Port Value-At-Risk Estimation Through Generalized Means

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Abstract

In many areas of application, like environment, finance, insurance and statistical quality control, and on the basis of a sample of either independent, identically distributed or possibly weakly dependent and stationary random variables from an unknown model F, it is a common practice to estimate the *value-at-risk* (VaR) at a small level q, i.e. a high quantile of probability $1-q$, i.e. a high enough value so that the chance of an exceedance of that value is equal to q, often smaller than $1/n$, where n is the size of the available sample. The semi-parametric estimation of these high quantiles depends heavily on a reliable estimation of the *extreme value index* (EVI), one of the primary parameters of extreme events, related to the heaviness of the right tail of F. It happens that most semi-parametric VaR-estimators available in the literature do not enjoy the adequate behaviour, i.e. they do not suffer the appropriate linear shift in the presence of linear transformations of the data, as does any theoretical quantile. For heavy tails, i.e. for a positive EVI, new VaR-estimators were introduced with such behaviour, the so-called PORT VaR-estimators, with PORT standing for *peaks over a random threshold*. Regarding EVI-estimation, new classes of PORT EVI-estimators, based on powerful generalizations of the Hill EVI-estimator were recently introduced. Now, also for heavy tails, we discuss the use of new classes of VaR-estimators with the aforementioned behaviour, using classes of EVI-estimators based on adequate generalized means related to the Hill EVI-estimators.

Keywords: extreme value theory; heavy tails; Monte-Carlo simulation; semi-parametric estimation.

1. Introduction

In many areas of application, it is a common practice to estimate the *value at risk at a level* q (VaR_q), a value, high enough, so that the chance of an exceedance of that value is equal to q, small, often smaller than $1/n$, with n the available sample size. Such a sample is denoted by $\underline{\mathbf{X}}_n = (X_1, \cdots, X_n)$ and their members are assumed to be either independent, identically distributed or stationary weakly dependent *random variables* (RVs) from a *cumulative distribution function* (CDF) F. Let us denote by $X_{1:n} \le \cdots \le X_{n:n}$ the associated ascending order statistics (OSs) and assume that there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the maximum, linearly normalized, i.e. $(X_{n:n} - b_n)/a_n$ converges in distribution to a non-degenerate RV. Then (Gnedenko, 1943), the limit CDF is necessarily of the type of the general *extreme value* (EV) CDF, given by

$$G_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}), \ 1+\xi x > 0, & \text{if } \ \xi \neq 0, \\ \exp(-\exp(-x)), \ x \in \mathbb{R}, & \text{if } \ \xi = 0. \end{cases} \tag{1.1}$$

The CDF F is said to belong to the max-domain of attraction of G_ξ and we write $F \in \mathcal{D}_\mathcal{M}(G_\xi)$. The parameter ξ is the *extreme value index* (EVI), the primary parameter of extreme events. This index measures the heaviness of the *right-tail function* $\overline{F}(x) := 1 - F(x)$, and the heavier the right-tail, the larger ξ is.

We here consider *heavy-tailed* models, i.e. *Pareto-type* underlying CDFs, with a positive EVI, working thus in $\mathcal{D}_\mathcal{M}^+ := \mathcal{D}_\mathcal{M}(G_{\xi>0})$. These *heavy-tailed* models are quite common in many areas of application, like biostatistics, computer science, finance, insurance, statistical quality control and telecommunications, among others. For *heavy-tailed* models, the classical EVI-estimators are the Hill

(H) estimators (Hill, 1975), which are the average of the log-excesses, $V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}$, $1 \le i \le k < n$, i.e.

$$\text{(1.2)} \qquad \mathrm{H}(k) \equiv \mathrm{H}\left(k; \underline{\mathbf{X}}_n\right) := \frac{1}{k}\sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right), \ 1 \le k < n.$$

