GENERIC HOMEOMORPHISMS HAVE FULL METRIC MEAN DIMENSION

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ABSTRACT. We prove that for C^0-generic homeomorphisms, acting on a compact smooth boundaryless manifold with dimension greater than one, the upper metric mean dimension with respect to the smooth metric coincides with the dimension of the manifold. As an application, we show that the upper box dimension of the set of periodic points of a C^0-generic homeomorphism is equal to the dimension of the manifold. In the case of continuous interval maps, we prove that each level set for the metric mean dimension with respect to the Euclidean distance is C^0-dense in the space of continuous endomorphisms of $[0, 1]$ with the uniform topology. Moreover, the maximum value is attained at a C^0-generic subset of continuous interval maps and a dense subset of metrics topologically equivalent to the Euclidean distance.

1. INTRODUCTION

The topological entropy is an invariant by topological conjugation and a very useful tool to either measure how chaotic is a dynamical system or to attest that two dynamics are not conjugate. It counts, in exponential scales, the number of distinguishable orbits up to arbitrarily small errors. Clearly, on a compact metric space, a Lipschitz map has finite topological entropy. However, if the dynamics is just continuous, the topological entropy may be infinite. Actually, K. Yano proved in [20] that, on compact smooth manifolds with dimension greater than one, the set of homeomorphisms having infinite topological entropy are C^0-generic. So the topological entropy is no longer an effective label to classify them.

In order to obtain a new invariant to distinguish maps with infinite entropy, E. Lindenstrauss and B. Weiss introduced in [13] the notions of *upper metric mean dimension* and *lower metric mean dimension* of an endomorphism f of a metric space (X, d), that we will denote by $\overline{\text{mdim}}_M (X, f, d)$ and $\underline{\text{mdim}}_M (X, f, d)$, respectively. Historically, the concepts of upper and lower metric mean dimension have been often useful to distinguish dynamical systems acting on infinite dimensional spaces. We highlight that these notions can also be meaningful and provide a new insight in the finite dimensional setting. For instance, in [10] the authors consider homeomorphisms homotopic to the identity on the torus and employed precisely the metric mean dimension as the finer scaling of complexity they needed to describe the

multifractal aspects of the sets of points with prescribed rotation vectors. Actually, while C^0-generic dynamics have infinite topological entropy [20], the metric mean dimension can be used to detect different rates of complexity at which this may grow to infinite. An extension of this notion to $\mathbb{Z}^k$-actions can be found in [4, 5, 7], and a variational principle for the metric mean dimension of free semigroup actions appeared in [1].

The upper and lower metric mean dimensions are metric versions of the *mean dimension*, a concept proposed by M. Gromov in [3] which may be viewed as a dynamical analogue of the topological dimension. In particular, it is known that the mean dimension of a homeomorphism $f : X \to X$ acting on a topological space X of finite dimension is zero. The upper and lower metric mean dimensions, unlike Gromov's concept, depend on the metric adopted on the space and are nonzero only if the topological entropy of the dynamics is infinite.

More recently, it was proved in [18] that, on a compact manifold with dimension greater than one, having positive upper metric mean dimension is a C^0-dense property in the whole class of homeomorphisms. Moreover, the authors established that the set of homeomorphisms with metric mean dimension equal to the dimension of the manifold is C^0-dense in the set of all the homeomorphisms with a fixed point. Unfortunately, the previous subset is not C^0-dense in the space of homeomorphisms. The existence of a fixed point is crucial due to the need of an adequate construction of separated sets using the pseudo-horseshoes introduced in [20]. If, instead, f admits a periodic point of period $p > 1$, then the argument of [18] ensures that $\mathrm{mdim}_M (f^p, X, d) = \dim X$ hence, as $\mathrm{mdim}_M (f^p, X, d) \leqslant p \, \mathrm{mdim}_M (f, X, d)$,

$$\mathrm{mdim}_M (X, f, d) \geqslant \frac{\dim X}{p}. \tag{1.1}$$

Therefore, in order to be able to consider homeomorphisms with periodic points of arbitrarily large periods (actually the C^0 generic case, as proved in [8]) and still obtain $\mathrm{mdim}_M (X, f, d) = \dim X$, one must compensate for the loss of metric mean dimension caused by their likely long periods. In this work we show that for C^0-generic homeomorphisms, acting on compact smooth boundaryless manifolds with dimension greater than one, not only the metric mean dimension is positive but it is equal to the dimension of the manifold. Our argument grew out of the results of [8], [20] and [11], to which we refer the reader for more background.

Let us be more precise. It is known, after [20, Proposition 2], that for any homeomorphism f, any scale $\delta > 0$ and any $N \in \mathbb{N}$ there exist a C^0-arbitrary small perturbation g of f and a suitable iterate g^k which has a compact invariant subset semi-conjugate to a subshift of finite type with N^k symbols. This ensures the existence of some scale $\varepsilon_0 > 0$ (depending on N and f) such that the largest cardinality of any (n, ε) separated subset of X with respect to g satisfies $s(g, n, \varepsilon) \geqslant N^n$ for every $\varepsilon \leqslant \varepsilon_0$ and all big enough n; so $h_{\mathrm{top}}(g) \geqslant \log N$. Although Yano's strategy succeeds in producing homeomorphisms C^0 close to f with arbitrarily large topological entropy, it fails to bring forth any lower bound on their metric mean dimension since there exists no explicit relation between ε_0 and N. To obtain better estimates than (1.1) for the metric mean dimension, we endeavored to find such a connection in $\mathbb{R}^{\dim X}$, and then forwarded the conclusions to the manifold X using the bi-Lipschitz nature of the charts. We have had to perform several C^0-small perturbations along the orbit of a periodic point (reminding the global changes done in the proof of Pugh's C^1 Closing Lemma [16]) in order to build a new version of the pseudo-horseshoes used in [20], now obliged to satisfy two conditions: to exist in all sufficiently small scales and to exhibit the needed separation in all moments of the construction. We will be back to this issue on Section 7.

The second setting we address here concerns the spaces $C^0([0,1],d)$ and $C^0([0,1],\rho)$ of continuous endomorphisms of the interval $[0,1]$, with the uniform metric, where d stands for the Euclidean distance and ρ denotes any metric topologically equivalent to d. Adjusting the construction of horseshoes done by M. Misiurewicz in [14], which paved the way to prove that the topological entropy of maps of the interval is lower semicontinuous and upper bounded by the exponential growth rate of the periodic points, the authors of [18] showed that the subset of those maps with maximal upper metric mean dimension (whose value is 1) is dense in $C^0([0,1],d)$. A finer construction allowed us to prove that, for every $0 \leqslant \beta \leqslant 1$, the level set of continuous maps for which the metric mean dimension exists and is equal to β is a dense subset of $C^0([0,1],d)$. Additionally, we show that the maximal value is attained at a C^0-generic subset of $C^0([0,1],\rho)$ and a dense subset of metrics topologically equivalent to the Euclidean distance. For more details we refer the reader to Section 3.

2. Upper and lower metric mean dimension

Most of the results we will use or prove require some mild homogeneity of the space so that local perturbations can be made. For simplicity we consider here only the case of smooth compact connected manifolds. Let X be such a manifold and d be a metric compatible with the topology on X. Given a continuous map $f \colon X \to X$ and a non-negative integer n, define the dynamical metric $d_n \colon X \times X \to [0,+\infty)$ by

$$d_n(x,y) = \max \left\{ d(x,y), \, d(f(x),f(y)), \, \ldots, \, d(f^{n-1}(x),f^{n-1}(y)) \right\}$$

and denote by $B_f(x,n,\varepsilon)$ the ball of radius ε around $x \in X$ with respect to the metric d_n. It is not difficult to check that d_n generates the same topology as d.

Having fixed $\varepsilon > 0$, we say that a set $A \subset X$ is (n,ε)-separated by f if $d_n(x,y) > \varepsilon$ for every $x, y \in A$. Denote by $s(f,n,\varepsilon)$ the maximal cardinality of all (n,ε)-separated subsets of X by f. Due to the compactness of X, the number $s(f,n,\varepsilon)$ is finite for every $n \in \mathbb{N}$ and $\varepsilon > 0$.

Definition 2.1. *The* lower metric mean dimension *of (f,X,d) is given by*

$$\underline{\mathrm{mdim}}_M (X,f,d) = \liminf_{\varepsilon \to 0} \frac{h(f,\varepsilon)}{|\log \varepsilon|}$$

where $h(f,\varepsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log s(f,n,\varepsilon)$. Similarly, the upper metric mean dimension *of (X,f,d) is the limit*

$$\overline{\mathrm{mdim}}_M (X,f,d) = \limsup_{\varepsilon \to 0} \frac{h(f,\varepsilon)}{|\log \varepsilon|}.$$

The upper/lower metric mean dimensions satisfy the following properties we may summon later:

(1) If the topological entropy $h_{\mathrm{top}}(f) = \lim_{\varepsilon \to 0} h(f,\varepsilon)$ is finite (as when f is a Lipschitz map on a compact metric space), then

$$\underline{\mathrm{mdim}}_M (X,f,d) = \overline{\mathrm{mdim}}_M (X,f,d) = 0.$$

(2) Given two continuous maps $f_1 : X_1 \to X_1$ and $f_2 : X_2 \to X_2$ on compact metric spaces (X_1,d_1) and (X_2,d_2), then

$$\overline{\mathrm{mdim}}_M (X_1 \times X_2, f_1 \times f_2, d_1 \times d_2) = \overline{\mathrm{mdim}}_M (X_1,f_1,d_1) + \overline{\mathrm{mdim}}_M (X_2,f_2,d_2).$$

(3) Given a continuous map $f : X \to X$ on a compact metric space (X, d), the upper box dimension of (X, d) is an upper bound for $\overline{\mathrm{mdim}_M}(X, f, d)$ (cf. Remark 4 of [18]).

(4) Let $f : X \to X$ be a continuous map on a compact metric space (X, d) and k be a positive integer. The inequality

$$\overline{\mathrm{mdim}_M}(X, f^k, d) \leqslant k \, \overline{\mathrm{mdim}_M}(X, f, d)$$

is always valid (the proof is similar to the one done for the entropy in [19]). The equality may fail (see the previous item), though it is valid whenever f is Lipschitz, in which case these values are zero for every $k \in \mathbb{N}$.

(5) For every continuous map $f : X \to X$ on a compact metric space (X, d), one has

$$\overline{\mathrm{mdim}_M}(\Omega(f), f \mid_{\Omega(f)}, d) = \overline{\mathrm{mdim}_M}(X, f, d)$$

where $\Omega(f)$ stands for the set of non-wandering points of f.

