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# ΣΥΜΜΕΤΡΙΑ - SYMMETRIA - SYMMETRY: ZOO- AND ETHNO-MATHEMATICS, BIRTH OF THE TERM IN GREECE, SURVIVAL IN THE THEORY OF RCHITECTURE, REBIRTH IN ART AND SCIENCE, AND THE FUTURE TASKS, WITH OUTLOOKS TO "PORTUGUESE SYMMETRIES" AND CAMÕES, AND TO THE GOLDEN SECTION WITH LESS "GOLD" AND WITHOUT LEONARDO.

## Celebrating the 33 years of SIS-Symmetry, its 12th Congress, which is the 5th European

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***Abstract:*** *The idea of the triennial congresses "Symmetry: Art and Science" is going back to the late 1980s, with a desire that the "Two Cultures" should have some bridges. Our goal was to provide a regular forum where representatives of different disciplines from East and West may come together for interdisciplinary exchange of ideas and shaping new forms of co-operations. We reached already five continents, and now we pay tribute to "Portuguese symmetries". We survey the development of the concept symmetry, starting from zoo- and ethno-mathematics (Platonic solids). We also discuss the related terminology of συμμετρία – symmetria – symmetry. We suggest that the Greek concept of συμμετρία (commensurability) vs. ἀσυμμετρία (incommensurability) played a relevant role at the beginnings of mathematics, then reached aesthetics (due proportion), and became a central concept in Vitruvius' theory of architecture. The term also appeared in the Greek Bible, but later disappeared in the translations. We present our view on how symmetry "returned" to the mainstream of art and then of science and gained a new meaning-family (bilateral, rotational, translatory symmetries, and invariances). Parallel with this, we present an "anti-golden-sectionism" because in some "well-known" cases the Golden Section (GS) had no role at all. On the other hand, the GS gained some importance in other cases that are less known. We discuss the Pacioli – Leonardo connection, adding that "Leonardo's code" in the case of his reconstruction of the "Vitruvian man" was not the GS, but perhaps the simple ratio 1/3. Finally, we turn to future tasks, including interdisciplinary education.*

Keywords: Symmetry, Greek Geometry, Septuaginta, Vitruvius and Leonardo, Anti-Golden-Sectionism.

## INTRODUCTION

Do we really have "Two Cultures", art and the humanities vs. science and technology, as was pointed out by Sir Charles P. Snow in 1959? What should we do for the unity of culture? This fact inspired me to organize two local conferences entitled "Symmetry in a Cultural Context" in 1987 and 1988 at Arizona State University, then the first international congress and exhibitions in 1989 in Budapest, Hungary, which led to the birth of the International Society for the Interdisciplinary Study of Symmetry (SIS-Symmetry). It is better to say that we have just *one culture*, which, in fact, has two hemispheres and many functional areas in these. On the other hand, the two sides of this "Split Culture" should be linked as the *corpus callosum* bridges the two hemispheres of the brain in the case of humans and primates. Of course, the concept of symmetry is not the only "bridge" between the two hemispheres, but it may inspire other connections. Our next triennial congresses reached many regions: Hiroshima, 1992; Washington, D.C., 1995; Haifa, 1998; Sydney, 2001; etc. The current 12$^{th}$ congress in Portugal, which is the 5$^{th}$ one in Europe is very special. (It is also interesting to think about the numbers 12 and 5, which present the key for answering my joking question: What is the reason that the right angle has 90 degrees and not 100 or other "comfortable" number? We should go back to Babylon for the answer.)

## 1 "PORTUGUESE SYMMETRIES"

Our host organization, the Faculdade de Arquitectura da Universidade do Porto links architecture and geometry at the highest level. It is really unique that two winners of the Pritzker-prize, which is considered the "Nobel prize in architecture", graduated from this faculty, and one of them, the architect Álvaro Siza Vieira designed the campus. Dean João Pedro Xavier, our honorary congress chair, was his long-time co-worker. Our congress chair Vera Viana is a researcher at the faculty, and also the director of Aproged (Associação dos Professores de Geometria e de Desenho), as well as the initiator and organizer of the *Geometrias* conferences, which were started in Porto, 2013.

