Universidade do Porto
Ph.D. Thesis Dissertation :
The Quotient Module , Coring Depth and Factorisation Algebras

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Pp y Morci,
Almas de mi vida,
Las amo.

Acknowledgments

A mis padres y hermanos que me han apoyado sin reservas en este proyecto. Los amo.

I would like to thank Professor Kadison for his support and friendship during this past four years, and for making this process not only enjoyable but memorable. I will never forget this years. Thank you.

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Resumo

Em Boltje-Danz-Külshammer, J. of Algebra, 03-019, (2011), mostra-se que para uma extensão de álgebras de grupos finitos sob cualquer anel comutativo a *profundidade* da extensão é sempre finita. A seguir, em Kadison, J. Pure and App Algebra, 218: 367-380, (2014), a profundidade da extensão obtém-se a partir de computações no módulo de permutações dos cosets direitos ou esquerdos. Isto, em geral é verdade para extensões de álgebras de Hopf de dimensão finita. Nós provámos que a profundidade da extensão de álgebras de Hopf de dimensão finita $R \subseteq H$ se relaciona com a profundidade do seu módulo generalizado de permutações $Q := H/R^+H$ na sua categoria modular. Além disso establecemos que a profundidade da extensão é finita se e só se Q for um módulo algebráico no anel de representações de H ou de R. Passamos a provar que a estabilização da cadeia descendente dos ideais aniquiladores das potências tensoriais de Q é uma condição necessária para a profundidade finita duma extensão. Em Danz, Comm of Alg, 39:5, 1635-1645, (2011), a autora fornece uma fórmula para a profundidade da extensão complexa torcida dos grupos simétricos de ordem n e $n+1$. Provemos aqui um cenário geral em que a profundidade duma extensão complexa de álgebras de produto cruzado $D\#_\alpha H \subseteq D\#_\alpha G$ duma extenção de grupo $H < G$ é sempre menor ou igual do que a profundidade da extensão de álgebras de grupo $kH < kG$. Para isso usámos o entrelaçado de um H-módulo álgebra A com um H-módulo coálgebra C, sob uma álgebra de Hopf H. Mostrámos que dita estrutura é um coanel de Galois quando $A = H$ e $C = Q$, numa extensão de álgebras de Hopf finitas $R \subseteq H$, e extendemo-la a uma extensão de álgebras de productos cruzados de grupos finitos. Provemos também um cenário para a profundidade de álgebras de factorização e uma fórmula para o valor da profundidade de uma sub álgebra numa álgebra de factorização em termos da sua profundidade modular.

Classificação matemática (2000) : 16S40, 16T05, 18D10, 19A22, 20C05

Palavras chave: profundidade, profundidade de grupo, profundidade modular, módulo de permutações, anelo de representações, coanel, coanel de Galois, álgebra de produto cruzado, álgebra de factorização, álgebra factorizável.

Abstract

In Boltje-Danz-Külshammer, J. of Algebra, 03-019, (2011) it was shown that for a finite group algebra extension over any commutative ring the depth is always finite. Later, in Kadison, J. Pure and App Algebra, 218: 367-380, (2014) depth of such a subgroup pair was obtained by computing on the permutation module of the left or right cosets. This holds more generally for finite dimensional Hopf algebra extensions. We show that the depth of the Hopf subalgebra pair $R \subseteq H$ is related to the depth of its generalised permutation module $Q := H/R^+H$ in its module category. Furthermore we establish that the pair is finite depth if and only if Q is an algebraic module in the representation ring of either H or R. A necessary condition for finite depth is provided as the stabilisation of the descending chain of annihilators of the tensor powers of Q. In Danz, Comm of Alg, 39:5, 1635-1645, (2011) the author provides a formula for the depth of a complex twisted group extension of the symmetric groups of order n and $n+1$. We provide a general setting in which the depth of the complex crossed product algebra extension $D\#_\alpha H \subseteq D\#_\alpha G$ of a group pair $H < G$ is always less or equal than the depth of the algebra extension $kH < kG$. For this we use the entwining of a left H-module algebra A with an H-module coalgebra C, over a Hopf algebra H. We show that such a structure is a Galois coring when $A = H$ and $C = Q$ for a finite dimensional Hopf algebra extension $R \subseteq H$, and that this extends to the crossed product algebra extension of a finite group extension. We also provide a setting for the depth of factorisation algebras and provide a formula for the value of depth of a subalgebra of a factorisation algebra in terms of its module depth.

Mathematics Subject Classification (2000) : 16S40, 16T05, 18D10, 19A22, 20C05

Key words: depth, subgroup depth, module depth, permutation module, representation ring, coring, Galois-coring, crossed product algebra, factorisation algebra, factorisable algebra.

Contents

Chapter 1

Introduction

1.1 Motivation

The *theory of depth* is the study of a similarity (finite depth), or lack thereof (infinite depth), between tensor powers of the regular representation of a ring extension. The earliest examples of this similarity and its implications in ring theory may be traced to the 1950's K. Hirata's similarity theory.

Considering finite group algebra extensions, the theory of depth has provided interesting and natural ways to link representation properties of a finite group algebra extension $kG \subseteq kH$ with structure characteristics of its group extension $H < G$, among others it has provided a characterisation of normality for group pairs in terms of their module representation theory.

In a more general setting, given a finite tensor category $\mathcal{C}$ one can define the depth of an object $X \in \mathcal{C}$. For example when H is a finite dimensional k-algebra then its category of finite dimensional right H-modules is a finite tensor category. This is useful since it brings to the game the representation ring (or Green ring) $A(H)$ of the algebra in question. Following this line of thought, concepts such as being an algebraic element in the representation ring become important and even central in the study of depth of ring extensions.

In 2011 it was shown that for any commutative ring R and a finite group algebra extension $RH \subseteq RG$ the depth of the extension is always finite (details are found in this work). Since Hopf algebras are a natural generalisation of group algebras it becomes primary to ask whether the same is true for finite dimensional Hopf algebra extensions. Other results such as the characterisation of normality have been extended to Hopf algebras in recent years providing evidence that, in terms of depth theory, group rings and Hopf algebras are even closer than they appear to be. Moreover, through the representation ring of a Hopf algebra extension, and its depth theory, one might be able to link Mackey's theorem (which is central in the study of group algebras and their modules) to Hopf algebra extensions by means of the quotient module of the extension which generalises permutation modules of group extensions.

In general, mathematicians dealing with representation problems in group theory or Hopf algebras, as well as cohomology, algebraic geometry and mathematical physics may find some of the results as well as some of the technics in depth theory useful for their own research.

1.2 Description

The work that is found in this thesis is divided in four chapters, in the first one we find the necessary tools and theoretical background that will allow the reader to familiarise himself with the ideas, the theory and in general with the problem that is our main concern. The next two chapters extend my contributions in two collaborations for publications and the last one is a closing chapter that provides a theoretical frame that would explain relations and inequalities that appear throughout the text. The core of this investigation revolves around the *quotient module* Q of a finite dimensional Hopf algebra pair $R \subseteq H$, and its role in the pursuit of a definite answer on whether for such a pair the depth is always finite.

For this we first look at Q within the finite tensor category of H-modules and use it to give conditions for finite depth. We develop the theory of depth for a module coalgebra in a module category and set a basis for the calculation of depth using Q. Secondly we describe the depth of a pair $R \subseteq H$ in terms of the depth of its coring extensions and use the module Q to give equalities as well as inequalities for depth, providing therefore a variety of situations in which the calculation of depth can be achieved by looking at Q in its entwined structures, and their module categories. Finally we provide a framework to describe *Smash products* and then use Q to provide theoretical results for the value of depth in the context of natural ring extensions.

Chapter 2 of this work is devoted to the definition of the *Generalised Permutation Module* Q of a finite dimensional Hopf algebra extension $R \subseteq H$

and its applications to the *Theory of Depth*. It turns out that such special module Q is a coalgebra in the finite tensor category of H-modules. This definition can be traced back to [25] and the results we discuss here are an extension of my contributions to my collaboration with Kadison and Young in [22].

In section 2.1 we start by defining the concept of depth of a module in a finite tensor category and move towards providing in corollary (2.9) an equivalent condition for the finiteness of depth of a finite dimensional H-module coalgebra.

Next, in section 2.2 we take the previous results and use them to provide a theoretical approach to the depth of a finite dimensional Hopf algebra pair $R \subseteq H$. In this case in the context of an *algebraic module* in the representation ring of either R or H. Theorem (2.22) tells us that depth is finite if and only if the quotient module coalgebra Q is algebraic. In corollary (2.26) we extend this to stating that depth is finite if and only if the non projective indecomposable constituents of Q are algebraic and compute the depth of the Sweedler algebra in its quantum double as an example in section 2.3.

The aim in section 2.4 is to provide a necessary condition for finite depth. For this we look at a descending chain of ideals in the Hopf algebra H, namely the annihilators of the tensor powers of the quotient module Q: $I_n := Ann_H(Q^{\otimes(n)})$. We note that the intersection of all such ideals is a maximal Hopf ideal and that this intersection is I_n for some $n \leq l(R)$, the length of the subalgebra R as an R^e-module. We denote l_Q the minimum n such that $I_{n+1} = I_n$. If $I_n = 0$ for some $n > 1$, we say Q is conditionally faithful. In theorem (2.43) we show that whenever Q is conditionally faithful and projective as an H-module then $d(Q, \mathcal{M}_H) = l_Q$. Finally in corollary (2.45) we give a formula for the depth of a Hopf algebra pair $R \subseteq H$ whenever R is semisimple.

Chapter 3 is the result of a second collaboration, in this case with L. Kadison and M. Szamotulski [23]. The purpose of this chapter is to use the definition of a *coring* found in [8] as well as in [9] to obtain results in depth theory. Namely we relate the h-depth of a Hopf subalgebra pair to that of the module Q in an *entwined structure*. Moreover we apply this idea to an extension of crossed products and obtain a refinement in the depth of a subgroup pair, that of a *twisted group extension* and use it to give a theoretical proof for an inequality found in [14].

Sections 3.1 and 3.2 deal with the definitions of corings and of entwining structures and apply those concepts to construct the coring $H \otimes Q$, as well as defining the depth of a coring C in a tensor category $\mathcal{M}_{A^e}$ (or ${}_A\mathcal{M}_A$) where A is a ring or a *quantum algebra*.

Next, in sections 3.3 and 3.4 we deal with the concept of a *Galois coring* and its depth in $\mathcal{M}_A$. Moreover we note that for a finite dimensional extension $B \subseteq A$ the Sweedler coring $A \otimes_B A$ is Galois and give a formula for its depth. In corollary (3.18) we use this approach to obtain a result for the depth of a finite dimensional left coideal subalgebra $R \subseteq H$.

We close the chapter with section 3.5. Here we use the notion of the *crossed product* of a twisted H-module algebra with a finite dimensional Hopf algebra H. We prove that for a finite dimensional Hopf algebra pair $R \subseteq H$ and a twisted H-module D the extension $B := D\#_\sigma R \subseteq D\#_\sigma H := A$ is a Galois coring extension and use it in theorem (3.27) to provide an inequality for the h-depth of the Hopf subalgebra pair. Finally in corollary (3.28) we extend the inequality (1.22) which appears in [4].

Chapter 4 is devoted to *factorisation algebras* and its main purpose is to provide a frame for the algebraic structures used throughout this thesis, for example, *smash products* , as well as to explain the role of depth from this point of view. Section 4.1 provides the theoretical framework as well as examples of such structures. Next in section 4.2 we define the concept of depth from the point of view of factorisation algebras and in theorem (4.4) we give an inequality for the value of depth of a factorisation algebra in its module category in terms of the depth of one of its subalgebras in its own module category. This together with a special condition yields a series of corollaries that provide equalities for the h-depth of smash products in their respective tensor categories in terms of the depth of the regular representation of the Hopf algebra H in its tensor category. In particular, in corollary (4.5) we give a theoretical explanation for equation (1.32). We finish this chapter with an example of a factorisable algebra and the depth of its generalised smash product in its module category.

Finally, in the appendix we give a sketch of a proof of the depth of the small quantum group $H_8 := U_q(sl_2)$ at the square root of unit in its Drinfeld double to make a case for the finiteness of depth, as the pair is infinite representation type. We finish the work with a short list of proposed problems that are a natural continuation to the work presented here.

1.3 Hopf Algebras

The aim of this subsection is to provide the reader with some basic definitions and results that appear throughout this work. Here all algebras H are finite dimensional as vector spaces over a field k of characteristic zero, unless specified otherwise. Given a ring R the category of finite dimensional right R-modules will be denoted $\mathcal{M}_R$ and the one for the left R-modules will be ${}_R\mathcal{M}$. The category of finite dimensional right R-comodules will be denoted $\mathcal{M}^R$ and that of the left R-comodules will be denoted ${}^R\mathcal{M}$. All rings are associative with 1 and for all ring extensions $B \subseteq A$ we have $1_B = 1_A$. Other notations will be specified in their own context when they appear. Through out this section we will follow [40] and [26] unless otherwise stated.

Definition 1.1. *Let k be a field. We say H is a k Hopf algebra if it is both, a k-algebra with unit and multiplication $(H, \mu, \cdot)$ and a k-coalgebra with coproduct and counit (H, Δ, ε) in which Δ and ε are algebra morphisms, endowed with an anti-homomorphism, called the antipode, $S : H \longrightarrow H$ satisfying for every $h \in H$:*

$$S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \varepsilon(h) \tag{1.1}$$

This is to say that S is the convolution inverse to the identity.

Example 1.2. *Let G be a group, k a field. Then kG is a Hopf algebra with the following structure:*

$$\begin{gathered}(\sum \alpha_i g_i)(\sum \beta_j h_j) = \sum\sum \alpha_i\beta_j g_i h_j \\ \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1 \\ S(g) = g^{-1}\end{gathered} \tag{1.2}$$

$\forall \alpha_i, \beta_i \in k$ and $\forall g_i, h_i \in G$.

Example 1.3. *The quantum plane, $\mathbf{O}_q = \langle x, y | xx^{-1} = 1 = x^{-1}x, xy = qyx\rangle$, $q \neq 0 \in k$, is a Hopf algebra with the following structure:*

$$\begin{gathered}\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes 1 + x \otimes y, \\ \varepsilon(x) = 1, \quad \varepsilon(y) = 0 \\ S(x) = x^{-1}, \quad S(y) = -x^{-1}y\end{gathered} \tag{1.3}$$

Other examples such as *enveloping algebras* of *Lie algebras*, *Taft algebras* and *small quantum groups* are Hopf algebras. For more examples of Hopf algebras one can check [38], [43] and [40].

Definition 1.4. *Let H be a Hopf algebra. A subset $R \subseteq H$ is called a Hopf subalgebra of H if:*

1. *R is a subalgebra of H*
2. *$\Delta(R) \subseteq R \otimes R$*
3. *$S(R) \subseteq R$*

That is to say R is itself a Hopf algebra with respect to the structure on H.

If R is a Hopf subalgebra of a Hopf algebra H we call the extension $R \subseteq H$ a Hopf algebra extension or Hopf algebra pair.

Definition 1.5. *Let $B \subseteq A$ be a ring extension, denote*

$$(A_B)^* = Hom(A_B, B_B) \tag{1.4}$$

and

$$^*({}_BA) = Hom({}_BA, {}_BB) \tag{1.5}$$

Definition 1.6. *[26, Theorem 1.2] Let $B \subseteq A$ be a ring extension. We say $B \subseteq A$ is a Frobenius extension if any of the the following equivalent conditions holds:*

1. *${}_BA_A \cong ({}_AA_B)^*$ and A_B is finite projective.*
2. *${}_AA_B \cong {}^*({}_BA_A)$ and ${}_BA$ is finite projective.*
3. *There is $E \in Hom_{B-B}(A, B)$, x_i, $y_i \in A$ such that for all $a \in A$*

$$\sum_i x_i E(y_i a) = a$$

and

$$\sum_i E(a x_i) y_i = a$$

We call (E, x_i, y_i) a Frobenius system, E a Frobenius homomorphism, (x_i, y_i) a dual basis and the map $a \longmapsto Ea$ a Frobenius isomorphism.

Example 1.7. *We say that a k algebra A is Frobenius if the extension $k \subseteq A$ is a Frobenius extension. Finite dimensional Hopf algebras are Frobenius algebras by Larson-Sweedler theorem (1969).*

This means that for a finite dimensional Hopf algebra H we have $H_H \cong ({}_H H)^*$.

Definition 1.8. *Let H be a Hopf algebra and $M \in {}_H\mathcal{M}$, the invariants of H on M are:*

$$M^H = \{m \in M \quad | h \cdot m = \varepsilon(h)m, \forall h \in H\} \tag{1.6}$$

If M is right H comodule, its coinvariants are:

$$M^{coH} = \{m \in M | \rho(m) = m \otimes 1\} \tag{1.7}$$

Definition 1.9. *Let H be a Hopf algebra, a right H-Hopf module is a vector space M such that:*

1. *M is a right H-module.*
2. *M is a right H-comodule.*
3. *The comodule map ρ is a right H-module map.*

Theorem 1.10. *[40, 1.9.4] Let H be a Hopf algebra, M a right H-Hopf module. Then:*

$$M \cong M^{coH} \otimes H \tag{1.8}$$

as right H-modules. In particular M is a free right H-module of rank $=$ $dim_k M^{coH}$.

Definition 1.11. *A left integral in H is an element t such that for every $h \in H$, $ht = \varepsilon(h)t$, i.e.$t \in H^H$.*

A theorem of Larson-Sweedler shows that the space of left integrals on H is just $(H^*)^{coH}$ where H^* can be defined as a right H-module. In the same manner we define *right integrals* in H.

For example, if $H = kG$ a group algebra, then $\sum_{g \in G} g$ generates both the spaces of left and right integrals.

Moreover, as an application of theorem (1.10) applied to H^* we can see that the subspace of left, (or right) integrals is one dimensional. Furthermore if t is a left integral then $S(t)$ is a right integral.

Theorem 1.12. *(Maschke's) Let H be a finite dimensional Hopf algebra, then H is semisimple if and only if there is a left integral t such that $\varepsilon(t) \neq 0$ if and only if there is a right integral l such that $\varepsilon(l) \neq 0$.*

Example 1.13. *If G is a finite group and k is a field such that its characteristic does not divide the order of the group; $char(k) \nmid |G|$, then the ring kG is semisimple.*

One important consideration to be taken into account when looking at H-modules is whether they are free, projective, injective or have any other property of this sort. The following theorem, known as *Nichols - Zoeller theorem* gives us a way of knowing when certain types of H-modules are free.

If in definition (1.9) we substitute H for any Hopf subalgebra $R \subseteq H$ we say that M is a (H, R)-Hopf-module. The category of (H, R)-Hopf-modules is denoted $\mathcal{M}_R^H$.