(6) Given a continuous map $f : X \to X$ on a compact metric space X,

$$\mathrm{mdim}(X, f) \leqslant \underline{\mathrm{mdim}_M}(X, f, d)$$

for every metric d on X compatible with the topology of X (cf. [13, Theorem 4.2]), where $\mathrm{mdim}(X, f)$ stands for the mean dimension of (X, f). The existence of a compatible metric for which the equality holds is conjectured for every maps (cf. [12]). This is known to be valid in the case of minimal systems (cf. [11, Theorem 4.3]) and to be true C^0-generically (cf. [6, Appendix]).

3. Main results

Let (X, d) be a compact smooth boundaryless manifold with dimension greater than one. Denote by $\mathrm{Homeo}(X, d)$ the set of homeomorphisms of (X, d). This is a complete metric space if endowed with the metric

$$D(f, g) = \max_{x \, \in \, X} \left\{ d(f(x), g(x)), \, d(f^{-1}(x), g^{-1}(x)) \right\}.$$

It is known from [18] that the upper metric mean dimension of every $f \in \mathrm{Homeo}(X, d)$ cannot be bigger than the dimension of the manifold X. Our first result states that typical homeomorphisms have the largest upper metric mean dimension with respect to any metric compatible with the smooth structure of the smooth manifold X.

Theorem A. *Let X be a compact smooth boundaryless manifold with dimension strictly greater than one and d be a metric compatible with the smooth structure of X. There exists a C^0-Baire residual subset $\mathfrak{R} \subset \mathrm{Homeo}(X, d)$ such that*

$$\overline{\mathrm{mdim}_M}(X, f, d) = \dim X \qquad \forall f \in \mathfrak{R}.$$

Some comments are in order. Firstly we note that homeomorphisms in one-dimensional smooth manifolds have zero entropy (thus their metric mean dimension is zero), so we cannot drop the requirement on the dimension of the manifold. A second remark concerns the use of a metric compatible with the smooth structure on the manifold. As a matter of fact, by employing smooth charts one guarantees that the induced metric on the Euclidean space is equivalent to the Euclidean metric, which is a product metric of one-dimensional length factors, enabling one to effectively make the computations.

The previous result also sheds light on the differences between the upper and lower metric mean dimensions. Since the manifold X has finite dimension (so its Lebesgue covering dimension is also finite), $\text{mdim}(X, f) = 0$ for every $f \in \text{Homeo}(X, d)$ (cf. [13]). Moreover, one always has

$$\text{mdim}(X, f) \leqslant \underline{\text{mdim}}_M(X, f, d).$$

Therefore, if $f \in \mathfrak{R}$ then

$$0 = \text{mdim}(X, f) \leqslant \inf_\rho \underline{\text{mdim}}_M(X, f, \rho) \leqslant \sup_\rho \overline{\text{mdim}}_M(X, f, \rho) = \dim X$$

where the infimum and supremum are taken on the space of distances ρ which induce the same topology on X as d. As the conjecture mentioned in [12] turns out to be true C^0-generically, that is, for generic homeomorphisms,

$$0 = \text{mdim}(X, f) = \inf_\rho \underline{\text{mdim}}_M(X, f, \rho)$$

(cf. [6, Appendix]), then C^0-generically one finds

$$\inf_\rho \underline{\text{mdim}}_M(X, f, \rho) < \sup_\rho \overline{\text{mdim}}_M(X, f, \rho).$$

It is natural to consider the upper metric mean dimension as a function of three variables, namely the dynamics f, the f-invariant non-empty compact set $Z \subset X$ and the metric d, and to ask whether it varies continuously. Concerning the first variable, within the space of homeomorphisms satisfying the assumptions of Theorem A the irregularity of the map $Z \mapsto \overline{\text{mdim}}_M(Z, f \mid_Z, d)$, with respect to the Hausdorff metric, is a consequence of property (5) in Section 2 together with the C^0-general density theorem [8]. Indeed, C^0-generically the non-wandering set is the limit (in the Hausdoff metric) of finite unions of periodic points, on which the upper metric mean dimension is zero, whereas Theorem A ensures that generically the upper metric mean dimension is positive. Regarding the variable f, in the case of smooth manifolds (X, d) where the C^1-diffeomorphisms are C^0-dense on the space of homeomorphisms (which is true if the dimension of the manifold X is smaller or equal to 3, cf. [15]), Theorem A implies that there are no continuity points of the map $f \mapsto \overline{\text{mdim}}_M(X, f, d)$. As far as we know, the dependence of the upper metric mean dimension on the distance d is still beyond utter understanding.

The study of the upper metric mean dimension for generic homeomorphisms in Theorem A leads to an unexpected generic feature of the set of periodic points, measured through their upper box-dimension.

Theorem B. *Let X be a compact smooth boundaryless manifold with dimension strictly greater than one and let d be a metric compatible with the smooth structure of X. There exists a C^0-Baire generic subset $\mathcal{R} \subset \text{Homeo}(X, d)$ such that*

$$\overline{\dim_B}(Per(f)) = \dim X \qquad \forall\, f \in \mathcal{R}.$$

The second problem we address in this paper is closely related to the previous one. Focusing on the space of continuous interval maps, we obtain stronger results both on the multifractal structure and the largest level set of the upper metric mean dimension with respect to any metric d topologically equivalent to the Euclidean distance (see Theorem C). Recall that two metrics d_1 and d_2 on a space X are topologically equivalent if the identity map from (X, d_1) to (X, d_2) is a homeomorphism. We show that, in the case of continuous maps on $([0, 1], d)$, actually each level set for the upper metric mean dimension is relevant, since it is C^0-dense

in $C^0([0, 1])$ with the uniform norm. Besides, we prove that the level set of the largest upper metric mean dimension is C^0-Baire generic independently of the chosen metric on the interval as long as it is topologically equivalent to the Euclidean distance, thus extending the statement of Theorem A to the one-dimensional context.

Theorem C. *Let $C^0([0, 1], d)$ be the space of continuous endomorphisms of the interval $([0, 1], d)$, where d stands for the Euclidean distance. For every $\beta \in [0, 1]$ there exists a dense subset $\mathcal{D}_\beta \subset C^0([0, 1], d)$ for the uniform metric such that*

$$\underline{\mathrm{mdim}}_M([0, 1], f, d) = \overline{\mathrm{mdim}_M}([0, 1], f, d) = \beta \qquad \forall\, f \in \mathcal{D}_\beta.$$

Moreover, if ρ denotes any metric topologically equivalent to the Euclidean distance then C^0-generically in $C^0([0, 1], \rho)$ one has $\overline{\mathrm{mdim}_M}([0, 1], f, \rho) = 1$.

Regarding the space of continuous endomorphisms of $([0, 1], d)$, we may summarize as follows what is now known:

(1) The subset of maps $f \in C^0([0, 1], d)$ satisfying $\overline{\mathrm{mdim}_M}([0, 1], f, d) = 1$ is C^0-generic (cf. Theorem C).

(2) The subset of maps of $f \in C^0([0, 1], d)$ such that $\underline{\mathrm{mdim}}_M([0, 1], f, d) = 0 = \mathrm{mdim}(X, f)$ is C^0-generic (cf. [6]).

(3) For each $0 < \beta < 1$, the subset of maps $f \in C^0([0, 1], d)$ satisfying $\overline{\mathrm{mdim}_M}([0, 1], f, d) = \underline{\mathrm{mdim}}_M([0, 1], f, d) = \beta$ is C^0-dense (cf. Theorem C).

If one endows the space $\mathcal{M}$ of metrics on $[0, 1]$ with the metric

$$d_1,\, d_2 \in \mathcal{M} \quad \mapsto \quad \mathbb{D}(d_1, d_2) = \min\left\{1,\ \max_{x,\,y \in X} \left|d_1(x, y) - d_2(x, y)\right|\right\}$$

then, though not complete due to the existence of pseudo-metrics on its closure, $\mathcal{M}$ is a Baire space (cf. [17]). Yet, the existence of discrete metrics, which form an open dense subset on $(\mathcal{M}, \mathbb{D})$, prevents this space to be separable (cf. [17, Theorem 4]). Besides, under these kind of metrics the interval $[0, 1]$ is not compact, hence the definition of metric mean dimension is meaningless. If, however, we restrict to the space $\mathcal{M}_E$ of metrics on $[0, 1]$ which are topologically equivalent to the Euclidean distance, then we get a subset of $C^0([0, 1] \times [0, 1], \mathbb{R}_+)$ with the uniform norm determined by the Euclidean distance in $[0, 1] \times [0, 1]$. This way, $\mathcal{M}_E$ is separable and Theorem C has the following direct consequence.

Corollary D. *There exists a dense subset $\mathfrak{D} \subset \mathcal{M}_E$ and a C^0-Baire generic subset $\mathcal{G}$ of $C^0([0, 1], d)$ such that*

$$\overline{\mathrm{mdim}_M}([0, 1], f, \rho) = 1 \qquad \forall\, f \in \mathcal{G} \quad \forall\, \rho \in \mathfrak{D}.$$

We observe that, in spite of Corollary D, it may happen that some metric on $\mathcal{M}_E$ is able to drastically drop the upper metric mean dimension. The construction of minimizing metrics for the upper metric mean dimension, and the estimation of the minimum value, are still open problems.

4. Absorbing disks

In this section we address some generic topological properties of homeomorphisms acting on smooth manifolds, aiming to check the existence of absorbing disks with arbitrarily small diameter.

Following M. Hurley in [8], if the dimension of the manifold X is $\dim X$ and $D_1^{\dim X}$ denotes the closed unit ball in $\mathbb{R}^{\dim X}$, call $B \subset X$ a *disk* if it is homeomorphic to $D_1^{\dim X}$. A closed subset K of X is called *k-absorbing* for a homeomorphism f of X if $f^k(K)$ is contained in the interior of K, and K is said to be *absorbing* if it is k-absorbing for some $k \in \mathbb{N}$. Note that if B is a k-absorbing disk, then, by Brouwer fixed point theorem, B contains a point periodic by f with period k. We say that a point $P \in X$ is a *periodic attracting point* for f if there is a p-absorbing disk B satisfying

(1) $\operatorname{diam}(f^i(B)) < \operatorname{diam}(B)$ for every $1 \leqslant i \leqslant p - 1$;

(2) $\bigcap_{j \geqslant 0} f^{jp}(B) = \{P\}$.

