The Depth Theory of Hopf Algebras and Smash Products

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November 25, 2016

To all my loved ones.

Abstract

The work done during this doctoral thesis involved advancing the theory of algebraic depth. Subfactor depth is a concept which had already existed for decades, but research papers discovered a purely algebraic analogue to the concept. The main uses of algebraic depth, which applies to a ring and subring pair, have been in Hopf-Galois theory, Hopf algebra actions and to some extent group theory.

In this thesis we consider the application of algebraic depth to finite dimensional Hopf algebras, and smash products. This eventually leads to a striking discovery, a concept of module depth. Before explaining this work the thesis will go through many know results of depth up to this point historically. The most important result of the thesis: we discover a strong connection between the algebraic depth of a smash product $A\#H$ and the module depth of an H-module algebra A. In separate work L. Kadison discovered a connection between algebraic depth $R \subseteq H$ for Hopf algebras and the module depth of V^*, another important H-module algebra. The three concepts are related.

Another important achievement of the work herein, we are able to calculate for the first time depth values of polynomial algebras, Taft algebras with certain subgroups and specific smash products of the Taft algebras. We also give a bound for the depth of $R/I \subseteq H/I$, which is extremely useful in general.

Contents

Background

The theory of depth originates from work in the book [10], where tower conditions on subfactors are considered. There is a notion of *subfactor depth* as it appears in, for example, the paper [39]. The body of work on subfactor depth certainly merits study from anyone interested in the underlying subjects. However the study of this subject will not directly lead to an understanding of depth theory as in this thesis, especially since subfactor depth involves Von Neuman algebras, including their topological properties which we do not treat.

The idea of *algebraic depth*, which we refer to throughout the thesis as *depth*, historically started with the definition of the depth 2 property. The first case of depth 2 was made with regards to strongly separable algebra extensions, which are extensions $S \subseteq R$ that have symmetric separability elements $e \in R \otimes_S R$ [17]. This purely algebraic notion of depth 2 generalises the idea of subfactor depth 2: papers by Kadison-Nikshych and a paper and preprint by Kadison-Szlachnyi further generalised the subfactor depth 2 condition to depth 2 [24], [23], [26], [25]. Given an arbitrary ring extension $R \subseteq H$ the depth 2 condition is when isomorphisms as below exist over R-H and H-R-bimodules:

$$(i)\ H \otimes_R H \oplus * \cong q \cdot H;$$

$$(ii)\ H \oplus * \cong p \cdot H \otimes_R H;$$

where $*$ represents an arbitrary R-H or H-R-bimodule.

Early work on depth 2 in invariant subalgebras was geared towards understanding Hopf-Galois actions, in this thesis Section 3.4 discusses a strong relationship of the two things. (One of the main reasons Jones developed a successful theory of subfactors was its application to noncommutative Galois theory [16].) Hence there has always been a good reason to study the depth 2 condition. Furthermore, after the original definition of depth 2 much effort was made to link depth 2 in Hopf algebras or Hopf algebroids with the normality condition. The equivalence was found and is presented for Hopf algebras in Subsection 3.2.2.

In [24] the depth n definition is made for Frobenius extensions, which generalises the notion of subfactor depth n. This concept of depth n is fully generalised to ring extensions in [19]. In this thesis we will write down this depth $n \in \mathbb{N}$ condition for general ring extensions, but we will break it into even and odd cases. This is effectively the starting point of the work in the thesis.

Contribution

Basic definitions are introduced in Chapter 1, which are an assortment of definitions and results appearing in other papers. See for example [4], [21], [20]. Notice that the concept of Hirata-depth, otherwise labelled as H-depth does not appear in the thesis, at no loss to the quality of results. The worked example on the polynomial algebra (Example 2) can be considered original work (**O.W**) in this thesis.

Chapter 2 serves as an introduction to some technical results on k-algebras and depth theory which are used later in the thesis. The section on groups explains the impressive fact that $d(kH, kG) < \infty$ for finite groups $H \subseteq G$, as in [4]. This fact was a strong motivator in studying Hopf algeba depth (Chapter 3), because there is a possibility that $d(R, H) < \infty$ for all finite dimensional Hopf algebras. Moreover the work on invariant subgroups is **O.W**. The sections on quivers (2.3) and morphisms which preserve depth (2.4) are taken from work in the joint paper [50], written by myself and L. Kadison. The work on both quiver examples was entirely down to L. Kadison but the work on morphisms which preserve depth is **O.W**.

In Chapter 3 Hopf algebras are introduced followed by smash products, which form two of the most important mathematical objects in this thesis. Hopf algebras H are necessary in defining smash products $A\#H$. Other than definitions and central concepts I work through the chain of results showing that $d(A, A\#H) \leq 2$. Moreover an equivalence between semidirect products and smash products when working over group algebras is explained. Out of Chapter 3 the idea of calculating $d(H, A\#H)$ for finite dimensional Hopf algebras arises, because we are interested in whether $d(H, A\#H)$ is finite, as with $d(A, A\#H)$ for finite dimensional H.

Two completely worked examples of smash products appear in Chapter 4. These examples remain unpublished but are **O.W**. The first examples deals with the family of Taft algebras T_n over $\mathbb{C}$, where each algebra is isomorphic to a particular smash product. The second example is based on forming a smash product between the Taft algebra T_2 over $\mathbb{R}$, with the complex numbers $\mathbb{C}\#T_2$. The depth values for each example are 3 and 4 respectively.

Module depth is a concept invented jointly with the supervisor of this thesis, L. Kadison. The language which appears in this thesis is slightly different than that which appears in for example the paper [21], which favours truncated tensor products of a module $T_n(W) := W \oplus (W \otimes W) \oplus \ldots \oplus (W^{\otimes n})$. Instead I defined module depth as when a module W satisfies $W^{\otimes n+1} \sim W^{\otimes n}$ for some $n \in \mathbb{N}$. In this chapter Hopf algebras are taken to be finite dimensional, and much progress is made in finding the value of depth $d(H, A\#H)$ by using the concept of module depth. Sections 5.4, 5.6 and 5.7 consist of **O.W**, unless otherwise stated. Some of the content in Chapter 5 appeared in a joint paper [11] with A. Hernandez and L. Kadison. Of particular note is the theorem which proves the depth result $d_{odd}(H, A\#H) = 2d(A, \mathcal{M}_H) + 1$. Consequently I was eventually able to write down a striking inequality:

$$2d(V, {}_H\mathcal{M}) - 1 \leq d_{odd}(R, H) \leq d_{odd}(H, H\#V^*),$$

for finite dimensional Hopf algebras $R \subseteq H$. This equality brings three of our central themes together: Hopf algebras, smash products and module depth.

Chapter 1

The Depth Theory of a Ring Extension

1.1 Tensor Products

In this thesis rings will always be unital with commutative addition and associative, distributive multiplication. We will similarly assume that all modules over rings are associative and unital. For background on rings and category theory see [2], [33], [34]. Aside from algebraic structures like groups, rings and modules the most commonly mentioned algebraic objects in this thesis are undoubtedly tensor products. In discussing tensor products we closely follow Section 3.7 of Jacobson's book [15]. The original concept of a tensor product was defined for vector spaces, who have bases, but here we deal with modules, which do not necessarily have bases.

For a set S define the *free Abelian group* on S, written as $F(S)$, to be the Abelian group with free generators the elements of S. Therefore an arbitrary element of $F(S)$ looks uniquely like $p_1s_1 + p_2s_2 + \ldots + p_ts_t$, with addition in this group $ps + qs = (p + q)s$.

Lemma 1.1.1. *Let S be a set and $F(S)$ the free Abelian group, define the map $\iota : S \to F(S)$ by $\iota(s) = s$. Let $f : S \to G$ be a mapping where G is an Abelian group, then there exists a unique group homomorphism $v : F(S) \to G$ making the diagram below commute:*

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F(S) \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle v} & \\ G & & \end{array}$$

Proof. Clearly the map $v(p_1s_1+\ldots+p_ts_t) = p_1f(s_1)+\ldots+p_tf(s_t)$ is a well-defined group homomorphism which satisfies $f = v \circ \iota$. Let $w : F(S) \to G$ be another group homomorphism such that $f = w \circ \iota$ then

$$\begin{aligned} w(p_1s_1 + \ldots + p_ts_t) &= w(p_1\iota(s_1) + \ldots + p_t\iota(s_t)) \\ &= p_1w(\iota(s_1)) + \ldots + p_tw(\iota(s_t)) \\ &= p_1f(s_1) + \ldots + p_tf(s_t) \\ &= v(p_1s_1 + \ldots + p_ts_t). \end{aligned}$$

Then $w = v$ and this homomorphism is unique. □

Let R be an associative unital ring. Let M be a right R-module and let N be a left R-module. We write $M \times N$ for the product set.

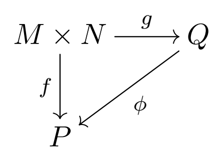
Definition 1.1.2. Suppose that M, N are as above, a balanced product of M and N is a pair (P, f), where P is an Abelian group and $f : M \times N \to P$ a set map, satisfying for all $m, m' \in M$, $n, n' \in N$ and $r \in R$

(i) $f(m + m', n) = f(m, n) + f(m', n)$.

(ii) $f(m, n + n') = f(m, n) + f(m, n')$.

(iii) $f(mr, n) = f(m, rn)$.

Given two balanced products $(P, f), (Q, g)$ we define a *morphism of balanced products* $(P, f) \to (Q, g)$ to be a group homomorphism $\phi : P \to Q$ such that $g = \phi \circ f$, diagrammatically this means we have the following commutative diagram:



Definition 1.1.3. A tensor product $(M \otimes_R N, h)$ is a balanced product which is universal in the sense that for any other balanced product (P, f) there is a unique morphism of balanced products $u : (M \otimes_R N, h) \to (P, f)$. In particular the underlying groups of any two tensor products will be isomorphic.

We will construct a concrete example of a tensor product. For any M, N as above define $F := F(M \times N)$, the free Abelian group on $M \times N$. We may write the elements of F as

$$p_1(m_1, n_1) + \ldots + p_t(m_t, n_t),$$

where $p_i \in \mathbb{Z}$ and $(m_i, n_i) \in M \times N$. In particular the group operation in F is such that

$$p(m,n) + q(m,n) = (p+q)(m,n).$$

Let us define the subgroup $I \subseteq F$ which is generated by elements of the form *(1)* $(m+m', n)-(m,n)-(m',n)$; *(2)* $(m, n+n')-(m,n)-(m,n')$ and; *(3)* $(mr, n)-(m, rn)$.

Proposition 1.1.4. *The pair $(F/I, h)$ forms a tensor product of M_R and ${}_RN$, where $h : M \times N \to F/I$ is the map $h(m,n) = (m,n) + I$.*

Proof. We must first check that $(F/I, h)$ is a balanced product, in other words that h satisfies conditions (i)-(iii) of Definition 1.1.2. The first condition is clearly satisfied because

$$\begin{aligned} h((m+m', n') - (m,n) - (m,n)) &= h(m+m', n') - h(m,n) - h(m,n) \\ &= ((m+m', n') + I) - ((m,n) + I) - ((m,n) + I) \\ &= (m+m', n') - (m,n) - (m,n) + I \\ &= I = 0. \end{aligned}$$

Conditions (ii) and (iii) are similarly satisfied, so that $(F/I, h)$ is a balanced product.

Suppose that we have a balanced product (P, f), by Lemma 1.1.1 there is a unique group homomorphism $v : F \to P$ such that $v(m,n) = f(m,n)$. Like the homomorphism h, f satisfies conditions $(i)-(iii)$ of Definition 1.1.2 so for all $m, m' \in M$, $n, n' \in N$

$$(m+m', n) - (m,n) - (m', n) \in kerv,$$

$$(m, n+n') - (m,n) - (m,n') \in kerv,$$

$$(mr, n) - (m, rn) \in kerv.$$

Since the three types of elements above generate the subgroup I it is clear that $I \subseteq kerv$, so $u : F/I \to P : (m,n) + I \mapsto f(m,n)$ is a well-defined homomorphism. Moreover this homomorphism satisfies $u(h(m,n)) = u((m,n)+I) = (m,n)$ in other words $f = u \circ h$. The homomorphism u is clearly the unique such one. □

Let us relabel the above tensor product example $(F/I, h)$ as $(M \otimes_R N, \otimes_R)$ where $\otimes_R(m,n) = (m,n) + I$ is written as $m \otimes n$. Therefore given $m, m' \in M$, $n, n' \in N$ and $r \in R$, *(i)* $(m+m') \otimes n = m \otimes n + m' \otimes n$; *(ii)* $m \otimes (n+n') = m \otimes n + m \otimes n'$ and; *(iii)* $mr \otimes n = m \otimes rn$.

Lemma 1.1.5. *Let M_R be an R-module and let ${}_R R$ be an R-module via left multiplication, then the map*

$$M \otimes_R R \to M,$$

$$m \otimes r \mapsto mr$$

is an isomorphism of groups. And for left R-modules ${}_R N$ there is similarly an isomorphism $R \otimes_R N \to N : r \otimes n \mapsto rn$.

Lemma 1.1.6. *Given a right module M and a family of left modules $\{N_i,\ i \in I\}$, then we have an isomorphism of groups*

$$M \otimes (\oplus_i N_i) \to \oplus_i (M \otimes N_i),$$

$$m \otimes (n_i)_{i\in I} \mapsto (m \otimes n_i)_{i\in I}.$$

1.1.1 Extension to Bimodules

Let M, M' be right modules and N, N' left modules, moreover let $f : M \to M'$ and $g : N \to N'$ be module homomorphisms. We may define a map $M \times N \to M' \otimes_R N' : (m, n) \mapsto f(m) \otimes g(n)$, which satisfies

(a) $f(m_1 + m_2) \otimes g(n) = f(m_1) \otimes g(n) + f(m_2) \otimes g(n)$,

(b) $f(m) \otimes g(n_1 + n_2) = f(m) \otimes g(n_1) + f(m) \otimes g(n_2)$,

(c) $f(mr) \otimes g(n) = f(m)r \otimes g(n) = f(m) \otimes g(rn)$.

This makes $M' \otimes_R N'$ a balanced product of M and N, so there is a unique group homomorphism $M \otimes_R N \to M' \otimes_R N'$ such that $m \otimes n \mapsto f(m) \otimes f(n)$. We call this homomorphism $f \otimes g$ in general. If $f' : M' \to M''$ and $g : N' \to N''$ are right and left module homomorphisms respectively then $(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g)$.

Assume that M is an S-R-bimodule, this is equivalent to M being a right R-module where for each $s \in S$ there is an R-module homomorphism $\nu_s : M \to M$. The notation $\nu_s(m) = sm$ is the one we are used to, however the mapping allows us to define a canonical left S-module structure in $M \otimes_R N$, where N is a left R-module. The group homomorphism $\nu_s \otimes 1 : M \otimes_R N \to M \otimes_R N$ is the unique one such that $m \otimes n \mapsto sm \otimes n$. In particular $M \otimes_R N$ is a left S-module.

Lemma 1.1.7. *Given an S-R-bimodule M and an R-T-bimodule N, then the tensor product $M \otimes_R N$ has one unique S-T-bimodule structure such that*
$s(m \otimes n)t = sm \otimes nt$, for all $s \in S$, $t \in T$.

Lemma 1.1.8. *Given an S-R-bimodule M, an R-T-bimodule N and a T-U-bimodule P. The tensor product is associative in the sense that there is an S-U-bimodule isomorphism*

$$\Xi : (M \otimes_R N) \otimes_T P \to M \otimes_R (N \otimes_T P).$$

Example 1. Let $n \in \mathbb{N}$ and let k be a field. We define the polynomial ring $k[X_1, \ldots, X_n]$ as the formal k-linear sums of a formal unit 1 and elements of the form $X_1^{p_1} \ldots X_n^{p_n}$, where $p_i \in \mathbb{N}$ and $X_k X_l = X_l X_k$ for all $1 \leq k, l \leq n$. We call the aforementioned generators the *monomials*, in particular $X_1^{p_1} \ldots X_n^{p_n}$ is a *monomial of degree* $p_1 + \ldots + p_n \in \mathbb{N}$. When $n > 0$ this is an infinite dimensional k-vector space.

Take natural numbers $1 \leq m < n$, and denote the rings $R := k[X_1, \ldots, X_m]$ and $H := k[X_1, \ldots, X_n]$. Via subring multiplication H is both a left and a right R-module, with which we can form the tensor product $H \otimes_R H$. Notice that as rings we have the following isomorphism:

$$H \cong k[x_1] \otimes \ldots \otimes k[x_n], \tag{1.1}$$

where on the right side we have componentwise multiplication $(x_1 \otimes x_2 \otimes \ldots \otimes x_n)(x_1' \otimes x_2' \otimes \ldots \otimes x_n') = x_1 x_1' \otimes x_2 x_2' \otimes \ldots \otimes x_n x_n'$. Indeed the isomorphism is given by identifying each monomial $X_1^{p_1} \cdots X_n^{p_n}$ with $x_1^{p_1} \otimes \ldots \otimes x_n^{p_n}$. We similarly express the subalgebra R as $k[x_1] \otimes \ldots \otimes k[x_m]$. Now apply (1.1) and the cancellation rule of Lemma 1.1.8 to deduce the following chain of H-R-bimodules isomorphisms:

$$\begin{aligned} H \otimes_R H &\cong k[X_1, \ldots, X_n] \otimes_R (k[x_1] \otimes \ldots \otimes k[x_n]) \\ &\cong k[X_1, \ldots, X_n] \otimes k[x_{m+1}] \otimes \ldots \otimes k[x_n] \\ &\cong k[X_1, \ldots, X_n, x_{m+1}, \ldots, x_n]. \end{aligned}$$

Where the bimodule structure of $k[X_1, \ldots, X_n, x_{m+1}, \ldots, x_n]$ is given by multiplication of $k[X_1, \ldots, X_n]$ as a subring. Note also that by symmetrical arguments the above vector spaces are isomorphic as R-H-bimodules.

1.2 Depth Theory

Throughout the thesis we are interested in two specific types of module categories *(a)* categories of bimodules ${}_R\mathcal{M}_S$ over rings or algebras R, S, and *(b)* the categories of the left (or right) modules ${}_H\mathcal{M}$ over a Hopf algebra H, as studied later in Chapter

5. Let $\mathcal{C}$ denote either of the above module categories, which should implicitly be understood to mean that all definitions and results which follow apply to categories of type (a) and (b).

The definitions of similarity and depth which appears in this chapter are already published by other authors. The entire thesis is built on the definitions which follow. In order to learn about depth some good papers to read are [4], [21], [20].

Definition 1.2.1. Let $n \cdot M$ denote the n-fold sum $M \oplus \ldots \oplus M$. Given two modules M, N in $\mathcal{C}$ we say that M *divides* N, and write $M \mid N$ when there exists a module M' such that

$$M \oplus M' \cong N.$$

Define a relation among modules in $\mathcal{C}$, by saying M *is similar to* N, and writing $M \sim N$, when there exist $p, q \in \mathbb{N}$ such that

$$M \mid q \cdot N \text{ and } N \mid p \cdot M.$$

Lemma 1.2.2. *There are two fundamental properties of similarity*

(a) $M \sim M$ for every module in $\mathcal{C}$,

(b) If $M \sim N$ and $N \sim P$, then $M \sim P$.

Proof. Given a module M then clearly $M \oplus 0 \cong M$ and so $M \sim M$. Assume now $M \oplus M' \cong N$ and $N \oplus N' \cong P$ for modules M, M', N, N', P, then certainly $M \oplus M' \oplus N' \cong P$ and we are done. □

Lemma 1.2.3. *Let M, N be modules in $\mathcal{C}$, then $M \mid N$ if and only if there are module homomorphisms $f : M \to N$ and $g : N \to M$ such that $g \circ f = id_M$.*

Proof. ($\Rightarrow$) Suppose $M \mid N$ so that there exists M' with $M \oplus M' \cong N$. Write $\pi : N \to M$ for the projection map $\pi(m \oplus m') = m$, write $\iota : M \to N$ for the canonical embedding $\iota(m) = m \oplus 0$, then $\pi \circ \iota = id$ as required.

($\Leftarrow$) Take f, g as in the assumption. Take $x \in N$, because $g \circ f = id$ we have the inclusion $x - fg(x) \in kerg$, therefore $x \in fg(x) + kerg$. It is clear now that $x \in imf + kerg$. Notice that $imf \cap kerg = \{0\}$ because $g \circ f = id$, note also that M is isomorphic to imf as a module. We may finally write $N \cong M \oplus kerg$ as modules. In particular this means $M \mid N$. □

The lemma above tells us that similarity ($\sim$) in $\mathcal{C}$ can be replaced by morphisms as in the commuting diagrams:

$$ id \circlearrowleft M \underset{f}{\overset{g}{\rightleftarrows}} q \cdot N \, , \qquad\qquad id \circlearrowleft N \underset{f'}{\overset{g'}{\rightleftarrows}} p \cdot M \, . $$

Corollary 1.2.4. *Let M, N be modules as above, then $M \mid q \cdot N$ if and only if there exist homomorphisms $f_i : M \to N$ and $g_i : N \to M$ $(1 \le i \le q)$ such that $\sum_i (g_i \circ f_i) = id$.*

1.2.1 Depth of Ring Extensions

Start by taking a ring extension $R \subseteq H$. Depth is a natural number calculated on the tensor bimodules

$$H^{\otimes_R n} := \underbrace{H \otimes_R H \otimes_R \ldots \otimes_R H}_{n \text{ factors}},$$

with $H^{\otimes_R 0} := R$. The sets defined above will be X-Y-bimodules for subrings $X, Y \subseteq H$, via the multiplication

$$x(h_1 \otimes h_2 \otimes \ldots h_{n-1} \otimes h_n)y = (xh_1) \otimes h_2 \otimes \ldots h_{n-1} \otimes (h_n y).$$

In defining depth we will consider the specific cases: *(a)* $X = Y = R$; *(b)* $X = R$, $Y = H$ and; *(c)* $X = H$, $Y = R$. Similarity as in Definition 1.2.1 applies to the bimodule categories ${}_X\mathcal{M}_Y$.

Definition 1.2.5. We say that the ring extension $R \subseteq H$ satisfies the *left depth* $2n$ condition, for $n \ge 1$ if

$$H^{\otimes_R n+1} \sim H^{\otimes_R n} \tag{1.2}$$

as R-H-modules. The *right depth* $2n$ condition is when (1.2) is satisfied as H-R-modules, and when both left and right depth $2n$ conditions hold we say $R \subseteq H$ satisfies *depth* $2n$. For $n \ge 0$ the condition of *depth* $2n+1$ is that (1.2) be satisfied as R-bimodules. Notice that by the way depth is defined we do not have a depth 0 condition.

Lemma 1.2.6. *If $\theta : H \to L$ is an isomorphism of rings, then*

$$d(R, H) = d(\theta R, \theta H).$$

This applies to infinite depth as well.

Lemma 1.2.7. *Assume we have an extension $R \subseteq H$ which satisfies the (left, right or both) depth $2n$ condition then it also satisfies the $2n+1$ condition. Similarly if the extension satisfies the $2n+1$ condition it satisfies the $2n+2$ condition (left and right).*

Proof. We show that if $R \subseteq H$ satisfies depth $2n+1$ then it satisfies depth $2n+2$. This case will make clear the idea of left or right depth $2n$ implying depth $2n+1$. Suppose $R \subseteq H$ has depth $2n+1$, in particular $H^{\otimes_R n} \sim H^{\otimes_R n+1}$ as R-bimodules. So for some $p, q \in \mathbb{N}$ we have $H^{\otimes_R n} \mid p \cdot H^{\otimes_R n+1}$ and $H^{\otimes_R n+1} \mid q \cdot H^{\otimes_R n}$. Apply the functor $- \otimes_R H$ to both divisions, so

$$H^{\otimes_R n+1} \cong H^{\otimes_R n} \otimes_R H \mid (p \cdot H^{\otimes_R n+1}) \otimes_R H \cong p \cdot H^{\otimes_R n+2},$$

$$H^{\otimes_R n+2} \mid q \cdot H^{\otimes_R n+1},$$

as R-H-bimodules. □

An implication of the lemma above is that we should consider a *minimum depth* (usually just called the depth of the ring extension), and we denote it by $d(R, H)$.

Write d_{odd} and d_{even} for the minimum odd and even depths. We can find examples of both even and odd minimum depth (>2 and >1 respectively). By the lemma directly above it is clear that $|d_{odd} - d_{even}| = 1$. Let us discuss one strategy for finding the minimum depth. In the even case $d(R, H) = 2n$ if and only if $d_{even}(R, H) = 2n$ but $d_{odd}(R, H) = 2n+1$; similarly $d(R, H) = 2n+1$ if and only if $d_{odd}(R, H) = 2n+1$ but $d_{even}(R, H) = 2n+2$.