In the history of Portugal, we may mention various important achievements related directly or indirectly to the concept of symmetry. For example, we may observe a fine interplay between symmetry and asymmetry in the *Estilo manuelino* or Portuguese late Gothic, which was developed during the reign of King Manuel I (1495-1521) in the Age of Discoveries (Vasco da Gama, Pedro Álvares Cabral, and others) and of intensive trade with Asia and Africa. This style also used semi-circular arches, not just pointed arches with bilateral symmetry, and introduced special motifs in stonework like navigational tools, including the armillary sphere, *esfera dos matemáticos*, or sea animals, centuries before the biologist Ernst Haeckel popularized such motifs for designers in his book *Kunstformen der*

*Natur* (Art Forms in Nature, 1904). A monumental example of this style is the *Mosteiro dos Jerónimos* in Lisbon. (St. Jerome was born in the mid-4$^{th}$ century in Central Europe and later we shall see his difficulties in how to translate the word σύμμετρος for the Latin Bible.). Long before the discovery of Australia, there were some ideas about its existence. The historian João de Barros, the "Portuguese Livius", suggested that sailors may find a path to the other world (*Décadas da Ásia*, 1552). The *poeta nacional de Portugal* Luís de Camões' referred to *antípodas* (antipodic symmetry of earth) in his epos entitled *Os Lusíadas* (Lisboa, 1572, Canto 8, estrofe 44). He also mentioned ancient authors like Strabon, Pomponius, Plinius, and Ptolemaios. (Incidentally, its Hungarian translation was published in 1865 by the poet and scientist Gyula Greguss, the director of Lutheran Secondary School in Pest, where later John von Neumann, Eugene P. Wigner, and some other famous scholars were educated; Greguss also dealt with the connections between aesthetics and science in a paper in 1868, where he rejected the overstatement of the golden section.)

Turning to recent times, Hermann Weyl's classical booklet entitled *Symmetry* (Princeton, 1952) was translated into Portuguese in both Brazil (Edusp, São Paulo, 1997) and Portugal (Gradiva, Lisboa, 2017). We may also mention N. M. L. Garcia's book *Simetria e periodicidade* (Lisboa, 1985) and the historic survey by Regina C. G. Pasquini and Humberto J. Bortolossi, entitled *Simetria, História de um conceito e suas implicações no contexto escolar* (São Paulo, 2015). Geraldo M. Rohde's book has a challenging title: *Simetria, Generalidades sobre simetria, geociências, biociências, ciências exatas, tecnologias e artes, filosofia* (São Paulo, 1982).

## 2 PRELIMINARIES: EXPLAINING THE SYMMETRIES IN NATURE, AS WELL AS "ZOO-MATHEMATICS" AND PRE-HISTORIC "ETHNOMATHATICS"

There are various studies explaining the emergence of symmetries in nature, from Pappus' (4$^{th}$ c. CE) study of the hexagonal honeycomb by an economy principle to D'Arcy Thompson's book *On Growth and Form* (1917), from Alan Turing mathematical model (1952) to the more recent approaches with new tools, like chaos theory (Thom), fractals (Mandelbrot), L-systems (Lindenmayer). It is still a challenging question (Garay and Thiemann) to find possible links, if any, among symmetry breaking at the levels of elementary particles (Lee and Yang), molecules (chirality), and living matter (many species). It could be also related to the question of the origins of life. Unfortunately, the Philae mission landing on a comet and collecting samples failed because of technical problems. Turning to the animal world, biologists gathered evidence that symmetry considerations play some role in mate selection in the case of many species from insects to primates (Møller, Thornhill, Gangestad, and others). Possibly the reason for this preference is related to the tendency that symmetric shape may indicate good genetic quality. On the other hand, we cannot simplify the process just for symmetry, since various signals are also used, including odors, special voices, songs, and dances. Interestingly, some animals have such abilities that we may call "zoo-mathematics", for example, several bird species are able to acquire number concepts (Koehler), and pigeons may discriminate symmetric forms from non-symmetric ones in a concept-like way (Delius at al.). Perhaps the latter ability is based neither on the redundancy of symmetric patterns, nor the bilateral symmetry of the birds' visual system, but on the practice of searching for food with symmetric shapes, like grains. Archaeological findings present various examples of symmetric tools and objects used by prehistoric people, but we also should observe a "dissymmetry principle", the lack of the too-perfect symmetries, parallel with the emerging handedness. There are various reasons for using symmetric symbols. For example, in the case of boustrophedon ("like the ox turn") writing with changing direction at each subsequent line, obviously, the vertically symmetric symbols were preferred, which can be considered from both left-to-right and right-to-left (like the letters A, H, I, etc.). We may also mention ethnomathematical topics and the mathematics-related knowledge of prehistoric and traditional people. An interesting example is the "discovery" of regular polyhedra more than a millennium before Plato. These were carved on stone spheres by Celtic craftsmen. We suggested an explanation of how they were able to find these structures. Simply they dealt with the densest packings of equal circles ("knobs") on spheres, and the optimal configurations lead to the regular polyhedra (but not to the cube in the case of 8 circles because the optimal configuration is the square antiprism).

