Wiggly Cosmic String Evolution

José Pedro Pinto Vieira

Mestrado em Física
Departamento de Física e Astronomia
2013/2014

Orientador
Carlos José Amaro Parente Martins, Investigador Coordenador,
Centro de Astrofísica da Universidade do Porto

Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Presidente do Júri,

Porto, ______/______/_________

Acknowledgements

First of all, my sincerest thanks to Dr. Carlos J. A. P. Martins, the most supportive supervisor a student can have, for all his patience and invaluable physical insight. If I ever do manage to become a cosmologist, I owe it to you.

Thanks also to Dr. Tasos Avgoustidis, who opened the door to extra dimensions. Without him, this thesis would be one chapter shorter.

In addition, a word of gratitude towards those who have taught me Physics and Mathematics for the last five years. It has been a pleasure to learn from (almost) all of you.

Furthermore, thanks to all the companions who have walked this path beside me, for all the pain and laughter we have shared, and for all the stimulating discussions since the Great Debate About The Nature Of Fire of 2009. In particular, special thanks to Artur Sousa, Catarina Cosme, Daniel Passos, Diogo Lopes, and Ester Simões.

And finally, a very special word of appreciation to the staunch supporters who have backed me up from behind the scenes. For your wholehearted dedication, thank you, mother, father, sister, grandparents, and Ana. God knows what you have to put up with.

I acknowledge the Gulbenkian Foundation for the significant financial support provided through *Programa de Estímulo à Investigação 2013*, grant number 132590. Needless to say, this thesis is much richer thanks to it.

Resumo

Cordas cósmicas são uma classe especial de defeitos topológicos que se podem formar num grande leque de cenários cosmológicos, incluindo extensões naturais do modelo padrão. Compreender a sua dinâmica e evolução é portanto uma questão de grande importância para aprender acerca dos mecanismos físicos desconhecidos que têm desempenhado papeis determinantes na História do Universo - especialmente agora que temos finalmente à nossa disposição um conjunto de dados experimentais que podem ajudar a constringir estes modelos (notavelmente de Planck [2] e BICEP2 [1, 17, 26]).

A enorme complexidade e não-linearidade destes sistemas, no entanto, faz com que mesmo os exemplos mais simples de redes de defeitos sejam impossíveis de descrever completamente por meios analíticos. Além disso, a grande gama de escalas de tempo e comprimento envolvidos limitam seriamente a nossa capacidade de retirar informação de simulações computacionais. Assim, a abordagem típica a este imbróglio resulta numa simbiose entre estas duas filosofias, usando-se dados de simulações para calibrar simplificações analíticas do problema. O desafio é então a construção de um modelo resolúvel que ainda assim capture a essência dos fenómenos mais importantes.

O modelo de Uma Escala dependente da Velocidade - "Velocity-dependent One-Scale model" (VOS), no original [19, 20, 21] - é o mais bem sucedido modelo deste género no que toca a modelar o comportamento de grande escala de uma rede simples de cordas cósmicas [25]. Em contrapartida, peca por ser fundamentalmente limitado em relação à informação que pode fornecer acerca da importante estrutura de pequena escala que se vai acumulando ao longo da História de uma rede.

Nesta dissertação, apresenta-se uma generalização natural e original do VOS que tem em conta a evolução da estrutura de pequena escala na rede [24]. Após uma discussão detalhada do formalismo matemático em causa, considera-se a sua aplicação a alguns limites fisicamente relevantes - em particular, deduzimos condições concretas sob as quais a estrutura de pequena escala na rede deve evoluir para um regime de "scaling".

Finalmente, é dado algum foco a um trabalho em desenvolvimento que explora a relação entre este modelo e uma versão do VOS com dimensões extra.

Abstract

Cosmic strings are a special class of topological defects which can be formed in a broad range of cosmological scenarios, including natural extensions of the standard model. Understanding their dynamics and evolution is thus of great importance to learn about the unknown physical mechanisms that have played significant roles in the History of the Universe - especially now that valuable data which may be used to constrain these models has finally become available (notably from Planck [2] and BICEP2 [1, 17, 26]).

The highly nonlinear complexity of these objects, however, makes even the simplest examples of these networks impossible to completely describe analytically. Moreover, the wide range of time and length scales involved severely limit how much can be learned from computational simulations. In the end, a typical approach is a compromise between these two philosophies in which simulation data is used to callibrate analytical simplifications of the problem. The challenge, then, is building a solvable model which still captures most of the important phenomena.

The Velocity-dependent One-Scale model (VOS) [19, 20, 21] is the most successful such model when it comes to modeling the large-scale behaviour of a "vanilla" network [25]. However, it is fundamentally limited in how much it can tell us about the important small-scale structure that builds up during the History of the network.

In this dissertation, we present an original natural generalization of the VOS which takes into account the evolution of small-scale structure in the network [24]. After a detailed discussion of the mathematical formalism involved, its application to a few physically relevant limits is considered - in particular, we deduce concrete conditions under which the small-scale structure in the network should evolve towards a scaling regime.

Finally, we also focus on on-going work which explores the relationship between this model and an extra-dimensional version of the VOS.

Contents

List of Figures

List of Abbreviations

EDVOS	Extra-Dimensional Velocity-dependent One-Scale model
FRW	Friedmann-Robertson-Walker
GUT	Grand Unification Theories
rms	root mean square
VOS	Velocity-dependent One-Scale model

1. Introduction to Cosmic Strings

In this chapter we shall briefly review some of the most basic notions about cosmic strings. Even though someone new to this topic can probably follow the work presented here with only this quick introduction, a reader interested in really getting acquainted with this field is advised to seek a more comprehensive introduction elsewhere - [30] probably being the best place to start. This chapter is meant to make this dissertation as self-contained as possible, but it simply cannot hope to cover all of the foundational results a newcomer needs.

1.1. Topological Defects and Spontaneous Symmetry Breaking in Phase Transitions

Spontaneous symmetry breaking in phase transitions is an important idea in modern physics which originated in condensed matter physics. In particular, it is the key principle behind the celebrated Englert–Brout–Higgs–Guralnik–Hagen–Kibble mechanism [11, 12, 13] (better known as simply the *Higgs mechanism*) of particle physics which won Higgs and Englert the 2013 Nobel Prize in Physics. Essentially, this occurs when a system which is ruled by laws invariant under certain transformations evolves into a state which is not invariant by those same transformations. One of the simplest examples of such a transition occurs in the Goldstone model for a complex field with the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \bar{\phi} - \frac{\lambda}{4}\left(|\phi|^2 - \eta^2\right)^2 \tag{1.1.1}$$

which is invariant under global transformations of the form

$$\phi(x) \rightarrow e^{i\alpha}\phi(x) \tag{1.1.2}$$

The "mexican hat" potential term here has a local maximum in $\phi = 0$ (which is favoured at high temperature) and a set of global minima lying on the circle $|\phi| = \eta$ (which are favoured at low temperature). If we start at high temperature and gradually decrease it, the favoured state will then change from a state invariant under 1.1.2 ($\phi = 0$) to one which clearly is not ($\phi = \eta e^{i\theta}$).

Topological defects (also known as topological solitons) are yet another concept which was born in the context of condensed matter physics and ended up being relevant for particle physics and cosmology - and, not so surprisingly, one which is intimately connected with spontaneous symmetry breaking. Their exact definition often depends on the person employing the term (and especially on the field in discussion) but a few generic characteristics are usually assumed. Essentially, for the purpose of this work, they are stable solutions to a field theory which cannot be continuously deformed into the vacuum solution and which correspond to localised (i.e., rapidly decreasing) energy concentrations. They are usually classified according to the effective dimensionality of this energy concentration configuration: a zero-dimensional (point-like) defect is a monopole, a one-dimensional (line-like) defect is a string (also called a vortex-line), a two-dimensional (membrane-like) defect is a domain wall, and a three-dimensional defect is a texture. Cosmic strings, in which we are interested, are simply strings which arise in cosmological scenarios (usually involving phase transitions with spontaneous symmetry breaking). In most realistic models, their thickness is comparable to the size of a proton whereas their length is comparable to the size of the observable Universe (hence their being considered "effectively" one-dimensional objects). They can be extremely massive - for example, a few metres of a string formed at a GUT scale should weigh about as much as the Earth.

The connection between topological defects and symmetry breaking is apparent in the so-called *Kibble Mechanism* [14], which describes the way topological defects are formed in these kinds of cosmological phase transitions. The idea is that, when a field must settle for a broken symmetry ground state, the choice must be dictated by random fluctuations (there is always more than one equivalent such state since a transformation that leaves $\mathcal{L}$ invariant must produce a ground state when acting on a ground state) and these choices must not be correlated in distant points (in particular, if the distance between two points is greater than the size of the cosmological horizon[1], causality implies that the two choices cannot be the least bit correlated). The consequence of having distant points settling for randomly different ground states is that in general continuity will force small regions of space to not be in any ground state - these will correspond to

[1]Defined as the maximum distance from which it is possible to have received light anywhere in the Universe.

topological defects and their dimensionality will be determined by topological features of the vacuum manifold of the theory in question. For example, a purely real version of Lagrangian 1.1.1 corresponds to a disconnected vacuum manifold (corresponding to the two points $\phi = \pm\eta$) and that alone is enough to say this theory must form domain walls in these transitions. In this case this was easy to see: if two three-dimensional regions in space have chosen different vacuum phases (i.e., one is positive and the other is negative), then there can only be continuity if they are bounded by effectively bi-dimensional regions where ϕ varies from $-\eta$ to η. Less obvious, but equally forced by topological considerations, is the fact that the complex version of Lagrangian 1.1.1 must form strings - now because the vacuum manifold (corresponding to the $|\phi| = |\eta|$ circunference) admits closed paths that cannot be smoothly deformed into a point. This happens because choosing a ground state is now tantamount to choosing a phase for ϕ. The lack of correlation in this choice over large distances implies that it may be possible to find closed loops in physical space along which this phase happens to vary by a non-zero multiple of 2π - and then continuity implies that there must be at least one point in any surface enclosed by that path where the phase cannot be defined - i.e., $\phi = 0$.

This relation between topological defects and these phase transitions has been extensively studied in the context of condensed matter systems such as magnetic materials (which usually form domain walls) and superconductors / superfluid helium (which usually forms vortices). So far, only topological defects in cosmological contexts remain to be observed (in fact, it has been quipped that they "have been seen everywhere but in the Universe"). Nevertheless, the fact that such general classes of cosmological models should form topological defects motivate the most optimistic while the most cautious use the same fact to argue that even an experimental rejection of the most popular defect scenarios would enable us to learn more about cosmological phase transitions.

1.2. The Abelian-Higgs: A Simple Model

There is a myriad of (arbitrarily complex) models other than the simple 1.1.1 which admit string solutions. Typically, a particular model is relevant in the context of a particular fundamental particle physics scenario, and each particular model will have strings behaving differently: their energy profile is different, as is the nature of the forces acting between strings, and some may even carry extra degrees of freedom (e.g. charges and currents, as in the case of *superconducting* strings). And in spite of all that, we are able to talk of strings without mention to the underlying physical picture because of one simple result: away from the string (i.e., when the distance to the string is much larger than its typical thickness) there is only a small number of relevant degrees of freedom. In this work, we are mainly interested in studying the simplest kinds of strings, which are simply characterized by their energy density. Not only are these simple (due to having only one degree of freedom), but they are also of great physical interest, since they usually arise as by-products of the breaking of local gauge symmetries.

Before focusing on the general properties of the dynamics of these strings, let us take a look at an important example of a specific realistic simple model: the Abelian-Higgs model. It is usually regarded as the archetype of such a "vanilla" stringy model, and it is used in most field theory simulations of cosmic strings available. It is a local version of 1.1.1 defined by the following Lagrangian density for a charged complex scalar field ϕ and gauge fields A^μ with the same "mexican hat" potential as before:

$$\mathcal{L} = \left(\partial_\mu + ieA_\mu\right)\overline{\phi}\left(\partial^\mu - ieA^\mu\right)\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\lambda\left(\left|\phi\right|^2 - \eta^2\right)^2 \tag{1.2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ as usual and η is some positive real number. Note that the non-relativistic limit of this Lagrangian density corresponds to the Ginzburg-Landau model of a superconductor if we interpret ϕ as the wave function of the Cooper pair.

A vortex configuration in a two-dimensional slice of space should have (in the Lorenz gauge) [30]

$$\begin{cases} \phi \longrightarrow \eta e^{i\vartheta(\theta)} & r \longrightarrow \infty \\ A_\theta \longrightarrow \frac{1}{er}\frac{d\vartheta}{d\theta}, & r \longrightarrow \infty \end{cases} \tag{1.2.2}$$

Note that the phase ϑ is being written as a differentiable function which must obey $\vartheta(0) = 0$ and $\vartheta(2\pi) = 2\pi n$ for some $n \in \mathbb{Z}$ (normally referred to as the winding number). One of the interesting consequences of this fact is that even in this semi-classical approach there is magnetic flux quantisation since (for a surface enclosed by a closed path away from $r = 0$)

$$\Phi_B \equiv \iint \mathbf{B}\cdot d\mathbf{S} = \oint \mathbf{A}\cdot d\mathbf{l} = \frac{1}{e}\int_0^{2\pi} \frac{d\vartheta}{d\theta}d\theta = \frac{2\pi}{e}n \tag{1.2.3}$$

The static cylindrically symmetric vortex solutions in this model are known as the

Abrikosov-Nielsen-Olesen vortices and can be written as [30]

$$\begin{cases} \phi(\mathbf{r}) = \eta e^{in\theta} f(r) \\ A_\theta(\mathbf{r}) = -n\frac{\alpha(r)}{er} \end{cases} \tag{1.2.4}$$

where the radial functions α and f tend to 1 as r goes to infinity and tend to 0 as r goes to 0. They can be found by solving

$$\begin{cases} \frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - n^2 \frac{f}{r^2}(\alpha-1)^2 - \frac{\lambda}{2} f(f^2-1) = 0 \\ \frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} - 2e^2 f^2(\alpha-1) = 0 \end{cases} \tag{1.2.5}$$

which has to be done numerically since exact solutions are not known. Once these have been calculated, one can for example calculate the string energy per unit length, μ_0, finding (for $n = \pm 1$, which is the non-trivial case with the lowest energy) [30]

$$\mu_0 = 2\pi\eta^2 g\left(\frac{\lambda}{2e^2}\right) \tag{1.2.6}$$

where $g(\beta)$ is a function of order unity which is exactly 1 if $\beta = 1$. It is interesting to note that, despite its small effect on μ_0, the value of this parameter $\beta = \lambda/2e^2$ influences the way in which vortices interact. For $\beta = 1$, there are no forces acting between two static vortices, whereas if $\beta > 1$ ($\beta < 1$) vortices experience repulsive (attractive) short-range forces (short-range in the sense that their intensity decreases exponentially with distance). Note that the short-ranged nature of these forces is a general feature of strings originated in the breaking of a local symmetry (because the particle mediating these forces is the massive boson associated with this breaking).