Recall that the definition of subring depth comes in two parts, $H^{\otimes_R n+1} \sim H^{\otimes_R n}$ if and only if (a) $H^{\otimes_R n} \mid p \cdot H^{\otimes_R n+1}$ and; (b) $H^{\otimes_R n+1} \mid q \cdot H^{\otimes_R n}$. The lemma below shows us that (a) is automatically satisfied.

Lemma 1.2.8. *For each $t \geq 1$, $H^{\otimes_R t} \mid H^{\otimes_R t+1}$ as H-R-bimodules, R-H-bimodules and R-bimodules. For the $t = 0$ case, if $H \mid q \cdot R$ as R-bimodules then $R \mid H$.*

Proof. For the final claim we refer to Lemma 1.2.10. Define maps $f : H^{\otimes_R t} \to H^{\otimes_R t+1}$ and $g : H^{\otimes_R t+1} \to H^{\otimes_R t}$ by $f(x_1 \otimes \ldots \otimes x_t) = x_1 \otimes \ldots \otimes x_t \otimes 1$ and $g(y_1 \otimes \ldots y_n \otimes y_{n+1}) = y_1 \otimes \ldots y_n y_{n+1}$. Clearly both these maps are H-R-bimodule homomorphisms, and moreover $g \circ f = id$. Therefore $H^{\otimes_R t} \mid H^{\otimes_R t+1}$ as H-R-bimodules. The R-H-bimodule case is proved using slightly different f and g. □

Corollary 1.2.9. *The following is a sufficient condition of depth $2n$ (depth $2n+1$):*

$$H^{\otimes_R n+1} \mid q \cdot H^{\otimes_R n}$$

as R-H-bimodules, H-R-bimodules (R-bimodules).

Example 2. Let k be a field and take $n, m \in \mathbb{N}$ such that $m < n$, then as in Example 1 we have the polynomial rings $R = k[X_1, \ldots, X_m]$ and $H = k[X_1, \ldots, X_n]$. This pair of rings satisfies $d(R, H) = 3$.

Without loss of generality let us consider $R := k[X]$ and $H := k[X, Y]$. First of all we ensure $d_{even}(R, H) > 2$. Recall that the right depth 2 condition is equivalent to $H \otimes_R H$ dividing a sum $q \cdot H$, as R-H-bimodules. From Example 1 that we have calculated $H \otimes_R H$ and provided an H-R-bimodule isomorphism

$$H \otimes_R H \cong k[X, Y, Y'].$$

In particular $k[X, Y, Y']$ is a free left H-module with basis $\{Y'^{\,i} \mid 1 \leq i\}$ in bijective correspondence with $\mathbb{N}$. Evidently H is also a free left H-module with basis $\{1\}$, meaning that there is no left H-module monomorphism $H \otimes_R H \to q \cdot H$. for any q. A similar argument shows us that $H \otimes_R H$ is a free left R-module with basis $\{Y^i, Y'^{\,j} \mid 1 \leq i, j\} \leftrightarrow \mathbb{N}^2$, moreover H is a free left R-module with basis $\{Y^i \mid 1 \leq i\} \leftrightarrow \mathbb{N}$. It is a well-known fact that $\mathbb{N}^2$ and $\mathbb{N}$ are in bijective correspondence [30, Apx.2, Sec.1], so we deduce that $H \otimes_R H$ and H are isomorphic free left R-modules, and therefore as R-bimodules.

Example 3. In the paper [6, Thm.6.19] the authors have discovered that for S_n the symmetric group on n letters the depth $d(\mathbb{C}S_n, \mathbb{C}S_{n+1})$ is exactly $2n - 1$. This result is extended to arbitrary fields k in [4, Prop.5.1].

1.2.2 Depth 1

The case of depth 1 is not difficult to deal with in general. There is a slight problem with depth 1 satisfying Corollary 1.2.9. The following proof, taken from [20], deals with this problem.

Lemma 1.2.10. *If $R \subseteq H$ is such that $H \mid q \cdot R$, for some $q \in \mathbb{N}$, then also as R-bimodules, $R \mid H$ and therefore $H \sim R$.*

Proof. If we start by assuming $H \mid q \cdot R$ then [20] explains how there is then the condition:

$$H \cong R \otimes_{Z(R)} H^R, \tag{1.3}$$

as R-$Z(R)$-bimodules, where $Z(R)$ is the centre of R and $H^R := \{h \in H \mid rh = hr, \forall r \in R\}$. The above equation on the elements of H tell us, restricting to H^R that as a $Z(R)$-module H^R is finitely generated projective. In particular, since $Z(R) \subseteq H^R$ is a commutative subring H^R will be a generator. (See [29, Sec.1.2]) Conveniently then we can write $Z(R)$ as a summand of $n \cdot H^R$. So follows the division $R \mid n \cdot H$, using (1.3). This implies the existence of n maps $\phi_i \in \mathrm{Hom}({}_RH_R, {}_RR_R)$ and n elements $r_i \in H^R$ such that $\sum \phi_i(r_i) = 1$. In order to simplify and write $R \mid H$ we define the function $E : R \to H$ by $E(h) = \sum \phi_i(hr_i)$. This map satisfies $E(r) = \sum \phi(rr_i) = r(\sum \phi(r_i)) = r$, so if $\iota : R \to H$ is the inclusion map, $E \circ \iota = id$ and so $R \mid H$. □

Example 4. For every ring $d(H, H) = 1$. For a finite dimensional k-algebra A $d(k, A) = 1$.

Example 5. Let Z be the centre of a ring H and let H be a Z-bimodule by multiplication. If H is a finitely generated projective left Z-module then ${}_ZH \mid q \cdot {}_ZZ$ by definition, and because Z commutes with elements of H the latter division is true as Z-bimodules. Applying Theorem 1.2.10 above we see that $Z \mid H$ too as Z-bimodules and so $H \sim Z$. Consequently $d(Z, H) = 1$.

Generally speaking for a ring extension $R \subseteq H$ where ${}_RH$ and H_R are both projective or even free, $d(R, H) > 1$ may occur. Later results show that we have $d(R, H) > 1$ for certain choices of finite dimensional Hopf algebra, even though ${}_RH$ and H_R are free by the Nichols-Zoeller theorem.

1.2.3 Finite Length Modules

Let $\mathcal{C}$ again represent one of the module categories ${}_R\mathcal{M}_S$ or ${}_H\mathcal{M}$ for rings R, S or Hopf algebra H.

Definition 1.2.11. Given a module M in $\mathcal{C}$, we say M is indecomposable if $M = M' \oplus M''$ implies either $M' = M$ or $M'' = M$.

Definition 1.2.12. A module M in $\mathcal{C}$ is called Noetherian if any ascending chain of submodules $M_1 \subseteq M_2 \subseteq \ldots$ stabilises, i.e. $M_i = M_{i+1}$ for some i. Similarly we call

M Artinian when descending chains $M_1 \supseteq M_2 \supseteq \dots$ of submodules stabilise. When a module is both Noetherian and Artianin we say that it has finite length.

Proposition 1.2.13 (Krull-Schmidt)**.** *Let M be a finite length module in $\mathcal{C}$ then there is a decomposition*

$$M \cong M_1 \oplus \dots \oplus M_r,$$

where each M_i is indecomposable. Furthermore given another decomposition into indecomposables $M \cong M_1' \oplus \dots \oplus M_t'$ then $r = t$ and there exists a permutation σ of $\{1, 2, \dots, r\}$ such that for each i, $M_{\sigma(i)} \cong M_i'$.

Definition 1.2.14. For a finite length module M let $\mathrm{Indec}(M)$ be the set of unique isomorphism classes of indecomposable submodules of M.

Lemma 1.2.15. *Given modules M, N in $\mathcal{C}$, then $M \sim N$ if and only if $Indec(M) = Indec(N)$.*

Proof. ($\Leftarrow$) First of all assume that $\mathrm{Indec}(M) = \mathrm{Indec}(N)$, then let $M \cong \bigoplus q_i \cdot M_i$ be the decomposition into non-isomorphic indecomposables with multiplicities. Since $\mathrm{Indec}(N) = \mathrm{Indec}(M)$ by assumption, $N \cong \oplus p_i \cdot M_i$ is the indecomposables decomposition of N. Take $q = max\{q_i\}$ then trivially $q_i \cdot M_i \mid q \cdot M_i \mid p_i q \cdot M_i$ for all i, and so $M \mid q \cdot N$. Similar reasoning tells us that $N \mid p \cdot M$ for $p = max\{p_i\}$. Subsequently $M \sim N$.

($\Rightarrow$) Suppose that $M \sim N$ so that for some $p, q \in \mathbb{N}$, firstly $M \mid q \cdot N$ and secondly $N \mid p \cdot M$. For the first property there must exist an M' such that $M \oplus M' \cong q \cdot N$. Decompose M, M' and N into indecomposable objects. So because of the uniqueness of indecomposables we may immediately see that $\mathrm{Indec}(M) \subseteq \mathrm{Indec}(q \cdot N)$. Again by the uniqueness of indecomposables $\mathrm{Indec}(N) =$
$\mathrm{Indec}(q \cdot N)$ and hence $\mathrm{Indec}(M) \subseteq \mathrm{Indec}(N)$. Similarly the second property will imply $\mathrm{Indec}(N) \subseteq \mathrm{Indec}(M)$. Therefore $M \sim N$ implies $\mathrm{Indec}(M) = \mathrm{Indec}(N)$. □

Lemma 1.2.16. *Let $R \subset H$ be a ring extension then for all $t \geq 1$ the following inclusion holds as H-R-bimodules, R-H-bimodules and R-bimodules: $Indec(H^{\otimes_R t}) \subseteq Indec(H^{\otimes_R t+1})$. This immediately implies the chain of bimodule inclusions:*

$$Indec(H) \subseteq Indec(H \otimes_R H) \subseteq Indec(H \otimes_R H \otimes_R H) \subseteq \dots.$$

Proof. On one hand we know from Lemma 1.2.8 that $H^{\otimes_R t} \mid H^{\otimes_R t+1}$ in general, on the other hand we know from the previous lemma that $\operatorname{Indec}(H^{\otimes_R t}) \subseteq \operatorname{Indec}(H^{\otimes_R t+1})$. $\square$

Chapter 2

Algebras

2.1 Definition

Algebras are rings with k-linear multiplication whose identity elements span a 1-dimensional k-vector space. In this thesis we will often focus on finite dimensional Hopf algebras and smash products over finite dimensional algebras, defined in Sections 3.1 and 3.3. See [14, Ch.7] for a fuller introduction to algebras and the basic results.

An algebra over a field k is a triple (A, μ, η) where A is a k-vector space and $\mu : A \otimes A \to A$ is a multiplication with unit map $\eta : k \to A$ which are both k-linear and satisfy commuting diagram conditions:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{id \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes id}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

$$\begin{array}{ccccc} & & A & & \\ & \nearrow{\scriptstyle \cong} & \uparrow{\scriptstyle \mu} & {\scriptstyle \cong}\nwarrow & \\ k \otimes A & \xrightarrow[\eta \otimes id]{} & A \otimes A & \xleftarrow[id \otimes \eta]{} & A \otimes k \end{array}$$

If A, B are algebras and $f : A \to B$ is a k-linear map, then we call f an *algebra homomorphism* when

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f\otimes f} & B \otimes B \\ \mu_A \downarrow & & \downarrow \mu_B \\ A & \xrightarrow{f} & B \end{array}$$

commutes and $f(1_A) = 1_B$.

Naturally we move on to modules over an algebra. A module M over an algebra A is a k-vector space with an action of A on M given by $m : A \otimes M \to M$ which satisfies commutativity of the following diagram:

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{id\otimes m} & A \otimes M \\ \mu\otimes id \downarrow & & \downarrow m \\ A \otimes M & \xrightarrow{m} & M \end{array}$$

There are two important open questions regarding the depth of algebra extensions. Both questions motivated much of the work in this thesis; the entire reason for considering smash product algebras later was to find examples of (2).

(1) *Does there exist a finite dimensional algebra extension with infinite depth?*

(2) *Does there exist an algebra extension with unequal left and right depth?*

2.1.1 Augmented Algebras

An *augmented algebra* is one with an algebra map $\rho : A \to k$. Given such an algebra we give some importance to the modules ${}_\rho k$ and k_ρ with multiplication $h \cdot \lambda = \rho(h)\lambda$ etc. Notice that $A \otimes_A {}_\rho k \cong {}_\rho k$ (easy exercise), for a subalgebra $B \subseteq A$ write $B^+ := ker\rho \cap B$, then

$$A \otimes_B {}_\rho k \cong A/AB^+.$$

The map $A \to A \otimes_B {}_\rho k : a \mapsto a \otimes 1$, has kernel AB^+, so we apply the isomorphism theorem for modules relating to homomorphisms and kernels [2, Cor.3.7]. The following result can be found in [20, Prop 1.3].

Proposition 2.1.1. *Suppose $B \subseteq A$ is an algebra extension such that A is augmented via ρ, if the extension has left depth 2 then $B^+A \subseteq AB^+$. Similarly, if the extension has right depth 2 then $AB^+ \subseteq B^+A$.*

Proof. Doing the left case is enough, so assume $A \otimes_B A \mid q \cdot A$ as B-A-bimodules. What we do is refer to the comment before the proposition and apply $- \otimes_A {}_\rho k$ to both sides:

$$A/AB^+ \mid q \cdot {}_\rho k$$

as left B-modules. Now the annihilator of ${}_\rho k$ is B^+ also annihilates A/AB^+ implying $B^+A \subseteq AB^+$ and we are done.

□

2.1.2 Finite Representation Type

The following concept has been well-discussed in [3], all related results and examples except those on depth come from this book. Representations (indecomposable modules) and depth have a close relationship, as we have already begun to understand in Subsection 1.2.3. Given an algebra A an *indecomposable* A-module M is one which cannot be written $M = M_1 \oplus M_2$ for two non-trivial A-submodules $M_1, M_2 \subseteq M$.

Definition 2.1.2. We say that an algebra A has finite representation type, f.r.t for short, when there are only finitely many isomorphism classes of indecomposables in ${}_H\mathcal{M}$ and $\mathcal{M}_H$. In any other case the algebra is said to have infinite representation type, i.r.t.

Given an algebra A and an A-module M a *composition series* for M is a chain of submodules $M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n$ such that M_i is a maximal submodule of M_{i+1}. We call A a *Nakayama algebra* if each indecomposable projective left or right module P has only one composition series. The proof of the result below can be found in [3, VI. Thm.2.1], and is proved using various detailed results which we do not treat here.

Proposition 2.1.3. *All Nakayama algebras have f.r.t.*

The classic example of an algebra with f.r.t is a finite dimensional semisimple algebra (defined later), but there are Nakayama algebras which are not semisimple. For example take k a field of characteristic p and G a cyclic p-group with order p^n, then $kG \cong k[X]/\langle X^{p^n} \rangle$ is Nakayama (and therefore has f.r.t). That this algebra is not semisimple can be seen from the existence of a non-trivial nilpotent ideal.

Let A^{op} be the *opposite algebra* of A, with multiplication $a \cdot_{op} b = ba$. Another way of talking about A-bimodules is as left $A \otimes A^{op}$-modules, for they are exactly the same thing: suppose M is an A-bimodule then define an action of $A \otimes A^{op}$ by $(a \otimes b) \cdot m = a \cdot m \cdot b$.

Proposition 2.1.4 ([50], Prop.2.1). *Suppose that $R \subseteq H$ is an extension of finite dimensional algebras where any of $R \otimes R^{op}$, $H \otimes R^{op}$, $R \otimes H^{op}$ or $H \otimes H^{op}$ have f.r.t then $d(R, H) < \infty$.*

Proof. Without loss of generality suppose that $R \otimes R^{op}$ has f.r.t. Recall Prop 1.2.16 which proves the inclusion $\text{Indec}(H^{\otimes_R t}) \subseteq \text{Indec}(H^{\otimes_R t+1})$ for all $t \in \mathbb{N}$, in this case we consider this inclusion over the category of R-bimodules (equivalently left $R \otimes R^{op}$-modules). Since there are finitely many isomorphism classes of indecomposables R-bimodules we deduce that $\text{Indec}(H^{\otimes_R n}) = \text{Indec}(H^{\otimes_R n+1})$ for some sufficiently large n. Therefore by Lemma 1.2.15, as R-bimodules $H^{\otimes_R n+1} \sim H^{\otimes_R n}$, the condition of depth $2n + 1$. □

Corollary 2.1.5. *Suppose that $R \subseteq H$ is an extension of finite dimensional algebras and let R lie in the centre of H. If R has f.r.t then $R \subseteq H$ has finite depth.*

Proof. Since R lies in the centre of H it commutes with every element of H, in particular it is a commutative algebra. As in the proof of Proposition 2.1.4 we consider the R-bimodules $H^{\otimes_R t}$ for increasing powers. Take an element $h_1 \otimes h_2 \otimes \ldots \otimes h_t \in H^{\otimes_R t}$ and $r \in R$, notice that

$$\begin{aligned} r(h_1 \otimes h_2 \otimes \ldots \otimes h_t) &= rh_1 \otimes h_2 \otimes \ldots \otimes h_t \\ &= h_1 r \otimes h_2 \otimes \ldots \otimes h_t \\ &= h_1 \otimes rh_2 \otimes \ldots \otimes h_t \\ &= \ldots \\ &= h_1 \otimes h_2 \otimes \ldots \otimes rh_t \\ &= (h_1 \otimes h_2 \otimes \ldots \otimes h_t)r. \end{aligned}$$

By the above property the left (or right) multiplicative R-modules $H^{\otimes_R t}$ will naturally be R-bimodules with their usual depth theory structure, and vice-versa. In particular because R has f.r.t the indecomposables of $H^{\otimes_R t}$ will stabilise as R-modules for increasing t and this is true as R-bimodules as well. For this reason depth will be finite. □

The notion of representation type fails to fully capture the notion of algebra depth. By [3, pp.111] the commutative algebra $k[X, Y]/\langle X, Y\rangle^2$ has i.r.t but we may show that $k[X, Y]/\langle X, Y\rangle^2 \subseteq k[X, Y, Z]/\langle X, Y\rangle^2$ has depth 3. (To prove this we can adapt the proof of Example 2: $k[X,Y,Z]/\langle X,Y\rangle^2$ as a left and right $k[X,Y]/\langle X,Y\rangle^2$-module has generators $\{1, Z^r \mid r \in \mathbb{N}\}$, in bijective correspondence with $\mathbb{N}$.)

2.2 Modules over a Finite Dimensional Algebra

Module decompositions over finite dimensional algebras are covered in numerous texts. I give a brief and dense treatment here in order to use the results in Chapter 4. Refer to [12, Chap.4] or [8, Sec.3.4] for a more systematic account of the results below.

Definition 2.2.1. Given an element of a ring $e \in H$ we call this element an idempotent when $e^2 = e$. An idempotent e is called *central* when e is in the centre of A. A pair of idempotents $e, f \in A$ are called *orthogonal* when $ef = fe = 0$. An idempotent e is *primitive* if when f, g are two orthogonal idempotents such that $e = f + g$ then $f = 0$ or $g = 0$.

Proposition 2.2.2 (Krull-Schmidt for Algebras)**.** *Every finite dimensional module M has finite length and therefore the Krull-Schmidt theorem for rings applies: $M \cong M_1 \oplus \ldots \oplus M_n$ for indecomposables M_i and $i \geq 1$. This decomposition is unique up to permutation.*

The following two results appear as [[8], Thm.1.7.2, Cor.1.7.1].

Proposition 2.2.3. *Given an A-module M then there is a bijective correspondence between decompositions of M into direct sums of submodules and sums of $id_A \in End({}_AM)$ into pairwise orthogonal idempotents.*

We say that an algebra is *local* when it contains a unique maximal left ideal. Notice that in a local algebra A the unique maximal left ideal will be the Jacobson radical (defined below) we deduce that this left ideal is a right ideal and is a two-sided ideal. Indeed we may have equivalently defined a local algebra to be one with a unique maximal right ideal.

Corollary 2.2.4. *A finite dimensional A-module M is indecomposable if and only if $End({}_AM)$ contains no non-trivial idempotents, in other words $End({}_AM)$ will be a local algebra.*

Proof. If $e \in End({}_AM)$ is a non-trivial idempotent then $e, (id_A - e)$ form a decomposition of id_A into orthogonal idempotents. $\square$

2.2.1 Semisimple Modules

Call a module S *simple* when it contains no non-trivial submodules $T \subsetneq S$. Simple modules are indecomposable, but not the other way around. We call a module M *semisimple* if it can be written $M = \oplus_{i \in I} S_i$ where each S_i is simple. An algebra is called semisimple when all of its modules are semisimple. This has many equivalences, and modules over semisimple algebras have a somewhat well-behaved structure. Indeed, without going into too many details in [6] the authors prove that finite dimensional semisimple (Hopf) algebra extensions $R \subseteq H$ always have finite depth.

Remark 2.2.5. An equivalent definition of a simple left (right) module is one isomorphic to $A/\mathcal{M}$ where $\mathcal{M}$ is a maximal left (right) ideal of A. For if S is simple then it is cyclic i.e. $S = R \cdot x$ for some (indeed all non-zero) $x \in S$, then set $\mathcal{M} := \text{Ann}(x)$. The isomorphism theorem for modules tell us $\text{Ann}(x)$ is maximal.

Definition 2.2.6. A strongly nilpotent element $e \in A$ is such that for a particular $m \in \mathbb{N}$ and any m elements $x_1, \ \ldots, \ x_m \in A$ we have $ex_1ex_2 \ldots ex_n = 0$.

Lemma 2.2.7. *An algebra is semisimple if and only if it contains no non-zero strongly nilpotent elements.*

Given an A-module M we define the *radical* of M to be

$$rad(M) := \bigcap \{K \subseteq A \mid K \text{ maximal submodule}\}.$$

Naturally extending we define the *Jacobson radical* of A to be $rad(A) := rad({}_AA)$, the radical of A as a left A-module. There is the following famous lemma to put minds at ease:

Lemma 2.2.8. *The radicals of ${}_AA$ and A_A are both equal. Therefore we may have equivalently defined the Jacobson radical using A_A.*

Lemma 2.2.9. *If A is semisimple then rad(A) $= 0$. If A is Artinian then it is semisimple if and only if rad(A) $= 0$.*

Lemma 2.2.10. *The radical rad(A) consists of elements $x \in A$ such that $1 - ax$ is a unit for all $a \in A$.*

Theorem 2.2.11 (Artin-Wedderburn Decomposition)**.** *Given a finite dimensional semisimple k-algebra A we may decompose it into matrix algebras*

$$A \cong M_{i_1}(D_1) \oplus \ldots \oplus M_{i_t}(D_t),$$

where $D_1, \ldots, D_t$ are all division algebras over k.

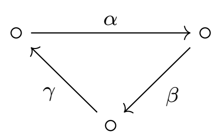
2.3 Path Algebras of Quivers

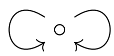
The depth theoretic results of this section appeared in a joint paper [50], where L. Kadison worked out the details of the examples personally. Introductory ideas on quivers are fully explored in the book dedicated to the subject [12].

In this section the notion of a quiver is explained, from the quiver we construct a special algebra called the path algebra. Quivers and path algebras have some nice applications to depth theory, in particular giving us the two main examples of this section. One of the interesting points of using quivers with depth theory is that they allow us to consider matrix algebras and find the depths of certain matrix subalgebras. Working with matrices and depth is generally complicated - one may attempt to compute the depth of some subalgebras in $M_n(A)$ and this becomes clear.

Definition 2.3.1. A quiver $Q = (V, A, s, t)$ is a set of vertices V and arrows A with $s, t : E \to V$ known as the source and target maps. If $\alpha \in A$ and $s(\alpha) = a$ and $t(\alpha) = b$ then we might better literally draw an arrow $a \xrightarrow{\alpha} b$.

A quiver Q is *finite* when $|V|$ and $|A|$ are finite. It is often easier to just reference a finite quiver by a picture, without even labeling the vertices or arrows. A few examples follow:





Definition 2.3.2. For a quiver Q a path of length l around the quiver is a series of arrows $(a|\alpha_1, \ldots, \alpha_l|b)$ such that $t(\alpha_i) = s(\alpha_{i+1})$ $(1 \leq i < l)$.

For each vertex $a \in V$ there is the trivial length 0 path $(a||a)$, usually denoted by ϵ_a. A cycle in Q is a path $(a|\ldots|a)$ of length ≥ 1. If the quiver has no cycles then it is called acyclic.