Observe that, since f is a bijection, the last equality implies that $f^p(P) = P$. We also remark that, given a periodic attracting point, it is possible to choose the disk B satisfying $f^j(B) \cap B = \emptyset$ for every $1 \leqslant j < p$. In the next sections we will always assume that absorbing disks satisfy this property.

Proposition 3 in [8] ensures that for every $F \in \operatorname{Homeo}(X, d)$ and every $\varepsilon > 0$ there is $f \in \operatorname{Homeo}(X, d)$ exhibiting a periodic attracting point and such that $D(F, f) < \varepsilon$. Notice that having a periodic attracting point is a C^0 *quasi-robust property*. More precisely, for every $g \in \operatorname{Homeo}(X, d)$ that is C^0 close enough to f the following conditions hold:

(a) if B is a p-absorbing disk for $f \in \operatorname{Homeo}(X, d)$ then B is p-absorbing for g;

(b) if B is a p-absorbing disk for $f \in \operatorname{Homeo}(X, d)$ then for every $1 \leqslant j < p$ the disk $f^i(B)$ is p-absorbing for g;

(c) for every $\delta > 0$ we may find some $J \geqslant 0$ such that $f^{Jp}(B)$ has diameter smaller than δ and is a p-absorbing disk for g.

Properties (a) and (b) are immediate consequences of the closeness in the uniform topology and the compactness of B. Property (c) is due to the attracting nature of the periodic point (that is, B is a p-absorbing disk satisfying $\bigcap_{j \geqslant 0} f^{jp}(B) = \{P\}$) and item (a). Unless stated otherwise, the p-absorbing disks we will use satisfy the aforementioned properties.

Altogether this shows that having a p-absorbing disk of diameter δ is a C^0-open and dense condition. Therefore, taking the intersection of the sets

$$\mathcal{H}_n = \Big\{ f \in \operatorname{Homeo}(X, d) \colon f \text{ has an absorbing disk with diameter at most } 1/n \Big\}$$

we conclude that:

Lemma 4.1. *C^0-generic homeomorphisms have absorbing disks of arbitrarily small diameter.*

5. Pseudo-horseshoes

In this section we introduce the class of invariants that will play the key role in the proof of Theorem A. They will be defined first on Euclidean spaces and afterwards conveyed to manifolds *via* charts.

5.1. **Pseudo-horseshoes on $\mathbb{R}^k$.** Consider in $\mathbb{R}^k$ the norm

$$\|(x_1, \cdots, x_k)\| := \max_{1 \leqslant i \leqslant k} |x_i|.$$

Given $r > 0$ and $x \in \mathbb{R}^k$, set

$$\begin{aligned}
D_r^k(x) &= \left\{ y \in \mathbb{R}^k \colon \|x - y\| \leqslant r \right\} \\
D_r^k &= D_r^k\big((0, \ldots, 0)\big).
\end{aligned}$$

For $1 \leqslant j \leqslant k$, let $\pi_j \colon \mathbb{R}^k \to \mathbb{R}^j$ be the projection on the first j coordinates.

Definition 5.1. *Consider $r > 0$, $x = (x_1, \cdots, x_k)$ and $y = (y_1, \cdots, y_k)$ in $\mathbb{R}^k$, and take an open set $U \subset \mathbb{R}^k$ containing $D_r^k(x)$. Having fixed a positive integer N, we say that a homeomorphism $\varphi \colon U \to \mathbb{R}^k$ has a pseudo-horseshoe of type N at scale r connecting x to y if the following conditions are satisfied:*

(1) $\varphi(x) = y$.

(2) $\varphi\big(D_r^k(x)\big) \subset \text{int}\big(D_r^{k-1}(\pi_{k-1}(y))\big) \times \mathbb{R}$.

(3) *For $i = 0, 1, \ldots, \left[\frac{N}{2}\right]$,*

$$\varphi\Big(D_r^{k-1}(\pi_{k-1}(x)) \times \Big\{ x_k - r + \frac{4ir}{N} \Big\}\Big) \subset \text{int}\Big(D_r^{k-1}(\pi_{k-1}(y))\Big) \times (-\infty, \, y_k - r).$$

(4) *For $i = 0, 1, \ldots, \left[\frac{N-1}{2}\right]$,*

$$\varphi\left(D_r^{k-1}(\pi_{k-1}(x)) \times \Big\{ x_k - r + \frac{(4i+2)r}{N} \Big\}\right) \subset \text{int}\Big(D_r^{k-1}(\pi_{k-1}(y))\Big) \times (y_k + r, \, +\infty).$$

(5) *For each $i \in \{0, \ldots, N - 1\}$, the intersection*

$$V_i = D_r^k(y) \cap \varphi\left(D_r^{k-1}(x) \times \Big[x_k - r + \frac{2ir}{N}, \, x_k - r + \frac{(2i+2)r}{N} \Big]\right)$$

is connected and satisfies:

(a) $V_i \cap (D_r^{k-1}(y) \times \{-r\}) \neq \emptyset$;

(b) $V_i \cap (D_r^{k-1}(y) \times \{r\}) \neq \emptyset$;

(c) *each connected component of $V_i \cup \partial D_r^k(y)$ is simply connected.*

The name *pseudo-horseshoe* seems adequate since, when $x = y$, the map φ does admit a compact invariant subset which is semi-conjugate to a subshift of finite type (cf. [9]). Each V_i is called a *vertical strip* of the pseudo-horseshoe φ, and we denote the collection of vertical strips of φ by $\mathcal{V}_\varphi$.

Notice that this definition is both topological and geometrical. Indeed, while we consider homeomorphisms, we also assume that certain scale is preserved and identify a preferable vertical direction by means of coordinates.

Definition 5.2. *Consider $\varepsilon > 0$ and a homeomorphism $\varphi \colon U \to \mathbb{R}^k$ with a pseudo-horseshoe of type N at scale r connecting x to y. The pseudo-horseshoe is said to be ε-separating if we may choose the collection $\mathcal{V}_\varphi$ so that the Hausdorff distance between distinct vertical strips is bigger than ε, that is, $\inf\{\|a - b\| \colon a \in V_i, \ b \in V_j\} > \varepsilon$ for every $i \neq j$.*

5.2. **Pseudo-horseshoes on manifolds.** So far, pseudo-horseshoes were defined in open sets of $\mathbb{R}^k$. Now we need to convey this notion to manifolds.

Definition 5.3. *Let (X, d) be a compact smooth manifold of dimension $\dim X$. Given $f \in$ Homeo(X, d) and constants $0 < \alpha < 1$, $\delta > 0$, $0 < \varepsilon < \delta$ and $p \in \mathbb{N}$, we say that f has a $(\delta, \varepsilon, p, \alpha)$-pseudo-horseshoe if we may find a pairwise disjoint family of open subsets $(\mathcal{U}_i)_{0 \leqslant i \leqslant p-1}$ of X so that*

$$f(\mathcal{U}_i) \cap \mathcal{U}_{(i+1) \bmod p} \neq \emptyset \qquad \forall\, i$$

and a collection $(\phi_i)_{0 \leqslant i \leqslant p-1}$ of homeomorphisms

$$\phi_i \colon D_\delta^{\dim X} \subset \mathbb{R}^{\dim X} \quad \rightarrow \quad \mathcal{U}_i \subset M$$

satisfying, for every $0 \leqslant i \leqslant p-1$:

(1) *$(f \circ \phi_i)\left(D_\delta^{\dim X}\right) \subset \mathcal{U}_{(i+1) \bmod p}$.*

(2) *The map*

$$\psi_i = \phi_{(i+1) \bmod p}^{-1} \circ f \circ \phi_i \colon \quad D_\delta^{\dim X} \to \mathbb{R}^{\dim X}$$

has a pseudo-horseshoe of type $\left\lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \right\rfloor$ at scale δ connecting $x = 0$ to itself and such that:

(a) *There are families $\{V_{i,j}\}_j$ and $\{H_{i,j}\}_j$ of vertical and horizontal strips, respectively, with $j \in \{1, 2, \ldots, \lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor\}$, such that $H_{i,j} = \psi_i^{-1}\left(V_{i,j}\right)$.*

(b) *For every $j_1 \neq j_2 \in \{1, 2, \ldots, \lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor\}$ we have*

$$\min\left\{ \inf\left\{\|a - b\| \colon a \in V_{i,j_1},\ b \in V_{i,j_2}\right\}, \quad \inf\left\{\|z - w\| \colon z \in H_{i,j_1},\ w \in H_{i,j_2}\right\}\right\} > \varepsilon.$$

Regarding the parameters $(\delta, \varepsilon, p, \alpha)$ that identify the pseudo-horseshoe, we note that δ is a small scale determined by the size of the p domains and the charts so that item (1) of Definition 5.3 holds; ε is the scale at which a large number (which is inversely proportional to ε and involves α) of finite orbits is separated to comply with the demand (2) of Definition 5.3; and α is conditioned by the room in the manifold needed to build the convenient amount of ε-separated points.

Definition 5.4. *We say that f has a* coherent $(\delta, \varepsilon, p, \alpha)$-pseudo-horseshoe *if the pseudo-horseshoe satisfies the extra condition*

(3) *For every $0 \leqslant i \leqslant p-1$ and every $j_1 \neq j_2 \in \{1, 2, \ldots, \lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor\}$, the horizontal strip H_{i,j_1} crosses the vertical strip $V_{(i+1) \bmod p, j_2}$.*

By crossing we mean that there exists a foliation of each horizontal strip $H_{i,j} \subset D_\delta^{\dim X}$ by a family $\mathcal{C}_{i,j}$ of continuous curves $c \colon [0,1] \to H_{i,j}$ such that $\psi_i(c(0)) \in D_\delta^{k-1} \times \{-\delta\}$ and $\psi_i(c(1)) \in D_\delta^{k-1} \times \{\delta\}$.

