ON RESOLUTION OF 1-DIMENSIONAL FOLIATIONS ON 3-MANIFOLDS

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Abstract. We prove a sharp resolution theorem for an important class of singularities of foliations containing in particular all singularities of complete holomorphic vector fields on complex manifolds of dimension 3. Roughly speaking our result relies on the combination of classical results about asymptotic expansions for solutions of differential equations with recent results on resolution of singularities for general foliations in dimension 3. With respect to the latter, starting from a by-product of Cano-Roche-Spivakovsky [5] we provide an otherwise self-contained elementary proof of a result paralleling McQuillan-Panazzolo [10] which is particularly well adapted to our needs.

1. Introduction

Recall that a singularity of a holomorphic vector field X is said to be *semicomplete* if the integral curves of X admit a maximal domain of definition in $\mathbb{C}$, cf. [13]. In particular whenever X is a *complete vector field* defined on a complex manifold M, every singularity of X is automatically semicomplete. On a different direction, the semicomplete property for vector fields/singularities is somehow similar to the Painlevé property for differential equations and it is also verified for several systems/vector fields appearing in the literature of Mathematical Physics. It is therefore interesting to consider the problem of resolution of singular points for 1-dimensional holomorphic foliations in the particular case where the foliations are associated with semicomplete vector fields. Along these lines the main result of this paper reads as follows:

Theorem A. *Let X be a semicomplete vector field defined on a neighborhood of the origin in $\mathbb{C}^3$ and denote by $\mathcal{F}$ the holomorphic foliation associated with X. Then one of the following holds:*

1. *The linear part of X at the origin is nilpotent non-zero and $\mathcal{F}$ admits a formal separatrix at the origin. In fact, $\mathcal{F}$ has a persistent nilpotent singularity at the origin (see Definition 3).*
2. *There exists a finite sequence of blow-ups maps along with transformed foliations*

$$\mathcal{F} = \mathcal{F}_0 \stackrel{\Pi_1}{\longleftarrow} \mathcal{F}_1 \stackrel{\Pi_2}{\longleftarrow} \cdots \stackrel{\Pi_r}{\longleftarrow} \mathcal{F}_r$$

such that all of the singular points of $\mathcal{F}_r$ are elementary, i.e. they possess at least one eigenvalue different from zero. Moreover each blow-up map Π_i is centered in the singular set of the corresponding foliation $\mathcal{F}_{i-1}$.

As indicated in the above statement, a singular point p of a (1-dimensional) holomorphic foliation $\mathcal{F}$ is called *elementary* if $\mathcal{F}$ possesses at least one eigenvalue different from zero at p. Similarly, by using standard terminology, we will say that $\mathcal{F}$ can be *resolved* (or desingularized, or reduced) if there is a sequence of blow-up maps leading to a foliation transforming $\mathcal{F}$ into a new foliation all of whose singularities are elementary. An immediate corollary of Theorem A which is worth point out is as follows:

Corollary B. *Let X be a semicomplete vector field defined on a neighborhood of $(0, 0, 0) \in \mathbb{C}^3$ and assume its linear part at the origin equals zero. Then item (2) of Theorem A holds.*

Let us make it clear that throughout the text the terminology *blow-up* always means standard (i.e. homogeneous) blow-ups. We will also have occasion of discussing blow-ups with weights (non-homogeneous) but these will be explicitly referred to as *weighted blow-ups*.

Theorem A asserts then that foliations associated with semicomplete vector fields in dimension 3 can be resolved by a sequence of blow-ups centered in the singular set except for a very specific case in which the vector field X (and hence the foliation $\mathcal{F}$) has non-zero nilpotent linear part. On the other hand, in view of the dichotomy provided by Theorem A, a few comments are needed to further clarify the role of item 1 in the statement in question. First note that more accurate normal forms are available for the vector fields in question: indeed Theorem 1 provide sharp normal forms for all persistent nilpotent singular points (see the discussion below). Moreover, *not all* nilpotent vector fields giving rise to persistent nilpotent singularities are semicomplete and, in this respect, the normal form provided by Theorem 1 will further be refined later on.

Next, it is natural to wonder if this phenomenon of existence of singular points that cannot be resolved by a sequence of standard blow-ups - as described in item (2) of Theorem A - can be found in genuinely global settings. To answer this question, it suffices to note that the polynomial vector field

$$Z = x^2\partial/\partial x + xz\partial/\partial y + (y - xz)\partial/\partial z$$

can be extended to a complete vector field defined on a suitable open manifold (see Section 5 for detail). As will be seen, the origin in the above coordinates constitutes a nilpotent singular point of Z that cannot be resolved by means of blow-ups centered at singular sets, albeit this nilpotent singularity can be resolved by using a blow-up centered at the (invariant) x-axis.

Finally the question raised above involving the existence of singularities as in item 1 of Theorem A in global settings can also be considered in the far more restrictive case of holomorphic vector field defined on *compact manifolds* of dimension 3. Owing to the compactness of the manifold every such vector field is necessarily complete. Hence the methods used in the proof of Theorem A easily yield:

Corollary C. *Let X denote a holomorphic vector field defined on a compact manifold M of dimension 3 and denote by $\mathcal{F}$ the singular holomorphic foliation associated with X. If $p \in M$ is a singular point of X, then there exists a finite sequence of blow-ups maps along with transformed foliations*

$$\mathcal{F} = \mathcal{F}_0 \stackrel{\Pi_1}{\longleftarrow} \mathcal{F}_1 \stackrel{\Pi_2}{\longleftarrow} \cdots \stackrel{\Pi_r}{\longleftarrow} \mathcal{F}_r$$

such that all of the singular points of $\mathcal{F}_r$ are elementary. Furthermore each blow-up map Π_i is centered in the singular set of the corresponding foliation $\mathcal{F}_{i-1}$.

Whereas the above theorems are somehow results about vector fields rather than about holomorphic foliations, it is not surprising that the corresponding proofs make substantial use of some general results involving resolution of singularities for the latter. Before stating a resolution result especially well suited to the proof of Theorem A (Theorem D below), it is then probably useful to provide a short summary of the main works in the area so as to put the statement of Theorem D in context.

To begin with, recall that every singular holomorphic foliation in dimension 2 can be resolved by a sequence of (one-point) blow-ups centered at non-elementary singular points as follows from a classical result due to Seidenberg, see [14] or [1], [7]. In dimension 3 the analogous question becomes much harder and, in particular, requires us to distinguish between *foliations of codimension* 1 and *foliations of dimension* 1. Whereas for codimension 1 foliations there is a decisive answer that can hardly be improved on, see [3], the story involving foliations of dimension 1 - the ones that appear in the present article - is longer and more elusive.

Resolution results for foliations of dimension 1 on $(\mathbb{C}^3, 0)$ started with [2] where the author proves his *formal local uniformization theorem*. Building on [2], Sancho and Sanz have found in a unpublished paper the first examples of (1-dimensional) foliations that cannot be resolved by means of blow-ups with invariant centers and detailed accounts of their result appear in [11] and in [4]. The next major result in the area is due to D. Panazzolo [11] who provided an algorithm based on the Newton diagram to resolve singularities of real foliations by using *weighted blow-ups* centered in the singular set of the foliation.

More recently and basically at the same time, two new important papers appeared in the area, namely [5] and [10]. In [5] the authors provide a strategy for reducing singular foliations which is based on (standard) blow-ups with invariant centers. Whereas they cannot ensure that elementary singularities are obtained at the end (cf. Sancho-Sanz's examples), they do prove that the "final models" are at worst quadratic. As explicitly pointed out in [4] (page 43), a by-product of their methods states that a singularity that cannot be resolved by means of blow-ups with centered in the singular set must admit a *formal separatrix* which, in fact, gives rise to a *sequence of infinitely near singular points* that cannot be resolved: this "weak" characterization of singularities that cannot be resolved by blow-ups as indicated above will be of particular importance for us. In turn, the existence of this formal separatrix follows from Theorem 3 and/or Proposition 3 in [5] although the former statement is significantly simpler than the full extent of the latter results. In fact, the use made in [4] of the mentioned characterization of singularities that cannot be resolved by blow-ups is comparable to the use that will be made in this paper.

Finally a more recent unpublished result of F. Cano provides a *formal* (global) desingularization theorem for foliations as above by means of blow-ups with formally invariant centers. In other words, this theorem is the global version of [2]. The proof is rather similar to the point of view of [5], in particular the globalization step also uses O. Piltant [12] gluing theorem. This result of Cano provides in particular the best setting for our Theorem D to be built upon and we shall return to this point later.

In a different direction, McQuillan and Panazzolo extended the algorithm of [11] to the general case of holomorphic foliations in [10], eventually obtaining a functorial resolution in the 2-category of Deligne-Mumford *champs*. For readers less comfortable with Deligne-Mumford champs, we will try to present a brief summary of the results obtained in [10] as explained to us by D. Panazzolo.

The first part of [10] is devoted to proving a resolution theorem corresponding to an accurate complex version of [11]. Namely, starting with a foliation $\mathcal{F} = \mathcal{F}_0$, there is a sequence of weighted blow-ups

$$\mathcal{F}_0 \stackrel{\Pi_1}{\longleftarrow} \mathcal{F}_1 \stackrel{\Pi_2}{\longleftarrow} \cdots \stackrel{\Pi_l}{\longleftarrow} \mathcal{F}_l$$

leading to a foliation $\mathcal{F}_l$ having only elementary singular points. Whereas the weighted blow-ups used here are all centered in the singular set of the corresponding foliations the nature of the resulting ambient space requires a comment. Indeed, if we want to keep the bimeromorphic character of the (total) resolution map then the ambient space must have orbifold-type singular points. In turn, this basically means that the singularities of $\mathcal{F}_l$ are elementary in suitable orbifold-charts and therefore that the resulting orbifold-action should be taken in consideration in their analysis.

Once the above result is established in [10], the authors go on to investigate the possibility of "resolving the orbifold-type singularities of the ambient space" in a way compatible with the foliation in question. This discussion accounts for the content of the last section in [10]. It is then proved that this desingularization is possible except in a specific case associated with orbifolds of type $\mathbb{Z}/2\mathbb{Z}$: this special case contains the examples provided by Sancho and Sanz and it can be thought of as an invariant formulation of our Theorem 1 (more on this below).

We can now state our Theorem D providing a resolution result related to the preceding ones but slightly better adapted to the proof of Theorem A.

Theorem D, cf. [9]. *Assume that $\mathcal{F}$ cannot be resolved by a sequence of blow-ups as in item 2 of Theorem A. Then there exists a sequence of one-point blow-ups (centered at singular points) leading to a foliation $\mathcal{F}'$ with a singular point p around which $\mathcal{F}$ is given by a vector field of the form*

$$(y + zf(x, y, z))\frac{\partial}{\partial x} + zg(x, y, z)\frac{\partial}{\partial y} + z^n \frac{\partial}{\partial z}$$

for some $n \geq 2$ and holomorphic functions f and g of order at least 1 with $\partial g/\partial x(0, 0, 0) \neq 0$. Furthermore we have:

- *The resulting foliation $\mathcal{F}'$ admits a formal separatrix at p which is tangent to the z-axis;*
- *The exceptional divisor is locally contained in the plane $\{z = 0\}$.*

The normal form of the nilpotent vector field representing $\mathcal{F}'$ around p above appears in our Theorem 1 where it is also shown that they are persistent under blow-ups centered in the singular set of the corresponding foliations. In other words, they generalize the examples of Sancho and Sanz and, in fact, Theorem 1 is equivalent to the material found in the last section of [10]: a straightforward computation shows that our normal forms correspond to the situation of orbifolds of type $\mathbb{Z}/2\mathbb{Z}$ that cannot be desingularized unless a weighted blow-up is performed. In other words, [10] formulates this result in an invariant way while we have opted for an explicit normal form.

Several comments are now required to properly place Theorem D with respect to the results in [5] and in [10]. The basic advantage of Theorem D with respect to the general resolution theorem of [10] lies in the fact that we only use (standard) blow-ups rather than weighted ones. Here we may also note that these difference is not about the birational character of the (total) resolution map, since this map is birational in both cases (think of the orbifold notion in the case of [10]). In fact, what is actually proved in [10] is the existence of a *birational model for $\mathcal{F}$* exhibiting a singularity as indicated in Theorem D. As far as Theorem A is concerned this last statement still has an issue since not only foliations but also vector fields have to be transformed under these maps. The issue in question arises from the observation that a birational image of a holomorphic vector field may be meromorphic: here we may bear in mind not only the observation that the structure of birational maps in dimension 3 is by no means as simple as in dimension 2 but also the fact that weighted blow ups - even if centered at singular sets - may reduce the order of holomorphic vector fields or even turn them into meromorphic ones. It turns out however that the order of our vector fields at singular points is an essential aspect of the proof of Theorem A and it is important that this order does not become smaller under a resolution procedure. Thus if Theorem A were to be proved out of the resolution result in [10], we would need to check through the algorithm in [11] that the orders of the vector fields in question behave well under the chosen weighted blow ups. Although this is conceivably true, this verification would require us to conduct a non-trivial discussion of Panazzolo's algorithm.

On the other hand, Theorem D as proved here has some merits on its own. First to prove this result we only need the observation that a foliation that cannot be resolved by blow-ups centered in the singular set must possess a formal separatrix ([4], [5]). Up to using the existence of this formal separatrix as starting point the proof of Theorem D is totally elementary and straightforward, thus dispensing with any sophisticated notion from algebraic geometry. Theorem D can also be seen as a nice complement to the results in [5] and even more so to Cano's formal (global) resolution theorem. Indeed, if Theorem D, Theorem 1 and Cano's formal resolution theorem are combined, there follows a resolution theorem claiming that in dimension 3 every foliation can be reduced to a foliation all of whose singularities are elementary except for finitely many singular points having the form indicated in Theorem D. Furthermore this reduction procedure uses only blow-ups centered in the corresponding singular sets. Naturally those final non-elementary singular points can be reduced by a single blow-up of weight 2 as in [10] (see also Section 4). In other words, the resulting theorem is comparable with the main result in [10]. The proofs however are very different. Furthermore this new theorem does not use weighted blow-ups unless they are absolutely indispensable and still resorts to them only at the final step of the resolution procedure: as pointed out above, this issue sometimes provides an additional element of comfort when studying singular points of foliations and/or vector fields.