## 3 THE BIRTH OF THE GREEK EXPRESSION "ΣΥΜΜΕΤΡΙΑ" AND THE BEGINNINGS OF MATHEMATICS AS A DEDUCTIVE SCIENCE

We should make a distinction between the evolution of the symmetry concept, as we understand it in modern time, and the development of the term "symmetry". Unlike various suggestions, including Weyl's approach at the beginning of his book *Symmetry*, we insist that the Greek term was coined not in connection with the proportions of sculptures since there existed appropriate expressions for that (λόγος = ratio, ἀναλογία = proportion). With great probability, the new term was introduced in connection with the discovery that there are not just commensurable (*syn* + *métron*) pairs of line segments, but also incommensurable ones (*a* + *syn* + *métron*), like the diagonal and the side of a square (or of a regular pentagon). This terminology could have reached very quickly aesthetics, too; indeed, Plato and Aristotle used it in both mathematical and aesthetical contexts:

συμμετρία = commensurability (in geometry) → due proportion (beauty)

ἀσυμμετρία = incommensurability (in geometry) → disproportion (ugliness, wrong size).

The term συμμετρία did not refer to bilateral symmetry or symmetric patterns. The ancient Greeks had expressions to describe such objects, see ἰσομετρία (*iso* + *métrētos*, "equal measure") isometry;

ἰσόρροπία (*iso* + *rhopē*, "equal weight") equilibrium; τάξις (*táxis*) order, regularity. The discovery of incommensurability significantly contributed to the birth of mathematics as a deductive science. The earlier ideas of "pre-mathematics", which were developed in connection with practical questions, mystical considerations, and even games, were transferred into a new compact body of knowledge. There are various competing hypotheses on the reasons behind the emergence of mathematics, which were formulated by, among others, two leading mathematicians and two historians of mathematics (see also the related studies by I. Vandoulakis):

*Table 1* Hypotheses on the emergence of mathematics in ancient Greece

| | | |
|---|---|---|
| H1 | B. L. van der Waerden | sorting out exact rules from approximate ones |
| H2 | A. N. Kolmogorov | the influence of social factors and the cultural context in the Greek polis |
| H3 | Á. Szabó | introducing the proof by reductio ad absurdum within Eleatic philosophy |
| H4 | I. G. Bashmakova | an exceptional phenomenon, which cannot be cognized |

Our reconstruction unites all of these. The Pythagoreans ($6^{th}$-$5^{th}$ cc. BCE) had success in describing musical harmonies with ratios of integers, by using the instrument monochord, and they also inspired sculptors to establish similar proportional systems (H2). But they failed to find an appropriate ratio for the diagonal (d) and the side of a square (a). They had good approximations (3/2, 7/5), but tried to find the exact ratio (H1), and they discussed this problem in groups (H2). Then, as an exceptional phenomenon (H4), it was discovered that supposing of the existence of the ratio $d/a$, where