There are two important main features of coherent $(\delta, \varepsilon, p, \alpha)$-pseudo-horseshoes. Firstly, $(\delta, \varepsilon, p, \alpha)$-pseudo-horseshoes associated to a homeomorphism f persist by C^0 perturbations of f. Secondly, if the $(\delta, \varepsilon, p, \alpha)$-pseudo-horseshoe is coherent, then the composition $\psi_{p-1} \circ \cdots \circ \psi_0$, on the suitable subdomain of $D_\delta^{\dim X}$, contains $\lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor^p$ horizontal strips which
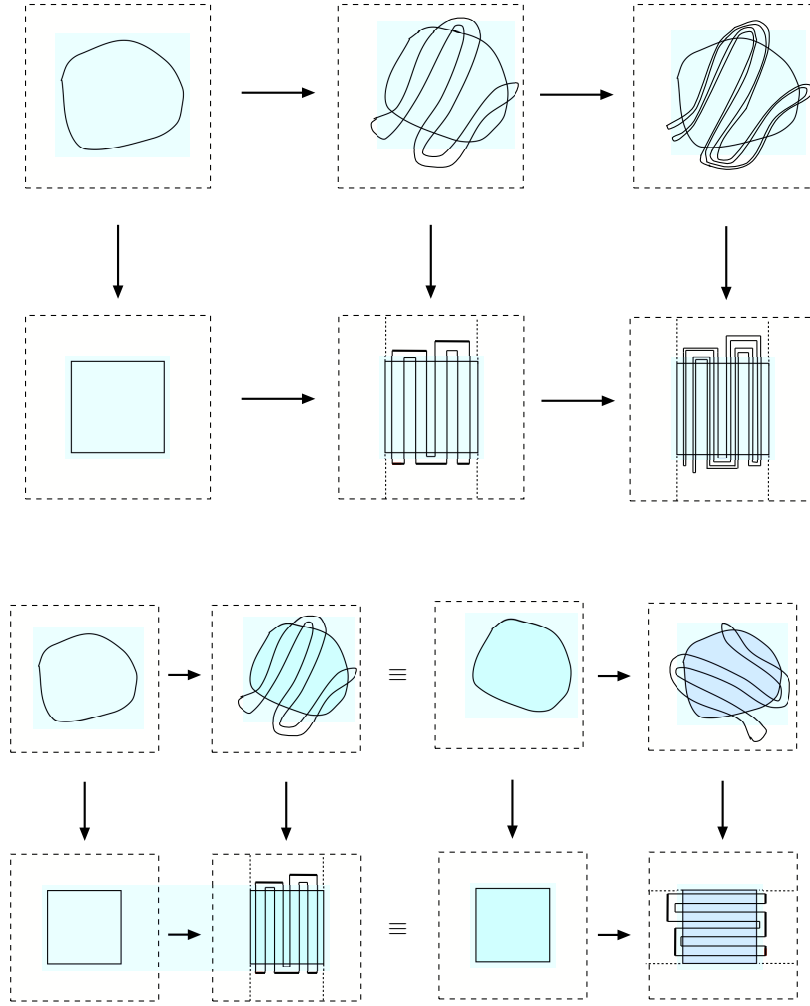
FIGURE 1. Illustration of a coherent (top) and a non-coherent (bottom) pseudo-horseshoe.

are mapped onto vertical strips, and are eventually ε-separated by f up to the pth iterate. In particular, any homeomorphism f which has a coherent $(\delta, \varepsilon, p, \alpha)$-pseudo-horseshoe also has a (p, ε)-separated set with $\lfloor \left(\frac{1}{\varepsilon} \right)^{\alpha \dim X} \rfloor^p$ elements (see Figure 1). It is precisely this type of characterization of the local behavior of vertical and horizontal strips in a neighborhood of a p-periodic point we will further select that compels the main differences between our argument and the ones used in [18, 20].

Remark 5.5. While vertical and horizontal strips in $\mathbb{R}^k$ can be defined in terms of Euclidean coordinates, the same notions on the manifold X are local and depend both on the dynamics of f and the smooth charts $(\phi_i)_{0 \leqslant i \leqslant p-1}$. On the manifold, the Intermediate Value Theorem ensures that $\widehat{H}_{i,j} := \phi_i(H_{i,j}) \subset \mathcal{U}_i$ crosses every vertical strip $\widehat{V}_{i,j} := f(\widehat{H}_{i,j})$ as well.

Remark 5.6. To estimate the metric mean dimension using local charts taking values in Euclidean coordinates, the separation scale in Euclidean coordinates (as in Definition 5.3) has to be preserved by charts. For this reason, we assume that the local charts $(\phi_i)_{0 \leqslant i \leqslant p-1}$ are bi-Lipschitz, and thereby we require the compact manifold to be smooth.

6. Separating sets

We start linking the existence of pseudo-horseshoes to the presence of big separating sets.

Proposition 6.1. *Assume that X is a smooth compact manifold. If $f \in \mathrm{Homeo}(X, d)$ then there exists $C > 1$ such that, if f has a coherent $(\delta, \varepsilon, p, \alpha)$-pseudo-horseshoe, then*

$$s\left(f, p\,\ell, C^{-1}\varepsilon\right) \geqslant \left(\left\lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \right\rfloor\right)^{p\,\ell} \qquad \forall \ell \in \mathbb{N}. \tag{6.1}$$

Proof. Let $N = \left\lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \right\rfloor$. By assumption, there are charts $(\phi_i)_{0 \leqslant i \leqslant p-1}$ such that each of the maps $\psi_i = \phi_{(i+1) \bmod p}^{-1} \circ f \circ \phi_i$ has an ε-separating pseudo-horseshoe of type N at scale δ. Moreover, the horizontal strips $(H_{i,j})_{j=1, \cdots, N}$ in the domain $D_\delta^{\dim X}$ of ψ_i are ε-separated and the same holds for the vertical strips $(V_{i,j})_{j=1, \cdots, N}$ in the image of ψ_i.

Define the horizontal and vertical strips, respectively, on the manifold X by

$$\widehat{H}_{i,j} := \phi_i(H_{i,j}) \qquad \text{and} \qquad \widehat{V}_{i,j} := f(\widehat{H}_{i,j}) = (f \circ \phi_i)(\widehat{H}_{i,j})$$

for $0 \leqslant i \leqslant p-1$ and $1 \leqslant j \leqslant N$. Observe that, by construction,

$$\phi_{(i+1) \bmod p}^{-1}\left(\widehat{V}_{i,j}\right) = \left(\phi_{(i+1) \bmod p}^{-1} \circ f \circ \phi_i\right)(H_{i,j}) = \psi_i(H_{i,j}) = V_{i,j}$$

is a vertical strip in the domain $D_\delta^{\dim X}$ of the pseudo-horseshoe ψ_i. Consider also the following non-empty compact subsets of X:

$$
\begin{aligned}
j \in \{1, \cdots, N\} &\quad \mapsto \quad \widehat{K}_{0,j} := \widehat{H}_{0,j} \\
j_1, j_2 \in \{1, \cdots, N\} &\quad \mapsto \quad \widehat{K}_{1,j_1,j_2} := f^{-1}(\widehat{V}_{0,j_1} \cap \widehat{H}_{1,j_2}) = f^{-1}(f(\widehat{K}_{0,j_1}) \cap \widehat{H}_{1,j_2}) \\
&\quad \vdots \qquad\qquad \vdots \\
j_1, j_2, \cdots, j_p \in \{1, \cdots, N\} &\quad \mapsto \quad \widehat{K}_{p-1,j_1,j_2,\ldots,j_p} := f^{-(p-1)}\left(f^{p-1}(\widehat{K}_{p-2,j_1,j_2,\ldots,j_{p-1}}) \cap \widehat{H}_{p-1,j_p}\right).
\end{aligned}
$$

Taking into account that X is a smooth manifold, we may assume that all the maps $\{\phi_i^{\pm 1} \colon 0 \leqslant i \leqslant p-1\}$ are Lipschitz with Lipschitz constant bounded by a uniform constant $C > 1$. In particular, by item 2(b) in Definition 5.3, there exist at least N points which are $(C^{-1}\varepsilon)$-separated by f in $\widehat{K}_{0,j}$.

Claim: *With the previous notation,*

$$
\begin{aligned}
(j_1, j_2) &\neq (J_1, J_2) \\
x &\in \widehat{K}_{1,j_1,j_2} \qquad \Rightarrow \qquad x \text{ and } y \text{ are } (2, C^{-1}\varepsilon)\text{-separated.} \\
y &\in \widehat{K}_{1,J_1,J_2}
\end{aligned}
$$

Indeed, as ϕ_2^{-1} is C-Lipschitz and $j_1 \neq J_1$, then

$$d_2(x, y) \geqslant d(f(x), f(y)) \geqslant \mathrm{dist}(\widehat{V}_{1,j_1}, \widehat{V}_{1,J_1}) \geqslant C^{-1}\mathrm{dist}(V_{1,j_1}, V_{1,J_1}) > C^{-1}\varepsilon$$

where

$$\operatorname{dist}(A, B) := \begin{cases} \inf\{\|a - b\| \colon a \in A, \, b \in B\}, & \text{if } A, B \subset \mathbb{R}^k \\[2mm] \inf\{d(a, b) \colon a \in A, \, b \in B\}, & \text{if } A, B \subset X. \end{cases}$$

On the other hand, if $j_1 = J_1$ and $j_2 \neq J_2$, then $f(x), f(y) \in \widehat{V}_{1,j_1}$ but lie in different horizontal strips; consequently, $f^2(x) \in \widehat{V}_{1,j_2}$ and $f^2(y) \in \widehat{V}_{1,J_2}$ and so

$$d_2(x, y) \geqslant d(f^2(x), f^2(y)) \geqslant C^{-1}\operatorname{dist}(\widehat{V}_{1,j_2}, \widehat{V}_{1,J_2}) > C^{-1}\varepsilon.$$

Recall that we have associated to $(j_1, j_2, \ldots, j_p) \in \{1, 2, \ldots, N\}^p$ the non-empty compact set

$$\widehat{K}_{p-1,j_1,j_2,\ldots,j_p} = f^{-(p-1)}\Big(f^{p-1}(\widehat{K}_{p-2,j_1,j_2,\ldots,j_{p-1}}) \cap \widehat{H}_{p-1,j_p}\Big)$$

and observe that, whenever $(j_1, j_2, \ldots, j_p) \neq (J_1, J_2, \ldots, J_p)$, one has

$$d_p(x, y) > C^{-1}\varepsilon \qquad \forall \, x \, \in \, \widehat{K}_{p-1,j_1,j_2,\ldots,j_p} \quad \forall \, y \, \in \, \widehat{K}_{p-1,J_1,J_2,\ldots,J_p}.$$

This proves that

$$s\left(f, p, C^{-1}\varepsilon\right) \geqslant N^p.$$

To show (6.1) for $\ell \in \mathbb{N} \setminus \{1\}$, we repeat ℓ times the previous recursive argument for the iterate f^p and the sets $\widehat{K}_{p-1,j_1,j_2,\ldots,j_p}$ instead of f and the sets $\widehat{K}_{0,j}$. $\qquad\square$

Corollary 6.2. *Under the assumptions of Proposition 6.1 one has*

$$\limsup_{n \to +\infty} \frac{1}{n} \log s\left(f, n, C^{-1}\varepsilon\right) \geqslant \alpha \dim X |\log \varepsilon|. \tag{6.2}$$

7. A C^0-PERTURBATION LEMMA ALONG ORBITS

We are interested in constructing coherent pseudo-horseshoes inside absorbing disks with small diameter. The argument depends on a finite number of C^0-perturbations of the initial dynamics on disjoint supports. Furthermore, the pseudo-horseshoes will be obtained inside a small neighborhood of an orbit associated to a suitable concatenation of homeomorphisms C^0-close to the initial dynamics.