Let us close this introduction with a brief outline of the structure of the paper. The reader is assumed to be familiar with basic material involving singular vector fields and foliations at the level of the references [1] and [7]. Section 2 and 3 are devoted to the proofs of Theorem D and of Theorem 1 which complements Theorem D. As mentioned these sections build on the previously established fact that a singularity which cannot be resolved by blow-ups centered in the singular set must have a formal separatrix giving rise to a sequence of infinitely near (non-elementary) singular points. Bar this observation the discussion in Sections 2 and 3 is self-contained with explicit calculations. Section 4 contains the proof of Theorem A and of its corollaries. In addition to the previously discussed material on resolution of singular foliations, the proof of Theorem A makes an important use of a classical theorem due to Malmquist about asymptotic expansions of solutions of certain systems of ordinary differential equations [9]. Finally in Section 5 we detail a couple of examples of foliations/vector fields which further illustrate the content of our main results. This section also contains some additional information on persistent nilpotent singularities associated with a *semicomplete vector field*.

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2. The multiplicity of a foliation along a separatrix

For background in the material discussed in the sequel, the reader is referred to [1] and to [7]. Consider a singular holomorphic foliation $\mathcal{F}$ of dimension 1 defined on a neighborhood of the origin in $\mathbb{C}^3$. By definition, $\mathcal{F}$ is given by the local orbits of a holomorphic vector field X whose singular set $\text{Sing}\,(X)$ has codimension at least 2. The vector field X is said to be a *vector field representing* $\mathcal{F}$. Albeit the representative vector field X is not unique, two of them differ by a multiplicative locally invertible function. The singular set $\text{Sing}\,(\mathcal{F})$ of a foliation $\mathcal{F}$ is defined as the singular set of a representative vector field X so that has codimension greater than or equal to 2.

Conversely with every (non-identically) zero germ of holomorphic vector field on $(\mathbb{C}^3, 0)$, it is associated a germ of *singular holomorphic foliation* $\mathcal{F}$. Up to eliminating any non-trivial common factor in the components of X, we can replace X with another holomorphic vector field Y whose singular set has codimension at least 2. The foliation $\mathcal{F}$ is then given by the local orbits of X. A global definition of singular (one-dimensional) holomorphic foliations can be formulated as follows.

Definition 1. Let M be a complex manifold. A singular (1-dimensional) holomorphic foliation $\mathcal{F}$ on M consists of a covering $\{(U_i, \varphi_i)\}$ of M by coordinate charts together with a collection of holomorphic vector fields Z_i satisfying the following conditions:

- For every i, Z_i is a holomorphic vector field having singular set of codimension at least 2 which is defined on $\varphi_i(U_i) \subset \mathbb{C}^n$.
- Whenever $U_i \cap U_j \neq \emptyset$, we have $\varphi_i^* Z_i = g_{ij} \varphi_j^* Z_j$ for some nowhere vanishing holomorphic function g_{ij} defined on $U_i \cap U_j$.

Throughout this section and the next one, all blow ups of foliations (and vector fields) are standard (homogeneous) and assumed to be centered in the singular set of the foliation in question (or of the foliation associated with the vector field).

Consider now a holomorphic foliation $\mathcal{F}$ defined on a complex manifold M of dimension 3 and let $p \in M$ be a singular point of $\mathcal{F}$. A *separatrix* (or analytic separatrix) for $\mathcal{F}$ at p is an irreducible analytic curve invariant by $\mathcal{F}$, passing through p, and not contained in the singular set $\mathrm{Sing}\,(\mathcal{F})$ of $\mathcal{F}$. Along similar lines, a *formal separatrix* for $\mathcal{F}$ at p is a formal irreducible curve S invariant by $\mathcal{F}$ and centered at p. In other words, in local coordinates (x, y, z) around p where $\mathcal{F}$ is represented by the vector field $X = F\partial/\partial x + G\partial/\partial y + H\partial/\partial z$, the formal separatrix S is given by a triplet of formal series $t \mapsto \varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$ satisfying the following (formal) equations

$$\varphi_1(t)(G \circ \varphi)(t) = \varphi_2(t)(F \circ \varphi)(t) \;\text{ and }\; \varphi_2(t)(H \circ \varphi)(t) = \varphi_3(t)(G \circ \varphi)(t) \tag{1}$$

where:

(1) $(F \circ \varphi)(t)$ (resp. $(G \circ \varphi)(t)$, $H \circ \varphi(t))$ stands for the formal series obtained by composing the Taylor series of F (resp. G, H) at the origin with the formal series of φ as indicated.
(2) In the preceding it is understood that at least one of the formal series $(F \circ \varphi)(t)$, $(G \circ \varphi)(t)$, and $(H \circ \varphi)(t)$ is not identically zero.

Note that an analytic separatrix of $\mathcal{F}$ can also be viewed as a *formal separatrix* up to considering the Puiseux parametrization associated with the separatrix in question. The corresponding terminology will be such that whenever we refer to a formal separatrix of $\mathcal{F}$ the possibility of having an actual (analytic) separatrix *is not* excluded. If we need to emphasize that a formal separatrix is not analytic, we will say that it is *strictly formal*. Finally note also that the third condition above is automatically satisfied whenever $\varphi(t)$ is a strictly formal curve satisfying Equation (1).

Consider again a singular point $p \in M$ of a holomorphic foliation $\mathcal{F}$. Choose local coordinates (x, y, z) around p and assume that $\mathcal{F}$ has a formal separatrix S at p which is given in the coordinates (x, y, z) by the formal series $\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))$. Consider now a local holomorphic vector field X defined around p and *tangent to $\mathcal{F}$ but not necessarily* representing $\mathcal{F}$. Note that the (formal) pull-back of the restriction of X to S by φ may be considered since S is a formal separatrix of $\mathcal{F}$. This pull-back is a formal vector field in dimension 1 given by

$$\varphi^*(X|_S) = g(T)\frac{\partial}{\partial T}$$

where g satisfies

$$(X \circ \varphi)(T) = g(T)\varphi'(T) \tag{2}$$

as formal series.

We recall the classical notion of multiplicity of a foliation along a separatrix which is well known also as basic example of valuation.

Definition 2. The multiplicity of X along S is the order of the formal series g at $0 \in \mathbb{C}$

$$\mathrm{mult}\,(X, S) = \mathrm{ord}(g, 0)\,.$$

In other words, setting $g(T) = \sum_{k\geq 1} g_k T^k$, $\text{mult}(\mathcal{F}, S)$ is the smallest positive integer $k \in \mathbb{N}^*$ for which $g_k \neq 0$. This multiplicity equals zero if and only if the series associated with $g(T)$ vanishes identically.

In turn, the multiplicity of $\mathcal{F}$ along S, $\text{mult}\,(\mathcal{F}, S)$, is defined as the multiplicity along S of a vector field X representing $\mathcal{F}$ around p. Since S as a separatrix for $\mathcal{F}$ is not contained in the singular set of $\mathcal{F}$, the multiplicity of $\mathcal{F}$ along S is never equal to zero.

It is immediate to check that the notions above are well defined in the sense that they depend neither on the choice of coordinates nor on the choice of the representative vector field X.

We begin with a simple albeit important lemma. To fix notation, we will say that S is a (formal) separatrix for a vector field X if S is a (formal) separatrix for the foliation $\mathcal{F}$ associated with X. We also recall that all blow-ups

Lemma 1. *The multiplicity of a vector field along a formal separatrix is invariant by blow-ups (centered at the singular set of the associated foliation).*

Proof. The statement means that the multiplicity of a vector field along a formal separatrix is invariant by blow-ups regardless of whether they are centered at a singular point or at a (locally) smooth analytic curve contained in the singular set of $\mathcal{F}$. We will prove the mentioned invariance in the case of blow-ups centered at a point. The case of blow-ups centered at analytic curves is analogous and thus left to the reader.

Let X be a holomorphic vector field defined on a neighborhood of the origin of $\mathbb{C}^3$ and admitting a formal separatrix S. Let $\pi : \widetilde{\mathbb{C}}^3 \to \mathbb{C}^3$ denote the blow-up map centered at the origin and denote by $\widetilde{S}$ the transform of S by π.

Consider an irreducible formal Puiseux parameterization φ (resp. $\widetilde{\varphi}$) for S (resp. $\widetilde{S}$) and set

$$\varpi = \varphi^{-1} \circ \pi \circ \widetilde{\varphi}\,.$$

We only need to prove that ϖ is a formal diffeomorphism at $0 \in \mathbb{C}$ since this clearly implies that the multiplicity of $\widetilde{X}$ on $\widetilde{S}$ coincides with the multiplicity of X on S.

If S is an analytic separatrix, then the above assertion can easily be checked as follows. The parameterizations φ and $\widetilde{\varphi}$ are defined (convergent) and one-to-one on neighborhoods U and V of $0 \in \mathbb{C}$. Similarly ϖ is defined from V to U and is clearly a diffeomorphism from $V \setminus \{0\}$ to $U \setminus \{0\}$. Furthermore, ϖ is bounded at $0 \in \mathbb{C}$ (in fact, ϖ is continuous at $0 \in \mathbb{C}$ sending the origin of $\mathbb{C}$ into the origin of $\mathbb{C}$) which implies that ϖ is holomorphic at $0 \in \mathbb{C}$. Now the one-to-one character of ϖ implies that ϖ is a diffeomorphism at the origin.

In the general case where S is strictly formal, we proceed as follows. First, note that ϖ is clearly a formal diffeomorphism provided that S is smooth in the formal sense, i.e. provided that $\varphi'(0) \neq (0,0,0)$. Hence the order of $\widetilde{X}$ on $\widetilde{S}$ coincides with the order of X on S. More generally, in suitable coordinates (x,y,z), we can assume without loss of generality that φ takes on the form $\varphi(T) = (T^m + \text{h.o.t}, T^n + \text{h.o.t}, T^p)$ for $p \leq m, n$. In turn the vector field X is given by

$$X = F(x,y,z)\frac{\partial}{\partial x} + G(x,y,z)\frac{\partial}{\partial y} + H(x,y,z)\frac{\partial}{\partial z}\,.$$

Since S is a formal solution of the differential equation associated with X, there follows that $\varphi'(T)$ and $X \circ \varphi$ satisfy Equation (1). Comparing the last component of $\varphi'(T)$ and of $(X \circ \varphi)(T)$ we conclude that the multiplicative function g appearing in Equation (2) is of order equal to $\text{ord}(H \circ \varphi, 0) - p + 1$. Thus

$$\text{mult}\,(X, S) = \text{ord}(H \circ \varphi, 0) - p + 1\,.$$

Let us now compute the order of $\widetilde{X}$ along $\widetilde{S}$. For this we consider affine coordinates (u,v,z) where the blow-up map is given by $\pi(u,v,z)=(uz,uz,z)=(x,y,z)$. The transform of X then becomes $\widetilde{X}=(1/z)Z$ where Z is the vector field given by

$$Z=(F(uz,vz,z)-uH(uz,vz,z))\partial/\partial x+(G(uz,vz,z)-vH(uz,vz,z))\partial/\partial y+zH(uz,vz,z)\partial/\partial z\,.$$

In turn, the transform of S by π is parameterized by $\psi(T)=\pi^*\varphi(T)=(T^{m-p}+\text{h.o.t}, T^{n-p}+\text{h.o.t}, T^p)$. Since $\widetilde{S}$ is a formal solution of the differential equation associated to $\widetilde{X}$, there follows again that $\psi'(T)$ and $(\widetilde{X}\circ\psi)(T)$ satisfy Equation (1). By comparing their last components, we conclude that

$$\begin{aligned}\text{mult}\,(\widetilde{X},\widetilde{S}) &= \text{ord}\,\big(H((T^{m-c}+\text{h.o.t})T^c,(T^{n-c}+\text{h.o.t})T^c,T^p),0\big)-p+1\\ &= \text{ord}\,(H(T^m+\text{h.o.t},T^n+\text{h.o.t},T^p),0)-p+1\\ &= \text{ord}(H\circ\varphi,0)\\ &= \text{mult}\,(X,S)\,.\end{aligned}$$

The lemma is proved. □

Consider again a foliation $\mathcal{F}$ defined on a neighborhood of the origin of $\mathbb{C}^3$ along with a formal separatrix S. Whereas Lemma 1 asserts that the multiplicity of a vector field along a formal separatrix is invariant by blow-ups, the analogous statement does not necessarily hold for a foliation. Indeed, let X be a vector field representing $\mathcal{F}$ around $(0,0,0)\in\mathbb{C}^3$ and denote by $\widetilde{X}$ the pull-back of X by the blowing-map centered at the origin. If X has order at least 2 at the origin then the singular set of $\widetilde{X}$ has codimension 1 since $\widetilde{X}$ vanishes identically on the corresponding exceptional divisor. More precisely, in the affine coordinates (u,v,z), where $x=uz$ and $y=vz$, we have $\widetilde{X}=z^{\alpha}Z$ for a certain holomorphic vector field Z having singular set of codimension at least 2 and a certain integer $\alpha\geq 1$. In fact, k denotes the order of X at $0\in\mathbb{C}^3$, then we have $\alpha=k$ or $\alpha=k-1$ according to whether or not the origin is a dicritical singular point. Here we remind the reader that a singular point is said to be *dicritical* if the exceptional divisor, given by $\{z=0\}$ in the above affine coordinates, is not invariant by $\widetilde{\mathcal{F}}$. Next note that the multiplicity of X along S coincides with the multiplicity of $\widetilde{X}$ along $\widetilde{S}$ (Lemma 1). However, the multiplicity of $\widetilde{\mathcal{F}}$ along $\widetilde{S}$ is not the multiplicity of $\widetilde{X}$ along $\widetilde{S}$ but rather the multiplicity of Z along $\widetilde{S}$. More precisely, we have

$$\begin{aligned}\text{mult}\,(\widetilde{\mathcal{F}},\widetilde{S}) &= \text{mult}\,(Z,\widetilde{S})\\ &= \text{mult}\,(\widetilde{X},\widetilde{S})-\text{ord}\,(z^{\alpha}\circ\varphi,0)\\ &= \text{mult}\,(\mathcal{F},S)-\text{ord}\,(z^{\alpha}\circ\varphi,0)\,,\end{aligned}$$

where φ stands for the Puiseux parametrization of $\widetilde{S}$. Summarizing, we have proved the following:

Proposition 1. *Let $\mathcal{F}$ be a holomorphic foliation on $(\mathbb{C}^3,0)$ admitting a formal separatrix S. If $\mathcal{F}$ has order at least 2 at the origin, then*

$$\text{mult}\,(\widetilde{\mathcal{F}},\widetilde{S})<\text{mult}(\mathcal{F},S)\,,$$

where $\widetilde{\mathcal{F}}$ (resp. $\widetilde{S}$) stands for the transform of $\mathcal{F}$ (resp. S) by the one-point blow-up centered at the origin. □

In order to state the analogue of Proposition 1 for blow-ups centered at smooth (irreducible) curves contained in Sing$\,(\mathcal{F})$, a notion of order for $\mathcal{F}$ with respect to the curves in question is needed. A suitable notion can be introduced as follows.

