{k-j})(x)+\sum_{j=a}^{k-b}\binom{k}{j}(\varphi\circ g_1^j\circ g_2^{k-j})(x)+\sum_{j=k-b+1}^{k}\binom{k}{j}(\varphi\circ g_1^j\circ g_2^{k-j})(x)\right].$$

The absolute values of the first and third terms in the previous sum are bounded above by

$$a\,\|\varphi\|_\infty\,\max\left\{\binom{k}{0},\cdots,\binom{k}{a}\right\}2^{-k}$$

and

$$b\,\|\varphi\|_\infty\,\max\left\{\binom{k}{k-b+1},\cdots,\binom{k}{k}\right\}2^{-k}$$

respectively, and both estimates converge to zero as k goes to $+\infty$. Thus, their Cesàro averages also converge to 0. Regarding the middle term, as g_1 and g_2 commute and z_0 is fixed by every element of G, one has

$$\begin{aligned}\frac{1}{2^k}\sum_{j=a}^{k-b}\binom{k}{j}(\varphi\circ g_1^j\circ g_2^{k-j})(x)&=\frac{1}{2^k}\sum_{j=a}^{k-b}\binom{k}{j}\big(\varphi\circ g_1^{j-a}\circ g_2^{k-j-b}\big)(g_1^a\circ g_2^b)(x)\\&=\frac{1}{2^k}\sum_{j=a}^{k-b}\binom{k}{j}(\varphi\circ g_1^{j-a}\circ g_2^{k-j-b})(z_0)=\left[\frac{1}{2^k}\sum_{j=a}^{k-b}\binom{k}{j}\right]\cdot\varphi(z_0)\end{aligned}$$

whose limit as k goes to $+\infty$ is precisely $\varphi(z_0)$. This proves that the sequence (11.8) converges to $\varphi(z_0)$, hence the same happens with its Cesàro averages.

Therefore, if $z_0\in\mathbb{S}^1$ and $z_1\in\mathbb{S}^1$ are the two common fixed points by g_1 and g_2, and we take $x\in\mathbb{S}^1$ in the pre-orbit by the semigroup action of z_0 (which is dense, since the pre-orbit of z_0 by g_1 is dense) and $y\in\mathbb{S}^1$ belongs to the (also dense) pre-orbit by the semigroup action of z_1, then the sequence $\big(\Phi_n(x)\big)_n$ converges to $\varphi(z_0)$ and the sequence $\big(\Phi_n(y)\big)_n$ converges to $\varphi(z_1)$. So, if we choose $\varphi\in C(\mathbb{S}^1,\mathbb{R})$ such that $\varphi(z_0)\neq\varphi(z_1)$ then, by Theorem 1.2, we conclude that the irregular set $\mathcal{I}(\Phi)$ is Baire generic in $\mathbb{S}^1$.

Remark 11.3. As the generators of the semigroup action described in Example 8 commute, one could consider, alternatively, the sequence $(\Psi_n)_n$ where

$$\Psi_n(\cdot) \;=\; \frac{1}{n^2} \sum_{k,\,\ell=0}^{n-1} (\varphi \circ g_1^k \circ g_2^\ell)(\cdot).$$

If $z_0 \in \mathbb{S}^1$ is a common fixed point for G and $x \in \mathcal{O}^-(z_0)$, then this sequence can be rewritten as

$$\frac{1}{n^2}\left[\sum_{(k,\ell)\,\in\,[a,n-1]\times[b,n-1]} (\varphi \circ g_1^\ell \circ g_2^{k-\ell})(x) + \sum_{(k,\ell)\,\notin\,[a,n-1]\times[b,n-1]} (\varphi \circ g_1^\ell \circ g_2^{k-\ell})(x) \right]$$

where the sum is taken over pairs of integers (k, ℓ). The first term is equal to

$$\frac{1}{n^2} \sum_{(k,\ell)\,\in\,[a,n-1]\times[b,n-1]} (\varphi \circ g_1^{\ell-a} \circ g_2^{k-b-\ell})(z_0) = \frac{(n-a)(n-b)}{n^2}\, \varphi(z_0)$$

and converges to $\varphi(z_0)$ as n goes to $+\infty$. The second term has absolute value bounded above by $\frac{a+b}{n}\,\|\varphi\|_\infty$, which goes to zero as n tends to $+\infty$. Thus, if $z_0 \in \mathbb{S}^1$ and $z_1 \in \mathbb{S}^1$ are two common fixed points and $\varphi(z_0) \neq \varphi(z_1)$ then, by Theorem 1.2, one concludes that $\mathcal{I}(\Psi)$ is Baire generic in $\mathbb{S}^1$.

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