Taking into account that X is a smooth compact boundaryless manifold, we may fix a finite atlas $\mathfrak{a}$ whose charts are bi-Lipschitz. If $r_{\mathfrak{a}} > 0$ denotes the Lebesgue covering number of the domains of the charts, up to a homothety we may assume that the image of every disk of radius $r_{\mathfrak{a}}$ in X contains a disk $D_1^{\dim X}(v) \subset \mathbb{R}^{\dim X}$ for some $v \in \mathbb{R}^{\dim X}$. Let $L > 0$ be an upper bound of the bi-Lipschitz constants of all the charts.

Proposition 7.1. *Given $\delta_0 > 0$ and $f \in \mathcal{H}$, there exist $p \in \mathbb{N}$ and $0 < \delta < \delta_0$ such that, for every $0 < \varepsilon \ll \delta$ and every $\alpha \in (0, 1)$, we may find $g \in \operatorname{Homeo}(X, d)$ satisfying:*

(a) *g has a coherent $(\delta, L\varepsilon, p, \alpha)$-pseudo-horseshoe;*

(b) *$D(g, f) \leqslant 2\delta_0$.*

Proof. We recall from Section 4 that C^0 generic homeomorphisms, belonging to the residual set $\mathcal{H}$ given by Lemma 4.1, have absorbing disks of arbitrarily small diameter which do not disappear under small C^0 perturbations. More precisely, given $\delta_0 > 0$, each $f \in \mathcal{H}$ has both a p-absorbing disk B with diameter smaller than δ_0, for some $p \in \mathbb{N}$, and an open neighborhood $\mathcal{W}_f$ in $\operatorname{Homeo}(X, d)$ such that for every $g \in \mathcal{W}_f$ the disk B is still p-absorbing for g. In what

follows we will always assume that $\mathcal{W}_f$ is inside the open ball in $(\mathrm{Homeo}(X,d), D)$ centered at f with diameter δ_0.

We start fixing coordinate systems. By Brouwer's fixed point theorem, f has a periodic point P of period p in B. For every $0 \leqslant i \leqslant p-1$, let ϕ_i be a bi-Lipschitz chart from $D_1^{\dim X} \subset \mathbb{R}^{\dim X}$ onto some open neighborhood of $f^i(P)$ contained in the disk $f^i(B)$ and such that $\phi_i((0, \cdots, 0)) = f^i(P)$. These charts are obtained by the composition of restrictions of the charts of the atlas $\mathfrak{a}$ and possible translations, which do not affect the value of L.

The next step is to choose $\delta > 0$ such that every C^0-perturbation $h \in \mathrm{Homeo}(X,d)$ of the identity whose support has diameter smaller than $3L\delta$ satisfies $h \circ f \in \mathcal{W}_f$, and so $D(h \circ f, f) \leqslant \delta_0$. The existence of such a δ is guaranteed by the uniform continuity of f^{-1}, since

$$D(h \circ f, f) = \max_{x \in X} \left\{ D(h(f(x)), f(x)), \, D(f^{-1}(h^{-1}(x)), f^{-1}(x)) \right\}.$$

We may assume, reducing δ if necessary, that the ball $B_{3L\delta}(f^i(P))$ is strictly contained in $f^i(B)$ for every $0 \leqslant i \leqslant p-1$. In fact, we may say more: the closeness in the uniform topology assures that the ball $B_{3L\delta}(g_i \circ \cdots \circ g_1(P))$ is contained in $f^i(B)$ for every g_i which is C^0-close enough to f and all $0 \leqslant i \leqslant p-1$.

Step 1: Let $N = \left\lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \right\rfloor$. Reducing δ if necessary, we may assume that the map

$$\phi_1^{-1} \circ f \circ \phi_0 : D_{3\delta}^{\dim X} \quad \rightarrow \quad D_1^{\dim X}$$

is well defined, fixes the origin and is a homeomorphism onto its image. A reasoning similar to the proof of [20, Proposition 1] provides a homeomorphism $\rho_1 \colon D_1^{\dim X} \to D_{2\delta}^{\dim X}$ isotopic to the identity and such that:

(1) $\left(\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0 \right) (D_\delta^{\dim X}) \subset \mathrm{int}(D_\delta^{\dim X - 1}) \times (-2\delta, \, 2\delta)$.

(2) For $i = 0, 1, \ldots, \left[\frac{N}{2}\right]$

$$\left(\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0 \right) \left(0, \ldots, 0, (-1 + \frac{4i}{N})\delta \right) \in \mathrm{int}(D_\delta^{\dim X - 1}) \times (-2\delta, -\delta).$$

(3) For $i = 0, 1, \ldots, \left[\frac{N-1}{2}\right]$

$$\left(\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0 \right) \left(0, \ldots, 0, (-1 + \frac{4i+2}{N})\delta \right) \in \mathrm{int}(D_\delta^{\dim X - 1}) \times (\delta, 2\delta).$$

By continuity of ρ_1, if $r > 0$ is small enough then the conditions (1)-(3) above imply that:

(1') $\left(\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0 \right) (D_r^{\dim X - 1} \times D_\delta^1) \subset \mathrm{int}(D_\delta^{\dim X - 1}) \times (-2\delta, \, 2\delta)$.

(2') For $i = 0, 1, \ldots, \left[\frac{N}{2}\right]$

$$\left(\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0 \right) (D_r^{\dim X - 1} \times \{(-1 + \frac{4i}{N})\delta\}) \subset \mathrm{int}(D_\delta^{\dim X - 1}) \times (-2\delta, -\delta).$$

(3') For $i = 0, 1, \ldots, \left[\frac{N-1}{2}\right]$

$$\left(\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0\right)\left(D_r^{\dim X - 1} \times \left\{(-1 + \frac{4i+2}{N})\delta\right\}\right) \subset \operatorname{int}(D_\delta^{\dim X - 1}) \times (\delta, 2\delta).$$
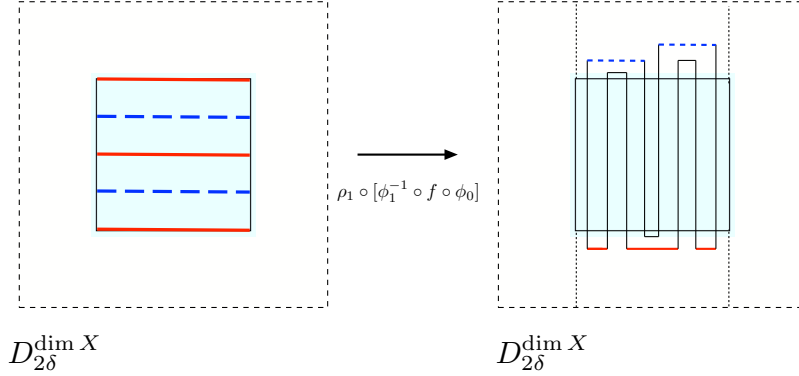

FIGURE 2. Illustration of the isotopy creating a pseudo-horseshoe.

Now, properties (1')-(3') imply that there exists a family $\mathcal{V} = (V_i)_{1 \leqslant i \leqslant N}$ of connected disjoint vertical strips such that

$$V_i = \left(\rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0\right)(K_i) \subset D_\delta^{\dim X}$$

for some connected subset

$$K_i \subset D_r^{\dim X - 1} \times \left[(-1 + \frac{2i}{N})\delta, \, (-1 + \frac{2i+2}{N})\delta\right].$$

The isotopic perturbation ρ_1 of the identity can be performed so that item (5) of Definition 5.1 holds, and we shall assume this is the case. See Figure 2. Making an extra C^0-perturbation supported in $D_\delta^{\dim X}$, if necessary, we ensure that the vertical strips V_i are ε-distant apart. This separability process is feasible because $\alpha \in (0, 1)$, so

$$N = \lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor < \left(\frac{1}{\varepsilon}\right)^{\dim X}.$$

Let $h_1 \in \operatorname{Homeo}(X, d)$ be a homeomorphism conveying ρ_1 to a neighborhood of $f(P)$ and such that

$$h_1(z) := \begin{cases} \phi_1 \circ \rho_1 \circ \phi_1^{-1}(z), & \text{if } z \in f(\phi_0(D_{2\delta}^{\dim X})) \\ \\ z, & \text{if } z \notin f(\phi_0(D_{3\delta}^{\dim X})). \end{cases}$$

By construction, the diameter of the support of h_1 is smaller than $3L\delta$. By the choice of δ, this ensures that the homeomorphism $f_1 = h_1 \circ f$ belongs to $\mathcal{W}_f$, and so $D(f_1, f) \leqslant \delta_0$.

Moreover, in $D_{2\delta}^{\dim X}$ one has

$$\phi_1^{-1} \circ f_1 \circ \phi_0 = \phi_1^{-1} \circ h_1 \circ f \circ \phi_0 = \rho_1 \circ \phi_1^{-1} \circ f \circ \phi_0$$

and, consequently, f_1 has a $L^{-1}\varepsilon$-separated pseudo-horseshoe of type N at scale δ connecting P to $f(P)$ (which may differ from $f_1(P)$). Thus, if $p = 1$, the proof of Proposition 7.1 is complete.

Step 2: Assume now that $p \geqslant 2$. By construction, the homeomorphism f_1 belongs to $\mathcal{W}_f$, and so $f_1(B)$ is a p-absorbing disk for f_1. Now, by a translation in the charts ϕ_1 and ϕ_2 in $\mathbb{R}^{\dim X}$, which does not change the Lipschitz constant L, we assume without loss of generality that $\phi_1(0, 0, \ldots, 0) = f(P)$ and $\phi_2(0, 0, \ldots, 0) = f_1(f(P))$. Therefore, $\left(\phi_2 \circ f_1 \circ \phi_1^{-1}\right)(0, 0, \ldots, 0) = (0, 0, \ldots, 0)$.