Theorem 1.14. *(Nichols-Zoeller) Let $R \subseteq H$ be a finite dimensional Hopf algebra extension. Then every $M \in \mathcal{M}_R^H$ is free as a R-module.*

Notice that as a consequence of this theorem one has that H_R is a free R-module. Of course all definitions can be given oppositely so that we end up with ${}_RH$ being a free R-module.

Let $R \subseteq H$ be a finite dimensional Hopf algebra extension. Let $R^+ := ker\varepsilon|_R$, Schneider showed (1990) that

1. $H \cong R \otimes H/R^+H$ as left R-modules and right H/R^+H-comodules.
2. $H \cong H/HR^+ \otimes R$ as right R-modules and left H/HR^+-comodules.

Definition 1.15. *Let H be a Hopf algebra, the left and right adjoint actions of H on itself are respectively:*

$$(ad_l h)(k) = \sum h_{(1)} k S(h_{(2)}) \tag{1.9}$$

$$(ad_r h)(k) = \sum S(h_{(1)}) k h_{(2)} \tag{1.10}$$

for all h,k $\in H$.

We say that $R \subseteq H$ is a normal Hopf subalgebra if:

$$(ad_l H)(R) \subseteq R, \quad and, \quad (ad_r H)(R) \subseteq R \tag{1.11}$$

and if only one of the conditions above is met we say the extension is left ad-stable or right ad-stable respectively.

It's not very hard to verify that a Hopf subalgebra $R \subseteq H$ is normal if and only if

$$R^+H = HR^+ \tag{1.12}$$

The author wishes to mention that bellow this point the summation sign on the Sweedler notation will be avoided, unless it is strictly necessary.

Definition 1.16. *Let H be a Hopf algebra, an algebra A is an H-module algebra if:*

1. *A is a (left) H-module via $h \otimes a \longmapsto h \cdot a$.*
2. *$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$.*
3. *$h \cdot 1_A = \varepsilon(h)1_A$. for all $h \in H$ and $a, b \in A$.*

   If only 2 and 3 are satisfied we say H measures A.

Definition 1.17. *Let H be a Hopf algebra, A an H-module algebra. Define the smash product of A and H, $A\#H$, as follows:*

1. *$A\#H = A \otimes H$ as k-spaces.*
2. *multiplication is given by:*

$$(a\#h)(b\#g) = a(h_{(1)} \cdot b)\#h_{(2)}g \tag{1.13}$$

In the next few paragraphs we will concentrate in the coalgebra structure of a Hopf algebra H, therefore we will make every statement referring to a coalgebra C. In particular we will mention the coradical C_0 and the coradical filtration $\{C_n\}$ of a coalgebra C.

Definition 1.18. *Let C be a coalgebra:*

1. *The coradical C_0 of C is the sum of all simple subcoalgebras of C.*
2. *C is pointed if every simple subcoalgebra is one-dimensional.*
3. *C is connected if C_0 is one-dimensional.*

Let C be a coalgebra, an element $g \in C$ is said to be a *grouplike element* of C if:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1. \tag{1.14}$$

In case $C = H$ a Hopf algebra, then a grouplike element also satisfies:

$$S(g) = g^{-1} \tag{1.15}$$

The collection of all grouplike elements in a coalgebra C is denoted by $G(C)$.

Notice that a one-dimensional subcoalgebra must be of the form kg where $g \in G(C)$ is a grouplike element of C. Then C is pointed if and only if $C_0 = kG(C)$. Moreover, a sum of simple subcoalgebras can be shown to be a direct sum, hence a coalgebra C is semisimple if and only if $C = C_0$.

To define the coradical filtration $\{C_n\}$ of C we define C_n inductively for $n \geq 1$ as follows:

$$C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C) \tag{1.16}$$

It is shown, for example in [40, Chapter 5], that $\{C_n\}_{n\geq 0}$ is a family of subcoalgebras satisfying $C_n \subseteq C_{n+1}$, $C = \bigcup_{n\geq 1} C_n$ and that $\Delta(C_n) = \sum_{i=0}^{n} C_i \otimes C_{n-i}$. A family of subspaces satisfying this conditions is called a *coalgebra filtration.* Moreover one shows that the coradical filtration $\{H_n\}$ of a Hopf algebra is a Hopf algebra filtration if and only if H_0 is a Hopf algebra itself. Recall that a set $\{A\}_n$ of subspaces of a Hopf algebra H is a Hopf algebra filtration if it is a coalgebra filtration , an algebra filtration (that is $A_m A_n \subseteq A_{m+n}$ for all m and n), $S(A_n) \subseteq A_n$.

The following is for the sake of completeness:

Lemma 1.19. *Let H be a Hopf algebra with cocommutative coradical H_0. Then the antipode of H is bijective.*

As we will see later different types of extensions provide different advantages for proving some results, one of the most important type of extensions are *Hopf-Galois extensions.*

Definition 1.20.

1. *Let A be a right H-comodule algebra (an algebra in $\mathcal{M}^H$) with structure map $\rho : A \longrightarrow A \otimes H$. The extension $A^{coH} \subseteq A$ is right H-Galois if the map $\beta : A \otimes_{A^{coH}} A \longrightarrow A \otimes H$ given by $\beta(a_1 \otimes a_2) = (a_1 \otimes 1)\rho(a_2)$ is bijective as an A-A bimodule map.*

2. *Let $B \subset A$ be an algebra extension. If moreover A is an H-comodule algebra such that $A^{coH} = B$, then $B \subseteq A$ is called an H-extension.*

3. *An H-extension $B \subseteq A$ is called a cleft extension if there is an H-comodule map $\gamma : H \longrightarrow A$ that is convolution invertible.*

4. *An H-extension $B \subseteq A$ is said to have the (right) normal basis property if $A \cong B \otimes H$ as left B-modules and right H-comodules.*

For example for a finite dimensional Hopf Algebra extension $R \subseteq H$ defining $Q := H/R^+H$ we get that $R \subseteq H$ is a Q-extension with normal basis property if Q is a Hopf algebra.

Theorem 1.21. *[40] Let $B \subseteq A$ be an H-extension. The following are equivalent:*

1. *$B \subseteq A$ is cleft.*

2. *$B \subseteq A$ is H-Galois with normal basis property.*

1.4 Background of Depth Theory

Throughout this section we revisit and explain in some detail the basic theory of depth and emphasise in those results that are either central to our discussion of the subject or that have been stepping stones in the development of the theory.

Definition 1.22. *Let R be a ring and N and M two left (equivalently right) R-modules. We say N and M are similar, and denote $N \sim M$ if N divides a multiple of M ($N|qM$) and M divides a multiple of N ($M|pN$), here q and p are natural numbers and $nV = V \oplus \cdots \oplus V$ n times. Note that this is equivalent to say that there is an isomorphism*

$$N \oplus * \cong qM \tag{1.17}$$

This is equivalent to saying that there is an split epimorphism

$$qM \twoheadrightarrow N$$

It is immediate that this first definition satisfies the axioms of an equivalence class. Moreover if we set ourselves in a context of finite dimensional module categories then this equivalence is preserved by additive functors. For example in a finite tensor category [18] the similarity is preserved by tensoring by any object in the category.

Let $B \subseteq A$ be a ring extension, denote the n-th tensor power of A over B, $A^{\otimes_B(n)} := A \otimes_B \cdots \otimes_B A$ (n times) as a natural X-Y-bimodule where $X, Y \in \{A, B\}$ whenever $n \geq 1$, and define $A^{\otimes_B(0)} = B$.

Definition 1.23. *[22, Definition 1.1] Let $B \subseteq A$ be a ring extension, we say B has minimum finite depth in A if there is a minimum natural number n satisfying one of the following:*

1. *Odd depth: $d_{odd}(B, A) = 2n+1$, if $A^{\otimes_B(n)} \sim A^{\otimes_B(n+1)}$ as B-B-bimodules for $n \geq 0$.*

2. *Left even depth: $d_{ev}(B, A) = 2n$, if $A^{\otimes_B(n)} \sim A^{\otimes_B(n+1)}$ as B-A-bimodules for $n > 0$.*

3. *Right even depth: $d_{ev}(B, A) = 2n$, if $A^{\otimes_B(n)} \sim A^{\otimes_B(n+1)}$ as A-B-bimodules for $n > 0$.*

4. *H depth: $d_h(B, A) = 2n - 1$, if $A^{\otimes_B(n)} \sim A^{\otimes_B(n+1)}$ as A-A-bimodules for $n > 0$.*

Since the similarity is preserved by tensor products, we notice that for every natural number m, $A^{\otimes_B(m)} \sim A^{\otimes_B(m+1)}$ implies $A^{\otimes_B(m)} \sim A^{\otimes_B(m+r)}$ for every $r \geq 1$. In addition, for the same reason the similarity is also preserved by module restriction $Res(- \downarrow_B^A)$. For these reasons is that we occupy ourselves with minimum depth only. If there is no such minimum m then we say the pair is infinite depth, $d(B, A) = \infty$. Denote $d(B, A) = min\{d_{odd}, d_{ev}\}$ when it exists.

An analogy from number theory would be to ask whether a sequence $\{a_n\}_{n\geq 1}$ of positive integers is affinely generated over the primes. (That is, generated by products of finitely many primes). For example, given a natural number c the series $a_n = c^n$ is finite depth whereas the series $a_n = p_n$ where p_i is sequence of increasing primes, is not finite depth.

Example 1.24. *Let $B \subseteq A$ be an extension of semisimple complex matrix algebras. Let $M : K_0(B) \longrightarrow K_0(A)$ be the $r \times s$ induction matrix where r and s are the number of irreducible representations of B in $K_0(B)$ and of A in $K_0(A)$ respectively. Let S be the order r symmetric matrix defined by $S := MM^T$. In [5] it is shown that the odd depth satisfies $d_{odd}(B, A) = 2n+1$ if $S^{(n)}$ and $S^{(n+1)}$ have an equal number of zero entries. The minimum even depth can be computed in the same manner by looking at the zero entries of* *the powers of $S^{(m)}M$. The overall minimum depth is given by the least $n \in \mathbb{N}$ such that*

$$M^{(n+1)} \leq qM^{(n-1)} \tag{1.18}$$

for some natural number q, each (i,j)-entry. $M^{(0)} := I_r$, $M^{(2n)} = S^{(n)}$, and $M^{(2n+1)} = S^{(n)}M$. Let N be the order s symmetric matrix defined by $N := M^T M$. The minimum H-depth of the extension $B \subseteq A$ is given by the least natural number n such that the zero entries of $N^{(n)}$ stabilise [29]. From matrix definitions it follows that a subalgebra pair of semisimple algebras $B \subseteq A$ over a field of characteristic zero is always finite depth. In characteristic p finite depth holds if either B or A is a separable algebra[33, Corollary 2.2] .

Notice that any ring extension $B \subseteq A$ satifies $A \oplus * \cong A \otimes_B A$ as an A-B or B-A-bimodule via the epi $\mu : A \otimes_B A \longrightarrow A$ which splits via $a \longmapsto a \otimes 1$ or $a \longmapsto 1 \otimes a$. A special type of extension, which we call *depth two* extension satisfies

$$A \otimes_B A \oplus * \cong qA \tag{1.19}$$

as either A-B or B-A-bimodules. This is a converse condition for $A \oplus * \cong A \otimes_B A$ in the sense of Hirata's similarity theory.

Example 1.25. *[24] Let $B \subseteq A$ be a ring extension, in any of the following cases we have a depth two extension:*

1. *Finite Hopf Galois extensions.*

2. *H-separable or centrally projective finite dimensional Hopf algebra extensions.*

Recall that an algebra extension $B \subseteq A$ is Hopf Galois with respect to a Hopf algebra H if A is a right H-comodule algebra, $B = A^{coH}$ and there is an A-A-bimodule bijection $\mu : A \otimes_B A \longrightarrow A \otimes H$, see definition (1.20). *Also, we say that a bimodule ${}_AM_B$ is centrally projective with respect to another bimodule ${}_AN_B$ if it satisfies ${}_AM_B \oplus * \cong q_AN_B$ for some natural number q. Finally we say $B \subseteq A$ is an H-separable extension if the tensor square $A \otimes_B A$ is centrally projective with respect to A.*

It is important to acknowledge that at first sight it is not easy to recognise a depth two ring extension. For this we need a setting in which we get hold of a variety of equivalent ways to describe this type of extension. Let

$B \subseteq A$ be a ring extension with centraliser denoted by $R := C_A(B) = A^B$, bimodule endomorphism ring $S := End_B A_B$ and B-central tensor-square $T := (A \otimes_B A)^B$. T has a ring structure induced from $T \cong End_A A \otimes_B A_A$ given by

$$tt' = t'_1 t_1 \otimes t_2 t'_2, \quad 1_T = 1 \otimes 1$$

Recall that an associative k-algebra A is said to be separable if for every field extension $L|k$ the algebra $A \otimes_K L$ is always semisimple.

Theorem 1.26. *[30, Theorem 2.1] The following are equivalent to the left depth two condition on a ring extension $B \subseteq A$ in (1)-(4). A left depth two condition on a Frobenius extension in (5). And a left depth two condition on a separable algebra A with separable subalgebra B over a field in (6):*

1. *The bimodules ${}_B A_A$ and ${}_B A \otimes_B A_A$ are similar.*
2. *There are $\{\beta_j\}_{j=1}^n \subset S$ and $\{t_j\}_{j=1}^n \subset T$ such that*
$$a \otimes a' = \sum_j t_j \beta_j(a) a'$$
3. *As natural B-A-bimodules $A \otimes_B A \cong Hom({}_R S, {}_R A)$ and ${}_R S$ is a f.g. projective module.*
4. *As natural B-A-bimodules $T \otimes_R A \cong A \otimes_B A$ and T_R is f.g. projective.*
5. *The endomorphism ring Frobenius extension has dual bases elements in T.*
6. *As a natural transformation between functors from the category of right B-modules into the category of right A-modules, there is a natural monic from $Ind_B^A Res_A^B Ind_B^A$ into $nInd_B^A$ for some positive integer n. In particular, for each pair of simple modules V_B and W_A, the number of isomorphic copies of W,*
$$\langle Ind_B^A Res_B^A Ind_B^A V, W\rangle \leq n\langle Ind_B^A V, W\rangle \tag{1.20}$$

We see that depth two can be linked to a necessary condition for Frobenius extensions via the dual basis property and to a necessary condition for an extension of separable algebras via induction and restriction of modules. Now we focus on an additional necessary condition of depth two that will allow us to construct an extension that is not depth two.

Lemma 1.27. *[30, 2.4] Let $B \subseteq A$ be a depth two ring extension. Then $R \otimes_T (A \otimes_B A) \cong A$ as A-A- bimodules and $End_R T \cong Z(A)$.*

The proof of the above lemma relies on noting that in the following diagram, γ is an isomorphism, where $m : T \otimes_R A \longrightarrow (A \otimes_B A)$; $t \otimes a \longmapsto ta$ is of course an isomorphism.

$$\begin{tikzcd} R \otimes_T T \otimes_R A \arrow[r, "R\otimes m"] \arrow[d, "\cong"'] & R \otimes_T (A \otimes_B A) \arrow[d, "\gamma"] \\ R \otimes_R A \arrow[r, "\cong"'] & A \end{tikzcd}$$

One then verifies that $\gamma^{-1}(a) = 1 \otimes_T (1 \otimes a)$. Since one sees that $Z(A) \hookrightarrow End_R T$ it suffices then to check that for every $f \in End_R T$ one has $f(1)$ satisfies a left integral condition and $f(1) \in E^T$ where $E := End_B A$.

Example 1.28. *Let A be the algebra of upper triangular 2 by 2 matrices over any field k, B the subalgebra of diagonal matrices, then $R = B$, T is spanned by $\{e_{11} \otimes e_{11}, e_{22} \otimes e_{22}\}$ in terms of matrix units e_{ij}, and it is easy to see that*

$$dimEnd(R_T) = dimEnd(R_R) = dimR = 2$$

while $dimZ(A) = 1$.

One of the most important achievements of depth two theory is the characterisation of normality of group algebra extensions in terms of the similarity of modules.

Theorem 1.29. *[30, Section 3] Let $H < G$ be a finite subgroup pair. Then for $B = \mathbb{C}H \subseteq A = \mathbb{C}G$ we have $d(B, A) = 2$ if and only if H is a normal subgroup of G.*

We give a sketch of the proof also following [30, Section 3]: First consider a normal finite subgroup pair $H \lhd G$ and let $\{g_1, \cdots, g_n\}$ be a set of representatives of the elements in G/H. Then the map

$$A^n \longrightarrow A \otimes_B A; (x_1, \cdots, x_n) \longmapsto \sum_{j=1}^{n} x_j g_j^{-1} \otimes_B g_j \tag{1.21}$$

is an isomorphism of A-B-bimodules. In the same way one gets an isomorphism of B-A-bimodules.

The converse is somewhat more tricky, first we prove that for every $\chi \in Irr(G)$ and $\psi \in Irr(H)$ irreducible characters there is a natural number n such that

$$\langle Ind_H^G(Res_H^G(Ind_H^G(\psi)))|\chi\rangle_G \leq n\langle Ind_H^G(\psi)|\chi\rangle_G$$

This is the Frobenius condition for depth two in terms of character representation given in equation (1.20).

Choosing $\chi = 1_G$ and $\psi \neq 1_H$ and using Frobenius reciprocity one can check that

$$\langle Ind_H^G(\psi)|1_G\rangle_G = \langle \psi|1_H\rangle_H = 0$$

From this and using Mackey's formula one sees that for every $g \in G$,

$$0 = \langle \psi|Ind^H_{g^{-1}Hg\bigcap H}(1_{g^{-1}Hg\bigcap H})\rangle_H$$

On the other hand

$$\langle 1_H|Ind^H_{g^{-1}Hg\bigcap H}(1_{g^{-1}Hg\bigcap H})\rangle_H = \langle 1_{g^{-1}Hg\bigcap H}|1_{g^{-1}Hg\bigcap H}\rangle_{g_{-1}Hg\bigcap H} = 1$$

Using Frobenius reciprocity one more time. This implies that

$$Ind^H_{g^{-1}Hg\bigcap H}(1_{g^{-1}Hg\bigcap H}) = 1_H$$

Finally we just compare degrees and conclude that $H = g^{-1}Hg\bigcap H$ and thus $H \lhd G$.

We notice that this proof relies on complex characters, but it can be extended to a more general situation where the extension of group rings is of the form $RH \subseteq RG$ and where R is an arbitrary commutative ring. In [6] R. Boltje and B. Külshammer give the details to accomplish that generalisation.

Now, assume $R \subseteq H$ is a finite dimensional normal Hopf algebra extension, then the quotient $Q = H/R^+H$ (where $R^+ = ker\varepsilon|_R$) is a Hopf algebra. One proves that the extension $R \subseteq H$ is Q-Galois and therefore depth two as we saw in example 1.25. In [6, 2.10] the authors give a series of equivalent conditions, separating the notions of right and left normality, that prove that the converse is also true.

