The Layzer-Irvine equation in theories with non-minimal coupling between matter and curvature

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Resumo

Neste trabalho, é derivada a equação de Layzer-Irvine no contexto de teorias alternativas da gravitação que envolvem um acoplamento não mínimo entre matéria e geometria. Como aplicação, é analisado o caso do enxame de galáxias Abell 586, por ser notoriamente esfericamente simétrico e livre de interações, recorrendo a alguns perfis de densidade.

Este trabalho baseia-se no trabalho desenvolvido na Ref. [1].

Palavras-chave: Relatividade Geral, teorias alternativas da gravitação, cosmologia, enxame A586.

Abstract

In this work, the Layzer-Irvine equation is derived in the context of alternative gravitational theories with non-minimal coupling between matter and geometry. As an application, the case of the spherically symmetric cluster Abell 586 is analysed, assuming some matter density profiles.

This work is based on work developed in Ref. [1].

Keywords: General Relativity, alternative theories of gravity, cosmology, A586 cluster.

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Chapter 1

Introduction

It is well established that General Relativity (GR) describes all known gravitational phenomena at the solar system with great accuracy, and predicts astrophysical objects like black holes [2, 3]. Nevertheless, there are several reasons, both theoretical and observational, to consider that GR might not be the full theory. For instance, GR is not compatible with Quantum Mechanics and Quantum Field Theory, which describe with great precision experimental results, in material science and high-energy Physics. Furthermore, on galactic and cosmological scales, matching observations requires two unknown components: a non-baryonic form of matter, dark matter, that explains galactic rotation curves and a dynamical mass on galaxy clusters; and an exotic form of energy to explain the late-time accelerated expansion of the Universe, namely dark energy. These two dark components constitute about 95% of the energy content of the Universe and their nature is still a mystery.

Consequently, several alternative gravitational theories have been proposed to account for the observations usually explained by the presence of dark matter and dark energy, such as, for instance, scalar-tensor theories, brane-world approaches and Einstein-aether models. There are also some so-called $f(\mathsf{R})$ theories [4–6], in which the scalar curvature term, in the Einstein-Hilbert action, is replaced by a generic non-linear function of it.

Recently, an interesting extension of the $f(\mathsf{R})$ theories has been proposed [7], in which the matter and curvature sectors are non-minimally coupled to each other. This model has a rich lore of theoretical and observational implications [8–10], and has bearings on issues such as stellar stability [11], preheating after inflation [12], mimicking of dark matter in galaxies [13] and clusters [14] and the large scale effect of dark energy [15].

An important tool to study gravitationally bound systems is the virial theorem. In the cosmological context, this theorem is associated with the Lazyer-Irvine equation [16–18], or the generalised cosmic virial theorem as it is often referred to. This equation can be directly applied to gravitationally collapsed astrophysical objects at different scales. If the gravitationally bound object is sufficiently relaxed, then that equation reduces to the usual virial theorem. From the deviation of the measured quantities such as mass, radius and velocity dispersion relatively to the virial ratio, we are able to test the existence of extra matter or the effect of modified gravity. These relations have been used: to study the interaction between the dark components of the Universe in galaxy clusters such as Abell 586 [19, 20] and Abell 1689 [19–22]; to assess the way dark components lead to structure formation [23,24]; in the context of $f(R)$ gravity [25] and scalar-tensor theories [26]; in modified gravitational potentials of the form $\varphi\left(a, |\overrightarrow{r_1} - \overrightarrow{r_2}|\right)$, where the cosmological evolution appears in terms of the scale factor, $a\left(t\right)$ [27].

In this work, we address the problem of adapting the Layzer-Irvine equation to theories with non-minimal coupling between matter and curvature.

The work is organised as follows. First, we shortly review the non-minimally coupled matter-curvature model [7] and some of its distinctive features. Then, we derive the Layzer-Irvine equation for these theories following up the procedure outlined in Refs. [17,23,26]. In Chapter 5, we apply the obtained Layzer-Irvine equation on the Abell 586, a relaxed spherically symmetric galaxy cluster, that has not undergone any relevant merging process in the last few Gyrs [28]. Finally, we show how to estimate the velocity dispersion potential assuming that the cluster is in hydrostatic and virial equilibrium, for different matter density profiles.

Chapter 2

Non-minimal curvature-matter coupling

In GR, the action functional is expressed as

$$S = \int \left[\kappa \mathsf{R} + \mathcal{L}_m\right] \sqrt{-g} d^4x \tag{2.1}$$

where R is the scalar curvature, $\mathcal{L}_m$ is the matter Lagrangian density, g is the metric determinant and $\kappa = c^4/16\pi G$, G being the Newton's gravitational constant.