$d^2 = 2a^2$ (Pythagorean theorem), $a$ and $d$ are relative primes (after simplification),

it leads to a contradiction. Here $d^2$ is an even number, thus $d$ is also even (since the square of an odd number is also odd), while $a$ should be an *odd* number because $a$ and $d$ are relative primes. If $d = 2k$ (here $k$ is integer), $d^2 = 4k^2$, then $4k^2 = 2a^2$ (original statement). Thus, $2k^2 = a^2$, therefore $a^2$ is an even number, consequently, $a$ is also *even*. But it is impossible since it was stated earlier that $a$ is an *odd* number. This means that there are no such $a$ and $d$. This method had its roots in philosophy (H3). Aristotle called it *reductio ad absurdum* (ἡ εἰς τὸ ἀδύνατον ἀπαγωγή). During this demonstration, they made operations in an "imaginary world" with such objects that are non-existing. (Using modern algebraic terminology: √2 is irrational; also see some recent geometric proofs by S. Tennenbaum, T. Apostol, and others). Very soon, around 300 BCE, it appeared the *Elements* by Euclid, or Εὐκλείδης, the "glorious", also using pieces of information from Babylon and Egypt. This work, and perhaps some similar ones that are lost, presented an abstract deductive science with axioms, definitions, theorems, and proofs. The Greeks dealt not only with the συμμετρία vs. ἀσυμμετρία of line segments, but also of areas, specifically squares, which can be considered as a preliminary of the modern concept of bilateral symmetry (with two equal halves) and the later generalization. Specifically, the diagonal and the side of a square are incommensurable in length (ἀσύμμετρος μήκει), but commensurable in square (σύμμετρος δυνάμει) by considering 1- and 2-unit squares. At the same time, there was no simple term for the golden section (GS). It was called ἀκρος καὶ μέσος λόγος (extreme and mean ratio, Euclid, Book 6, Definition 3), and all known surviving references to this concept are in mathematical texts. Note that the GS is ἀσυμμετρία from the geometrical point of view; the golden number is irrational. Interestingly, Euclid's approach to the regular polyhedra is not perfect; in his sense, there are more than five ones (he did not require that the vertex figures should be equivalent).

## 4 THE LATIN "SYMMETRIA": SURVIVAL IN THE THEORY OF ARCHITECTURE

The Greeks established city-states that occupied small territories and made great achievements in geometry and philosophy. Later the Romans conquered large territories and focused on ruling the waste empire and shaped a legal system, the Roman law, but they had much less interest in geometry. On the other hand, a general idea of equilibrium and logical reasoning, including the method of *reductio ad absurdum*, were also important for shaping the legal system.

*Table 2* The Greek συμμετρία and ἀναλογία in Latin "doubling"

| G R E E K | LATIN ADOPTION (TRANSLITERATION) | LATIN TRANSLATION (INTERPRETATION) |
|---|---|---|
| συμμετρία (common measure, proportion) | symmetria (Varro, Vitruvius, Plinius) | commensus (Vitruvius, 1st c. BCE) commensuratio (Boethius, 5th-6th cc.) |
| ἀναλογία (proportion) | analogia (more general meaning) | proportio (Cicero, 1st c. BCE) (he coined it during translating Plato) |

The original meanings of the Greek συμμετρία were interpreted by appropriate Latin terms (Table 2), and it was more comfortable to use native words. Since the study of geometry declined, even *commensus* was used rather in an aesthetical sense as "due proportion". In the surviving body of ancient and medieval literature, there is just one remarkable work where the expression *symmetria* is frequently used: Vitruvius' *De architectura libri decem* (Ten Books on Architecture, 1$^{st}$ c. BCE). According to Vitruvius, all buildings should have *firmitas*, *utilitas*, and *venustas*, that is strength, utility, and beauty (Book 1, Chap. 3, Sec. 3). Vitruvius gave a special emphasis to aesthetical questions, and frequently used the expressions *symmetria* and *proportio* (85 and 32 times, respectively), with the following distinction (Book 6, Chap. 2, Sec. 5):

(1) the first step of design is considering the general method of *symmetria* (theory),

(2) then select the concrete proportions (practice).