In this article, dealing only with heavy tails, we suggest ways to improve the performance of the existent VaR-estimators. High quantiles depend on the EVI, ξ in (1.1), and recently, new classes of reliable EVI-estimators based on adequate *generalized means* (GMs) have appeared in the literature, and will be introduced in Section 2. But these GM EVI-estimators are NOT location-invariant, contrarily to the PORT-GM EVI-estimators, which depend on an extra *tuning* parameter s, and where PORT stands for *peaks over a random threshold*. The use of the GM EVI-estimators enables us to introduce interesting classes of GM VaR-estimators, as can be seen in Section 3. But again, the associated GM VaR-estimators do not enjoy the adequate behaviour, i.e. they do not suffer the appropriate linear shift in the presence of linear transformations of the data, as does any theoretical quantile. We now discuss, also in Section 3, the use of new classes of VaR-estimators with the aforementioned behaviour, the so-called PORT-GM VaR-estimators, using classes of EVI-estimators based on adequate GMs related to the H EVI-estimators, in (1.2). In Section 4 we provide information on the possibly normal asymptotic behaviour of the aforementioned classes of EVI and VaR-estimators and on the adaptive choice of the most reliable VaR-estimate, jointly with some overall comments.

2. Classes of GM EVI-estimators

Hölder's *mean-of-order-p* (MO_p) EVI-*estimators.* First note that we can write

$$\mathrm{H}(k) = \sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} = \ln\left(\prod_{i=1}^{k}\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k}.$$

The H EVI-estimators are thus the logarithm of the *geometric mean* (or *mean-of-order*-0) of the statistics $U_{ik} := X_{n-i+1:n}/X_{n-k:n}, 1 \le i \le k < n$. Brilhante *et al.* (2013), and almost simultaneously Paulauskas and Vaičiulis (2013), and Beran *et al.* (2014), considered as basic statistics, the MO_p of U_{ik}, $1 \le i \le k < n$, for $p \ge 0$. More generally, Gomes and Caeiro (2014), and also Caeiro *et al.* (2016), considered those same statistics for any $p \in \mathbb{R}$ and the associated class of MO_p EVI-estimators:

$$\text{(2.1)} \qquad \mathrm{H}_p(k) = \mathrm{H}_p\left(k; \underline{\mathbf{X}}_n\right) := \begin{cases} \left(1 - \left(\frac{1}{k}\sum_{i=1}^{k} U_{ik}^p\right)^{-1}\right)/p, & \text{if } p < 1/\xi, p \ne 0, \\ \frac{1}{k}\sum_{i=1}^{k} \ln U_{ik} = \mathrm{H}(k), & \text{if } p = 0. \end{cases}$$

Lehmer's mean-of-order-p (L_p) EVI-*estimators.* Beyond the average, the *p*-moments of log-excesses, i.e. $M_{k,n}^{(p)} := \frac{1}{k}\sum_{i=1}^{k}\{ln\, X_{n-i+1:n} - ln\, X_{n-k:n}\}^p$, $p \ge 1$ [$M_{k,n}^{(1)} = \mathrm{H(k)}$], introduced in Dekkers *et al.* (1989), have also played a relevant role in the EVI-estimation, and can more generally be parameterized in $p \in \mathbb{R}\backslash\{0\}$. And another simple generalization of the average is Lehmer's mean-of-order-*p*: Given a set of positive numbers $\underline{\mathbf{a}} = (a_1, \cdots, a_k)$ such a mean generalizes both the arithmetic mean ($p = 1$) and the harmonic mean ($p = 0$). Lehmer's mean-of-order-*p* is defined as

$$L_p(\underline{\mathbf{a}}) := \sum\nolimits_{i=1}^{k} a_i^p \Big/ \sum\nolimits_{i=1}^{k} a_i^{p-1}, \quad p \in \mathbb{R}.$$

The H EVI-estimators can thus be considered as the Lehmer's mean-of-order-*p* of the *k* log-excesses $\underline{\mathbf{V}} := (V_{ik}, 1 \le i \le k < n)$, for $p = 1$. Following Penalva *et al.* (2016) (see also, Gomes *et al.*, 2016c), note now that $V_{ik} = \xi E_{k-i+1:k}(1 + o_p(1))$ with E denoting a standard exponential RV and the $o_p(1)$-term uniform in $i, 1 \le i \le k$. Since $\mathbb{E}(E^p) = \Gamma(p+1), \forall p > -1$, with $\Gamma(.)$ denoting the Gamma function, the law of large numbers enables us to say that, as $n \to \infty$, $\frac{1}{k}\sum_{i=1}^{k} V_{ik}^p$ converges in probability to $\Gamma(p+1)\xi^p$. Hence the reason for the class of L_p EVI-estimators, consistent for all $\xi \ge 0, p > 0$, and given by