1.3. Basics of String Dynamics

Let us now focus on the general case of local or gauge strings (also known as Goto-Nambu strings), which are free of long-range interactions. For problems in which the typical distances involved are much greater than the thickness of the string, its dynamics can be derived from the effective action known as the Goto-Nambu action [30]

$$S = -\mu_0 \int \sqrt{-\gamma}\, d^2\zeta \tag{1.3.1}$$

where γ is the determinant of the induced worldsheet metric defined in terms of a background metric $g_{\mu\nu}$ and a bidimensional parametrization $x^\mu = x^\mu(\zeta^a)$ of the string (where it is canonically chosen that ζ^0 be timelike and ζ^1 spacelike) by

$$\gamma_{ab} = g_{\mu\nu} x^\mu_{,a} x^\nu_{,b} \tag{1.3.2}$$

It is interesting to note that this action is exactly proportional to the area swept out by the motion of the string.

The equations of motion for x^μ are then simply

$$\nabla^2 x^\mu + \Gamma^\mu_{\nu\lambda} \gamma^{ab} x^\nu_{,a} x^\lambda_{,b} = 0 \tag{1.3.3}$$

where $\Gamma^\mu_{\nu\lambda}$ are the Christoffel symbols for the background metric, γ^{ab} is the inverse of γ_{ab}, and ∇^2 is the covariant laplacian given by

$$\nabla^2 x^\mu = \frac{1}{\sqrt{-\gamma}} \partial_a \left(\sqrt{-\gamma} \gamma^{ab} x^\mu_{,b} \right) \tag{1.3.4}$$

Considering the flat FRW metric

$$ds^2 = a^2 \left(d\tau^2 - d\mathbf{x}^2 \right) \tag{1.3.5}$$

and the usual transverse temporal gauge[2]

$$\begin{cases} \zeta^0 = \tau \\ \dot{\mathbf{x}} \cdot \mathbf{x}' = 0 \end{cases} \tag{1.3.6}$$

the induced metric becomes a diagonal matrix

$$[\gamma_{ab}] = \begin{bmatrix} a^2 \left(1 - \dot{\mathbf{x}}^2\right) & 0 \\ 0 & -a^2 \mathbf{x}'^2 \end{bmatrix} \tag{1.3.7}$$

and the energy per unit ζ^1 (usually also called σ now that ζ^0 has been identified with the conformal time τ) is

$$\epsilon = -\frac{x'^2}{\sqrt{-\gamma}} = \sqrt{\frac{\mathbf{x'^2}}{1 - \dot{\mathbf{x}}^2}} \tag{1.3.8}$$

meaning that the energy in a segment of string is simply

[2]Using the standard notation with $\dot{Q} \equiv dQ/d\tau$ and $Q' \equiv dQ/d\zeta^1$.

$$E = \mu_0 a \int \epsilon d\sigma \tag{1.3.9}$$

The zero-component of eq. 1.3.3 then becomes

$$\dot{\epsilon} + 2\epsilon\frac{\dot{a}}{a}\dot{\mathbf{x}}^2 = 0 \tag{1.3.10}$$

while its i-components give

$$\ddot{\mathbf{x}} + 2\frac{\dot{a}}{a}\dot{\mathbf{x}}\left(1 - \dot{\mathbf{x}}^2\right) = \frac{1}{\epsilon}\left(\frac{\mathbf{x}'}{\epsilon}\right)' \tag{1.3.11}$$

Note also that averages in a string can be conveniently defined at the expense of ϵ by

$$\langle Q \rangle \equiv \frac{\int Q\epsilon d\sigma}{\int \epsilon d\sigma} \tag{1.3.12}$$

1.4. String Intersections

As the configuration of a string network evolves according to eq. 1.3.3 it often happens that two string segments (either from different strings or from separate locations of the same string) travel towards each other and get very close. Then, clearly, the zero-width approximation which enables us to write the effective action 1.3.1 breaks down and we momentarily need to consider the full microscopic model in order to know what happens. Fortunately, when this kind of crossing takes place, there are generaly only three possible outcomes (the probability of each happening depends on the particular model we are working with, and can usually only be estimated by numerical means due to the nonlinear complexity of these phenomena) [30]:

1. The segments *exchange partners* or *intercommute*, i.e., their ends detach from one segment and get stuck to another at the intersection. This kind of process can often lead to loop production as illustrated in figure 1.4.1. A generic consequence of this phenomenon is the appearance of discontinuities in the direction of the string tangent (called *kinks*). This is typically the most likely outcome and Goto-Nambu simulations usually just assume this happens every time segments cross (as is the case for the Abelian-Higgs model).

2. The strings become connected by a *bridge* which can be point-like (*junction*) or string-like (*zipper*), depending on the specific model. This kind of consequence is

especially relevant in the study of *cosmic superstrings*.

3. The segments do not interact and come out of the "collision" intact - by far the least interesting and least likely result.

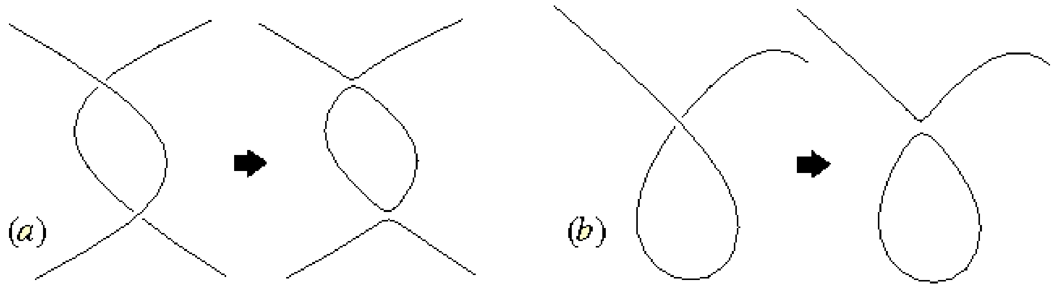

Figure 1.4.1.: (figure borrowed from [30]) A loop is produced due to exchange of pairs in (a) a crossing between two colliding strings (b) self-intersection in a single string.

2. Cosmic String Evolution

In section 1.3 we discussed the motion of a single Goto-Nambu string. In this chapter, we shall review standard ways to make use of these results in the context of network evolution.

2.1. Scaling Solutions: A Simple Picture

The first assumption that is canonically made about cosmological string networks, and which can be seen to hold approximately in high-resolution simulations [23], is that they look like random walks on large scales (i.e., they are *Brownian* networks). In particular, their large-scale properties are assumed to be described by a single characteristic length L which determines the typical distance between neighbouring strings (moreover, L is usually also identified with the radius of curvature of strings - yet another assumption which can be tested numerically [25]). On average, there is about one segment of length L per volume L^3, meaning that the energy density of the network can be written as

$$\rho = \frac{\mu_0}{L^2} \tag{2.1.1}$$

As a first approximation, one might be tempted to assume that cosmological expansion would stretch L as $L \propto a$ (which essentially corresponds to focusing a limit in which the strings are "frozen" and thus do not interact). That scenario, however, is not cosmologically viable as that would mean $\rho \propto a^{-2}$, implying that the string energy density would decay slower than both radiation and matter (which decay as a^{-4} and a^{-3}, respectively) and thus soon dominate the Universe. Clearly then, the interactions between strings and consequent loss of energy into loops (which are expected to decay both gravitationally and into smaller loops) are of paramount importance when studying the fate of a network.

Interestingly, when these losses into loops are taken into account, they are expected to balance the evolution equation for ρ in such a way that the network approaches a *scaling* regime with L proportional to the horizon size $d_H \propto t$. This kind of asymptotic behaviour is observed in simulations as well as in the standard models for network evolution which will be discussed in subsequent sections. This is because a feature which is typically taken into account in these models (and which, once again, is in reasonable agreement with numerical simulations) is that the energy loss into loops goes as $\left.\frac{d\rho}{dt}\right|_{loops} \sim -\frac{\mu_0}{L^3} = -\frac{\rho}{L}$ (which roughly corresponds to having a segment travel a distance of L before encountering another segment and producing a loop of size L). This means that the evolution equation for ρ can be written approximately as the sum of a contribution from eq. 1.3.10 with this term, becoming

$$\frac{d\rho}{dt} \approx -2H\rho - \frac{\rho}{L} \tag{2.1.2}$$

which can be rewritten in terms of $\gamma \equiv L/t$ and $\lambda = Ht \equiv \frac{t}{a}\frac{da}{dt}$ (meaning $\lambda = 1/2$ in the radiation era and $\lambda = 2/3$ in the matter era) as

$$\frac{1}{\gamma}\frac{d\gamma}{dt} = \frac{1}{2t}\left(2\left(\lambda - 1\right) + \gamma^{-1}\right) \tag{2.1.3}$$

which has the scaling solution

$$\gamma = \frac{1}{2\left(1 - \lambda\right)} \tag{2.1.4}$$

as an attractor.

2.2. The One-Scale Model

The One-Scale model, due to works by Kibble [16] and Bennet [6] in the mid-eighties, is among the first efforts to describe the cosmological evolution of a system of cosmic strings and make some quantitative predictions about the attractor scaling solution. Even though it is not the most accurate model available, it is worth some consideration since its basic assumptions are the starting point of the standard more reliable alternatives (such as the velocity-dependent one-scale model, which will be discussed in section 2.3).

This model is essentially a more rigorous version of the picture in section 2.1 - one in which both the average rms velocity of strings and energy losses into loops are taken into account. The main simplification in this model (and the one in the origin of its name) is that the only relevant length scale in a string network is the characteristic length L. In particular, the fraction of energy lost into loops of size between l and $l + dl$ per correlation volume is determined by some loop production function $f\left(l/L\right)$ (which we write as an undefined function we expect to peak around $l = L$, but whose exact shape can be sought numerically) so that the energy loss into loops becomes

$$\left.\frac{d\rho}{dt}\right|_{loops} = -\frac{\mu_0}{L^3}\int_0^\infty f\left(l/L\right)\frac{dl}{L} \equiv -c\frac{\rho}{L} \tag{2.2.1}$$

If to this we add the contribution from eq. 1.3.10 (this time not neglecting the velocity) the analogue of eq. 2.1.2 becomes

$$\frac{d\rho}{dt} = -2H\rho\left(1+v^2\right) - c\frac{\rho}{L} \tag{2.2.2}$$

where v^2 is just the rms velocity of the strings defined as

$$v^2 \equiv \left\langle \dot{\mathbf{x}}^2 \right\rangle \equiv \frac{\int \epsilon \dot{\mathbf{x}}^2 d\sigma}{\int \epsilon d\sigma} \tag{2.2.3}$$

which is treated as a constant in this equation (depending only on whether the Universe is dominated by radiation or matter).

As expected, there is an attractor scaling solution given by (once again making $\gamma \equiv L/t$ and $H = \lambda/t$)

$$\gamma = \frac{c/2}{1-\lambda\left(1+v^2\right)} \tag{2.2.4}$$

(note that c and v^2 should both depend on λ).

2.3. The Velocity-dependent One-Scale Model

As has been mentioned before, the One-Scale model is not the most accurate of models. That is because it fails to capture some important physical phenomena due to two main limitations: the fact that the rms velocity is being considered a constant, which is not expected to hold but in a scaling regime, and the fact that it is completely "blind" to the presence of small-scale structure in the network (i.e., kinks and wiggles formed by intercommutation). The latter is a more fundamental difficulty in the sense that it is an unavoidable consequence of the assumption that there is only one relevant length scale L, so we will postpone its discussion until chapter 3, where we focus on original work on how to get around this problem. The former, however, is basically due to a purely mathematical simplification which can be dropped if the resulting additional terms can all be written in terms of v and L. This can be achieved by noting the evolution of v in eq. 2.2.2 depends on $\ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}$ and using the identities (see [19])

$$\frac{1}{\epsilon\left(1-\dot{\mathbf{x}}^2\right)}\left(\frac{\mathbf{x}'}{\epsilon}\right)' \cdot \dot{\mathbf{x}} = -\frac{\mathbf{x}' \cdot \dot{\mathbf{x}}'}{\mathbf{x}'^2} = \frac{a}{R}\left(\dot{\mathbf{x}} \cdot \hat{\mathbf{u}}\right) \tag{2.3.1}$$

where $\hat{\mathbf{u}}$ is the unit vector parallel to the curvature radius vector.