In the so-called $f(\mathsf{R})$ theories [4–6], the scalar curvature term in the previous action is replaced by an arbitrary function of it:

$$S = \int \left[\frac{1}{2} f(\mathsf{R}) + \mathcal{L}_m\right] \sqrt{-g} d^4x \tag{2.2}$$

More generally, one can think in a non-minimal coupling between curvature and matter [7]:

$$S = \int \left[\frac{1}{2} f_1(\mathsf{R}) + (1 + f_2(\mathsf{R})) \mathcal{L}_m\right] \sqrt{-g} d^4x, \tag{2.3}$$

where $f_1(\mathsf{R})$ and $f_2(\mathsf{R})$ are arbitrary functions of the scalar curvature, R. One notes that, by setting $f_1(\mathsf{R}) = 2\kappa\mathsf{R}$ and $f_2(\mathsf{R}) = 0$, General Relativity is recovered. Varying the action with respect to the metric yields the field equations [7]:

$$F \mathsf{R}^{\mu}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} f_1 - \left(g^{\mu\sigma}\nabla_\sigma\nabla_\nu - \delta^{\mu}_{\nu}\Box\right) F = (1 + f_2)\, T^{\mu}_{\nu}, \tag{2.4}$$

with $F_i \equiv df_i/d\mathsf{R}$ $(i = 1, 2)$, $F \equiv F_1 + 2F_2\mathcal{L}_m$, and $T_{\mu\nu}$ is the energy-momentum tensor of matter defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m\right)}{\delta\left(g^{\mu\nu}\right)} \tag{2.5}$$

The Bianchi identities for the Einstein tensor, $\nabla_\mu G^\mu_\nu = 0$, and the identity

$$\left(\Box\nabla_\nu - \nabla_\nu\Box\right)F_i = \mathsf{R}_{\mu\nu}\nabla^\mu F_i \tag{2.6}$$

imply for the expression of Eq.(2.4):

$$\nabla_\mu T^\mu_\nu = \left(\mathcal{L}_m\delta^\mu_\nu - T^\mu_\nu\right)\nabla_\mu \ln\left(1 + f_2\right). \tag{2.7}$$

This is one of the fundamental features of the model (2.3) - the non-conservation of the energy-momentum tensor. This property induces an extra force acting on a test particle, which is orthogonal to the fluid four-velocity and can be expressed for a perfect fluid as:

$$f^\mu = \frac{1}{\rho + p}\left[\frac{F_2}{1 + f_2}(\mathcal{L}_m + p)\nabla_\nu\mathsf{R} + \nabla_\nu p\right]h^{\mu\nu}, \tag{2.8}$$

where $h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is the projection operator.

Chapter 3

Perturbed Friedmann-Lemaître-Robertson-Walker model

We now aim to achieve the prime objective of this work, which is to derive the Layzer-Irvine equation for the non-minimal coupling model described by the action Eq. (2.3). To do so, we follow closely the derivation performed in Refs. [17,19,23,26].

First, we shall consider that the Universe is well described by a perfect fluid, whose energy-momentum tensor reads $T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu}$, where $u^\mu = (1, u^i)$ is the four-velocity under the condition $u^\mu u_\mu = -1$. We also admit an homogeneous and isotropic spacetime described by the Robertson-Walker metric,γ_{ij}, whose perturbations are given by the line element

$$ds^2 = -\left(1 + 2\Phi\right) dt^2 + a^2\left(t\right)\left(1 - 2\Psi\right)\gamma_{ij} dx^i dx^j. \tag{3.1}$$

From now on, we consider the choice of the Lagrangian density as $\mathcal{L}_m = -\rho$, which is the most suitable for describing bound systems as discussed in Ref. [29]. The other possible choice, $\mathcal{L}_m = p$, is not very useful, since we assume a pressureless Universe.[1] Defining the potential velocity in terms of the components of the 4-