Since normality implies symmetry in terms of the left or right quotient modules one has the following:

Theorem 1.30. *Let $R \subseteq H$ a finite dimensional Hopf algebra extension, then R has left depth 2 in H if and only if R has right depth 2 in H if and only if R is a normal Hopf subalgebra of H.*

Now, we know that for every n, depth n implies depth $n+1$, hence for a finite dimensional Hopf algebra pair $R \subseteq H$, $d(R,H) = 1$ implies R is normal in H. Moreover in [10] it is shown that when such extension is semisimple then $Z(H) = Z(R)$ and that $A(H^*)$ act trivially on $A(R)$, the Green ring of H^* and R respectively. See definition (2.17) .

In [4] the authors define the combinatorial depth $d_c(H,G)$ for a group algebra extension $RH \subseteq RG$ over any commutative ring R. This is accomplished by comparing sums of combinatorial modules over either G or H. They prove that such depth is always finite and find that $d_c(H,G) \leq 2|G : N_G(H)| < \infty$. Afterwards they provide inequalities for the ring extensions in case R is a ring of any characteristic and prove the following chain of inequalities.

$$d_0(H,G) \leq d_p(H,G) \leq d_R(H,G) \leq d_{\mathbb{Z}}(H,G) \leq d_c(H,G) \tag{1.22}$$

Here from left to right we have depth of a group algebra extension over a field of characteristic zero, over a field of any characteristic p, over any commutative ring R, over the integers and finally combinatoric depth.

As examples the authors also provide the following formulas:

$$d(\mathbb{S}_n, \mathbb{S}_{n+1}) = 2n - 1 \tag{1.23}$$

$$d(\mathbb{A}_n, \mathbb{A}_{n+1}) \leq 2(n - \lceil \sqrt{n} \rceil) + 1 \tag{1.24}$$

Here $\mathbb{S}_n$ and $\mathbb{A}_n$ are the symmetric and alternating groups on n letters respectively. Moreover in [14] it is proved that given $\alpha \in H^2(\mathbb{S}_n, C^x)$ a non trivial two cocycle then,

$$d(\mathbb{C}_\alpha \mathbb{S}_n, \mathbb{C}_\alpha \mathbb{S}_{n+1}) = 2(n - \lceil \frac{\sqrt{8n+1}-1}{2} \rceil) + 1, \tag{1.25}$$

the depth of the twisted complex group algebra extension of the symmetric group on n letters in the twisted complex group algebra of the symmetric group of degree $n+1$. In [21] the authors find formulas for the Young subgroups of symmetric groups.

In [25] Kadison defines *module depth* within a finite tensor category (details are provided in chapter (2)) and uses this definition to relate the depth of a finite dimensional Hopf algebra extension $R \subseteq H$ to that of its *quotient module*:

$$Q = H/R^+H \tag{1.26}$$

A proof of the following is presented in proposition (2.14). The result is presented for H-H-bimodules, but the argument holds for this case as well.

Lemma 1.31. *[25, 3.1]: Let A be an arbitrary algebra, M an A-H-bimodule, then*

$$M \otimes_R H \cong M \otimes Q \tag{1.27}$$

as A-H-bimodules, via

$$m \otimes h \longmapsto mh_{(1)} \otimes \overline{h_{(2)}} \tag{1.28}$$

This is the key for linking depth of a finite dimensional Hopf algebra pair $R \subseteq H$ to the depth of the quotient module Q in its R or H-module category. As we will see later, projective modules play a fundamental role in the theory of module depth in finite tensor categories; for this reason the following is relevant to our work:

Theorem 1.32. *[25, 3.5] The quotient module Q is projective as a right H-module if and only if the subalgebra R is semisimple if and only if Q is projective as a right R-module.*

Lemma 1.33. *[25, 4.7] The R-module Q is semisimple if one of the following three conditions is met:*

1. *R is an ad-stable Hopf subalgebra of H.*
2. *The Jacobson radical of R, $J(R)$, is a left ad-stable ideal in H.*
3. *$J(R) \subseteq J(H)$ and Q_H is semisimple.*

The following proposition allows us to pass from a β Frobenius extension [26, Chapter 7] $R \subseteq H$ to a regular Frobenius extension $Q^* \hookrightarrow H^*$.

Proposition 1.34. *Given $R \subseteq H$ a finite dimensional Hopf algebra extension, the canonical epimorphism of right H-module coalgebras $H \longrightarrow Q \longrightarrow 0$ induces a monomorphism of left H-module algebras*

$$0 \longrightarrow Q^* \longrightarrow H^* \tag{1.29}$$

which is an ordinary free Frobenius extension of rank $dim(R)$ and a Hopf-Galois extension.

Finally in [25, 5.5] it is proven that the depth of $R \subseteq H$ is related to the depth of the H-module algebra Q^* in its smash product with either H or R in the following way:

$$d(R, H) - d(R, Q^* \# R) \leq 2 \tag{1.30}$$

and

$$d_h(R, H) = d_{odd}(H, Q^* \# H) \tag{1.31}$$

Furthermore [48] relates the depth of $H \subseteq A\#H$ to that of the depth of the left H-module algebra A in the module category ${}_H\mathcal{M}$:

$$d(H, A\#H) = 2d(A,{}_H \mathcal{M}) + 1 \tag{1.32}$$

Chapter 2

The Quotient Module Q

2.1 Module Depth

Throughout this section Q is the quotient module of a finite dimensional Hopf algebra extension $R \subseteq H$ over a field k. We define the idea of an algebraic module in the representation ring of H, $A(H)$ and use it to give an equivalent condition for finite depth of the pair $R \subseteq H$. As a result of this, we give a purely theoretical explanation of the result in [4] on subgroup depth.

We start with the concept of module depth in a finite tensor category. Definitions and theory about module depth can be found in [25]. Finite tensor categories are defined and explained exhaustively in [18]. The concept of an algebraic module in the representation ring of a group G can be traced back to [20, Chapter 5].

A tensor category is an abelian rigid category in which the unit object 1 is simple, one can find a complete definition in [3]. Moreover it is known that in such a category the tensor product bifunctor is exact in both terms.

Definition 2.1. *Let $\mathcal{C}$ be a tensor category. We say $\mathcal{C}$ is a finite tensor category over k if every object X has finite length, $\mathcal{C}$ has finitely many simple objects up to isomorphism, and every simple object S has a projective cover $P(S)$.*

Example 2.2. *If H is a finite dimensional Hopf algebra (or more generally a quasi-Hopf algebra) then the category $\mathcal{M}_H$ of finite dimensional right H-modules is a finite tensor category via the diagonal map on H induced from* *the coproduct $\Delta : H \longrightarrow H \otimes H$. Moreover in a finite tensor category $\mathcal{C}$ the tensor product $P \otimes X$ as well as $X \otimes P$ of a projective object with any other object in the category is again a projective object of $\mathcal{C}$. [18]*

Definition 2.3. *Let $\mathcal{C}$ be an additive category, we say $\mathcal{C}$ is a Krull-Schmidt category if every object decomposes uniquely as a finite direct sum of objects having a local endomorphism ring.*

Let R be a ring and M an R-module. The Krull-Schmidt theorem states that if M is both Noetherian and Artinian (or equivalently finite length) then M is a unique direct sum of indecomposable modules $M_1 \oplus \cdots \oplus M_k$. Uniqueness here is up to permutation of the unique indecomposable elements.

Examples of Krull-Schmidt categories include abelian categories in which every object has finite length. By the Krull-Schmidt theorem if H is a finite dimensional k algebra then $\mathcal{M}_H$ is a Krull-Schmidt category.

We now consider $\mathcal{C}$ a finite tensor category. Let W be a nonzero object in $\mathcal{C}$. Denote $W^{\otimes(n)} = W \otimes \cdots \otimes W$ (n times) and $W^{\otimes(0)} = 1_{\mathcal{C}}$.

Definition 2.4. *Let $\mathcal{C}$ be a finite tensor category, let W be a nonzero object in $\mathcal{C}$. Define the n-th truncated tensor algebra of W, $T_{(n)}(W) = W \oplus W^{\otimes(2)} \oplus \cdots \oplus W^{\otimes(n)}$ for $n \geq 1$, and $T_{(0)}(W) = 1_{\mathcal{C}}$.*

Definition 2.5. *[25, 4.1] Let W be a non zero object in a finite tensor category $\mathcal{C}$. We say that W has finite depth $n \geq 0$ in $\mathcal{C}$ and denote it by $d(W, \mathcal{C}) = n$ if*

$$T_{(n+1)}(W) \sim T_{(n)}(W) \tag{2.1}$$

We note that if W has depth n then it also has depth $n+1$, for this reason we are interested in minimum depth, $d(W, \mathcal{C}) = n$ that is the smallest n that would satisfy the relation. If there is no such n then we write $d(W, \mathcal{C}) = \infty$.

As we know, the indecomposable representations of H, present as the building blocks of the unique decomposition of its modules, play a critical role in the representation theory of the algebra. The next lemma tells us about the part indecomposable representations play in depth theory.

Let M be an H-module. Define $Indec(M)$ to be the set of indecomposable constituents of M as an H-module.

Lemma 2.6. *[25, 4.4] Let W be an H-module, then for all $n \geq 1$*

$$Indec(T_{(n)}(W)) \subseteq Indec(T_{(n+1)}(W)) \tag{2.2}$$

And with equality if and only if $d(W, \mathcal{M}_H) \leq n$

Proof. Since W is in a Krull-Schmidt category and $T_{(n)}(W)|T_{(n+1)}(W)$ the first inclusion follows. The opposite inclusion arises when we observe that by definition $d(W, \mathcal{M}_H) = n$ implies $T_{(n)}(W) \sim T_{(n+1)}(W)$ which in turn is equivalent to $T_{(n+1)}(W) \oplus * \cong qT_{(n)}(W)$. This says that there is a split injection $i : T_{(n+1)}(W) \longrightarrow qT_{(n)}(W)$. Again since we are in a Krull-Schmidt category the inclusion is granted. □

Definition 2.7. *Let $\mathcal{C}$ be a module category. An object M in $\mathcal{C}$ is a module coalgebra if it is a coalgebra in the module category $\mathcal{C}$. This is to say, the coproduct Δ_M and the counit ε_M are both module homomorphisms.*

Lemma 2.8. *[25, Prop 3.8] Let H be a finite dimensional Hopf algebra and W a finite dimensional right H-module coalgebra, then $W^{\otimes(n)}|W^{\otimes(n+1)}$ as H-modules for $n \geq 1$. If moreover H is semisimple then it is also true for $n = 0$.*

Corollary 2.9. *Let H be a finite dimensional Hopf algebra and let $W \in \mathcal{M}_H$ be a module coalgebra. Then*

$$d(W, \mathcal{M}_H) \leq n \iff W^{\otimes(n)} \sim W^{\otimes(n+1)} \tag{2.3}$$

Proof. It suffices to notice that for an H-module coalgebra the coproduct Δ_W splits via the counit ε_W. Moreover since we are in a Krull-Schmidt category all indecomposable components in the different summands $W^{\otimes(k)}$ for $1 \leq k \leq n$ in $T_{(n)}(W)$ are contained in $W^{\otimes(n)}$. □

As we saw above, indecomposable representations of an algebra H play a role in the finiteness of depth. We say an algebra H has *finite representation type* if the set of indecomposable isoclasses of representations of H is finite. The following theorem links finite representation type algebras to finite module depth. We provide the proof here.

Theorem 2.10. *[25, Proposition 4.8] Let H be a finite dimensional k-algebra of finite representation type. Let W be an H-module. Then*

$$d(W, \mathcal{M}_H) < \infty \tag{2.4}$$

Proof. Recall that if W is in a Krull-Schmidt category, then for every $n \geq 1$ lemma (2.6) tells us that $Indec(T_{(n)}(W)) \subseteq Indec(T_{(n+1)}(W))$. Now since we only have a finite number of indecomposable representations of H it follows that at some natural number n the $(n + 1)$ tensor power of W has no new indecomposable elements in $\mathcal{M}_H$, and hence equality is achieved $Indec(T_{(n)}(W)) = Indec(T_{(n+1)}(W))$. This implies that as H modules $T_{(n+1)}(W)|mT_{(n)}(W)$ for some natural number m. Since the opposite is always true we have $d(W, \mathcal{M}_H) \leq n$ □

2.2 Q as an Algebraic Module

Definition 2.11. *Let $R \subseteq H$ be a finite dimensional Hopf algebra pair. Denote $R^+ := ker(\varepsilon|_R)$, the kernel of the counit restricted to R. Let $Q = H/R^+H$ be a right quotient module, and we call it the generalised permutation module.*

Proposition 2.12. *Let $R \subseteq H$ be a finite dimensional Hopf algebra pair, then Q is a finite dimensional right H-module coalgebra.*

Proof. That Q is finite dimensional is straightforward. By the normal basis property one deduces that $dim(Q) = dim(H)/dim(R)$.

Note that there is an epimorphism of H-module coalgebras

$$H \longrightarrow Q; \quad h \longmapsto \overline{h} = h + R^+H. \tag{2.5}$$

Let $h, g \in H$. The right action of H on Q is given by:

$$\overline{h} \cdot g = \overline{hg}. \tag{2.6}$$

Furthermore the coalgebra structure is also induced by the canonical epimorphism $H \twoheadrightarrow Q$:

$$\Delta_Q(\overline{h}) = \overline{h_{(1)}} \otimes \overline{h_{(2)}}. \tag{2.7}$$

That the coalgebra structure is compatible with the H-module structure is given by the fact that the coproduct in H is an algebra map. □

Corollary 2.13. *Let $R \subseteq H$ be a finite dimensional Hopf algebra pair, then*

$$d(Q, \mathcal{M}_H) \leq n \Leftrightarrow Q^{\otimes(n+1)} \sim Q^{\otimes(n)} \Leftrightarrow Q^{\otimes(n+1)}|qQ^{\otimes(n)}.$$

for some natural number q.

Proof. It's just combining corollary (2.9) and proposition (2.12) □

The next proposition is also found in [25], here we state in its H-bimodule form for our purposes. The proof relies on the same arguments though.

Proposition 2.14. *[25, Theorem 3.6] Let $R \subseteq H$ a finite dimensional Hopf algebra pair and W be an H-H-bimodule, then as H-H-bimodules*

$$W \otimes_R H^{\otimes_R (n)} \cong W \otimes Q^{\otimes (n)}.$$

Proof. First consider the following H-H-bimodule map.

$$w \otimes_R h_1 \otimes_R \cdots \otimes_R h_n \longmapsto w h_{1(1)} \cdots h_{n(1)} \otimes \overline{h_{1(2)} \cdots h_{n(2)}} \otimes \cdots \otimes \overline{h_{n(n+1)}} \quad (2.8)$$

We claim that the following is the H-H-bimodule inverse to the previous one.

$$u \otimes \overline{v_1} \otimes \cdots \otimes \overline{v_n} \longmapsto$$

$$uS(v_{1(1)}) \otimes_R v_{1(2)} S(v_{2(1)}) \otimes_R \cdots \otimes_R v_{n-1(2)} S(v_{n(1)}) \otimes_R v_{n(2)}. \quad (2.9)$$

We will proceed by induction on n :Consider $w \otimes h \in W \otimes_R H$, then applying the map (2.8) and composing with the map (2.9) we get the following: $w \otimes h \longmapsto wh_{(1)} \otimes \overline{h_{(2)}} \longmapsto wh_{(1)} S(h_{(2)}) \otimes h_{(3)} \cong w\varepsilon(h_{(1)}) \otimes h_{(2)} \cong w \otimes h$. Hence we get the identity map in $W \otimes_R H$. The fact that is H-H-bimodule map results from the diagonal action of H from both the left and the right. Furthermore, R linearity arises from the fact that for every $r \in R$ one has $(r - \varepsilon(r)) \in R^+$ and hence in Q we have that $\overline{r} = \varepsilon(r)$.

Now let $v \otimes \overline{u} \in W \otimes Q$. Applying map (2.9) and composing with map (2.8) one gets the following: $v \otimes \overline{u} \longmapsto vS(u_{(1)}) \otimes_R u_{(2)} \longmapsto vS(u_{(1)}) u_{(2)} \otimes \overline{u_{(3)}} \cong v\varepsilon(u_{(1)}) \otimes \overline{u_{(2)}} \cong v \otimes \overline{u}$. Hence we get the identity map in $W \otimes Q$. H-H-bimodule map and R linearity follow from the same remarks as above.

Assume now that the isomorphism holds for a certain natural number n, and consider $W \otimes_R H^{\otimes_R (n+1)}$. Notice that this is isomorphic to $W \otimes_R H \otimes_R H^{\otimes_R (n)}$. Since H is naturally an H-H-bimodule we apply the induction hypothesis to obtain $W \otimes_R H^{\otimes_R (n+1)} \cong W \otimes_R H \otimes Q^{\otimes (n)}$ applying the theorem to $W \otimes_R H$ we get $W \otimes Q \otimes Q^{\otimes (n)} \cong W \otimes Q^{(n+1)}$. Hence we get the result. $\square$

Note that it is a direct consequence of this that as H-H-bimodules

$$H^{\otimes_R (n+1)} \cong H \otimes Q^{\otimes (n)}. \quad (2.10)$$

Theorem 2.15. *[25, 5.2] Let $R \subseteq H$ be a finite dimensional Hopf algebra pair. Then:*

$$d_h(R, H) = 2d(Q, \mathcal{M}_H) + 1.$$

Proof. ($\leq$) Let $d(Q, \mathcal{M}_H) = n$. By corollary (2.13) this implies that $Q^{\otimes(n+1)} \sim Q^{\otimes(n)}$ in $\mathcal{M}_H$. Tensoring from the left on both sides of the relation with $(H \otimes -)$ leaves us with $H \otimes Q^{\otimes(n+1)} \sim H \otimes Q^{\otimes(n)}$. By the previous remark we then get $H^{\otimes_R(n+2)} \sim H^{\otimes_R(n+1)}$ as H-H-bimodules. This by definition is just $d_h(R, H) \leq 2n + 1$. Then $d_h(R, H) \leq 2d(Q, \mathcal{M}_H) + 1$.

($\geq$) Notice that $k \otimes_H H \cong k$ and that for every $X \in \mathcal{M}_H$ we have $k \otimes X \cong X$. Of course the diagonal action of H accounts for the action over tensor products of Q. It is now clear that the steps in the previous proof can be reversed and hence $d_h(R, H) = 2n + 1$ implies $d(Q, \mathcal{M}_H) \leq n$ and so $2d(Q, \mathcal{M}_H) + 1 \leq d_h(R, H)$. □

Even though it seems like a trivial observation it is important, for the sake of completeness and for the clarity of concepts to come, to point out that we proved that

$$d_h(R, H) < \infty \iff d(Q, \mathcal{M}_H) < \infty. \tag{2.11}$$

It is also important to notice, in order to consider the restrictions of Q as an R-module and to get a similar result for minimum depth, that by considering H-R-bimodules in the proof above we get:

$$2d(Q, \mathcal{M}_R) + 1 \leq d(R, H) \leq 2d(Q, \mathcal{M}_R) + 2 \tag{2.12}$$

As a result of this we get the following.