A connected quiver is one such that the underlying graph is connected. We understand this as follows: if there is either a path $a \to b$ or $b \to a$ then we say a and b

are related vertices. Generate an equivalence relation by this relation, then we say the quiver is connected if every vertex lies in the equivalence class. In the remainder of this section we will focus entirely on connected quivers.

Definition 2.3.3. A subquiver $Q' = (V', A', s', t')$ of Q is a quiver in its own right such that $V' \subseteq V$, $A' \subseteq A$ and $s' = s|_{A'}$, $t' = t|_{A'}$ where applicable.

Given a finite quiver Q define its *path algebra* kQ over field k by first writing kQ_l for the k-span of paths of length l and summing:

$$kQ := \bigoplus_{i=0}^{\infty} kQ_i.$$

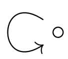
This takes care of the underlying set but multiplication is easily defined by concatenation, given $\alpha = (a|\alpha_1, \ldots, \alpha_l|b)$ and $\beta = (c|\beta_1, \ldots, \beta_p|d)$ write

$$\alpha\beta = \begin{cases} (a|\alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_p|d) & b = c \\ 0 & b \neq c \end{cases}$$

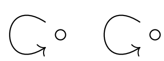
By this definition kQ is an $\mathbb{N}$-graded k-algebra. We need to take Q finite because otherwise we won't have an identity element according to this definition. The identity as it stands will be $\sum_{a \in V} \epsilon_a$.

Example 6. The path algebras of the quivers

Q



Q'



are $k[X]$ and $k[X, Y]/\langle XY \rangle$ respectively, both being commutative.

2.3.1 Depth of the Diagonal Subalgebra

Suppose that we have a finite connected acyclic quiver Q with n vertices, then there is always a way of labeling the vertices $1, 2, \ldots, n$ such that if there exists an arrow $i \to j$ then $i > j$ ([40], Cor 8.6). With such a labeling of vertices notice that $\epsilon_i kQ \epsilon_j$ denotes the k-linear span of paths $i \to j$, call this set $Q_{i,j}$, then $Q_{i,j} Q_{j,k} \subseteq Q_{i,k}$. In particular $Q_{i,i} = k\epsilon_i \cong k$ because Q is acyclic.

Lemma 2.3.4 ([12], 1.12)**.** *Let Q be a finite connected, acyclic quiver as above then kQ is isomorphic to a lower triangular matrix algebra*

$$\begin{pmatrix} Q_{1,1} & 0 & \cdots & 0 \\ Q_{2,1} & Q_{2,2} & \cdots & 0 \\ & & \ddots & \\ Q_{n,1} & Q_{n,2} & \cdots & Q_{n,n} \end{pmatrix}$$

This result is intuitively clear since $kQ = \bigoplus Q_{i,j}$, the trickiest part is showing that the matrix multiplication is the same as the algebra multiplication. Notice that for all $i \in V$ $Q_{i,i} \cong k$, because there are no loops, that means we are writing the algebra as:

$$\begin{pmatrix} k & 0 & \cdots & 0 \\ Q_{2,1} & k & \cdots & 0 \\ & & \ddots & \\ Q_{n,1} & Q_{n,2} & \cdots & k \end{pmatrix}$$

Example 7 (Kronecker Algebra)**.** Let Q be the quiver:

$$\circ \underset{\beta}{\overset{\alpha}{\leftleftarrows}} \circ$$

This is known as the *Kronecker quiver*, naturally we then call kQ the *Kronecker algebra.* Now because this quiver is connected, finite and acyclic we may apply the theorem above, noting that $Q_{2,1} \cong k^2$:

$$kQ \cong \begin{pmatrix} k & 0 \\ k^2 & k \end{pmatrix}.$$

In [3, VIII.7] the Kronecker algebra is shown to have infinite representation type.

A path algebra $A = kQ$ over a finite, connected, acyclic quiver will be augmented n times, with $\rho_i : A \to k$ given by $x \mapsto \epsilon_i x \epsilon_i$, which is sending the matrix of x to the $(i,i)^{th}$ entry. Write ${}_{\rho_i}k_{\rho_j}$ for the simple A-bimodule such that $a \cdot 1 \cdot d = \rho_i(a)\rho_j(d)1$. Think back to Proposition 2.1.1, which tells us that if an augmented algebra (A, ρ) has a subalgebra B with left depth 2 then $B^+A \subseteq AB^+$. We use this fact to prove a result on depth.

Theorem 2.3.5 ([50], Thm.4.2)**.** *Consider $A = kQ$ as a triangular matrix algebra of a finite, connected, acyclic quiver Q with $|V| > 1$. Then the subalgebra $\bigoplus^n k$, representing the main diagonal, has depth* 3 *in A.*

Proof. Write $B := \oplus^n k (= \oplus_{i=1}^n \epsilon_i A \epsilon_i)$, we show that $B \subseteq A$ doesn't have left depth 2, this in turn shows that depth 2 is not a possibility. Write $B_i^+ := B \cap ker\rho_i$, then it is sufficient to show that $B_i^+ A \not\subseteq AB_i^+$. Note that AB_i^+ are the matrices with the i^{th} column entries all zero; similarly $B_i^+ A$ are the matrices with zero i^{th} row entries. For $j > i$ the matrices in $Q_{j,i}$ have i^{th} row zero, so by our assumption the i^{th} column is zero and indeed $Q_{j,i} = 0$. This is a contradiction because $Q_{j,i}$ are linear spans of all paths $j \to i$ and we are saying there are no non-trivial arrows, but A is connected.

We show now the depth 3 condition being satisfied. Let $n_{ij} = dim(Q_{i,j})$ then $Q_{i,j}$ are closed under left and right B multiplication, indeed since B is the diagonal matrix algebra $Q_{i,j} \cong n_{ij} \cdot {}_{\rho_i}k_{\rho_j}$ as B-bimodules. Thus we have the expression

$$A = \bigoplus_{i \geq j} Q_{i,j} \cong \bigoplus_{i \geq j} n_{ij} \cdot {}_{\rho_i}k_{\rho_j}.$$

Notice that we may write $Q_{i,j} = Q_{i,j}Q_{j,j}$ for all i, j, so now as B-bimodules

$$\begin{aligned} A \otimes_B A = \bigoplus_{i,j=1}^{n} \bigoplus_{i \geq k \geq j}^{n} Q_{i,k} \otimes_B Q_{k,j} \\ \cong \bigoplus_{i \geq j} m_{ij} \cdot {}_{\rho_i}k_{\rho_j}, \end{aligned}$$

where $m_{ij} = \sum_{i \geq k \geq j} n_{ik} n_{kj}$. Note that if $m_{ij} \neq 0$ then $n_{ik}n_{kj} \neq 0$ for some k, and this implies there will be at least one path $i \to j$, so that $n_{ij} \neq 0$ too. Comparing the two expressions shows us that both $A \otimes_B A$ and A have the same indecomposables (which are the simple bimodules ${}_{\rho_i}k_{\rho_j}$) and so $A \otimes_B A \sim A$ as B-bimodules. □

2.3.2 Depth of the Arrow Subalgebra

Let A be the path algebra of a finite (with n vertices), acyclic, connected quiver, we calculate its Jacobson radical. Let A_t be the linear span of paths in Q of length $t \in \mathbb{N}$. (Basic graph theory tells us that if Q is finite and acyclic then for a certain number N the set $A_t = 0$ for $t \geq N$. This is because we run out of vertices to join with paths.)

Lemma 2.3.6 ([12], II.1.10)**.** *Let N be the maximal length of a path in Q then the radical $radA$ is equal to $A_1 \oplus A_2 \oplus \ldots \oplus A_N$.*

Consider the algebra $B := k1 \oplus A_1 \oplus A_2 \oplus \cdots \oplus A_N = k1 \oplus radA$, we call this the *primary arrow subalgebra*. Since $\epsilon_i \notin B$ for all i the only path of length zero contained is 0, we conclude that $radA \subset B$ is the unique maximal left, right and two-sided ideal. It is easy to deduce the isomorphism $B/radB = B/radA \cong k1$. We may conclude that B has one unique simple right module by Remark 2.2.5, which we denote by k_ρ. As in the discussion before Theorem 2.3.5 A has n augmentations, allowing us to define n simple A-B-bimodules ${}_{\rho_i}k_\rho$, $1 \leq i \leq n$. (It is also clear that ${}_\rho k_\rho$ is a simple B-bimodule.)

Lemma 2.3.7. *A is an indecomposable B-bimodule.*

Proof. By Lemma 2.2.4 it is enough to show that $End_B A_B$ is a local ring. Let $F \in End_B A_B$ and let $I = \langle \epsilon_1, \ldots, \epsilon_n, \alpha_1, \ldots, \alpha_m \rangle$ be an ordered basis of A such that lengths of successive basis elements are ascending, i.e. length$(\alpha_i) \leq$ length(α_{i+1}), all $1 \leq i \leq m-1$. Put F into matrix form with respect to I, $(M^\alpha_\beta)_{\alpha,\beta \in I}$ so that $F(\alpha) = \sum_{\beta \in I} M^\alpha_\beta \beta$ for coefficients $M^\alpha_\beta \in k$, for any $\alpha \in A$. If α is not a basis element sum the matrices of its basis decomposition.

Given a path $\alpha : i \to j$ of length $r \geq 1$, note that $\alpha \in B$ and so

$$F(\alpha) = \alpha F(\epsilon_j) = F(\epsilon_i)\alpha,$$

so we may write

$$\sum_{\beta \in I} M^\alpha_\beta \beta = \sum_{\gamma \in I} M^{\epsilon_j}_\gamma \alpha\gamma = \sum_{\delta \in I} M^{\epsilon_i}_\delta \delta\alpha.$$

Recall that Q has no non-trivial cycles, so when comparing the last equality of the above it follows that $M^{\epsilon_j}_\gamma = 0$ for paths $\gamma : j \to k$ and $M^{\epsilon_i}_\delta = 0$ for all paths $\delta : \ell \to i$. In other words $F(\alpha) = M^{\epsilon_i}_{\epsilon_i}\alpha = M^{\epsilon_j}_{\epsilon_j}\alpha$, and this applies to basis elements $\alpha_1, \ldots, \alpha_m$ so that the matrix of F has a constant diagonal on those elements.

Now for each $\epsilon_i \notin B$ we analyse the action of F. Suppose that $i < j$, take $\alpha : k \to j$ to be a path for any vertex k. Notice that $\alpha F(\epsilon_i) = F(\alpha\epsilon_i) = 0$, so that $\sum_{\beta \in I} M^{\epsilon_i}_\beta \alpha\beta = 0$ and therefore $M^{\epsilon_i}_\beta = 0$ whenever $s(\beta) = j$. In particular, $M^{\epsilon_i}_{\epsilon_j} = 0$. It follows that the set of $F \in End_B A_B$ has the form of a triangular matrix algebra with constant diagonal, like B, and is a local algebra. □

Remark 2.3.8. The k-algebra of upper triangular matrices with constant diagonals is a local algebra. An element in this algebra is invertible if and only if the diagonal entries are non-zero. We can explicitly view the inverses in the 2×2 matrices case:

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix},$$

such that $a \neq 0$, b is arbitrary, will have inverse

$$\begin{pmatrix} a^{-1} & -ba^{-2} \\ 0 & a^{-1} \end{pmatrix}.$$

In particular this means that the subset of upper triangular matrices with zero diagonals form the unique maximal ideal, because any element not belonging to this ideal will be invertible.

Theorem 2.3.9 ([50], Thm.5.2)**.** *The depth of the primary arrow subalgebra B in the path algebra A defined above is $d(B, A) = 4$.*

Proof. We first compute $A \otimes_B A$ and show $d(B, A) > 3$. Note that two paths of nonzero length, α, β where $s(\alpha) = i$ satisfy $\alpha \otimes_B \beta = \epsilon_i \otimes_B \alpha\beta$, which is zero unless $t(\alpha) = s(\beta)$. It follows that

$$A \otimes_B A = \oplus_{i=1}^{n} k\epsilon_i \otimes_B \epsilon_i \oplus_{i=2}^{n} \oplus_{j=1}^{i-1} \epsilon_i \otimes_B \epsilon_i A \epsilon_j \oplus_{i \neq j} k\epsilon_i \otimes_B \epsilon_j.$$

It is obvious that the first two summations above are isomorphic as B-B-bimodules to ${}_BA_B$. Note that when $i \neq j$, for all paths α, β,

$$\alpha\epsilon_i \otimes_B \epsilon_j = 0 = \epsilon_i \otimes_B \epsilon_j \beta$$

since $\alpha\epsilon_i \in B$ is either zero or a path ending at i, whence $\alpha\epsilon_i\epsilon_j = 0$. It follows that $A \otimes_B A \cong A \oplus n(n-1)_\rho k_\rho$ as B-B-bimodules; moreover, as A-B-bimodules, we note for later reference

$$ {}_AA \otimes_B A_B \cong {}_AA_B \oplus \oplus_{i=1}^{n} (n-1)_{\rho_i} k_\rho \tag{2.1}$$

By Lemma 2.3.7, ${}_BA_B$ is an indecomposable, but the B-B-bimodule $A \otimes_B A$ contains another nonisomorphic indecomposable, in fact ${}_\rho k_\rho$, so that as B-bimodules, $A \otimes_B A \not\sim q \cdot A$ for any multiple q by Krull-Schmidt decomposition.

Now we establish that the subalgebra $B \subseteq A$ has right depth 4 by comparing (2.1) with the computation below:

$$A \otimes_B A \otimes_B A = \oplus_{i=1}^{n} \ k\epsilon_i \otimes \epsilon_i \otimes \epsilon_i \oplus_{i=2}^{n} \oplus_{j=1}^{i-1} \ \epsilon_i \otimes \epsilon_i \otimes \epsilon_i A \epsilon_j \oplus_{i \neq j \neq k} \ k\epsilon_i \otimes \epsilon_j \otimes \epsilon_k$$

$$\cong A \oplus (n^2 - 1)_{\rho_1} k_\rho \oplus \cdots \oplus (n^2 - 1)_{\rho_n} k_\rho$$

as A-B-bimodules, where $i \neq j \neq k$ symbolizes $i \neq j$, $j \neq k$ or $i \neq k$. It is clear that since no new bimodules appear in a decomposition of ${}_AA \otimes_B A \otimes_B A_B$ as compared with ${}_AA \otimes_B A_B$, that there is $q \in \mathbb{N}$ (in fact $q = n + 1$ will do) such that $A \otimes_B A \otimes_B A \mid q \cdot A \otimes_B A$ as A-B-bimodules. It follows that the minimum depth

$d(B, A) = 4$. □

2.4 Morphisms which Preserve Depth

Let A and B be algebras, a *short exact sequence* of A-B-bimodules is a pair of A-B-bimodule homomorphisms $f : M' \to M$ and $g : M \to M''$ such that f is a monomorphism ($ker f = 0$), $im f = ker g$ and g is an epimorphism ($img = M''$). We often put this information in a diagram:

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \ ,$$

where the rule is *the image of every map is equal to the kernel of the proceeding one*. Now we may define a *right exact sequence* to be a pair of A-B-bimodule homomorphisms $f : M' \to M$ and $g : M \to M''$ such that $im f = ker g$ and g is an epimorphism. Graphically displayed as before:

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \ .$$

Definition 2.4.1. Given algebras A, B, C, D and a functor $F : {}_A\mathcal{M}_B \to {}_C\mathcal{M}_D$ we say that F is exact when it preserves exact sequences. This means that given an exact sequence of A-B-bimodules

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0 \ ,$$

then the below sequence of C-D-bimodules is exact

$$0 \longrightarrow FM' \xrightarrow{Ff} FM \xrightarrow{Fg} FM'' \longrightarrow 0 \ .$$

Moreover we call F right exact when it sends every right exact sequence to a right exact sequence.

The following lemma is well-known and has proofs in numerous textbooks, in particular [15, Thm.3.15].

Lemma 2.4.2 (Tensor Products are Right Exact)**.** *Let M be an A-B-bimodule then the functor $- \otimes_B N : {}_A\mathcal{M}_B \to {}_A\mathcal{M}_C$ sending a bimodule X to $M \otimes_B X$ is right*

exact. Similarly, given a B-C-bimodule N the functor
$M \otimes_B - : {}_B\mathcal{M}_C \to {}_A\mathcal{M}_C$ *is right exact.*

One consequence of the above result is that $ker(id \otimes g) = id \otimes imf = id \otimes kerg$. For the purposes of the main results in this section we require a better estimate of the kernel of a tensor product of maps:

Lemma 2.4.3. *Let $M \xrightarrow{\gamma} M'$ be an A-B-bimodule epimorphism and let $N \xrightarrow{\delta} N'$ be a B-C-bimodule epimorphism. Then $\gamma \otimes \delta : M \otimes_B N \to M' \otimes_B N'$ is an epimorphism, moreover $ker(\gamma \otimes \delta)$ is generated by elements of the form $x \otimes y$ where either $x \in ker\gamma$ or $y \in ker\delta$.*

Proof. Firstly let U be the submodule of $M \otimes N$ generated by elements of the form $x \otimes y$ where $x \in ker\gamma$ or $y \in ker\delta$. When $\sum x_i \otimes y_i \in U$ then clearly $(\gamma \otimes \delta)(\sum x_i \otimes y_i) = 0$ and therefore $U \subseteq ker(\gamma \otimes \delta)$. This allows us to define a bimodule map

$$\Phi : M \otimes_B N/U \to M' \otimes_B N'$$

defined by $\Phi(x \otimes y + U) = \gamma(x) \otimes \delta(y)$. Showing that this is an isomorphism proves the lemma. Now the fact that γ and δ are surjective allows us to write any element in $M' \times N'$ as a sum of elements of the form $(\gamma(m), \delta(n))$, $m \in M, n \in N$. We define an bimodule map $\Psi_0 : M' \times N' \to M \otimes_B N/U$ by $\Psi_0(\gamma(m), \delta(n)) = m \otimes n + U$. This map is well-defined: let $\gamma(m_1) = \gamma(m_2)$ and $\delta(n_1) = \delta(n_2)$ so $m_1 = m_2 + \ell$ and $n_1 = n_2 + k$, for $\ell \in ker\gamma,\ k \in ker\delta$, thus

$$\begin{aligned}\Psi_0(m_1, n_1) &= m_1 \otimes n_1 + U = (m_2 + \ell) \otimes (n_2 + k) \\ &= m_2 \otimes n_2 + \ell \otimes n_2 + m_2 \otimes k + \ell \otimes k \\ &= m_2 \otimes n_2 + U.\end{aligned}$$

Notice that $(M' \times N', \Psi_0)$ is a balanced product according to Definition 1.1.2, therefore there is a bimodule map $\Psi : M' \otimes N' \to M \otimes N/U$ given by $\gamma(m) \otimes \delta(n) = m \otimes n + U$. Evidently $\Psi \circ \Phi = id$ and $\Phi \circ \Psi = id$ and we are done. □

2.4.1 Kernel Condition

Let $R \subseteq H$ be an extension of algebras and let $\theta : H \to L$ be a homomorphism of algebras. We will assume that θ is surjective, and also we shall write $\theta(R)$ as R to cut down on notation later.

Definition 2.4.4. We say that the homomorphism $\theta : H \to L$ preserves the depth of $R \subseteq H$ when $d(R, L) \leq d(R, H)$.

Notice that we may give L an H-bimodule structure by θ pullback:

$$h \cdot x \cdot h' = \theta(h)x\theta(h').$$

In particular L can be an R- or R-H- or H-R-bimodule by restricting the above. This immediately tells us that we can form tensor products $L \otimes_R L$ and indeed $L^{\otimes_R n}$ for any $n \in \mathbb{N}$. All such tensor products will be bimodules. Throughout the section the following notation is used:

$$\theta^{\otimes n} : H^{\otimes_R n} \to L^{\otimes_R n},$$

defined in the obvious way $\theta^{\otimes n}(x \otimes y \otimes \ldots \otimes z) = \theta(x) \otimes \theta(y) \otimes \ldots \otimes \theta(z)$. We will write $q \cdot \theta_n = \oplus^q \theta_n$ to denote the canonical morphism $q \cdot H^{\otimes_R n} \to q \cdot L^{\otimes_R n}$. All of the maps defined above are H-bimodule morphisms.

Suppose the minimum depth is $d(R, H) = 2n$ or $2n + 1$, with the appropriate bimodule conditions in each case. By Lemmas 1.2.3 and 1.2.8, $H^{\otimes_R n+1} \sim H^{\otimes_R n}$ is equivalent to finding two bimodule maps as in the diagram:

$$id \circlearrowleft H^{\otimes_R n+1} \underset{f}{\overset{g}{\rightleftarrows}} q \cdot H^{\otimes_R n}\,. \tag{2.2}$$

Notice that neither f nor g are necessarily unique. In the next result we ask that one particular pair be found and satisfy the property:

Theorem 2.4.5. *Let f, g be as in (2.2) and assume the condition*

$$f(ker\theta^{\otimes n+1}) \subseteq q \cdot ker(\theta^{\otimes n}) \text{ and } ker\theta^{\otimes n+1} \supseteq g(q \cdot ker(\theta^{\otimes n}))$$

holds, then θ preserves depth.

Before proving the theorem let us clarify the core idea of the proof. Assume that we are in the depth 2 or 3 case so that $H \otimes_R H \sim H$, then we would like to define homomorphisms as in the diagram:

$$\begin{array}{ccc} H\otimes_R H & \underset{f}{\overset{g}{\leftrightarrows}} & q\cdot H \\ {\scriptstyle\theta\otimes\theta}\downarrow & & \downarrow{\scriptstyle q\cdot\theta} \\ L\otimes_R L & \underset{\overline{f}}{\overset{\overline{g}}{\leftrightarrows}} & q\cdot L \end{array},$$

such that $\overline{g}\circ\overline{f} = id$. This would prove that $d(R, L) \leq 2$ (or ≤ 3). Define $\overline{f}$ by lifting each element in $L\otimes_R L$ by $(\theta\otimes\theta)^{-1}$ and acting on the resulting set by f, and finally acting by $q\cdot\theta$. Defining $\overline{g}$ is the same but in the opposite direction. As it turns out these maps can be well-defined, as long as the kernel condition of the theorem is satisfied.

Proof of 2.4.5. Assume without loss of generality that $d(R, H) = 2n+1$. (The even case is almost identical.) The core idea of the proof is to produce two R-bimodule maps as in the diagram

$$\overset{id}{\circlearrowleft}\; L^{\otimes_R n+1} \underset{\overline{f}}{\overset{\overline{g}}{\leftrightarrows}} q\cdot L^{\otimes_R n}\ .$$

This is in line with the remark before the proof deploying Lemmas 1.2.3 and 1.2.8. Take $y \in L^{\otimes_R n+1}$ then $y = \theta^{\otimes n+1}(x)$, where $x \in H^{\otimes_R n+1}$. Now define $\overline{f}(y) = (q\cdot\theta^{\otimes n})f(x)$. Similarly if $(y_i) \in q\cdot L^{\otimes_R n}$ then $(y_i) = (q\cdot\theta^{\otimes n})(x_i)$, define $\overline{g}(y_i) = (\theta^{\otimes n+1})g(x_i)$. If both maps are well-defined then $\overline{g}\circ\overline{f} = id$ because

$$(\overline{g}\circ\overline{f})(y) = \overline{g}(q\cdot\theta_n(f(x))) = \theta^{\otimes n+1}(g\circ f(x)) = y.$$

We must prove that our maps $\overline{f}$ and $\overline{g}$ are well-defined. We prove the case of $\overline{f}$ because the idea for $\overline{g}$ is clear after that. It goes without saying that $\theta^{\otimes n+1}(x) = \theta^{\otimes n+1}(x')$ if and only if $x - x' \in \ker\theta^{\otimes n+1}$. Indeed write $y = \theta^{\otimes n+1}(x)$, $y' = \theta^{\otimes n+1}(x')$ and assume that $y = y'$, thus

$$\begin{aligned} \overline{f}(y) - \overline{f}(y') &= \overline{f}(\theta^{\otimes n+1}(x)) - \overline{f}(\theta^{\otimes n+1}(x')) \\ &= (q\cdot\theta^{\otimes n})f(x) - (q\cdot\theta^{\otimes n})f(x') \\ &= (q\cdot\theta^{\otimes n})(f(x) - f(x')). \end{aligned}$$

Therefore $\overline{f}(y) = \overline{f}(y')$ if and only if

$$f(ker\theta_{n+1}) \subseteq ker(q \cdot \theta_n) = q \cdot ker\theta_n.$$

$\square$

Theorem 2.4.6. *Given an H-ideal $I \subseteq R$ then write $\pi : H \to H/I$ for the canonical H-bimodule quotient map. Then π preserves depth i.e.*

$$d(R/I,\ H/I) \leq d(R, H).$$

Proof. Since $I \subseteq R$, take $x \in I = ker\pi$ so $xy,\ yx \in I$ for any $y \in H$. Now $x \otimes y = 1 \otimes xy \in 1 \otimes I$ for any $y \in H$. Similarly $y \otimes x \in I \otimes 1$, and indeed $I \otimes 1 = 1 \otimes I = (1 \otimes 1)I$. Therefore $ker\pi^{\otimes_R n} \subseteq ker\pi(1 \otimes \ldots \otimes 1)$. Now we apply Theorem 2.4.5 where $f((1 \otimes \ldots \otimes 1)ker\pi) = f(1 \otimes \ldots \otimes 1)ker\pi \subseteq q \cdot \ker \pi^{\otimes_R n+1}$ holds, and similarly for g, and thus π preserves depth. $\square$

Corollary 2.4.7. *If $I_1 \subseteq I_2 \subseteq \ldots$ is a chain of H-ideals contained in R then*

$$1 \leq \ldots \leq d(R/I_2,\ H/I_2) \leq d(R/I_1,\ H/I_1) \leq d(R, H).$$

Proof. Use the isomorphism theorem for modules (see [2, Cor.3.7]) which tells us that $(H/I_r)/(I_{r+1}/I_r)$ is isomorphic to H/I_{r+1}. $\square$

2.5 Groups and Depth

Groups have quite a prominent place in depth theory. Group theory is rich in examples which inform us about our theories, In particular, later we will talk about Hopf algebras of finite dimension, group algebras being well-documented examples of Hopf algebras.