It is interesting to quote the Roman polymath Plinius (Pliny the Elder, 1st c. CE), who died at the age of 55-56 when he led a rescue operation with ships during the eruption of Mount Vesuvius in 79. In his encyclopedic work *Naturalis Historia,* he also discussed the works of ancient Greek sculptors and added that "there is no Latin word for *symmetria*" (*non habet Latinum nomen symmetria*), which could explain the fine details observable in those statues (Book 34, Chap. 19, Sec. 65). The preferred expression in the later period was *proportio*, while *symmetria* and *cummensura* remained available for a later return to the mathematical terminology.

## 5 THE GREEK "ΣYMMETPIA" IN THE BIBLE AND ITS DISAPPEARANCE IN THE LATIN AND THE LATER MODERN TRANSLATIONS

According to a legend, 72 Jewish scholars, 6 from each of 12 tribes were asked by Ptolemy II, the Greek Pharaoh of Egypt, to translate the Torah (Tōrā) from Hebrew to Greek. Modern textual analysis shows that it was translated during a longer period in the 3rd and 2nd centuries BCE. This translation is named *Septaginta*, referring to the 70 translators, and abbreviated with the Roman numeral LXX. In its text, the word σύμμετρος occurs, but just once. Let us see its context:

*Table 3* Bible, Old Testament, Jeremiah 22:14

| **Hebrew Bible / Ta-Na-Kh** תַּנַ"ךְ<br>the Tōrā (Instruction), the Nəḇīʾīm (Prophets), the Kəṯūvīm (Writings) | | מִדּוֹת<br>mid·dō·wṯ<br>wide | בֵּית<br>bêṯ<br>a house | לִּי<br>lî<br>myself | אֶבְנֶה־<br>'eḇ·neh-<br>I will build | ← | |
|---|---|---|---|---|---|---|---|

| | |
|---|---|
| Septuaginta (LXX, 3rd-2nd cc. BCE) | … ᾠκοδόμησας σεαυτῷ οἶκον **σύμμετρον** … |
| St. Jerome Vulgata (Latin, 382-405) | … aedificabo mihi domum **latam** … |
| King James Version (1611) | … I will build me a **wide** house … |
| Sir Brenton's LXX (1851) | … Thou hast built for thyself a **well-proportioned** house … |
| New American Standard Bible (NAS) | … I will build myself a **roomy** house … |
| New International Version (NIV) | … I will build myself a **great** palace … |
| Luther German Bible (1534) | … Ich will mir ein **großes** Haus bauen … |
| Louis Segond French Bible | … Je me bâtirai une maison **vaste** … |
| Nova Versão Internacional, Portugal | … Construirei para mim um **grande** palácio … |
| Károli Biblia, Hungary (1590) | … **Nagy** házat építek magamnak … |
| Japanese Contemporary Bible | … **大な**宮殿を建 … |

| | |
|---|---|
| **מִדּוֹת** (mid·dō·wṯ)<br>5 occurrences | = of great size [men] – see Numbers 13:32; great – see Jeremiah 22:14.<br>measurements [temple] – see Ezekiel 41:17, 42:15, 43:13 |
| **כַּמִּדּוֹת (**kam·mid·dō·wṯ)<br>6 occurrences | = same measurements [temple]<br>see Ezekiel 40:24, 40:28, 40:29, 40:32, 40:33, 40:35 |
| **latus (**Latin)<br>Oxford Latin Dict., 2016 | = (with measurements) having a specified transverse extent, broad, wide; to a great width; extending over a large area; etc. |

What was the possible reason that מִדּוֹת was translated as σύμμετρος? Jeremiah 22:13 and 14 present an opposition: The earlier king of Judah built "his palace by *unrighteousness* (…) making his own people work for nothing", but his son, the new king promised that he himself will build "a *great* palace" (NIV, our emphases). The original Hebrew word may refer to both "great size" and "measurements" (Table 3, part 2), and also used for the detailed description of the new Temple according to the prophet Ezekiel's vision, during his exile in Babylon. The expression σύμμετρος is a good choice to express that the new palace would be a right one by using "common measures" during the construction. St. Jerome rendered it by using the Latin word *latus*, which expresses the size and also has an indirect link to measurement (Table 3, part 2). The modern translators had difficulties presenting these fine details, although Sir Brenton tried to reflect the Greek word by using the expression "well-proportioned". We may say, however, that the Greek συμμετρία and the Latin *symmetria* did not become Biblical terms. The Greek word was used occasionally by the Greek Church Fathers as "due proportion" (Clement of Alexandria, 2nd-3rd cc.; St. Basil the Great, 4th c.; Athanasius, the Patriarch of Alexandria, 4th), then from the 5th century as "keeping to the same measure" (Pseudo-Dionysius, 5th c.; Eulogius of Alexandria, 7th c.).