Proposition 2.16.

$$|d(Q, \mathcal{M}_H) - d(Q, \mathcal{M}_R)| \leq 1 \tag{2.13}$$

Proof. Notice that $|d_h(R, H) - d(R, H)| \leq 2$ this combined with theorem (2.15) and equation (2.12) yields the result. □

It is also worth noting that from equation (2.12) one gets that minimum even depth is given by:

$$d_{ev}(R, H) = 2d(Q, \mathcal{M}_R) + 2 \tag{2.14}$$

The finiteness of the different depths are related by:

$$d(R, H) < \infty \Leftrightarrow d(Q, \mathcal{M}_R) < \infty \Leftrightarrow d(Q, \mathcal{M}_H) < \infty \Leftrightarrow d_h(R, H) < \infty \tag{2.15}$$

In [20, Chapter 5] we find the concept of algebraic modules in the representation ring $A(kG)$ of a group algebra kG where G is a group and k a ring. Here we aim to provide certain conditions to set this construct in the frame of a finite dimensional Hopf algebra pair $R \subseteq H$ and relate it to the depth of its quotient module Q in $\mathcal{M}_H$. For the rest of this section $k = \overline{k}$ and is of characteristic zero.

Definition 2.17. *Let H be a finite dimensional Hopf algebra over a field k. Denote the Green Ring of H as:*

$$A(H) = \{(V) | \ (V) \ is \ an \ isoclass \ of \ H - modules\}$$

The ring structure in $A(H)$ is given by:

$$(V) + (W) = (V \oplus W).$$
$$(V) \cdot (W) = (V \otimes W). \tag{2.16}$$

It is also important to notice that as a ring or k-algebra

$$A(H) = \langle (L_i) \rangle,$$

where (L_i) are isoclasses of indecomposable H-modules.

Recall that given an algebra A we define $K_0(A)$ as the abelian group generated by the projective A-modules under direct summation. Also $G_0(A)$ is the abelian group under direct summation generated by all A-modules such that for every short exact sequence

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

one has the following relation:

$$[B] - [C] + [D] = 0$$

Example 2.18. *$K_0(H) \subseteq A(H)$ is a finite rank ideal since $P \otimes X \in K_0(H)$ for all $P \in K_0(H)$ and all $X \in A(H)$. This a fact well known for finite tensor categories. [18, Proposition 2.1] as well as [20].*

Example 2.19. *Let H be a finite dimensional semisimple algebra, such that $\mathcal{M}_H$ is a tensor category. Then $K_0(H) = G_0(H) = A(H)$.*

Definition 2.20. *Let $X \in A(H)$, we say that X is an algebraic H-module if it satisfies a non-zero polynomial in $A(H)$ with integer coefficients. That is to say, there is a natural number n and integers $a_0, \cdots, a_n$ not all of them zero such that*

$$\sum_{k=0}^{n} a_k(X^{\otimes(k)}) = 0 \tag{2.17}$$

Where we identify the element with its class.

Theorem 2.21. *[20, 5.1] Let H be a finite dimensional Hopf algebra, W an H-module. The following are equivalent:*

1. *W is algebraic*

2. *There exists a finite number of indecomposable H-modules $L_1, \cdots, L_m$ such that for every indecomposable L that satisfies $L|W^{\otimes(n)}$ for some n then $L \in \{L_1, \cdots, L_m\}$.*

Proof. (1$\Rightarrow$ 2)
Let W be algebraic, then it satisfies a non-zero polynomial in $A(H)$, that is equivalent to say there are two sets of positive integers $a_0, \cdots, a_k$ and $b_0, \cdots, b_j$ such that $j < k$ and such that $a_k \neq 0$ for which

$$\sum_{i=0}^{k} a_i(W^{\otimes(i)}) = \sum_{i=0}^{j} b_i(W^{\otimes(i)})$$

Now, since W is finite dimensional then every indecomposable L_r occurring in $W^{\otimes(k)}$ is also occurring in $W^{\otimes(i)}$ for some $i < k$. Then by induction on s, is clear that for every $s \geq k$ every indecomposable L_r occurring in $W^{\otimes(s)}$ is also occurring in some $W^{\otimes(i)}$ for some $i < k$, this of course suffices to get 2.
(2 $\Rightarrow$ 1)
The statement is equivalent to the following decomposition for some integers a_{ij} with $i \leq m+1$ and $j \leq m$.

$$\begin{array}{ccccc}
W & = & a_{11}(L_1) & \cdots & a_{1m}(L_m) \\
W^{\otimes(2)} & = & a_{21}(L_1) & \cdots & a_{2m}(L_m) \\
. & . & . & . & . \\
. & . & . & . & . \\
. & . & . & . & . \\
W^{\otimes(m+1)} & = & a_{(m+1)1}(L_1) & \cdots & a_{(m+1)m}(L_m)
\end{array}$$

The linear independence of the (L_i), guaranteed since $\mathcal{M}_H$ is a Krull-Schmidt category, implies that W must satisfy a polinomial of degree of at most $m+1$. □

Theorem 2.22. *Let H be a finite dimensional Hopf algebra and W in $\mathcal{M}_H$. Then:*

$$W \textit{ is an algebraic } H-\textit{module} \Leftrightarrow d(W, \mathcal{M}_H) < \infty.$$

Proof. ($\Leftarrow$) Let $d(W, \mathcal{M}_H) = n$ for some natural number. Then by definition we have

$$T_{n+1}(W)|rT_n(W)$$

for some natural number r. In a Krull-Schmidt category this is equivalent to saying that every indecomposable H-module occuring in $T_n(W)$ is also occuring in $T_{n+1}(W)$ up to isomorphism. Then $T_{n+1}(W)$ and all its summands $W^{\otimes k}$ for every $k < n+1$ are expressible in terms of these summands. By theorem (2.21) W must satisfy some polinomial in $A(H)$.
($\Rightarrow$) This direction is straightforward. □

Corollary 2.23. *Let H be a finite dimensional Hopf algebra and $P \in \mathcal{M}_H$ a finite dimensional projective H-module. Then P is algebraic.*

Proof. First note that by theorem (2.21) it is immediate that if V is an algebraic H-module and $W|V$ in $\mathcal{M}_H$ then W is itself algebraic. Secondly, note that $d(H, \mathcal{M}_H) = 1$ since all projective indecomposable H-modules are contained in H_H, hence by theorem (2.22) we have that H is an algebraic H-module. Finaly let $P \in \mathcal{M}_H$ be a projective module, since we are in a finite tensor category this implies $P|H^m$ for some m and hence by the previous remark P must be algebraic. □

In [20, Chapter 9] it is shown readily that via Mackey's theorem, permutation modules of group algebras over commutative rings are algebraic.

Example 2.24. *Let $H < G$ be a finite group extension, k a commutative ring and $Q \cong k[G/H]$. Notice that as right G-modules, left H-modules we have $Q \cong Ind(k \uparrow_H^G)$. Consider now the induction of Q from the trivial H-module to a G-module. Then Mackey's tensor theorem implies*

$$Q_H^G \otimes Q_H^G \cong \bigoplus_{x \in H\backslash G/H} Q^G_{H \bigcap H^x} \tag{2.18}$$

Then, by iterating this process one finds that the depth of Q as a kG-module is the number of conjugates intersecting at the core.

Corollary 2.25. *[4] Let $H < G$ be a finite group pair, k a commutative ring and $kH \subseteq kG$ a finite dimensional ring extension, then*

$$d(kH, kG) < \infty$$

Proof. Just recall that as mentioned above permutation modules over group algebras over commutative rings are algebraic. Notice that for a finite group pair $H < G$ one has that $Q \cong k(H\backslash G)$ (the right cosets) is a permutation module. Apply equation (2.15) and theorem (2.22) □

It is worth to note that since finite depth and being algebraic in a finite tensor category are equivalent conditions and that since projective modules are always algebraic the following is interesting:

Proposition 2.26. *Let H be a finite dimensional Hopf algebra and let W be a finite dimensional H-module, then $d(W, \mathcal{M}_H) < \infty$ if and only if the non-projective indecomposable H-module constituents of W are algebraic.*

2.3 The Depth of the Sweedler Algebra in its Drinfeld Double

In this section we calculate the depth of the Sweedler algebra in its Drinfeld double. This is a nice example since we know that all Taft algebras are non semisimple, hence by lemma (**??**) we have that Q is not projective and therefore it must have at least one non-projective indecomposable constituent.

Definition 2.27. *Let n be a natural number and ω an n-th primitive root of unity. Denote the n-th Taft algebra over the complex numbers $\mathbb{C}$*

$$H_\omega(n) =: \langle a, b | a^n = 0, b^n = 1, ab = \omega ba \rangle_{\mathbb{C}} \tag{2.19}$$

an n^2-dimensional algebra with Hopf algebra structure given by:

$$\Delta(b) = b \otimes b, \quad \varepsilon(b) = 1, \quad S(b) = b^{-1} = b^{(n-1)}$$

$$\Delta(a) = 1 \otimes a + a \otimes b, \quad \varepsilon(a) = 0, \quad S(a) = (-\omega b)^{(n-1)} a. \tag{2.20}$$

Let $n = 2$ and $\omega = -1$, the Taft algebra $H_{-1}(2) =: R$ is called the Sweedler algebra, introduced by M. Sweedler in 1969.

It is easily seen that $\{a^i b^j | 0 \leq i, j \leq 1\}_{\mathbb{C}}$ is a basis for R which of course is 4-dimensional.

For future calculations and for the sake of completeness we will mention certain aspects of the representation ring of R which as usual we will denote $A(R)$. For more information on the subject refer to [12].

First note that R has two orthogonal, primitive idempotents,

$$e_1 = \frac{1}{2}(1+b), \quad e_2 = \frac{1}{2}(1-b) \tag{2.21}$$

Moreover, notice that $\langle b \rangle \subseteq R$ is a group algebra of dimension 2, and hence a semisimple Hopf algebra, this implies that J, the Jacobson radical of R, is contained in $\langle a \rangle$. On the other hand since $a^2 = 0$ we have $\langle a \rangle \subseteq J$.

Lemma 2.28. *R has exactly two non isomorphic 2-dimensional projective indecomposables $P_1 = e_1 R$, $P_2 = e_2 R$ and two one-dimensional non isomorphic non projective simples $S_1 = P_1/P_1 J$, $S_2 = P_2/P_2 J$.*

The action of R on P_1 and P_2 can be presented in the following diagrams respectively:

$$\begin{array}{cc} P_1 \cdot R & P_2 \cdot R \\ \begin{tikzcd} e_1 \arrow[r, "b"] \arrow[d, "a"] & e_1 \\ e_1 a \arrow[r, "b"] \arrow[d, "a"] & -e_1 a \\ 0 & \end{tikzcd} & \begin{tikzcd} e_2 \arrow[r, "b"] \arrow[d, "a"] & -e_2 \\ e_2 a \arrow[r, "b"] \arrow[d, "a"] & e_2 a \\ 0 & \end{tikzcd} \end{array} \tag{2.22}$$

And the action on S_1 and S_2 is as follows:

$$\begin{array}{cc} S_1 \cdot R & S_2 \cdot R \\ \begin{tikzcd} e_1 \arrow[r, "b"] \arrow[d, "a"] & e_1 \\ 0 & \end{tikzcd} & \begin{tikzcd} e_2 \arrow[r, "b"] \arrow[d, "a"] & -e_2 \\ 0 & \end{tikzcd} \end{array} \tag{2.23}$$

In [12, Corollary 3.7] it is proven that the representation ring of every Taft algebra is commutative, with this in mind and following again [12] we get the next result:

Lemma 2.29. *The tensor products of the indecomposable R-modules are as follows:*

$$P_1 \otimes P_1 = P_1 \otimes P_2 = P_2 \otimes P_2 = P_1 \oplus P_2$$

$$P_1 \otimes S_1 = P_2 \otimes S_2 = P_1, \quad P_1 \otimes S_2 = P_2 \otimes S_1 = P_2$$

$$S_1 \otimes S_1 = S_1 \quad S_1 \otimes S_2 = S_2 \quad S_2 \otimes S_2 = S_1 \tag{2.24}$$

For the description of $D(R)$, the Drinfeld double of R, we will follow both [13] and [37].

Definition 2.30. *Denote $D(R) =: H$ the Drinfeld double of the Sweedler algebra R and define it as a $\mathbb{C}$-algebra as follows:*

$$H = \langle a, b, c, d \rangle_{\mathbb{C}} \tag{2.25}$$

With algebra relations:

$$ba = -ab, \quad db = -bd, \quad ca = -ac \quad dc = -cd \quad bc = cb$$

$$da + ad = 1 - bc, \quad a^2 = 0, \quad b^2 = 1, \quad c^2 = 1, \quad d^2 = 0 \tag{2.26}$$

And Hopf structure given by:

$$\Delta(a) = a \otimes b + 1 \otimes a, \quad \varepsilon(a) = 0, \quad S(a) = -ab$$

$$\Delta(b) = b \otimes b, \quad \varepsilon(b) = 1, \quad S(b) = b^{-1}$$

$$\Delta(c) = c \otimes c, \quad \varepsilon(c) = 1 \quad S(c) = c^{-1}$$

$$\Delta(d) = d \otimes c + 1 \otimes d, \quad \varepsilon(d) = 0, \quad S(d) = -dc \tag{2.27}$$

Note that $\{a^i b^j c^k d^l | 0 \leq i, j, k, l \leq 1\}_{\mathbb{C}}$ is a basis for H and that it is a 16-dimensional algebra, and that $R^+ = ker(\varepsilon|_R) = \langle 1 - b, a \rangle_{\mathbb{C}}$. Then

$$Q = H/R^+H \cong \overline{c^i d^j} \tag{2.28}$$

with $0 \leq i, j \leq 1$, a 4-dimensional R-module coalgebra which has as elements of its basis:

$$\overline{c}, \quad \overline{d}, \quad \overline{1} \quad and \quad \overline{cd} \tag{2.29}$$

Now, using the relations from equation (2.26) we get the following diagrams representing the action of R over Q on the right:

$$\begin{array}{cc} (1) & (2) \\ \begin{tikzcd} \overline{cd} \arrow[r, "b"] \arrow[d, "a"] & \overline{-cd} \\ \overline{c-1} \arrow[r, "b"] \arrow[d, "a"] & \overline{c-1} \\ 0 & \end{tikzcd} & \begin{tikzcd} \overline{d} \arrow[r, "b"] \arrow[d, "a"] & \overline{-d} \\ \overline{1-c} \arrow[r, "b"] \arrow[d, "a"] & \overline{1-c} \\ 0 & \end{tikzcd} \end{array} \tag{2.30}$$

$$\begin{array}{cc} (3) & (4) \\ \begin{tikzcd} \overline{1} \arrow[r, "b"] \arrow[d, "a"] & \overline{1} \\ 0 & \end{tikzcd} & \begin{tikzcd} \overline{c} \arrow[r, "b"] \arrow[d, "a"] & \overline{c} \\ 0 & \end{tikzcd} \end{array} \tag{2.31}$$

Now notice that (1) $\cong$ (2) $\cong P_2$, (3) $\cong$ (4) $\cong S_1$. Notice also that:

$$\begin{array}{c} (5) \\ \begin{tikzcd} \overline{cd+d} \arrow[r, "b"] \arrow[d, "a"] & \overline{-(cd+d)} \\ 0 & \end{tikzcd} \end{array} \tag{2.32}$$

Then we get (5) $\cong S_2$. Putting all this together we get:

$$Q \cong P_2 \oplus S_2 \oplus S_1 \tag{2.33}$$

Then using the isomorphisms in lemma (2.29) we get the following:

$$\begin{aligned} Q^{\otimes(2)} &\cong (P_2 \oplus S_2 \oplus S_1) \otimes (P_2 \oplus S_2 \oplus S_1) \\ &\cong 3P_1 \oplus 3P_2 \oplus 2S_1 \oplus 2S_2 \end{aligned} \tag{2.34}$$

This tells us that Q and $Q^{\otimes(2)}$ are not similar as R modules since $Indec_R(Q) \neq Indec_R(Q^{\otimes(2)})$. On the other hand $Q^{\otimes(2)}$ has all possible indecomposable R-modules as constituents, notice that all of them must appear in the next tensor $Q^{\otimes(3)}$, since $Q^{\otimes(n)}|Q^{\otimes(n+1)}$ in general, and then using formula (2.12) we get

$$5 \leq d(R,H) \leq 6 \tag{2.35}$$

Then the minimum even depth is

$$d_{ev}(R,H) = 6 \tag{2.36}$$

2.4 A Descending Chain of Annihilators of the Tensor Powers of Q

Definition 2.31. *Let R be a ring and $M \in \mathcal{M}_R$. Denote the annihilator ideal of M in R:*

$$Ann_R(M) = \{r \in R \quad | \quad m \cdot r = 0 \quad \forall m \in M\} \tag{2.37}$$

The following lemma is well known.

Lemma 2.32. *Let R be a ring and V, W two R-modules, suppose there is a monomorphism of R-modules, $\phi : V \hookrightarrow W$. Then $Ann_R(W) \subseteq Ann_R(V)$*

Proof. Let $\phi : V \longrightarrow W$ be a monic of R-modules, let $r \in Ann_R(W)$ then for every $v \in V$ we have $0 = \phi(v)r = \phi(vr)$ but since ϕ is monic we have that $vr = 0 \in V$ hence $r \in Ann_R(V)$. □

In [41, Lemma 6] it is shown that for a finite dimensional Hopf algebra H a bi-ideal I is always a Hopf ideal. That is two say that any two sided ideal such that is also a coideal satisfying $\Delta(I) \subseteq H \otimes I + I \otimes H$ can be proven to satisfy $S(I) = I$.

Recall that $Q = H/R^+H$ is a right H-module coalgebra and that $Q^{\otimes(n)} = Q \otimes \cdots \otimes Q$ n times, with $Q^{\otimes(0)} = k_\varepsilon$, the trivial R-module. Denote by$I_n = Ann_R(Q^{\otimes(n)})$ for $n \geq 1$ and $I_0 = R^+$.

Now as we proved in corollary (2.9) the H-module coalgebra structure of Q implies that $Q^{\otimes(n)}|Q^{\otimes(n+1)}$ for all $n \geq 1$. This means there exists a split monic $Q^{\otimes(n)} \hookrightarrow Q^{\otimes(n+1)}$. By lemma (2.32) we then have the following descending chain of ideals:

$$R^+ = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n := Ann_R(Q^{\otimes(n)}) \supseteq \cdots \tag{2.38}$$

Denote $l(R)$ the length of R as an R^e-module, where $R^e = R \otimes R^{op}$. Note that if t is the number of non-isomorphic simples of R then $l(R_R) \geq t$ with equality if and only if R is semisimple over a field k.