The resulting system of equations is then (after identifying $L = R$ and substituting c for cv to take into account that the rate at which intersections occur is proportional to the velocity of the string segments involved)

$$\begin{cases} 2\frac{dL}{dt} = 2HL\left(1+v^2\right) + cv \\ \frac{dv}{dt} = \left(1-v^2\right)\left[\frac{k(v)}{L} - 2Hv\right] \end{cases} \tag{2.3.2}$$

where k, called the *momentum parameter*, is defined as

$$k = \frac{\left\langle\left(1-\dot{\mathbf{x}}^2\right)\left(\dot{\mathbf{x}} \cdot \hat{\mathbf{u}}\right)\right\rangle}{v\left(1-v^2\right)} \tag{2.3.3}$$

and in most relevant regimes (both relativistic and non-relativistic) can be written as (see [22])

$$k\left(v\right) = \frac{2\sqrt{2}}{\pi}\frac{1-8v^6}{1+8v^6} \tag{2.3.4}$$

In doing so, one ends up with a model that retains most of the simplistic features of the one-scale model but is still valid in regimes with varying v (notably in friction-dominated regimes) and can thus be used to make predictions across the whole history of a network (i.e., there is no need to change parameters as we change cosmological eras). In particular, we find the attractor scaling solution

$$\begin{cases} \gamma^2 = \frac{k(k+c)}{4\lambda(1-\lambda)} \\ v^2 = \frac{k(1-\lambda)}{\lambda(k+c)} \end{cases} \tag{2.3.5}$$

when $a \propto t^\lambda$ (note that the second equation is an implicit equation for the velocity which fixes k). Nevertheless, scaling solutions do not seem to exist when the scale factor is not a power law - as during the transition between the radiation and the matter epoch or after the onset of dark energy domination (around the present time). During these transition epochs, eqs. 2.3.2 have to be treated numerically - and it is found that realistic cosmic string networks should not have enough time to reach a scaling regime during the matter-dominated era [22].

It is this improved version of the one-scale picture, first developed by Martins and Shellard [19, 20, 21], which is commonly known as the Velocity-dependent One-Scale model (VOS). Despite its simplicity, it performs remarkably well when tested against high-resolution simulations [25], which makes its use the most reliable method when it comes to making quantitative predictions about the evolution of the large-scale properties (i.e., L and v) of a network. It is also interesting to note that, even though L is still the only length scale playing a role in this model, the momentum parameter, as defined in eq. 2.3.3, clearly depends on the shape of small-scale structure.

3. Wiggly Cosmic String Evolution

As was briefly mentioned in section 2.3, one of the most serious problems with models assuming a one-scale approximation is that they are intrinsically limited in how much they can tell us about what happens in length scales below the characteristic length L. This problem is especially relevant since it has been realised that realistic networks should build up a significant amount of structure on these scales - mainly as a consequence of kinks formed by intercommutations [3, 23].

There have been previous attempts to tackle this problem. A significant step was taken by Kibble and Copeland in 1991 with the introduction of a two-scale model [15] which successfully shed some light into the fate of wiggly networks in spite of a few quantitative shortcomings (briefly addressed in [30]). Subsequently, the authors, together with Austin, suggested a three-scale model [4] to overcome these difficulties - however, the additional degree of freedom this implied seriously compromised the predictive power of the model.

With this in mind, in this chapter we present a generalisation of the VOS model which explicitly takes into account the evolution of small-scale structure in the network while still preserving the main victories of the VOS. The basic formalism introduced herein is the result of original work done in collaboration with C. J. A. P. Martins (the supervisor of this thesis) and E. P. S. Shellard which can be found in [24].

3.1. Elastic String Dynamics

In the zero-width approximation, a generic string model can be defined by an action

$$S = -\int \mathcal{L}\sqrt{-\gamma}d^2\zeta \tag{3.1.1}$$

where the Lagrangian density can depend on the background metric $g_{\mu\nu}$, background fields such as Maxwell-type gauge potentials A_μ or a Kalb-Ramond gauge field $B_{\mu\nu}$ (but not their gradients) and relevant internal fields encoded in a function Λ (usually called the *master function*) [10, 9]

$$\mathcal{L} = \Lambda + J^{\mu} A_{\mu} + \frac{1}{2} W^{\mu\nu} B_{\mu\nu} + ... \tag{3.1.2}$$

It is easy to see that the action for the Goto-Nambu strings we have focused on can be obtained by simply setting $\Lambda = -\mu_0$ and making all else null. Superconducting strings, for example, correspond to the case in which the Maxwell field is important while the term with the Kalb-Ramond field is ideal for describing global strings (i.e., strings arising from the breaking of a global symmetry, which typically have long-range interactions mediated by the massless Goldstone boson associated with this breaking). A well known feature arising from the Goto-Nambu action is that the string tension and energy density coincide

$$U = T = \mu_0 \tag{3.1.3}$$

An *elastic string* model is one for which U and T do not necessarily coincide (and can even vary). These models can be described [9, 10] by a function $\Lambda = \Lambda(\chi)$ where for a simple case in which external fields are not relevant (such as the one we are interested in)

$$\chi \equiv \gamma^{ab} \phi_{,a} \phi_{,b} \tag{3.1.4}$$

ϕ being a scalar field identified with a *stream function* defined on the worldsheet which is constant along the flow lines of a conserved current.

At this point, the attentive reader might be wondering about the relevance of this little detour. If we are only interested in Goto-Nambu strings, why should we look at these more general models? The answer is related to the property 3.1.3 of Goto-Nambu strings: if we look at a very wiggly Goto-Nambu string from afar, we are not able to make out the small-scale details of its configuration - instead, it seems as though we are looking at a much smoother string which happens to have a much higher energy density and a much lower tension! This means that a macroscopic description of a wiggly Goto-Nambu string is equivalent to the microscopic description of an elastic string. In particular, it makes sense to think of wiggly strings as carrying a mass current which "renormalizes" the bare energy per unit length μ_0. This interpretation motivates the use of the Lagrangian density

$$\mathcal{L} = -\mu_0 \sqrt{1 - \gamma^{ab} \phi_{,a} \phi_{,b}} \tag{3.1.5}$$

which has the equation of state

$$UT = \mu_0^2 \tag{3.1.6}$$

which has been shown to be the exact equation of state for a macroscopic (averaged) wiggly Goto-Nambu string [8, 18]. Consistently with this physical interpretation, ϕ is treated as a mesoscopic quantity which depends only on the worldsheet time and some "renormalization scale" ℓ which must be large enough for the spatial dependences of U to be negligible but still much smaller than the correlation length.

It is useful to introduce the dimensionless parameter

$$w \equiv \sqrt{1-\chi} \tag{3.1.7}$$

in terms of which the local string tension and energy density can be written:

$$T = \mu_0 w, \; U = \frac{\mu_0}{w} \tag{3.1.8}$$

Note that w must be between 0 and 1 ($w = 1$ corresponding to the Goto-Nambu case).

The equations of motion now become

$$\dot{\left(\frac{\epsilon}{w}\right)} + \left(\frac{\epsilon}{w}\right)\frac{\dot{a}}{a}\left[2w^2\dot{\mathbf{x}}^2 + \left(1+\dot{\mathbf{x}}^2\right)\left(1-w^2\right)\right] = 0 \tag{3.1.9}$$

$$\ddot{\mathbf{x}} + \dot{\mathbf{x}}\left(1-\dot{\mathbf{x}}^2\right)\frac{\dot{a}}{a}\left(1+w^2\right) = \frac{w^2}{\epsilon}\left(\frac{\mathbf{x}'}{\epsilon}\right)' \tag{3.1.10}$$

$$\frac{\dot{w}}{w} = \left(1-w^2\right)\left(\frac{\dot{a}}{a} + \frac{\mathbf{x}'\cdot\dot{\mathbf{x}}'}{\mathbf{x}'^2}\right) \tag{3.1.11}$$

where 3.1.9 is the general (wiggly) form of 1.3.10, 3.1.10 is the general form of 1.3.11, and 3.1.11 comes simply from varying the action with respect to ϕ and using the definition of w.

3.2. Averaged Evolution

There are now two independent measures of energy one can work with. These are the *total energy* in a string segment

$$E = a\int \epsilon U d\sigma = \mu_0 a \int \frac{\epsilon}{w} d\sigma \tag{3.2.1}$$

and the energy in a Goto-Nambu string segment with the same configuration (without wiggles), called the *bare energy*

$$E_0 = \mu_0 a \int \epsilon d\sigma \tag{3.2.2}$$

And even though they are not independent of these two, it may also be convenient to introduce the *energy in small-scale wiggles*

$$E_w = E - E_0 = \mu_0 a \int \frac{1-w}{w} \epsilon d\sigma \tag{3.2.3}$$

and the renormalization factor

$$\mu \equiv \frac{E}{E_0} > 1 \tag{3.2.4}$$

Naturally, both energies can be used to define different characteristic lengths

$$\rho \equiv \frac{\mu_0}{L^2} \tag{3.2.5}$$

$$\rho_0 \equiv \frac{\mu_0}{\xi^2} \tag{3.2.6}$$

which are related via

$$\xi^2 = \mu L^2 \tag{3.2.7}$$

Since in general a network may be composed of strings with different "wiggliness" (and two very distant segments in the same string may even, for some reason, have acquired slightly different amounts of small-scale structure[1]) it is important to distinguish between two averaging procedures. It seems logical that the natural average on this string be defined over the total energy as

$$\langle Q \rangle = \frac{\int QU\epsilon d\sigma}{\int U\epsilon d\sigma} = \frac{\int Q\frac{\epsilon}{w} d\sigma}{\int \frac{\epsilon}{w} d\sigma} \tag{3.2.8}$$

but in some instances it may happen that the most convenient average to take into account be the complementary "bare" average

$$\langle Q \rangle_0 = \frac{\int Q\epsilon d\sigma}{\int \epsilon d\sigma} \tag{3.2.9}$$

[1]This formalism still holds if w varies very slowly with σ - the effects of this variation in the equations of motion are proportional to w' so it is feasible for this variation to be slow enough not to be significant in these local equations but still fast enough to have a relevant effect over cosmological length scales.

Note that the two procedures are simply related by

$$\langle Q \rangle = \frac{\langle QU \rangle_0}{\langle U \rangle_0} = \frac{\langle Q/w \rangle_0}{\mu} \tag{3.2.10}$$

where we have used the useful (and trivial) identity

$$\mu = \langle w \rangle^{-1} = \left\langle w^{-1} \right\rangle_0 \tag{3.2.11}$$

In particular, we now define the rms velocity v by

$$v^2 \equiv \left\langle \dot{\mathbf{x}}^2 \right\rangle \tag{3.2.12}$$

We can now use the equations of motion (3.1.9, 3.1.10, and 3.1.11) to deduce evolution equations for these quantities - with exactly the same kind of calculations that enabled us to write eqs. 2.3.2. For the energies these are

$$\frac{\dot{E}}{E} = \frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a} = \left[\left\langle w^2 \right\rangle - v^2 - \left\langle w^2 \dot{\mathbf{x}}^2 \right\rangle\right] \frac{\dot{a}}{a} \tag{3.2.13}$$

$$\frac{\dot{E}_0}{E_0} = \frac{\dot{\rho}_0}{\rho_0} + 3\frac{\dot{a}}{a} = \left[1 - \mu \left\langle w \left(1 + w^2\right) \dot{\mathbf{x}}^2 \right\rangle\right] \frac{\dot{a}}{a} - \frac{a\mu}{R} \left\langle w \left(1 - w^2\right) (\dot{\mathbf{x}} \cdot \hat{\mathbf{u}}) \right\rangle \tag{3.2.14}$$

where R is the curvature radius of the bare string.

So far in our analysis the renormalization scale ℓ has been deliberately omitted. That is simply because it is easier to insert it in the evolution equations starting with the equation for μ. From the definition of μ (eq. 3.2.4) one expects the simple relation

$$\frac{\dot{\mu}}{\mu} = \frac{\dot{E}}{E} - \frac{\dot{E}_0}{E_0} \tag{3.2.15}$$

In fact, if we substitute eqs. 3.2.13 and 3.2.14 here we find the correct expression for the evolution of μ on a fixed scale ℓ. If, however, we want ℓ to vary as a function of time - and in general we do, lest the small-scale structures we try to model be stretched to scales above ℓ by Hubble expansion - then we must be more careful. Generically, we have

$$\frac{\dot{\mu}}{\mu} = \left.\frac{\dot{\mu}}{\mu}\right|_{fixed\ \ell} + \frac{1}{\mu}\frac{\partial \mu}{\partial \ell}\dot{\ell} \tag{3.2.16}$$

where the new scale drift term can be conveniently expressed in terms of the multifractal dimension of the string segments at the scale ℓ, noted $d_m(\ell)$. If we assume (as we should in order to make physical sense) that $\ell \ll R$ then we can use (see appendix A)

$$\frac{1}{\mu}\frac{\partial\mu}{\partial\ell} = \frac{d_m(\ell) - 1}{\ell} + \mathcal{O}\left\{\frac{\ell}{R^2}\right\} \tag{3.2.17}$$

and write

$$\frac{\dot{\mu}}{\mu} = \frac{a\mu}{R}\left\langle w\left(1 - w^2\right)(\dot{\mathbf{x}}\cdot\hat{\mathbf{u}})\right\rangle + \frac{\dot{a}}{a}\left[\left\langle w^2\right\rangle - 1 + \left\langle(\mu w - 1)\left(1 + w^2\right)\dot{\mathbf{x}}^2\right\rangle\right] + [d_m(\ell) - 1]\frac{\dot{\ell}}{\ell} \tag{3.2.18}$$

Because E cannot depend on ℓ, eq. 3.2.15 then implies

$$\frac{\dot{E}_0}{E_0} = \frac{\dot{\rho}_0}{\rho_0} + 3\frac{\dot{a}}{a} = \left[1 - \mu\left\langle w\left(1 + w^2\right)\dot{\mathbf{x}}^2\right\rangle\right]\frac{\dot{a}}{a} - \frac{a\mu}{R}\left\langle w\left(1 - w^2\right)(\dot{\mathbf{x}}\cdot\hat{\mathbf{u}})\right\rangle - [d_m(\ell) - 1]\frac{\dot{\ell}}{\ell} \tag{3.2.19}$$

Note that this makes physical sense since changing ℓ is tantamount to redefining what is a wiggle and what is not and thus transfering energy between E_0 and E_w. Note also that these terms are irrelevant in the case with $d_m = 1$ (straight lines look like straight lines across all scales).