[1] In the context of theories with non-minimal coupling between matter and curvature, one breaks the degeneracy on the Lagrangian density choice that existed in General Relativity [29]. We can

velocity as $u_i = -\partial_i v$ and computing the first order perturbation in the components δT^i_0 of the stress tensor for a matter dominated epoch, $\rho \approx \rho_m$, we get [30]:

$$\dot{v} + \dot{\Phi}_c v = \Phi + \delta\Phi_c, \tag{3.3}$$

where $\Phi_c = \ln(1 + f_2)$. This expression can be rewritten in terms of the four-velocity as

$$\dot{u}_i = -\nabla_r(\Phi + \delta\Phi_c - v\dot{\Phi}_c). \tag{3.4}$$

We shall make the assumption that the flow velocity associated to the expansion rate of the Universe is much smaller than the typical peculiar velocities of cosmic structures. Then $u_i \approx a\dot{x}_i \equiv v_{m\,i}$. Under this condition, Eq. (3.4) can be expressed in a more convenient form

$$\frac{\partial}{\partial t}(av_m) = -a\nabla_r\left(\Phi + \delta\Phi_c - \dot{\Phi}_c v\right). \tag{3.5}$$

The evolution of matter density perturbations is given in the Fourier space by [30]

$$\dot{\delta\rho_m} + 3H\delta\rho_m = 3\dot{\Psi}\rho_m - \left(\frac{k^2}{a^2}\frac{v}{a}\right)\rho_m, \tag{3.6}$$

where $H = \dot{a}/a$ is the expansion rate. In the configuration space, using the notation $\sigma_m \equiv \delta\rho_m$, only considering peculiar velocities and in the subhorizon approximation $(k/a > H)$ we can write

$$\dot{\sigma}_m + 3H\sigma_m = -\frac{1}{a}\nabla_x \cdot \left(\rho_m \overrightarrow{v_m}\right). \tag{3.7}$$

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easily see that the non-conservation of the energy-momentum tensor, Eq. (2.7), strongly depends on the Lagrangian density. For instance, for a pressureless Universe, $p \simeq 0$, such that $\nabla_\nu p \simeq 0$, the extra force, Eq. (2.8], vanishes for $\mathcal{L}_m = p$, whilst for the other possible, $\mathcal{L}_m = -\rho$, gives

$$f^\mu = -\nabla_\nu\left(1 + f_2\left(\mathsf{R}\right)\right)h^{\mu\nu}, \tag{3.2}$$

which is, in general, different from zero.

Finally, from the time component of the non-conservation of the energy-momentum tensor, for a pressureless ($w = 0$) Universe with Lagrangian density $\mathcal{L} = -\rho_m$, then[2]

$$\dot{\rho}_m + 3H\rho_m = 0. \tag{3.9}$$

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[2]The generalisation of the previous result is as follows [31]

$$\dot{\rho}_m + 3H\left(1 + w\right)\rho_m = \frac{F_2}{1 + f_2}\left(\alpha - 1\right)\rho_m \dot{\mathsf{R}}, \tag{3.8}$$

where $\alpha = \begin{cases} 1 \;\;, & \mathcal{L} = -\rho_m \\ -w, & \mathcal{L} = p \end{cases}$ so that the Lagrangian density has the form $\mathcal{L} = -\alpha\rho_m$ (see Ref. [29] for a thorough discussion) and $w = p/\rho_m$ is the equation of state parameter.

Chapter 4

The Layzer-Irvine equation

We are now able to derive the Layzer-Irvine equation. We start by contracting Eq. (3.5) with $a\overrightarrow{v}_m\rho_m d^3r$, for $r = ax$, and then integrating over the volume, we get:

$$\int \rho_m a\overrightarrow{v}_m \frac{\partial}{\partial t}\left(a\overrightarrow{v_m}\right) d^3r = -\int a^2\overrightarrow{v_m}\rho_m \nabla_r \left(\Phi + \delta\Phi_c - \dot{\Phi}_c v\right) d^3r. \qquad (4.1)$$

Using Eq. (3.9), the left hand side of Eq. (4.1) can be expressed as $\frac{\partial}{\partial t}\left(a^2K\right)$, where $K \equiv 1/2\int \rho_m v_m^2 d^3r$ is the kinetic energy associated with the peculiar velocity.

In its turn, the right hand side can be evaluated performing an integration by parts:

$$\begin{aligned}
&-\int a^2\overrightarrow{v_m}\rho_m \nabla_r \left(\Phi + \delta\Phi_c - \dot{\Phi}_c v\right) d^3r = \\
&= -\int \nabla_r \left(a^2\overrightarrow{v_m}\rho_m \left(\Phi + \delta\Phi_c - \dot{\Phi}_c v\right) d^3r\right) + \int \left(\Phi + \delta\Phi_c - \dot{\Phi}_c v\right) \nabla_r \cdot \left(a^2\overrightarrow{v_m}\rho_m\right) d^3r \\
&= -\int \left(\Phi + \delta\Phi_c - \dot{\Phi}_c v\right) a^2 \left(\dot{\sigma}_m + 3H\sigma_m\right) d^3r.
\end{aligned} \qquad (4.2)$$

In the last equality, the first integral corresponds to a total derivative, which therefore vanishes. Moreover, we have resorted to Eq. (3.7).