Recall first that the order of the foliation $\mathcal{F}$ at the origin is defined to be the degree of the first non-zero homogeneous component of a vector field X representing $\mathcal{F}$. The mentioned degree, as well as all of the corresponding non-zero homogeneous component, may be recovered through the family of homotheties $\Gamma_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)$. More precisely, the degree is simply the unique positive integer $d \in \mathbb{N}$ for which

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{d-1}} \Gamma_\lambda^* X \tag{3}$$

is a non-trivial vector field. Furthermore the non-trivial vector field obtained as this limit is exactly the first non-zero homogeneous component of X. We shall adapt this construction to define the order of $\mathcal{F}$ over a curve.

Let then C be a smooth curve contained in $\mathrm{Sing}\,(\mathcal{F})$. Our purpose is to define the order of $\mathcal{F}$ with respect to C. Clearly there are local coordinates (x, y, z) in which the curve in question coincides with the z-axis, i.e. it is given by $\{x = y = 0\}$ (as usual we only perform blow-ups centered at smooth curves; naturally this is not a very restrictive condition since every curve can be turned into smooth by the standard resolution procedure). The blow-up centered at $\{x = y = 0\}$ is equipped with affine coordinates (x, t, z) and (u, y, z) where the corresponding blow-up map is given by $\pi_z(x, t, z) = (x, tx, z)$ (resp. $\pi_z(u, y, z) = (uy, y, z)$). Consider now the family of automorphisms given by

$$\Lambda_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, z).$$

The *order of $\mathcal{F}$ with respect to C* (or the order of $\mathcal{F}$ over C) is defined as the unique integer $d \in \mathbb{N}$ for which

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{d-1}} \Lambda_\lambda^* X \tag{4}$$

yields a non-trivial vector field. Note that this integer d may be seen as the degree of X with respect to the variables x, y. In fact, assume that in coordinates (x, y, z) the vector field X is given by $X = X_1(x, y, z)\partial/\partial x + X_2(x, y, z)\partial/\partial y + X_3(x, y, z)\partial/\partial z$. The pull-back of X by Λ_λ becomes

$$\Lambda_\lambda^* X = \frac{1}{\lambda}\left[X_1(\lambda x, \lambda y, z)\frac{\partial}{\partial x} + X_2(\lambda x, \lambda y, z)\frac{\partial}{\partial y}\right] + X_3(\lambda x, \lambda y, z)\frac{\partial}{\partial z}$$

Denote by k (resp. l) the maximal power of λ that divides $X_1(\lambda x, \lambda y, z)\partial/\partial x + X_2(\lambda x, \lambda y, z)\partial/\partial y$ (resp. $X_3(\lambda x, \lambda y, z)\partial/\partial z$). The order d defined above is simply the minimum between k and $l + 1$.

The analogue of Proposition 1 for blow-ups centered at smooth (irreducible) curves can now be stated as follows.

Proposition 2. *Let $\mathcal{F}$ be a holomorphic foliation on $(\mathbb{C}^3, 0)$ admitting a formal separatrix S. Let $\widetilde{\mathcal{F}}$ (resp. $\widetilde{S}$) stands for the strict transform of $\mathcal{F}$ (resp. S) by the blow-up centered at a smooth (irreducible) curve contained in* $\mathrm{Sing}\,(\mathcal{F})$*. If $\mathcal{F}$ has order at least 2 with respect to the blow-up center, then*

$$\mathrm{mult}\,(\widetilde{\mathcal{F}}, \widetilde{S}) < \mathrm{mult}(\mathcal{F}, S)\,.$$

□

Let us close this section with a first application of Proposition 1 to the reduction of singular points (a slightly more general discussion involving Proposition 2 appears in Section 3). Let $\mathcal{F}$ be a holomorphic foliation defined on a neighborhood of the origin of $\mathbb{C}^3$ and let X be a holomorphic vector field representing $\mathcal{F}$. Recall that a singular point p of $\mathcal{F}$ is said to be *elementary* if the linear part of X at p, $DX(p)$, has at least one eigenvalue different from zero.

Similarly the singular point p is said to be *nilpotent* if the linear part of X at p is nilpotent and non-zero.

Consider now a singular foliation $\mathcal{F}_0$ along with a formally smooth separatrix S_0 at the origin ($(0,0,0) \simeq p_0$). Consider the blow-up $\mathcal{F}_1$ of $\mathcal{F}_0$ centered at the origin. The transform S_1 of S_0 selects a singular point p_1 of $\mathcal{F}_1$ in the exceptional divisor $\Pi_1^{-1}(0,0,0)$. In fact, if the point $p_1 \in \Pi_1^{-1}(0,0,0)$ selected by S_1 were regular for $\mathcal{F}_1$, then $\Pi_1^{-1}(0,0,0)$ would not be invariant by $\mathcal{F}_1$ and the formal separatrix S_1 (and hence S) would actually be analytic and $\mathcal{F}_1$ would be regular on a neighborhood of p_1: this situation is excluded in what follows.

Next let $\mathcal{F}_2$ be the blow-up of $\mathcal{F}_1$ at p_1. Again the transform S_2 of S_1 will select a singular point $p_2 \in \Pi_2^{-1}(p_1)$ of $\mathcal{F}_2$. The procedure is then continued by induction so as to produce a (infinite) sequence of foliations $\mathcal{F}_n$

$$\mathcal{F}_0 \stackrel{\Pi_1}{\longleftarrow} \mathcal{F}_1 \stackrel{\Pi_2}{\longleftarrow} \cdots \stackrel{\Pi_n}{\longleftarrow} \mathcal{F}_n \stackrel{\Pi_{n+1}}{\longleftarrow} \cdots \tag{5}$$

along with singular points p_n and formal separatrices S_n.

Lemma 2. *Consider a sequence of foliations $\mathcal{F}_n$ as in (5) along with a sequence of formal separatrices S_n and singular points p_n. Assume that for every $n \in \mathbb{N}$, p_n is not an elementary singular point of $\mathcal{F}_n$. Then there exists $n_0 \in \mathbb{N}$ such that p_n is nilpotent singularity of $\mathcal{F}_n$ for every $n \geq n_0$.*

Proof. The statement follows from Proposition 1. Indeed, by assumption, p_n is not an elementary singular point of $\mathcal{F}_n$ (for every $n \in \mathbb{N}$). Assume, in addition, that $\mathcal{F}_1$ is not nilpotent at p_1. This means that the order of $\mathcal{F}_1$ at p_1 is at least 2 so that the multiplicity of $\mathcal{F}_2$ along S_2 is strictly smaller than the multiplicity of $\mathcal{F}_1$ along S_1. If $\mathcal{F}_2$ is again non-nilpotent at p_2, then the order of $\mathcal{F}_2$ at p_2 is again at least 2. There follows that the multiplicity of $\mathcal{F}_3$ along S_3 is strictly smaller than the multiplicity of $\mathcal{F}_2$ along S_2. When the procedure is continued, the multiplicity of $\mathcal{F}_{n+1}$ along S_{n+1} will be strictly smaller than the multiplicity of $\mathcal{F}_n$ along S_n whenever p_n is not a nilpotent singularity of $\widetilde{\mathcal{F}}_n$. Hence we obtain a decreasing - though not necessarily strictly decreasing - sequence of non-negative integers. This sequence must eventually become constant. If n_0 is the index for which the sequence is constant for $n \geq n_0$, then Proposition 1 ensures that $\mathcal{F}_n$ has order 1 at p_n for every $n \geq n_0$. Since by assumption p_n is not an elementary singularity of $\mathcal{F}_n$, we conclude that p_n must be a nilpotent singularity of $\mathcal{F}_n$ for $n \geq n_0$. The lemma is proved. □

3. On persistent nilpotent singularities

In this section we are going to discuss nilpotent singular points that are persistent under blow-up transformations and this will lead to the two main results of the section, namely Theorem 1 and Theorem D. As mentioned Theorem 1 generalizes the celebrated examples of vector fields obtained by Sancho and Sanz and constitutes a slightly more explicit formulation of the previously established result that can be found in the last section of [10]. The corresponding proofs are however totally different.

First let us make it clear what is meant by being *persistent under blow-up transformations.* In the sequel the centers of the blow-ups maps are always *contained in the singular set of the foliation* and either are a single point or a smooth analytic curve. Similarly, the reader is again reminded that all blow-ups are assumed to be standard for the entirety of the section.

Let $\mathcal{F}_0$ denote a singular foliation along with an irreducible formal separatrix S_0 at a chosen singular point p_0. Consider a sequence of blow-ups and transformed foliations which is obtained as follows. First we choose a center C_0 with $p_0 \in C_0$ and contained in the singular set of $\mathcal{F}_0$. Then let $\mathcal{F}_1$ denote the blow-up of $\mathcal{F}_0$ with center C_0. The transform S_1 of S_0 selects a point

p_1 in the exceptional divisor $\Pi_1^{-1}(C_0)$. In the case where p_1 is regular for $\mathcal{F}_1$ the sequence of blow-ups stops at this level. Otherwise p_1 is a singular point for $\mathcal{F}_1$ and another blow-up will be performed. In this case we denote by $\mathcal{F}_2$ the blow-up of $\mathcal{F}_1$ with a center C_1 contained in the singular set of $\mathcal{F}_1$ and such that $p_1 \in C_1$. Again the transform S_2 of S_1 will select a point $p_2 \in \Pi_2^{-1}(C_1)$. If p_2 is a regular point for $\mathcal{F}_2$, then the sequence of blow-ups stops. Otherwise we consider the blow-up of $\mathcal{F}_2$ with a center C_2 passing through p_2. The procedure is then continued by induction so as to produce a sequence of foliations $\mathcal{F}_n$

$$\mathcal{F}_0 \xleftarrow{\Pi_1} \mathcal{F}_1 \xleftarrow{\Pi_2} \cdots \xleftarrow{\Pi_n} \mathcal{F}_n \xleftarrow{\Pi_{n+1}} \cdots \tag{6}$$

along with singular points p_n and formal separatrices S_n. The mentioned sequence is finite if there exists $n \in \mathbb{N}$ such that p_n is regular for $\mathcal{F}_n$. A sequence of points p_n obtained from a formal separatrix S_0 as above is often called a *sequence of infinitely near singular points.*

Definition 3. With the preceding notation, assume that p_0 is a nilpotent singular point for $\mathcal{F}_0$. The point p_0 is said to be a persistent nilpotent singularity if there exists a formal separatrix S_0 of $\mathcal{F}_0$ such that for every sequence of blowing-ups as in (6) the following conditions are satisfied:

- ($\imath$) The singular points p_n (selected by the transformed separatrices S_n) are all nilpotent singular points for the corresponding foliations;
- ($\imath\imath$) The multiplicity $\mathrm{mult}(\mathcal{F}_n, S_n)$ of $\mathcal{F}_n$ along S_n does not depend on n.

Remark 1. Note that Condition ($\imath$) implies Condition ($\imath\imath$) if the blow-up is centered at the nilpotent singular point in question. In the case of blow-ups centered at smooth curves, however, it is possible to have a strictly smaller multiplicity, cf. the proof of Lemma 4.

In view of the mentioned Seidenberg's theorem persistent nilpotent singularities do not exist in dimension 2. In dimension 3 however the examples of singularities that cannot be resolved by blow-ups with invariant centers found by Sancho and Sanz correspond to persistent nilpotent singularities. In fact, Sancho and Sanz have shown that the foliation associated with the vector field

$$X = x\left(x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta z\frac{\partial}{\partial z}\right) + xz\frac{\partial}{\partial y} + (y - \lambda x)\frac{\partial}{\partial z}$$

possesses a strictly formal separatrix $S = S_0$ such that for every sequence of blowing-ups as above, the corresponding sequence of infinitely near singular points consists of nilpotent singularities. Furthermore the foliations $\mathcal{F}_n$ also satisfy $\mathrm{mult}\,(\mathcal{F}_n, S_n) = 2$ for every $n \in \mathbb{N}$, where S_n stands for the transform of S. The set of persistent nilpotent singular points is thus non-empty. Most of this section will be devoted to the characterization of these persistent singularities and the final result will be summarized by Theorem 1. We begin with the following proposition:

Proposition 3. *Let $\mathcal{F}$ be a singular holomorphic foliation on $(\mathbb{C}^3, 0)$ and assume that the origin is a persistent nilpotent singularity of $\mathcal{F}$. Then, up to finitely many one-point blow-us, there exist local coordinates and a holomorphic vector field X representing $\mathcal{F}$ and having the form*

$$(y + f(x, y, z))\frac{\partial}{\partial x} + g(x, y, z)\frac{\partial}{\partial y} + z^n\frac{\partial}{\partial z} \tag{7}$$

for some $n \geq 2 \in \mathbb{N}$ and some holomorphic functions f and g of order at least 2 at the origin. Moreover the orders of the functions $z \mapsto f(0, 0, z)$ and of $z \mapsto g(0, 0, z)$ can be made arbitrarily large (in particular greater than $2n$).