Finally, we can write the evolution equation for the rms velocity

$$\dot{(v^2)} = \frac{2a}{R}\left\langle w^2\left(1 - \dot{\mathbf{x}}^2\right)(\dot{\mathbf{x}}\cdot\hat{\mathbf{u}})\right\rangle - \frac{\dot{a}}{a}\left\langle\left(v^2 + \dot{\mathbf{x}}^2\right)\left(1 + w^2\right)\left(1 - \dot{\mathbf{x}}^2\right)\right\rangle + \frac{1 - v^2}{1 + \langle w^2\rangle}\frac{\partial\langle w^2\rangle}{\partial\ell}\dot{\ell} \tag{3.2.20}$$

where the (last) scale-drift term is included for analogous reasons (just imposing $\dot{E}/E$ to be scale-invariant).

The presence of this scale-drift term in the equation for velocity may be rather unexpected since in the VOS v is interpreted as a microscopic quantity. However, clearly now it has to be regarded as a mesoscopic one, which behaves more like a coherent velocity than like a microscopic rms velocity. It is interesting to note that this is actually a consequence of the renormalization procedure that generically cannot act on $\mathbf{x}$ without having some effect on v. In fact, when we imposed that $U' = 0$ we were implicitly imposing another mesoscopic-like property to $\dot{\mathbf{x}}$: $\left(\dot{\mathbf{x}}^2\right)' = 0$! Intriguingly, this can be seen (using the gauge condition 1.3.6) to be equivalent to assuming $\ddot{\mathbf{x}}\cdot\mathbf{x}' = 0$, which is to be expected if the acceleration is essentially due to curvature effects.

It is also worthwhile to notice that Hubble stretching and string curvature seem to act on wiggliness and velocity in analogous ways, the most important difference being that the curvature term for wiggliness vanishes in both the Goto-Nambu ($w = 1$) and the tensionless ($w = 0$) limits, while the analogous term for velocity vanishes only in the tensionless limit.

3.3. Network Dynamics

The discussion in section 3.2 merely describes the evolution of a network in the absence of intercommutation. As has been discussed in chapter 2, energy loss into loops is crucial to the cosmological fate of string networks. In addition to that, intercommutation events which do not produce loops are also important in any wiggly evolution model since they still produce kinks - which should be modeled as an energy transfer from the bare string to the wiggles.

In order to address this energy loss phenomenology it is necessary to identify one (or a combination) of the two characteristic length scales of the problem (3.2.5 and 3.2.6) with the string correlation length which is important for loop production. If the renormalization scale ℓ is being chosen such that the bare network looks Brownian, then we can recover the kind of argument employed to write eq. 2.1.1 and identify the correlation length (and the curvature) with ξ.

We then write, in analogy with what was done to model losses to loops in section 2.3,

$$\frac{1}{\rho_0}\frac{d\rho_0}{dt}\bigg|_{loops} = -cf_0\left(\mu\right)\frac{v}{\xi} \tag{3.3.1}$$

$$\frac{1}{\rho_w}\frac{d\rho_w}{dt}\bigg|_{loops} = -cf_1\left(\mu\right)\frac{v}{\xi} \tag{3.3.2}$$

$$\frac{1}{\rho}\frac{d\rho}{dt}\bigg|_{loops} = -cf\left(\mu\right)\frac{v}{\xi} \tag{3.3.3}$$

where the f parameters are allowed to explicitly depend on μ because numerical simulations suggest that small-scale structure enhances loop production. They should be related by

$$f\left(\mu\right) = \frac{f_0\left(\mu\right)}{\mu} + \left(1 - \frac{1}{\mu}\right) f_1\left(\mu\right) \tag{3.3.4}$$

and $f_0\left(\mu\right)$ and $f\left(\mu\right)$ should both be unity in the Goto-Nambu limit if c is to be interpreted as the c appearing in eq. 2.3.2 (note that L in those equations should be identified with ξ rather than L because of its interpretation as a correlation length).

And we can now model energy transfers from E_0 to E_w as

$$\frac{1}{\rho_0}\frac{d\rho_0}{dt}\bigg|_{wiggles} = -cs\left(\mu\right)\frac{v}{\xi} \tag{3.3.5}$$

where s should be a non-trivial function that vanishes in the Goto-Nambu limit and encodes the effects of kink formation due to intercommutation and kink decay due to gravitational radiation.

Note that it is easier to "guess" a form for the *loop-chopping* parameters than for s. It is always possible to find some reasonable $f\left(\mu\right)$ and $f_0\left(\mu\right)$ that fit simulations, but it is insufficient to do the same for s since gravitational radiation in long strings is usually not taken into account in simulations.

A natural guess for the loop-chopping parameters, motivated by the fact that the VOS seems to perform well when tested against high-resolution simulations, is one that makes both L and ξ increase in the same way as in the Goto-Nambu case due to losses to loops. This would simply correspond to

$$f_0\left(\mu\right) = 1 \tag{3.3.6}$$

$$f\left(\mu\right) = \sqrt{\mu} \tag{3.3.7}$$

$$f_1\left(\mu\right) = \frac{\mu^{3/2} - 1}{\mu - 1} \tag{3.3.8}$$

Of course there is no reason why these should hold in the most general situation, but the success of the VOS justifies their use as a first approximation in situations with small μ. For a better motivated ansatz which coincides with this in this limit please see appendix B.

3.4. Perturbative Limits

Before going into a full analysis of the wiggly model it is instructive to consider two natural limits: the tensionless limit (when there is much small-scale structure and thus $T \ll U$) and the linear limit (when small-scale structure is scarce and the dynamics should differ only slightly from the VOS case). Both limits substantially simplify the dynamical equations while allowing a clear insight into the physical phenomena taking place.

3.4.1. The Tensionless Limit

The tensionless limit occurs when there is a lot of small scale structure - i.e., $\mu \gg 1$ or, more tractably, $w \to 0^+$. This situation is not necessarily the most realistic (simulations suggest a scenario that is closer to the linear limit) but its study will hopefully provide useful intuition for future studies. Since w is so small, the natural way to simplify the evolution equations in this limit is to treat w as a perturbation and neglect $\mathcal{O}\{w^2\}$.

First of all, we should realise that this limit is necessarily non-relativistic since v can only decrease in a very simple way (because the curvature and the scale-drift term in eq. 3.2.20 both vanish in this limit)

$$v \propto a^{-1} \tag{3.4.1}$$

One of the welcome consequences of this further simplification is that it enables us to neglect the energy loss terms discussed in section 3.3 (since they are all proportional to v). Simply put, because the strings are moving so slowly, there are barely any intercommutations. Of course one can always consider some energy loss parameters which increase tremendously with μ, but the natural guesses do not seem to increase fast enough for this to compensate the low frequency of intercommuting.

It then immediately follows that

$$E = const, \ L \propto a^{3/2} \tag{3.4.2}$$

The situation is physically obvious: the network is effectively frozen and being conformally stretched. As a result, we should expect the wiggliness at a fixed scale to be decreasing. In fact, neglecting scale-drift in eq. 3.2.19 we arrive at

$$E_0, \xi \propto a \tag{3.4.3}$$

which confirms conformal stretching (recall that E_0 is proportional to the length of the bare string), while doing the same thing in eq. 3.2.18 yields

$$\mu \propto a^{-1} \tag{3.4.4}$$

confirming the decrease in wiggliness at a fixed scale.

In the general case with a varying renormalization scale

$$\frac{\dot{E}_0}{E_0} \sim -\frac{\dot{\mu}}{\mu} \sim \frac{\dot{a}}{a} - [d_m(\ell) - 1]\frac{\dot{\ell}}{\ell} \tag{3.4.5}$$

which generically makes μ decay slower than a^{-1}. In fact, if we were to assume a simple fractal network (i.e., with a constant d_m) we could find some suggestive solutions for a few natural choices of ℓ. Following a scale proportional to the scale factor, for instance, would result in

$$\mu \propto a^{d_m-2}, \; \xi \propto a^{(1+d_m)/2} \tag{3.4.6}$$

while following a scale proportional to the correlation length would lead to

$$\mu \propto a^{4/(3-d_m)-3}, \; \xi \propto a^{2/(3-d_m)} \tag{3.4.7}$$

It is pertinent to note that 3.4.6 implies that μ (and thus E_0) is constant for a large-scale Brownian network ($d_m = 2$) while 3.4.7 means this only happens when $d_m = 5/3$ (which, interestingly, is the fractal dimension of a self-avoiding random walk [29]).

3.4.2. The Linear Limit

The most useful perturbative limit is without a doubt the linear limit, when there is only a little small-scale structure and therefore both w and μ are very close to 1 - which simulations predict should be the case [23]. At the mesoscopic level we just define

$$w = 1 - y \tag{3.4.8}$$

where $y \ll 1$; macroscopically this corresponds to

$$\mu \sim 1 + \langle y \rangle \equiv 1 + Y \tag{3.4.9}$$

where Y is also very small and positive.

Curiously, there is now a more intuitive relation between the two averaging procedures

$$\langle Q \rangle \sim \frac{\int Q(1+y)\,\epsilon d\sigma}{(1+y)\,\epsilon d\sigma} \sim \langle Q \rangle_0 + corr_0(y, Q) \tag{3.4.10}$$

(where the correlator is just $corr_0(z, w) = \langle zw \rangle_0 - \langle z \rangle_0 \langle w \rangle_0$) and in particular they are equivalent for quantities independent of w. Assuming that is the case for $\dot{\mathbf{x}}^2$ then

$$\left\langle w^{\alpha_1}\dot{\mathbf{x}}^{2\alpha_2}\right\rangle \sim (1-\alpha_1 Y)\, v^{2\alpha_2} \tag{3.4.11}$$

which greatly simplifies our evolution equations. In what follows we assume $k\,(v)$ maintains its standard form (eq. 2.3.4). This assumption is related to the ubiquitous expectation that our undefined renormalization procedure should transform our complex strings in strings for which the VOS assumptions apply. Still, since we know that the VOS assumptions are not always obeyed everywhere in realistic networks, a generalization of eq. 2.3.4 might be needed to study relativistic scenarios with very high wiggliness.

Linearising the averaged evolution equations (3.2.13, 3.2.18, 3.2.19, and 3.2.20) one finds

$$\frac{\dot{E}}{E} = \left[\left(1-2v^2\right) - 2Y\left(1-v^2\right)\right]\frac{\dot{a}}{a} \tag{3.4.12}$$

$$\frac{\dot{E}_0}{E_0} = \left[\left(1-2v^2\right) + 2Yv^2\right]\frac{\dot{a}}{a} - 2\frac{kaYv}{R} - \left[d_m\left(\ell\right)-1\right]\frac{\dot{\ell}}{\ell} \tag{3.4.13}$$

$$\dot{(v^2)} = 2v\left(1-v^2\right)\left[\frac{ka}{R}\left(1-2Y\right) - 2v\left(1-Y\right)\frac{\dot{a}}{a} - \frac{\left[d_m\left(\ell\right)-1\right]}{2v}\frac{\dot{\ell}}{\ell}\right] \tag{3.4.14}$$

$$\dot{Y} = 2Y\left(\frac{kav}{R} - \frac{\dot{a}}{a}\right) + \left[d_m\left(\ell\right)-1\right]\frac{\dot{\ell}}{\ell} \tag{3.4.15}$$

Finally, switching from conformal to physical time and introducing the energy loss terms discussed in section 3.3, we end up with the following generalised (linear wiggly) VOS model evolution equations

$$2\frac{dL}{dt} = 2\left[1+v^2+Y\left(1-v^2\right)\right]HL + cfv\left(1-\frac{1}{2}Y\right) \tag{3.4.16}$$

$$2\frac{d\xi}{dt} = 2\left[1+\left(1-Y\right)v^2\right]H\xi + \left[2kY + c\left(f_0+s\right)\right]v + \left[d_m\left(\ell\right)-1\right]\frac{\xi}{\ell}\frac{d\ell}{dt} \tag{3.4.17}$$

$$\frac{dv}{dt} = \left(1-v^2\right)\left[\frac{k}{\xi}\left(1-2Y\right) - 2Hv\left(1-Y\right) - \frac{\left[d_m\left(\ell\right)-1\right]}{2v\ell}\frac{d\ell}{dt}\right] \tag{3.4.18}$$

$$\frac{dY}{dt} = \left[2kY + c\left(f_0+s-f\right)\right]\frac{v}{\xi} - 2HY + \frac{\left[d_m\left(\ell\right)-1\right]}{\ell}\frac{d\ell}{dt} \tag{3.4.19}$$

which naturally reduce to the VOS equations for $Y=0$ (when necessarily $d_m=1$). Recall that f_0, f, and s are in principle functions of Y such that f_0+s-f is linear in Y (just like d_m-1).