Proposition 2.33. *Let $R \subseteq H$ be a finite dimensional Hopf algebra pair. Then $I_0 = I_1$ if and only if $d(R, H) \leq 2$.*

Proof. ($\Leftarrow$) Suppose $d(R, H) \leq 2$, then by definition $H \sim H \otimes_R H$ both as R-H and H-R-bimodules, in particular it does so as R-R-bimodules. Then we have by inequality (2.12) that $d(Q, \mathcal{M}_R) = 0$ but this is equivalent to say that $Q | nk_\varepsilon$ for some natural number n. Then we know there is a split monic $Q \hookrightarrow nk_\varepsilon$ which by lemma (2.32) implies that $I_0 \subseteq I_1$. But equation (2.38) tells us that $I_1 \subseteq I_0$.

($\Rightarrow$) Assume $I_0 \subseteq I_1$ then we have that $R^+ \subseteq Ann_R(Q)$ so for all $h \in H$ and for all $r \in R^+$ we have that $\overline{h} \cdot r = hr + R^+H = 0$ this is equivalent to say that $hr \in R^+H$ which then implies $HR^+ \subseteq R^+H$. Hence R is left ad-stable, which implies left depth 2, and then by theorem (1.30) $d(R, H) \leq 2$. $\square$

Definition 2.34. *Denote*

$$I_Q = \bigcap_{i=1}^{\infty} I_i \tag{2.39}$$

Since we are in finite dimensional case we get that $I_Q = Ann_R(Q^{\otimes(n)})$ for some n.

The next theorem shows that $I_Q \subseteq I_1$ is the maximal Hopf ideal in $Ann_R(Q)$. This is true in general for any H-module and was first proposed by Rieffel.

Theorem 2.35. *Let $I \subseteq I_1$, be a Hopf ideal. Then $I \subseteq I_Q$. Moreover $I_Q = I_n$ for some $n \leq l(R)$*

Proof. Let $x \in I$. Then since I is an R-coideal, we have that $\Delta(x) \in I \otimes R + R \otimes I$ and hence $(Q \otimes Q) \cdot x = 0$ so we have $x \in I_2$. Now suppose $\Delta(x) = x_{(1)} \otimes r_{(1)} + r_{(2)} \otimes x_{(2)}$, then by coassociativity of the coproduct we have that $\Delta^2(x) = \Delta(x_{(1)}) \otimes r_{(1)} + \Delta(r_{(2)}) \otimes x_{(2)} = [x_{(1)} \otimes r_{(1)} + r_{(2)} \otimes x_{(2)}] \otimes r_{(3)} + r_{(4)} \otimes r_{(5)} \otimes x_{(3)} = x_{(1)} \otimes r_{(1)} \otimes r_{(3)} + r_{(2)} \otimes x_{(2)} \otimes r_{(3)} + r_{(4)} \otimes r_{(5)} \otimes x_{(3)}$ and then $\Delta^2(x) \in I \otimes R \otimes R + R \otimes I \otimes R + R \otimes R \otimes I$. Following this thought and always taking into account the coassociativity of the coproduct one comes to

$$\Delta^n(I) \subseteq \sum_{i=0}^{n} R^{\otimes(i)} \otimes I \otimes R^{\otimes(n-i)} \tag{2.40}$$

Then is immediate that for every n and any $x \in I$ one has that $Q^{\otimes(n)} \cdot x = 0$ so $x \in I_n$ for every n. So we conclude that I_Q is a maximal ideal contained in I_1.

We now want to prove that I_Q is a Hopf ideal. Since I_Q is an ideal it suffices to prove that I_Q is a coideal in R. First note that the Jordan-Hölder theorem implies that there exists an $n \leq l(R)$ such that $I_n = I_{n+1}$ so that

$$I_Q = \bigcap_{i=1}^{n} I_i = I_n. \tag{2.41}$$

Now we prove that whenever $I_n = I_{n+1}$ then for every $r > 1$ we have $I_n = I_{n+r}$.

Let $x \in I_n = I_{n+1}$ then $Q^{\otimes(n+1)} \cdot x = 0$. This implies that $\Delta(x) \subseteq I_n \otimes R + R \otimes I_1$ since $Q^{\otimes(n+1)} = Q^{\otimes(n)} \otimes Q$. By the coassociativity of the coproduct we have then that $(\Delta \otimes R) \circ \Delta(x) \in (\Delta \otimes R)(I_n \otimes R + R \otimes I_1) \subseteq [I_n \otimes R + R \otimes I_1] \otimes R + R \otimes R \otimes I_1 \subseteq I_n \otimes R \otimes R + R \otimes I_1 \otimes R + R \otimes R \otimes I_1$. Clearly $(\Delta \otimes R) \circ \Delta(x)$ annihilates $Q^{\otimes(n+2)} = Q^{\otimes(n)} \otimes Q \otimes Q$. Now suppose $x \in I_n = I_{n+r}$, then by previous step and by the definition of $\Delta^r(x)$ we have that $(\Delta^r \otimes R) \circ \Delta(x) \in I_{n+r+1}$.

Finally suppose $I_n = I_{n+1}$ for some n as we just proved this implies in particular that $I_n = I_{2n}$ so by definition $0 = Q^{\otimes(2n)} \cdot x = (Q^{\otimes(n)} \otimes Q^{\otimes(n)}) \cdot x$ which of course means that $\Delta(x) \in I_n \otimes R + R \otimes I_n$. Then we conclude that $I_Q = I_n$ is a coideal and hence a Hopf ideal. $\square$

Definition 2.36. *Denote l_Q the smallest integer n for which $I_n = I_{n+1}$, i.e. the length of the descending chain of annihilator ideals I_k.*

Recall also that for a ring R, an R-module W is said to be faithful if $I_W = Ann_R(W) = 0$.

Definition 2.37. *[22] We say that an H-module W is conditionally faithful if $I_{W^{\otimes(n)}} = 0$, i.e. if $W^{\otimes(n)}$ is faithful for some $n > 1$.*

It is also important to mention that if W is conditionally faithful then I_W contains no non-zero Hopf ideals. Consider the finite dimensional Hopf algebra extension $R \subseteq H$ and its quotient module $Q = H/R^+H$. If $R = H$, then Q is not conditionally faithful. On the other hand if $R = k_\varepsilon$ then Q is conditionally faithful.

Definition 2.38. *Let R be a ring and W an R-module. We say W is a (R) generator if there is a natural number $n \geq 1$ and a monomorphism of R-modules ϕ such that:*

$$\phi : R_R \hookrightarrow n \cdot W \tag{2.42}$$

The following lemma is well known.

Lemma 2.39. *Let R be a ring, an R-module W is faithful if and only if it is a generator.*

Proof. ($\Leftarrow$) Suppose W is an R generator. Then there is $n \geq 1$ and a monomorphism of R modules $\phi : R_R \hookrightarrow n \cdot W$. By lemma (2.32) then we have that $Ann_R(W) \subseteq Ann_R(R_R) = 0$.

($\Rightarrow$) Suppose W is a faithful R-module, let $\{w_1, \cdots, w_n\}$ be a k-basis for W, consider $\psi : R \longrightarrow nW; r \longmapsto (w_1 r, \cdots, w_n r)$. Then since W is faithful ψ is obviously an R-monomorphism. Then W is a generator. $\square$

If in addition, R is a Frobenius algebra, then R_R is an injective module, then the monomorphism $R_R \hookrightarrow n \cdot W_R$ is a split mono in $\mathcal{M}_R$ and then we have that $R_R | nW$. In particular we get the following.

Theorem 2.40. *If H is a finite dimensional Hopf algebra and W_H is a faithful module then every indecomposable projective H-module P satisfies $P|W$.*

Proof. By previous Lemma W_H is a generator and by previous remark we have $H | n \cdot W$ for some n. Since $P|H$ it follows that $P|W$. $\square$

Example 2.41. *Let H be a finite dimensional Hopf algebra of dimension ≥ 2, H_H is faithful projective H-module. This says that for every $n \geq 1$ $H^{\otimes(n)}$ is also faithful projective, in particular theorem* (2.40) *above, implies that $H^{\otimes(2)} \sim H$, as H-modules and that says $d(H, \mathcal{M}_H) = 1$. In the same way every faithful projective H-module W_H satisfies $d(W, \mathcal{M}_H) = 1$.*

In the context of conditionally faithful H-modules this means the following:

Proposition 2.42. *Let H be a finite dimensional Hopf algebra and W_H a conditionally faithful projective H-module. Then:*

$$d(W, \mathcal{M}_H) = l_W \tag{2.43}$$

Proof. Since $W^{\otimes(l_W)}$ is a projective faithful H-module then for all $r \geq 1$ $W^{\otimes(l_W+r)}$ has all projective indecompossables contained in $W^{\otimes(l_W)}$ as summands. Then $W^{\otimes(l_W)} \sim W^{\otimes(l_W+r)}$. Therefore $d(W, \mathcal{M}_H) \leq l_W$. Suppose now that there is $m \leq l_W$ such that $d(W, \mathcal{M}_H) = m$. This implies $W^{\otimes(m)} \sim W^{\otimes(l_W)}$, then $W^{\otimes(m)}$ is faithful. Since by definition (2.36) l_W is minimum we get $m = l_W$. □

Theorem 2.43. *Let $R \subseteq H$ be a finite dimensional Hopf algebra pair. Suppose Q is a conditionally faithful projective H-module. Then*

$$d_h(R, H) = 2l_Q + 1$$

Moreover, if Q is conditionally faithful and R-projective, then

$$2l_{Q_R} + 1 \leq d(R, H) \leq 2l_{Q_R} + 2$$

Proof. The first equation is the result of combining proposition (2.42) and theorem (2.15). The second equation is proposition (2.42) and equation (2.12). □

We have seen through this section that for a finite dimensional Hopf algebra pair $R \subseteq H$ the chain of annihilators

$$R^+ = I_0 \supseteq I_1 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

stabilises for some n where $I_Q = I_n$. Furthermore we proved that if Q is conditionally faithful projective then $d_h(R, H) = 2l_Q + 1$.

Is then desirable to know whether the conditions of conditional faithfulness and or of projectivity of Q can be relaxed so that we can get a more general result on depth; in the case of the following theorem we see that given semisimplicity of a ring R , the condition of stabilisation of the chain of annihilators is enough for finite depth.

Theorem 2.44. *Let R be a semisimple ring. Let W_R and V_R be two R-modules such that $Ann_R(W) = Ann_R(V)$. Then $V \sim W$.*

Proof. Let R be a semisimple ring. The Wedderburn-Artin theorem says that
$$R \cong R_1 \times R_2 \times \cdots \times R_t$$
where R_i is the matrix algebra $M_{n_i}(D_i)$ and D_i is a division algebra. Moreover by semisimplicity we can express as R-modules

$$R \cong n_1 S_1 \oplus \cdots \oplus n_t S_t$$

where the S_i are the simple R_i-modules. As a consequence of the same structure theorem we also have that both W and V are semisimple R-modules. We express them in terms of multiples of the simple components of R as follows,
$$W = k_1 S_1 \oplus \cdots \oplus k_t S_t$$
$$V = m_1 S_1 \oplus \cdots \oplus m_t S_t$$
here k_i, m_i are all natural numbers, not all of them zero.

If $Ann_R(W) = Ann_R(V) = 0$ then necessarily all k_i and m_i are strictly positive. Then the result follows.

Let $Ann_R(W) = Ann_R(V) \neq 0$ and denote this 2-sided ideal by I. Suppose by contradiction that W and V are not similar. By definition of similarity this means that the sets of indecomposable constituents $Indec(W)$ and $Indc(V)$ must differ by at least one element. Without loss of generality assume that there is one i such that $k_i = 0$ and $m_i \neq 0$ in the decomposition of W and V. Then, clearly $Ann_R(W) \supseteq Ann_R(V)$, but since the S_i are non isomorphic simples and annihilator ideals are maximal then $Ann_R(W) \nsubseteq Ann_R(V)$. Hence the result follows. □

Corollary 2.45. *Let $R \subseteq H$ be a finite dimensional Hopf algebra pair such that R is semisimple. Then $d_h(R, H) = 2l_Q + 1$.*

Proof. By definition of l_Q one has that $Ann_R(Q^{\otimes(l_Q)}) = Ann_R(Q^{\otimes(l_Q+1)})$. Since R is semisimple theorem (2.44) tells us that $Q^{\otimes(l_Q)} \sim Q^{\otimes(l_Q+1)}$. By definition this implies $d_h(R, H) \leq 2l_Q + 1$. Since l_Q is the least integer to satisfy this property one gets the result. □

Chapter 3

Coring Depth

3.1 Corings

A coring is a generalisation of a coalgebra introduced by M. Sweedler in 1975 [47]. As pointed out by Takeuchi [28] one can construct examples of corings given an entwined structure which, together with their modules generalise the notion of Doi-Koppinen modules introduced in [16] and in [35].

Definition 3.1. *[8] Let A be a ring. An A-coring is an A-A-bimodule C, with A-A-bimodule maps $\Delta_C : C \longrightarrow C \otimes_A C$ (called coproduct) and $\varepsilon : C \longrightarrow A$ (called the counit), such that the following diagrams commute:*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_A C \arrow[d, "\Delta \otimes C"] \\ C \otimes_A C \arrow[r, "\Delta \otimes C"'] & C \otimes_A C \otimes_A C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_A C \arrow[d, "\varepsilon \otimes C"] \\ C \otimes_A C \arrow[r, "C \otimes \varepsilon"'] & A \otimes_A C \arrow[ul, "\cong"] \end{tikzcd} \tag{3.1}$$

Together they imply that for all $c \in C$, in usual Sweedler notation:

$$c_{(11)} \otimes c_{(12)} \otimes c_{(2)} = c_{(1)} \otimes c_{(21)} \otimes c_{(22)} = c_{(1)} \otimes c_{(2)} \otimes c_{(3)} \tag{3.2}$$

$$\varepsilon(c_{(1)})c_{(2)} = c_{(1)}\varepsilon(c_{(2)}) = c \tag{3.3}$$

where the summation over c is understood.

Definition 3.2. *A right C-comodule $M \in \mathcal{M}^C$ is a right A-module together with a right A-module map $\rho^M : M \longrightarrow M \otimes_A C$ such that the following diagrams commute:*

$$\begin{tikzcd} M \arrow[r, "\rho^M"] \arrow[d, "\rho^M"'] & M \otimes_A C \arrow[d, "\rho^M \otimes C"] \\ M \otimes_A C \arrow[r, "M\otimes\Delta_C"'] & M \otimes_A C \otimes_A C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\rho^M"] \arrow[dr, "\cong"'] & M \otimes_A C \arrow[d, "M\otimes\varepsilon"] \\ & M \otimes_A A \end{tikzcd} \tag{3.4}$$

Let M, N be right C-comodules, we say $f : M \longrightarrow N$ is a C-comodule map if it is a right A-module map and

$$\rho^N \circ f = (f \otimes C) \circ \rho^M. \tag{3.5}$$

i.e.

$$\rho^N(f(m)) = f(m_{(0)}) \otimes m_{(1)}$$

The category of right C-comodules is denoted by M^C and their morphisms as $Hom^C(-,-)$.

Example 3.3. *Let $B \subseteq A$ a ring extension. The natural A-A-bimodule $C = A \otimes_B A$ is an A-coring, called the Sweedler coring, where $\Delta_C : C \longrightarrow C \otimes_A C \cong A \otimes_B A \otimes_B A$ is defined by $\Delta_C(a \otimes b) = a \otimes 1 \otimes b$ and $\varepsilon_C : C \longrightarrow A$ by $\varepsilon_C(a \otimes b) = ab$.*

Example 3.4. *Let $B \subseteq A$ a Frobenius extension with $F : A \longrightarrow B$ a Frobenius homomorphism and $e = \sum_i x_i \otimes_B y_i$ a dual basis tensor satisfying $ae = ea$ for all $a \in A$. A is a coring over its subalgebra B with coproduct $\Delta : A \longrightarrow A \otimes_B A$ defined by $\Delta(a) = ea$ and $\varepsilon : A \longrightarrow B$ such that $\varepsilon = F$.*

3.2 Entwining Structures

Interesting examples of corings arise from entwining structures [11].

Definition 3.5. *[7, 2.1] An entwined structure over (a field) k is a triplet $(A, C)_\psi$ consisting of a k-algebra A and a k-coalgebra C together with a k-module map $\psi : C \otimes A \longrightarrow A \otimes C$ satisfying two commutative pentagons and two commutative triangles given by the following equations.*

$$\psi \circ (C \otimes \mu) = (\mu \otimes C) \circ (A \otimes \psi) \circ (\psi \otimes A) \tag{3.6}$$

where μ is the multiplication in A.

$$(A \otimes \Delta_C) \circ \psi = (\psi \otimes C) \circ (C \otimes \psi) \circ (\Delta_C \otimes A) \tag{3.7}$$

$$\psi \circ (C \otimes u_A) = u_A \otimes C \tag{3.8}$$

$$(A \otimes \varepsilon_C) \circ \psi = \varepsilon_C \otimes A, \tag{3.9}$$

where u_A is the unit in A.

A morphism of entwining structures is a pair $(f,g) : (A,C)_\phi \longrightarrow (\widehat{A},\widehat{C})_{\widehat{\phi}}$ where $f : A \longrightarrow \widehat{A}$ is an algebra map and $g : C \longrightarrow \widehat{C}$ is a coalgebra map, together with

$$(f \otimes g) \circ \phi = \widehat{\phi} \circ (g \otimes f) \tag{3.10}$$

We write $\phi(c \otimes a) = a_\alpha \otimes c^\alpha$ and the summation over the index is understood.

Example 3.6. *Let $A = H$ be a finite dimensional Hopf algebra. Let C be a right H-module coalgebra. The entwining map is given by*

$$\psi : C \otimes H \longrightarrow H \otimes C, \quad \psi(c \otimes h) = h_{(1)} \otimes ch_{(2)}. \tag{3.11}$$

Proof. That ψ is a k-module map is straightforward. We just check equations (3.6) and (3.8) are satisfied since the other two equations are similar.