Proof. Let $\mathcal{F}$ be a nilpotent persistent singular point and denote by S a formal separatrix giving rise to a sequence of infinitely near singular points as in Definition 3. Up to finitely many one-point blow-ups the formal separatrix S can be assumed to be smooth in the formal sense. Up to performing an additional one-point blow-up, we may also assume that $\mathcal{F}$ admits an analytic smooth *invariant surface* which is, in addition, transverse to the formal separatrix S. In fact, the resulting exceptional divisor is necessarily invariant under the transformed foliation since the previous singular point is nilpotent and non-zero (recall that the exceptional divisor is not invariant by the blown-up foliation if and only if the first non-zero homogeneous component is a multiple of the radial vector field).

Note also that even if we consider the blow-up of $\mathcal{F}$ centered at a smooth analytic curve contained in the singular set of the foliation (rather than the one-point blow-up) the resulting exceptional divisor is still invariant by the transformed foliation. The argument is similar to the preceding one: if this component were not invariant, then the first non-zero homogeneous component of the foliation *with respect to this curve* would be a multiple of vector field $x\partial/\partial x + y\partial/\partial y$ at generic points in this center. Again this cannot happen since the origin is a nilpotent singular point. Finally, denoting by E_n the total exceptional divisor associated with the birational transformation $\Pi_{\circ n} = \Pi_n \circ \cdots \circ \Pi_2 \circ \Pi_1$. The argument above also applies to ensure that every irreducible component of the exceptional divisor is invariant by the corresponding foliation $\mathcal{F}_n$.

Summarizing, we can assume without loss of generality that all of the following holds:

- the formal separatrix S is smooth;
- $\mathcal{F}$ possesses a (smooth) analytic invariant surface E;
- the formal separatrix S is transverse to E (in the formal sense).

In view of the preceding, consider local coordinates (x, y, z) around p_0 where the smooth invariant surface E is given by $\{z = 0\}$. Denote by H a formal change of coordinates preserving $\{z = 0\}$ as invariant surface and taking the formal separatrix S to the z-axis given by $\{x = 0,\, y = 0\}$. Since the vector field obtained by conjugating X through H is merely formal, let H_m denote the polynomial change of coordinates obtained by truncating H at order m. We then set

$$Y_m = (DH_m)^{-1}(X \circ H_m)\,.$$

The map H_m is holomorphic and so is the vector field Y_m. Denote by $\mathcal{F}_m$ the foliation associated with Y_m. The foliation $\mathcal{F}_m$ clearly admits a formal separatrix S_m whose order of tangency with the z-axis goes to infinity with m and thus can be assumed arbitrarily large.

Under the above conditions, the vector field Y_m has the form

$$Y_m = A(x, y, z)\frac{\partial}{\partial x} + B(x, y, z)\frac{\partial}{\partial y} + C(x, y, z)\frac{\partial}{\partial z}$$

with

(1) $C(x, y, z) = z^n + g(z) + xP(x, y, z) + yQ(x, y, z)$, for some $n \in \mathbb{N}$, some holomorphic functions P and Q divisible by z and some holomorphic function g divisible by z^{n+1};
(2) $A(0, 0, z)$ and $B(0, 0, z)$ having order arbitrarily large, say greater than $2n$.

Note that the value of $n = \operatorname{ord}(C(0, 0, z)) \geq 2$ depends only on the initial foliation $\mathcal{F}$ and not on the choice of $m \in \mathbb{N}^*$. In fact, the value of n is nothing but the multiplicity of $\mathcal{F}$ along S and hence it is invariant by (formal) changes of coordinates. The orders of $A(0, 0, z)$ and of $B(0, 0, z)$ depend however on m. Note that the orders in question are related to the contact order between S_m and the z-axis. In particular these orders can be made arbitrarily large as well.

Naturally the foliation $\mathcal{F}$ and $\mathcal{F}_m$ are both nilpotent at the origin. Next we have:
Claim. Up to a linear change of coordinates in the variables x, y, the linear part of Y_m is given by $y\partial/\partial x$.

Proof of the Claim. The formal Puiseux parametrization φ of S_m has the form $\varphi(T) = (T^r + \text{h.o.t.}, T^s + \text{h.o.t.}, T)$ where the integers r and s are related to the contact order between S_m and the z-axis. In particular both r and s can be made arbitrarily large. Now it is clear that both $\partial A/\partial z$ and $\partial B/\partial z$ must vanish at the origin provided that φ is invariant by the vector field Y_m. On the other hand, $\partial C/\partial x$ and $\partial C/\partial y$ are both zero at the origin since P and Q are divisible by z (cf. condition (1) above). It is also clear that $\partial C/\partial z$ equals zero at the origin since $n \geq 2$. Thus both the third line and the third column in the matrix representing the linear part of Y_m at the origin are entirely constituted by zeros. Using again the fact that this matrix is nilpotent, the standard Jordan form ensures that a linear change of coordinates involving only the variables x, y brings the linear part of Y_m to the form $y\partial/\partial x$. It is also immediate to check that this linear change of coordinates does not affect the previously established conditions and/or normal forms. The claim is proved. □

Consider now the blow-up of $\mathcal{F}$ centered at the origin. In coordinates (u, v, z) where $(x, y, z) = (uz, vz, z)$, the transform $\widetilde{Y}_m$ of Y_m by the mentioned blow-up is given by

$$\widetilde{Y}_m = \widetilde{A}(u, v, z)\frac{\partial}{\partial x} + \widetilde{B}(u, v, z)\frac{\partial}{\partial y} + \widetilde{C}(u, v, z)\frac{\partial}{\partial z}$$

where

$$\widetilde{A}(u, v, z) = \frac{A(uz, vz, z) - uC(uz, vz, z)}{z} \quad \text{and} \quad \widetilde{B}(u, v, z) = \frac{B(uz, vz, z) - vC(uz, vz, z)}{z}$$

and where $\widetilde{C}(u, v, z) = C(uz, vz, z)$. In particular $\widetilde{\mathcal{F}}_m$ is nilpotent at the origin, with the same linear part as $\mathcal{F}_m$. Furthermore the above formulas easily imply all of the following:

(a) the order of $\widetilde{C}(0, 0, z)$ coincides with the order of $C(0, 0, z)$;
(b) the maximal power of z dividing $\widetilde{C}(u, v, z) - z^n - g(z)$ is strictly greater than the maximal power of z dividing $C(x, y, z) - z^n - g(z)$;
(c) $\operatorname{ord} \widetilde{A}(0, 0, z) = \operatorname{ord} A(0, 0, z) - 1$ and $\operatorname{ord} \widetilde{B}(0, 0, z) = \operatorname{ord} B(0, 0, z) - 1$.

Also the transform $\widetilde{S}_m$ of S_m is a formal separatrix tangent to the z-axis. The tangency order is still large (at least $2n$) since the order in question is related to the orders of $\widetilde{A}(0, 0, z)$ and of $\widetilde{B}(0, 0, z)$ and these orders fall only by one unity (item (c)). In turn, the multiplicity of $\widetilde{\mathcal{F}}_m$ along $\widetilde{S}_m$ coincides with the multiplicity of $\mathcal{F}_m$ along S_m (from item (a)). Finally the function C was divisible by z. Now, according to item (b), $\widetilde{C}$ is divisible by z^2. In fact, item (b) ensures that after at most n one-point blow-ups, the corresponding singular point is still a nilpotent singularity for which the component of the representative vector field in the direction transverse to the exceptional divisor (given in local coordinates by $\{z = 0\}$) has the form $z^n I(u, v, z)$ where $I(u, v, z)$ is a holomorphic function satisfying $I(0, 0, 0) \neq 0$. Dividing all the components of the vector field in question by I then yields another representative vector field with the desired normal form. The proposition is proved. □

An additional simplification can be made on the normal form (7) of Proposition 3. Namely:

Lemma 3. *Up to performing an one-point blow-up, the functions f and g in (7) become divisible by z.*

Proof. Again let π denote the blow-up map centered at the origin and set $\widetilde{X} = \pi^* X$. In the above mentioned affine coordinates (u, v, z), we have

$$\widetilde{X} = (y + \widetilde{f}(x, y, z))\frac{\partial}{\partial x} + \widetilde{g}(x, y, z)\frac{\partial}{\partial y} + z^n \frac{\partial}{\partial z},$$

where $\widetilde{f} = (f(uz, vz, z) - uz^n)/z$ and $\widetilde{g} = (g(uz, vz, z) - vz^n)/z$. The functions $\widetilde{f}$, $\widetilde{g}$ are thus divisible by z since f and g have order at least 2 at the origin. The lemma follows. $\square$

Next we are going to determine conditions on f and g for the singular point $p_0 \simeq (0,0,0)$ to be a persistent nilpotent singularity under blow-up transformations. Let then $\mathcal{F}$ be the foliation associated with a vector field X having the normal form provided by Proposition 3 and Lemma 3. Namely X is given by

$$X = (y + zf(x,y,z))\frac{\partial}{\partial x} + zg(x,y,z)\frac{\partial}{\partial y} + z^n\frac{\partial}{\partial z}\,, \tag{8}$$

where f and g are holomorphic functions of order at least 1 and $n \in \mathbb{N}$, with $n \geq 2$. Let S be a smooth formal separatrix of $\mathcal{F}$ giving rise to the persistent nilpotent singular points (see Definition 3). Without loss of generality, the contact order $k_0 (\geq 4)$ between S and the z-axis is assumed to be large and, similarly, $f(0,0,z)$ and $g(0,0,z)$ are assumed to have order bigger than $2n \geq 4$.

Note that the curve locally given by $\{y = 0, z = 0\}$ coincides with the singular set of $\mathcal{F}$. We are then allowed to perform either an one-point blow-up centered at $p_0 \simeq (0,0,0)$ or a blow-up centered at the mentioned curve. Now we have:

Lemma 4. *Assume that $\mathcal{F}$ has a persistent nilpotent singularity at the origin and let S denote the corresponding formal separatrix. Then $g(x,0,0) = \lambda x + \text{h.o.t.}$ for some constant $\lambda \in \mathbb{C}^*$.*

Proof. Denote by X_1 (resp. $\mathcal{F}_1$, S_1) the transform of X (resp. $\mathcal{F}$, S) by the blow-up map π_1 centered on $\{y = 0, z = 0\}$. In local coordinates (x, v, z) where $y = vz$ the vector field X_1 is given by

$$X_1 = (vz + zf(x,vz,z))\frac{\partial}{\partial x} + (g(x,vz,z) - vz^{n-1})\frac{\partial}{\partial v} + z^n\frac{\partial}{\partial z}\,. \tag{9}$$

Note that $g(x,0,0)$ does not vanish identically, otherwise $g(x,vz,z)$ would be divisible by z and hence the vector field X_1 would vanish identically over the exceptional divisor (locally given by $\{z = 0\}$). This is impossible since the multiplicity of $\mathcal{F}_1$ along the transform of S would be strictly smaller than the multiplicity of $\mathcal{F}$ along S, hence contradicting Condition ($\imath$) in Definition (3). In particular, the singular set of $\mathcal{F}_1$ is locally given by $\{x = 0, z = 0\}$.

On the other hand, the formal separatrix S_1 is still tangent to the (transform of the) z-axis since the contact between S and the (initial) z-axis was greater than 2 (in fact the contact between S_1 and the z-axis is at least $k_0 - 1 \geq 3$). Hence, in the affine coordinates (x, v, z), the foliation $\mathcal{F}_1$ must have a nilpotent singularity at the origin. Combining the conditions that $f(0,0,0) = 0$, $n \geq 2$, and the fact that the order of $g(0,0,z)$ is greater than $2n$, the preceding implies that $\partial g/\partial x$ does not vanish at the origin. In other words, $g(x,0,0) = \lambda x + \text{h.o.t.}$ as desired. $\square$

Note that the above proof also yields the following sort of converse to Lemma 4.

Lemma 5. *Keeping the preceding notation, let $\mathcal{F}$ be given by a vector field X as in (8) and assume that S is a formal separatrix of $\mathcal{F}$ with contact at least 3 with the z-axis. Assume that $g(x,0,0) = \lambda x + \text{h.o.t.}$, with $\lambda \neq 0$. Then the blow-up $\mathcal{F}_1$ of $\mathcal{F}$ centered at the curve $\{y = 0, z = 0\}$ has a nilpotent singularity at the point of the exceptional divisor selected by S_1.* $\square$

Continuing the discussion of Lemma 4, consider again the vector field X_1 in (9). We perform the blow-up centered at the curve locally given by $\{x = 0, z = 0\}$ - which is contained in the

the singular set of $\mathcal{F}_1$ - and denote by X_2 (resp. $\mathcal{F}_2$, S_2) the transform of X_1 (resp. $\mathcal{F}_1$, S_1). In affine coordinates (u, v, z) with $x = uz$ we have

$$X_2 = (v + f(uz, vz, z) - uz^{n-1})\frac{\partial}{\partial x} + (g(uz, vz, z) - vz^{n-1}))\frac{\partial}{\partial v} + z^n\frac{\partial}{\partial z}\,. \tag{10}$$

The contact of S_2 with the z-axis equals $k_0 - 1 \geq 3$ as follows from a simple computation (cf. also Remark 1). In particular the formal separatrix S_2 is still based at the origin of the coordinates (u, v, z) and it is tangent to the z-axis. Since the above formula shows that $\mathcal{F}_2$ has a nilpotent singularity at the origin, we conclude:

Lemma 6. *The foliation $\mathcal{F}_2$ (resp. vector field X_2) has a nilpotent singularity at the point of the exceptional divisor selected by S_2 (identified with the origin of the coordinates (u, v, z)).* □

The reader will also note that the singular set of $\mathcal{F}_2$ is still locally given by the curve $\{v = 0, z = 0\}$ which clearly contains the origin. As already mentioned, S_2 is tangent to the z-axis.

Remark 1. Let us point out that X_2 locally coincides with the transform of X by the one-point blow-up centered at the origin. In this sense, to include blow-ups centered at curves in the current discussion does not lead us to additional conditions to have nilpotent singular points.