In order to demonstrate the utility of this simplified model, an analysis of its scaling solutions can be carried out. Given the range of natural choices of ℓ we may resort to, it makes sense to consider $\ell \propto t$ in addition to $\xi = \gamma_\xi t$, $v = const$, and $Y = const$. Moreover, we assume $d_m \sim 1 + 2Y$, which numerical simulations [23] suggest may be a reasonable approximation in expanding space. As expected the resulting solution for (γ_ξ, v) is only a linear perturbation in Y around the VOS solution (γ_{GN}, v_{GN}) which can be found by solving the system 2.3.5. It can be obtained (for $c \neq 0$) by solving the algebraic system

$$\begin{cases} \frac{v^2}{v_{GN}^2} \sim 1 + \beta Y \\ \frac{\gamma_\xi^2}{\gamma_{GN}^2} \sim 1 + \left[\beta + 2B - A + \frac{2\lambda\left(1+v_{GN}^2\right)(\beta-1)}{1-\lambda\left(1+v_{GN}^2\right)}\right] Y \\ Y \sim \frac{2(\lambda-1)[(2A+1)k+(A+1)c]}{[1-2A+2D+(A-2D)\lambda]k+[2D(1-\lambda)-A(2-\lambda)]c} \end{cases} \tag{3.4.20}$$

where $\beta = \frac{c\lambda(2-A-2B)-2k(1-2\lambda)}{2\lambda(k+c)}$, $a \propto t^\lambda$ as before, and the energy loss terms have been conveniently rewritten as

$$\begin{cases} f_0 + s - f \sim AY \\ f_0 + s + f \sim 2\left(1 + BY\right) \\ f_0 + s \sim 1 + DY \end{cases} \tag{3.4.21}$$

Focusing on the third equation in the system 3.4.20, we can use the physical requirement that Y be positive to impose constraints on the linear term in the expansion of $s\left(Y\right)$ and even on A and B. For example, if we assume eqs. 3.3.6 and 3.3.7 (meaning $A \sim -1/2 + D$, $B \sim 1/4 + D/2$, and $DY \sim s$), then

$$Y \sim \frac{2\left(\lambda-1\right)\left[4kD + \left(1+2D\right)c\right]}{\left(4 - \left[1+2D\right]\lambda\right)k + \left(2 - \lambda\left(1+2D\right)\right)c} \tag{3.4.22}$$

and if $k > 0$ (which is favoured by simulations [23]) requiring Y to be positive is equivalent to imposing (recall that D must also be positive)

$$D > \frac{\left(4-\lambda\right)\frac{k}{c} + 2 - \lambda}{2\lambda\left(\frac{k}{c}+1\right)} \tag{3.4.23}$$

while demanding that $Y < 1$ implies

$$D < \frac{\left(3+\frac{k}{c}\right)\lambda - 4\left(\frac{k}{c}+1\right)}{4 + 8\frac{k}{c} - \left(10\frac{k}{c}+2\right)\lambda} \tag{3.4.24}$$

and surprisingly these two conditions are incompatible in both the matter and the radiation epoch (considering $c = 0.23$, as found in numerical simulations of Abelian-Higgs strings [25]), which shows that it is not necessarily trivial to find natural energy loss parameters which enable small-scale scaling - and also that eqs. 3.3.6 and 3.3.7 may not be valid, since simulations suggest small-scale scaling can happen in the linear limit [23].

A more complete discussion of the consistency relation the energy loss functions must obey is left for section 3.5.

3.5. Scaling In The Full Model

First of all, let us write the full evolution equations for a network with uniform wiggliness at a scale $\ell(t)$.

$$2\frac{dL}{dt} = HL\left[3 + v^2 - \frac{(1-v^2)}{\mu^2}\right] + \frac{cfv}{\sqrt{\mu}} \tag{3.5.1}$$

$$2\frac{d\xi}{dt} = H\xi\left[2 + \left(1 + \frac{1}{\mu^2}\right)v^2\right] + v\left[k\left(1 - \frac{1}{\mu^2}\right) + c\left(f_0 + s\right)\right] + \left[d_m(\ell) - 1\right]\frac{\xi}{\ell}\frac{d\ell}{dt} \tag{3.5.2}$$

$$\frac{dv}{dt} = \left(1 - v^2\right)\left[\frac{k}{\xi\mu^2} - Hv\left(1 + \frac{1}{\mu^2}\right) - \frac{1}{1+\mu^2}\frac{\left[d_m(\ell) - 1\right]}{v\ell}\frac{d\ell}{dt}\right] \tag{3.5.3}$$

$$\frac{1}{\mu}\frac{d\mu}{dt} = \frac{v}{\xi}\left[k\left(1 - \frac{1}{\mu^2}\right) - c\left(f - f_0 - s\right)\right] - H\left(1 - \frac{1}{\mu^2}\right) + \frac{\left[d_m(\ell) - 1\right]}{\ell}\frac{d\ell}{dt} \tag{3.5.4}$$

As before, we assume the multifractal dimension can be written as a function of μ. For expanding space, simulations [23] suggest

$$d_m(\ell) = 2 - \frac{1}{\mu^2} \tag{3.5.5}$$

which, as expected, approaches the Brownian case ($d_m \sim 2$) at large scales.

Once again the scaling solutions[2] can be found by solving an algebraic system (for $\mu \neq 1$)

$$\begin{cases} v^2 = \frac{\left[4X^2 - 2\lambda X(1+X)\right](k/c) - X(1-X)(f_0+s)}{\lambda(1+X)^2\left[(k/c) + f_0 + s\right]} \\ \gamma_\xi = v\frac{k(1-X) + c(f_0+s)}{1 + X - \lambda\left[2 + (1+X)v^2\right]} \\ \frac{v}{\gamma_\xi}\left[k(1-X) - c(f - f_0 - s)\right] + (1-\lambda)(1-X) = 0 \end{cases} \tag{3.5.6}$$

where we have used the useful quantity $X \equiv 1/\mu^2$.

[2] Consistently with our previous approach, we are assuming that ℓ is also scaling.

Now we are interested in answering the general question: what conditions must f, f_0, s, and c be subjected to so that scaling solutions exist? If we are only interested in mathematical existence (i.e., if we are not worried about non-physical results such as speeds above the speed of light) then there is an unexpected result: for any specific shape of f, f_0, and s, given any X, there exist at most two values of c, c_X, such that there is a scaling solution with that constant value of X. It is easy to find such a c_X, if it exists: first just compute

$$v^2 = \frac{\left[4X^2 - 2\lambda X\left(1+X\right)\right]\varphi_X - X(1-X)\left(f_0+s\right)}{\lambda\left(1+X\right)^2\left[\varphi_X + f_0 + s\right]} \tag{3.5.7}$$

where φ_X is a real solution of the quadratic equation

$$A\varphi_X^2 + B\varphi_X + C = 0 \tag{3.5.8}$$

where

$$A = (1-\lambda)(1-X)\left(1-X^2\right) - (1-X)\left[4X^2 - 2\lambda(1+X)X\right] + \left(1-X^2\right)\left[1+X-2\lambda\right] \tag{3.5.9}$$

$$\begin{aligned} B = {} & (1-\lambda)(1-X^2)(2-X)(f_0+s) + (f-f_0-s)(4X^2 - 2\lambda(1+X)X) + \\ & + (f_0+s)X(1-X)^2 + \left[(f_0+s)(1-X) - f + f_0 + s\right]\left[(1+X)^2 - 2\lambda(1+X)\right] \end{aligned} \tag{3.5.10}$$

$$C = (f_0+s)^2(1-\lambda)\left(1-X^2\right) - (f_0+s)(f-f_0-s)\left[X(1-X) + (1+X)^2 - 2\lambda(1+X)\right] \tag{3.5.11}$$

(of course, if there are no real solutions to 3.5.8 that just means that scaling is impossible for that X), then compute $k(v)$ using eq. 2.3.4 and the c_X we are after is simply

$$c_X = \frac{k(v)}{\varphi_X} \tag{3.5.12}$$

if it is positive and less than 1 (otherwise there is no scaling).

In order to understand what this purely mathematical consistency condition implies, let us focus on a specific kind of shape for the loop-chopping functions. In the absence of solid evidence in favour of a specific shape[3] (and given that the discussion in the previous section shows that the "guesses" 3.3.6 and 3.3.7 may not be the most convenient to use if we are interested in studying scaling), we illustrate the kind of analysis that can be carried out using the following family of loop-chopping functions (see appendix B for a

[3]Which, as has been mentioned, should in principle be obtainable from high-resolution simulations.

heuristic derivation)

$$\begin{cases} f_0 = 1 \\ f = 1 + \eta\left(1 - \frac{1}{\sqrt{\mu}}\right) \end{cases} \tag{3.5.13}$$

where η is some positive number which is related to how much energy is lost by the string to small-scale loops. Since the equations we are solving are simple but rather lenghty, we choose to do our analysis simply by computationally solving them for a uniform sampling of the interval $X \in (0,1)$ (testing about 1000 points for each η).

Firstly, let us consider only the matter era (which is when simulations seem to suggest it is the easiest to achieve scaling [23]), with $\lambda = 2/3$. For the sake of simplicity let us assume that $s = 0$ for the time being. The corresponding results are summarised in figure 3.5.1. In this era only one type of solution seems to be acceptable (i.e., results in physical values of c): the solution associated with the greater root of eq. 3.5.8 (called c_{X+} in the plots). Even this kind of solution does not seem to be physical for $\eta \lesssim 7$ (note that $\eta = 1$ makes eqs. 3.5.13 equivalent to eqs. 3.3.6 and 3.3.7). However, as we increase η, there is a clear tendency: the c_X necessary to have scaling with a "fixed" X decreases (which makes sense if we interpret an increase in μ as an increase in energy lost to small-scale loops) and scaling becomes possible for increasing values of μ (smaller X). Based on available simulations [23], which predict an increasing μ during the radiation era, one would expect this kind of tendency to remain in this epoch - since for scaling to occur in these simulations it would be necessary for the small-scale component of the network to lose more energy in the form of small-scale loops.

Interestingly, it does not seem to be possible to have scaling for $\mu \gtrsim 2.2$ ($X \lesssim 0.2$), suggesting that networks in the matter era cannot stabilise in the tensionless limit regardless of how much energy they lose to loops (although we shall see that accounting for kink formation by intercommutation, via $s \neq 0$, can change this picture).

The effect of adding $s > 0$ is not so straightforward. Supposing (for simplicity) a simple linear dependence $s = D\,(1 - X)$ (with $D > 0$) and fixed $\eta = 15$ (which seems to give reasonable results for the Abelian-Higgs model, when $c \sim 0.23$ [25]) we find that at first an increase in D seems to have the same effect as a decrease in η in the previous discussion, until scaling becomes impossible after $D \sim 2$. After that, however, for $D \sim 5$, scaling becomes possible again and increasing D now seems to have an effect similar to increasing η in the case with $D = 0$. This behaviour is captured in figure 3.5.2.

It is also worthwhile to note that we can now have scaling closer to the tensionless limit.

Looking at the radiation era ($\lambda = 1/2$) we find the same kind of qualitative behaviour with a few important differences. Firstly, there is now a second kind of solution, which is rather insensitive to η and is consistently suppressed by an increase in D (and even for $D = 0$ only applies to values of c below 0.2). Secondly, the solution we were following earlier does not allow scaling near the Goto-Nambu limit unless D is high enough[4] ($D \sim 1.7$). And finally, there does not seem to exist any kind of mechanism stopping us from getting scaling in the tensionless limit. These results can be seen in figure 3.5.3.

If we are interested in physical solutions then we must also impose $0 < v^2 < 1$ and $0 < \gamma_\xi < \frac{1}{1-\lambda}$ (where the last inequality just expresses the causal requirement that the correlation length must be below the horizon length). These requirements naturally discard some of the situations permitted above. In particular, γ_ξ seems to behave in a rather chaotic manner for the c_{X-} solution[5]. Since the specifics of the dynamics of each of these quantities make this analysis much more complex, and keeping in mind that our goal in this section is simply to illustrate the new paths this formalism uncovers and provide some intuition regarding the physical processes taking place, we will not pursue a complete analysis. Instead, we just show the results for a specific choice of energy loss functions which seems to give plausible results for $c \sim 0.23$ (and which was "guessed" using the intuition from the discussion so far): $\eta = 17$, $D = 0.2$. Figure 3.5.4 relates to the matter era while figure 3.5.5 is about the radiation epoch.

Naturally, future work will have to focus on determining conditions of stability for these scaling regimes.

One interesting feature which may also warrant further study is the fact that some values of c seem to admit more than one solution (we have seen some examples with three possible solutions in the radiation era, but there is even one example of two possible solutions in the matter epoch - see figure 3.5.2 for $D = 5$). It may be that the existence of more than one solution to a given model gives rise to "cross-over" phenomena which would be interesting to study - in particular, it would be interesting to determine whether a clever choice of parameters could relate a momentary stabilisation of μ in radiation era simulations [23] to this kind of behaviour.

[4]Note that this kind of feature is especially welcome since simulations seem to suggest small-scale scaling in the matter era but not in the radiation era. Bearing this in mind, one would expect D to be small enough to retain it.

[5]This is probably at least partly due to numerical errors. Since both the numerator and the denominator of the second equation in 3.5.6 are rapidly oscillating about zero with very small amplitudes, there is not a simple way to present these results free of these errors.