(2.2) $$\mathrm{L}_p(k) \equiv \mathrm{L}_p\left(k;\ \underline{\mathbf{X}}_n\right) := \frac{L_p(\underline{\boldsymbol{V}})}{p} = \frac{1}{p}\frac{\sum_{i=1}^{k} V_{ik}^{p}}{\sum_{i=1}^{k} V_{ik}^{p-1}} = \frac{M_{k,n}^{(p)}}{pM_{k,n}^{(p-1)}} \qquad [\mathrm{L}_1 \equiv \mathrm{H}(k)].$$

Classes of PORT-GM EVI-*estimators*. The classes of GM EVI-estimators, in (2.1) and (2.2), depend on this *tuning parameter* $p \in \mathbb{R}$, are highly flexible, but, as often desired, they are not location-invariant, depending strongly on possible shifts in the model underlying the data, contrarily to what happens to the EVI, which is independent of shifts in the data. It is thus sensible to suggest the use of the classes of PORT-GM EVI-estimators. They are similar in spirit to the PORT-H EVI-estimators, studied in Araújo Santos *et al.* (2006), and further considered in Gomes *et al.* (2008). Classes of PORT estimators are based on a *sample of excesses* over a random threshold $X_{n_s:n}$, with $n_s := \lfloor ns \rfloor + 1,\ 0 \le s < 1$,

(2.3) $$\underline{\mathbf{X}}_n^{(s)} := \left(X_{n:n} - X_{\lfloor ns \rfloor+1:n}, \cdots, X_{\lfloor ns \rfloor+2:n} - X_{\lfloor ns \rfloor+1:n}\right).$$

For $0 \le s < 1$ and $k < n - n_s$, the PORT-GM class of EVI-estimators has the same functional form of the GM class of EVI-estimators, but with $\underline{\mathbf{X}}_n = (X_1, \cdots, X_n)$ replaced by the sample of excesses $\underline{\mathbf{X}}_n^{(s)}$, in (2.3). With GM denoting either H or L, respectively defined in (2.1) and (2.2), they are thus given by

$$\mathrm{GM}_p^{(s)}(k) \equiv \mathrm{GM}_p\left(k; \underline{\mathbf{X}}_n^{(s)}\right).$$

These estimators are now invariant for both changes of scale and location in the data, and depend on the extra *tuning parameter s*, which provides a highly flexible class of EVI-estimators. Indeed, as shown in Gomes *et al.* (2016b), for the MO_p EVI-estimation, these estimators may compare favorably with the PORT versions of the second-order *minimum-variance reduced-bias* (MVRB) EVI-estimators in Caeiro *et al.* (2005), provided that we adequately choose (p, s).

3. VaR-estimation

Just as we did before for the EVI-estimation, we are going to base inference on the largest k top OSs. Let us denote $U(t) := F^{\leftarrow}(1 - 1/t) = \inf\{x: F(x) \ge 1 - 1/t\}$. Using the notation $a(t) \sim b(t)$ if and only if $\lim_{t\to\infty} a(t)/b(t) = 1$, most heavy-tailed parents are such that $U(t) \sim Ct^{\xi}$ as $t \to \infty$. The, and since $\chi_{1-q} \equiv \mathrm{VaR}_q$ is such that $1 - F(\mathrm{VaR}_q) = q$,

$$\mathrm{VaR}_q = U(1/q) \sim Cq^{-\xi}, \text{ as } q \to 0,$$

and an obvious estimator of VaR_q is $\widehat{\mathrm{VaR}}_q = \hat{C}q^{-\hat{\xi}}$, with $\hat{C}$ and $\hat{\xi}$ any consistent estimators of C and ξ, respectively. Denoting Y an RV from a standard Pareto model, with CDF $F_Y(y) = 1 - 1/y, y \ge 1$, $X_{n-k:n} \overset{d}{=} U(Y_{n-k:n}) \overset{p}{\sim} CY_{n-k:n}^{\xi} \overset{p}{\sim} C(n/k)^{\xi}$, as $n \to \infty$. An obvious estimator of C is thus $\hat{C} = (k/n)^{\hat{\xi}}\, X_{n-k:n}$, and the obvious VaR_q-estimator was introduced in Weissman (1978), being given by