Let us follow the definition of a group as in [13]. A group is a quadruple (G, $\circ$, e, $^{-1}$) such that $\circ$ is an associative action, e is an identity for this action, and $^{-1}$ is an inverse for this action. Given a ring R and a group G the *group ring* RG has underlying set formal pairs $\{rg \mid r \in R,\ g \in G\}$ with formal addition and multiplication defined by $(rg)(r'g') = (rr')(gg')$. When R is a k-algebra we call RG the *group algebra* with respect to R and G.

Definition 2.5.1. We say that a set S is a *left G-set* when there is a set map

$G \times S \to S$, with action denoted by $\cdot$, such that $(gh) \cdot s = g \cdot (h \cdot s)$ and $e \cdot s = s$ for all $s \in S$. We define a right G-set in a similar way.

The main results which follow appear in the paper [4]. In their paper the authors show that extensions of finite group rings over commutative rings have finite depth. Throughout this section let R denote a commutative ring. The idea of the proof of the theorem is broken down after the statement:

Theorem 2.5.2. *Let $H \subseteq G$ be an extension of finite groups then the group ring extension $RH \subseteq RG$ has finite depth.*

Given two groups $H \subseteq G$ one may define a (G, H)-set, which is a left G-set and a right H-set X such that

$$(g \cdot x) \cdot h = g \cdot (x \cdot h).$$

In a very natural way, if $K \subseteq H \subseteq G$ are finite groups and if we have a (G, H)-set X and an (H, K)-set Y one can say that $X \times Y$ has a (G, K)-set structure. We have a left H-action on $X \times Y$ given for $(x, y) \in X \times Y$ by $h \cdot (x, y) = (xh^{-1}, hy)$. Denote the H-orbit of an element (x, y) by $[x, y]$ then we have a set $X \times_H Y$ which is generated as a subgroup by all H-orbits $\{[x, y] \mid (x, y) \in X \times Y\}$.

Lemma 2.5.3. *Let $K \subseteq H \subseteq G$ be a finite group extension with respective (G, H)- and (H, K)-sets X, Y:*

(i) There is an equivalence between the category of RG-RH-bimodules and the categories of left $R[G \times H]$-modules and right $R[H \times G]$-modules.

(ii) There is an isomorphism of RG-RK-bimodules
$R[X \times_H Y] \cong R[X] \otimes_{RH} R[Y]$.

Definition 2.5.4. Combinatorial depth comes in two flavours, odd and even, very much similar to ring depth. First of all define $\Theta_1 := G$ which is a (G, G)-set (and so a (G, H)- and (H, G)-set), and generally $\Theta_{i+1} := \Theta_i \times_H G$.

- For $i \geq 1$ we say $H \subset G$ has left combinatorial depth $2i$ when there is a monomorphism of (H, G)-sets $\Theta_{i+1} \hookrightarrow m \cdot \Theta_i$. We say it has right combinatorial depth $2i$ when this is true for (G, H)-sets.

- Odd combinatorial depth $2i + 1$ means the monomorphism above is between (H, H)-sets.

The next step towards understanding the theorem is to apply the functor $R[\ ]$ to the sets Θ_i. Using Lemma 2.5.3 we see that $R[\Theta_i] \cong R[G]^{\otimes_{R[H]} i}$ as X-Y-bimodules, where $X, Y \in \{R[G], R[H]\}$. So now there is a hint that ring depth might be related to combinatorial depth when working with groups. This is true:

Lemma 2.5.5. *Suppose the extension $H \subseteq G$ has left combinatorial depth $2i$ then it also has right combinatorial depth $2i$. The opposite is also true. Moreover the combinatorial depth of finite group extensions is always finite.*

The lemma above implies the following, but we omit the pages of details involved in the proofs:

Theorem 2.5.6. *The ring extension $RH \subseteq RG$ has left depth $2i$ if and only if it has right depth $2i$. Moreover, when finite, $d(RH, RG)$ is bounded by the combinatorial depth of $H \subseteq G$.*

Corollary 2.5.7. *For an extension of finite groups $H \subseteq G$ and a ring R, the depth $d(RH, RG)$ is finite.*

2.5.1 Invariant Subalgebras

Much of this discussion on invariant subalgebras was motivated by a question posed in [4]. Specifically the authors ask for an estimate on the value of $d(k[V]^G, k[V])$, where $k[V]$ is defined below. We at least make some progress in answering this question.

Suppose G acts faithfully on a finite dimensional vector space V. This is equivalent to G being isomorphic to a subgroup of $GL(V)$ (the invertible matrices over the basis of V). For our immediate purposes let $k[V]$ denote the polynomial algebra over the basis of V^*. If V has basis $\{e_1, \ldots, e_n\}$ then V^* has basis $\{\delta_1, \ldots, \delta_n\}$, where the deltas are Kronecker deltas. We may consider $f \in k[V]$ as a linear operator on V by acting on the underlying basis. Given that V is a G-set there is a G-action for $k[V]$ given by $(g \cdot f)(v) = f(g^{-1} \cdot v)$. The *$G$-invariants* of $k[V]$ are defined by

$$k[V]^G := \{p \in k[V] \mid g \cdot p = p, \ \forall g \in G\}.$$

We may ask immediately what it means for $k[V]^G \subseteq k[V]$ to have depth 1. What about depth 2? In general does this extension have finite depth, and will that tell us something about the action of G on V? The depth 1 case is a corollary of the Shephard-Todd-Chevalley theorem, in fact it is an equivalent condition:

Theorem 2.5.8. *Let G be a subset of $GL(V)$, for V finite dimensional. We will say that G is generated by pseudo-reflections when the generators of G fix a codimension 1 subspace of V. The following are equivalent.*

- ($\star$) *$k[S]^G \subseteq k[S]$ has depth 1.*
- (a) *G is generated by pseudo-reflections.*
- (b) *The algebra of invariants $k[S]^G$ is a (free) polynomial algebra.*
- (c) *The algebra $k[S]$ is a free module over $k[S]^G$.*
- (d) *The algebra $k[S]$ is a projective module over $k[S]^G$.*

Proof. The equivalence of (a) - (d) are part of the Shephard-Todd-Chevalley result, so we fit ($\star$) into this framework.

($\star \Rightarrow$ d). For $R \subseteq H$ an arbitrary ring extension one of the necessary conditions of this extension being depth 1 is that H is a projective R-module (left and right), easily seen by $H \mid q \cdot R$. Now set $R = k[S]^G$ and $H = k[S]$.

(c $\Rightarrow \star$). The first four conditions are equivalent, so (a) is equivalent to (c) and we have that $k[S]$ is a free $k[S]^G$-module (and bimodule by commutativity of $k[S]$) so $k[S] \cong q \cdot k[S]^G$ which implies $k[S] \sim k[S]^G$ in other words ($\star$).

□

Subalgebra depth 1 is a fairly stringent condition and so we would like to have groups not generated by pseudo-reflections and see which values of depth they achieve. In particular what does depth 2 say about the action of G? Shephard and Todd, the namesakes of the theorem, completed more work in deciding which groups are generated by pseudo-reflections in characteristic 0, see [43]. (When reading this paper note that in characteristic 0 pseudo-reflections are the same as reflections.)

Example 8. Any finite simple group which is not cyclic will not be generated by pseudo-reflections. One well-known class of examples are the alternating groups A_n with $n \geq 5$. In general A_n acts on the letters $\{1, \dots, n\}$ and so if V is a vector space of dimension n we may consider $k[V]^{A_n}$ which is determined in [44]. This invariant subalgebra is generated by $\nabla, e_1, \dots e_n$ where $e_l := x_1^l + \dots + x_n^l$ are the elementary symmetric functions and $\nabla := \sum_{\sigma \in A_n} \sigma \cdot x_1 x_2^2 \dots x_{n-1}^{n-1}$. Can we express $k[V] \otimes_{k[V]^{A_n}} k[V]$ in simpler terms? Is depth finite?

Chapter 3

Depth Theory for Hopf Algebras and Smash Products

3.1 Coalgebras

Recall our definitions of an algebra from Section 2.1. We might have defined a k-algebra as a monoidal object in the category Vect_k of k-vector spaces, and this would have been equivalent [33]. A coalgebra then is a monoidal object in the opposite category ${\text{Vect}_k}^{op}$, or a comonoidal object in Vect_k.

Definition 3.1.1. A k-coalgebra is a triple (C, Δ, ϵ) where C is a k-vector space and $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ such that the following diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\quad\Delta\quad} & C \otimes C \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle id\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes id} & C \otimes C \otimes C \end{array}$$

$$\begin{array}{ccccc} & {\scriptstyle\cong}\swarrow & C & \searrow{\scriptstyle\cong} & \\ & & {\scriptstyle\Delta}\big\downarrow & & \\ k \otimes C & \xleftarrow[\epsilon\otimes id]{} & C \otimes C & \xrightarrow[id\otimes\epsilon]{} & C \otimes k \end{array}$$

We call Δ the *comultiplication* map and ϵ the *counit*. We call a coalgebra C cocommutative if $(\tau \circ \Delta)(c) = \Delta(c)$ for each $c \in C$, where $\tau : C \otimes C \to C \otimes C$ is the twist map: $\tau(x \otimes y) = y \otimes x$.

Analogously to the algebra case we define a homomorphism of coalgebras $g : C \to D$ to be a k-linear map which makes the square commute:

$$\begin{array}{ccc} C & \xrightarrow{g} & D \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes C & \xrightarrow{g \otimes g} & D \otimes D \end{array}$$

as well as satisfying $g\epsilon_C(x) = \epsilon_D g(x)$ for all $x \in C$.

Definition 3.1.2 (Sweedler Notation)**.** Assume that we have a coalgebra C with comultiplication Δ then for any $x \in C$, $\Delta(x) \in C \otimes C$ and so we may write

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

The notation "(x)" on the bottom of the sum serves to remind us which element we are comultiplying; we drop this notation when the element is clearly known. The elements $x_{(1)}, x_{(2)} \in C$ do not have a subscript telling them apart from other terms in the sum. However these elements are symbolic and it should be implicitly understood from this point that we do not need to see differences between terms in the sum.

Lemma 3.1.3. *Given an algebra A then $A^\circ := \{f \in A^* \mid f(I) = 0,\ I \subseteq A$ some ideal s.t. $dim(A/I) < \infty\}$ is a subset of A^* and has a coalgebra structure given by $\Delta(f)(x, y) = xy$. (The counit is given by $\epsilon(f) = f(1)$.) Furthermore $A^* = A^\circ$ when A is finite dimensional.*

Proof. The complete proof can be found in [36, Chap.2]. □

Corollary 3.1.4. *Let A be finite dimensional then $A^* = A^o$ and A^* is a coalgebra.*

When we have a coalgebra and an algebra C and A then the set $A^C := Hom_k(C, A)$ has a natural algebra structure, called *convolution*, which is denoted by $*$ and defined as below:

$$(f * g)(c) = \sum f(c_{(1)})g(c_{(2)}). \tag{3.1}$$

The identity element will be $\epsilon 1$.

Corollary 3.1.5. *For every coalgebra C the dual C^* is an algebra.*

3.2 Hopf Algebras

We define a bialgebra to be quintuple $(B, \mu, \eta, \Delta, \epsilon)$ such that (B, μ, η) is an algebra, (B, Δ, ϵ) is a coalgebra and one of the following conditions hold:

(a) μ and η are coalgebra homomorphisms.

(b) Δ and ϵ are algebra homomorphisms.

Lemma 3.2.1. *The two pairs of conditions above are equivalent*

Remark 3.2.2. A subbialgebra D of a bialgebra B is a linear subspace which is a subalgebra and subcoalgebra of B. Notice that the compatibility conditions directly above are automatically implied by virtue of $D \subseteq B$.

Lemma 3.2.3. *Given a bialgebra B with comultiplication Δ recall the Sweedler notation for an $x \in B$ given by $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$. Let $y \in B$ then the Sweedler notation of xy is given by*

$$\Delta(xy) = \sum_{(xy)} (xy)_{(1)} \otimes (xy)_{(2)} = \sum_{(x),(y)} x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}.$$

Proof. Remember that Δ is an algebra homomorphism, so $\Delta(xy) = \Delta(x)\Delta(y)$. Put this fact together with the knowledge of the algebra structure of $B \otimes B$, $(x \otimes y)(x' \otimes y') = xx' \otimes yy'$, and we are done. $\square$

Definition 3.2.4. A Hopf algebra is a bialgebra H along with k-linear map $S : H \to H$ satisfying:

$$\sum h_{(1)} S(h_{(2)}) = \sum S(h_{(1)}) h_{(2)} = \epsilon(h) 1$$

The map S above is known as the *antipode*. Some important properties of this map are:

(a) S is an algebra anti-homomorphism: $S(1) = 1$ and $S(hk) = S(k)S(h)$, for all $h, k \in H$.

(b) S is a coalgebra anti-homomorphism: $S(\epsilon) = \epsilon$ and $\Delta S(h) = \sum S(h_{(2)}) \otimes S(h_{(1)})$.

(c) S is the convolution inverse of id_H in the sense of (3.1) applied to $\mathrm{Hom}(H, H)$.

The properties of S above combined with convolution give us $S * id(h) = \sum S(h_{(1)})h_{(2)} = \epsilon(h)$ by definition, similarly $id * S = \epsilon 1$ and $\epsilon 1$ is the unit element.

Definition 3.2.5. Let K be a subbialgebra of a Hopf algebra H, then we say K is a Hopf subalgebra when it satisfies $S(K) \subseteq K$.

Given a Hopf algebra H notice in particular that H^* has an algebra structure using the convolution product above. Let H be finite dimensional then Corollary 3.1.5 implies a coalgebra structure on H^*, thusly H^* is a bialgebra. Additionally we define an antipode $T : H^* \to H^* : T(f) = f \circ S$. Thus there is a canonical Hopf algebra structure on H^* whenever H is finite dimensional. (In the infinite dimensional case H^o as in Corollary 3.1.5 will be a Hopf algebra and H^* a module coalgebra; an object which is introduced later.) The Hopf algebra structure in the lemma below is well known.

Lemma 3.2.6. *Given a finite dimensional Hopf algebra H then H^* will be a Hopf algebra with the structure defined above.*

Example 9. Given a group G then the group algebra is a Hopf algebra with coalgebra structure given by $\Delta(g) = g \otimes g$, which is then extended k-linearly. The antipode exists and is defined by $S(g) = g^{-1}$. By Corollary 2.5.7 we have already shown that the depth of some Hopf algebra extensions is finite.

3.2.1 Integral Elements

Throughout let H denote a Hopf algebra, as above.

Definition 3.2.7. A left integral element $t \in H$ satisfies $ht = \epsilon(h)t$ for all $h \in H$. Similarly a right integral is $s \in H$ with $sh = \epsilon(h)s$.

We denote the subalgebra of left and right integrals by $\int_H^l$ and $\int_H^r$ respectively. The following result is attributed to Larson and Sweedler, see [31].

Theorem 3.2.8. *Let H be finite dimensional then:*

(a) The left and right integrals are 1-dimensional i.e. $dim(\int_H^l) = dim(\int_H^r) = 1$

(b) The antipode S is a bijection with $S(\int_H^l) = \int_H^r$ and vice-versa.

(c) H is a cyclic left and right H^*-module, meaning that there exists an $h \in H$ such that $H = H^* \cdot h$ for some module structure.

Corollary 3.2.9. *Both $\int_H^l$ and $\int_H^r$ are two-sided H-ideals.*

Proof. For the left case take $t \in \int_H^l$ and $h, k \in H$ so that $k(th) = (kt)h = \epsilon(k)(th)$. This means that $\int_H^l$ is stable under right multiplication, and trivially under left. $\square$

Definition 3.2.10. Taking $0 \neq t \in \int_H^l$ let $\alpha : H \to k$ be the map such that $th = \alpha(h)t$. We call α *the modular function* on H.

One need not define say an α' in the case of $s \in \int_H^r$ for which $hs = \alpha'(h)s$, because $\alpha' = \alpha \circ S^{-1}$. For an arbitrary $h \in H$ notice the following:

$$S(h)s = S(h)S(t) = S(th) = \alpha(h)S(t) = \alpha(h)s.$$

Theorem 3.2.11 (Maschke's Theorem). *Let H be finite dimensional and $t \in \int_H^l$, $s \in \int_H^r$ both non-zero then H is semisimple if and only if $\epsilon(t) \neq 0$ and $\epsilon(s) \neq 0$.*

Corollary 3.2.12 (Maschke's Theorem for Groups). *Let G be a finite group and k a field then kG is a semisimple if and only if $char(k) \nmid |G|$*

3.2.2 Normal Hopf Subalgebras

In group theory we have the concept of a normal subgroup, that is $N \subseteq G$ with $gN = Ng$ for all $g \in G$. Normal subgroups K are those for which the canonical quotient G/K is again a group. Since Hopf algebras are a generalisation of group algebras, we would expect a generalisation of normality.

Definition 3.2.13 (Normality). Given a Hopf algebra H we define the adjoint action in left and right types, for $h, x \in H$ write

$$ad_l(h)(x) = \sum h_{(1)} x S(h_{(2)}),$$

$$ad_r(h)(x) = \sum S(h_{(1)}) x h_{(2)}.$$

We say that a sub-Hopf algebra K of H is left normal when $ad_l(h)(K) \subseteq K$ and right normal when $ad_r(h)(K) \subseteq K$ for all $h \in H$. If it is left and right normal we will call it normal.

Example 10. Let $L \subseteq G$ be finite groups, k an arbitrary field. Then $kL \subseteq kG$ is an extension of finite dimensional Hopf algebras. Moreover kL is a normal Hopf subalgebra if and only if L is a normal subgroup. We need only notice that for $x \in L$ and $g \in G$, $ad_l(g)(x) = ad_r(g^{-1})(x) = gxg^{-1}$, which belongs to kL if and only if L is normal.

Lemma 3.2.14 ([47]). *If $K \subseteq H$ is an extension of Hopf algebras such that S is bijective then K is left normal if and only if it is right normal.*

Proof. ($\Rightarrow$). Suppose that K satisfies left normality, so for $h \in H$ and $x \in K$, $ad_l(h)(x) \in K$. We want to evaluate $ad_r(h)(x) = \sum S(h_{(1)}) x h_{(2)}$ and show that it is an element of K. Using bijectivity choose $h' \in H$ such that $S(h') = h$. Note that S and thus S^{-1} is a coalgebra antihomomorphism and so we can write $\Delta(h') = \sum h'_{(1)} \otimes h'_{(2)} = \sum S^{-1}(h_{(2)}) \otimes S^{-1}(h_{(1)})$. Now

$$ad_r(h')(k) = \sum S(h'_{(1)})kh'_{(2)} = \sum SS^{-1}(h_{(2)})kh'_{(2)} = \sum h_{(2)}kS^{-1}(h_{(1)})$$

This tells us that $S(ad_r(h')(k)) = \sum(h_{(1)})S(k)S(h_{(2)}) = ad_l(h)(S(k))$, which since S is bijective and K is closed under S and S^{-1} implies that K is right normal.
($\Leftarrow$). Similarly proved. □

This lemma applies in the case of finite dimensional Hopf algebras because by Theorem 3.2.8 each one has an invertible antipode.

Suppose that $K \subseteq H$ is left normal then $HK \subseteq KH$. Take $k \in K$, $h \in H$ then

$$hk = \sum h_{(1)}k\epsilon(h_{(2)}) = \sum h_{(1)}kS(h_{(2)})h_{(3)} = \sum ad_l(h_{(1)})(k)h_{(2)},$$

meaning that $hk \in KH$. One may naively assume that the $HK \subseteq KH$ condition is equivalent to left normality, and $KH \subseteq HK$ to right normality, but this is not true in general. Write $K^+ := ker\epsilon \cap K$:

Lemma 3.2.15 ([47], Thm.4.4(a))**.** *Let H be finite dimensional, then $K \subseteq H$ is a left normal Hopf subalgebra if and only if $HK^+ \subseteq K^+H$. Similarly K is right normal if and only if $K^+H \subseteq HK^+$.*

Proof. ($\Rightarrow$) We use the argument above the lemma and restrict to K^+.
($\Leftarrow$) This is a trickier argument and made well in [36, Lem.3.4.2]. □

Corollary 3.2.16. *Let H be finite dimensional, then $K \subseteq H$ is a normal Hopf subalgebra if and only if $HK^+ = K^+H$.*

Definition 3.2.17. Given a Hopf algebra H and a vector subspace J we say that J is a Hopf ideal when it is an ideal (respecting the algebra structure) and a coideal ($\Delta_H(J) \subseteq H \otimes J + J \otimes H$) and furthermore $S(J) \subseteq J$.

Remark 3.2.18. If $J \subseteq H$ is a Hopf ideal then $\overline{H} := H/J$ has a canonical Hopf algebra structure.

Going back to a normal Hopf subalgebra K, we can see that $HK^+ = K^+H$ is an ideal:

$$H(HK^+)H = (HK^+)H = (K^+H)H \subseteq K^+H = HK^+.$$

Moreover by Lemma 5.5.1 proved later, HK^+ is a coideal. Indeed HK^+ will be a Hopf ideal because

$$S(HK^+) \subseteq S(K^+)S(H) \subseteq K^+H \subseteq HK^+,$$

where $S(ker\epsilon) \subseteq ker\epsilon$ and thus $S(K^+) \subseteq K^+$ are clear from the properties of S on page 35. We have just showed that HK^+ is a Hopf ideal and H/HK^+ is a Hopf algebra.

Normality has a strong relation to the depth 2 condition, and the point of discussing normality in such detail was to arrive at this result:

Theorem 3.2.19 ([5], Prop.2.8)**.** *Given a finite dimensional Hopf algebra extension $K \subseteq H$ then this extension satisfies the left depth 2 condition if and only if K is right normal. The same is also true of right depth 2 and left normality.*

Proof. ($\Rightarrow$) Assume that $R \subseteq H$ has depth 2. Notice that H is an augmented algebra (see page 14 for the definition) using the counit $\epsilon : H \to k$, so we can apply Proposition 2.1.1. Using this result left depth 2 implies the condition $K^+H \subseteq HK^+$ and because H is finite dimensional Lemma 3.2.15 says this is equivalent to $K \subseteq H$ being right normal. Proving that right depth 2 implies left normality is similar.

($\Leftarrow$) This more difficult direction can be found in the reference. □

3.3 Smash Products

Definition 3.3.1. Given a Hopf algebra H we define an H-module algebra to be an algebra A with an H-module structure which satisfies

(a) $h \cdot (xy) = \sum (h_{(1)} \cdot x)(h_{(2)} \cdot y)$

(b) $h \cdot 1 = \epsilon(h)1$.

In other words the algebra maps $\mu : A \otimes A \to A$ and $\eta : k \to A$ are H-module maps.

Remark 3.3.2. In the definition above we have used a left H-module, so explicitly we have defined a *left H-module algebra.* If A were a right H-module we would have defined a *right H-module algebra.* By default we use the left variety.

Remark 3.3.3. We call C an H-comodule algebra when it is an algebra and H-comodule such that the coaction $\rho : C \to H \otimes C$ is an algebra morphism. There is a form of Sweedler's notation to express the coaction of ρ on $x \in C$, that is

$\rho(x) = \sum x_{(0)} \otimes x_{(1)}$ where $x_{(0)} \in H$ and $x_{(1)} \in C$. Notice the order of H and C in the coaction ρ, this implies that C is a left H-comodule, thus called a left H-comodule algebra. The right H-comodule algebra case would be where C is a right H-comodule. If H is finite dimensional then A is a left H-module algebra if and only if A is a right H^*-comodule algebra: switching left and right gives an equally true result.