Proceeding as in Step 1, we find homeomorphisms $\rho_2 : D_1^{\dim X} \to D_{2\delta}^{\dim X}$ and

$$h_2(z) := \begin{cases} \phi_2 \circ \rho_2 \circ \phi_2^{-1}(z), & \text{if } z \in f_1(\phi_1(D_{2\delta}^{\dim X})) \\[2mm] z, & \text{if } z \notin f_1(\phi_1(D_{3\delta}^{\dim X})) \end{cases}$$

such that

- the support of h_2 is contained in a ball with diameter $3L\delta$ centered at $f_1(f(P))$;

- $f_2 = h_2 \circ f_1$ has a $L^{-1}\varepsilon$-separated pseudo-horseshoe of type N at scale δ connecting $f(P)$ to $f_1(f(P))$.

The support of the perturbation h_2 is disjoint from the one of the homeomorphism h_1 and has diameter smaller that $3L\delta$; thus $f_2 \in \mathcal{W}_f$, and so $D(f_2, f) \leqslant \delta_0$.

Let us summarize what we have obtained so far. Under the two previous perturbations we have built a homeomorphism $f_2 \in \mathcal{W}_f$ exhibiting two pseudo-horseshoes, one connecting P to $f(P)$ and another connecting $f(P)$ to $f_1(f(P))$. Since these perturbations are performed in Euclidean coordinates (using either the charts ϕ_i or their modifications by rigid translations, which do not change the notions of horizontal and vertical strip), and then conveyed to the manifold X using the fixed charts, we are sure that these pseudo-horseshoes are coherent.

Step 3: The recursive argument. Set $f_0 = f$. Using the previous argument recursively we obtain homeomorphisms $\{f_0, f_1, f_2, \ldots, f_{p-1}\}$ such that $f_i \in \mathcal{W}_f$, so clearly $D(f_i, f) \leqslant \delta_0$ for every $1 \leqslant i \leqslant p - 1$; besides, f_{p-1} has $L^{-1}\varepsilon$-separated pseudo-horseshoes connecting the successive points of the finite piece of the random orbit

$$\{P, \, f_0(P), \, (f_1 \circ f_0)(P), \, (f_2 \circ f_1 \circ f_0)(P), \ldots, (f_{p-1} \circ \cdots \circ f_2 \circ f_1 \circ f_0)(P)\}.$$

If the points $(f_{p-1} \circ \cdots \circ f_2 \circ f_1 \circ f_0)(P)$ and P are distinct, to end the proof of Proposition 7.1 we need an extra perturbation to identify them. This last perturbation is performed in the interior of the disk B, so the resulting homeomorphism g satisfies $D(g, f_{p-1}) \leqslant \delta_0$ and $g = f$ in $X \setminus \bigcup_{0 \leqslant j \leqslant p-1} f^j(B)$. Therefore, $D(g, f) \leqslant D(g, f_{p-1}) + D(f_{p-1}, f) \leqslant 2\delta_0$ and g has a $L^{-1}\varepsilon$-separated pseudo-horseshoe of type N at scale δ connecting the point P to itself. $\qquad\square$

Remark 7.2. For the construction of the pseudo-horseshoes it is essential that α is strictly smaller than 1. Indeed, only if $0 < \alpha < 1$ are we able to create $\lfloor (1/\varepsilon)^{\alpha \dim X} \rfloor$ points that are

ε-separated inside a ball with diameter 2δ, since this obliges $\varepsilon > 0$ to satisfy the condition $\sqrt[\dim X]{\lfloor (1/\varepsilon)^{\alpha \dim X} \rfloor}\, \varepsilon < 4\delta$ or, equivalently, $0 < \varepsilon < \sqrt[1-\alpha]{4\delta}$.

8. Proof of Theorem A

Firstly, we note that $\overline{\mathrm{mdim}_M}(X, f, d) \leqslant \dim X$ for every $f \in \mathrm{Homeo}(X, d)$ (cf. [18, §5]). We are left to prove the converse inequality in a residual subset of $\mathrm{Homeo}(X, d)$.

Fix a strictly decreasing sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ in the interval $(0, 1)$ which converges to zero. For any $\alpha \in (0, 1)$ and $k \in \mathbb{N}$, consider the C^0-open set $\mathcal{O}(\varepsilon_k, \alpha)$ of the homeomorphisms $g \in \mathrm{Homeo}(X, d)$ such that g has a coherent $(\delta, L\varepsilon_k, p, \alpha)$-pseudo horseshoe, for some $\delta > 0$ and $p \in \mathbb{N}$ and $L > 0$. Observe that, given $\alpha \in (0, 1)$ and $K \in \mathbb{N}$, the set

$$\mathcal{O}_K(\alpha) := \bigcup_{\substack{k \in \mathbb{N} \\ k \geqslant K}} \mathcal{O}(\varepsilon_k, \alpha)$$

is C^0-open and, by Proposition 7.1, nonempty. Besides, it is C^0-dense in $\mathrm{Homeo}(X, d)$ since the residual $\mathcal{H}$ (cf. Lemma 4.1) is C^0-dense in the Baire space $\mathrm{Homeo}(X, d)$ and Proposition 7.1 holds for every $f \in \mathcal{H}$. Define

$$\mathfrak{R} := \bigcap_{\alpha \in (0,1) \cap \mathbb{Q}} \bigcap_{K \in \mathbb{N}} \mathcal{O}_K(\alpha).$$

This is a C^0-Baire residual subset of $\mathrm{Homeo}(X, d)$ and

Lemma 8.1. $\overline{\mathrm{mdim}_M}(X, g, d) = \dim X$ *for every* $g \in \mathfrak{R}$.

Proof. Take $g \in \mathfrak{R}$. Given a rational number $\alpha \in (0, 1)$ and a positive integer K, the homeomorphism g has a coherent $(\delta, L\varepsilon_{j_K}, p, \alpha)$-pseudo-horseshoe for some $j_K \geqslant K$, $\delta > 0$, $p \in \mathbb{N}$ and $L > 0$. Therefore, by Corollary 6.2,

$$\limsup_{n \to +\infty} \frac{1}{n} \log s(g, n, L\varepsilon_{j_K}) \geqslant \alpha \dim X \,|\log \varepsilon_{j_K}|$$

for a subsequence $(\varepsilon_{j_K})_{K \in \mathbb{N}}$ of $(\varepsilon_k)_{k \in \mathbb{N}}$. Thus,

$$\overline{\mathrm{mdim}_M}(X, g, d) \geqslant \limsup_{k \to +\infty} \frac{\limsup_{n \to +\infty} \frac{1}{n} \log s(g, n, L\varepsilon_k)}{|\log \varepsilon_k|} \geqslant \alpha \dim X.$$

As $\alpha \in (0, 1) \cap \mathbb{Q}$ is arbitrary, Theorem A is proved. $\square$

Remark 8.2. The assumption that the manifold X has no boundary is not essential. Allowing boundary points we need to alter the argument to prove Proposition 7.1 on two instances. Firstly, absorbing disks must be considered with respect to the induced topology. Secondly, the role of Brouwer fixed point theorem is transferred to the C^0-closing lemma, which also ensures the existence of a periodic point. In case this periodic point lies at the boundary of the manifold, an additional C^0-arbitrarily small perturbation yields a close homeomorphism with an interior periodic point. Accordingly, we are obliged to change the closeness estimate on the statement of Proposition 7.1, by replacing $2\delta_0$ by $3\delta_0$.

9. Proof of Theorem B

Let X be a compact smooth boundaryless manifold with dimension strictly greater than one and d be a metric compatible with the smooth structure of X. It is known that there exist C^0-Baire generic subsets $\mathcal{R}_i \subset \mathrm{Homeo}(X, d)$ $(i = 1, 2)$ such that:

(1) The closure of the set $\mathrm{Per}(f)$ of periodic points coincides with both the non-wandering set $\Omega(f)$ and the chain recurrent set $CR(f)$ of every homeomorphism $f \in \mathcal{R}_1$ (cf. [8]).

(2) $\overline{\mathrm{mdim}_M}(X, f, d) = \dim X$ for every $f \in \mathcal{R}_2$ (by Theorem A).

Therefore, taking into account that $\overline{\mathrm{mdim}_M}(X, g, d) = \overline{\mathrm{mdim}_M}(\Omega(g), g\mid_{\Omega(g)}, d)$ for all homeomorphisms $g \in \mathrm{Homeo}(X, d)$, we obtain

$$\overline{\mathrm{mdim}_M}(\overline{\mathrm{Per}(f)}, f\mid_{\overline{\mathrm{Per}(f)}}, d) = \overline{\mathrm{mdim}_M}(\Omega(f), f\mid_{\Omega(f)}, d) = \dim X$$

for every homeomorphism f in the C^0-Baire generic subset $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$. Now, noticing that $\overline{\mathrm{mdim}_M}(\Omega(f), f\mid_{\Omega(f)}, d) \leqslant \overline{\dim_B}(\Omega(f)) \leqslant \dim X$ we conclude that

$$\overline{\dim_B}(\overline{Per(f)}) = \overline{\dim_B}(\Omega(f)) = \dim X \qquad \forall f \in \mathcal{R}.$$

The theorem now follows from the fact that the upper box dimensions of a set and its closure coincide (cf. [2, Proposition 3.4]).

10. Proof of Theorem C

We begin presenting the proof for the case of the Euclidean distance. Afterwards we will stress the necessary adaptations for more general metrics.

10.1. **Euclidean metric.** The first step of the proof is the construction of piecewise affine continuous models with any prescribed metric mean dimension. To complete the argument we use surgery in the space of continuous maps on the interval.

10.1.1. *Piecewise affine models.* Recall that d stands for the Euclidean metric on $[0, 1]$ and that $C^0([0, 1], d)$ is the space of continuous maps on the interval $[0, 1]$ with the uniform metric. We start describing examples in $C^0([0, 1], d)$ with metric mean dimension equal to any prescribed value $\beta \in [0, 1]$.

Proposition 10.1. *For every $\beta \in [0, 1]$ there exists a piecewise affine function $f_\beta \in C^0([0, 1], d)$ such that $f_\beta(0) = 0$, $f_\beta(1) = 1$ and $\mathrm{mdim}_M([0, 1], f_\beta, d) = \beta$.*

Proof. If $\beta = 0$, the assertion is trivial: take for instance $f_\beta =$ identity map. Now, fix $\beta \in (0, 1]$, take $a_0 = a_{-1} = 1$ and consider a sequence $(a_k)_{k \in \mathbb{N}}$ of numbers in $(0, 1)$ strictly decreasing to zero. For any $k \geqslant 0$, consider the interval

$$J_k = [a_{2k+1}, a_{2k}]$$

denote by γ_k the diameter $a_{2k} - a_{2k+1}$ of J_k and fix a point b_{k+1} of the interval (a_{2k+2}, a_{2k+1}). Let $G_k := [a_{2k+2}, a_{2k+1}]$ be the closed interval gap between J_k and J_{k+1}.