(3.1) $$\mathrm{Q}_{\hat{\xi}}^{(q)}(k) := X_{n-k:n}(k/(nq))^{\hat{\xi}}.$$

For heavy-tailed models, the 'classical' EVI-estimators, usually the ones which are plugged in the previous formula, are the H EVI-estimators, already defined in (1.2), the average of the log-excesses. We thus get the so-called 'classical' VaR-estimators, based on the H EVI-estimators, with the obvious notation, $Q_{\mathrm{H}}^{(q)}(k)$.

GM VaR-*estimation*. The high asymptotic bias of the H EVI-estimators, for small up to moderate k-values, has recently led researchers to consider the possibility of dealing with the bias term in an appropriate way, building new estimators, $\hat{\xi}_R(k)$ say, the so-called *second-order reduced-bias* (SORB) estimators (see Gomes and Guillou, 2015, for an overview of the topic). Caeiro *et al.* (2005), considered *corrected-Hill* (CH) MVRB EVI-estimators,

(3.2) $$\mathrm{CH}_{\hat{\beta},\hat{\rho}}(k) := H(k)\left(1 - \hat{\beta}(n/k)^{\hat{\rho}}/(1 - \hat{\rho})\right),$$

with $(\hat{\beta},\hat{\rho})$ adequate consistent estimator of (β,ρ), a vector of the second-order parameters, so that the asymptotic variance is kept at the same level of the variance of the H EVI-estimators. Gomes and Pestana (2007) considered then, as a possible alternative to the classical VaR-estimator, $Q_{\mathrm{H}}^{(q)}(k)$, the estimator in (3.1) based upon the EVI-estimators in (3.2), i.e. $Q_{\mathrm{CH}}^{(q)}(k)$, a reference for the VaR-estimation.

With GM_p denoting either H_p or L_p, respectively given in (2.1) and (2.2), we now think sensible to work with the new VaR_q-estimators $Q_{\mathrm{GM}_p}^{(q)}(k)$, with the obvious functional form

$$\mathrm{Q}_{\mathrm{GM}_p}^{(q)}(k) := X_{n-k:n}(k/(nq))^{\mathrm{GM}_p(k)}.$$

The Monte-Carlo simulations in Gomes *et al.* (2015) show the potentiality of the MO_p VaR_q semi-parametric estimators, being still under development the study of the L_p VaR_q-estimators.

PORT-GM VaR-*estimation*. Most of the semi-parametric VaR-estimators in the literature (see the functional equation in (3.1), Beirlant *et al.*, 2004, and de Haan and Ferreira, 2006), do not enjoy the adequate behaviour in the presence of linear transformations of the data, a behaviour related to the fact that for any high-quantile,

(3.3) $$\mathrm{VaR}_q(\lambda+\delta X) = \lambda + \delta \mathrm{VaR}_q(X),$$

for any model X, real λ and positive δ. Recently, and for $\xi > 0$, Araújo Santos *et al.* (2006) provided VaR-estimators with the linear property in (3.3), based on a *sample of excesses* over the random threshold $\underline{\mathbf{X}}_n^{(s)}$, $n_s := \lfloor ns \rfloor + 1, 0 \le s < 1$, given in (2.3), being s possibly null only when the underlying parent has a finite left endpoint (see Gomes *et al.*, 2008b, for further details on this subject). They were named PORT VaR-estimators, and were based on the PORT-H, $\mathrm{H}(k;\underline{\mathbf{X}}_n^{(s)})$, $k < n - n_s$, with $\mathrm{H}(k;\underline{\mathbf{X}}_n)$ provided in (1.2).
Now, we further suggest for an adequate VaR-estimation, the use of the PORT-GM_p EVI-estimators,