The first H-module algebra we want to discuss is what we call *trivial*, this being an A such that $h \cdot a = \epsilon(h)a$ for all $h \in H$. Given any algebra we may give it a trivial H-module structure, and it will automatically become an H-module algebra. Let ${}_\epsilon k$ be the underlying field of H, with trivial module structure. The following lemma is well-known.

Lemma 3.3.4. *Given a trivial H-module A with finite dimension then $A \cong {}_\epsilon k^{dimA}$ as H-modules.*

Definition 3.3.5. let C be a k-coalgebra which has an H-module structure, then we call C an H-module coalgebra when Δ and ϵ are H-module homomorphisms.

Remark 3.3.6. As with module algebras we can define left and right H-module coalgebras, the one defined above was of the left variety.

Given a Hopf algebra H and an H-module algebra A we can form the smash product $A\#H$ with underlying set $A \otimes H$ and with multiplication:

$$(a\#h)(b\#k) = a(h_{(1)} \cdot b)\#h_{(2)}k. \tag{3.2}$$

This is an interesting structure already because even when A and H are commutative the smash product may not be: $(a\#1)(1\#h) = a\#h$ but $(1\#h)(a\#1) = \sum h_{(1)} \cdot a\#h_{(2)}$.

Notice that we can imbed both A and H in $A\#H$ via the algebra maps $a \mapsto a\#1$ and $h \mapsto 1\#h$. We shall consider A and H as subsets of $A\#H$ directly so we can write ah for $a\#h$; this extends to $AH = A\#H$.

We might have also taken a right H-module algebra B and formed the smash product $H\#B$. (In Chapter 5 we explicitly make use of a particular right smash product.)

Example 11 (The Heisenberg Double)**.** Let H be a finite dimensional Hopf algebra, then we can describe a left H-module algebra structure on H^* by

$$x \rightharpoonup f(h) = f(hx).$$

The smash product $H^*\#H$ is generally called the *Heisenberg double.* Indeed $H^* \subseteq H^*\#H$ is a H-Galois extension. Galois actions are explained in Section 3.4 with the main result there implying $d(H^*, H^*\#H) \leq 2$.

Now follow two technical lemmas, which we will reference in Chapter 4. Recall the definition of the distinguished grouplike element α from Definition 3.2.10.

Lemma 3.3.7 ([36], Lem.4.4.3)**.** *Suppose A is an H-module algebra, where H is a Hopf algebra with invertible antipode. Take $t \in \int_H^l$, then for all $a \in A$ and $h \in H$:*

(1) $ah = \sum h_{(2)}(S^{-1}h_{(1)} \cdot a)$.

(2) $hat = (h \cdot a)t$ and $tah = t(S^{-1}(\alpha \rightharpoonup h) \cdot a)$, where the action $\rightharpoonup$ is as in the Heisenberg double above.

(3) $(t) := AtA$ is an ideal in $A\#H$.

We adapt the above result to right integrals:

Lemma 3.3.8. *Take $A\#H$ as above and $s \in \int_H^r$ then $(s) = AsA$ is an $A\#H$-ideal.*

Proof. Given $h \in H$ and $a \in A$

$$has = (h_{(1)} \cdot a)h_{(2)}s = \alpha(h_{(2)})(h_{(1)} \cdot a)s \in AsA.$$

Now we use part (2) of Lemma 3.3.7 above:

$$sah = s(\sum h_{(2)}(S^{-1}h_{(1)} \cdot a)) = \alpha(h_{(2)})s(S^{-1}h_{(1)} \cdot a),$$

this element belongs to AsA as well. Thus $H(As) \subseteq (s)$ and $(sA)H \subseteq (s)$ and we are done. □

3.3.1 Semidirect Products

In this section we will consider semidirect products as defined in [34]. Let G be a group and let N be a normal subgroup, and H another subgroup. We say that G is an *inner semidirect product* of N with H if one of the following conditions hold:

(a) $G = NH$ and $N \cap H = \{e\}$;

(b) Every element of G can be written uniquely as nh for some $n \in N$ and $h \in H$;

(c) Every element can be written as hn.

It goes without saying that the conditions above are equivalent. When one of the above conditions hold we will write: $G = N \rtimes H$. This concept generalises to the one of *outer semidirect product*. This is given for two groups N and H unrelated to those above. We say that H acts on N via automorphisms when there exists a map $\phi : H \to \mathrm{Aut}(N)$ and thus an associated group action of H on N. We then form a new group $H \rtimes_\phi N$ with underlying set $N \oplus H$ and multiplication

$$(n_1 h_1)(n_2 h_2) = (n_1(h_1 \cdot n_2))(h_1 h_2). \tag{3.3}$$

The concepts of inner and outer semidirect products are naturally equivalent. That is to say given an inner semidirect decomposition of a group, we automatically have a canonical outer semidirect decomposition, the opposite is also true. For this reason we henceforth use the term *semidirect product*.

Astute readers may notice a similarity in the form of smash product multiplication (3.2) and with (3.3). Take k to be any field, the two results below is well-known.

Lemma 3.3.9. *Let N and H be groups where H acts on N by automorphisms, then kN is a kH-module algebra and the smash product $kN\#kH$ exists.*

The action of kH on kN is canonically given by $(\sum \lambda_i h_i)\cdot(\sum \mu_j n_j) = \sum \lambda_i \mu_j (h_i \cdot n_j)$ $(\lambda_i, \mu_j \in k)$. Write $h \cdot 1 = 1$ and of course $h \cdot (nm) = (h \cdot n)(h \cdot m)$, this gives us a kH-module algebra.

Proposition 3.3.10. *Take N and H as above then there is the following isomorphism of algebras*

$$kN\#kH \cong k(N \rtimes H).$$

In particular the subalgebras kN and kH are preserved under this isomorphism.

Proof. Write an algebra homomorphism $\psi : k(N \rtimes H) \to kN\#kH$ as follows $\psi(nh) = n\#h$. This is easily seen to be an algebra homomorphism

$$\begin{aligned} \psi((n_1 h_1)(n_2 h_2)) &= \psi[(n_1(h_1 \cdot n_2))(h_1 h_2)] \\ &= n_1(h_1 \cdot n_2))\#(h_1 h_2) \\ &= (n_1\#h_1)(n_2\#h_2) \\ &= \psi(n_1 h_1)\psi(n_2 h_2) \end{aligned}$$

The inverse to this map is clearly $n\#h \mapsto nh$. □

Recall that isomorphic ring extensions have the same depth (Lemma 1.2.6). This allows us to write

$$d(kH, kN\#kH) = d_k(kH, k(N \rtimes H)).$$

Notice that $k(N \rtimes H)$ is a group algebra, so if H, N are finite groups $N \rtimes H$ is finite too and Corollary 2.5.7 tells us that the value of depth above is finite.

Example 12. The symmetric group S_3 is well-known to be generated by (12) and (123). There are the two subgroups $A_3 = \{(1), (123), (132)\}$ and $H = \{(1), (12)\}$, where alternating subgroup A_3 is normal and $H \cong S_2$. We can write $S_3 = A_3 \rtimes H$ because the intersection $N \cap H$ is trivial and $HA_3H = A_3H = S_3$ by normality and A_3, H containing the generators. Now it is clear $S_3 \cong A_3 \rtimes S_2$, and this means by Proposition 3.3.10 and Example 3

$$d(kS_2, kA_3\#kS_2) = d_k(S_2, A_3 \rtimes S_2) = d_k(S_2, S_3) = 3.$$

3.4 Galois Extensions

The main point of this section is to explain what Galois actions are and their relationship to depth. A more technical, but an essentially equivalent explanation can be found [22] where the theorem below appears. First of all let us remember the coinvariants of a comodule algebra. So let C be a coalgebra which has a comodule algebra A (with $\rho : A \to C \otimes A$) associated, then the coinvariants are defined by $A^{coC} := \{a \in A \,|\rho(a) = a \otimes 1\}$. (All of these things are over the arbitrary field k.)

Definition 3.4.1. Given a Hopf algebra H and a right H-comodule algebra A with coaction ρ and $B := A^{coH}$. Then we say that $B \subseteq A$ is a Hopf-Galois extension (H-Galois) if the following k-linear map is an isomorphism:

$$\begin{aligned} \beta : A \otimes_B A &\to A \otimes H \\ : a \otimes b &\mapsto (a \otimes 1)\rho(b). \end{aligned}$$

Remark 3.4.2. What stops us from instead defining $\beta'(a \otimes b) = \rho(a)(b \otimes 1)$ and basing the definition of a Hopf-Galois extension on this? The answer is nothing, indeed if the antipode S is bijective then β is an isomorphism $\Leftrightarrow \beta'$ is too.

Lemma 3.4.3 (Kreimer and Takuchi, [28], Thm.1.7)**.** *Let A and B be as above with H finite dimensional then if β is surjective it is also injective.*

Lemma 3.4.4. *The map β above is A-B-linear and β' is B-A-linear.*

Proof. Given $a \in A, b \in B$ recall that $B = A^{coH}$ so that $\rho(b) = b \otimes 1$. Now for $x \otimes y \in A \otimes_B A$

$$\begin{aligned} a\beta(x \otimes y)b &= a(x \otimes 1)\rho(y)b \\ &= a(xy_{(0)}b \otimes y_{(1)}) \\ &= a(xy_{(0)} \otimes y_{(1)})(b \otimes 1) \\ &= a(xy_{(0)} \otimes y_{(1)})\rho(b) \\ &= a(x \otimes 1)\rho(y)\rho(b) \\ &= a(x \otimes 1)\rho(yb) \\ &= \beta(ax \otimes yb). \end{aligned}$$

The case of β' is very similar. □

One of the main motivations for talking Galois extensions in the case of smash products is the following lemma. This is the immediate corollary of [36, Thm 1.8.4]:

Lemma 3.4.5 ([36], Cor.8.2.5)**.** *Let $B \subseteq A$ be an extension of right H-comodule algebras such that $B = A^{coH}$ then $B \subseteq A$ is Galois with the normal basis property ($A \cong A \otimes H$ as left A-modules and right H-modules) if and only if $A \cong B\#_\sigma H$ as left A-modules.*

Of course the $\#_\sigma$ notation has yet to appear. It is known as a crossed product (as opposed to a smash product) and is defined in [36], also in the definition to come.

Suppose that A is a k-algebra, we say that H measures A when there is a map $H \otimes A \to A : h \otimes a \to h \cdot a$ with *(i)* $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$; and *(ii)* $h \cdot 1 = \epsilon(h)1$.

Lemma & Definition 3.4.6. *Suppose that as above H measures A and that there is a map $\sigma \in Hom_k(H \otimes H, A)$ which is invertible in the sense of convolution stated in Equation 3.1. Then we define $A\#_\sigma H$ to have underlying set $A \otimes H$ with a multiplication*

$$(a\#h)(b\#k) = \sum a(h_{(1)} \cdot b)\sigma(h_{(2)}, k_{(1)})\#h_{(3)}k_{(2)}.$$

The lemma content follows: This multiplication for $A\#_\sigma H$ is associative if and only if the following conditions hold *(a) A is a H-module with a σ-twist that is*

$$h \cdot (k \cdot a) = \sum \sigma(h_{(1)}, k_{(1)})(h_{(2)}k)(2) \cdot a)\sigma^{-1}(h_{(3)}, k_{(3)}),$$

for all $h, k \in H$ and $a \in A$.

(b) σ [is normal and so] satisfies $\sigma(1, h) = \sigma(h, 1) = \epsilon(h)1$, and

$$\sum(h_{(1)} \cdot \sigma(k_{(1)}, l_{(1)})\sigma(h_{(2)}, k_{(2)}l_{(2)}) = \sum \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}k_{(2)}, l) \qquad (3.4)$$

for all $h, k, l \in H$

There are a few sources of information on crossed products, in particular the paper of [37] explains crossed products in section 4. The proof of the exercise involves showing that $1\#1$ is the identity if and only if σ is normal. Associativity is equivalent to the 2-cocycle equation (3.4) above.

Note in particular that $A\#H$ is a crossed product with $\sigma = \epsilon \otimes \epsilon$. The following result comes from [18, Thm 4.1], as a combination of results from [26] and [24].

Proposition 3.4.7. *With $A^{coH} = B \subseteq A$ an H-Galois extension and H finite dimensional then $d(B, A) \leq 2$.*

Proof. Give $A \otimes_B A$ the usual A-bimodule structure, for $x \otimes y \in A \otimes_B A$ and $a, a' \in A$ write $a(x \otimes y)a' = ax \otimes ya'$. Give $A \otimes H$ the A-bimodule structure which ignores H: $a(x \otimes h)a' = axa' \otimes h$. It is not hard to see that $A \otimes H \cong A^{dimH}$ as A-bimodules, call this isomorphism ϕ.

Assume that $\beta : A \otimes_B A \to A \otimes H$ is bijective and k-linear, the same goes for β' as in Remark 3.4.2. Recall by Lemma 3.4.4 that β is A-B-linear and β' is B-A-linear. Using Lemma 3.4.3 for finite dimensional H and putting together the pieces:

$$A \otimes_B A \underset{\beta'}{\overset{\beta}{\rightrightarrows}} A \otimes H \xrightarrow{\phi} A^{dimH} .$$

Where $\phi \circ \beta$ is a B-A-bimodule isomorphism and $\phi \circ \beta'$ is an A-B-bimodule isomorphism. Indeed this is both left and right parts of the depth 2 condition by Lemma 1.2.9.

□

Note that parts (a) and (b) of the theorem below appear in [36]. Part (c) is where the depth theory relates to the known results.

Theorem 3.4.8. *Let A be an H-module algebra, then we have the smash product $A\#H$. Furthermore this smash product will have a comodule algebra structure given by $\rho(a\#h) = \sum a\#h_{(1)} \otimes h_{(2)}$. The following are true:*

(a) $(A\#H)^{coH} = A$.

(b) $A \subseteq A\#H$ *is H-Galois.*

(c) When H is finite dimensional $d(A, A\#H) \leq 2$.

Proof. We check that $A\#H$ is in fact an H-comodule algebra. This is easily done, for if $a\#h,\ d\#k \in A\#H$

$$\begin{aligned} \rho((a\#h)(d\#k)) &= \rho(a(h_{(1)} \cdot d)\#h_{(2)}k) = a(h_{(1)} \cdot d)\#h_{(2)}k_{(1)} \otimes h_{(3)}k_{(2)} \\ &= (a\#h_{(1)} \otimes h_{(2)})(d\#k_{(1)} \otimes k_{(2)}) \\ &= \rho(a\#h)\rho(d\#k). \end{aligned}$$

To prove the claim in (a) suppose that $\sum_i a_i\#h_i \in (A\#H)^{coH}$, then $\rho(\sum_i a_i\#h_i) = (\sum_i a_i\#h_i) \otimes 1$ which by definition of ρ is equal to $\sum_i a_i\#\Delta(h_i) = \sum_i \sum_{(h_i)} a_i\#(h_i)_{(1)}\otimes(h_i)_{(2)}$. Recall the counit ϵ of H, now apply the map $(id\otimes\epsilon\otimes id)$ to the previous equality, the resulting equality is $\sum_i a_i\#1\otimes\epsilon(h_i)1 = \sum_i a_i\#1\otimes h_i$. Applying $id\otimes\mu$ to this equality tells us that $\sum_i a_i\#h_i = \sum_i a_i\#\epsilon(h_i)1$. Therefore $(A\#H)^{coH} = A$.

For the proof of (b) we provide an inverse to β, showing the bijectivity of this map. I claim this inverse is $\delta : A\#H\otimes H \to A\#H\otimes_A A\#H$ defined by $\delta(a\#h\otimes k) = (a\#hS(k_{(1)})) \otimes (1\#k_{(2)})$.

$$\begin{aligned} \delta \circ \beta(a\#h \otimes d\#k) &= \delta(a(h_{(1)} \cdot d)\#h_{(2)}k_{(1)} \otimes k_{(2)}) \\ &= a(h_{(1)} \cdot d)\#h_{(2)}k_{(1)}S(k_{(2)}) \otimes k_{(3)} \\ &= a(h_{(1)} \cdot d)\#h_{(2)} \otimes \epsilon(k_{(1)}k_{(2)} \\ &= (a\#h)(d\#1) \otimes (1\#k) \\ &= (a\#h) \otimes (d\#k). \end{aligned}$$

And similarly:

$$\begin{aligned}\beta \circ \delta(a\#h \otimes k) &= \beta[(a\#hS(k_{(1)}) \otimes (1\#k_{(2)})]\\ &= (a\#hS(k_{(1)}) \otimes 1)\rho(1\#k_{(2)})\\ &= (a\#hS(k_{(1)})k_{(2)}) \otimes k_{(3)}\\ &= a\#h \otimes k.\end{aligned}$$

To show (c) we invoke Proposition 3.4.7. □

3.4.1 Galois Extension from Depth Two

In their paper [26] the authors have worked with various algebraic structures arising from the depth 2 condition (and other interesting details). One major result from the paper, on Hopf-Galois extensions, is stated below - although we omit much of the details of the paper. It provides a thought-provoking converse to Proposition 3.4.7.

Given a pair of k-algebras $B \subseteq A$, we say that this extension is *Frobenius* when there is a *Frobenius homomorphism* $E : {}_BA_B \to {}_BA_B$ and so-called *dual bases* $x_i, y_i \in A$ $(1 \leq i \leq n)$ such that for all $a \in A$

- $\sum_i E(ax_i)y_i = a$;
- $E(a'a) = 0$ for all $a' \in A$ implies $a = 0$.

We call the algebra extension $B \subseteq A$ *irreducible* when $A^B = k$, where A^B are the elements $a \in A$ such that $ba = ab$ for all $b \in B$. The following result is taken from [26, Cor.8.1.4].

Proposition 3.4.9. *Let $B \subseteq A$ be an irreducible Frobenius extension with $d(B, A) \leq 2$, then $B \subseteq A$ is a Hopf-Galois extension.*

The extension $B \subseteq A$ can only be Hopf-Galois with respect to some Hopf algebra: the set $(A \otimes_B A)^B$ can be shown (after a long and intricate proof) to have the Hopf algebra structure we require.

Chapter 4

Smash Product Depth Examples

During the last chapter we made the point that for a finite dimensional Hopf algebra H and an H-module algebra A the depth $d(A, A\#H) \leq 2$ (Theorem 3.4.8). In Example 12 it was made clear that there exists more than one finite group algebra kG and kG-module algebra A such that

$$d(kG, A\#kG) > 2.$$

In this chapter we will provide two examples of a finite dimensional Hopf algebras H which are neither group algebras nor their duals with an H-module algebra such that $d(H, A\#H) > 2$. Beforehand we provide a well-known property of group algebras:

Lemma 4.0.1. *Given any group algebra kG then it is cocommutative, consequently the dual Hopf algebra $(kG)^*$ is commutative.*

Proof. By definition the coproduct in kG is defined by $\Delta(x) = x \otimes x$ which is clearly equal to Δ^{op}. □

4.1 The Taft Algebras

Let us work with a class of well-known $\mathbb{C}$-algebras known as the *Taft algebras.* For background details one might read Taft's paper, where this class of algebras were discovered [46]. As in the paper we can replace $\mathbb{C}$ with any field k which has the necessary primitive roots of unity. The case of $n = 2$ is called Sweedler's algebra, which predates Taft's set of examples and was the first historical case of a Hopf algebra neither commutative nor cocommutative [41],[45]. Since group algebras kG

are cocommutative and their duals $(kG)^*$ are commutative the Taft algebras are neither group algebras nor their duals.

For some $n \geq 2$ let $\psi \in \mathbb{C}$ be a primitive n^{th} root of unity, we define the n^{th} *Taft algebra* as

$$H_n := \mathbb{C}\langle 1,\ g,\ x,\ xg |\ g^n = 1,\ x^n = 0,\ xg = \psi gx\rangle.$$

Notice the Hopf algebra structure given by:

$$\Delta(x) = 1 \otimes x + x \otimes g,\ \Delta(g) = g \otimes g,$$
$$S(x) = -xg^{-1},\ S(g) = g^{-1} \text{ and } \epsilon(x) = 0,\ \epsilon(g) = 1$$

Fix a particular $n \geq 2$ and write $H := H_n$. The subalgebra $\mathbb{C}\langle g\rangle \subseteq H$ generated by the group-like element g is isomorphic to $\mathbb{C}\mathbb{Z}_n$, indeed this is an isomorphism of Hopf algebras. Write $B := \mathbb{C}\langle g\rangle$ for nicer reading. Notice that $\mathbb{C}\langle x\rangle \subseteq H$ is also a Hopf subalgebra, and write $X := \mathbb{C}\langle x\rangle$.

Lemma 4.1.1. *Taking B and X as above we may express H as a smash product:*

$$H \cong X\#B.$$

Proof. One must demonstrate first that X is a B-module algebra, in other words a $\mathbb{C}\mathbb{Z}_n$-module algebra. Define an action of g on x by $g \cdot x = \psi^{-1}x$, then we extend the action so that $g^r \cdot x^s = \psi^{-rs}x^s$. So the smash product $X\#B$ does exist and both $X \subseteq H$ and $B \subseteq H$ imbed in $X\#B$ naturally via $x \mapsto x\#1$ and $g \mapsto 1\#g$. In particular the multiplication of the smash product is $(1\#g)(x\#1) = g \cdot x\#g = \psi^{-1}x\#g = \psi^{-1}(x\#1)(1\#g)$. This looks exactly like the multiplication property for x, g in H and then H imbeds in $X\#B$ as an algebra. Since the dimensions are equal $(dimX dimB = n^2)$ the embedding is surjective. □

Given that H is a Hopf algebra $X\#B$ will have exactly the same Hopf algebraic structure on its basis elements.

Lemma 4.1.2. *The Hopf algebra extension $B \subseteq X\#B$ does not satisfy the depth 2 condition.*

Proof. We show that left depth 2 is not satisfied. By Theorem 3.2.19 we need only show that $B \subseteq H$ is not left normal:

$$\begin{aligned}(ad_l x)(g) &= \sum x_{(1)} g S(x_{(2)}) \\ &= gS(x) + xgg^{-1} \\ &= -gxg^{-1} + x \\ &= -\psi^{-1} xgg^{-1} + x \\ &= (\psi^{-1} - 1)x\end{aligned}$$

Notice that $\psi^{-1} - 1 \neq 0$ and thus $(\psi^{-1} - 1)x \notin B$ and as such $(ad_l H)(B) \nsubseteq B$. Then B is not left ad-stable i.e. it is not left normal. $\square$

So depth 2 wasn't satisfied, B fails to have the necessary properties of such an extension. The next case to look at is depth 3, after depth 2 there is not a strong classification of Hopf subalgebras satisfying the condition; we need to evaluate each example on a case-by-case basis, and learn as we go along. Demonstrating the depth 3 condition means showing that $H \otimes_B H \sim q \cdot H$ as B-bimodules. Using Lemmas 1.2.3 and 1.2.8 we must now find B-bimodule maps satisfying the diagram:

$$id \circlearrowleft H \otimes_B H \rightleftarrows q \cdot H \ .$$

Lemma 4.1.3. *Both H and $H \otimes_B H$ are free left B-modules with the following bases:*

$$\{1,\ x^r \mid 1 \leq r < n\},$$
$$\{1 \otimes 1,\ x^r \otimes 1,\ 1 \otimes x^s,\ x^r \otimes x^s \mid 1 \leq r, s < n\}.$$

In particular note that $dim({}_B H) = n$ and $dim({}_B H \otimes_B H) = n^2$. These sets are also bases for the right modules H_B and $H \otimes_B H_B$.

Proof. First of all we show linear independence of the $1, x, \ldots, x^{n-1}$. To this end take $f_0, f_1, \ldots, f_{n-1} \in B$ and suppose

$$f_0 + f_1 x + \ldots + f_{n-1} x^{n-1} = 0.$$

Recall that $x^n = 0$, by definition, and multiply the equation above on the right by x^{n-1}, so that we have the new equation

$$f_0 x^{n-1} + f_1 x^n + \ldots + f_{n-1} x^{2n-2} = f_0 x^{n-1} = 0.$$

We may deduce that $f_1 = 0$, and we can repeat the process multiplying by $n-2$, $n-3$ etc. Finally we deduce that $f_0 = f_1 = \ldots = f_{n-1} = 0$ and thus $1, x, \ldots, x^{n-1}$ is a linearly independent set of B-module elements. It is clear that ${}_BH$ is generated by these elements, so they form a basis.