Consider again a vector field X having the form (8). In the course of the preceding discussion, it was seen that the vector fields obtained through two successive blow-ups centered over the corresponding curves of singular points are respectively given by

$$X_1 = zr(x, v, z)\frac{\partial}{\partial x} + (x + zs(x, v, z))\frac{\partial}{\partial v} + z^n\frac{\partial}{\partial z}\,, \tag{11}$$

and by

$$X_2 = (v + zf_{(1)}(u, v, z))\frac{\partial}{\partial u} + zg_{(1)}(u, v, z)\frac{\partial}{\partial v} + z^n\frac{\partial}{\partial z}\,, \tag{12}$$

where r, s, $f_{(1)}$, and $g_{(1)}$ are all holomorphic functions vanishing at the origin of the corresponding coordinates. As usual the coordinates (x, v, z) are determined by $(x, y, z) = (x, vz, z)$ while $(x, y, z) = (uz, vz, z)$. Furthermore the functions $f_{(1)}$ and $g_{(1)}$ satisfy

$$f_{(1)}(u, v, z) = \frac{f(uz, vz, z) - uz^{n-1}}{z} \quad \text{and} \quad g_{(1)}(u, v, z) = \frac{g(uz, vz, z) - vz^{n-1}}{z}\,.$$

The following relations arise immediately:

(1) $\operatorname{ord} r(0, 0, z) = \operatorname{ord} f(0, 0, z)$ and $\operatorname{ord} s(0, 0, z) = \operatorname{ord} g(0, 0, z)$;
(2) $\operatorname{ord} f_{(1)}(0, 0, z) = \operatorname{ord} f(0, 0, z) - 1$ and $\operatorname{ord} g_{(1)}(0, 0, z) = \operatorname{ord} g(0, 0, z) - 1$;
(3) $\partial g_{(1)}/\partial u(0, 0, 0) = \partial g/\partial x(0, 0, 0) = \lambda \neq 0$.

Assume now that the above procedure is continued, i.e. assume X_2 is blown-up at the origin (identified with the singular point selected by the transform of the initial formal separatrix). As pointed out in Remark 1, here it is convenient to keep in mind that two consecutive blow-ups centered at curves contained in the singular set of the corresponding foliation can be replaced by a single one-point blow-up. To continue the procedure requires us to introduce new affine coordinates for each of these blow-ups and, in doing so, notation is likely to become cumbersome. To avoid this, and since the computations are similar to the previous ones, let us abuse notation and write (x, y, z) for the coordinates (u, v, z): naturally these "new" coordinates (x, y, z) have little to do with the initial ones. Similarly, coordinates for the first blow-up are then given by (x, v, z) while the second blow-up possesses coordinates (u, v, z). Assuming these identifications are made at every step - i.e. at every pair of blow-ups as indicated above - let X_{2i} denote the vector field obtained after i-steps where i satisfies $i < k_0$ (recall that k_0 stands for the contact order of the formal separatrix with the "initial z-axis"). In the (final) coordinates (u, v, z), the vector field X_{2i} takes on the form (8)

$$X_{2i} = (v + zf_{(i)}(u, v, z))\frac{\partial}{\partial u} + zg_{(i)}(u, v, z)\frac{\partial}{\partial v} + z^n\frac{\partial}{\partial z},$$

with $\partial g_{(i)}/\partial u(0, 0, 0) = \partial g/\partial x(0, 0, 0) = \lambda \neq 0$. In more accurate terms, recall that the orders of $f_{(i)}(0, 0, z)$ and of $g_{(i)}(0, 0, z)$ are directly related to the contact order of the transform of the formal separatrix S with the corresponding z-axis. At every step (consisting of a pair of blow-ups), the orders of $f_{(i)}(0, 0, z)$ and of $g_{(i)}(0, 0, z)$ decrease by one unity so that we have

$$\text{ord}\,(f_{(i)}(0, 0, z)) = \text{ord}\,(f(0, 0, z)) - i \quad \text{and} \quad \text{ord}\,(g_{(i)}(0, 0, z)) = \text{ord}\,(g(0, 0, z)) - i\,.$$

Thus, for $i \geq k_0 = \min\{\text{ord}\,(f(0, 0, z)), \text{ord}\,(g(0, 0, z))\}$, the vector field X_{2i} no longer takes on the form (8). At first sight this might suggest that the initial nilpotent singularity may fall short of being persistent, yet it is exactly the opposite that is true: the singularity is necessarily persistent.

To explain the last claim above, we begin by observing that the z-axis is not intrinsically determined by Formula (8). In fact, the z-axis is only subject to have some *high* contact order with the formal smooth separatrix S and it is S - rather than the z-axis - that has an intrinsic nature in our discussion. In particular, if S were analytic, we could make S coincide with the z-axis which, in turn, would imply that all the functions $f_{(i)}(0, 0, z)$ and $g_{(i)}(0, 0, z)$ vanish identically. It would then follow at once that the singularity is persistent.

Remark 2. It should be emphasized that our definition of persistent singularities requires the centers of all the blow-ups to be contained in the singular set of the corresponding foliations. This accounts for the difference between choosing centers that are contained in the singular set and the slightly weaker condition of allowing *invariant centers*. An analytic separatrix of a foliation is a legitimate invariant center so that, if we are allowed to perform blow-ups with invariant centers, the preceding singularity would be turned into an elementary one by blowing-up the foliation along the separatrix in question. This explains why in the example of Sancho and Sanz they want the corresponding separatrix to be strictly formal. This further illustrate the analogous comments made in the Introduction.

We now go back to the vector field X_{2i} which no longer has the form (8). To show that this vector field corresponds to a persistent nilpotent singularity when the separatrix S is strictly formal we will construct a change of coordinates where the vector field X_2 still takes on the form (8) but where the orders of the "new" functions $z \mapsto f_{(1)}(0, 0, z)$ and $z \mapsto g_{(1)}(0, 0, z)$ increase strictly so as to restore the values of the initial orders. The desired change of coordinates can be made polynomial by truncating a certain *formal* change of coordinates as in the proof of Proposition 3. This is the content of Lemma 7 below.

Consider the vector field X_2 given in (u, v, z) coordinates by Formula (12) along with the initial vector field X given in (x, y, z) coordinates by Formula (8).

Lemma 7. *There exists a polynomial change of coordinates H having the form $(u, v, z) = H(\tilde{x}, \tilde{y}, z) = (h_1(\tilde{x}, z), h_2(\tilde{y}, z), z)$ where the vector field X_2 becomes*

$$X_2 = (\tilde{y} + z\bar{f}(\tilde{x}, \tilde{y}, z))\frac{\partial}{\partial \tilde{x}} + z\bar{g}(\tilde{x}, \tilde{y}, z)\frac{\partial}{\partial \tilde{y}} + z^n\frac{\partial}{\partial z}$$

with

(a) $\text{ord}\,(\bar{f}(0, 0, z)) \geq \text{ord}\,(f(0, 0, z))$ *and* $\text{ord}\,(\bar{g}(0, 0, z)) \geq \text{ord}\,(g(0, 0, z))$*;*
(b) $\partial\bar{g}/\partial\tilde{x}(0, 0, 0) = \partial g/\partial x(0, 0, 0)$.

Proof. Denote by S_2 the transform of S through the one-point blow-up centered at the origin which is therefore a formal separatrix for the foliation associated with X_2. Since S_2 is smooth and tangent to the z-axis, it can be (formally) parametrized by the variable z. In other words, S_2 is given by $\varphi(z) = (f(z), g(z), z)$ for suitable formal series f and g with zero linear parts.

Consider now the formal map given in local coordinates $(\tilde{x}, \tilde{y}, z)$ by $H(\tilde{x}, \tilde{y}, z) = (\tilde{x} - f(z), \tilde{y} - g(z), z)$. The linear part of H at the origin is represented by the identity matrix so that H is a formal change of coordinates which, in addition, preserves the plane $\{z = 0\}$. Furthermore, H takes the formal separatrix S_2 to the z-axis. As previously mentioned, the formal vector field obtained by conjugating X_2 through H is strictly formal if S_2 is strictly formal. So, let H_m stand for the polynomial change of coordinates obtained from H by truncating it at order m and let $Y_m = (DH_m)^{-1} (X \circ H_m)$. Clearly the map H_m is holomorphic and so is the vector field Y_m. Moreover the foliation $\mathcal{F}_m$ associated to Y_m possesses a formal separatrix T_m, whose tangency order with the z-axis goes to infinity with m.

It is straightforward to check that the vector field Y_m has the form

$$Y_m = (\tilde{y} + zf_m(\tilde{x}, \tilde{y}, z))\frac{\partial}{\partial \tilde{x}} + zg_m(\tilde{x}, \tilde{y}, z)\frac{\partial}{\partial \tilde{y}} + z^n\frac{\partial}{\partial z}$$

with $\partial g_m/\partial \tilde{x}(0,0,0) = \partial g/\partial x(0,0,0)$, for every $m \in \mathbb{Z}$. Furthermore, for m sufficiently large we have $\operatorname{ord}(f_m(0,0,z)) \geq \operatorname{ord}(f(0,0,z))$ and $\operatorname{ord}(g_m(0,0,z)) \geq \operatorname{ord}(f(0,0,z))$ as well. The lemma is then proved. □

The results of these section can now be summarized as follows:

Theorem 1. *Let $\mathcal{F}$ be a singular holomorphic foliation on $(\mathbb{C}^3, 0)$ and assume that the origin is a persistent nilpotent singularity of $\mathcal{F}$. Let S denote the corresponding formal separatrix of $\mathcal{F}$. Then, up to finitely many one-point blow-ups, the foliation $\mathcal{F}$ is represented by a vector field X having the form*

$$(y + zf(x,y,z))\frac{\partial}{\partial x} + zg(x,y,z)\frac{\partial}{\partial y} + z^n\frac{\partial}{\partial z} \tag{13}$$

for some $n \in \mathbb{N}$, $n \geq 2$, and some holomorphic functions f and g of order at least 1 such that

(a) *The separatrix S is tangent to the z-axis. In fact, the contact order of S and the z-axis can be made arbitrarily large. Equivalently the orders of $f(0,0,z)$ and of $g(0,0,z)$ are arbitrarily large.*

(b) $\partial g/\partial x(0,0,0) \neq 0$.

Conversely, every nilpotent foliation $\mathcal{F}$ represented by a vector field X as above and possessing a (smooth) formal separatrix S tangent to the z-axis gives rise to a persistent nilpotent singularity. □

We close the section with the proof of Theorem D.

Proof of Theorem D. Assume that $\mathcal{F}$ is singular foliation on $(\mathbb{C}^3, 0)$ whose singularity cannot be resolved by blow-ups centered at the singular set of $\mathcal{F}$. Proceeding as in [4], cf. the discussion in the introduction, let S denote a formal separatrix of $\mathcal{F}$ giving rise to a sequence of infinitely near singular points. Next apply a sequence of one-point blow-ups to S and to its transform. Since the multiplicity of the corresponding foliations along the transforms of S form a monotone decreasing sequence, this sequence becomes constant after a finite number of steps. Denoting by $\mathcal{F}_k$ (resp. S_k) the corresponding foliation (resp. transform of S), there follows that $\mathcal{F}_k$ has a nilpotent singular point at the point in the exceptional divisor selected by S_k. Furthermore, owing to Remark 1, the multiplicity in question does decrease even if blow-ups centered at (smooth) singular curves are allowed. Thus the mentioned nilpotent singular point of $\mathcal{F}_k$ is a persistent one, i.e. it satisfies the conditions in Definition 3. The results of the present section can now be applied to this singular point and the statement follows from Theorem 1. □

4. Proof of Theorem A

This section is devoted to the proof of Theorem A and of its corollaries while the last section of the paper will contain some examples complementing our main results as well as a sharper version of the normal form given in Theorem 1 which is valid for foliations tangent to semicomplete vector fields.

Let X be a holomorphic vector field defined on an open set U of some complex manifold, we begin by recalling the definition of semicomplete vector field [13]. The reference [13] also contains the basic properties of semicomplete vector fields needed in the sequel.

Definition 4. Let X and U be as above. The vector field X is said to be semicomplete on U if for every point $p \in U$ there exists a connected domain $V_p \subseteq \mathbb{C}$ with $0 \in V_p$ and a holomorphic map $\phi_p : V_p \to U$ satisfying the following conditions:

- $\phi_p(0) = 0$ and $\frac{d\phi}{dt}|_{t=t_0} = X(\phi_p(t_0))$
- for every sequence $\{t_i\} \subseteq V_p$ such that $\lim_{i \to +\infty} t_i \in \partial V_p$, the sequence $\{\phi_p(t_i)\}$ leaves every compact subset of U

Clearly a vector field that is semicomplete on U is semicomplete on every open set $V \subseteq U$ as well. In particular the notion of *germ of semicomplete vector field* makes sense. Furthermore if X is a complete vector field on a complex manifold M, then the germ of X at every singular point is necessarily a germ of semicomplete vector field.

There is a useful criterion (Proposition 4) to detect vector fields that fail to be semicomplete which is as follows. Let X be a holomorphic vector field defined on an open set U and denote by $\mathcal{F}$ its associated (singular) holomorphic foliation. Consider a leaf L of $\mathcal{F}$ which *is not* contained in the zero set of X. Then leaf L is then a Riemann surface naturally equipped with a meromorphic abelian 1-form dT dual to X in the sense that $dT.X = 1$ on L. The 1-form dT is often referred to as the *time-form* induced by X on L. The following proposition is taken from [13].

Proposition 4. *Let X be a holomorphic semicomplete vector field on an open set U. Let L be a leaf of the foliation associated with X on which the time-form dT is defined (i.e. L is not contained in the zero set of X). Then we have*

$$\int_c dT \neq 0$$

for every path $c : [0, 1] \to L$ (one-to-one) embedded in L. □

In the sequel we say that X (resp. $\mathcal{F}$) is a vector field (resp. foliation) defined on $(\mathbb{C}^3, 0)$ meaning that X (resp. $\mathcal{F}$) is defined on a neighborhood of the origin in $\mathbb{C}^3$. The main result of this section is the following theorem.