(3.4) $$\mathrm{GM}_p(k;s) := \mathrm{GM}_p(k;\,\underline{\mathbf{X}}_n^{(s)}),\ \ k < n - n_s \qquad [\mathrm{GM{=}H\ and\ GM{=}L}],$$

with H_p, L_p and $\underline{\mathbf{X}}_n^{(s)}$ respectively given in (2.1), (2.2) and (2.3). Such PORT-GM_p VaR-estimators are given by

$$\widehat{\mathrm{VaR}}_q(k;p,s) := \left(X_{n-k:n} - X_{n_s:n}\right)(k/(nq))^{\mathrm{GM}_p(k,s)} + X_{n_s:n}.$$

4. Asymptotic behaviour of estimators, adaptive choice of the tuning parameters and overall comments

First and second-order frameworks for heavy tails. Let $\mathcal{R}_{\mathrm{a}}$ denote the class of regularly varying functions with an index of regular variation equal to a $\in \mathbb{R}$, i.e. measurable function $g(.)$ such that $\forall\, x > 0, g(tx)/g(t) \to x^{\mathrm{a}}$, as $t \to \infty$. A model F is said to be heavy-tailed if $\xi > 0$, in (1.1), and we have the first-order condition,

$$F \in \mathcal{D}_{\mathcal{M}}^{+} \iff U \in \mathcal{R}_{\xi} \iff 1 - F \in \mathcal{R}_{-1/\xi}.$$

To obtain information on the non-degenerate *normal* behaviour of the estimators, it is usual to assume the following second-order condition,

(4.1) $$\lim_{t\to\infty} (\ln U(tx) - \ln U(t) - \xi \ln x)/A(t) = \begin{cases} (x^{\rho}-1)/\rho, & \text{if } \rho < 0, \\ \ln x, & \text{if } \rho = 0, \end{cases}$$

valid for all $x > 0$, where $\rho \le 0$ is a second-order parameter. Slightly more restrictively, and essentially for VaR-estimation, we shall assume to be working in Hall-Welsh class of models (Hall and Welsh, 1985), where $\exists\ \xi > 0, \rho < 0, C > 0$ and $\beta \neq 0$ such that

(4.2) $$U(t) = Ct^{\xi}(1 + \xi\beta\, t^{\rho}/\rho + o(t^{\rho})), \text{ as } t \to \infty.$$

Asymptotic behaviour of EVI-*estimators*. To have consistency of the aforementioned EVI-estimators in all $\mathcal{D}_{\mathcal{M}}^{+}$, we need to work with *intermediate* values of k, i.e. a sequence of positive integers $k = k_n, 1 \le k < n$, such that $k = k_n \to \infty$ and $k_n = o(n)$, as $n \to \infty$. Under the aforementioned second-order framework in (4.1), the asymptotic behaviour of the H EVI-estimator was derived in de Haan and Peng (1998). More generally (Brilhante *et al.*, 2013, and Gomes and Caeiro, 2014, for the H_p EVI-estimators, and Penalva *et al.*, 2016, for the L_p EVI-estimators), and again using the notation GM_p for both H_p and L_p, the asymptotic distributional representation

$$\text{GM}_p(k) \stackrel{d}{=} \xi + \sigma_{\text{GM}_p}(\xi) Z_k^{(\text{GM}_p)}/\sqrt{k} + b_{\text{GM}_p}(\xi,\rho)\, A(n/k) + o_p(A(n/k))$$

holds with $Z_k^{(\text{GM}_p)}$ asymptotically standard normal RVs. Then, when $\sqrt{k}A(n/k) \to \lambda$, finite, as $n \to \infty$, $\sqrt{k}(\text{GM}_p(k) - \xi)$ converges in distribution to a $\mathcal{N}\left(\lambda b_{\text{GM}_p}, \sigma^2_{\text{GM}_p}\right)$.