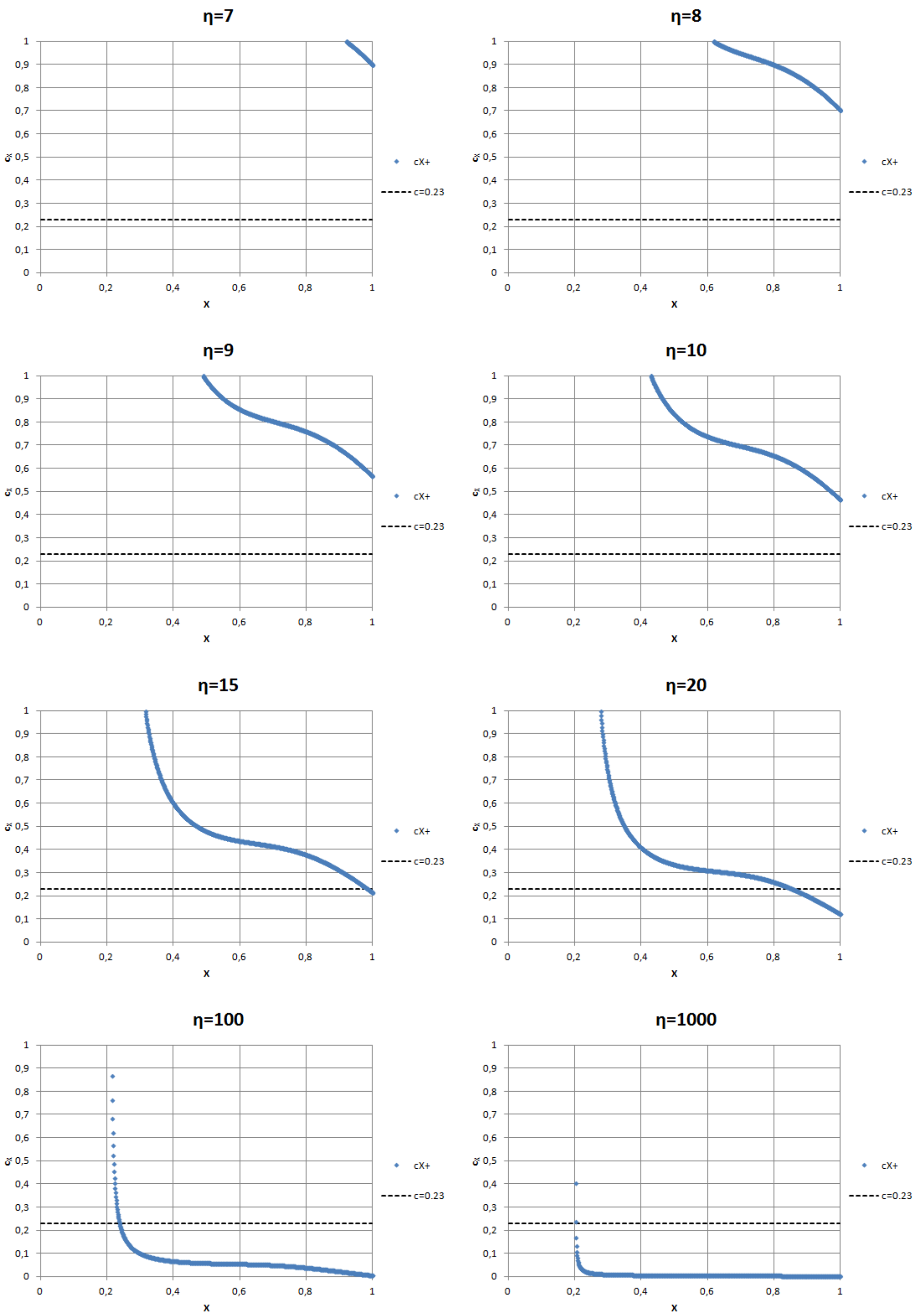

Figure 3.5.1.: Possible scaling regimes in the matter era for increasing values of η and $s = 0$. The Abelian-Higgs case corresponds to the intersection with the dashed line (for $c = 0.23$).

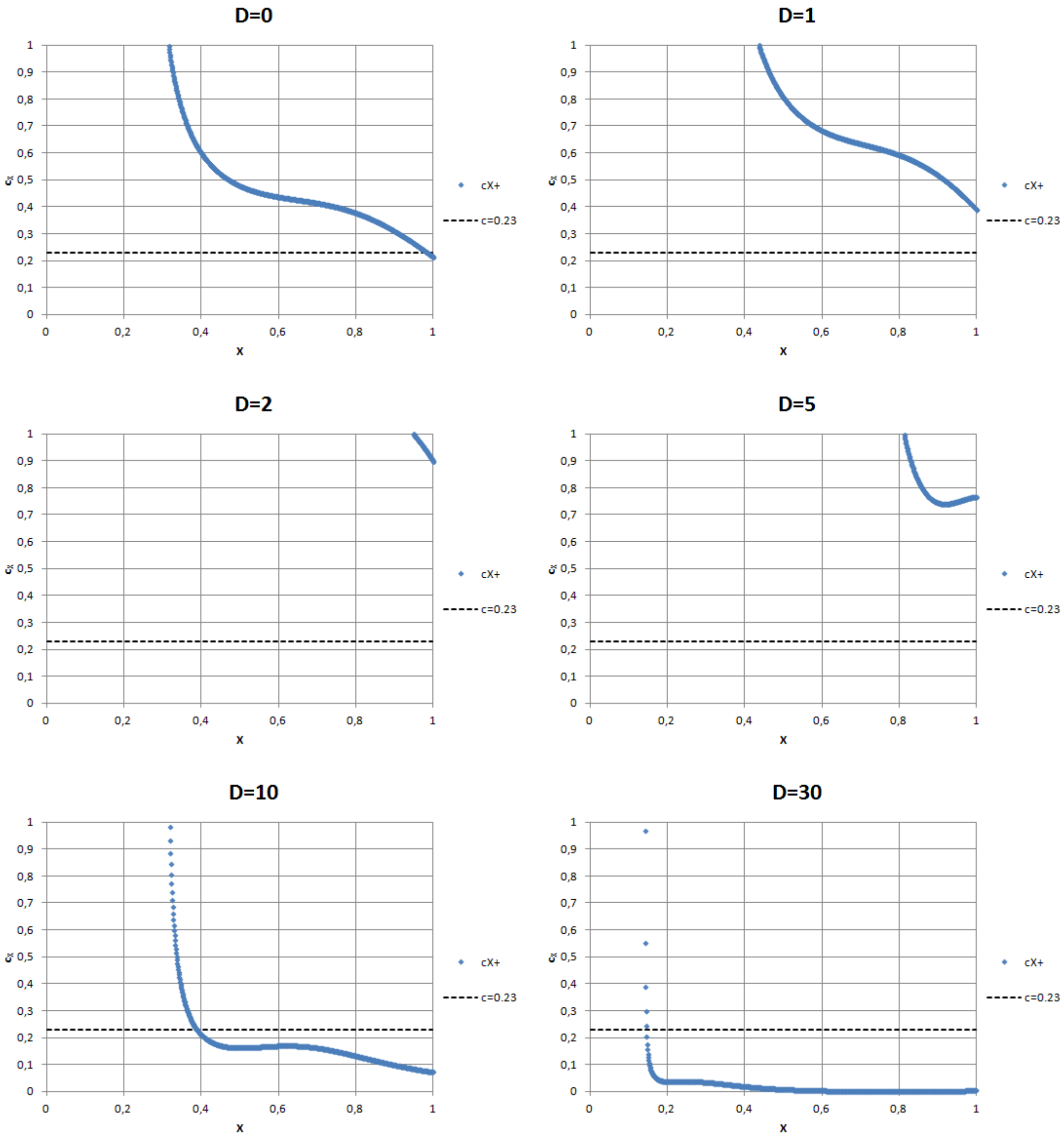

Figure 3.5.2.: Possible scaling regimes for $\eta = 15$ and $s = D\,(1 - X)$ in the matter era. The Abelian-Higgs case corresponds to the intersection with the dashed line (for $c = 0.23$).

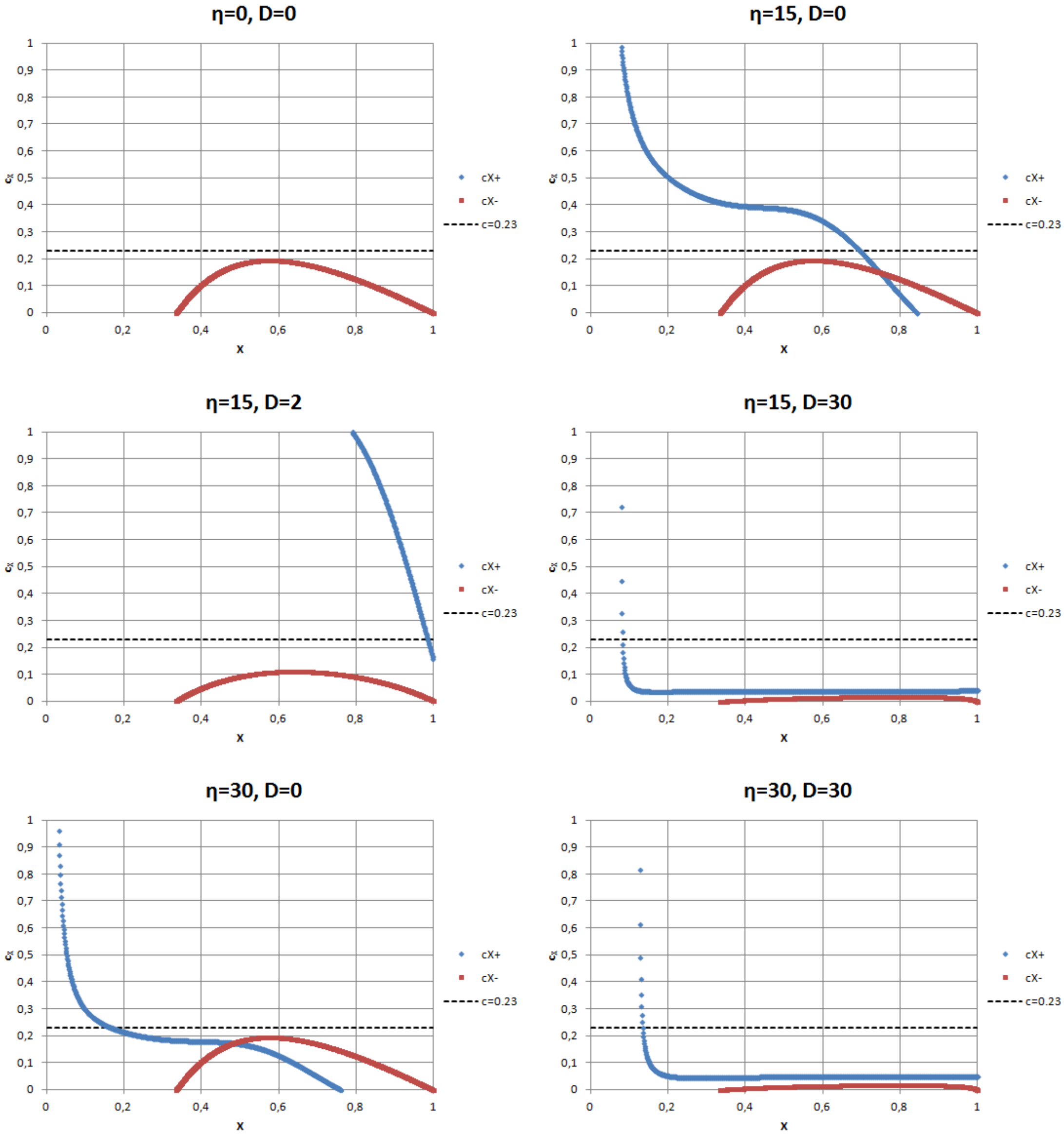

Figure 3.5.3.: Possible scaling regimes in the radiation epoch for varying values of η and D. The Abelian-Higgs case corresponds to the intersection with the dashed line (for $c = 0.23$).

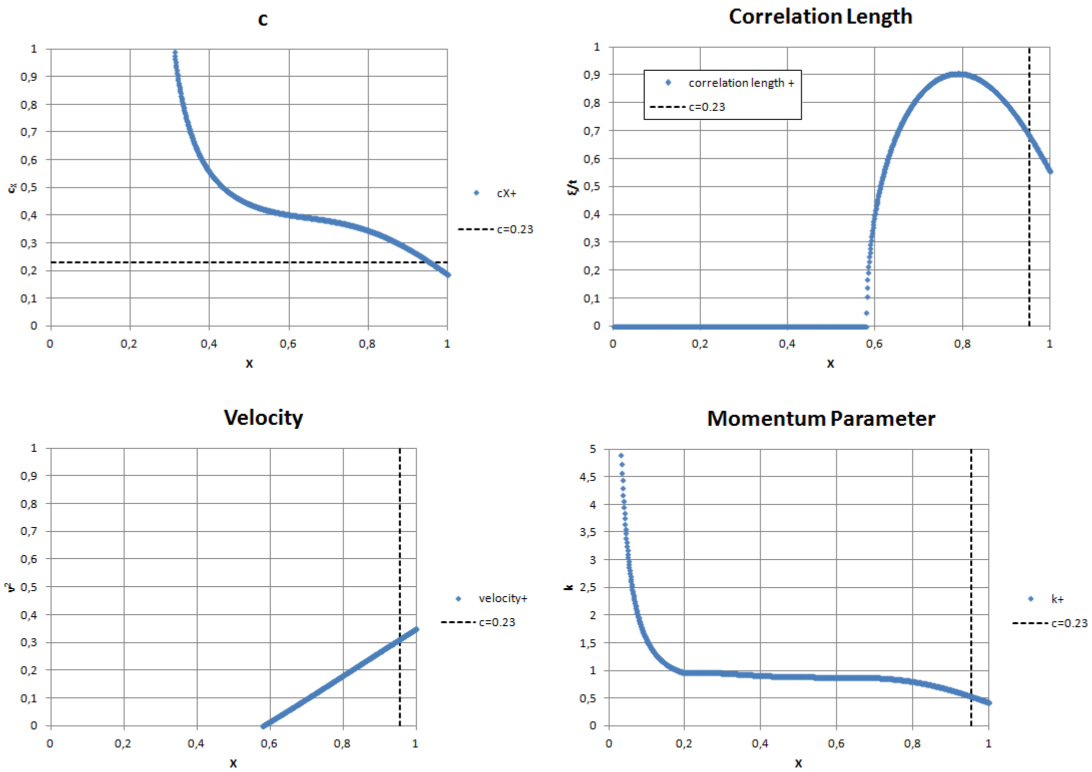

Figure 3.5.4.: Results for $\eta = 17$ and $D = 0.2$ in the matter era. The Abelian-Higgs case corresponds to the intersection with the dashed line (for $c = 0.23$).