Collecting the results, we get

$$\frac{\partial K}{\partial t} + 2HK = -\int (\Phi + \delta\Phi_c - \dot{\Phi}_c v)\frac{\partial}{\partial t}\left(\sigma_m d^3r\right). \qquad (4.3)$$

We will require that each potential satisfies Poisson's equation. We shall define the autocorrelation function $f\left(\overrightarrow{r}\right)$ of the matter density perturbation field, σ_m, as in Ref. [17] as

$$\left\langle \sigma_m(\overrightarrow{r},t)\sigma_m(\overrightarrow{r'},t)\right\rangle = \left\langle \sigma_m^2\right\rangle f\left(|\overrightarrow{r}-\overrightarrow{r'}|\right). \tag{4.4}$$

From which we can define some astrophysical and cosmological scales. We should also note that $\langle \sigma_m(\overrightarrow{r},t)\rangle = 0$. Additionally, we use that

$$\frac{\partial}{\partial t}\frac{1}{|r-r'|} = -\frac{H}{|r-r'|}. \tag{4.5}$$

Since we require that the potentials satisfy the Poisson's equation, then any of them can be expressed in terms of the matter density perturbation as

$$\varphi = -G\int \frac{\sigma_m(r',t)}{|r-r'|}d^3r'. \tag{4.6}$$

Bearing this in mind, the right hand side of Eq. (4.3) can be expressed as

$$\begin{aligned} -\int \varphi\frac{\partial}{\partial t}(\sigma_m d^3r) &= G\int\frac{\partial}{\partial t}(\sigma_m d^3r)\int\frac{\sigma'_m}{|r-r'|}d^3r' \\ &= G\int\frac{\partial}{\partial t}(\sigma'_m d^3r')\int\frac{\sigma_m}{|r-r'|}d^3r, \end{aligned} \tag{4.7}$$

where $\sigma_m \equiv \sigma_m(\overrightarrow{r},t)$ and $\sigma'_m \equiv \sigma_m(\overrightarrow{r'},t)$. Now, recalling the result (4.5), the expression (4.7) can be written as

$$G\int\frac{\partial}{\partial t}(\sigma'_m d^3r')\int\frac{\sigma_m}{|r-r'|}d^3r = -(\dot{U}_\varphi + HU_\varphi), \tag{4.8}$$

where

$$U_\varphi \equiv -\frac{G}{2}\int\int\frac{\sigma_m\sigma'_m}{|r-r'|}d^3r d^3r' = \frac{1}{2}\int\varphi\,\sigma_m d^3r. \tag{4.9}$$

Note that the non-minimal coupling effects on the gravitational coupling in the case of clusters are negligible, so that the effective gravitational constant, as defined in Ref. [30], obeys $G_{eff} \approx G$. Now we can write the Layzer-Irvine equation in the form

$$\frac{\partial}{\partial t}(K + U_\Phi + U_{\delta\Phi_c-\dot{\Phi}_c v}) + H(2K + U_\Phi + U_{\delta\Phi_c-\dot{\Phi}_c v}) = 0, \tag{4.10}$$

which can be rearranged into a more convenient form:

$$\frac{\partial}{\partial t}(K + U + U_{NMC}) + H(2K + U + U_{NMC}) = 0, \tag{4.11}$$

with $U \equiv U_\Phi$ and

$$U_{NMC} \equiv U_{\delta\Phi_c - \dot{\Phi}_c v} = \frac{1}{2}\int \left(\delta\Phi_c - \dot{\Phi}_c v\right)\, \sigma_m d^3 r. \tag{4.12}$$