Remark 1. *At optimal levels, in the sense of minimal root mean squared error* (RMSE)*, the optimal* MO_p (OMO_p) *class,* $\text{H}^*(k) = \text{H}_{p_\text{M}}(k)$*, outperforms the* H EVI-*estimator in the whole* (ξ,ρ)*-plane. And, again at optimal levels, the optimal Lehmer* EVI-*estimator, say* L^**, beats on its turn* H^**, also in the whole* (ξ,ρ)*-plane.*

To derive the asymptotic properties of the PORT-GM EVI-estimators, it is worth noting that since $X_{\lfloor ns\rfloor+1:n} - U(1/(1-s)) = O_p(1/\sqrt{n})$, the EVI-estimator $\text{GM}_p(k,s)$, in (3.4), has the same asymptotic behaviour of $\widetilde{\text{GM}}_p(k,s)$, defined as $\text{GM}_p(k,s)$, but with $X_{n-i+1:n}$ replaced everywhere by $X_{n-i+1:n} - U(1/(1-s)), 1 \le i \le n$.

The PORT-MO_p EVI-estimators were studied in Gomes *et al.* (2016b). Similar results, still under development, are expected for the PORT-L_p EVI-estimators. The PORT methodology leads to no change in the asymptotic variance. There is only a change in the asymptotic bias, no longer ruled by $A(t)$, but ruled by

$$B(t) = \begin{cases} \xi\, \chi_s/U_0(t), & \text{if } \xi+\rho_0 < 0 \;\wedge \chi_s \ne 0, \\ A_0(t) + \xi\, \chi_s/U_0(t), & \text{if } \xi+\rho_0 = 0 \;\wedge \chi_s \ne 0, \\ A_0(t), & \text{otherwise,} \end{cases}$$

where $\chi_s = F^{\leftarrow}(s)$, and (A_0, U_0) are the functions (A, U) associated with a shift $s = 0$.

Asymptotic behaviour of the VaR*-estimators.* For all the aforementioned classes of VaR-estimators, generally denoted $Q_{\hat\xi}^{(q)}(k)$, and with $r_n = k/(nq)$, we can write

$$Q_{\hat\xi}^{(q)}(k) - \text{VaR}_q \stackrel{d}{=} \ln r_n\, \text{VaR}_q(\hat\xi - \xi)(1 + o_p(1)),$$

whenever working in Hall-Welsh class of models, in (4.2). For intermediate k, and whenever $q = q_n \to 0, \ln(nq_n) = o(\sqrt{k})$, and $nq_n = o(\sqrt{k})$, a similar normal behaviour appears for the EVI and associated VaR-estimators, but with a rate of convergence which is no longer $1/\sqrt{k}$ but $1/(\sqrt{k}\, \ln r_n\, \text{VaR}_q)$. For a PORT VaR-estimation, see Henriques-Rodrigues and Gomes (2009) and Figueiredo *et al.* (2016).

Adaptive choice. The adaptive choice of the tuning parameters (k, p, s) can be done through heuristic sample-path stability algorithms, like the ones in Gomes *et al.* (2013) and Neves *et al.* (2015). Alternatively, it is also sensible to use a bootstrap algorithm of the type of the ones in Gomes *et al.* (2011; 2012), Brilhante *et al.* (2013), Caeiro and Gomes (2015) and Gomes *et al.* (2016a), where R-scripts are provided.

Overall comments. For all k, there is a clear reduction in RMSE, as well as in bias, with the attainment of estimates closer to the target value ξ. At optimal levels, even the PORT-H^* beats the MVRB estimators. Indeed, the PORT-H_p, considered as a function of p, can even beat the PORT-MVRB EVI-estimators. The patterns of the estimates are always of the same type, in the sense that, for all k, the MVRB clearly beat the Hill, the H^* moderately beat the MVRB, regarding minimal MSE, and adequate MO_p and PORT-MO_p strongly beat the MVRB EVI-estimators.

For recent overviews on statistics of univariate extremes, see Beirlant *et al.* (2012), Scarrot and McDonald (2012) and Gomes and Guillou (2015).

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