Note that $Bx = xB$ and so the basis $\{1,\ x^r \mid 1 \leq r < n\}$ of ${}_BH$ is also a basis for H_B. Using basic tensor product theory we see $\{1, x^r\} \otimes_B \{1, x^s\}$ is a basis for $H \otimes_B H$.

Obviously we may flip all of the prior arguments and get the same bases for H_B and $H \otimes_B H_B$. □

Theorem 4.1.4. *The depth $d(B, H)$ is 3.*

Proof. Let m be a natural number and denote by (m) the modulo value $m(\text{mod } n)$. By this convention $\psi^m = \psi^{(m)}$ for all $m \in \mathbb{N}$ because it is an n^{th} root of unity. Define a map $F : H \otimes_B H \to n \cdot H$ on the basis (and extend by left and right B-action) as follows:

$$x^r \otimes x^s \mapsto (x^{(r+s)})_{r+1}.$$

We may take $r, s = 0$ to ensure the whole basis is considered. F is a homomorphism of B-bimodules, for

$$g(x^r \otimes x^s) = \psi^{r+s}(x^r \otimes x^s)g$$

and $g(x^{(r+s)})_{r+1} = \psi^{r+s}(x^{(r+s)})_{r+1}g$. It is not hard to see that F is surjective, because $(x^i)_j$ is a basis of $n \cdot H$ and if $i \geq j-1$ then $F(x^{j-1} \otimes x^{i-j+1}) = (x^i)_j$ and if $i < j-1$ then $F(x^{j-1} \otimes x^{n-j+i+1}) = (x^i)_j$. We prove that F is a monomorphism by showing that it is non-zero on the basis elements. Notice that $F(x^r \otimes x^s) = (x^{(r+s)})_{r+1}$ can not take a zero value because $x^m = 0$ only for $m \geq n$ whereas $0 \leq (r+s) < n$ □

4.2 The Algebra $\mathbb{C}\#H_2$

In this example we work with the coopposite of Sweedler's algebra H_2, considered as an $\mathbb{R}$-algebra, where $\mathbb{R}$ obviously has primitive square roots of unity. We may consequently consider $\mathbb{C}$ as an H_2^{cop}-module algebra, so the smash product $\mathbb{C}\#H_2^{cop}$ exists. A definition and details of this object are given in [36], [35].

The definition of Sweedler's algebra is explicitly given by

$$H_2 := \mathbb{R}\langle 1,\ g,\ x,\ gx |\ g^2 = 1,\ x^2 = 0,\ gx = -xg\rangle.$$

Now, following page 52 of the book [36] we define $H := H_2^{cop}$, that is the coopposite Hopf algebra. The Hopf algebra structure of the Taft algebras is given in the previous example, so the coopposite case is with reversed comultiplication: $\Delta(x) = x \otimes 1 + g \otimes x$ and $\Delta(g) = g \otimes g$, then $S(g) = g$ and $S(x) = xg$, also $\epsilon(g) = 1,\ \epsilon(x) = 0$.

Notice that H is neither commutative nor cocommutative. Furthermore H is not semisimple by Lemmas 2.2.9 and 2.2.10, and because $x \in$rad(H).

The complex numbers $\mathbb{C}$ form a 2-dimensional $\mathbb{R}$-algebra in the most natural way; basis $\{1, i\}$. Furthermore H above acts on $\mathbb{C}$ in the following manner:

$$g \cdot 1 = 1,\ g \cdot i = -i$$
$$x \cdot 1 = 0,\ x \cdot i = 1$$

Thus $\mathbb{C}$ is an H-module algebra and we can form a smash product $\mathbb{C}\#H$. Write

$$B := \mathbb{C}\#H.$$

This is of course an 8-dimensional algebra with generators those of H plus i, and with additional relations:

$$gi = -ig,\ xi = 1 - ix.$$

In [36, Sec.4.4.8] Montgomery discusses some important technical details of H. In particular she produces left and right integrals $t = (1+g)x \in \int_H^l$ and $t' = (1-g)x \in \int_H^r$. Abstractly she shows that this is not a simple algebra, but more concretely she is able to decompose the algebra into 2 ideals:

$$\mathbb{C}\#H = (t) \oplus (t'), \tag{4.1}$$

where $(t) := BtB$ and $(t') := Bt'B$ are ideals because Lemmas 3.3.8 and 3.3.7 prove this generally. These ideals are 4-dimensional and can be written concisely as

$$(t) = \mathbb{C}t\mathbb{C} = \mathbb{R}\{t, it, ti, iti\},$$

$$(t') = \mathbb{C}t'\mathbb{C} = \mathbb{R}\{t', it', t'i, it'i\}.$$

In particular we know the $\mathbb{R}$-bases explicitly.

Lemma 4.2.1. *The H-bimodules (t) and (t') are indecomposable, and consequently are indecomposable B-bimodules.*

Proof. We show that $E := End(_H(t)_H) \cong \mathbb{R}$ and thus this algebra has no non-trivial idempotents and we apply Lemma 2.2.4. Take an arbitrary $f \in E$ then we can express f by its action on the basis of (t), which is given above. For example suppose $f(t) = \alpha t + \beta it + \gamma ti + \delta iti$, for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. By f being a B-bimodule homomorphism the element $f(t)$ must satisfy $tf(t) = f(t)t = 0$ and thus $f(t) = \alpha t$. Now $f(iti) \in (t)$ and must satisfy $xf(iti)x = f(t) = \alpha t$ and $gf(iti) = -f(iti)g = -f(iti)$, which ensure that $f(iti) = \alpha(iti)$. Similar processes for $f(it)$, $f(ti)$ show $f = \alpha \cdot id$ and since f was arbitrary we are done.

Finally, any B-bimodule M which can be written as $M = M_1 \oplus M_2$ for B-bimodules M_1, M_2, has this property as an H-bimodule because $H \subseteq B$. □

Proposition 4.2.2. *As H-bimodules (t) and (t') are not isomorphic. In particular $Indec(_HB_H) = \{(t), (t')\}$.*

Proof. We show that any B-bimodule map $(t) \to (t')$ must be zero. Take such a map ϕ then $\phi(t)t' = \phi(tt') = 0$ suggests that $\phi(t) = \alpha t' + \beta it' + \gamma t'i + \delta it'i$, however we also have $t'\phi(t) = 0$ and thus $\phi(t) = \alpha t'$. The only valid case is where $\alpha = 0$, for analyse $ti\phi(t)$. □

Corollary 4.2.3. *The isomorphism classes of indecomposables $Indec(_BB_B)$, $Indec(_HB_B)$ and $Indec(_BB_H)$, all consist of the bimodules $\{(t), (t')\}$.*

One can show that $\mathbb{C}\#H$ is semisimple using Lemma 2.2.7 because $\mathbb{C}\#H$ contains no strongly nilpotent element. The only candidate would be nilpotent x but the following theorem shows it is not strongly nilpotent.

Lemma 4.2.4. *Element xi is an idempotent. In particular $(xi)^k \neq 0$ for all $k \in \mathbb{N}$.*

Proof. $(ix)(ix) = i(xix) = i(1 - ix)x = ix$ since $x^2 = 0$. □

Given that $\mathbb{C}\#H$ is semisimple we apply the Krull-Schmidt theorem to decomposition (4.1) and this tells us that (t) and (t') are simple H, H-B and B-H-bimodules. Moreover because $\mathbb{C}\#H$ is finite dimensional semisimple the Artin-Wedderburn theorem gives a decomposition as $\mathbb{R}$-algebras into $M_{i_1}(D_{i_1}) \oplus \ldots, \oplus M_{i_t}(D_{i_t})$, where $D_{i_1}, \ldots, D_{i_t}$ are division rings over k.

Theorem 4.2.5. *As $\mathbb{R}$-algebras $\mathbb{C}\#H$ is isomorphic to*

$$M_2(\mathbb{R}) \oplus M_2(\mathbb{R}),$$

where $(t) \cong (t') \cong M_2(\mathbb{R})$ as algebras.

Proof. First of all write $e_1 = \frac{it}{2}$ and $e_2 = \frac{ti}{2}$, then $(t) = Be_1 \oplus Be_2$. This is easily verified by writing $Be_1 = \mathbb{R}\{t, it\}$ and $Be_2 = \mathbb{R}\{iti, ti\}$. If we define $e_3 = \frac{it'}{2}$ and $e_4 = \frac{t'i}{2}$ then $(t') = Be_3 \oplus Be_4$ in a very similar way.

Now, $(t) = \mathbb{R}\{t, it, iti, ti\}$ has an algebra structure [given by multiplication of basis elements]. In particular note the two definitive properties:

$$tit = (1+g)xi(1+g)x = (1+g)^2xix = (1+2g+g^2)x(1-xi) = 2t \qquad (4.2)$$

$$t^2 = (1+g)x(1+g)x = (1+g)(1-g)x^2 = 0. \qquad (4.3)$$

Therefore we have an identity element $e_1 + e_2 \in (t)$, and another $e_3 + e_4 \in (t')$; indeed (t') is also an algebra.

The [first] map we need is $(t) \to M_2(\mathbb{R})$ defined by

$$e_1 = \frac{it}{2} \mapsto e_{11}, \ \frac{iti}{2} \mapsto e_{12},$$
$$e_2 = \frac{ti}{2} \mapsto e_{22}, \ \frac{t}{2} \mapsto e_{21}.$$

where $\{e_{ij}\}$ is the canonical basis for the 2×2 matrices. This is seen to be an algebra map by going through the matrix unit multiplication property $e_{ij}e_{kl} = \delta_{jk}e_{il}$ and noting for example

$$\frac{it}{2}\frac{iti}{2} = \frac{ititi}{4} = \frac{2iti}{4} = \frac{iti}{2},$$
$$\frac{t}{2}\frac{ti}{2} = \frac{tti}{4} = 0.$$

This is a bijection because it is clearly injective and dimensions are equal.

We may write $(t') = \mathbb{R}\{t', it', t'i, it'i\}$ and analogues (4.2) and (4.3) hold: $it'i = 2t'$ and $t'^2 = 0$. Thus as algebras $(t) \cong (t')$ and our second map $(t') \to M_2(\mathbb{R})$ is given by $e_3 \mapsto \bar{e}_{11}$, $e_4 \mapsto \bar{e}_{22}$, $\frac{it'i}{2} \mapsto \bar{e}_{21}$ and $\frac{t'}{2} \mapsto \bar{e}_{12}$. (We write $\bar{e}_{ij}$ purely to distinguish from e_{ij}.) $\square$

One immediately corollary of this theorem is that $\mathbb{C}\#H$ is not a Hopf algebra. For Hopf algebras K we have the counit map $\epsilon : K \to k$, this makes (H, ϵ) an augmented algebra.

Corollary 4.2.6. *With the given algebra structure* $\mathbb{C}\#H$ *does not have any Hopf algebra structure.*

Proof. As mentioned above if $\mathbb{C}\#H \cong M_2(\mathbb{R}) \oplus M_2(\mathbb{R})$ had a Hopf algebra structure it would have a counit. We simply notice that every non-zero linear map $\epsilon : M_2(\mathbb{R}) \to \mathbb{R}$ will not be multiplicative. Write $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ for the canonical basis, taking $e_{ij} \notin ker\epsilon$ we may write $e_{ij} = e_{ik}e_{kj}$ and so for any $1 \leq k \leq 2$, $e_{ik} \notin ker\epsilon$. Take $k \neq i$ so $e_{ik}e_{ij} = 0$ and $\epsilon(e_{ik}e_{ij}) = 0$ whereas $\epsilon(e_{ik})\epsilon(e_{ij}) \neq 0$. □

Lemma 4.2.7. *Some important multiplication properties are:* $tt' = tit' = t'it = t't = 0$, *and moreover*

$$\begin{aligned} ti &= (1+g) - it' \\ it &= (1-g) - t'i \\ t'i &= (1-g) - it \\ it' &= (1+g) - ti. \end{aligned}$$

4.2.1 Indecomposables of $B \otimes_H B$

Lemma 4.2.8. *Given that* $B = Be_1 + Be_2 + Be_3 + Be_4$ *there are* 6 *generators of the* B*-bimodule* $B \otimes_H B$*:*

$$\{e_1 \otimes e_1,\ e_2 \otimes e_2,\ e_3 \otimes e_3,\ e_4 \otimes e_4,\ e_2 \otimes e_3,\ e_4 \otimes e_1\}$$

Proof. We know that $B = \bigoplus^4 Be_i = \bigoplus^4 e_iB$ then $B \otimes_H B$ can be written as $\bigoplus Be_i \otimes e_jB$. Now there are certain obvious $k, l \in \mathbb{N}$ for which $e_k \otimes e_l$ will be zero, for we use the previous lemma to eliminate some. Notice that

$$iX \otimes iY = 0 = Ui \otimes Vi$$

if $(X, Y) \in \{(t, t'),\ (t', t)\}$ and $(U, V) \in \{(t, t'),\ (t', t)\}$. Similarly $iR \otimes Si = 0$ for all $R = t, t'$ and $S = t, t'$.

We've discovered 8 elements $e_k \otimes e_l$ who are definitely zero, that leaves 8 unchecked. The potential non-zero $e_k \otimes e_l$ are

$$\begin{aligned} &e_1 \otimes e_1,\ e_2 \otimes e_2,\ e_3 \otimes e_3,\ e_4 \otimes e_4, \\ &e_2 \otimes e_3,\ e_2 \otimes e_1,\ e_4 \otimes e_1,\ e_4 \otimes e_3. \end{aligned}$$

Now we must strike $e_2 \otimes e_1$ and $e_4 \otimes e_3$ from our list because of the properties in the previous lemma: $ti \otimes it = ((1+g) - it') \otimes it = (1+g) \otimes it - it' \otimes it$, expanding the first term, $(1+g) \otimes it = 1 \otimes (1+g)it = 0$. All the other elements are verifiably non-zero and we are done. □

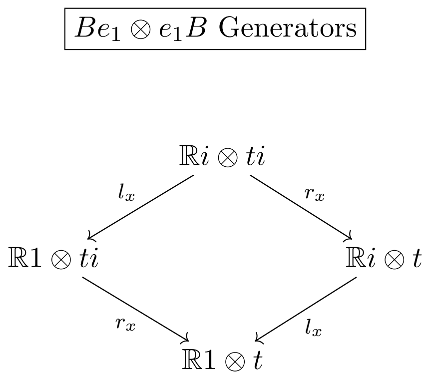
One final addition which can be made to the above lemma (which leads into the following proposition) is that $Be_1 \otimes e_1 B = Be_2 \otimes e_2 B$ and similarly for e_3, e_4. Now we have 4 distinct sets closed under left and right B multiplication.

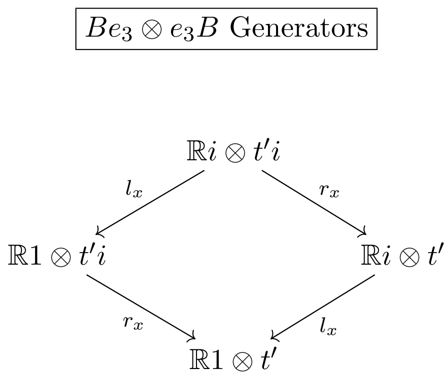
Proposition 4.2.9. *As a vector space* $B \otimes_H B$ *is* 16*-dimensional and has basis* $X_1 \cup X_2 \cup X_3 \cup X_4$ *where*

$$
\begin{aligned}
X_1 &:= \{1 \otimes t,\ 1 \otimes ti,\ i \otimes t,\ i \otimes ti\},\\
X_2 &:= \{1 \otimes t',\ 1 \otimes t'i,\ i \otimes t',\ i \otimes t'i\},\\
X_3 &:= \{ti \otimes it',\ iti \otimes it',\ iti \otimes it'i,\ ti \otimes it'i\},\\
X_4 &= \{t'i \otimes it,\ it'i \otimes it,\ it'i \otimes iti,\ t'i \otimes iti\}.
\end{aligned}
$$

Proof. As stated just before the lemma we have 4 distinct sets invariant under left and right B multiplication: they are $Be_1 \otimes e_1 B$, $Be_3 \otimes e_3 B$, $Be_2 \otimes e_3 B$ and $Be_4 \otimes e_1 B$.

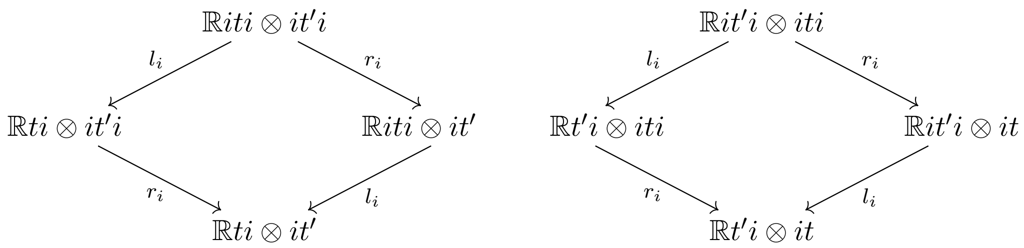
Let r_x and l_x denote right and left multiplication by $x \in B$. Rewrite $e_1 \otimes e_1 = ti \otimes ti = i \otimes t$ and the same for $e_3 \otimes e_3$ then the following diagrams commute:





Moreover these diagrams show that $Be_1 \otimes e_1 B$ and $Be_3 \otimes e_3 B$ are generated by 4 elements as a $\mathbb{R}$-vector space. Similarly the diagrams corresponding to $Be_2 \otimes e_3 B$ and $Be_4 \otimes e_1 B$ are

$\boxed{Be_2 \otimes e_3B \text{ Generators}}$ $\boxed{Be_4 \otimes e_1B \text{ Generators}}$



Notice that we easily deduce the diagram of $Be_4 \otimes e_1B$ by swapping t and t' in the diagram of $Be_2 \otimes e_3B$. So there are 4 distinct sets, one for each diagram, with the property of being closed up to scalar multiple under left or right B-multiplication. We've proved the lemma. □

If we write the B-bimodule generated by X_i as Y_i, then $B \otimes_H B = \bigoplus_{i=1}^4 Y_i$. I state without proof that that $Y_i \ncong Y_j$ for $i \neq j$. (A simple proof involves taking arbitrary $f : Y_i \to Y_j$ an H-bimodule map and showing that f is not an isomorphism.)

Remark 4.2.10. Looking at the diagrams in the previous proof, we see that $BX_iB = \mathbb{C}X_i\mathbb{C}$. Furthermore $\mathbb{C}X_1\mathbb{C} = \mathbb{C}e_1 \otimes e_1\mathbb{C}$ and similarly every other X_j is a cyclic $\mathbb{C}$-bimodule. However it is a fact that because $\mathbb{C}$, as a division ring it always acts faithfully, thus every cyclic module or bimodule is simple. So every Y_i is a simple B-bimodule, but what about their H-bimodule structure? H is not even semisimple, so indecomposables won't always be simple modules.

Proposition 4.2.11. *The H-bimodules Y_i are indecomposable and in particular $Indec({}_HB \otimes_H B_H) = \{Y_1, Y_2, Y_3, Y_4\}$.*

Proof. The method we would use is that of Proposition 4.2.2, showing that $End({}_HY_{iH}) = \mathbb{R}id$. The cases of Y_1 and Y_2 are taken care of by Proposition 4.2.2 and noting the H-bimodule isomorphisms

$$(t) = \mathbb{R}\{t, it, ti, iti\} \cong \mathbb{R}\{1 \otimes t, it \otimes 1, 1 \otimes ti, it \otimes i\} = Y_1 \text{ and } (t') \cong Y_2$$

We handle the case of Y_3 in a careful manner, and show that its endomorphism ring is the algebra $\mathbb{R}id$. Take any $f \in End({}_HY_{3H})$ then $tf(ti \otimes it') = f(ti \otimes it')t = 0$ implies $f(ti \otimes it') = \alpha ti \otimes it'$. □

Naturally this will tell us that $B \otimes_H B \cong Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$ as B-H and H-B-bimodules, thus we can ensure the extension $H \subseteq B$ does not satisfy depth 3 as follows:

Proposition 4.2.12. *The extension $H \subseteq B$ does not satisfy the depth 3 condition. In other words $d(H, B) > 3$.*

Proof. Proposition 4.2.11 above allows us to write $\mathrm{Indec}({}_HB_H) = \{(t), (t')\}$ but the previous proposition proves the equality $\mathrm{Indec}({}_HB \otimes_H B_H) = \{Y_1, Y_2, Y_3, Y_4\}$. Since the sets of isomorphism-unique indecomposables are unequal, Lemma 1.2.15 implies $d(H, B) > 3$. □

4.2.2 Indecomposables of $B \otimes_H B \otimes_H B$.

We start as before by decomposing $B^{\otimes_H 3}$ using our generators e_1, e_2, e_3, e_4 and iti. What we must do is write B as three different sums: $B = \bigoplus_{i=1}^4 Be_i = \bigoplus_{i=1}^4 e_iB$ and $B = H(iti)H \oplus H(it'i)H$. (Notice that $H(iti)H = (t)$ and thus is indecomposable.) Explicitly:

$$B^{\otimes_H 3} = (\bigoplus_{i,j=1}^{4} Be_i \otimes H(iti)H \otimes e_jB) \oplus (\bigoplus_{i,j=1}^{4} Be_i \otimes H(it'i)H \otimes e_jB),$$

where we write $\otimes$ to denote $\otimes_H$. Now because we may commute the H across the $\otimes$ and also because $Be_iH = Be_i$ we may write $B^{\otimes_H 3}$ as:

$$(\bigoplus_{i,j=1}^{4} Be_i \otimes iti \otimes e_jB) \oplus (\bigoplus_{i,j=1}^{4} Be_i \otimes it'i \otimes e_jB).$$

Now the next step is to see for which $1 \leq i, j \leq 4$ the elements $e_i \otimes iti \otimes e_j$ and $e_i \otimes it'i \otimes e_j$ are zero. But recall Lemma 4.2.8 from before, which tells us exactly how to do this. Notice that $iti = 2e_1i = 2ie_2$ and $it'i = 2e_3i = 2ie_4$.

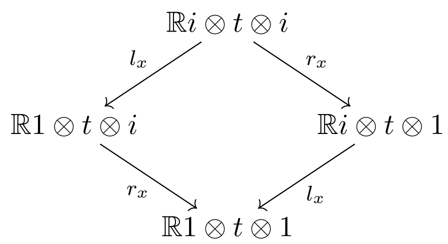
Lemma 4.2.13. *Upon elimination we are left with 8 unique non-zero elements:*

$$e_1 \otimes iti \otimes e_2,\ e_3 \otimes it'i \otimes e_4,$$
$$e_1 \otimes iti \otimes e_3,\ e_3 \otimes it'i \otimes e_1,$$
$$e_4 \otimes iti \otimes e_2,\ e_2 \otimes it'i \otimes e_4,$$
$$e_4 \otimes iti \otimes e_3,\ e_2 \otimes it'i \otimes e_1.$$

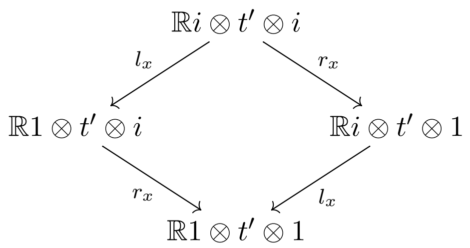
Label these elements $V_1, \ldots, V_8$ (left to right, top to bottom) and write $W_i := BV_iB$ then $B^{\otimes_H 3} = \bigoplus_{i=1}^{8} W_i$.

We now may explicitly write down diagrams for each indecomposable bimodule; or rather the generators thereof.