Theorem 2. *Let X be a holomorphic vector field on $(\mathbb{C}^3, 0)$ and denote by $\mathcal{F}$ its associated foliation. Assume that the origin is a persistent nilpotent singularity for $\mathcal{F}$ and let S denote the corresponding formal separatrix of $\mathcal{F}$. Assume at least one the following holds:*

- *The multiplicity* $\mathrm{mult}\,(\mathcal{F}, S)$ *of $\mathcal{F}$ along S is at least* 3*;*
- *The linear part $J^1_0 X$ of X at the origin equals zero.*

then X is not semicomplete on a neighborhood of the origin.

We start our approach to Theorem 2 with a couple of remarks involving blow-ups. First note that the transform of a vector field by a single or by a finite sequence of blow-ups may have non-trivial common factors among its components (i.e. the transformed vector field may have a non-empty divisor of zeros even if the zero set of the initial vector field has codimension 2 or greater). Here - as far as foliations are concerned - these common factors can always be *eliminated* though they have to be taken into account whenever the *vector field* is the object of study. In other words, when a sequence of blow-ups is applied to resolve the singular points of the foliation $\mathcal{F}$ associated with a vector field X, a local expression for the corresponding transform of X will in general involve a holomorphic vector field Y, as a local representative of the foliation in question, and a local multiplicative function h take may take on the value zero at certain points. This remark applies also to the statement of Theorem 2: under the conditions in question X has the general form $X = hY$ where Y is a vector field representing $\mathcal{F}$ - and thus having singular set of codimension at least 2 and non-zero nilpotent linear part at the origin - and h is a holomorphic function possibly satisfying $h(0) = 0$. In particular if $h(0) = 0$ then $J_0^1 X$ is necessarily zero.

Finally we also point out that the semicomplete character of a holomorphic vector field is preserved under birational transformations. In particular it is preserved under blow-ups. It is however not necessarily preserved under *weighted* blow-ups if these *are regarded as finite-to-one maps* rather than from the birational point of view associated with the orbifold action. Incidentally, blow-ups with weight 2 will be needed in what follows.

Let us now fix a holomorphic vector field X on $(\mathbb{C}^3, 0)$ and assume that the origin is a persistent nilpotent singularity for the associated foliation $\mathcal{F}$. Let S denote a formal separatrix of $\mathcal{F}$ giving rise to a sequence of infinitely near (nilpotent) singular points. Since blow-ups preserve the semicomplete character of vector fields, up to transforming X through finitely many one-point blow-ups, we can assume the existence of local coordinates (x, y, z) where X is given by (cf. Sections 3 and 4)

$$X = z^k h(x, y, z) \left[(y + zf(x, y, z)) \frac{\partial}{\partial x} + zg(x, y, z) \frac{\partial}{\partial y} + z^n \frac{\partial}{\partial z} \right] \tag{14}$$

for suitable nonnegative integers k, n and holomorphic functions f, g, and h satisfying all of the following:

- $n \geq 2$ and $h(0, 0, 0) \neq 0$;
- $f(0, 0, 0) = g(0, 0, 0) = 0$. Furthermore the orders at $0 \in \mathbb{C}$ of $f(0, 0, z)$ and $g(0, 0, z)$ are arbitrarily large, say larger than $2n$ (in particular S is tangent to the z-axis);
- $\partial g(0, 0, 0)/\partial x = \lambda \neq 0$.

It is convenient to add a comment about the condition $h(0, 0, 0) \neq 0$ since the zero-divisor of the initial vector field X may be non-empty (the condition of having a persistent nilpotent singularity concerns solely the foliation associated with X). An easy consequence of Theorem 1 which was previously pointed out in [5], [4], asserts that the formal separatrix S cannot be contained in a surface invariant by the foliation. Thus by using blow-ups as before, S can be separated from the irreducible components of the zero-divisor of X: since the zero divisor of the transformed vector field consists of the transform of the zero-divisor of X and of irreducible components of the exceptional divisor, the claim follows. Note also that a sort of converse to the above claim holds in the sense that if the zero-divisor of the initial vector field is not empty, then we will necessarily have $k \geq 1$ in Formula (14).

Theorem 2 is thus reduced to proving that X as given in Formula (14) is not semicomplete on a neighborhood of the origin provided that at least one of the following conditions holds:

the linear part of X at the origin $J_0^1 X$ equals zero (equivalently $k \geq 1$) or $n = \text{mult}\,(\mathcal{F}, S) > 2$, where $\mathcal{F}$ stands for the foliation associated with X.

Let us begin by showing that $\mathcal{F}$ can be resolved by using a single blow-up of weight 2. Here these weight 2 blow-ups will be viewed as a two-to-one map. Note also that the lemma below includes some useful explicit formulas for the transformed vector field as well.

Lemma 8. *Let X be as in Formula (14) and denote by Π the blow-up of weight 2 centered at the curve $\{y = z = 0\}$ (the curve of singular points of $\mathcal{F}$). Let $\Pi^*\mathcal{F}$ be the transform of $\mathcal{F}$. Then the singular point of $\Pi^*\mathcal{F}$ selected by S in the exceptional divisor is elementary and the corresponding eigenvalues of $\Pi^*\mathcal{F}$ are 0, 1, and -1. Furthermore the transform Π^*X of X is a holomorphic vector field vanishing with order $2k+1$ on the exceptional divisor.*

Proof. Let (x, y, z) be the local coordinates where X is given by Formula (14) and consider the indicated weight 2 blow-up map Π. In natural coordinates (u, v, w) the map Π is given by

$$\Pi(u, v, w) = (u, vw, w^2)\,,$$

where $\{w = 0\}$ is contained in the exceptional divisor. Now a straightforward computation shows that Π^*X is given in the (u, v, w) coordinates by

$$\begin{aligned}\Pi^*X &= w^{2k} h \left[\left(vw + w^2 f(u, vw, w^2)\right) \frac{\partial}{\partial u} + \left(wg(u, vw, w^2) - \frac{1}{2} vw^2 \right) \frac{\partial}{\partial v} + \frac{1}{2} w^{2n-1} \frac{\partial}{\partial w} \right] \\ &= w^{2k+1} h \left[(v + wf(u, vw, w^2)) \frac{\partial}{\partial u} + \left(g(u, vw, w^2) - \frac{1}{2} vw \right) \frac{\partial}{\partial v} + \frac{1}{2} w^{2n-2} \frac{\partial}{\partial w} \right] ,\end{aligned}$$

where the function h is evaluated at the point (u, vw, w^2). Since $h(0, 0, 0) \neq 0$, there follows that the zero-divisor of Π^*X locally coincides with the exceptional divisor (given by $\{w = 0\}$). Moreover the order of vanishing of Π^*X at the exceptional divisor is $2k+1$. In turn, the foliation $\Pi^*\mathcal{F}$ is represented by the vector field

$$Y = (v + wf(u, vw, w^2)) \frac{\partial}{\partial u} + \left(g(u, vw, w^2) - \frac{1}{2} vw \right) \frac{\partial}{\partial v} + \frac{1}{2} w^{2n-2} \frac{\partial}{\partial w}$$

whose linear part at the origin is given by $v\partial/\partial u + \lambda u \partial/\partial u$ since $f(0, 0, 0) = 0$ and $n \geq 2$ (here $\lambda = \partial g(0, 0, 0)/\partial x \neq 0$). Thus the eigenvalues of $\mathcal{F}$ at the origin are 0 and the two square roots of λ which is clearly equivalent to having eigenvalues 0, 1, and -1. The lemma is proved. □

Recall that our purpose is to show that X is not semicomplete on a neighborhood of $(0, 0, 0)$ provided that $k \neq 0$ or $n > 2$. Note that, in general, this conclusion does not immediately follow by proving that the vector field Π^*X *is not* semicomplete on a neighborhood of the origin of the coordinates (u, v, w) since the map Π is not one-to-one. It is in fact easy to construct examples of semicomplete vector fields whose transforms under ramified coverings are no longer semicomplete. Yet, in our context, the situation can be described in a more accurate form. Consider a regular leaf L of the foliation $\Pi^*\mathcal{F}$ which is equipped with the time-form dT_{Π^*X} induced by Π^*X. Assume that $c : [0, 1] \to L$ is open path over which the integral of dT_{Π^*X} equals zero so that, in particular, Π^*X is not semicomplete (Proposition 4). If X happens to be semicomplete, then we must necessarily have $\Pi(c(0)) = \Pi(c(1))$. Hence the idea to prove Theorem 2 will be to find open paths c on L, $c : [0, 1] \to L$ satisfying $\Pi(c(0)) \neq \Pi(c(1))$ and over which the integral of the time-form dT_{Π^*X} is equal to zero. If c is one of these paths, its projection by Π is still an open path contained in a leaf of $\mathcal{F}$ and over which the integral of the time-form *induced by* X is zero so that X cannot be semicomplete.

At this point it is convenient to recall the notion of function asymptotic to a formal series. Let then $t \in \mathbb{C}$ be a variable and considers a formal series $\psi(t)$. Consider also a circular sector

V of angle θ, vertex at $0 \in \mathbb{C}$, and small radius. A holomorphic function ψ_V defined on $V \setminus \{0\}$ is said to be *asymptotic (on V)* to the formal series $\psi(t)$ if for every $i \in \mathbb{N}$ and for every sector $W \subset V$, of angle strictly smaller than θ and sufficiently small radius, there exists a constant $\mathrm{Const}_{i,W}$ such that

$$\|\psi_V(t) - \psi_i(t)\| \leq \mathrm{Const}_{i,W} \, \|t\|^{i+1} \, ,$$

where ψ_i stands for the i^{th}-jet of ψ at $0 \in \mathbb{C}$. The adaptation of the above definition to vector-valued formal series $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ and functions $\psi : V \to \mathbb{C}^n$ is straightforward and thus left to the reader.

The following lemma appears in [6] (Lemma 3.12).

Lemma 9. *Let $V \subset \mathbb{C}$ denote a circular sector with vertex at $0 \in \mathbb{C}$ and angle $2\pi/l$, where l is a strictly positive integer. Assume that ρ is a holomorphic function on $V \setminus \{0\}$ such that*

$$\|\rho(x) - x^{l+2}\| \leq \mathrm{Const} \, \|x^{l+3}\|$$

for a suitable constant Const*. Then for every $r > 0$, there exists an open path c embedded in the intersection of V with the disc of radius r and center at $0 \in \mathbb{C}$ such that the integral of the 1-form $dx/\rho(x)$ equals zero.*

Proof. It suffices to sketch the argument and refer to [13] for the detail concerning the effect of higher order terms. Consider first the special case where $\rho(x) = x^{l+2}$. In this case the 1-form $dx/\rho(x)$ admits the function $x \mapsto -1/(l+1)x^{l+1}$ as primitive. Thus it is enough to choose a path c of the form $c(t) = x_0 e^{2i\pi t/(l+1)}$ where x_0 has small absolute value and is such that the resulting path c is still contained in V.

In the general case, the leading term of $\rho(x)$ is x^{l+2}. In fact, for $\|x\|$ small, the difference $\|\rho(x) - x^{l+2}\|$ is bounded by Const $\|x^{l+3}\|$ which is of order larger than x^{l+2} itself. The statement then follows by using the "perturbation" technique in [13]. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. With the notation of Lemma 8, consider the vector field $\Pi^* X$ and note that $\Pi^* X = w^{2k+1} h(u, uw, w^2) Y$ where Y is given by

$$Y = (v + wf(u, vw, w^2)) \frac{\partial}{\partial u} + \left(g(u, vw, w^2) - \frac{1}{2} vw \right) \frac{\partial}{\partial v} + \frac{1}{2} w^{2n-2} \frac{\partial}{\partial w} \, ,$$

for suitable k, n, f, g, and h as above. The vector field Y is in particular a representative of the foliation $\Pi^* \mathcal{F}$. Fixed a neighborhood U of the origin, we look for leaves L of $\Pi^* \mathcal{F}$ along with open paths $c : [0, 1] \to L$ contained in U such that the two conditions below are satisfied:

- $\int_c dT_{\Pi^* X} = 0$;
- $\Pi(c(0)) \neq \Pi(c(1))$.

The existence of the desired paths c will be obtained with the help of a theorem due to Malmquist in [9] (*Théorème 1*, page 95) provided that $n \geq 3$ or $k \geq 1$.

To begin we can assume that $\lambda = \partial g(0, 0, 0)/\partial x = 1$, up to a multiplicative constant, so that the linear part of Y at the origin has eigenvalues 0, 1, and -1. Consider then the linear change of coordinates $(\bar{u}, \bar{v}, \bar{w}) \mapsto (\bar{u} + \bar{v}, \bar{u} - \bar{v}, \bar{w})$. The pull-back $\overline{Y}$ of Y in the coordinates $(\bar{u}, \bar{v}, \bar{w})$ becomes

$$\overline{Y} = \left[(\bar{u} + \bar{w} A(\bar{u}, \bar{v}, \bar{w})) \frac{\partial}{\partial \bar{u}} + (-\bar{v} + \bar{w} B(\bar{u}, \bar{v}, \bar{w} + C(\bar{u} + \bar{v}))) \frac{\partial}{\partial \bar{v}} + \frac{1}{2} \bar{w}^{2n-2} \frac{\partial}{\partial \bar{w}} \right]$$

for suitable holomorphic functions A and B of order at least 1 and a holomorphic function C of order at least 2. Similarly the vector field $\overline{\Pi^* X}$ corresponding to the pull-back of $\Pi^* X$ in the coordinates $(\bar{u}, \bar{v}, \bar{w})$ satisfies $\overline{\Pi^* X} = \bar{w}^{2k+1} h(\bar{u} + \bar{v}, \bar{u} - \bar{v}, \bar{w}) \overline{Y}$.