We see that the non-minimal coupling between matter and geometry induces an extra term in the standard generalised cosmic virial theorem, which can account for the "dark components" effects on several systems. For a relaxed astrophysical system which no longer evolves in time, we get a generalised version of the virial theorem for these gravitational theories:

$$2K + U + U_{NMC} = 0. \tag{4.13}$$

From this equation we can proceed to analyse clusters of galaxies and impose some constraints on the non-minimal model. Clearly, any deviation from the usual virial ratio $K/U = -1/2$ can be expressed in terms of the quotient:

$$\frac{U_{NMC}}{U} = -2\frac{K}{U} - 1. \tag{4.14}$$

Chapter 5

The Abell 586 cluster

We consider now the well known relaxed cluster Abell 586, following up the procedure developed in Refs. [19, 22]. We assume the obvious cases of the top-hat and isothermal spheres density profiles. In order to test the sensitivity of the results, we adopt tentatively the Navarro-Frenk-White (NFW) density profile [32], even though this is known to be a profile obtained from N-body simulations for galaxies within the Cosmological Standard Model, ΛCDM (which assumes that dark energy is parametrised by a cosmological constant, Λ, dark matter is taken to be non-relativistic). The NFW model is, therefore, somewhat inacurate for clusters. As we shall see, results for U_{NMC} are dependent on the density profile choice, even though not strongly so. It is relevant to bear in mind that the considered density is exclusively baryonic.

5.1 Top-hat density profile

In this case, one assumes that the kinetic and potential energy densities are well described by [19]

$$\rho_K \simeq \frac{9}{8\pi}\frac{M}{R^3}\sigma_v^2, \tag{5.1}$$

$$\rho_W \simeq -\frac{3}{8\pi}\frac{G\,M^2}{\langle R\rangle\,R^3}, \tag{5.2}$$

where M e R are the total baryonic mass and radius of Abell 586, which include galaxies and intra-cluster gas, σ_v is the velocity dispersion and $\langle R\rangle$ is the mean intergalactic radius. Since the case we are studying has spherical symmetry, the total volume is simply $V = 4\pi R^3/3$, and the ratio between total peculiar kinetic and potential energies is the same as the ratio of the energy densities, thus:

$$\frac{K}{U} \equiv \frac{\rho_K}{\rho_W} = -3\frac{\sigma_v^2 \langle R\rangle}{G\, M}. \tag{5.3}$$

5.2 Navarro-Frenk-White density profile

The Navarro-Frenk-White model is very useful in realistic N-body simulations within the ΛCDM paradigm. It is characterised by the energy density [32]:

$$\rho(r) = \frac{\rho_0}{\frac{r}{r_0}\left(1+\frac{r}{r_0}\right)^2}, \tag{5.4}$$

where r is the distance from the centre, ρ_0 and r_0 are the density and shape parameters, respectively. As described in Ref. [22], the total mass and mean radius can be calculated by integrating Eq. (5.4) over the volume:

$$M = 4\pi\int_0^R \rho(r)r^2dr = 4\pi r_0^3\rho_0\left[\ln\left(1+\frac{R}{r_0}\right) - \frac{R}{R+r_0}\right], \tag{5.5}$$

$$\langle R\rangle = r_0\frac{\left[\frac{R}{r_0} - 2\ln\left(1+\frac{R}{r_0}\right) + \frac{R}{R+r_0}\right]}{\left[\ln\left(1+\frac{R}{r_0}\right) - \frac{R}{R+r_0}\right]}. \tag{5.6}$$

We point out that r_0 can be numerically calculated from the mean radius, $\langle R\rangle$. Thus, the density parameter, ρ_0, is immediatly solved numerically. From these quantities we can now estimate the kinetic and potential energy densities assuming a constant average velocity, σ_v [22] :

$$\rho_K = \frac{9}{8\pi}\frac{M}{R^3}\sigma_v^2, \tag{5.7}$$

$$\rho_W = -\frac{3GM^2}{4\pi R^3 r_0} \frac{\left[\left(1+\frac{R}{r_0}\right)\left[\frac{1}{2}\left(1+\frac{R}{r_0}\right) - \ln\left(1+\frac{R}{r_0}\right)\right] - \frac{1}{2}\right]}{\left[\left(1+\frac{R}{r_0}\right)\ln\left(1+\frac{R}{r_0}\right) - \frac{R}{r_0}\right]^2}. \tag{5.8}$$

Thus, we can compute the virial ratio as:

$$\frac{K}{U} \equiv \frac{\rho_K}{\rho_W} = -\frac{3}{2}\frac{\sigma_v^2 r_0}{G\,M} \frac{\left[\left(1+\frac{R}{r_0}\right)\ln\left(1+\frac{R}{r_0}\right) - \frac{R}{r_0}\right]^2}{\left[\left(1+\frac{R}{r_0}\right)\left[\frac{1}{2}\left(1+\frac{R}{r_0}\right) - \ln\left(1+\frac{R}{r_0}\right)\right] - \frac{1}{2}\right]}. \tag{5.9}$$