Let $c \otimes h \otimes g \in C \otimes H \otimes H$ now, $\psi \circ (id_C \otimes \mu)(c \otimes h \otimes g) = \psi(c \otimes hg) = (hg)_{(1)} \otimes c(hg)_{(2)} = h_{(1)}g_{(1)} \otimes ch_{(2)}g_{(2)}$ since Δ is an algebra map.
On the other hand $(\mu \otimes id_C) \circ (id_A \otimes \psi) \circ (\psi \otimes id_A)(c \otimes h \otimes g) = (\mu \otimes id_C) \circ (id_A \otimes \psi)(h_{(1)} \otimes ch_{(2)} \otimes g) = (\mu \otimes id_C)(h_{(1)} \otimes g_{(1)} \otimes ch_{(2)}g_{(2)}) = h_{(1)}g_{(1)} \otimes ch_{(2)}g_{(2)}$ hence we have equation (3.6).
Now consider $c \otimes 1 \cong 1 \otimes c \in C \otimes k \cong k \otimes C$, then $\psi \circ (id_C \otimes u_H)(c \otimes 1) = \psi(c \otimes 1_H) = 1 \otimes c$. More over $(u_H \otimes C)(1 \otimes c) = 1 \otimes c$. So we have equation (3.8) as well. □

Another interesting structure related to entwining structures is that of an entwining module.

Definition 3.7. *Given $(A,C)_\psi$ an $(A,C)_\psi$- module M is a right A-module right C-comodule for which ρ_M is the module action and ρ^M the comodule coaction, satisfying*

$$\begin{tikzcd} M \otimes A \arrow[r, "\rho^M \otimes A"] \arrow[d, "\rho^M \circ \rho_M"'] & M \otimes C \otimes A \arrow[d, "M \otimes \psi"] \\ M \otimes C & M \otimes A \otimes C \arrow[l, "\rho_M \otimes C"] \end{tikzcd} \tag{3.12}$$

Equivalently, for every $m \otimes a \in M \otimes A$ we have

$$\rho^M(ma) = m_0 a_\alpha \otimes m_1^\alpha. \tag{3.13}$$

As expected, is easy to compute that $A \otimes C$ is an example of an $(A, C)_\psi$-entwined module with right C coaction given by $A \otimes \Delta_C$ and a right A-module action

$$(a' \otimes c)a = a'\psi(c \otimes a) = a' a_\alpha \otimes c^\alpha. \tag{3.14}$$

Example 3.8. *Following the previous example one sets $A = H$ a Hopf algebra and C an H-module coalgebra. Then $H \otimes C$ is an entwined module with C coaction given by $H \otimes \Delta_C$ and a right H-module action*

$$(h \otimes c) \cdot g = h\psi(c \otimes g) = hg_{(1)} \otimes cg_{(2)}, \tag{3.15}$$

the diagonal action of H on the right.

For an entwined structure $(A, C)_\psi$ we can view $A \otimes C$ as an A-A-bimodule with left action being left multiplication in A: $a(b \otimes c) = ab \otimes c$ and the right action as defined before, $(a \otimes c) \cdot b = ab_\alpha \otimes c^\alpha$ for all a , b in A and c in C. Then $\mathcal{C} = A \otimes C$ is an A-coring with coproduct $\Delta_{\mathcal{C}} = A \otimes \Delta_C$ and counit $\varepsilon_{\mathcal{C}} = A \otimes \varepsilon_C$.

Conversely if $A \otimes C$ is an A-coring then we can define an entwined structure $(A, C)_\psi$ with entwining map given by $\psi : C \otimes A \longrightarrow A \otimes C$ such that $\psi(c \otimes a) = (1 \otimes c) \cdot a$. In this way one can prove the following:

Proposition 3.9. *[9, 2.8.1] Entwining structures $(A, C)_\psi$ and A-corings $\mathcal{C} = A \otimes C$ are in one-to-one correspondence.*

3.3 Galois Corings

Let C be a coalgebra. A ring extension $B \subset A$ is called a *C-Galois extension* if A is a right C-comodule via $\rho^A : A \longrightarrow A \otimes C$, where

$$B := A^{coC} = \{b \in A | \rho^A(ba) = b\rho^A(a) \forall a \in A\}$$

are the C-coinvariants in A and there is a canonical bijective left A-module map, right C-comodule map $can : A \otimes_B A \longrightarrow A \otimes C$ such that

$$can(a \otimes_B a') = a\rho^A(a') = aa'_{(0)} \otimes a'_{(1)} \tag{3.16}$$

It is important to notice that ρ^A is a left B-module map and that *can* is a coring homomorphism with respect to the Sweedler coring, since:

$$\varepsilon_{A\otimes C} \circ can(a \otimes_B a') = \varepsilon_{A\otimes C}(aa'_{(0)} \otimes a'_{(1)}) =$$

$$aa'_{(0)} \otimes \varepsilon_C(a'_{(1)}) = aa' = \varepsilon_{A\otimes_B A}(a \otimes_B a') \tag{3.17}$$

Moreover:

$$\Delta_{A\otimes C} \circ can(a \otimes_B a') = \Delta_{A\otimes C}(aa'_{(0)} \otimes a'_{(1)}) = aa'_{(0)} \otimes a'_{(1)} \otimes a'_{(2)} \tag{3.18}$$

On the other hand we have:

$$can \otimes can \circ \Delta_{A\otimes_B A}(a \otimes_B a') = can \otimes can(a \otimes_B 1 \otimes_B a') = can \otimes_A can(a \otimes_B 1 \otimes_A 1 \otimes_B a')$$

$$= a \otimes 1 \otimes_A a'_{(0)} \otimes a'_{(1)} = aa'_{(0)} \otimes a'_{(1)} \otimes a'_{(2)} \tag{3.19}$$

since $A \otimes C$ is a right A-module via ρ^A.

Now we consider $b \in B$. Notice that since $can : A \otimes_B A \longrightarrow A \otimes C$ is a coring homomorphism and $1 \otimes_B 1$ is a grouplike element in $A \otimes_B A$ one has that $can(1 \otimes_B 1)$ is a grouplike element in $A \otimes C$. Also $can(b \cdot (1 \otimes_B 1)) = b \cdot can(1 \otimes_B 1) = b \cdot (1 \otimes 1_C)$ and then $can(b \cdot (1 \otimes_B 1)) = can(1 \otimes_B b) = b_{(0)} \otimes b_{(1)} = (1 \otimes 1_C) \cdot b$. So that $b \cdot (1 \otimes 1_C) = (1 \otimes 1_C) \cdot b$ since *can* is a bijection.

Lemma 3.10. *[8, Lemma 5.1] Let $\mathcal{C}$ be an A coring. Then A is a right $\mathcal{C}$-comodule if and only if there exists a grouplike element $g \in \mathcal{C}$.*

For example if $\mathcal{C}$ is the Sweedler coring $A \otimes_B A$ associated to a ring extension $B \subseteq A$, then one verifies that $1 \otimes_B 1$ is a grouplike in $\mathcal{C}$ and hence $A \in \mathcal{M}^C$.

Definition 3.11. *Let C be an A coring , $g \in C$ a grouplike element in C, $B = A^g = \{b \in A \mid bg = gb\}$. Then the coring C is Galois if the coring homomorphism*

$$\overline{can} : A \otimes_B A \longrightarrow C, \quad \overline{can}(a \otimes a') = aga' \tag{3.20}$$

is bijective.

We point out that the last equation implies $\overline{can}(1 \otimes 1) = g$.

Proposition 3.12. *[31, Lemma 3.9] Let $B \subseteq A$ be an extension of rings. Then the Sweedler coring $A \otimes_B A$ is a Galois coring.*

Proposition 3.13. *Let $R \subseteq H$ be a left coideal subalgebra of a finite dimensional Hopf Algebra; that is, $\Delta(R) \subseteq H \otimes R$. Let $R^+ = ker\varepsilon|_R$ the kernel of the counit restricted to R. Then we have that R^+H is a right H-submodule of H and a left H-coideal. This yields then that $Q = H/R^+H$ is a right H-module coalgebra.*

We of course have an epimorphism of right H-module coalgebras $p : H \longrightarrow Q$; $h \longmapsto \overline{h} = h + R^+H$. The H coring $H \otimes Q$ has a grouplike element $1 \otimes \overline{1}$ and it is a Galois coring since $H \otimes_R H \cong H \otimes Q$ via

$$x \otimes_R y \longmapsto xy_{(1)} \otimes \overline{y_{(2)}} = x \cdot \rho^H(1) \cdot y \tag{3.21}$$

and with inverse given by

$$xS(z_{(1)}) \otimes z_{(2)} \longleftarrow x \otimes z \tag{3.22}$$

3.4 Depth of Galois Corings

Definition 3.14. *Let C be an A-coring, we say that C has depth n if $C^{\otimes_A (n+1)} \sim C^{\otimes_A (n)}$ as A-A-bimodules and we denote the minimum depth by $d(C,_A \mathcal{M}_A)$.*

Example 3.15. *As we saw in example (3.3) for a ring extension $B \subseteq A$ we define the Sweedler coring as $C = A \otimes_B A$. It is straightforward that in $\mathcal{M}_A$ the n-fold tensor power $C^{\otimes_A (n)} \cong A^{\otimes_B (n+1)}$ from cancelations of the type $M \otimes_A A \cong M$. So we come to the equality*

$$d_h(B, A) = 2d(C,_A \mathcal{M}_A) + 1 \tag{3.23}$$

Proof. Let $d(C,_A \mathcal{M}_A) = n$ this means by definition that $C^{\otimes_A (n+1)} \sim C^{\otimes_A (n)}$ which as we just noted above in turn is equivalent to saying $A^{\otimes_B (n+2)} \sim A^{\otimes_B (n+1)}$. By definition we get $d_h(B, A) = 2n + 1$. ☐

Another easy consequence of the definitions in the above subsections is the following.

Example 3.16. *Let $B \subseteq A$ be a Frobenius extension. By example* (3.4), *A is a B-coring and therefore*

$$2d(A,_B \mathcal{M}_B) + 1 = d(B, A) \tag{3.24}$$

Proposition 3.17. *For the H-coring $H \otimes C$ defined in example* (3.6)

$$d(H \otimes C,_H \mathcal{M}_H) = d(C, \mathcal{M}_H). \tag{3.25}$$

Proof. First note that in $\mathcal{M}_H$

$$(H \otimes C)^{\otimes_H n} \cong H \otimes C^{\otimes n} \tag{3.26}$$

This comes from cancelations of the type $H \otimes C \otimes_H H \cong H \otimes C$ where the map is defined as

$$\bigotimes_i (h_i \otimes c_i) \longmapsto$$

$$h_1 h_{2(1)} \ldots h_{n(1)} \otimes c_1 h_{2(2)} \ldots h_{n(2)} \otimes \ldots c_{n-1} h_{n(n-1)} \otimes c_n \tag{3.27}$$

and inverse the map is

$$h \otimes c_1 \otimes \ldots \otimes c_n \longmapsto h \otimes c_1 \otimes_H 1_H \otimes c_2 \otimes_H \ldots \otimes_H 1_H \otimes c_n \tag{3.28}$$

Now consider $d(H \otimes C, \mathcal{M}_H) = n$. This means $(H \otimes C)^{\otimes_H (n+1)} \sim (H \otimes C)^{\otimes_H (n)}$ as H-H-bimodules. By using the isomorphism above we get $H \otimes C^{\otimes(n+1)} \sim H \otimes C^{\otimes(n)}$. Then applying the additive functor $(k \otimes_H -)$ on the left on both sides of the relation we get $C^{\otimes(n+1)} \sim C^{\otimes(n)}$ as right H-modules, which tells us that $d(C, \mathcal{M}_H) \leq n$.

For the other inequality and starting from $d(C, \mathcal{M}_H) = n$ we apply the additive functor $(H \otimes -)$ to the relation $C^{\otimes(n+1)} \sim C^{\otimes(n)}$ and reverse the process using the inverse isomorphism to get $d(H \otimes C,_H \mathcal{M}_H) \leq n$. Thus obtaining the result. □

Corollary 3.18. *Let H be a finite dimensional Hopf algebra, $R \subseteq H$ a left coideal subalgebra and $Q = H/R^+H$. Then*

$$d_h(R, H) = 2d(Q, \mathcal{M}_H) + 1. \tag{3.29}$$

Proof. First we recall from proposition (3.13) that $H \otimes Q$ is a Galois coring, then by proposition (3.17) $d(H \otimes_R H,_H \mathcal{M}_H) = d(H \otimes Q,_H \mathcal{M}_H)$. The result follows from equation (3.23) and proposition (3.17). Then $d_h(R, H) = 2n + 1$. □

Example 3.19. *[24, 5.1] Let $R \subseteq H$ be a finite dimensional Hopf subalgebra pair such that $d(R,H) \leq 2$ then $H \otimes_B T$ is Galois coring, where $T = End_H(H \otimes_R H)_H$ is a right bialgebroid over B with H a right T-comodule algebra and $B = H^R$ the centraliser of R in H.*

3.5 Crossed Products

For this section we will consider the crossed product $D\#_\sigma H$ of a twisted H-module algebra with a finite dimensional Hopf Algebra H.

Definition 3.20. *Let H be a finite dimensional Hopf algebra and let D be a twisted H-module algebra. Consider a twisted two-cocycle $\sigma : H \otimes H \longrightarrow D$ that is convolution invertible, and such that for all $h, k, m \in H$ and all $a \in D$ the following are satisfied:*

$$h \cdot (k \cdot a) = \sum \sigma(h_{(1)}, k_{(1)})(h_{(2)}k_{(2)} \cdot a)\sigma^{-1}(h_{(3)}, k_{(3)}) \tag{3.30}$$

$$\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)1_H \tag{3.31}$$

$$\sum [h_{(1)} \cdot \sigma(k_{(1)}, m_{(1)})]\sigma(h_{(2)}, k_{(2)}m_{(2)}) = \sum \sigma(h_{(1)}, k_{(1)})\sigma(h_{(2)}k_{(2)}, m) \tag{3.32}$$

Then $D\#_\sigma H$ is an associative algebra over the vector space $D \otimes H$ and with multiplication

$$(d\#h)(v\#k) = \sum d(h_{(1)} \cdot v)\sigma(h_{(2)}, k_{(1)})\#h_{(3)}k_{(2)} \tag{3.33}$$

With identity $1\#1$.

Throughout the section we omit the summation sign for brevity unless it leads to confusion.

Example 3.21. *Let $H = kG$ the group algebra of a finite group G over (a field) k and $A = D\#_\sigma kG$ the group crossed product, then the definition simplifies to*

$$g \cdot (h \cdot d) = \sigma(g, h)(gh \cdot d)\sigma^{-1}(g, h) \tag{3.34}$$

$$(g \cdot \sigma(h, s))\sigma(g, hs) = \sigma(g, h)\sigma(gh, s) \tag{3.35}$$

equation (3.35) *is the* 2*-cocycle condition. The product then becomes*

$$(d\#g)(v\#h) = d(g \cdot v)\sigma(g,h)\#gh \tag{3.36}$$

for all $g, h, s \in kG$ *and* $d, v \in D$.

Example 3.22. *Let* $\sigma = 1_D$ *in the previous example, then* $D\#_\sigma kG$ *becomes* $D \star kG$ *the skew group algebra with product*

$$(d\#g)(v\#h) = d(g \cdot v)\#gh$$

moreover if we set $\sigma(x,y) = \varepsilon(x)\varepsilon(y)1_H$ *then* $D\#_\sigma H = D\#H$ *the regular smash product of* D *and* H.

Example 3.23. *Suppose the action of* G *over* D *is trivial, that is* $g \cdot d = d$ *for all* $g \in G$ *and all* $d \in D$. *Then the crossed product becomes the twisted group algebra* $D\#_\sigma kG = D_\sigma[G]$ *with product*

$$(d\#g)(v\#h) = dv\sigma(g,h)\#gh$$

As a concrete example of a twisted group algebra let $D = \mathbb{R}$ and $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\sigma : G \otimes G \longrightarrow \mathbb{R}$ given by

σ	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$(0,0)$	1	1	1	1
$(0,1)$	1	-1	1	-1
$(1,0)$	1	-1	-1	1
$(1,1)$	1	1	-1	-1

Then $\mathbb{R}_\sigma[G] \cong \mathbb{H}$, the quaternions: $\mathbb{H} = \langle 1, i, j, k\rangle_{\mathbb{C}}$ with relations $i^2 = j^2 = k^2 = -1$, $ij = k, ji = -k$, $jk = i, kj = -i$, $ik = -j, ki = j$.

Example 3.24. *Let* $N \triangleleft G$ *be a normal finite group extension, let* $Q = G/N$ *and for every* $q \in Q$ *let* $\gamma(q)$ *be a coset representative of* q, $\gamma(1) = 1_G$. *Define* $\sigma(q,v) = \gamma(q)\gamma(v)\gamma(qv)^{-1}$ *and the action* $q \cdot n = \gamma(q)n\gamma(q)^{-1}$. *Then*

$$kG = kN\#_\sigma kQ \tag{3.37}$$

Proof. Since $G = \bigcup_{q\in Q} N\gamma(q)$, $kG = (kN)\gamma(Q)$ and we may multiply in kG in the following way:

$$(n\gamma(q))(m\gamma(v)) = n[\gamma(q)m\gamma(q^{-1})][\gamma(q)\gamma(v)\gamma(qv)^{-1}]\gamma(qv) =$$

$$= n(q \cdot m)\sigma(q,v)\gamma(qv) \tag{3.38}$$

one easily verifies that all the equations in the definition (3.20) are satisfied and therefore $kG = kN\#_\sigma kQ$. $\square$

We shall now return to our primary setting. Let $R \subseteq H$ be a finite dimensional Hopf algebra extension over a field k. Let $Q = H/R^+H$. Let D be a twisted H-module algebra together with a two-cocycle $\sigma : H \otimes H \longrightarrow D$ and consider

$$B := D\#_\sigma R \subseteq D\#_\sigma H =: A \tag{3.39}$$

a crossed product algebra extension. Consider now the canonical right H-module coalgebra epimorphism $\pi : H \longrightarrow Q$, $h \longmapsto h + R^+H = \overline{h}$. Since A is naturally an H-comodule via $\rho = D \otimes \Delta$, we can induce a Q-comodule structure via

$$A \xrightarrow{D\otimes\Delta} A \otimes H \xrightarrow{A\otimes\pi} A \otimes Q \tag{3.40}$$

the coaction then becomes

$$\rho(d\#h) = d\#h_{(1)} \otimes \overline{h}_{(2)} \tag{3.41}$$

Using the entwining

$$\psi : Q \otimes A \longrightarrow A \otimes Q, \quad \overline{h} \otimes d\#g \longmapsto d\#g_{(1)} \otimes \overline{hg_{(2)}} \tag{3.42}$$

and setting the right and left A action as

$$b(a \otimes \overline{h})c = bac_{(0)} \otimes \overline{hc_{(1)}} \tag{3.43}$$

for $a, b, c \in A$ and $h \in H$, we endow then $A \otimes Q$ with an A coring structure.

Moreover let $r \in R$, then $\rho(d\#r)d\#r_{(1)} \otimes \overline{r}_{(2)} = d\#r \otimes \overline{1}$ since for all $r \in R$ one has $\overline{r} = \varepsilon(r)\overline{1}$. Then we have that $B \subseteq A^{coQ}$.