On each interval G_k, define f_β as a continuous piecewise affine map which maps the interval $[a_{2k+2}, a_{2k+1}]$ onto itself, fixes the boundary points and has an attracting fixed point at b_k whose topological basin of attraction contains all points in the interval (a_{2k+2}, a_{2k+1}). By construction the set $\bigcup_{k \geqslant 0} G_k$ is f_β-invariant, restricted to which f_β has zero topological entropy; hence this compact set will not contribute to the metric mean dimension of f_β.

We now define the map f_β on the set $\bigcup_{k \geqslant 0} J_k$. Let $(\ell_k)_{k \geqslant 0}$ be a strictly increasing sequence of positive odd integers such that $\ell_0 \geqslant 3$. Fix $k \geqslant 0$ and subdivide the interval J_k in ℓ_k sub-intervals $(J_{k,i})_{1 \leqslant i \leqslant \ell_k}$ of equal size γ_k/ℓ_k, where $\gamma_k = a_{2k} - a_{2k+1}$. For each $0 \leqslant i < \ell_k$, set

$$c_{k,i} := a_{2k+1} + i\,\frac{\gamma_k}{\ell_k}.$$

Afterwards define

$$f_\beta(x) := \begin{cases} \frac{\ell_k}{\gamma_k}\,(x - c_{k,i}) + a_{2k+1}, & \text{if } x \in J_{k,1+4i}, \quad 0 \leqslant i \leqslant i_k \\[2mm] -\frac{\ell_k}{\gamma_k}\,(x - c_{k,i}) + a_{2k}, & \text{if } x \in J_{k,3+4i}, \quad 0 \leqslant i \leqslant i_k - 1 \end{cases} \tag{10.1}$$

where $1 \leqslant i_k \leqslant \ell_k$ is given by

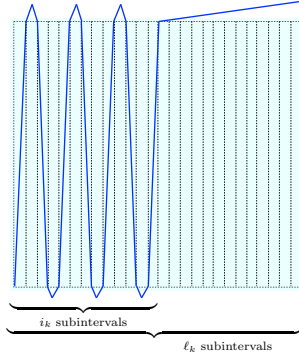$$i_k := \left\lfloor \left(\frac{\ell_k}{\gamma_k}\right)^\beta \right\rfloor. \tag{10.2}$$

FIGURE 3. Selection procedure of piecewise linear components of f_β.

In rough terms, we have defined f_β on each interval J_k as a piecewise affine self map taking values on $G_k \cup J_k \cup G_{k-1}$ in such a way that it has a metric mean dimension close to β at a certain scale. See Figure 3. Notice that this construction is entirely analogous to the generation process of a $(\delta, \varepsilon, p, \alpha)$-pseudo-horseshoe in Section 5, taking $\delta = \gamma_k$, $\varepsilon = \gamma_k/\ell_k$, $p = 1$ and $\alpha = \beta$. In particular, having such a pseudo-horseshoe is a C^0-open condition.

In the remaining sets

$$\left(\bigcup_{0 \leqslant i < \frac{4i_k-1}{4}} J_{k,2+4i} \right) \bigcup \left(\bigcup_{1+4i_k < i \leqslant \ell_k} J_{k,i} \right) \bigcup \left(\bigcup_{0 \leqslant i < \frac{4i_k-3}{2}} J_{k,4+4i} \right) \tag{10.3}$$

we define f_β as a piecewise affine map preserving the boundary points in such a way that the sets (10.3) are mapped inside the regions G_{k-1} and G_k, respectively (see e.g. Figure 4). By construction, the map f_β is continuous, piecewise affine and fixes the points 0 and 1.

Claim: *If the sequences $(a_k)_{k \in \mathbb{N}}$ and $(\ell_k)_{k \in \mathbb{N}}$ satisfy the additional condition*

$$a_{2k} = \frac{a_{2k-2} - a_{2k-1}}{\ell_{k-1}} \qquad \forall k \in \mathbb{N} \tag{10.4}$$

then $\mathrm{mdim}_M\left([0,1], f_\beta, d\right) = \beta$.

Indeed, given $\varepsilon > 0$ smaller than $\frac{a_2 - a_3}{\ell_1}$, let $k = k(\varepsilon) \in \mathbb{N}$ be the largest positive integer such that

$$\varepsilon < \varepsilon_k := \frac{a_{2k} - a_{2k+1}}{\ell_k}.$$

Thus $\varepsilon_{k+1} \leqslant \varepsilon < \varepsilon_k$, and so the assumption (10.4) ensures that $a_{2k+2} = \varepsilon_{k+1} \leqslant \varepsilon$. Therefore,

$$f_\beta([0, a_{2k+2}]) \subset [0, a_{2k+2}] \subset [0, \varepsilon]$$

and, as $\varepsilon < \varepsilon_k$, for every $n \in \mathbb{N}$ one has

$$s(f_\beta, n, \varepsilon) \geqslant s(f_{\beta | J_k^\infty}, n, \varepsilon) \geqslant s(f_{\beta | J_k^\infty}, n, \varepsilon_k) = \left\lfloor \left(\frac{\ell_k}{\gamma_k}\right)^\beta \right\rfloor^n = \left\lfloor \frac{1}{\varepsilon_k^\beta} \right\rfloor^n \geqslant \left\lfloor \frac{1}{\varepsilon^\beta} \right\rfloor^n \qquad (10.5)$$

where $J_k^\infty := \bigcap_{i \geq 0} f_\beta^{-i}(J_k)$. Consequently, as n is arbitrary

$$\underline{\mathrm{mdim}}_M([0, 1], f_\beta, d) \geqslant \beta.$$

Before proceeding, we note that the sequences $(a_k)_{k \in \mathbb{N}}$ and $(\ell_k)_{k \in \mathbb{N}}$ may be chosen complying with the condition (10.4).
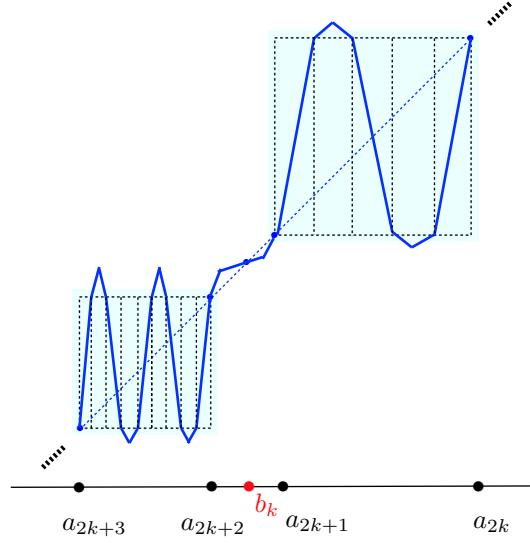

FIGURE 4. Local construction of an attractor between two consecutive pseudo-horseshoes.

On the other hand, by construction the derivative of f_β at the points of the intersection $J_k \cap f_\beta^{-1}(J_k) \cap \ldots \cap f_\beta^{-(n-1)}(J_k)$ is constant and equal to γ_k / ℓ_k. Thus, this set is formed by $(\frac{\gamma_k}{\ell_k})^n$ disjoint and equally spaced subintervals. Moreover, any such subinterval is the $(n, \varepsilon_k/2)$-dynamical ball associated to its mid-point. Therefore, every (n, ε)-dynamical ball of f_β which is contained in an (n, ε_k)-dynamical ball inside $J_k \cap f_\beta^{-1}(J_k) \cap \ldots \cap f_\beta^{-(n-1)}(J_k)$ has diameter smaller or equal to $\varepsilon (\frac{\gamma_k}{\ell_k})^n$ (actually equal when dynamical balls do not intersect

the boundary of the connected components of $J_k \cap f_\beta^{-1}(J_k) \cap \ldots \cap f_\beta^{-(n-1)}(J_k))$. This implies in particular that

$$s(f_{\beta|J_k^\infty}, n, \varepsilon) \leqslant s(f_{\beta|J_k^\infty}, n, \varepsilon_k) \cdot \frac{\varepsilon_k \left(\frac{\gamma_k}{\ell_k}\right)^n}{\varepsilon \left(\frac{\gamma_k}{\ell_k}\right)^n} = s(f_{\beta|J_k^\infty}, n, \varepsilon_k) \cdot \frac{\varepsilon_k}{\varepsilon} = \left\lfloor \frac{1}{\varepsilon_k^\beta} \right\rfloor^n \cdot \frac{\varepsilon_k}{\varepsilon} \qquad (10.6)$$

and so

$$\limsup_{n \to \infty} \frac{1}{n} \log s(f_{\beta|J_k^\infty}, n, \varepsilon) \leqslant \beta |\log \varepsilon_k| \leqslant \beta |\log \varepsilon|.$$

Furthermore, if $1 \leqslant t < k$ then (10.6) also implies that

$$s(f_{\beta|J_t^\infty}, n, \varepsilon) \leqslant s(f_{\beta|J_t^\infty}, n, \varepsilon_t) \cdot \frac{\varepsilon_t}{\varepsilon}$$

which yields

$$\limsup_{n \to +\infty} \frac{1}{n} \log s(f_{\beta|J_t^\infty}, n, \varepsilon) \leqslant \beta |\log \varepsilon_t| \leqslant \beta |\log \varepsilon|.$$

Since ε may be taken arbitrarily small, we conclude that

$$\overline{\mathrm{mdim}_M}\,([0,1], f_\beta, d) \leqslant \beta.$$

Thus, $\mathrm{mdim}_M\,([0,1], f_\beta, d) = \beta$. This completes the proofs of the claim and of the proposition. $\qquad\square$

10.1.2. *Level sets of the metric mean dimension.* Let us now show that for every $\beta \in [0,1]$ there exists a C^0-dense subset $\mathcal{D}_\beta \subset C^0([0,1], d)$ such that $\mathrm{mdim}_M\,([0,1], f, d) = \beta$ for every $f \in \mathcal{D}_\beta$.

When $\beta = 0$ it is enough to take $\mathcal{D}_0 = C^1([0,1])$, which is a C^0-dense subset of $C^0([0,1], d)$. Indeed, for any C^1 interval map f one has $h_{\mathrm{top}}(f) \leqslant \log \|f'\|_\infty < +\infty$ and, consequently, $\mathrm{mdim}_M\,([0,1], f, d) = 0$.