## 6 THE REBIRTH OF "SYMMETRIA": VITRUVIUS'S HELP, SCIENTISTS' ADVANCES

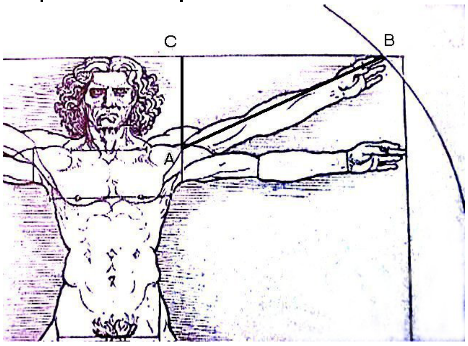
During a long period, it was believed that the first occurrence of the term *symmetria* in modern languages was the French *symmetrie* in 1529, which became later *symétrie*. Even the 20-volume *Oxford English Dictionary* (1989) gives this example as the earliest one. But in 1995, during the Washington congress of SIS, we presented some earlier ones. Our strategy was to check the early interpretations of the Vitruvian text in the Italian Renaissance, with an idea that the Latin *symmetria* cannot be replaced by the Italian *proporzione* because there was some distinction between the two concepts, as we have seen earlier. In this way, we located some early examples (Lorenzo Ghiberti, ca. 1450, which survives in his grandson's notebook; Francesco di Giorgio Martini's manuscripts, ca. 1475 and ca. 1485; Cesare Cesariano, 1521, the first printed Italian edition of Vitruvius). The expression spread from Italy, as well as with the translations of Vitruvius' text in Spanish (Diego de Sagredo, Madrid, 1542), French (Martin, Paris, 1547), and German (Rivius, 1548). It is often stated that the Renaissance artist, especially Leonardo, frequently used the golden section (GS), but this is not so.

### 6.1 THE "GS" WITH LESS "GOLD"; LEONARDO'S "VITRUVIAN CODE" IS 1/3

The "Golden sectionism" – it is our term referring to the overstatements – spread from Germany, following Zesing's book of 1854, which has many attractive illustrations. He presented the GS as a "new theory of the proportions of the human body" and a "hitherto unrecognized basic law of morphology penetrating the whole nature and art", as emphasized in the title of the book. The GS as a unifying principle was well received in a time when the unification of the German states was a target and later had similar importance in the Russian Empire, then in the former Soviet Union, as we discussed these social-cultural-political aspects in an earlier paper published in Ukraine. The "strong golden sectionism" was relaxed to a "weak" one by the psychologist Fechner's statistical approach in the 1870s, while Ghyka presented the GS in a mystical-philosophical context in his books from 1927 in many languages. Enthusiastic people "recognized" the usage of the GS elsewhere, from Egyptian pyramids and Greek temples to Renaissance painting and modern art, despite the fact that in some cases there were documents against this. For example, we demonstrated that the sculptor Polykleitos' canon was not based on the GS by using the surviving fragments of his book. The case of Le Corbusier is special, because he, after seeing Ghyka's book, "modified" his earlier drawings in order to present the usage of the GS in his book *Modulor* (as it was discussed by Herz-Fischler). It is not surprising that Leonardo became the main target of the "golden sectionists", because of his interest in human proportion and his friendship with the mathematician Luca Pacioli, who introduced the expression "divine proportion" instead of the complicated term "extreme and mean ratio" (GS). Turning to Leonardo's proportional study of the "Vitruvian man" (ca. 1490), which illustrates Vitruvius' text (Book 3, Chap. 1, Sec. 2-3), we demonstrated that it is not based on the GS (Figure 1).