Note that the singularity of the foliation associated with $\overline{Y}$ at the origin is a codimension 1 saddle-node, i.e. it has exactly one eigenvalue equal to zero. In fact, it is a *resonant* codimension 1 saddle-node in the sense that the non-zero eigenvalues, 1 and -1, are resonant. This type of singularity is closely related to a classical result due to Malmquist involving systems of differential equations with an irregular singular point, cf. [9]. We will state a slightly simplified version of Malmquist results which is adapted to our problem. For $\delta \in \{0,1\}$, assume we are given a system of differential equations of the form

$$\begin{cases} \bar{w}^{l+1}\frac{d\bar{u}}{d\bar{w}} = s_1\bar{u} + \beta_1(\bar{u},\bar{v},\bar{w}) \\ \bar{w}^{l+1}\frac{d\bar{v}}{d\bar{w}} = s_2\bar{v} + \delta\bar{u} + \beta_2(\bar{u},\bar{v},\bar{w}) \end{cases} \tag{15}$$

where $s_1.s_2 \neq 0$ and where β_1, β_2 are convergent series (in particular conditions (A) and (B) of [9] are necessarily verified). Now let $\overline{\Psi}(\bar{w}) = (\overline{\psi}_1(\bar{w}), \overline{\psi}_2(\bar{w}))$ be a formal solution for the system in question. Malmquist then shows that for every $\varepsilon > 0$, there exist circular sectors of angle $2\pi/k - \varepsilon$ in the space of the $\bar{w}$-variable with respect to which the system (15) admits a unique solution which is asymptotic to the formal solution $\overline{\Psi}(\bar{w})$.

The system (15) is naturally related to saddle-nodes singularities as those given by the vector field $\overline{Y}$. In fact, the vector field $\overline{Y}$ is essentially equivalent to the system of differential equations

$$\begin{cases} \bar{w}^{2n-2}\frac{d\bar{u}}{d\bar{w}} = \bar{u} + \bar{w}A(\bar{u},\bar{v},\bar{w}) \\ \bar{w}^{2n-2}\frac{d\bar{v}}{d\bar{w}} = -\bar{v} + \bar{w}B(\bar{u},\bar{v},\bar{w}) + C(\bar{u}+\bar{v}) \, . \end{cases}$$

Thus we have $s_1 = 1$, $s_2 = -1$ and $l = 2n-3$ and the formal solution $\overline{\Psi}(\bar{w}) = (\overline{\psi}_1(\bar{w}), \overline{\psi}_2(\bar{w}))$ is obtained out of the (initial) formal separatrix S (whose formal parameterization is simply $\bar{w} \mapsto (\bar{w}, \overline{\psi}_1(\bar{w}), \overline{\psi}_2(\bar{w}))$). Since these statements are clearly invariant by change of coordinates, we can return to the variables (u,v,w) where the vector field Π^*X is defined. Keeping in mind that $w = \bar{w}$, the angle of the sector V remains unchanged and the formal parameterization of S will simply be denoted by $\Psi(w) = (w, \psi_1(w), \psi_2(w))$ (where $\psi_1 = \overline{\psi}_1 + \overline{\psi}_2$ and $\psi_2 = \overline{\psi}_1 - \overline{\psi}_2$).

Fix then an arbitrarily small neighborhood of the origin of the coordinates (u,v,w). Owing to Malmquist theorem, we can choose a solution of Y asymptotic to the formal series $\Psi(w) = (w, \psi_1(w), \psi_2(w))$ of S on the above mentioned sector V (recall that $w = \bar{w}$). In particular there are points $w_0 \in \mathbb{C}$ with $\|w_0\|$ arbitrarily small, and there are leaves of $\Pi^*\mathcal{F}$ to which paths of the form $c(t) = (0, 0, w_0 e^{2\pi i t/(2n-3)})$ can be lifted (with respect to the fibration given by projection on the w-axis). Furthermore these lifted paths are contained in arbitrarily small neighborhoods of the origin provided that $\|w_0\|$ is small enough. In other words, once one convenient circular sector V of angle $2\pi/(2n-3)$ is chosen, we can "parameterize" an open set of a certain leaf L of $\Pi^*\mathcal{F}$ by a map of the form $w \mapsto (w, \psi_{1,V}(w), \psi_{2,V}(w))$, $w \in V$, where the holomorphic functions $\psi_{i,V}$ are asymptotic on V to the formal series $\psi_i(w)$, $i = 1,2$.

The restriction to L of $\Pi^*X = w^{2k+1}h(u,v,w)Y$ can be considered in the w-coordinate so as to become identified with a certain one-dimensional vector field $Z(w) = \rho(w)\partial/\partial w$ defined on V. Since $h(0,0,0) \neq 0$ and the formal series $\psi_i(w)$ have zero linear terms (S is tangent to the w-axis), there follows that ρ has an asymptotic expansion of the form

$$w^{2n+2k-1} + \text{h.o.t.}$$

up to a multiplicative constant, where h.o.t. stands for terms of order higher than $2n+2k-1$. Since $k \geq 0$ and since V is a sector of angle $2n-3$, Lemma 9 implies the existence of an open embedded path $c \subset V$ over which the integral of the time-form associated with $Z(w)$ equals zero. Hence the vector field Π^*X is *never semicomplete* (even if $n = 2$ and $k = 0$).

What precedes shows that Π^*X is not semicomplete but we still need to show that the initial vector field X is not semicomplete. It is in this part of the argument that the assumption $n \geq 3$

unless $k \geq 1$ will play a role. To conclude that X is not semicomplete we need to consider the possibility of having $c(0)^2 = c(1)^2$ in the above mentioned path $c \subset V$. If this happens, it means that the difference of argument between $c(0)$ and $c(1)$ is π. However, in the preceding discussion (cf. also Lemma 9), it was seen that the constructed path c is such that the difference of argument between $c(0)$ and $c(1)$ can be made arbitrarily close to $2\pi/(2n+2k-2) = \pi/(n+k-1)$. Recalling that $n \geq 2$, there immediately follows that the desired path c as above satisfying in addition $c(0)^2 \neq c(1)^2$ can be found provided that $n \geq 3$ or $k \geq 1$. Theorem 2 is proved. □

Let us close this section with the proof of Theorem A and its corollaries.

Proof of Theorem A. Let X be a semicomplete vector field on $(\mathbb{C}^3, 0)$ and denote by $\mathcal{F}$ its associated foliation. Assume that item (2) in the statement of Theorem A does not hold, i.e. that $\mathcal{F}$ cannot be turned into a foliation all of whose singular points are elementary by means of blow-ups centered at singular sets. Thus owing to Theorem D there is a sequence of one-point blow-ups starting at the origin which leads to a transform $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ exhibiting a persistent nilpotent singular point p along with a formal separatrix S of $\widetilde{\mathcal{F}}$ at p. The corresponding transform of X will be denoted by $\widetilde{X}$.

Next assume aiming at a contradiction that the linear part of X at the origin is equal to zero. The the transform $\Pi_1^*(X)$ of X under the first blow-up map vanishes identically over the exceptional divisor. Since the subsequent blow-ups will always be performed at *singular points of the foliation* which are contained in the zero-divisor of the vector fields in question, there follows that the zero-divisor of $\widetilde{X}$ is not empty on a neighborhood of p. Therefore the linear part of X at p is equal to zero so that Theorem 2 implies that X is not semicomplete. The resulting contradiction proves Theorem A. □

Remark 2. The preceding proof makes it clear why Theorem D is better suited to the proof of Theorem A than the resolution theorem proved by McQuillan and Panazzolo in [10]. Since their statement uses weighted blow-ups, it is not clear that the transform of X will still be holomorphic with a non-empty zero-divisor on a neighborhood of a persistent nilpotent singularity p. In our case this claim is clear since only (standard) blow-ups are performed and they are all centered in the singular set of the associated foliation (actually we have only used one-point blow-ups). A similar result relying on the arguments in [10] is possibly true but it would require us to discuss in detail the resolution algorithm provided in [11] upon which the work in [10] is built so as to check whether the transformed vector field still satisfies the above conditions.

Corollary B is an immediate consequence of Theorem A while Corollary C requires additional explanation.

Proof of Corollary C. This statement is actually more of a by-product of the proof of Theorem 2 than a corollary of Theorem A itself. Consider a compact manifold M and a holomorphic vector field X defined on M. Let $\mathcal{F}$ denote the singular foliation associated with X and assume for a contradiction that $\mathcal{F}$ possesses a singular point p which cannot be resolved by a sequence of blow-ups as in Theorem A. In particular the linear part of X at p is nilpotent non-zero and X is a representative of $\mathcal{F}$ on a neighborhood of p. We also denote by S a formal separatrix of $\mathcal{F}$ at p giving rise to a sequence of infinitely near (nilpotent) singular points.

Up to performing a finite sequence of one-point blow-ups, which changes neither the compactness of M nor the holomorphic nature of X, we can assume that X admits the normal form (14), as previously used in this section. Consider then the curve of singular points of $\mathcal{F}$ locally given by $\{y = z = 0\}$. Since M is compact this curve of singular points $\mathcal{C}$ is global and compact on M. Furthermore, up to resolving its singular points (as curve), we can assume $\mathcal{C}$ to be smooth. Thus $\mathcal{C}$ can globally be blown-up with weight 2 as in Lemma 8. Again this weighted blow-up is viewed as a two-to-one map as opposed to a birational one. Yet, the resulting manifold $\widetilde{M}$ is still compact. Similarly the computations in Lemma 8 show that the transform $\Pi^* X$ of X is still holomorphic. Hence $\Pi^* X$ is complete on $\widetilde{M}$. Thus the restriction of $\Pi^* X$ to an open set $U \subset \widetilde{M}$ is a semicomplete vector field. A contradiction now arises from noting that it was seen in the proof of Theorem 2 that the vector field $\Pi^* X$ is *never* semicomplete on a neighborhood of the codimension 2 saddle-node appearing in connection with the transform of S. This ends the proof of Corollary C. □

5. Examples and complements

The first part of this section is devoted to detailing a couple of examples respectively related to Theorem A and to Theorem D. The remainder of the section will be devoted to a refinement of Theorem 2 which, of course, can also be used to make Theorem A slightly more accurate.

5.1. **A couple of examples.** As the title indicates, this subsection consists of a couple of interesting examples. We will begin with the example of complete vector field mentioned in the introduction and then we will provide a very simple and explicit example of persistent nilpotent singularity that cannot be reduced to the examples of Sancho and Sanz. The argument used here to show that the later example cannot be reduced to those of Sancho and Sanz is elementary and different from [10].

• **Example**: The vector field

$$Z = x^2 \partial/\partial x + xz \partial/\partial y + (y - xz)\partial/\partial z\,.$$

Owing to the discussion in Sections 2 and 3, it is clear that the foliation $\mathcal{F}$ associated with Z has a persistent nilpotent singular point at the origin which is associated with the *convergent* separatrix $\{y = z = 0\}$. Here it is convenient to remind the reader that our notion of persistent nilpotent singular point only takes into consideration blow-ups centered in the singular set of foliations. Thus the fact that the separatrix giving rise to a sequence of infinitely near (nilpotent) singular points is convergent and can hence be blown-up to resolve the singularity in question is of relatively little importance for us.

The main point for us here is to substantiate the claim made in the introduction that Z becomes a complete vector field on a suitable open manifold M. To begin with, note that the coordinate $x(T)$ satisfies

$$x(T) = \frac{x_0}{1 - Tx_0}$$

so that $x(T)$ is defined for every $T \neq 1/x_0$. In turn we have $d^2y/dt^2 = zdx/dt + xdz/dt$ so that the vector field Z yields

$$\frac{d^2y}{dt^2} = xy = \frac{x_0}{1 - Tx_0} y$$

which has a regular singular point at $T = 1/x_0$ and is non-singular otherwise. It then follows from the classical theory of Frobenius (see for example [8]) that $y(T)$ is holomorphic and defined for all $T \in \mathbb{C}$. Now the vector field Z also gives us

$$\frac{dz}{dt} = y - xz = -\frac{x_0}{1 - Tx_0} z + y(T)\,.$$

Since $y(T)$ is holomorphic on all of $\mathbb{C}$, there follows that $z(T)$ is holomorphic on all of $\mathbb{C}$ as well.

Summarizing the preceding, the integral curve $\phi(T) = (x(T), y(T), z(T))$ of vector field Z satisfying $\phi(0) = (x_0, y_0, z_0)$ is defined for all $T \in \mathbb{C} \setminus \{1/x_0\}$. Furthermore as $T \to 1/x_0$, the coordinate $x(T)$ goes off to infinity while $y(T)$ and $z(T)$ are holomorphic at $T = 1/x_0$. In particular the vector field Z is semicomplete on $\mathbb{C}^3$.

To show that Z can be extended to a complete vector field on a suitable manifold M is slightly more involved. Denote by $\mathcal{F}$ the foliation associated with Z on $\mathbb{C}^3$. Note that the plane $\{x = 0\}$ is invariant by $\mathcal{F}$ and that $\mathcal{F}$ is transverse to the fibers of the projection $\pi_1(x, y, z) = x$ away from $\{x = 0\}$. The x-axis is also invariant by $\mathcal{F}$ and $\mathcal{F}$ can be seen as a linear system over the variable x, namely we have $dy/dx = z/x$ and $dz/dx = y/x^2 - z/x$, cf. Chapter III of [7].

Let L be a leaf of $\mathcal{F}$ which is not contained in $\{x = 0\}$. The restriction of π_1 to L is a local diffeomorphism from L to the x-axis. In view of the previous discussion, this local diffeomorphism can, in fact, be used to lift paths contained in $\{y = z = 0\} \setminus \{(0, 0, 0)\}$. Similarly, owing to the description of $\mathcal{F}$ as a linear system, the parallel transport along leaves of $\mathcal{F}$ induces linear maps between the fibers of π_1 (isomorphic to $\mathbb{C}^2$). Finally the holonomy (monodromy) arising from the invariant x-axis coincides with the identity (cf. Lemma 12). Thus we have proved the following:

Lemma 10. *Away from $\{x = 0\}$, the leaves of $\mathcal{F}$ are graphs over the punctured x-axis. In particular the space of these leaves is naturally identified to $\mathbb{C}^2$ with coordinates (y, z).* □

The restriction of Z to the invariant plane $\{x = 0\}$ being clearly complete, to obtain an extension of Z as a complete vector field on a suitable open manifold M we proceed as follows. Fix a leaf L of $\mathcal{F}$ with $L \subset \mathbb{C}^3 \setminus \{x = 0\}$ and denote by Z_L the restriction of Z to L. Consider the parameterization of L having the form $x \mapsto (x, A(x), B(x))$ where $x \in \mathbb{C}^*$ and where A and B are holomorphic functions. In the coordinate x, the one-dimensional vector field Z_L becomes $x^2\partial/\partial x$ and thus can be turned in a complete vector field by adding the "point at infinity" to L (i.e. $\{u = 0\}$ in the coordinate $u = 1/x$). Thus to obtain the manifold M we simply add the "point at infinity" to every leaf L of $\mathcal{F}$ ($L \not\subset \{x = 0\}$). The description of the leaves of $\mathcal{F}$ as a linear system and the holomorphic behavior of the functions $y(T), z(T)$ as $T \to 1/x_0$ makes it clear the resulting space can be equipped with the structure of a complex manifold M. Moreover Z is naturally complete on M as desired.