Fix $0 < \beta \leqslant 1$ and $f \in C^0([0,1], d)$, and let $\varepsilon > 0$ be arbitrary. We claim that there exists $h \in C^0([0,1], d)$ such that $D(f, h) < \varepsilon$ and $\mathrm{mdim}_M\,([0,1], h, d) = \beta$. The proof is done through a local perturbation starting at the space of C^1-interval maps as we will explain. Firstly, by the denseness of the C^1-interval maps we may choose $h_1 \in C^1([0,1])$ so that $D(h_1, h) < \frac{\varepsilon}{3}$. Secondly, if P denotes a fixed point of h_1 (which surely exists), let $h_2 \in C^0([0,1], d)$ be such that $D(h_2, h_1) < \frac{\varepsilon}{3}$ and whose set of fixed points in a small neighborhood of P consists of an interval J centered at P. This C^0-perturbation can be performed in such a way that h_2 is C^1 at all points except, possibly, the extreme points of J. Finally, if $\widehat{J} \subsetneq \tilde{J} \subsetneq J$ and $\widehat{J}, \tilde{J}$ are intervals of diameter smaller than $\varepsilon/3$, we take a C^1 map χ such that $\chi \equiv 1$ on $\widehat{J}$ and $\chi \equiv 0$ on $[0,1] \setminus \tilde{J}$.

Let T_λ denote the homothety of parameter $\lambda \in (0,1)$ and $|\widehat{J}|$ stand for the diameter of the interval $\widehat{J}$. Since $\{\chi, 1 - \chi\}$ is a partition of unity, the map

$$h_3 := h_{3,\beta} = (1 - \chi) \cdot h_2 + \chi \cdot T_{|\widehat{J}|} \circ f_\beta \circ T_{|\widehat{J}|^{-1}} \qquad (10.7)$$

is continuous, coincides with h_2 on $[0,1] \setminus \tilde{J}$ and is linearly conjugate to f_β on the interval $\widehat{J}$. Moreover, by the uniform continuity of h_2 we can choose h_3 so that $D(h_3, h_2) < \frac{\varepsilon}{3}$ provided that $\widehat{J}, \tilde{J}$ are small enough. This guarantees that $D(h_3, h) < \varepsilon$ and, since all maps in the combination (10.7) but f_β are smooth (except possibly at two points), then

$$\mathrm{mdim}_M\,([0,1], h_3, d) = \mathrm{mdim}_M\,([0,1], f_\beta, d) = \beta.$$

This ends the proof of the first part of Theorem C.

Regarding the last statement of Theorem C, we might argue as in the proof of Theorem A. However, as we have established that $\mathcal{D}_1$ is C^0-dense in $C^0([0,1], d)$, the reasoning can be simplified (observe that the case $\alpha = 1$ was not considered in Theorem A).

Take a strictly decreasing sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ in the interval $(0, 1)$ converging to zero. Given $K \in \mathbb{N}$, consider the non-empty C^0-open set

$$\mathcal{D}_K = \Big\{ g \in C^0([0,1]) \colon \ g \text{ has a } (\gamma, \varepsilon_k, 1, 1)\text{-pseudo-horseshoe, for some } k \geqslant K \text{ and } \gamma > 0 \Big\}.$$

Notice that $\mathcal{D}_K$ is C^0-dense in $C^0([0,1], d)$ by the first part of Theorem C. Define

$$\mathfrak{D} := \bigcap_{K \in \mathbb{N}} \mathcal{D}_K.$$

This is a C^0-Baire residual subset of $C^0([0,1], d)$. Besides, $\overline{\mathrm{mdim}_M}([0,1], g, d) = 1$ for every $g \in \mathfrak{D}$. Indeed, given a positive integer K, such a map g has a $(\gamma_{j_K}, \varepsilon_{j_K}, 1, 1)$-pseudo-horseshoe for some $j_K \geqslant K$ and $\gamma_{j_K} > 0$. Therefore, an estimate analogous to (10.5) indicates that, for a subsequence $(\varepsilon_{j_K})_{K \in \mathbb{N}}$ of $(\varepsilon_k)_{k \in \mathbb{N}}$, one has

$$\limsup_{n \to +\infty} \frac{1}{n} \log s(g, n, \varepsilon_{j_K}) \geqslant |\log \varepsilon_{j_K}|.$$

Thus,

$$\overline{\mathrm{mdim}_M}([0,1], g, d) \geqslant \limsup_{k \to +\infty} \frac{\limsup_{n \to +\infty} \frac{1}{n} \log s(g, n, \varepsilon_k)}{|\log \varepsilon_k|} \geqslant 1$$

and so $\overline{\mathrm{mdim}_M}([0,1], g, d) = 1$.

10.2. **Metrics in $\mathcal{M}_E$.** Let ρ be any metric on $[0,1]$ topologically equivalent to d, and $C^0([0,1], \rho)$ denote the space of continuous endomorphisms of the interval $([0,1], \rho)$. The argument used in the case of the Euclidean distance does not apply directly due to the possible absence of homogeneity. More precisely, the surgery described in equation (10.7) may not preserve the upper metric mean dimension of the piecewise affine model. Notwithstanding, we will show that a suitable C^0-local perturbation replaces the previous role of the homothety.

10.2.1. *Piecewise affine models.* The construction is similar to the one in Subsection 10.1.1, in local domains.

Proposition 10.2. *For every closed interval $I \subset [0,1]$ there exists a piecewise affine function $g_1 \in C^0(I, \rho)$ such that g_1 fixes the boundary points of I and $\mathrm{mdim}_M(I, g_1, \rho\mid_{I \times I}) = 1$.*

Proof. The proof is similar to the one of Proposition 10.1. Write $I = [a, b]$. Take $a_0 = a_{-1} = a$ and consider a sequence $(a_k)_{k \in \mathbb{N}}$ of numbers in I strictly decreasing to a. For any $k \geqslant 0$, consider the interval

$$J_k = [a_{2k+1}, \, a_{2k}]$$

denote by γ_k the d-diameter of J_k and fix a point b_{k+1} of the interval (a_{2k+2}, a_{2k+1}). Let $G_k := [a_{2k+2}, a_{2k+1}]$ be the closed interval gap between J_k and J_{k+1}. As before, on each interval G_k, define g_1 as a continuous piecewise affine map which maps the interval $[a_{2k+2}, a_{2k+1}]$ onto itself, fixes the boundary points and has an attracting fixed point at b_k whose topological basin of attraction contains all points in the interval (a_{2k+2}, a_{2k+1}). The set $\bigcup_{k \geqslant 0} G_k$ does not contribute to the metric mean dimension of g_1.

We now define the map g_1 on the set $\bigcup_{k\geqslant 0} J_k$. Let $(\ell_k)_{k\geqslant 0}$ be a strictly increasing sequence of positive odd integers such that $\ell_0 \geqslant 3$. Fix $k \geqslant 0$ and subdivide the interval J_k in ℓ_k sub-intervals $(J_{k,i})_{1\leqslant i\leqslant \ell_k}$ of equal d-length, each of them with length γ_k/ℓ_k. As before, it is enough to define g_1 affine increasing on intervals of the form $J_{k,1+4i}, 0 \leqslant i \leqslant i_k$, and affine and decreasing on intervals of the form $J_{k,3+4i}, 0 \leqslant i \leqslant i_k - 1$, where

$$i_k := \left\lfloor \frac{\ell_k}{\gamma_k} \right\rfloor. \tag{10.8}$$

In the remaining sets

$$\left(\bigcup_{0\leqslant i < \frac{4i_k-1}{4}} J_{k,2+4i} \right) \bigcup \left(\bigcup_{1+4i_k < i \leqslant \ell_k} J_{k,i} \right) \bigcup \left(\bigcup_{0\leqslant i < \frac{4i_k-3}{2}} J_{k,4+4i} \right) \tag{10.9}$$

we define g_1 as a piecewise affine map preserving the boundary points of I in such a way that the sets (10.9) are mapped inside the regions G_{k-1} and G_k, respectively. By construction, the map g_1 is continuous, piecewise affine and fixes the points a and b.

We claim that if the sequences $(a_k)_{k\in\mathbb{N}}$ and $(\ell_k)_{k\in\mathbb{N}}$ satisfy the additional condition

$$a_{2k} = \frac{\mathrm{diam}_\rho([a_{2k-2}, a_{2k-1}])}{\ell_{k-1}} \qquad \forall k \in \mathbb{N} \tag{10.10}$$

then $\mathrm{mdim}_M\,(I, g_1, \rho) = 1$. Indeed, an estimate identical to (10.5) (which at this point involves a counting argument and does not make use of the metric) ensures that

$$\underline{\mathrm{mdim}_M}\,(I, g_1, \rho\,|_{I\times I}) \geqslant 1.$$

This proves the proposition. $\qquad\qquad\square$

10.2.2. *Maximal level set of the upper metric mean dimension.* In order to complete the proof of the theorem, as before it is enough to show that there exists a C^0-dense subset $\mathcal{D}_1 \subset C^0([0,1], \rho)$ such that $\overline{\mathrm{mdim}_M}\,([0,1], f, \rho) = 1$ for every $f \in \mathcal{D}_1$. Take any continuous map f of the interval. Since it has a fixed point, by a C^0-small perturbation we produce a C^0-close continuous map exhibiting an interval $I \subset [0,1]$ of fixed points. Let $g_1 : I \to I$ be given by Proposition 10.2 with $\mathrm{mdim}_M\,(I, g_1, \rho\,|_{I\times I}) = 1$. The map $f_1 \in C^0([0,1], \rho)$ given by

$$f_1(x) = \begin{cases} g_1(x) & \text{if } x \in I \\ f(x) & \text{otherwise} \end{cases}$$

is C^0-close to f and satisfies $\mathrm{mdim}_M\,([0,1], f_1, \rho) = 1$. This completes the proof of Theorem C.

Remark 10.3. A final word concerning the different range of the results obtained for homeomorphisms on high-dimensional manifolds and the ones established for continuous maps of the interval. While the use of a metric compatible with the smooth structure on the manifold may turn to be not essential, the fact is that our proof of Theorem A depends on the property that the cardinality of an ε-separated set on the cube $[0,1]^{\dim X}$ is approximately equal to $(\frac{1}{\varepsilon})^{\dim X}$, an estimate valid for the Euclidean metric since it is strongly equivalent to the metric given by the maximum of the coordinates.

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