5.3 Isothermal spheres density profile

Another useful density profile is the isothermal spheres density profile, given by

$$\rho\left(r\right) = \frac{\rho_0}{\left(\frac{r}{r_0}\right)^2}. \tag{5.10}$$

Since there is no characteristic scale in this case, we set the fiducial parameters, r_0 and $M_0 = 4\pi\rho_0 r_0^3/3$, as the total mass and radius of the halo. Therefore, the mass and the mean radius are [22]:

$$M = 4\pi \int_0^R \frac{\rho_0 r^2}{\left(\frac{r}{r_0}\right)^2} dr = M_0 \frac{R}{r_0}, \tag{5.11}$$

$$\langle R\rangle = \frac{R}{2}. \tag{5.12}$$

With these quantities, we can now get the expressions for the peculiar kinetic and potential energy densities, assuming constant average velocity dispersion [22]

$$\rho_K = \frac{9}{8\pi}\frac{M}{R^3}\sigma_v^2, \tag{5.13}$$

$$\rho_W = -\frac{3GM^2}{4\pi R^4}. \tag{5.14}$$

The virial ratio can be then straightforwardly obtained

$$\frac{K}{U} \equiv \frac{\rho_K}{\rho_W} = -\frac{3}{2}\frac{\sigma_v^2 R}{G\,M}. \tag{5.15}$$

Method	$\sigma_v(km/s)$
X-ray Luminosity	1015 ± 500
X-ray Temperature	1174 ± 130
Weak Lensing	1243 ± 58
Velocity distribution	1161 ± 196

Table 5.1: Velocity dispersion data from different observational methods of Abell 586 as given by Ref. [28].

5.4 Analysis

For the analysis of Abell 586 we use data from Ref. [28], namely:

- total baryonic mass is given by $M_{bar} = M_{gas}\,(1 + 0.16\,h_{70}^{0.5})$, where $M_{gas} = 0.48 \times 10^{14}\,M_\odot$ is the intracluster gas mass, and $h_{70} = H_0/70$ is the reduced Hubble parameter at present time;
- radius, $R = 422\,kpc$;
- velocity dispersion obtained from different methods as shown in Table 5.1.

From the 31 galaxies of A586, we can compute the averaged distance from a galaxy i with equatorial coordinates (α_i, δ_i) to the centre of the cluster (α_c, δ_c) throughout the formula [22]

$$r_i^2 = 2d^2[1 - cos(\alpha_i - \alpha_c)cos(\delta_c)cos(\delta_i) - sin(\delta_c)sin(\delta_i)]. \tag{5.16}$$

where d is the radial distance from the centre of the cluster to Earth.

From these distances, we get

$$\langle R \rangle = 223.6\,kpc. \tag{5.17}$$

Furthermore, the errors are computed through the propagation uncertainties formula for $f\,(x_i) = \prod_i x_i^{n_i}$,

$$\Delta f = |f| \sqrt{\sum_i \left(\frac{n_i \Delta x_i}{x_i}\right)^2}, \tag{5.18}$$

U_{NMC} / U	Top-hat	NFW	Isothermal
X-ray Luminosity	4.8 ± 5.7	5.3 ± 6.2	4.5 ± 5.4
X-ray Temperature	6.7 ± 1.8	7.4 ± 2.0	6.3 ± 1.7
Weak Lensing	7.7 ± 1.1	8.5 ± 1.2	7.2 ± 1.0
Velocity distribution	6.6 ± 2.6	7.3 ± 2.9	6.1 ± 2.5

Table 5.2: NMC constraints for different density profiles in terms of various observational methods.

where the Δ symbol denotes the standard deviation for each measurable quantity.

With the density profiles described above, we can now compute the ratio U_{NMC}/U and detect the deviations from the standard virial ratio $K/U = -1/2$. In Table 5.2 we exhibit several values for the mentioned ratio according to different density profiles and observational sources. As in Ref. [22], the non-minimal ratio, and the ensued virial ratio, yields higher values for weak lensing velocity dispersion. This observational method is not the most reliable one since it introduces correlation between estimated of mass and velocity, as pointed out in Ref. [22]. In our case, we want to identify deviations from the baryonic virial ratio and interpret them as an effect from the non-minimal coupling between curvature and matter.