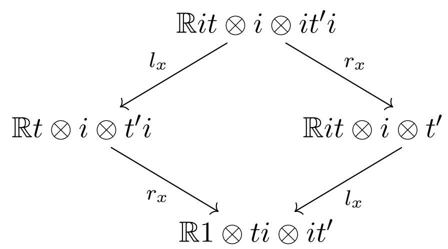
W_1 Generators



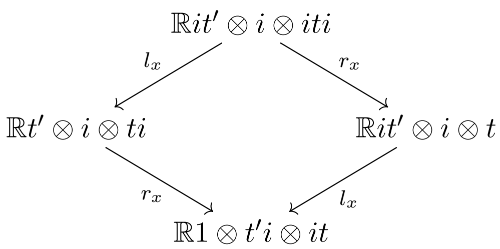
W_2 Generators



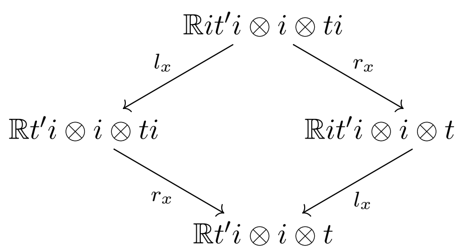
W_3 Generators



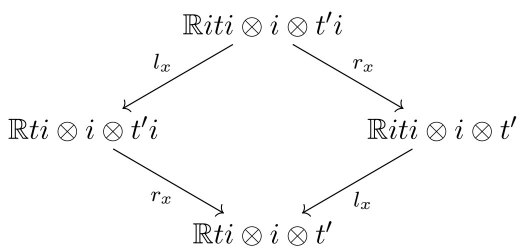
W_4 Generators



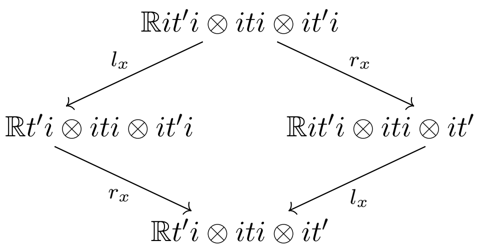
W_5 Generators



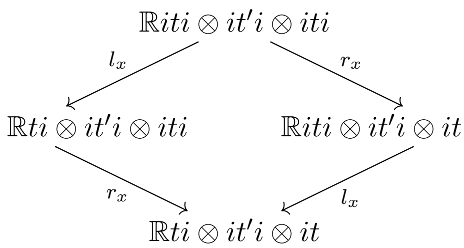
W_6 Generators



$\boxed{W_7 \text{ Generators}}$



$\boxed{W_8 \text{ Generators}}$



Lemma 4.2.14. *Each W_i is an indecomposable H-bimodule.*

Proof. This proof relies on checking each diagram above, comparing properties of all the generators (which are $\mathbb{R}$-basis elements). In particular we notice that for each W_i and every pair of basis elements e, e' either $\text{Ann}({}_He) \neq \text{Ann}({}_He')$ or $\text{Ann}(e_H) \neq \text{Ann}(e'_H)$. (It is enough to check for example $l_t(e) = 0$ but $l_t(e') \neq 0$.) The net effect of this procedure is to ensure that all H-bimodule maps $f : W_i \to W_i$ belong to $\mathbb{R}id$ and we are done by Lemma 1.2.15. $\square$

Remark 4.2.15. The irreducible H-bimodules $W_1, \ldots, W_8$ have B-H and H-B-bimodule structures to consider as well, indeed they will be irreducible.

Theorem 4.2.16. *B-bimodule isomorphisms: $W_1 \cong W_8$ and $W_2 \cong W_7$ and $W_3 \cong W_6$ and $W_4 \cong W_5$.*

Proof. Define maps as follows (and expand by H-multiplication):

$$W_1 \to W_8 : \quad i \otimes t \otimes i \mapsto iti \otimes it'i \otimes iti.$$

$$W_2 \to W_7 : \quad i \otimes t' \otimes i \mapsto it'i \otimes iti.$$

$$W_3 \to W_6 : \quad it \otimes i \otimes it'i \mapsto iti \otimes i \otimes t'i.$$

$$W_4 \to W_5 : \quad it' \otimes i \otimes iti \mapsto it'i \otimes i \otimes ti \otimes it'i.$$

Now it is clear that those maps are bijections, for we are mapping one $\mathbb{R}$-basis element to another - moreover all W_i are cyclic B-bimodules, and we are mapping one cyclic generator to another, thus the maps are B-bilinear. Recall the indecomposable B- and H-bimodules of $B \otimes_H B$, which we called Y_1, Y_2, Y_3, Y_4. $\square$

Theorem 4.2.17. *We have B-bimodule isomorphisms:* $Y_1 \cong W_1$, $Y_2 \cong W_2$, $Y_3 \cong W_3$ *and* $Y_4 \cong W_4$.

Proof. The map $Y_1 \to W_1$ is given by $i \otimes ti \mapsto i \otimes t \otimes i$ and extending H-bilinearly. It clearly satisfies the conditions of a B-bimodule map because, on the elements given it is linear in x, g and i. The other maps are defined identically, mapping the top-most element in the respective diagram to the top-most in the other diagram. □

Corollary 4.2.18. *The unique indecomposable B-H-bimodules Indec*$({}_B B^{\otimes_H 3}{}_H)$ *are the modules* $\{W_1, W_2, W_3, W_4\} = \{Y_1, Y_2, Y_3, Y_4\}$. *The same set is given for the H-B-bimodule structure.*

Theorem 4.2.19. *The minimum depth* $d(H, B)$ *is* 4.

Proof. The indecomposables $\text{Indec}({}_B B^{\otimes_H 2}{}_H)$ and $\text{Indec}({}_B B^{\otimes_H 3}{}_H)$ are the same modules. By Lemma 1.2.15 we have shown left depth 4. The right depth 4 condition is similarly satisfied. □

Chapter 5

Module Depth

The idea of algebraic depth for ring extension is one which involves bimodules of the form

$$H \otimes_R \ldots \otimes_R H.$$

Our choice of R and H change the depth value in ways we cannot completely describe. In the beginning of this chapter another type of depth will be defined, which depends on Hopf algebras and modules rather than algebra extensions and bimodules. This new type of depth is given the name *module depth*, and is fascinating in its own right, but more importantly will be used later in the chapter to link two major results.

In this chapter we only consider finite dimensional Hopf algebras unless stated otherwise. We are very interested in gaining intuition on whether or not finite dimensional Hopf algebra extensions have finite depth. Notice that very many of the results in Chapter 1 apply to the module categories used below.

5.1 Definitions

Given an Hopf algebra H then the category ${}_H\mathcal{M}$ of left H-modules is a finite tensor category: which means, given two modules $A, B \in {}_H\mathcal{M}$, then $A \otimes B$ is a module via

$$h \cdot (a \otimes b) = \sum (h_{(1)} \cdot a) \otimes (h_{(2)} \cdot b).$$

This action is better know as the *diagonal action*. By repeated application of $\otimes$, in a finite tensor category we may form the tensor product of n arbitrary modules, for any $n \in \mathbb{N}$.

Notice that ${}_H\mathcal{M}$ is an additive monoidal category as in Section 1.2. Now we use the tensor product $\otimes$ over the field k and the diagonal action described above. This means we have similarity between H-modules and in particular:

Definition 5.1.1. Let $A \in {}_H\mathcal{M}$, we say that A has depth n (for $n \in \mathbb{N}$) when $A^{\otimes n+1} \sim A^{\otimes n}$ as H-modules.

Very importantly let us define $A^0 := k$, so that there exists a module depth 0. Some basic results are immediate:

Lemma 5.1.2. *Let M, N be H-modules, and assume that $M \mid N$ then this is equivalent to there being two H-module maps $f : M \to N$ and $g : N \to M$ such that $g \circ f = id$.*

Proof. We follow exactly the same logic as in Lemma 1.2.3 for ring depth. □

Lemma 5.1.3. *Let $n, m \in \mathbb{N}$, if an H-module V has depth nm, then $V^{\otimes n}$ has depth m.*

Proof. We note first of all that if V satisfies $V^{\otimes nm+1} \sim V^{\otimes nm}$ then it also satisfies $V^{\otimes nm+1} = V \otimes V^{\otimes nm} \sim V \otimes V^{\otimes nm+1} = V^{\otimes nm+2}$ and so on. Then $V^{\otimes nm} \sim V^{\otimes nm+1} \sim \ldots \sim V^{\otimes nm+(n-1)} \sim V^{\otimes nm+n} = (V^{\otimes n})^{\otimes m+1}$. □

Lemma 5.1.4. *Every H-module algebra A satisfies $A \mid A \otimes A$ via the map*

$$\begin{aligned} \phi : & A \to A \otimes A, \\ & a \mapsto a \otimes 1. \end{aligned}$$

Similarly, for any H-module coalgebra C we get $C \mid C \otimes C$ as H-modules, via the comultiplication $\Delta : C \to C \otimes C$.

Proof. This is a H-module mapping because if $h \in H$ then $h = \sum h_{(1)}\epsilon(h_{(2)})$ by basic Hopf algebra properties and moreover

$$\begin{aligned} \phi(h \cdot a) = (h \cdot a) \otimes 1 &= (\sum h_{(1)}\epsilon(h_{(2)}) \cdot a) \otimes 1 \\ &= \sum h_{(1)} \cdot a \otimes \epsilon(h_{(2)})1 \\ &= \sum h_{(1)} \cdot a \otimes h_{(2)} \cdot 1 \\ &= h \cdot (a \otimes 1). \end{aligned}$$

For the module coalgebra C basic coalgebra properties tell us that $(1 \otimes \epsilon_C) \circ \Delta = id$, where ϵ_C is the counit of C. By definition of module coalgebra Δ is a H-homomorphism and we are done. □

Supposing A is an H-module algebra then $A^{\otimes n}$ is also an H-module algebra, for all $n \in \mathbb{N}$, via the canonical multiplication. By a slight extension of the proof above one deduces $A^{\otimes n} \mid A^{\otimes n+1}$ as H-modules. On the other hand if M is an H-module which is not necessarily an H-module algebra $M^{\otimes n}$ may theoretically not divide $M^{\otimes n+1}$. We would then have to verify $M^{\otimes n} \mid q \cdot M^{\otimes n+1}$ (and of course $M^{\otimes n+1} \mid p \cdot M^{\otimes n}$) for the depth n condition.

Lemma 5.1.5. *If A is an H-module algebra or coalgebra then $A^{\otimes t} \mid A^{\otimes t+1}$, for every $t \geq 1$. In particular if $A \mid q \cdot k$ for some $q \in \mathbb{N}$, then $k \mid dim(A) \cdot A$.*

Proof. The first part is clear. Suppose that $A \mid q \cdot k$, by Lemma 5.1.2 there is an injective H-module map $f : A \to q \cdot k$. Now because k is a trivial H-module and f in injective we deduce that A must be a trivial module as well. In particular if $e_1, \ldots, e_n$ is k-basis of A then $h \cdot e_i = \epsilon(h) e_i$, it follows that $A \cong dim(A) \cdot k$. □

Recall the Krull-Schmidt theorem for modules (Proposition 2.2.2). Consequently we may write every finite dimensional H-module as a direct sum of unique indecomposable modules. For two H-modules M and N, $M \sim N$ if and only if $\text{Indec}(M) = \text{Indec}(N)$, this is a usage of Lemma 1.2.15. One can use this fact in relation to finite representation type as follows:

Proposition 5.1.6 ([21], Prop.4.8)**.** *Suppose that H has finite representation type then every finite dimensional module algebra A and every finite dimensional module coalgebra C in $\mathcal{M}_H$ has finite depth.*

Proof. Given a module algebra A (with the same proof applied to C), use Lemma 5.1.4 to see that $A \mid A \otimes A$, and by tensoring $A^{\otimes t} \mid A^{\otimes t+1}$, $t \in \mathbb{N}$. By Lemma 1.2.15 this is equivalent to $\text{Indec}(A^{\otimes t}) \subseteq \text{Indec}(A^{\otimes t+1})$. At some point this inclusion chain must stabilise because H has finite representation type, so the indecomposable H-modules (up to isomorphism) are finitely many. □

Proposition 5.1.7 ([21], Prop.3.3)**.** *Given a Hopf algebra H then there are finitely many projective indecomposable H-modules up to isomorphism. Suppose P is a projective H-module algebra (or coalgebra), then $d(P, {}_H\mathcal{M})$ is finite.*

Proof. Write $H_H = \bigoplus_{r=1}^{n} I_r$ as a sum of its indecomposables, then $\{I_1, \ldots, I_n\}$ describe all the projective indecomposables up to isomorphism. Each I_r is projective by being a summand of the free module H_H. Let P be a general projective module then by definition there is another module X such that $P \oplus X \cong p \cdot H_H$ for some $p \in \mathbb{N}$. Clearly then $\mathrm{Indec}(P) \subseteq \mathrm{Indec}(H)$. Now by [9, Prop 2.1] (and the original idea in [27]) we are able to prove that $P^{\otimes t}$ are projective modules for all $t \in \mathbb{N}$ therefore $\mathrm{Indec}(P^{\otimes t}) \subseteq \mathrm{Indec}(H)$. Since there are finitely many projective indecomposable modules we are done. □

5.2 Direct Sums in Braided Subcategories

Given modules A, B in ${}_H\mathcal{M}$, then we show below that for certain H the depth of A, B and $A \oplus B$ are related. Notice the following result for module algebras.

Proposition 5.2.1. *Let A and E be finite dimensional H-module algebras such that $A \mid E$ and E has module depth n in ${}_H\mathcal{M}$, then A has module depth less than or equal to n.*

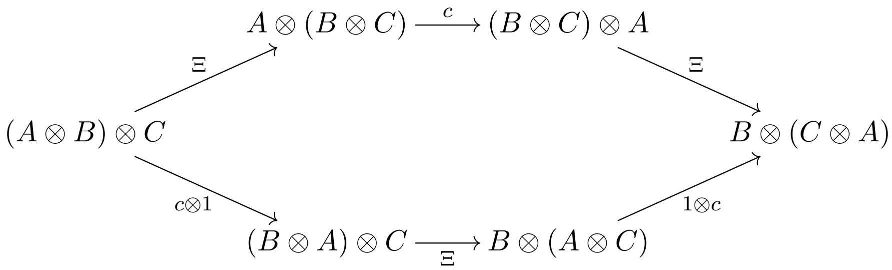
Proof. Since A is a module algebra we have already established that $A^{\otimes t} | A^{\otimes t+1}$ for all $t \in \mathbb{N}$. Notice that because $A|E$ we also have $A^{\otimes n+t} | E^{\otimes n+t}$ and thus $\mathrm{Indec}(A^{\otimes n+t}) \subseteq \mathrm{Indec}(E^{\otimes n+t})$ for every $t \in \mathbb{N}$. By basic properties $dim E = dim A^2$ and so E has a Krull-Schmidt decomposition (Theorem 2.2.2). Given that E has depth n specifically Lemma 1.2.15 applies which implies $\mathrm{Indec}(E^{\otimes n}) = \mathrm{Indec}(E^{\otimes n+t})$ for any $t \in \mathbb{N}$. Therefore the indecomposables of $A^{\otimes n+t}$ must stabilise for some t. □

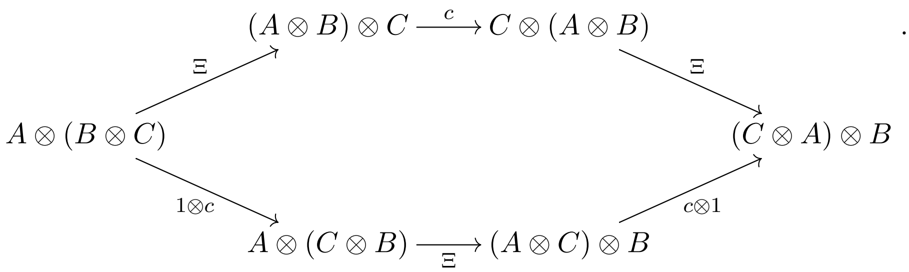
Recall the binomial formula for complex numbers x, y

$$(x+y)^n = \sum_{r=1}^{n} \binom{n}{r} x^r y^{n-r}.$$

The reason we remember this formula is so that it becomes an analogy for modules. First of all we limit our attention to braided subcategories:

Definition 5.2.2. Given a subcategory $\mathcal{C}$ of ${}_H\mathcal{M}$ we say it is braided when there exists a natural isomorphism $c_{A,B} : A \otimes B \to B \otimes A$ for each pair of modules A, B in $\mathcal{C}$, writing Ξ for the associativity isomorphisms the following diagrams should also commute:





Let H be a Hopf algebra with braided $\mathcal{C}$ as above, and A, B be modules in $\mathcal{C}$ then using an induction proof as with the binomial formula (and considering $c_{A,B} : A \otimes B \to B \otimes A$ in place of commutative multiplication):

$$(A \oplus B)^{\otimes n} \cong \bigoplus_{r=0}^{n} \binom{n}{r} \cdot A^{\otimes r} \otimes B^{\otimes n-r}. \tag{5.1}$$

Theorem 5.2.3. *Let H be a Hopf algebra with a braided subcategory $\mathcal{C} \subseteq {}_H\mathcal{M}$, given A and B in $\mathcal{C}$ write $E = A \oplus B$. Suppose that A and B have depth n and m respectively, then E satisfies depth $2M$ where $M := max\{n, m\}$.*

Proof. Suppose that A has depth n and B depth m then both satisfy the depth M condition. We may write the following expressions, based on (5.1) above:

$$\begin{aligned} E^{\otimes 2M} &\cong (A \oplus B)^{\otimes 2M} \cong \bigoplus_{r=0}^{2M} \binom{2M}{r} \cdot A^{\otimes(2M-r)} \otimes B^{\otimes r}, \\ E^{\otimes 2M+1} &\cong (A \oplus B)^{\otimes 2M+1} \cong \bigoplus_{r=0}^{2M+1} \binom{2M+1}{r} \cdot A^{\otimes(2M+1-r)} \otimes B^{\otimes r}. \end{aligned} \tag{5.2}$$

What we do now is look at the summands $A^{\otimes(2M+1-r)} \otimes B^{\otimes r}$ of (5.2):

- When $r = 0$ and $r = 2M + 1$ respectively, by the module depths of A and B,

$A^{\otimes 2M+1} \sim A^{\otimes 2M}$ and $B^{\otimes 2M+1} \sim B^{\otimes 2M}$.

- While $r < M$ the summands satisfy $A^{\otimes 2M+1-r} \otimes B^{\otimes r} \sim A^{\otimes 2M-r} \otimes B^{\otimes r}$.
- When $r = M$ the summand satisfies $A^{\otimes M+1} \otimes B^{\otimes M} \sim A^{\otimes M} \otimes B^{\otimes M}$.
- While $r \geq M+1$ the summands satisfy $A^{\otimes 2M+1-r} \otimes B^{\otimes r} \sim A^{\otimes 2M+1-r} \otimes B^{\otimes r-1}$.

In conclusion we have shown that in each line of (5.2) every summand is similar to a term of the other line and so $E^{\otimes 2M} \sim E^{\otimes 2M+1}$ and we are done. □

To specifically illustrate the idea of the proof let us do a small example. If we are considering that both A and B have depth 1 then $M = 2$ and

$$(A \oplus B)^{\otimes 2} \cong A^{\otimes 2} \oplus 2(A \otimes B) \oplus B^{\otimes 2}.$$

Then as in the proof

$$(A \oplus B)^{\otimes 3} \cong \underbrace{A^{\otimes 3}}_{\sim A^{\otimes 2}} \oplus 3(\underbrace{A^{\otimes 2} \otimes B}_{\sim A \otimes B}) \oplus 3(\underbrace{A \otimes B^{\otimes 2}}_{\sim A \otimes B}) \oplus \underbrace{B^{\otimes 3}}_{\sim B^{\otimes 2}} \sim (A \oplus B)^{\otimes 2}.$$

Finding braided subcategories of ${}_H\mathcal{M}$ is not too prohibitive a task. For instance if H is cocommutative then $M \otimes N \cong N \otimes M$ as H-modules, for all modules M,N. We can see this from the twist map $\tau : M \otimes N \to N \otimes M$ defined by $\tau(m \otimes n) = n \otimes m$, which is a k-linear bijection in general. When H is cocommutative we see that τ is an H-module mapping and therefore is an isomorphism. Another class of examples which are of interest to us are the Taft algebras H_n. In their paper [7, Corollary 3.7] the authors have calculated all tensor products of the finite dimensional indecomposable modules of the Taft algebras. Subsequently they have shown that $M \otimes N \cong N \otimes N$ for all such modules. Althought this does not strictly prove that the subcategory of finite dimensional modules is braided it is enough to apply (5.1) and the Theorem. For the class of quasitriangular Hopf algebras [36, pg.180] the whole module category ${}_H\mathcal{M}$ is braided.

5.3 Functors

Given a functor $F : {}_K\mathcal{M} \to {}_H\mathcal{M}$ between module categories, where K, H are Hopf algebras, we say F *preserves similarity* if when $M \sim N$ in ${}_K\mathcal{M}$ then $F(M) \sim F(N)$ in ${}_H\mathcal{M}$. This concept may be worth exploring in generality, there are published results about functors and algebraic depth bounds [4], [21], here we only introduce important examples for later use. Let $R \subseteq H$ be the usual finite dimensional Hopf algebra extension.

Lemma 5.3.1. *Given three Hopf algebras H, K and L, and two functors $F : {}_H\mathcal{M} \to {}_K\mathcal{M}$ and $G : {}_K\mathcal{M} \to {}_L\mathcal{M}$, then if F and G preserve similarity so does $G \circ F$.*

Example 13 (Dual)**.** Consider the contravariant functor $D : {}_H\mathcal{M} \to \mathcal{M}_H$, sending a left H-module $(M, \triangleright)$ to $(M^*, \triangleleft)$, where $M^* = \mathrm{Hom}_k(M, k)$ and which has right H-module structure defined by $(f \triangleleft h)(x) = f(h \triangleright f)$. A morphism $\sigma : M \to N$ under D becomes $D\sigma : N^* \to M^*$ in a canonical way: $D\sigma(h) = h \circ \sigma$.

Example 14. Given a morphism of algebras $f : H \to K$ we define a covariant functor $\Psi_f : \mathcal{M}_K \to \mathcal{M}_H$, sending an H-module $(M, \triangleleft)$ to $(M, \triangleleft_f)$, such that $m \triangleleft_f h = m \triangleleft f(h)$. Moreover given a morphism of K-modules $\sigma : M \to N$ we define $\Psi_f(\sigma) = \sigma$ as k-linear maps. Now if f above is an isomorphism we may find $f^{-1} : K \to H$ such that $f \circ f^{-1} = id_K$ and $f^{-1} \circ f = id_H$ so that $\Psi_f \circ \Psi_{f^{-1}}$ and $\Psi_{f^{-1}} \circ \Psi_f$ are the identity functors of their respective categories. In other words $\mathcal{M}_H$ and $\mathcal{M}_K$ become isomorphic as categories.

Write $\Psi := \Psi_S : \mathcal{M}_H \to \mathcal{M}_{H^{op}}$ where $S : H \to H^{op}$ is the antipode. Since canonically the categories $\mathcal{M}_{H^{op}}$ and ${}_H\mathcal{M}$ are isomorphic we may consider Ψ as taking right H-modules to left H-modules. By the properties after the definition of S, it is an algebra algebra anti-isomorphism whenever H is finite dimensional, in this case $\mathcal{M}_H$ and ${}_H\mathcal{M}$ are isomorphic as categories.

The following lemma is quite obvious but useful later, as is the fact that given a left H-module $(M, \triangleright)$ then Ψ sends this module to $(M, \triangleleft_S)$ with action $m \triangleleft_S h = S(h) \triangleright m$.

Lemma 5.3.2. *The contravariant functor $\Psi \circ D : {}_H\mathcal{M} \to {}_H\mathcal{M}$ preserves similarity.*

Proof. By the lemma above, if D and Ψ preserve similarity then so does $\Psi \circ D$. The case of D is taken care of in Lemma 5.7.2. Suppose that $M \mid N$, so there exist $f : M \to N$ and $g : N \to M$ which satisfy $g \circ f = id$. By definition of a functor $\Psi(id) = id$ and $\Psi(g \circ f) = \Psi(g) \circ \Psi(f)$. Therefore $\Psi(M) \mid \Psi(N)$. □

Notice that a similar proof also works for functors F which respect direct sums of modules: $F(X \oplus Y) \cong F(X) \oplus F(Y)$. So that such functors also preserve similarity. Let ${}_\epsilon k$ be the 1-dimensional left H-module defined by $h \cdot \lambda = \epsilon(h)\lambda$, the analogous 1-dimensional right module is given by k_ϵ, and similarly define ${}_\epsilon k_\epsilon$ in the bimodule case. Define a functor $\mathcal{Y}_R : {}_H\mathcal{M} \to {}_H\mathcal{M}$ by $\mathcal{Y}_R(M) = M \otimes_{R\,\epsilon} k_\epsilon$. By basic properties of tensor products $\mathcal{Y}_R(M \oplus N) \cong \mathcal{Y}_R(M) \oplus \mathcal{Y}_R(N)$ and the functor $\mathcal{Y}_R$ preserves similarity.