Theorem 3.25. *Let $R \subseteq H$, Q, D, A and B as described above. Then the A coring $A \otimes Q$ is a Galois coring.*

To prove this we need the following lemma which appears as a remark in [43].

Lemma 3.26. *Let $B \subseteq A$ a faithfully flat extension of rings such that A is an H-comodule algebra and $B \subseteq A^{coQ}$. Then if*

$$can : A \otimes_B A \longrightarrow A \otimes Q; \quad a \otimes a' \longmapsto aa'_{(0)} \otimes \overline{a}_{(1)}$$

is a bijection we have that $B = A^{coQ}$

Proof. Just consider the following commutative diagram:

$$\begin{array}{ccc} B \subseteq A & \longrightarrow & A \otimes_B A \\ {\scriptstyle\subseteq}\big\downarrow & & \big\downarrow{\scriptstyle can} \\ A^{coQ} \subseteq A & \underset{\rho^A}{\longrightarrow} & A \otimes Q \end{array} \tag{3.44}$$

Where the upper right arrow represents either $A \otimes 1_A$ or $1_A \otimes A$, and the bottom is the defining exact sequence of A^{coQ}, since *can* is a bijection and A_B and ${}_BA$ are faithfully flat modules we have that the upper sequence is also exact, then $B = A^{coQ}$ as desired. □

Proof. [**Theorem 3.25**] We first notice that A is a unitary algebra with unit $1_D \# 1_H$ and that $\rho(1_D \# 1_H) = 1_A \otimes \overline{1} := g$ is a unit element in $A \otimes Q$. Consider the following map:

$$\begin{aligned} \beta : A \otimes_B A &\longrightarrow A \otimes Q \\ a \otimes a' &\longmapsto aga' = a(d\#h_{(1)}) \otimes \overline{h}_{(2)} \end{aligned} \tag{3.45}$$

where $a' = d\#h$

Now to see that β is well defined consider $a, b, c \in D$, $r \in R$ and $h, g \in H$, then:

$$\beta[d\#h \otimes_B (b\#r)(c\#g)] = \beta[d\#h \otimes_B b(r_{(1)} \cdot c)\sigma(r_{(2)}, g_{(1)})\#r_{(3)}g_{(2)}] =$$

$$\begin{aligned} &= (d\#h)[b(r_{(1)} \cdot c)\sigma(r_{(2)}, g_{(1)})\#r_{(3)}g_{(2)}] \otimes \overline{r_{(4)}g_{(3)}} = \\ &(d\#h)[b(r_{(1)} \cdot c)\sigma(r_{(2)}, g_{(1)})\#r_{(3)}g_{(2)}] \otimes \overline{g}_{(3)} \end{aligned} \tag{3.46}$$

since $\overline{rg} = \varepsilon(r)\overline{g}$.

On the other hand we have:

$$\begin{aligned} \beta[(d\#h)(b\#r) \otimes_B c\#g] &= (d\#h)(b\#r)(c\#g_{(1)}) \otimes \overline{g}_{(2)} = \\ &(d\#h)[b(r_{(1)} \cdot c)\sigma(r_{(2)}, g_{(1)})\#r_{(3)}g_{(2)}] \otimes \overline{g}_{(3)} \end{aligned} \tag{3.47}$$

as desired.

Next we consider

$$\begin{aligned} \beta^{-1} : A \otimes Q &\longrightarrow A \otimes_B A \\ a \otimes \overline{h} &\longmapsto a[\sigma^{-1}(S(h_{(2)}), h_{(3)})\#S(h_{(1)})] \otimes_B 1_D\#h_{(4)} \end{aligned} \tag{3.48}$$

Again we call attention to [40, 7.27] and notice that

$$\gamma : H \longrightarrow A; \quad h \longmapsto 1_D \# h \tag{3.49}$$

has a convolution inverse given by

$$\mu : H \longrightarrow A; \quad h \longmapsto \sigma^{-1}(S(h_{(2)}), h_{(3)}) \# S(h_{(3)})$$

then we can set to calculate

$$\beta \circ \beta^{-1}(a \otimes \overline{h}) = \beta\{[a\sigma^{-1}(S_{(2)}, h_{(3)}) \# S(h_{(1)})] \otimes 1_D \# h_{(4)}\} =$$

$$a[\sigma^{-1}(S(h_{(2)}), h_{(3)}) \# S(h_{(1)})](1_D \# h_{(4)}) \otimes \overline{h}_{(5)} = a\varepsilon_(h_{(4)}) \otimes \overline{h}_{(5)} = a \otimes \overline{h}$$

and then we have

$$\beta \circ \beta^{-1} = 1_{A \otimes Q} \tag{3.50}$$

To check that β^{-1} is well defined set $h = rg$ where $r \in R^+$ and $g \in H$, then $\beta^{-1}(a \otimes \overline{h}) = \beta^{-1}(a \otimes \overline{rg}) = \varepsilon(r)\beta^{-1}(a \otimes \overline{g}) = 0$ since $\overline{rg} = \varepsilon(r)\overline{g}$ for all $r \in R$ and $g \in G$ and ofcourse β^{-1} is k-linear. So β is a surjection.

To check that β is one-to-one is somewhat more complicated and for that we follow [43, Chapter 3] to a certain extent, with some modifications to match our case properly.

Recall that $H \longrightarrow Q$ is quotient coalgebra and right H-module morphism and that Q^* is an augmented algebra via

$$\varepsilon_{Q^*}(\phi) = \phi(\overline{1}) \tag{3.51}$$

Then the space of left integrals $\mathcal{I}_l^{Q^*}$ over Q^* is well defined.

More over Q^* is a left H-module via

$$h \cdot \phi(\overline{x}) = \phi(\overline{xh}) \tag{3.52}$$

Now let λ_{R^*} and λ_{H^*} be left integrals in R^* and H^* respectively. There exists an element $\Gamma \in R$ such that $\lambda_{R^*} \cdot \Gamma = \varepsilon$, that is to say Γ is a right integral in R and $\lambda_{R^*}(\Gamma) = 1$.

Let

$$\lambda : Q \longrightarrow k; \quad \overline{h} \longmapsto \lambda_{H^*}(\Gamma h) \tag{3.53}$$

for $h' \in R^+H$ and $h \in H$ we have $\lambda(h'h) = \lambda_{H^*}(\Gamma h'h) = 0$ since $\varepsilon(h') = 0$ and we see that λ is a left integral in Q^* since for all $h \in H$ we have $\overline{1}\lambda(\overline{h}) =$

$\lambda_{H^*}(\Gamma h) = \overline{1}\Gamma_{(1)}h_{(1)}\lambda_{H^*}(\Gamma_{(2)}h_{(2)})$ since λ_{H^*} is an integral in Q^*, then the equality follows as $= \overline{h}_{(1)}\lambda_{H^*}(\Gamma h_{(2)}) = \overline{h}_{(1)}\lambda(\overline{h}_{(2)})$.

The reader will remember that as a consequence of Nichols-Zoeller Theorem given a left integral Λ in H, there exists an element $h \in H$ such that $\Lambda = h\Gamma$ now applying the Nakayama automorphism α to $\lambda_{H^*}(xy) = \lambda H^*(\alpha(y)x)$.

Now define $\Lambda = \alpha(h)$ and notice that since Λ is an integral in H we have $\varepsilon(x) = \lambda_{H^*}(\Gamma x\Lambda) = \lambda_{H^*}(h\Gamma x) = \Lambda\lambda(x)$ so we have $\varepsilon_{Q^*} = \Lambda\lambda$.

Define

$$f :_B A \longrightarrow_B A; \quad a \longmapsto a_{(0)}\lambda(\overline{a}_{(1)}) \tag{3.54}$$

Notice that $Im(f) \subseteq A^{coQ}$ since for all $a \in A$ $a_{(0)} \otimes a_{(1)}\lambda(\overline{a}_{(2)}) = a_{(0)} \otimes \lambda(\overline{a}_{(1)}) = a_{(0)}\lambda(\overline{a}_{(1)}) \otimes \overline{1}$, besides f is A^{coQ}-linear and by restiction is B-linear.

Since $A \otimes A \longrightarrow A \otimes H; \quad a \otimes a' \longmapsto aa'_{(0)} \otimes a'_{(1)}$ is a surjection then by the bijectivity of the antipode one also has that $A \otimes A \longrightarrow A \otimes H; \quad a \otimes a' \longmapsto a_{(0)}a' \otimes a_{(1)}$ is a bijection.

Then choose

$$\sum r_i \otimes l_i \in A \otimes_B A$$

such that

$$\sum r_{i0}l_i \otimes r_{i1} = 1_A \otimes \Lambda \tag{3.55}$$

This yields

$$a = \sum_i f(ar_i)l_i \tag{3.56}$$

for all $a \in A$

Then assume $x \otimes y \in A \otimes_B A$ is such that $\beta(x \otimes y) = xy_{(0)} \otimes \overline{y}_{(1)} = 0 \in A \otimes Q$, now since $y \in A$ we have that $x \otimes y = \sum x \otimes f(yr_i)l_i = \sum x \otimes y_{(0)}r_{i0}\lambda(\overline{y_{(1)}r_{i1}})l_i = \sum x \otimes y_{(0)}r_{i0}y_{(1)}r_{i1}\lambda(\overline{y_{(2)}r_{i2}})l_i = 0$ since $xy_{(0)} \otimes \overline{y}_{(1)} = 0$.

Then β is an injection and therefore a bijection.

We also remark that by theorem (1.14) both $_RH$ and H_R are free modules and hence faithfully flat, this in turn by descend over the tensor product implies $_BA$ and A_B are faithfully flat. By the lemma (3.26) preceding this proof we conclude then that $B = A\#_\sigma H = A^{coQ}$ and hence $A \otimes Q$ is Galois. $\square$

This setting is appropriate for the following theorem which is the main one in this subsection.

Theorem 3.27. *Let $R \subseteq H$ be a finite dimensional Hopf algebra extension, $Q = H/R^+H$ and D a twisted H-module algebra, define $B = D\#_\sigma R \subseteq D\#_\sigma H = A$. Then*

$$d_h(B, A) \leq d_h(R, H) \tag{3.57}$$

Proof. Suppose $d_h(R, H) = 2n+1$. By theorem (2.15) we have $d(Q, \mathcal{M}_H) = n$. By definition this means $Q^{\otimes(n)} \sim Q^{\otimes(n+1)}$. Apply $A \otimes -$ on the left to get $A \otimes Q^{\otimes(n)} \sim A \otimes Q^{\otimes(n+1)}$. Thus $A \otimes Q$ is Galois. By definition (3.11) we get then $A^{\otimes_B(n+1)} \cong A^{\otimes_B(n+2)}$ as A-A-bimodules. The result follows from this. □

Corollary 3.28. *Let $H < G$ be a finite group extension, σ a 2-cocycle and D a twisted G-module. Then inequality* (1.22) *extends to:*

$$\begin{aligned} d(D\#_\sigma H, D\#_\sigma G) \leq d_0(H, G) \leq d_p(H, G) \leq \\ d_R(H, G) \leq d_{\mathbb{Z}}(H, G) \leq d_c(H, G) \end{aligned} \tag{3.58}$$

Example 3.29. *Let $\mathbb{S}_n < \mathbb{S}_{n+1}$ be the symmetric groups of order n and $n+1$ respectively. Let α be a non trivial two-cocycle and consider the twisted group algebra extension $\mathbb{C}_\alpha\mathbb{S}_n \subseteq \mathbb{C}_\alpha\mathbb{S}_{n+1}$ as well as the regular complex group algebra extension $\mathbb{C}\mathbb{S}_n \subseteq \mathbb{C}\mathbb{S}_{n+1}$. Let $1 \leq n$, then: $2(n - \lceil\frac{\sqrt{8n+1}-1}{2}\rceil) + 1 \leq 2n-1$. Finally, we combine equations* (1.23) *and* (1.25) *together with this to get:*

$$\begin{aligned} d(\mathbb{C}_\alpha\mathbb{S}_n, \mathbb{C}_\alpha\mathbb{S}_{n+1}) = 2(n - \lceil\frac{\sqrt{8n+1}-1}{2}\rceil) + 1 \leq \\ d(\mathbb{C}\mathbb{S}_n, \mathbb{C}\mathbb{S}_{n+1}) = 2n-1 \end{aligned} \tag{3.59}$$

which is the left hand side of the string of inequalities (3.58).

Chapter 4

Factorisation Algebras

For this chapter we will always consider finite dimensional algebras over a field k.

4.1 Definition and an example

Definition 4.1. *Let A and B be finite dimensional algebras over a field k, m_A and m_B their respective multiplication, and let*

$$\psi : B \otimes A \longmapsto A \otimes B; \quad b \otimes a \longmapsto a_\alpha \otimes b^\alpha \tag{4.1}$$

and such that

$$\psi(1_B \otimes a) = a \otimes 1_B, \quad \psi(b \otimes 1_A) = 1_A \otimes b \tag{4.2}$$

for all $a \in A$ and $b \in B$.

Now suppose ψ satisfies the following octagon:

$$\begin{aligned}(A \otimes m_B) \circ (\psi \otimes B) \circ (B \otimes m_A \otimes B) \circ (B \otimes A \otimes \psi) = \\ (m_A \otimes B) \circ (A \otimes \psi) \circ (A \otimes m_B \otimes A) \circ (\psi \otimes B \otimes A)\end{aligned} \tag{4.3}$$

that is to say, for all $a, d \in A$ and $b, c \in B$

$$(ad_\alpha)_\beta \otimes b^\beta c^\alpha = a_\beta d_\alpha \otimes (b^\beta c)^\alpha \tag{4.4}$$

Then $A \otimes B$ becomes a unital associative algebra with product

$$(a \otimes b)(c \otimes d) = a\psi(b \otimes c)d = ac_\alpha \otimes b^\alpha d \tag{4.5}$$

where $a, c \in A$ and $b, d \in B$ and the unit element is $1_A \otimes 1_B$. Furthermore, A and B are subalgebras of $A \otimes B$ via the inclusion.

We call such ψ a factorisation of A and B and $A \otimes B$ a factorisation algebra and denote it $A \otimes_\psi B$.

Of course setting $\psi(b \otimes a) = a \otimes b$ yields the tensor algebra $A \otimes B$. We can construct more sophisticated algebras with underlying set $A \otimes B$ as in the following examples.

Example 4.2. *Let H be a Hopf algebra and A a left H-module algebra. Suppose H measures A:*

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b), \quad h \cdot 1_A = \varepsilon(h)1_A$$

Let ψ be defined as

$$\psi : H \otimes A \longrightarrow A \otimes H; \quad h \otimes a \longmapsto h_{(1)} \cdot a \otimes h_{(2)} \tag{4.6}$$

We show that the octagon (4.3) is satisfied. On the left hand side of the equation we have:

$$h \otimes a \otimes g \otimes b \overset{H\otimes A\otimes\psi}{\longmapsto} h \otimes a \otimes g_{(1)} \cdot b \otimes g_{(2)} \overset{H\otimes m_A\otimes H}{\longmapsto}$$

$$h \otimes ag_{(1)} \cdot b \otimes g_{(2)} \overset{\psi\otimes H}{\longmapsto} h_{(1)} \cdot (ag_{(1)} \cdot b) \otimes h_{(2)} \otimes g_{(2)} \overset{A\otimes m_H}{\longmapsto}$$

$$h_{(1)} \cdot (ag_{(1)} \cdot b) \otimes h_{(2)}g_{(2)} = (h_{(1)} \cdot a)(h_{(2)}g_{(1)} \cdot b) \otimes h_{(3)}g_{(2)} \tag{4.7}$$

Last equality coming from measuring axioms. Then on the right hand it goes as follows:

$$h \otimes a \otimes g \otimes b \overset{\psi\otimes H\otimes A}{\longmapsto} h_{(1)} \cdot a \otimes h_2 \otimes g \otimes b \overset{A\otimes m_H\otimes A}{\longmapsto}$$

$$h_{(1)} \cdot a \otimes h_{(2)}g \otimes b \overset{A\otimes\psi}{\longmapsto} h_{(1)} \cdot a \otimes (h_{(2)}g)_{(1)} \cdot b \otimes (h_{(2)}g)_{(2)} \overset{m_A\otimes H}{\longmapsto}$$

$$(h_{(1)} \cdot a)(h_{(2)}g)_{(1)} \cdot b \otimes (h_{(2)}g)_{(2)} = (h_{(1)} \cdot a)(h_{(2)}g_{(1)} \cdot b) \otimes h_{(3)}g_{(2)} \tag{4.8}$$

So we get the desired result.

Note also that

$$\psi(1_H \otimes a) = 1_H \cdot a \otimes 1_H = a \otimes 1_H$$

and

$$\psi(h \otimes 1_A) = h_{(1)} \cdot 1_A \otimes h_{(2)} = \varepsilon(h_{(1)})1_H \otimes h_{(2)} = 1_A \otimes h$$

Finally the product is given by:

$$(a \otimes h)(b \otimes g) = a\psi(h \otimes b)g = ah_{(1)} \cdot b \otimes h_{(2)}g \tag{4.9}$$

Then both A and H are subalgebras via the inclusion.

We conclude that via ψ as defined in this example the factorisation algebra $A \otimes_\psi H$ is exactly the smash product $A\#H$.

4.2 Depth of a Factorisation Algebra

Let $A \otimes_\psi B$ be a factorisation algebra via $\psi : B \otimes A \longrightarrow A \otimes B$, denote this as $S_\psi = A \otimes_\psi B$.