A simple calculation shows that the navel is not at the golden section point (0.618…) of the height, but a bit lower (0.605...). On the other hand, 1/3 for BC could be "Leonardo's code" that he kept secret. It is true that Leonardo used the expression "divine proportion" in the "Paragone" of his *Trattato della pittura*, which is formally in a quote by the Hungarian king Mathias Corvinus, but it is a metaphorical expression and not a reference to the GS. We demonstrated that this part of Leonardo's book was written earlier than he met Luca Pacioli in 1497 and started collaborating with him. Thus, not Leonardo adopted Pacioli's mathematical term, but perhaps Pacioli took Leonardo's expression and gave this a mathematical meaning (GS). It is also true that Leonardo drew the skeleton models of the polyhedra for Pacioli's book, but they used glass models and there is no need to study the GS to draw a dodecahedron or an icosahedron. Of course, we do not say that Leonardo had no interest in the GS at all, but perhaps it was not a central concept for him. In the case of the human body and human face, he was interested in variety and also drew many deformed ones.

| ← 1/4 → |



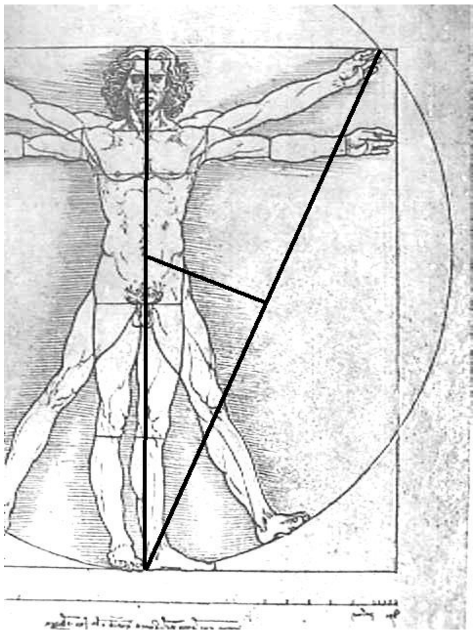

B is given, A and C are available by extending the appropriate line segments marked by Leonardo. According to him:

(1) AC = 1/6 ("from the top of the breast to the crown of the head" is "the sixth of the man", i.e., the height of the man)

(2) AB = 1/4 + 1/8 = 3/8 (the length of the arm, which is "from the elbow to the tip of the hand [middle finger] is the fourth part; from this elbow to the end of the shoulder is the eighth part")

Using the Pythagorean theorem, we have:

$BC = \sqrt{65}/24 = 0.335… \approx 1/3$



The added bold lines are just helping the calculation; these are not regulating lines.

$R = (325 + 3\sqrt{65})/576 = 0.606…$

*Figure 1* The ratio 1/3 for BC (see on the left side) is either the "secret code" of Leonardo or, at least, the key to a good method to approximate his construction. Leonardo stated that "the maximum width of the shoulders is in itself the fourth part of a man" and marked the ends of this distance with two short vertical line segments (we wrote this 1/4 above the drawing). The centre of the circle is at the navel, as Vitruvius wrote. If we know the exact position of point B, where the circle intersects the upper side of the square, it is easy to calculate the height of the navel from the ground, which is also the radius R of the circle. We just should consider the added bold lines on the right-side figure and the similarity between the big and the small right-angled triangles. There was no need to introduce new regulating lines, as it is frequently made in the literature. We just used the numerical data given by Leonardo in the text around the figure (1/2, 1/4, 1/6, 1/7, 1/8, 1/10 times the height of the man). The calculation led to the followings: $BC = \sqrt{65}/24 = 0.335… \approx 1/3$, and R = 0.605… (using 1/3) or R = 0.606… (using $\sqrt{65}/24$). This value differs from the "golden number" $(\sqrt{5} - 1)/2 = 0.618…$ One may argue that 0.605… is close to 3/5 = 0.6, but no other ratios of neighboring Fibonacci numbers were mentioned by Leonardo, with the exception of 1/2. He used here just ratios in the form 1/n (see the list above). The fact that we rounded $\sqrt{65}/24$ as 1/3 is perhaps not against Leonardo's thinking because he himself used a similar approximation in order to have the ratio 1/6 ( "the distance from the top of the breast to the crown of the head" is 1/6 = 0.166…, but it is redundant because this distance can be calculated by using some other ratios in the text: 1/8 + 1/7 –1/10 = 47/280 = 0.167…).[1]