• **Example**: The (germ of) foliation $\mathcal{F}_\lambda$ given by

$$X_\lambda = (y - \lambda z)\frac{\partial}{\partial x} + zx\frac{\partial}{\partial y} + z^3\frac{\partial}{\partial z},$$

with $\lambda \in \mathbb{C}$.

As mentioned the first examples of persistent nilpotent singularities were supplied by Sancho and Sanz and many others are now known (cf. [10] and/or Theorem 1 in the present paper). Still it seems interesting to provide an additional explicit example along with a self-contained proof that it cannot be reduced to the examples of Sancho and Sanz. We begin by observing that, when $\lambda \neq 0$, neither the vector field X_λ nor the vector fields considered by Sancho and Sanz (see Section 3) are in the normal form indicated in Theorem 1. Nonetheless we have:

Lemma 11. *The foliation associated to X_λ possesses a formal separatrix through the origin. The formal separatrix is, in fact, strictly formal if $\lambda \neq 0$.*

Proof. The leaves of the foliation associated with X_λ can be viewed as the solutions of the following system of differential equations:

$$\begin{cases} \frac{dx}{dz} = \frac{y - \lambda z}{z^3} \\ \frac{dy}{dz} = \frac{x}{z^2}\,. \end{cases} \tag{16}$$

We look for a formal solution $\varphi(z) = (x(z), y(z))$ of the system (16) in the form $x(z) = \sum_{k\geq 0} a_k z^k$ and $y(z) = \sum_{k\geq 0} b_k z^k$. By substituting these expressions in the first equation of (16) and comparing both sides, we obtain

$$b_0 = 0\,, \qquad b_1 = \lambda\,, \qquad b_2 = 0\,, \qquad \text{and} \qquad b_{k+3} = (k+1)a_{k+1} \quad \text{for} \quad k \geq 0\,.$$

In turn, substitution and comparison in the second equation of (16) yields

$$a_0 = a_1 = 0 \qquad \text{and} \qquad a_{k+1} = kb_k \quad \text{for} \quad k \geq 1\,.$$

Therefore $b_0 = b_2 = 0$, $b_1 = \lambda$, and

$$b_{k+3} = k(k+1)b_k$$

for $k \geq 0$. It then follows that the coefficients of $y(z)$ having the form b_{3l} and b_{3l+2} are zero for all every $l \geq 0$. Furthermore, for $l \geq 1$ we have

$$b_{3l+1} = \lambda \Pi_{j=1}^{l} \frac{(3j-1)!}{(3j-3)!}\,.$$

In the particular case where $\lambda = 0$, the series in question vanishes identically. This means that the curve, given in coordinates (x, y, z) by $\{x = 0,\ y = 0\}$ is a convergent separatrix for the foliation $\mathcal{F}_0$. Thus we assume from now on that $\lambda \neq 0$.

We want to check that the series $y = y(z) = \sum_{k\geq 0} b_k z^k$ diverges so as to ensure that $z \mapsto (x(z), y(z), z)$ constitutes a *strictly formal* separatrix for $\mathcal{F}_\lambda$. To do this, just note that the series of $y(z)$ can be reformulated as $z \sum_{k\geq 0} c_k z^{3k}$, where $c_0 = 0$ and $c_k = \lambda \Pi_{j=1}^{k} \frac{(3j-1)!}{(3j-3)!}$. Up to considering the new variable $w = z^3$, the radius of convergence of this later series is given by

$$\lim_{k\to\infty} \frac{c_k}{c_{k+1}} = \lim_{k\to\infty} \frac{1}{(3k-1)(3k-3)} = 0$$

and the lemma follows. □

Summarizing the preceding, the z-axis is invariant by $\mathcal{F}_\lambda$ if $\lambda = 0$. When $\lambda \neq 0$, $\mathcal{F}_\lambda$ admits a strictly formal separatrix S_λ parameterized by a triplet of formal series

$$z \to \varphi(z) = (\sum_{k\geq 1} a_k z^k, \sum_{k\geq 1} b_k z^k, z)\,.$$

Clearly $\varphi'(z) \neq (0,0,0)$ so that S_λ is formally smooth. In the case $\lambda = 0$, the foliation $\mathcal{F}_0$ satisfies the conditions in Theorem 1 and thus has a persistent nilpotent singularity at the origin. If $\lambda \neq 0$, we note that S_λ *is not* tangent to the z-axis. However, arguing as in Lemma 7, there exists a polynomial change of coordinates H, of form $H(\tilde{x}, \tilde{y}, z) = (h_1(\tilde{x}, z), h_2(\tilde{y}, z), z)$, and such that the formal separatrix $S\lambda$ becomes tangent (with arbitrarily large tangency order) to the z-axis. This gives the foliation $\mathcal{F}_\lambda$ the normal form indicated in Theorem 1 and ensure that $\mathcal{F}_\lambda$ gives rise to a persistent nilpotent singularity with a strictly formal separatrix.

Regardless of whether or not $\lambda = 0$, the multiplicity $\text{mult}\,(\mathcal{F}_\lambda, S_\lambda)$ of $\mathcal{F}_\lambda$ along S_λ is equal to 3. This contrasts with the examples of Sancho and Sanz where the corresponding multiplicity is always 2. Since the multiplicity along a formal separatrix is clearly invariant by (formal) change of coordinates, there follows that the singularities $\mathcal{F}_\lambda$ are not conjugate to the singularities of Sancho and Sanz. Furthermore, as shown in Sections 2 and 3, the value of $\text{mult}\,(\mathcal{F}_\lambda, S_\lambda)$ is invariant by blow-ups centered in the singular set of the corresponding foliations. Therefore the singularities of $\mathcal{F}_\lambda$ cannot give rise to a singularity in the family of Sancho and Sanz by means of any finite sequence of blow-ups as above.

5.2. **Local holonomy and semicomplete persistent nilpotent singularities.** To close this paper we again turn our attention to semicomplete vector fields. Assume that X is a vector field whose associated foliation $\mathcal{F}$ possesses a persistent nilpotent singularity at $(0,0,0) \in \mathbb{C}^3$. Assume also that X is semicomplete. Owing to Theorem 2, the vector field X has the normal form in Theorem 1 with $n = 2$. In fact, denoting by S the formal separatrix of $\mathcal{F}$ giving rise to a sequence of infinitely near nilpotent singularities, we have $\mathrm{mult}\,(X, S) = \mathrm{mult}\,(\mathcal{F}, S) = 2$. These vector fields are thus very close to the examples of Sancho and Sanz.

This raises the problem of classifying semicomplete vector fields in the Sancho and Sanz family. In what follows we will conduct this classification only in the special case $\lambda = 0$, i.e. when the formal separatrix S is actually convergent. Our purpose in doing this is to point out the role played by the holonomy of this separatrix. Furthermore, by dealing only with the case of convergent separatrices, we avoid some technical difficulties that would require a longer discussion: this discussion however is not really indispensable from the point of view of this paper. Finally we also note that the material developed below includes Lemma 12 already used in the study of the vector field $Z = x^2\partial/\partial x + xz\partial/\partial y + (y - xz)\partial/\partial z$.

We begin by recalling the context of the proof of Theorem 2. After performing the weight 2 blow-up, we have found an open path c contained in a leaf of the blown-up foliation $\Pi^*\mathcal{F}$ over which the integral of the corresponding time-form is equal to zero. Whereas this implies that Π^*X is not semicomplete, in the case $n = 2$ and $k = 0$ we were not able to conclude that $\Pi(c(0)) \neq \Pi(c(1))$: the possibility of having X semicomplete cannot be ruled out. Let us then consider this problem for the Sancho and Sanz family with $\lambda = 0$, i.e. for the family of vector fields having the form

$$X = x^2 \frac{\partial}{\partial x} + (xz - \alpha xy)\frac{\partial}{\partial y} + (y - \beta xz)\frac{\partial}{\partial z}\,.$$

We note once and for all that the case $\alpha = 0$ and $\beta = 1$ correspond to the previously discussed vector field Z.

Let V be a neighborhood of the origin where X is assumed to be semicomplete. Denote by S the separatrix of $\mathcal{F}$ given by the invariant axis $\{y = z = 0\}$. Fix a local transverse section Σ_r through a base point $(r, 0, 0) \in V$. Denote by L_p the leaf of $\mathcal{F}$ passing through the point (r, p) with $p \in \Sigma_r$ (with the evident identifications). If p is close enough to $(0,0)$, then the closed path $c(t) = (re^{2\pi i t}, 0, 0)$ can be lifted, with respect to the projection on the x-axis, into a path c_p contained in L_p. Furthermore we have

$$\int_{c_p} dT_L = \int_c \frac{dx}{x^2} = 0\,,$$

where dT_L stands for the time-form induced on L_p by X. Thus the vector field X cannot be semicomplete *unless the holonomy map associated with $\mathcal{F}$ and S coincides with the identity.* Next we have:

Lemma 12. *Assume that X and $\mathcal{F}$ are as above. Then the holonomy map associated with $\mathcal{F}$ and S coincides with the identity if and only if α, $\beta \in \mathbb{Z}$ with $\alpha \neq \beta$.*

Proof. With the preceding notation, let $c_p(t) = (x(t), y(t), z(t))$ so that $x(t) = re^{2\pi i t}$. The functions $y(t)$ and $z(t)$ satisfy the following differential equations:

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} = \frac{xz - \alpha xy}{x^2}\, 2\pi i x = 2\pi i(z - \alpha y)\\
\frac{dz}{dt} &= \frac{dz}{dx}\frac{dx}{dt} = \frac{y - \beta xz}{x^2}\, 2\pi i x = 2\pi i(e^{-2\pi i t} y - \beta z)\,.
\end{aligned}$$

In terms of matrix representations, this system becomes

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -2\pi i\alpha & 2\pi i \\ 2\pi i e^{-2\pi i t} & -2\pi i\beta \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} .$$

The solution of this (non-autonomous) system can easily be obtained in terms of the coefficient matrix (denoted by $A(t)$ in the sequel). In particular,

$$\begin{bmatrix} y(1) \\ z(1) \end{bmatrix} = e^{\int_0^1 A(s)\, ds} \begin{bmatrix} y(0) \\ z(0) \end{bmatrix} ,$$

where

$$\int_0^1 A(s)\, ds = \begin{bmatrix} -2\pi i\alpha & 2\pi i \\ 0 & -2\pi i\beta \end{bmatrix} .$$

Hence the matrix $B = \int_0^1 A(s)\, ds$ has two distinct eigenvalues if and only if $\alpha \neq \beta$. When $\alpha = \beta$, the matrix e^B has the form

$$\begin{bmatrix} e^{-2\pi i\alpha} & 2\pi i e^{-2\pi i\alpha} \\ 0 & e^{-2\pi i\alpha} \end{bmatrix}$$

so that the holonomy map is given by $(y, z) \mapsto e^{-2\pi i\alpha}(y + 2\pi i z, z)$ and hence never coincides with the identity.

Suppose now that $\alpha \neq \beta$. We then have $B = PDP^{-1}$ where

$$D = \begin{bmatrix} -2\pi i\alpha & 0 \\ 0 & -2\pi i\beta \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ 0 & \alpha - \beta \end{bmatrix} .$$

Therefore

$$e^B = \begin{bmatrix} e^{-2\pi i\alpha} & \frac{1}{\alpha-\beta}\left(e^{-2\pi i\beta} - e^{-2\pi i\alpha}\right) \\ 0 & e^{-2\pi i\beta} \end{bmatrix} .$$

This matrix (and thus the holonomy) coincides with the identity if and only if α, $\beta \in \mathbb{Z}$. The lemma follows. □

Lemma 12 ensures that X is not semi-complete if $\alpha = \beta$ or if one of the two parameters α or β is not an integer. The converse is provided by Lemma 13 below.

Lemma 13. *The vector field $X = x^2\partial/\partial x + (xz - \alpha xy)\partial/\partial y + (y - \beta xz)\partial/\partial z$ is semicomplete for every pair α, β in $\mathbb{Z}$ with $\alpha \neq \beta$.*

Proof. The argument is very much similar to the one employed for the vector field Z ($\alpha = 0$ and $\beta = 1$). Consider an integral curve $(x(T), y(T), z(T))$ of X. Clearly $x(T) = x_0/(1 - x_0T)$ which is a uniform function on $\mathbb{C} \setminus \{1/x_0\}$ (here we use the word uniform as opposed to multi-valued). Thus we need to check that $y = y(T)$ and $z = z(T)$ are also uniform functions of T. This being clear for the integral curves contained in the invariant set $\{x = 0\}$, consider the remaining orbits of X. These remaining orbits, or rather the leaves of the associated foliation, can locally be parameterized by x, i.e. by a map of the form $x \mapsto (x, y(x), z(x))$. Since x is a uniform function of T, becomes reduced to showing that $y(x)$ and $z(x)$ are uniform functions of x. To do this, note that dy/dx and dz/dx are solutions of the linear system

$$\begin{cases} \frac{dy}{dx} = \frac{z}{x} - \alpha\frac{y}{x} \\ \frac{dz}{dx} = \frac{y}{x^2} - \beta\frac{z}{x} . \end{cases}$$

This system has no singularities for $x \neq 0$. Furthermore the parallel transport along leaves gives rise to linear maps. In particular the holonomy map arising from moving around the point $\{x = 0\}$ is linear itself. This last map however is the identity thanks to Lemma 12. The functions $y(x)$ and $z(x)$ are thus uniform functions of $x \in \mathbb{C}^*$ (for fuller details see Chapter III of [7]). The lemma is proved. □

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