Keeping in mind Eqs. (4.9) and (4.11), we can express the non-minimal potential in terms of the function $f_2(R)$ of Eq. (2.3) as

$$U_{NMC} = \frac{1}{2} \int d^3 r \sigma_m \left(\frac{F_2}{1+f_2} \right) \left[\delta \mathsf{R}\, c^2 - \dot{\mathsf{R}} v \right] . \tag{5.19}$$

The scalar curvature can be computed by performing the trace of Eq. (2.4), assuming a general power law coupling function $f_2(\mathsf{R}) = (\mathsf{R}/\mathsf{R}_n)^n$, yielding [33,34]:

$$\mathsf{R} = \frac{1}{2\kappa} \left[1 + (1-2n) \left(\frac{\mathsf{R}}{\mathsf{R}_n} \right)^n \right] \rho - \frac{3n}{\kappa} \Box \left[\left(\frac{\mathsf{R}}{\mathsf{R}_n} \right)^n \frac{\rho}{\mathsf{R}} \right] . \tag{5.20}$$

Considering the weak coupling regime, $(\mathsf{R}/\mathsf{R}_n)^n \ll 1$, the above expression simplifies to

$$\mathsf{R} \approx \frac{\rho}{2\kappa}, \tag{5.21}$$

$v_0\,(kpc^2 s^{-1})$	Top-hat	NFW	Isothermal
X-ray Luminosity	-2.377×10^{16}	-1.581×10^{16}	-1.095×10^{15}
X-ray Temperature	-3.347×10^{16}	-2.216×10^{16}	-1.548×10^{15}
Weak Lensing	-3.812×10^{16}	-2.520×10^{16}	-1.765×10^{15}
Velocity distribution	-3.263×10^{16}	-2.160×10^{16}	-1.509×10^{15}

Table 5.3: NMC constraints on the parameter v_0 for different density profiles in terms of the various observational methods.

which is consistent with the assumption of the subhorizon approximation, where $\mathsf{R} \sim H^2$. The Ricci scalar fluctuation is then $\delta\mathsf{R} \approx \delta\rho/2\kappa$, whilst the time derivative, $\dot{\mathsf{R}}$, can be calculated using Eq. (3.9), yielding $\dot{\mathsf{R}} \approx \dot{\rho}/2\kappa \approx -3H\rho/2\kappa$.

Hence, under these conditions, the non-minimal coupling can be expressed as

$$U_{NMC} = \frac{1}{2}\int d^3r\sigma_m \frac{n}{\rho}\left(\frac{\rho}{2\kappa\mathsf{R}_n}\right)^n \left[\sigma_m\, c^2 + 3H\rho v\right]. \qquad (5.22)$$

For each different density profile, we have an estimate for the value of the non-minimal coupling potential energy. Additionally, from Ref. [14] the best fit value of the index n for Abell 586 is $n = 0.43$. Thus, it results that $\mathsf{R}_{0.43} \equiv 1/\sqrt{r_{0.43}} \approx 5.69 \times 10^{-8}\ \mathrm{m}^{-1/2}$, since the characteristic lenght for A586 is $r_{0.43} \sim 0.01$ pc [14]. From here, we are able to estimate the velocity potential of the cluster.

Since the cluster is already virialised, we shall assume that each constituent galaxy has the same peculiar velocity, $\langle v_m \rangle = v_m$. And from the previous definition and the isotropy and spherical symmetry of the cluster, it follows that

$$v_m = -\partial_r v \implies v = -\sigma_v r + v_0, \qquad (5.23)$$

where v_0 is the velocity potential at $r = 0$. From this expression, and for each value of velocity dispersion given by the various observational methods and for each density profile we obtain a well defined value for v_0. The results are shown in Table 5.3. Clearly, the value v_0 depends on the choice of the density profile. We note that this quantity is merely an integration constant, which has no particular physical meaning.

Another interesting issue to analyse is the $f_2(\mathsf{R})$ behaviour for A586 cluster in terms of the distance r from the cluster's centre. This coupling function can be written as:

$$f_2(\mathsf{R}) = f_2(r) = \left(\frac{\rho(r)}{2\kappa \mathsf{R}_{0.43}}\right)^{0.43}, \tag{5.24}$$

where $\rho(r)$ is the density profile. In Fig. 5.1, we show the plot of the function f_2 in terms of r.