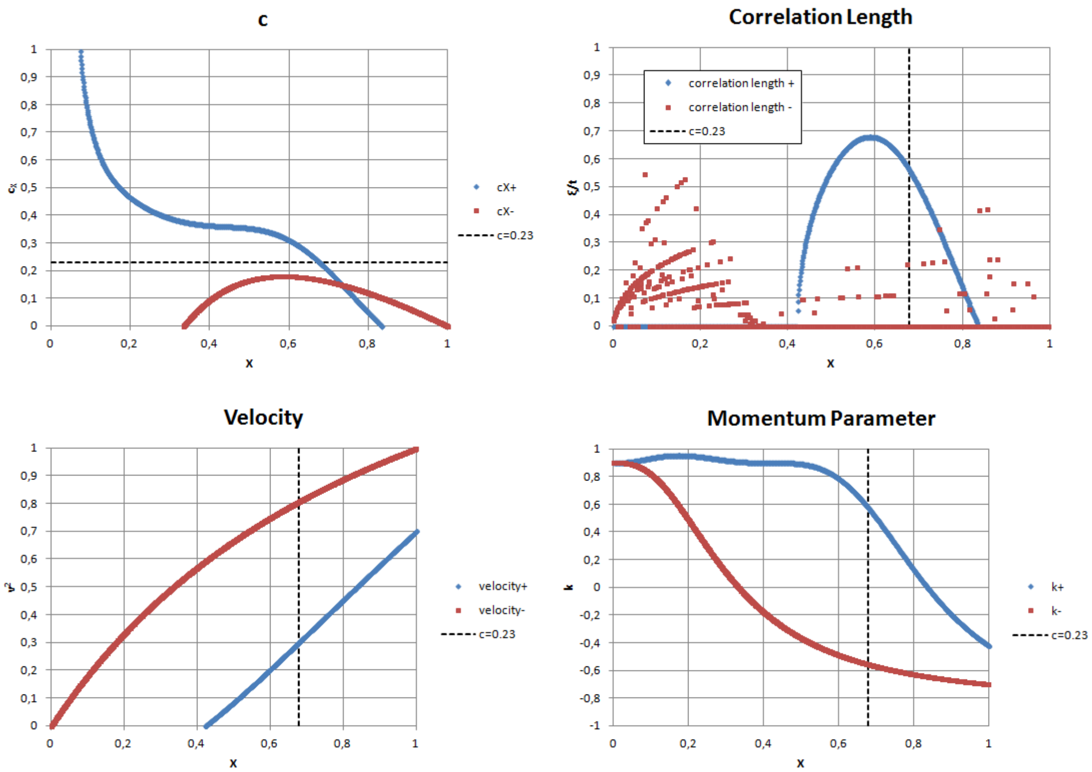

Figure 3.5.5.: Results for $\eta = 17$ and $D = 0.2$ in the radiation era. The Abelian-Higgs case corresponds to the intersection between the blue points and the dashed line (for $c = 0.23$).

4. Bonus Chapter: Extra-dimensional Analogies!

The idea of theories with extra dimensions has been around since the 1920's, when Kaluza and Klein proposed a $(4+1)-$dimensional spacetime in an attempt to unify Gravity and Electromagnetism. Even though their original idea did not work, the formalism they started developing has been useful to others until the present day [7] - string theory being a good example of a popular framework which requires extra spatial dimensions.

Cosmic strings can also exist in these extra-dimensional scenarios - in particular, cosmic string formation is unavoidable at the end of a brane inflation epoch [28]. So far we have only focused on strings evolving in three spatial dimensions, but the dynamics of strings in extra-dimensional spacetimes is also fairly well studied - there even being an extra-dimensional VOS model (commonly called the EDVOS) [5].

After the development of the formalism introduced in chapter 3, an interesting connection between our wiggly model and the EDVOS was realised. This realisation was a product of an on-going collaboration with A. Avgoustidis and has yet to be communicated elsewhere. This short chapter is devoted to briefly unveiling this unexpected property.

4.1. Strings in Extra Dimensions

Just like the Goto-Nambu action, as written in eq. 1.3.1, is the starting point for the study of string dynamics in a $(3+1)-$dimensional spacetime, the study of strings in a general $(D+1)-$ dimensional background starts with its natural extension

$$S_D = -\mu_0 \int \sqrt{-\gamma^{(D)}} d^2\zeta \tag{4.1.1}$$

where μ_0 is the string tension (or linear energy density), $x^\mu = x^\mu(\zeta^a)$ is the same kind of parametrization as before, and $\gamma_{ab}^{(D)}$ is just the $(D+1)-$dimensional analogue of 1.3.2, the induced metric which can be written at the expense of a background metric $g^{(D)}_{\mu\nu}$ as

$$\gamma^{(D)}_{ab} = g^{(D)}_{\mu\nu} x^{\mu}_{,a} x^{\nu}_{,b} \tag{4.1.2}$$

As one would expect, the resulting equations of motion are expressible as in eq. 1.3.3 if all the terms are identified with the respective higher-dimensional analogues.

We are interested in the generalised FRW metric

$$ds^2 = a^2 d\tau^2 - a^2 d\mathbf{x}^2 - b^2 d\mathbf{l}^2 \tag{4.1.3}$$

where $\mathbf{x}^i = x^i$ for $i = 1, 2, 3$ and $\mathbf{l}^m = x^m$ for $m = 4, ..., D$. For compact extra dimensions the coordinates in $\mathbf{l}$ are periodically identified. The new scale factor b can be allowed to evolve independently of a.

The energy per unit coordinate length is now

$$\epsilon_{(D)} = -\frac{x'^2}{\sqrt{-\gamma}} = \sqrt{\frac{a^2\mathbf{x'^2} + b^2\mathbf{l'^2}}{a^2 - a^2\dot{\mathbf{x}}^2 - b^2\dot{\mathbf{l}}^2}} \tag{4.1.4}$$

and in the transverse gauge

$$\begin{cases} \zeta^0 = \tau \\ \dot{x} \cdot x' = 0 \end{cases} \tag{4.1.5}$$

the equations of motion can be written as [5]

$$\frac{\dot{\epsilon}_{(D)}}{\epsilon_{(D)}} + \frac{\dot{a}}{a}\left[1 + \dot{\mathbf{x}}^2 - \left(\frac{\mathbf{x}'}{\epsilon_{(D)}}\right)^2\right] + \frac{b\dot{b}}{a^2}\left[\dot{\mathbf{l}}^2 - \left(\frac{\mathbf{l}'}{\epsilon_{(D)}}\right)^2\right] = 0 \tag{4.1.6}$$

$$\ddot{\mathbf{x}} + \dot{\mathbf{x}}\left[\frac{\dot{a}}{a}\left(1 - \dot{\mathbf{x}}^2 + \left(\frac{\mathbf{x}'}{\epsilon_{(D)}}\right)^2\right) - \frac{b\dot{b}}{a^2}\left(\dot{\mathbf{l}}^2 - \left(\frac{\mathbf{l}'}{\epsilon_{(D)}}\right)^2\right)\right] = \frac{1}{\epsilon_{(D)}}\left(\frac{\mathbf{x}'}{\epsilon_{(D)}}\right)' \tag{4.1.7}$$

$$\ddot{\mathbf{l}} + \dot{\mathbf{l}}\left[\frac{\dot{a}}{a}\left(1 - \dot{\mathbf{x}}^2 + \left(\frac{\mathbf{x}'}{\epsilon_{(D)}}\right)^2\right) - \frac{b\dot{b}}{a^2}\left(\dot{\mathbf{l}}^2 - \left(\frac{\mathbf{l}'}{\epsilon_{(D)}}\right)^2\right)\right] = \frac{1}{\epsilon_{(D)}}\left(\frac{\mathbf{l}'}{\epsilon_{(D)}}\right)' \tag{4.1.8}$$

We will not discuss the VOS-type model that results from the averaging of these equations, the EDVOS (Extra-Dimensional VOS), because it is not relevant for the point we want to make and its derivation is analogous to what has already been done for the VOS in section 2.3. We will, however, note that there is one key difference between the VOS and the EDVOS equations: the loop-chopping term. If we keep in mind that the space swept out by the motion of a string is always two-dimensional, then strings randomly distributed in a higher-dimensional space should have much lower collision probabilities than strings in three-dimensional space. For an isotropic Universe (with $b = a$) this effect is taken into account by introducing a suppression term in the loop-chopping efficiency c [5]

$$c \longrightarrow c_D = c\left(\frac{\delta}{L}\right)^{D-3} \tag{4.1.9}$$

where δ is the *capture radius* of the strings, which quantifies how close they need to be in order to interact. This extra time dependence generally causes networks to not reach scaling solutions. The situation is different for the case of stabilised compact extra dimensions (with $b = 1$ and the coordinates in the "$\mathbf{l}$ sector" being periodically identified), if the compactification radius R_c is smaller than L (as it typically is), when this relation becomes

$$c \longrightarrow c_D = c\left(\frac{\delta}{R_c}\right)^{D-3} \tag{4.1.10}$$

and scaling solutions are still possible. For the reader whose interest has been piqued, a complete derivation and thorough discussion of the EDVOS can be found in [5].

4.2. Extra Dimensions and Wiggles

At last, the point of this chapter is revealed: what is this unexpected relation between wiggly strings and extra-dimensional strings?

Firstly, the way we describe strings in a spacetime with small compact extra dimensions is similar to the way we describe strings with small-scale wiggles. In both cases we can think of $\mathbf{x}$ as parametrizing a string configuration that somehow approximates the true string configuration and whose dynamics is deviated from the $((1+3)-$ dimensional) Goto-Nambu behaviour by the presence of "hidden" energy components. The difference is just in the interpretation of the terms: in the extra-dimensional case $\mathbf{x}$ is the projection of a $(1+D)-$dimensional string onto a $(1+3)-$ dimensional space, while in the wiggly case it is a "renormalized" version of a more complex $(1+3)-$ dimensional configuration; conversely, in the extra-dimensional case the extra energy is hiding in the extra dimensions, while in the wiggly case it is hiding in the wiggles which cannot be seen in $\mathbf{x}$.

Given this formal resemblance, it is pertinent to ask how far this parallelism can be taken. If we focus only on the evolution of $\mathbf{x}$ and the extra energy, does the interpretation we choose make a difference? The answer can be found by rewriting the higher-dimensional Goto-Nambu action (4.1.1) in a form that resembles the elastic action with the Lagrangian density 3.1.5. In order to do this, we use

$$\gamma_{ab}^{(D)} = g_{\mu\nu}^{(D)} x^{\mu}_{,a} x^{\nu}_{,b} = \gamma_{ab}^{(3)} - \hat{g}_{mn}^{(D-3)} l^m_{,a} l^n_{,b} = \gamma_{ac}^{(3)} \left[\delta^c_b - \gamma^{(3)cd} \hat{g}_{mn}^{(D-3)} l^m_{,d} l^n_{,b}\right] \tag{4.2.1}$$

(where we have introduced the $(D-3)-$ dimensional euclidean metric $\hat{g}_{mn}^{(D-3)} = -g_{mn}^{(D)}$ for $m, n = 4, ..., D$) to factor out the determinant of the $(4+1)-$ dimensional induced metric

$$-\gamma^{(D)} = -\det \gamma_{ab}^{(D)} = -\gamma^{(3)} \det \left(\delta^a_b - \gamma^{(3)ac} \hat{g}_{mn}^{(D-4)} l^m_{,c} l^n_{,b}\right) \tag{4.2.2}$$

and the $(D+1)-$ dimensional Goto-Nambu action becomes

$$S_D = -\mu_0 \int \sqrt{-\gamma^{(3)}} \sqrt{\det \left(\delta^a_b - \gamma^{(3)ac} \hat{g}_{mn}^{(D-4)} l^m_{,c} l^n_{,b}\right)} d^2\zeta \tag{4.2.3}$$

which makes the analogy with 3.1.5 much more apparent. If we now focus only on the case with four spatial dimensions it is straightforward to show that, in the transverse gauge,

$$S_4 = -\mu_0 \int \sqrt{-\gamma^{(3)}} \sqrt{1 - b^2 \gamma^{(3)ab} l_{,a} l_{,b}} d^2\zeta \tag{4.2.4}$$

(where l^4 is just renamed l) which is the same as the action one obtains from 3.1.5 with the correspondence $l \leftrightarrow \phi$ as long as $b = 1$ (as can be chosen for a stabilised compact extra dimension) and $l' = 0$ - note also that, in this limit, this correspondence means $\epsilon_{(4)} \leftrightarrow \epsilon/w$. Simply put, if we ignore intersections, the dynamics of elastic strings in normal spacetime is the same as the dynamics of Goto-Nambu strings moving in a $(1+4)-$ dimensional background where the extra dimension is stabilised and compact, in the limit in which the strings do not wrap around the extra dimension (even though they are allowed to move there).

It is appropriate to keep in mind that the same kind of trick was used before [27] to show that the dynamics of a string in the same kind of higher-dimensional background in the limit with very small compactification radius (which makes physical sense, for us to not realise the existence of the extra dimensions) corresponds in an equivalent way to that of a Witten superconducting string with zero charge, as described by the action

$$S_W = \int \sqrt{-\gamma} \left[-\mu_0 + \frac{1}{2} \gamma^{ab} \phi_{,a} \phi_{,b}\right] d^2\zeta \tag{4.2.5}$$

which means that the correspondence between wiggly strings and higher-dimensional strings can also be translated into a correspondence between wiggly strings in the linear limit and superconducting strings with zero charge in a limit with $\phi'^2 \ll \epsilon^2\dot{\phi}^2$.

These kinds of analogies are a good way to gain insight about particular limits of very complex models. In this case, they might be particularly useful since simulations with superconducting or wiggly strings are much easier to run than simulations with strings in extra dimensions. Further work should clarify how much this connection allows us to learn about extra-dimensional strings from the study of wiggly (or realistic Goto-Nambu) strings and vice-versa.

5. Conclusions and Further Work

In this thesis, we have successfully developed a novel formalism to model cosmic string network evolution. Building upon solid work already established in the literature (notably the foundations of the VOS laid down by Martins and Shellard [19, 20, 21, 22] and the rigorous work on elastic string models carried out by Carter and Martin [8, 9, 10, 18]), we have managed to devise a relatively simple model which retains the main advantages of the VOS while enabling us to describe the evolution of small-scale structure in the network.