Thanks to multiplication in S_ψ and the fact that A and B are subalgebras of S_ψ we have that $S_\psi^{\otimes_B(n)}$ is an A-B bimodule via

$$a(a_1 \otimes b_1 \otimes_A \cdots \otimes_A a_n \otimes b_n)b =$$

$$aa_1 \otimes b_1 \otimes \cdots \otimes a_n \otimes b_n b \tag{4.10}$$

This is then enough to state the next proposition

Proposition 4.3. *Let $S_\psi = A \otimes_\psi B$ be a factorisation algebra between A and B as described above, then*

$$S_\psi^{\otimes_B(n)} \cong A^{\otimes(n)} \otimes B \tag{4.11}$$

as A-B- bimodules via

$$\theta_n(a_1 \otimes b_1 \otimes_B \cdots \otimes_B a_n \otimes b_n) =$$

$$a_1 \otimes a_{2_{\alpha 1}} \otimes \cdots \otimes a_{n_{\alpha(n-1)}} \otimes b_1^{\alpha 1} b_2^{\alpha 2} \cdots b_n \tag{4.12}$$

and inverse

$$\theta_n^{-1}(a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes b) = a_1 \otimes 1_B \otimes_B \cdots 1_B \otimes_B a_n \otimes b \tag{4.13}$$

Proof. First of all A-B linearity is given by multiplication either in A or B. Then notice that using multiplication in S_ψ and equation (4.4) one gets θ_n in the following way:

$$a_1 \otimes b_1 \otimes_B a_2 \otimes b_2 \otimes_B a_3 \otimes b_3 \otimes_B \cdots \otimes_B a_n \otimes b_n \longmapsto$$
$$a_1 \otimes (1_A \otimes b_1)(a_2 \otimes b_2) \otimes_B a_3 \otimes b_3 \otimes_B \cdots \otimes_B a_n \otimes b_n =$$
$$a_1 \otimes a_{2(\alpha 1)} \otimes b_1^{\alpha 1} b_2 \otimes_B a_3 \otimes b_3 \otimes_B \cdots \otimes_B a_n \otimes b_n \longmapsto$$
$$a_1 \otimes a_{2(\alpha 1)} \otimes (1_A \otimes b_1^{\alpha 1} b_2)(a_3 \otimes b_3) \otimes_B \cdots \otimes_B a_n \otimes b_n =$$
$$a_1 \otimes a_{2(\alpha 1)} \otimes a_{3(\alpha 2)} \otimes (b_1^{\alpha 1} b_2)^{\alpha 2} b_3 \otimes_B \cdots \otimes_B a_n \otimes b_n =$$
$$a_1 \otimes a_{2(\alpha 1)} \otimes a_{3(\alpha 2)} \otimes b_1^{\alpha 1} b_2^{\alpha 2} b_3 \otimes_B \cdots \otimes_B a_n \otimes b_n$$

Repeat this process to the right $n-1$ times to get

$$a_1 \otimes a_{2(\alpha 1)} \otimes \cdots a_{n(\alpha n-1)} \otimes b_1^{\alpha 1} b_2^{\alpha 2} \cdots b_n$$

Now it is easy to check that:

$$\theta_n^{-1} \circ \theta_n(a_1 \otimes b_1 \otimes_B \cdots \otimes_B a_n \otimes b_n) = a_1 \otimes b_1 \otimes_B \cdots \otimes_B a_n \otimes b_n$$

Indeed, notice that by multiplication and equation (4.4) for the last tensor powers in $\theta_n(a_1 \otimes b_1 \otimes_B \cdots \otimes_B a_n \otimes b_n)$ one has:

$$\cdots 1_B \otimes_B a_{n(\alpha(n-1))} \otimes b_1^{\alpha 1} b_2^{\alpha 2} \cdots b_n = \cdots 1_B \otimes_B (1_A \otimes b_1^{\alpha 1} \cdots b_{n-1})(a_n \otimes b_n)$$

but $1_A \otimes b_1^{\alpha 1} \cdots b_{n-1} \in B$ and moves along to the left over the B-tensor, so we get $= \cdots \otimes b_1^{\alpha 1} \cdots b_{n-1} \otimes (a_n \otimes b_n)$ Repeat this process to the left $n-1$ times and we get what we want, that is $\theta_n^{-1} \circ \theta_n = id_{S_\psi^{\otimes_B (n)}}$. Checking that $\theta \circ \theta^{-1} = Id_{A \otimes B^{\otimes n}}$ is straightforward. So we have what we wanted. □

Theorem 4.4. *Let $A \otimes_\psi B$ be a factorisation algebra with A and B finite dimensional algebras, (${}_B\mathcal{M}$ a finite tensor category) and A a left B-module. Then*

$$d_{odd}(B, S_\psi) \leq 2d(A,_B \mathcal{M}) + 1 \qquad (4.14)$$

Proof. Let $d(A,_B \mathcal{M}) = n$. Then n is the smallest integer such that $A^{\otimes(n+1)} \sim A^{\otimes(n)}$. Tensoring on the right by $(- \otimes B)$ one gets $A^{\otimes(n+1)} \otimes B \sim A^{\otimes(n)} \otimes B$. By previous theorem this is equivalent to $(A \otimes_\psi B)^{\otimes_B (n+1)} \sim (A \otimes_\psi B)^{\otimes_B (n)}$. This by definition is $d_{odd}(B, S_\psi) \leq 2n + 1$. □

Note that if B is also an augmented algebra then the inequality above becomes an equality since one can apply $(-\otimes_B k)$ on both sides of $A^{\otimes(n+1)} \otimes B \sim A^{\otimes(n)} \otimes B$ to get $A^{\otimes n+1} \sim A^{\otimes n}$.

Corollary 4.5. *[22, 6.2] Let H be a finite dimensional Hopf algebra and A a left H-module algebra measured by H, then*

$$d_{odd}(H, A\#H) = 2d(A,_H \mathcal{M}) + 1 \tag{4.15}$$

4.3 A Factorisable Algebra and the Depth of Q

Let C be a k-coalgebra, recall that its k-dual C^* is a k-algebra via the convolution product and evaluation: $C \otimes C^* \longrightarrow k$; $c \otimes \theta \longmapsto \theta(c)$.

Definition 4.6. *Let A be an algebra and C a coalgebra. Let $(A, C)_\psi$ be an entwining structure as defined in* (3.5)*. We say $(A, C)_\psi$ is factorisable if there exists a unique*

$$\overline{\psi} : A \otimes C^* \longrightarrow C^* \otimes A \tag{4.16}$$

such that the following diagram commutes:

$$\begin{array}{ccc} C \otimes A \otimes C^* & \xrightarrow{C\otimes\overline{\psi}} & C \otimes C^* \otimes A \\ {\scriptstyle \psi\otimes\ C^*}\big\downarrow & & \big\downarrow{\scriptstyle ev_{C^*}\otimes A} \\ A \otimes C \otimes C^* & \xrightarrow[A\otimes ev_{C^*}]{} & A \end{array} \tag{4.17}$$

We write $\overline{\psi}(a \otimes \theta) = \theta_i \otimes a^i$ for every $a \in A$ and $\theta \in C^*$, summation over i is understood.

Furthermore entwining structures $(A, C)_\psi$ are factorisable provided C is k projective [7] when k is a general commutative ring.

Let now $R \subseteq H$ be a finite dimensional Hopf algebra extension, let Q be their quotient module coalgebra.

By example (3.5) applied to $C = Q$ we get that $(H, Q)_\psi$ defined by $\psi(\overline{h} \otimes g) = g_{(1)} \otimes \overline{hg_{(2)}}$ for all $h, g \in H$ is an entwining structure. Since Q is a vector space over k it is k-projective. Now consider

$$\overline{\psi} : H \otimes Q^* \longrightarrow Q^* \otimes H$$

$$h \otimes \theta \longmapsto (h_{(2)} \rightharpoonup \theta) \otimes h_{(1)} \tag{4.18}$$

here the action is

$$(h \rightharpoonup \theta)(-) = \theta(\overline{-h})$$

For $h, - \in H$.

Then the diagram (4.17) is satisfied and $(H, Q)_\psi$ is factorisable via $\overline{\psi}$ as defined above.

Definition 4.7. *Let H be a Hopf algebra, A a finite dimensional left H-module algebra measured by H. Let $\psi : H \otimes A \longrightarrow A \otimes H$; $h \otimes a \longmapsto a_\alpha \otimes h^\alpha$ be a factorisation. Define the generalised smashed product of A and H as follows:*

$$A \#_\psi H$$

with product,

$$(a\#h)(b\#g) = a\psi(h\#b)g = a(b_\alpha \# h^\alpha)g \tag{4.19}$$

Consider now the algebra Q^{*op}, that is to say, for all $\theta, \gamma \in Q^{*op}$ and $h \in H$ we have $< \overline{h}, \theta\gamma >=< \overline{h_{(2)}}, \theta >< \overline{h_{(1)}}, \gamma >$. Then we form the generalised factorised smash product algebra $Q^{*op}\#_{\overline{\psi}}H$ with product given by

$$(\theta\#h)(\gamma\#g) = \theta(h_{(2)} \rightharpoonup \gamma)\#h_{(1)}g \tag{4.20}$$

Of course we identify Q^{*op} with $Q^{*op}\#1_H$ and H with $\varepsilon_H\#H$. Let $h, g \in H$ and $\theta, \gamma \in Q^{*op}$, the multiplication yields

$$(\varepsilon_H\#h)(\varepsilon_H\#g) = \varepsilon_H(h_{(2)} \rightharpoonup \varepsilon_H)\#h_{(1)}g = \varepsilon_H\#hg \tag{4.21}$$

and

$$(\theta\#1_H)(\gamma\#1_H) = \theta(1_H \rightharpoonup \gamma)\#1_H 1_H = \theta\gamma\#1_H \tag{4.22}$$

Hence both H and Q^{*op} are subalgebras in $Q^{*op}\#_{\overline{\psi}}H$. Moreover $Q^{*op}\#_{\overline{\psi}}H$ is an H and Q^{*op} left and right module via multiplication.

Theorem 4.8. *Let $R \subseteq H$ be a finite dimensional Hopf algebra extension and $Q = H/R^+H$ its generalised permutation module. Let $Q^{*op}\#_{\overline{\psi}}H$ be the generalised factorised smash product. Then*

$$d_{odd}(H, Q^{*op}\#_{\overline{\psi}}H) = d_h(R, H) \tag{4.23}$$

Proof. First notice that $Q^{*op}\#_{\overline{\psi}}H$ is a factorisation algebra via $\psi(\gamma \otimes h) = h_{(2)} \rightharpoonup \gamma \otimes h_{(1)}$ with unit element $\varepsilon_H \otimes 1_H$. Assume then that $d_{odd}(H, Q^{*op}\#_{\overline{\psi}}H) = 2n+1$. Apply theorem (4.4) and use the fact that H is augmented to get $d(Q^{*op}, {}_H\mathcal{M}) = n$. Since $Q^* \longrightarrow H^*$ is a Frobenius extension [25] one can see that this implies by self duality and opposing categories $d(Q, \mathcal{M}_H) = n$. By theorem (2.15) we get $d_h(R, H) = 2n+1$ which is what we wanted. $\square$

Chapter 5

Appendix

5.1 A Case in Favour of Finite Depth

Let $R \subseteq H$ be a finite dimensional Hopf algebra pair. Recall that given R or H of finite representation type then the pair is necessarily of finite depth.

It is then necessary to focus on infinite representation type finite dimensional Hopf algebra pairs if we want to give a definite answer to the problem of whether a given extension $R \subseteq H$ is always of finite depth.

To provide with a case in favour of this hypothesis we will sketch a proof of the depth of the 8-dimensional small quantum group H_8 in its Drinfeld double $D(H_8)$. As we will see both are of infinite representation type.

Definition 5.1. *The 8-dimensional small quantum group H_8 is defined as an algebra over the complex numbers $\mathbb{C}$ by:*

$$H_8 = \langle K, E, F \quad | \quad K^2 = 1, E^2 = F^2 = 0, EK = -KE, FK = -KF, EF = FE \rangle \tag{5.1}$$

with coalgebra structure given by:

$$\Delta(K) = K \otimes K, \quad \varepsilon(K) = 1$$

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \varepsilon(E) = 0$$
$$\Delta(F) = F \otimes K + 1 \otimes F, \quad \varepsilon(F) = 0 \tag{5.2}$$

Furthermore the antipode S is defined by:

$$S(K) = K, \quad S(E) = -EK, \quad S(F) = -KF \tag{5.3}$$

Notice that $ker\varepsilon = H_8^+ = \langle 1-K, E, F\rangle_{\mathbb{C}}$. Moreover, the Jacobson radical $J_{H_8} = \langle E, F\rangle_{\mathbb{C}}$, $J_{H_8}^2 = \langle EF\rangle_{\mathbb{C}}$ and $J_{H_8}^3 = \{0\}$.

As it is seen in [28] H_8 has two orthogonal primitive idempotents:

$$e_1 = \frac{1+K}{2}, \quad e_2 = \frac{1-K}{2} \tag{5.4}$$

Proposition 5.2. *Some important indecomposable modules of H_8 for our computations are:*
Two 4-dimensional projective indecomposables:

$$P_1 = e_1 H_8, \quad P_2 = e_2 H_8, \tag{5.5}$$

and two non projective simples:

$$S_1 = soc(P_1) \cong \mathbb{C}, \quad S_2 = soc(P_2) \tag{5.6}$$

Other indecomposables are:
3-dimensional non projective indecomposables:

$$M_1 = \mathbb{C}e_1E \oplus \mathbb{C}e_1F \oplus \mathbb{C}e_1EF, \quad M_2 = \mathbb{C}e_2E \oplus \mathbb{C}e_2F \oplus \mathbb{C}e_2EF, \tag{5.7}$$

2-dimensional non projective indecomposables:

$$N_1 = \mathbb{C}e_1E + \mathbb{C}e_1EF, \quad N_2 = \mathbb{C}e_1F + \mathbb{C}e_1EF$$

$$V_1 = \mathbb{C}e_2E + \mathbb{C}e_2EF, \quad V_2 = \mathbb{C}e_2F + \mathbb{C}e_2EF, \tag{5.8}$$

Definition 5.3. *Let M be an R-module. The Loewy length, or radical length of M, is the smallest m such that $J^mM = 0$, we denote it by $L(M) = m$.*

Definition 5.4. *Let A be a k-algebra. We say A is Nakayama if the principal modules P_i of A have a unique composition series.*

Definition 5.5. *Let A be a k-algebra. Let $\{e_1, \cdots, e_m\}$ be a complete set of primitive orthogonal idempotents. We say A is a basic algebra if $e_iA \ncong e_jA$ for all $i \neq j$.*

Theorem 5.6. *[36, 3.1] Let H be a finite dimensional basic Hopf algebra. Then H is of finite representation type if and only if H is a Nakayama algebra.*

Theorem 5.7. *H_8 is of infinite representation type.*

Proof. Clearly H_8 is basic. The Loewy length of P_1 is $L(P_1) = 3$, on the other hand, the length is $l(P_1) = 4$. Hence P_1 has two different composition series and then H_8 is not Nakayama. The result follows from previous theorem. □

We now consider the Drinfeld double of H_8.

Definition 5.8. *The Drinfeld double of H_8, denoted by $D(H_8)$ is a 64-dimensional Hopf algebra defined as an algebra over the complex numbers by:*

$$D(H_8) = \langle K, E, F, L, D, G \quad | \quad K^2 = L^2 = 1, E^2 = F^2 = D^2 = G^2 = 0$$

$$EK = -KE, FK = -KF, DK = -DK, GK = -KG, EL = -LE, FL =$$
$$-LF, DL = -LD, GL = -LG, KL = LK, EF = FE, DG = GD,$$
$$FD = -DF, EG = GE, ED = -DE, FG + GF = 1 - LK\rangle \tag{5.9}$$

The coalgebra structure is given by: K, E, F as in H_8

$$\Delta(L) = L \otimes L, \quad \varepsilon(L) = 1,$$

$$\Delta(D) = L \otimes D + D \otimes 1, \quad \varepsilon(D) = 0,$$
$$\Delta(G) = G \otimes L + 1 \otimes G, \quad \varepsilon(G) = 0. \tag{5.10}$$

and the antipode S is given by: K, E, F as in H_8,

$$S(L) = L, \quad S(D) = -DL, \quad S(G) = -GL \tag{5.11}$$

Proposition 5.9. [1]

1. *the Jacobson radical $J_{D(H_8)} = \langle E, F, D, L\rangle_{\mathbb{C}}$.*

2. *$D(H_8)$ is basic and has 4 16-dimensional projective indecomposables.*

3. *$D(H_8)$ is not Nakayama and hence of infinite representation type.*

We will now consider $Q = D(H_8)/H_8^+ D(H_8)$. Recall that $H_8^+ = \langle 1 - K, E, F\rangle$. It is then easy to compute Q as a 8-dimensional $D(H_8)$ module coalgebra with basis:

$$Q = \mathbb{C}\{\overline{L^i}, \overline{D^j}, \overline{G^k}\} \tag{5.12}$$

where $0 \leq i, j, k \leq 1$.

Furthermore, let $X \in D(H_8)$, using the relations given in (5.8) and the fact that $H^+ = \langle 1 - K, E, F\rangle$ we see that in Q:

$$K \cdot \overline{X} = \overline{X}, \quad E \cdot \overline{X} = F \cdot \overline{X} = 0 \tag{5.13}$$

Using this and the relations above we get that as a right H_8 module

$$Q \cong V_1 \oplus N_1 \oplus 2S_2 \oplus 2S_1 \tag{5.14}$$

Recall that both H_8 and $D(H_8)$ have commutative representation rings [39].

Proposition 5.10. *The tensor products of the indecomposable summands of Q are as follows:*

$$V_1 \otimes V_1 \cong V_1 \otimes N_1 \cong N_1 \otimes N_1 \cong V_1 \oplus N_1$$

$$V_1 \otimes S_1 \cong N_1 \otimes S_2 \cong V_1, \quad N_1 \otimes S_1 \cong V_1 \otimes S_2 \cong N_1$$
$$S_2 \otimes S_2 \cong S_1, \quad S_2 \otimes S_1 \cong S_2, \quad S_1 \otimes S_1 \cong S_1 \tag{5.15}$$

Theorem 5.11.
$$3 \leq d(H_8, D(H_8)) \leq 4 \tag{5.16}$$

Proof. By the previous proposition $Q \sim Q^{\otimes(2)}$. Apply equation (2.12). □

[1] The proof of this proposition relies exactly in the same arguments as theorem (5.7). Full details are expected to be published as a preprint soon.

5.2 Further Research

As it was first exposed at the beginning of this work, one of the most important questions in depth theory in recent years is whether given a finite dimensional Hopf algebra extension $R \subseteq H$, the pair is always of finite depth. As it has been exposed in these pages, the quotient module coalgebra $Q = H/R^{+}H$ has proven to be useful in understanding depth, as well as it is a natural vehicle to intertwine the concept of depth of a module in its module category with the main definition in terms of tensor powers of the regular representation of the Hopf algebra H. This gives us the power to determine the depth of a pair by just looking into the representation rings $A(H)$ or $A(R)$ of the Hopf algebras $H \supseteq R$.

With this in mind, and taking into account the advancements made in this current work I propose the following research problems for the immediate future, either to enrich our understanding of depth and its relationship to Q or to pursue a definite answer for our main question:

1. As it is pointed out in chapter 2, in [20, Chapter 9] it is shown that Mackey's decomposition theorem is used to show that permutation modules of finite group algebras are always algebraic. This in turn implies that finite group algebra extensions over commutative rings are always finite depth. Since for a finite dimensional Hopf algebra pair $R \subseteq H$, their quotient module Q is a natural generalisation of the permutation module of a finite group extension, it is natural to ask if for such an extension $R \subseteq H$ it is true that their quotient module Q is always algebraic, and hence the extension of finite depth.

2. We know finite representation type implies finite depth, then semisimple pairs are always finite depth, it is then natural to look into finite dimensional extensions of pointed Hopf algebras.

3. Study finite dimensional Hopf algebra extensions of infinite representation type to look for counterexamples to finite depth.

4. Give a formula for the depth of a n-th Taft algebra at the root of unity $H_\omega(n)$ in its Drinfel double $D(H_\omega(n))$. (Work in progress. Preprint soon to appear).

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