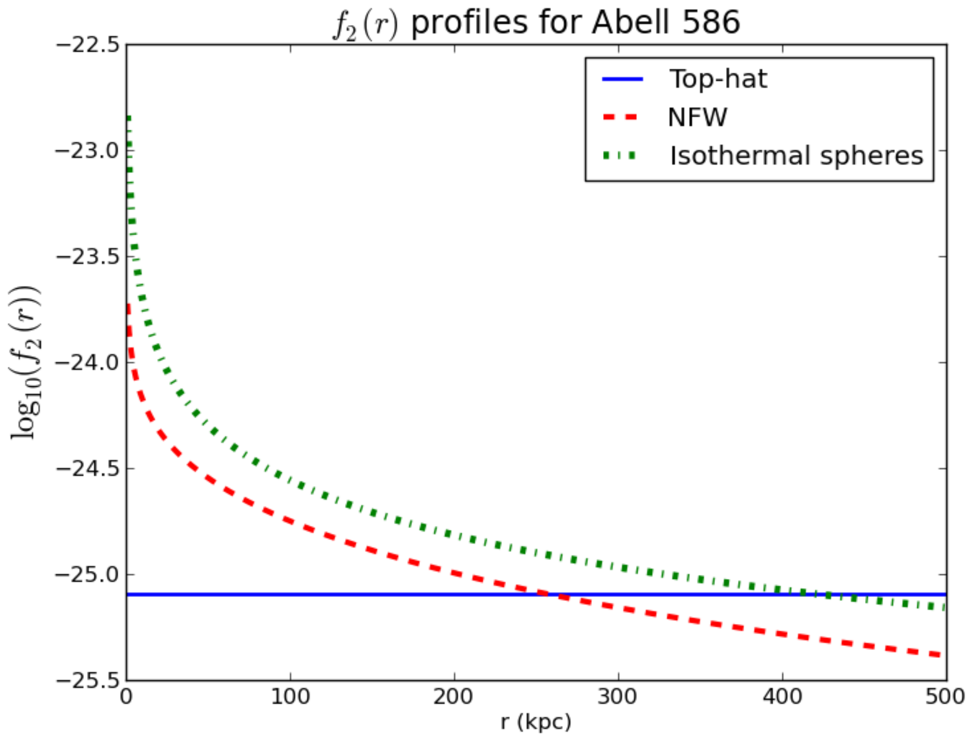

Figure 5.1: Function $f_2(r)$ for the cluster A586, in terms of the discussed density profiles.

We notice that the isothermal spheres and the Navarro-Frenk-White density profiles are singular at the cluster's centre, $r = 0$, which results in a much stronger coupling in this region, whilst the top-hat profile exhibits a constant effect of the matter-curvature coupling all over the cluster's size.

Chapter 6

Conclusions

In this work, the Layzer-Irvine equation has been derived in the context of alternative gravitational theories with non-minimal coupling between curvature and matter. These theories have some distinctive signatures, namely the non-conservation of the energy-momentum tensor and the deviation from the geodesic motion due to an extra force term.

The Layzer-Irvine equation allow us to study bound systems and can be derived from General Relativity. Nevertheless, in order to match the observations, one must admit the existence of dark matter. Thus, the goal of this work was to match the observations with only baryonic matter and a coupling between the matter and curvature.

To do so, it was considered a pressureless matter dominated Universe and the power-law function, $f_2(\mathsf{R}) = (\mathsf{R}/\mathsf{R}_n)^n$, with $n = 0.43$ [14], in the sub-horizon approximation $k^2 \gg a^2H^2$. Since the NMC theories break the Lagrangian density degeneracy, it was adopted the choice $\mathcal{L} = -\rho$ [29].

It was found that in these theories, as far as the generalised cosmic virial theorem is concerned, an extra potential energy term appears as a result of the non-minimal coupling.

Applying that equation to the spherically symmetric and relaxed cluster A586, the virial ratio was computed for the observed baryonic matter, finding that the extra potential energy arising from the non-minimal coupling is crucial. Indeed, using the velocity dispersion values obtained from various observational methods and three different density profiles (top-hat, Navarro-Frenk-White, and isothermal spheres), the ratio between the non-minimal coupling potential energy and the baryonic energy potential, U_{NMC}/U, is of the order of ~ 7. One can also conclude from the values in Table 5.2 that the velocity dispersion value from X-ray luminosity is not very reliable, as discussed before in Ref. [22].

Since the new potential energy term can be expressed in terms of the non-minimal coupling and the velocity potential, the latter was estimated assuming that it is a linear function of the distance from the cluster's centre. This assumption is consistent with the fact that the A586 cluster has already virialised and reached hydrostatic equilibrium, since it has not undergone any merging process within the last Gyrs.

Finally, it was also analysed the $f_2\left(\mathsf{R}\right)$ function over the distance from the cluster's centre for the different density profiles used in this work. From the summarised plot, one concludes that for singular density profiles at $r = 0$, as the NFW and the isothermal spheres profiles, the coupling function is naturally stronger. Whilst for the top-hat, one find a constant function over the distance.

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