Naturally, bringing out the full potential of this new approach will take further work. In particular, it is important to use simulations to gain a more quantitative understanding of the way energy is lost by the small-scale component of the network - as we have shown how the details of this process determine which kinds, if any, of small-scale scaling regimes are possible.

In the end we also point at an unexpected connection between elastic and extra-dimensional strings as a potential way of using our model to learn more about different kinds of string models. Exploring exactly how that can be done is the subject of work in progress.

Appendices

A. Multifractal Analysis

Given an arbitrary one-dimensional shape and some undefined renormalization procedure (that smoothens it so that the renormalized shape has no structure below some predefined renormalization length scale), its multifractal dimension at a scale ℓ, $d_m(\ell)$, is defined as [29]

$$d_m(\ell) = \frac{d \log M(\ell)}{d \log \ell} \tag{A.0.1}$$

where $M(\ell)$ is the length of the renormalized figure for a renormalization length ℓ. This quantity generalizes the usual notion of fractal dimension in the sense that it allows us to assign scale-dependent fractal dimensions to physical objects which cannot have a "normal" fractal dimension because physical constraints limit the minimum size of structure - $d_m(\ell)$ can be thought of as the fractal dimension a shape seems to have when we cannot make out details whose scale is below ℓ. The definition of fractal dimension that is usually used in these kinds of problems can be simply recovered by

$$D = \lim_{\ell \to 0^+} d_m(\ell) \tag{A.0.2}$$

which is trivially unity for any physical one-dimensional shape. As ℓ increases, however, complex shapes which approximate fractal structures should see an increase in this value. In particular, for the case of cosmic strings, the fact that they look Brownian on large scales implies that at large scales $d_m(\ell)$ should approach 2.

For now, let us keep in mind the simple renormalization procedures illustrated in figure A.0.1 when thinking about cosmic strings. The algorithm is simple: pick a point on the string, find the first point at a distance ℓ (first in the sense that you will reach it first if you are traveling on the string starting from the point you have picked), and substitute the segment uniting them for a simpler smoother one. In our case, since it is convenient for the network to have a characteristic finite curvature R, we should choose a segment with constant curvature R (as shown in red) - the straight (green) case is shown because numeric simulations usually use that kind of simpler renormalized shape to calculate μ, but we will see that our results are not significantly affected by that choice provided that ℓ is much smaller than R (as it should in order for us to be modeling small-scale structure).

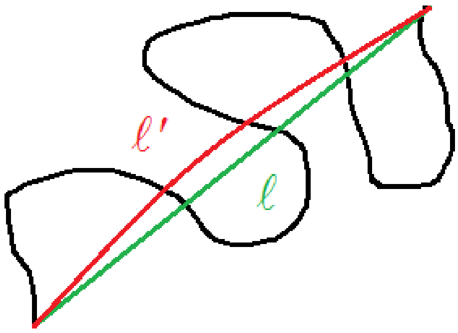

Figure A.0.1.: Schematic picture of a renormalization procedure. ℓ is the renormalization scale below which the details of the original shape (in black) are being smoothed while ℓ' is the length of the renormalized shape with finite curvature (in red). In numeric simulations, the renormalization procedure being used is usually the version which does not assign curvature to the renormalized segments (in green).

We can define the "computational" energy renormalization factor

$$\bar{\mu} \equiv \frac{M(\ell)}{\ell} = \frac{E/\mu_0}{\ell} \tag{A.0.3}$$

(where E obviously refers to the segment, not the network) and it is trivial to verify that

$$\frac{\ell}{\bar{\mu}} \frac{\partial \bar{\mu}}{\partial \ell} = d_m(\ell) - 1 \tag{A.0.4}$$

Using

$$\mu = \bar{\mu} \frac{\ell}{\ell'} \tag{A.0.5}$$

and the relation between ℓ and ℓ' (which can be found using the fact that the red segment in figure A.0.1 is an arch with radius R)

$$\sin \frac{\ell'}{2R} = \frac{\ell}{2R} \tag{A.0.6}$$

we can write

$$\mu = \bar{\mu}\left(1 + \mathcal{O}\left\{\left(\frac{\ell}{R}\right)^2\right\}\right) \tag{A.0.7}$$

which then implies

$$\frac{\ell}{\mu}\frac{\partial\mu}{\partial\ell} = \frac{\ell}{\bar{\mu}}\frac{\partial\bar{\mu}}{\partial\ell} + \mathcal{O}\left\{\left(\frac{\ell}{R}\right)^2\right\} = d_m\left(\ell\right) - 1 + \mathcal{O}\left\{\left(\frac{\ell}{R}\right)^2\right\} \tag{A.0.8}$$

which is basically eq. 3.2.17.

B. Guessing Energy Loss Terms

A discussion of the full wiggly model requires some assumption regarding the form of the energy loss functions. As has been said, it should be possible to investigate the dependence of these functions on μ by computational means. For the purpose of this discussion, though, we propose a plausible ansatz for the loop-chopping terms which relies on the same kind of logic as 2.2.1.

Firstly, we argue that the argument used in the writing of 2.2.1 should also apply to the smooth-looking renormalised string. Therefore the loop terms in the equation of ρ_0 should be the same as their VOS counterparts, with the correlation length being identified with ξ. This means[1]

$$\left.\frac{d\rho_0}{dt}\right|_{loops} = -\frac{\mu_0 v}{\xi^3}\int_0^\infty g\left(l/\xi\right)\frac{dl}{\xi} \equiv -cv\frac{\rho_0}{\xi} \tag{B.0.1}$$

and thus

$$f_0\left(\mu\right) = 1 \tag{B.0.2}$$

If the total energy lost to loops were merely the energy contained in the large loops (with size $\sim \xi$) considered to carry the energy lost by the bare string, then the analogue of eq. B.0.1 for ρ could be written simply multiplying everything by μ and in the end we would have $f\left(\mu\right) = 1$ as well. However, when a loop is formed by intercommutation in a wiggly string, there is usually the possibility that a class of much smaller loops will be formed - for that reason, for $\mu > 1$, one would expect $f\left(\mu\right) > 1$. The typical length of these smaller loops, we conjecture, can be related to a characteristic length scale which can be written as a combination of L and ξ such that it is zero when $L = \xi$ - and clearly the simplest scale with these properties is simply $\xi_\star = \xi - L$. Bearing this in mind, we

[1] Notation warning: we are now noting the loop production function as $g\left(l/\xi\right)$ instead of $f\left(l/\xi\right)$ to avoid confusion with the energy loss function $f\left(\mu\right)$.

try the substitution $g(l/\xi) \to g(l/\xi) + g_\star(l/\xi_\star)$ and get

$$\left.\frac{d\rho}{dt}\right|_{loops} = -\frac{\mu_0\mu v}{\xi^3}\int_0^\infty g(l/\xi)\,\frac{dl}{\xi} - \frac{\mu_0\mu v}{\xi^3}\int_0^\infty g_\star(l/\xi_\star)\,\frac{dl}{\xi} \equiv -cv\frac{\rho}{\xi}\left[1+\eta\frac{\xi_\star}{\xi}\right] \tag{B.0.3}$$

(where η is a positive number defined as $\eta = c^{-1}\int g_\star(x)\,dx$), which is clearly only acceptable in the Goto-Nambu limit if $\xi_\star \to 0^+$ as μ goes to unity. Using the simple $\xi_\star = \xi - L$ this leads to

$$f(\mu) = 1 + \eta\left(1 - \frac{1}{\sqrt{\mu}}\right) \tag{B.0.4}$$

which by eq. 3.3.4 implies

$$f_1(\mu) = 1 - \eta\frac{\sqrt{\mu}}{\sqrt{\mu}+1} \tag{B.0.5}$$

Incidentally, in the linear limit $f(Y) \sim 1 + \frac{\eta}{2}Y$ and if $\eta = 1$ this ansatz becomes indistinguishable from eqs. 3.3.6 and 3.3.7.

This derivation, we emphasise, is everything but rigorous. However, faced with the lack of a rigorously obtained set of energy loss functions, we use these to illustrate the kind of useful calculations that can be carried out with our wiggly formalism. If further study unveils a more reliable set of functions, all we have to do is redo the calculations in section 3.5 with the corresponding substitutions.

Bibliography

[1] P.A.R. Ade, W. Aikin, R. D. Barkats, J. Benton, S. A. Bischoff, C. J. Bock, J. A. Brevik, J. I. Buder, E. Bullock, D. Dowell, C. L. Duband, P. Filippini, J. S. Fliescher, R. Golwala, S. M. Halpern, M. Hasselfield, R. Hildebrandt, S. C. Hilton, G. V. Hristov, V. D. Irwin, K. S. Karkare, K. P. Kaufman, J. G. Keating, B. A. Kernasovskiy, S. M. Kovac, J. L. Kuo, C. M. Leitch, E. M. Lueker, P. Mason, B. Netterfield, C. T. Nguyen, H. R. O'Brient, W. Ogburn, R. A. Orlando, C. Pryke, D. Reintsema, C. S. Richter, R. Schwarz, D. Sheehy, C. K. Staniszewski, Z. V. Sudiwala, R. P. Teply, G. E. Tolan, J. D. Turner, A. G. Vieregg, A. L. Wong, C. and W. Yoon, K.˙Detection of *b*-mode polarization at degree angular scales by bicep2. *Phys. Rev. Lett.*, 112:241101, Jun 2014.

[2] P.A.R. Ade et al. Planck 2013 results. XXV. Searches for cosmic strings and other topological defects. *arXiv:1303.5085*, 2013.

[3] B. Allen and R. R. Caldwell. Generation of structure on a cosmic-string network. *Phys. Rev. Lett.*, 65:1705–1708, Oct 1990.

[4] Daren Austin, E. J. Copeland, and T. W. B. Kibble. Evolution of cosmic string configurations. *Phys. Rev. D*, 48:5594–5627, Dec 1993.

[5] A. Avgoustidis and E. P. S. Shellard. Cosmic string evolution in higher dimensions. *Phys. Rev. D*, 71:123513, Jun 2005.

[6] David P. Bennett. Evolution of cosmic strings. *Phys. Rev. D*, 33:872–888, Feb 1986.

[7] J. Beringer et al. Review of particle physics. *Phys. Rev. D*, 86:010001, Jul 2012.

[8] Brandon Carter. Integrable equation of state for noisy cosmic string. *Phys. Rev. D*, 41:3869–3872, Jun 1990.

[9] Brandon Carter. Transonic elastic model for wiggly goto-nambu string. *Phys. Rev. Lett.*, 74:3098–3101, Apr 1995.

[10] Brandon Carter. Brane dynamics for treatment of cosmic strings and vortons. *arXiv:hep-th/9705172*, 1997.

[11] F. Englert and R. Brout. Broken symmetry and the mass of gauge vector mesons. *Phys. Rev. Lett.*, 13:321–323, Aug 1964.

[12] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble. Global conservation laws and massless particles. *Phys. Rev. Lett.*, 13:585–587, Nov 1964.

[13] Peter W. Higgs. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.*, 13:508–509, Oct 1964.

[14] T W B Kibble. Topology of cosmic domains and strings. *Journal of Physics A: Mathematical and General*, 9(8):1387, 1976.

[15] T W B Kibble and E Copeland. Evolution of small-scale structure on cosmic strings. *Physica Scripta*, 1991(T36):153, 1991.

[16] T.W.B. Kibble. Evolution of a system of cosmic strings. *Nuclear Physics B*, 252(0):227 – 244, 1985.

[17] Joanes Lizarraga, Jon Urrestilla, David Daverio, Mark Hindmarsh, Martin Kunz, and Andrew R. Liddle. Can topological defects mimic the bicep2 *b*-mode signal? *Phys. Rev. Lett.*, 112:171301, Apr 2014.

[18] Xavier Martin. Cancellation of longitudinal contribution in wiggly string equation of state. *Phys. Rev. Lett.*, 74:3102–3104, Apr 1995.

[19] C. J. A. P. Martins. *Quantitative String Evolution.* PhD thesis, Cambridge University, 1997.

[20] C. J. A. P. Martins and E. P. S. Shellard. Quantitative string evolution. *Phys. Rev. D*, 54:2535–2556, Aug 1996.

[21] C. J. A. P. Martins and E. P. S. Shellard. Scale-invariant string evolution with friction. *Phys. Rev. D*, 53:R575–R579, Jan 1996.

[22] C. J. A. P. Martins and E. P. S. Shellard. Extending the velocity-dependent one-scale string evolution model. *Phys. Rev. D*, 65:043514, Jan 2002.

[23] C. J. A. P. Martins and E. P. S. Shellard. Fractal properties and small-scale structure of cosmic string networks. *Phys. Rev. D*, 73:043515, Feb 2006.

[24] C.J.A.P. Martins, E.P.S. Shellard, and J.P.P. Vieira. Models for Small-Scale Structure on Cosmic Strings: I. Mathematical Formalism. *arXiv:1405.7722*, 2014.

[25] J. N. Moore, E. P. S. Shellard, and C. J. A. P. Martins. Evolution of abelian-higgs string networks. *Phys. Rev. D*, 65:023503, Dec 2001.

[26] Adam Moss and Levon Pogosian. Did bicep2 see vector modes? first *b*-mode constraints on cosmic defects. *Phys. Rev. Lett.*, 112:171302, Apr 2014.

[27] M.F. Oliveira, A. Avgoustidis, and C.J.A.P. Martins. Cosmic string evolution with a conserved charge. *Phys. Rev. D*, 85:083515, Apr 2012.

[28] Saswat Sarangi and S.H.Henry Tye. Cosmic string production towards the end of brane inflation. *Physics Letters B*, 536:185 – 192, 2002.

[29] H. Takayasu. *Fractals in the Physical Sciences.* Manchester University Press, Manchester, UK, 1990.

[30] A. Vilenkin and E. P. S. Shellard. *Cosmic Strings And Other Topological Defects.* Cambridge University Press, 1994.