5.4 Endomorphism Module Algebras

The theorem in this section is original work. Start with an H-module algebra A over a field K and let $E := End_k(A)$ be the k-linear endomorphisms. It is common knowledge that E has an algebra structure given by $(gf)(x) = g(f(x))$. What is more interesting is the H-module structure of E where the action of $h \in H$ on $f \in E$ is given by

$$(h \cdot f)(x) = \sum h_{(1)} \cdot f(Sh_{(2)} \cdot x),$$

for all $x \in A$. This makes E an H-module algebra, for given $f, g \in E$ and $h \in H$

$$\begin{aligned}\sum (h_{(1)} \cdot g)(h_{(2)} \cdot f)(x) &= \sum (h_{(1)} \cdot g)(h_{(2)} \cdot f(Sh_{(3)} \cdot x)) \\ &= \sum h_{(1)} \cdot g(Sh_{(2)} \cdot (h_{(3)} \cdot f(Sh_{(4)} \cdot x))) \\ &= \sum h_{(1)} \cdot g(\epsilon h_{(2)} f(Sh_{(3)} \cdot x)) \\ &= \sum h_{(1)} \cdot g(f(Sh_{(2)} \cdot x)) \\ &= \sum (h \cdot gf)(x).\end{aligned}$$

In other words $h \cdot fg = (h_{(1)} \cdot f)(h_{(2)} \cdot g)$. Moreover $h \cdot id = \epsilon(h) id$.

Now we present an embedding $A \hookrightarrow E$, namely the map $\phi : A \to E$ where $\phi(a)$ is the endomorphism sending x to ax. That this is an embedding is clear, $\phi(a)(1) = a$. In fact $\phi(A)$ is a subalgebra of E:

$$\phi(a)\phi(b)(x) = a(bx) = (ab)x = \phi(ab)(x).$$

Proposition 5.4.1. *$A \mid E$ as H-modules.*

Proof. To prove this proposition we use Lemma 1.2.3. The map ϕ has a left inverse (is a split monomorphism) $\psi : E \to A$ defined by $\psi(f) = f(1)$. Clearly $(\psi\phi)(a) = a1 = a$. All we need show is that ϕ and ψ are H-module maps. Take $h \in H$ so $\phi(h \cdot a)(x) = (h \cdot a)x$ and $h \cdot \phi(a)$ sends $x \in A$ to

$$\begin{aligned}\sum h_{(1)} \cdot a(Sh_{(2)} \cdot x) &= \sum (h_{(1)} \cdot a)(h_{(2)} \cdot Sh_{(3)} \cdot x) \\ &= \sum (h_{(1)} \cdot a)(\epsilon(h_{(2)}) \cdot x) \\ &= (h \cdot a)x.\end{aligned}$$

Checking that ψ is an H-module map is easy:

$$\psi(h \cdot f) = (h \cdot f)(1) = \sum h_{(1)} \cdot f(Sh_{(2)} \cdot 1) = \sum h_{(1)} \cdot f(\epsilon(Sh_{(2)})) = h \cdot f(1).$$

□

Corollary 5.4.2. *Let A, E be as above, and suppose that E has depth n then A has finite depth and $d(A, {}_H\mathcal{M}) \leq n$.*

Proof. Directly apply Proposition 5.2.1. □

We are able to construct a converse to the corollary above, but as with the last section the correct setting is in braided subcategories. In the well known lemma below A^* is in fact the module $(\Psi \circ D)(A)$ as described in Section 5.3.

Lemma 5.4.3. *The H-module $A \otimes A^*$ is isomorphic to E via the map $v(a \otimes f)(x) = af(x)$.*

Proof. Write the k-basis of A as $\{a_1, \ldots, a_t\}$, then by basic linear algebra there is the basis $\{a_1^*, \ldots, a_t^*\}$ of the dual A^* satisfying $a_i^*(a_j) = \delta_{ij}$, the Kronecker deltas. Now v is injective because given any basis element $a_i \otimes a_j^* \in A \otimes A^*$ we have $v(a_i \otimes a_j^*)(a_j) = a_i a_j^*(a_j) = a_i$, by linear independence of the basis of A we deduce that $kerv = 0$. It follows that v is a linear isomorphism because $dim(A \otimes A^*) = dim(A)^2 = dim(E)$.

All that remains is to show that v is a H-module homomorphism. Given $h \in H$ and $a \otimes f \in A \otimes A^*$ we have the following:

$$\begin{aligned} v(h \cdot (a \otimes f))(x) &= v(h_{(1)} \cdot a \otimes h_{(2)} \cdot f)(x) \\ &= (h_{(1)} \cdot a)(h_{(2)} \cdot f)(x) \\ &= (h_{(1)} \cdot a) f(Sh_{(2)} \cdot x) \\ &= (h \cdot v(a \otimes f))(x). \end{aligned}$$

This completes the proof. □

Proposition 5.4.4. *Suppose that A and A^* belong to a braided subcategory of ${}_H\mathcal{M}$ then A and E have the same depth (even in the infinite case).*

Proof. Suppose that A has depth n, so that $A^{\otimes n+1} \sim A^{\otimes n}$. By 5.3.2 $(\Psi \circ D)$ preserves similarity, in other words $(A^*)^{\otimes n+1} \sim (A^*)^{\otimes n}$ also. By the braided property we may write $(A \otimes A^*)^{\otimes n+1} \cong A^{\otimes n+1} \otimes (A^*)^{\otimes n+1}$ which is clearly similar to $A^{\otimes n} \otimes (A^*)^{\otimes n}$, which shows that $E^{\otimes n+1} \sim E^{\otimes n}$. Thus $d(E, {}_H\mathcal{M}) \leq n$

Suppose that E has depth m, then by Corollary 5.4.2 we have $d(A, {}_H\mathcal{M}) \leq m$. Putting the two inequalities together proves the proposition. $\square$

5.5 Hopf Algebra Depth

The results in this section can be found in [21] and link very closely to our later results on smash product depth bounds, as well as being interesting in themselves. Let $R \subseteq H$ be an extension of finite dimensional Hopf Algebras. Write $R^+ := ker\epsilon \cap R$, this vector space is not generally a left or right H-ideal, but $HR^+ = H(ker\epsilon \cap R)$ will be a left H-ideal, clarified in Subsection 3.2.2. Our space HR^+ moreover is an H-coideal which can be verified using the lemma below:

Lemma 5.5.1. *Given a homomorphism of coalgebras $f : C \to D$ the kernel of the map $kerf$ is a coideal, in other words $\Delta(kerf) \subseteq kerf \otimes C + C \otimes kerf$.*

Proof. A well-known result in vector space theory is as follows: given vector spaces V, W and linear maps $g : V \to V'$ and $h : W \to W'$ then $ker(g \otimes h) = kerg \otimes W + V \otimes kerh$.

Take some $x \in kerf$ then by consequence of f being a coalgebra map $\Delta_D(f(x)) = (f \otimes f)(\Delta_C(x)) = 0$ and so $\Delta_C(x) \in ker(f \otimes f)$. $\square$

Now clearly since HR^+ is both a left ideal and a coideal the quotient $V := H/HR^+$ has the structure of a left H-module coalgebra. In addition to the aforementioned module and coalgebra structures, we can see that V is a trivial right R-module. HR^+ is a right module by virtue of R^+ being an R-ideal. Moreover $r - \epsilon(r) \in R^+$ and therefore for $\overline{h} \in V$, $\overline{h}(r - \epsilon(r)) = 0$ and so $\overline{h}r = \overline{h}\epsilon(r)$. In summary, V is a left H-module coalgebra and a trivial right R-module.

Theorem 5.5.2. *With $R \subseteq H$ an extension of Hopf algebras then H is a free left and free right R-module.*

The above theorem was originally proved by W. Nichols and M. Zoeller in [38]. A well-known corollary of this result is the following, where the lecture series [42] gives a good account of the details leading to the proof:

Lemma 5.5.3. *If H is finite dimensional then $dimV = \frac{dim(H)}{dim(R)}$.*

Proof. By the theorem above we are able to write ${}_RH = Re_1 \oplus \ldots \oplus Re_p \cong \bigoplus_{i=1}^p {}_RR$ which tells us that $dim(H) = (dimR)p$ so that $p = \frac{dimH}{dimR}$. We show that $dimV = p$.

As before every $r \in R$ satisfies $\overline{r} = \epsilon(r)$. In particular for every $rx \in R^+H$ write $x = x_1e_1 + \ldots + x_pe_p$, so that $rx = rx_1e_1 + \ldots + rx_pe_p$ and $rx_ie_i \in R^+H$. Now $\{\overline{e_i} \mid 1 \le i \le p\}$ is a k-basis in V, for every $h \in H$ can be written as $r_1e_1 + \ldots + r_pe_p$ thus $\overline{h} = \epsilon(r_1)\overline{e_1} + \ldots + \epsilon(r_p)\overline{e_p}$. The conclusion is that if we write $\lambda_1\overline{e_1} + \ldots + \lambda_p\overline{e_p} = 0$ then this is true if and only if for $\epsilon(r_i) = \lambda_i$ any choice $r_1e_1 + \ldots + r_pe_p \in HR^+$. Then we are done because $\overline{r_ie_i} = 0$ for all i. □

We could also have discussed $V' := H/R^+H$ which is a right module coalgebra. Similar results hold for V' such as $dimV' = \frac{dimH}{dimR}$. In general we do not need to use V' because there is enough information encoded in V for our important theorems. We will call V the *generalised quotient module* associated to $R \subseteq H$.

The following result has been proved in numerous texts, for example [42], [48], [1], although credit of the proof goes to E. Abe. The result provides us with breaking insight, so let us be clear on the module actions, we give $H^{\otimes_R n}$ the usual H-bimodule structure

$$h(a \otimes b \otimes \ldots \otimes c)k = ha \otimes b \otimes \ldots \otimes ck.$$

Take the module $V^{\otimes n} \otimes H$ with a kind of left diagonal action of H:

$$h(\overline{x_1} \otimes \ldots \otimes \overline{x_n} \otimes y)k = h_{(1)}\overline{x_1} \otimes \ldots \otimes h_{(n)}\overline{x_n} \otimes yk. \tag{5.3}$$

Proposition 5.5.4. *$H \otimes_R H \cong V \otimes H$ as H-bimodules.*

Proof. Define the linear map

$$\begin{aligned} \phi : H \otimes_R H &\to V \otimes H \\ : x \otimes y &\mapsto \overline{x_{(1)}} \otimes x_{(2)}y. \end{aligned}$$

This is well-defined, for it can be written as the composition of the following linear maps:

$$H \otimes_R H \xrightarrow{\Delta \otimes id} (H \otimes H) \otimes_R H \xrightarrow{id \otimes \mu_R} H \otimes H \xrightarrow{\pi_V \otimes id} H \otimes H,$$

where $\mu_R : H \otimes_R H \to H$ is defined by $x \otimes y \mapsto xy$ and $\pi_V : H \to V$ is canonical.

This is an H-bimodule map, for take $h, k \in H$

$$\begin{aligned} h\phi(x \otimes y)k &= h(\overline{x_{(1)}} \otimes x_{(2)}y)k \\ &= h_{(1)}\overline{x_{(1)}} \otimes h_{(2)}x_{(2)}yk \\ &= \overline{h_{(1)}x_{(1)}} \otimes h_{(2)}x_{(2)}yk \\ &= \phi(hx \otimes yk). \end{aligned}$$

The inverse of ϕ is given by $\psi : V \otimes H \to H \otimes_R H : \overline{w} \otimes z \mapsto w_{(1)} \otimes S(w_{(2)})z$. This is well-defined, it is enough to consider when $\overline{w} = \overline{w'}$ so that $\overline{w} \otimes z = \overline{w'} \otimes z$ (let z range through the basis elements of H). In this case $w' = w + hr$, for $hr \in HR^+$, and so

$$\begin{aligned} \psi(\overline{w'} \otimes z) &= w'_{(1)} \otimes S(w'_{(2)})z \\ &= w_{(1)} \otimes S(w_{(2)})z + h_{(1)}r_{(1)} \otimes S(h_{(2)}r_{(2)})z \\ &= w_{(1)} \otimes S(w_{(2)})z + h_{(1)}r_{(1)} \otimes S(r_{(2)})S(h_{(2)})z \\ &= w_{(1)} \otimes S(w_{(2)})z + h_{(1)}r_{(1)}S(r_{(2)}) \otimes S(h_{(2)})z \\ &= w_{(1)} \otimes S(w_{(2)})z = \psi(\overline{w} \otimes z). \end{aligned}$$

We note above that $\sum r_{(1)}S(r_{(2)}) = \epsilon(r) = 0$. It is clear that $\phi \circ \psi = id$ and $\psi \circ \phi = id$. $\square$

Corollary 5.5.5. *With the H-bimodule structures defined above, there is an isomorphism $H^{\otimes_R n} \cong V^{\otimes n-1} \otimes H$, for $n \geq 1$.*

Proof. We show the $n = 3$ case for simplicity, using the proposition above to find a chain of isomorphisms $H \otimes_R H \otimes_R H \cong (V \otimes H) \otimes_R H \cong V \otimes (H \otimes_R H) \cong V \otimes V \otimes H$. $\square$

Take an extension of Hopf algebras $R \subseteq H$ with depth $2n$, by definition this means we have a similarity of R-H-bimodules and H-R-bimodule: $H^{\otimes_R n+1} \sim H^{\otimes_R n}$. A restriction of the isomorphism above gives us an equivalent condition:

$$V^{\otimes n} \otimes H \sim V^{\otimes n-1} \otimes H$$

as R-H- and H-R-bimodules, with the action as in (5.3). The same idea can be used for a depth $2n + 1$ extension. Now below when we ascertain that $d(V, {}_H\mathcal{M})$ is finite we shall denote it as d_V for quickness. Credit of the theorem below goes to L. Kadison.

Theorem 5.5.6 ([21], Thm.4.1)**.** *Let $R \subseteq H$ be a Hopf algebra extension with algebraic quotient module V. Then $d(V, {}_R\mathcal{M})$ is finite if and only if $d(R, H)$ is finite.*

Proof. Proved using Propositions 5.5.7 and 5.5.8 both below. □

Proposition 5.5.7. *Suppose that $d(R, H)$ is finite then $d(V, {}_H\mathcal{M})$ is finite and $2d_V \leq d_{even}(R, H)$.*

Proof. Assume that $R \subseteq H$ has depth $2n \geq 2$, then $H^{\otimes_R n+1} \sim H^{\otimes_R n}$ as R-H-bimodules in particular. If we expand this similarity using the corollary above we get $V^{\otimes n} \otimes H \sim V^{\otimes n-1} \otimes H$ as H-R-bimodules. Recall that $\mathcal{Y}_R$ of Section (5.3) preserves similarity, we apply it to both sides of the above similarity:

$$V^{\otimes n} \otimes H \otimes_R {}_\epsilon k_\epsilon \sim V^{\otimes n-1} \otimes H \otimes_R {}_\epsilon k_\epsilon \tag{5.4}$$

as left H-modules (ignore the trivial right module structure). Now by a well-known lemma as demonstrated in ([21], Lem 2.1), $H \otimes_R k \cong V$ and therefore (5.4) becomes $V^{\otimes n+1} \sim V^{\otimes n}$. □

Proposition 5.5.8. *If $d(V, {}_H\mathcal{M})$ is finite then $d(R, H) \leq 2d_V + 2$.*

Proof. Suppose V has depth n as an H-module, then $V^{\otimes n+1} \sim V^{\otimes n}$ as H-modules. Tensoring the similarity by $- \otimes H$ with rightmost module multiplication, which preserves similarity, this gives us $V^{\otimes n+1} \otimes H \sim V^{\otimes n} \otimes H$ as H-bimodules. By Corollary 5.5.5 the following H-bimodule condition is satisfied: $H^{\otimes_R n+2} \sim H^{\otimes_R n+1}$. □

Corollary 5.5.9. *Let $R \subseteq H$ and V be as above where one or the other has finite depth, then the following inequality holds:*

$$2d_V - 1 \leq d_{odd}(R, H) \leq 2d_V + 1.$$

With Theorem 5.5.6 we get a result about representation type, more powerful than Proposition 2.1.4.

Corollary 5.5.10 ([21], Cor.5.8)**.** *Suppose that either R or H have finite representation type, then $R \subseteq H$ has finite depth.*

Proof. Assume R has finite representation type then by Proposition 5.1.6 all module coalgebras have finite depth. Now we apply the previous theorem to the module coalgebra V. □

Example 15. In the paper [32] the Taft algebras T_n are shown to be Nakayama algebras and to have finite representation type. Therefore for any Hopf subalgebra $R \subseteq T_n$ and any containing Hopf algebra $H \supseteq T_n$ the depths $d(R, T_n)$ and $d(T_n, H)$ will be finite.

5.6 Depth of a Hopf Algebra in a Smash Product

Let H be a finite dimensional Hopf algebra, furthermore take A a finite dimensional H-module algebra.

Lemma 5.6.1. *As right H-modules $A\#H$ is isomorphic to $A \otimes H$.*

Proof. For $a\#h \in A\#H$ and $k \in H$ $(a\#h)k = (a\#h)(1\#k) = a\#hk$. □

Proposition 5.6.2. *A is a trivial H-module if and only if $H \subseteq A\#H$ has depth* 1.

Proof. ($\Leftarrow$). Suppose $H \subseteq A\#H$ has depth 1, then $A\#H \sim H$ as H-bimodules or equivalently $A\#H \mid q \cdot H$ for some nonzero $q \in \mathbb{N}$. Now apply the functor $- \otimes_H k_\epsilon$ and the lemma above, so we have $A \otimes H \otimes_H k_\epsilon | \; q \cdot (H \otimes_H k_\epsilon)$. After identifying $H \otimes_H -$ with the identity functor we end up with the division $A \mid q \cdot k_\epsilon$. This means that $A \oplus A' \cong q \cdot k$, for some H-module A', so that A is isomorphic to some H-submodule of $q \cdot k$ and thus is trivial.

($\Rightarrow$). Suppose that A is a trivial H-module, then we may immediately write $A \cong q \cdot k$ as H-modules for some nonzero $q \in \mathbb{N}$. Notice that $(q \cdot k) \otimes H \cong q \cdot H$ as H-bimodules with left diagonal and rightmost actions. Indeed using the prior facts and the previous lemma it is clear that $q \cdot H \cong A \otimes H \cong A\#H$ as H-bimodules. □

Proposition 5.6.3. *With $A\#H$ as above, there is an isomorphism of H-bimodules*

$$(A\#H)^{\otimes_H n} \cong A^{\otimes n} \otimes H,$$

where $A\#H$ is an H-bimodule via left- and right-most multiplication and $A \otimes H$ with left diagonal action and right-most H-multiplication.

Proof. Recall from the introduction on smash products that $A\#H = (A\#1)(1\#H)$. Thus we can write $(1\#H)(A\#H) = (A\#H)$, and moreover

$$\begin{aligned}(A\#H) \otimes_H (A\#H) &= (A\#1)(1\#H) \otimes_H (A\#H) \\ &= (A\#1) \otimes_H (1\#H)(A\#H) \\ &= (A\#1) \otimes_H (A\#H).\end{aligned} \tag{5.5}$$

This idea extends to n-fold tensor products. Define a map $\phi : (A\#H)^{\otimes_H n} \to A^{\otimes n} \otimes H$ by

$$(a\#1) \otimes (b\#1) \otimes \ldots \otimes (c\#k) \mapsto a \otimes b \otimes \ldots \otimes c \otimes k$$

That this is a bijection is demonstrated in (5.5), so all that remains is to verify that it is a H-bimodule homomorphism. Since it is clearly a right H-module homomorphism (with rightmost H action) we verify the left H action: $h\phi(a\#1 \otimes b\#k) = h(a \otimes b \otimes k) = (h_{(1)} \cdot a) \otimes (h_{(2)} \cdot b) \otimes h_{(3)}k$ and this is equal to

$$\begin{aligned}\phi(h(a\#1 \otimes b\#k)) &= \phi((h_{(1)} \cdot a)\#h_{(2)} \otimes b\#k) \\ &= \phi((h_{(1)} \cdot a)\#1 \otimes (h_{(2)} \cdot b)\#h_{(3)}k) \\ &= (h_{(1)} \cdot a) \otimes (h_{(2)} \cdot b) \otimes h_{(3)}k.\end{aligned}$$

□

Now comes one of the main results of the research.

Theorem 5.6.4. *Let A be a left H-module as above then*

$$d_{odd}(H, A\#H) = 2d(A, {}_H\mathcal{M}) + 1$$

Proof. ($\leq$) For the first part we assume that A has module depth n so that $A^{\otimes n+1} \sim A^{\otimes n}$ as H-modules. Now apply $- \otimes H$ to both sides of the equivalence and get $A^{\otimes n+1} \otimes H \sim A^{\otimes n} \otimes H$. The isomorphism of Proposition 5.6.3 implies

$$(A\#H)^{\otimes_H n+1} \sim (A\#H)^{\otimes_H n},$$

as H-bimodules.

($\geq$) Assume that $d_{odd}(H, A\#H) = 2n+1$ so that as H-bimodules $(A\#H)^{\otimes_H n+1} \sim (A\#H)^{\otimes_H n}$ then apply Proposition 5.6.3 to see that $A^{\otimes n+1} \otimes H \sim A^{\otimes n} \otimes H$ as H-

bimodules. Apply $- \otimes_H k$ to both sides of this equivalence to get

$$A^{\otimes n+1} \otimes k \sim A^{\otimes n} \otimes k.$$

Since $- \otimes k$ is isomorphic to the identity functor it follows that $A^{\otimes n+1} \sim A^{\otimes n}$ as H-modules, which is the depth n condition for A. $\square$

Corollary 5.6.5. *Taking A, H as above, the following inequality holds:*

$$2d(A, {}_H\mathcal{M}) \leq d_{even}(H, A\#H) \leq 2d(A, {}_H\mathcal{M}) + 2.$$

Proof. Recall that $|d_{odd} - d_{even}| = 1$, so we can just apply Theorem 5.6.4 above. $\square$

5.7 Hopf Algebra Depth II

The following results link the depth of a Hopf algebra extension $R \subseteq H$ to the depth of certain smash products.

Lemma 5.7.1 ([36])**.** *Let C be a right H-module coalgebra, then C^* is a left H-module algebra. The same is true with left and right reversed.*

If A is a finite dimensional right H-module algebra then A^* is a left H-module coalgebra. We can reverse left and right again.

Proof. We apply Lemma 3.1.5 for C^* and Corollary 3.1.4 for A^* then it is easy to check that the colagebra and algebra structures are compatible with the module structures. $\square$

Proposition 5.7.2. *Suppose A is a finite dimensional H-module algebra, then the module depths of A and A^* are equal.*

Proof. We prove this in two steps. First of all assume that M and N are two arbitrary H-modules such that $M|N$ as H-modules, then $M^*|N^*$ as H-modules. For take maps as in the diagram:

$$\mathrm{id} \circlearrowright M \underset{f}{\overset{g}{\rightleftarrows}} N \ ,$$

then we have the dual of the diagram:

$$\overset{id}{\circlearrowleft} M^* \underset{f^*}{\overset{g^*}{\rightleftarrows}} N^* \ .$$

Furthermore $f^* \circ g^* = (g \circ f)^* = (id_M)^* = id_{M^*}$.

For the final step we see that $M^* \otimes M^* \cong (M \otimes M)^*$. In ([49], Prop 4.1.6) the map $\psi : M^* \otimes M^* \to (M \otimes M)^*$ defined by $\psi(f \otimes g)(x \otimes y) = f(x)g(y)$ is proven to be a k-linear isomorphism. We will see that it is a H-homomorphism as well:

$$\begin{aligned}
\psi(f \otimes g \cdot h)(x \otimes y) &= \psi(f \cdot h_{(1)} \otimes g \cdot h_{(2)})(x \otimes y) \\
&= (f \cdot h_{(1)})(x)(g \cdot h_{(2)})(y) \\
&= f(h_{(1)} \cdot x)g(h_{(2)} \cdot y) = (\psi(f \otimes g) \cdot h)(x \otimes y)
\end{aligned}$$

It is a general fact that if a module homomorphism θ has an inverse as a set map, then θ^{-1} is a module homomorphism too. This proves the proposition. □

This result implies that if A is a left H-module then $d(A, {}_H\mathcal{M}) = d(A^*, \mathcal{M}_H)$.

Theorem 5.7.3 (Hopf Algebra and Smash Product Depth). *Let $R \subseteq H$ be finite dimensional Hopf algebras, if any of the depth values below are finite (or infinite) then all of them are, and the inequalities hold:*

$$2d(V, {}_H\mathcal{M}) - 1 \leq d_{odd}(R, H) \leq d_{odd}(H, H\#V^*),$$

where $V = H/HR^+$ is the generalised quotient module.

Proof. Write $d := d(V^*, \mathcal{M}_H)$ for brevity. Recall from Proposition 5.7.2 that $d = d(V, {}_H\mathcal{M})$. Now by Theorem 5.6.4 and Corollary 5.5.9 respectively

$$d_{odd}(H, H\#V^*) = 2d + 1, \tag{5.6}$$

$$d_{odd}(R, H) \leq 2d + 1. \tag{5.7}$$

Morover Corollary 5.5.9 completes the inequalities.

□

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