It would be better to focus not on questionable or false examples, but on such cases where the GS was really used or at least studied. For example, we wrote about the composer Liszt's correspondence on the GS in the late 1850s, then van der Schoot made further studies. The film director Eisenstein also dealt with the GS and discussed it as an organizing principle in the 1930s. The sculptor Beothy worked out his *Série d'Or* in the 1930s and 40s. We may also mention contemporary works, the composer Schulze's "Fibonacci Style", the architect Hizume's Fibonacci- and quasicrystalline structures, and the subway station *Saldanha II* in Lisbon decorated by the geometric construction of the GS. We may refer to some scientific fields where the GS plays an important role, including the search problem in computing, the theory of polyhedra, Penrose tiling and quasicrystals, fullerene molecules, etc. A mathematical question led us to deal with the golden *n*-section, where a line segment is divided into *n* part so that $a_1 / a_2 = a_2 / a_3 = … = a_{n-1} / a_n = a_n / (a_1 + a_2 + … + a_n)$.

The development of the terminology of the GS is also an interesting topic. It was believed for a long period that the term GS, specifically the German *der goldene Schnitt* appeared first in 1835. In 1997 and 2007, we gave two slightly earlier examples and predicted that there are further ones. Now the earliest known example is from 1717, which was presented by Becker in 2019. Our guess is that the term GS was coined and popularized by German scholars and teachers in order to avoid confusion with the usage of too many expressions for the same concept (extreme and mean ratio, divine proportion, divine section, continuous division). They chose the adjective "golden", which can be linked indirectly to the adjectives in the earlier terms. They also translated the new German expression into Latin (*sectio aurea*) and Greek (*khrusē tomē*) to demonstrate that the very concept is ancient. As we remarked earlier (Chap. 3), it was used in Greek mathematics, but there is no surviving evidence for artistic applications from that time. Now the term *sectio aurea* is widely used, but it is misleading.

## 6.2 SCIENTISTS' AND ARTISTS' ADVANCES FOR NEW "SYMMETRIES"

Since symmetry became an "empty niche" in mathematics, it could gain new meanings. As we hinted earlier, the Greek concept of common measure of areas can be linked to the idea of bilateral symmetry. Claude Perrault, the French physician, who also became the designer of the eastern facade of the Louvre (1667-1670), remarked in his French translation of Vitruvius that symmetry has two meanings, an old one and a new. The idea of bilateral (mirror) symmetry, and later its generalization like rotational and translatory symmetries appeared in the geometry books. Often the French mathematician Legendre's book of 1794 is credited for the precise geometrical definition of symmetry. Since Legendre was a leading mathematician of his age, his book became obviously influential, but there were some earlier works, too. The new geometrical concept of symmetry was introduced into other fields very quickly (Table 4).

*Table 4* The emergence of the geometric symmetry concept in other fields with appropriate modifications

| | |
|---|---|
| crystallography | Haüy's symmetry law (1815) |
| chemistry | Pasteur's molecular dissymmetry (1848), cf. chirality of molecules |
| theoretical physics | P. Curie's dissymmetry principles (1894) |
| mathematical physics | Noether's theorem on symmetries (invariances) and conservation laws (1918),<br>Wigner: Events, laws of nature, and invariance principles (Nobel Lecture, 1963) |

Pasteur introduced the concept of dissymmetry, between symmetry and asymmetry, as the lack of some possible elements of symmetry that makes it possible to have different left-handed and right-handed molecules (the "too symmetric" objects, like a cube, have no two different versions). The concept of symmetry became especially fruitful in geometrical crystallography to describe all the possible ideal crystal structures step by step. It was also helpful to introduce the algebraic group theory for considering geometric symmetries and dealing with symmetry groups (Table 5).

*Table 5* History of symmetry in crystallography

| | |
|---|---|
| Haüy | symmetry law (1815) |
| Frankenheim (1826), Hessel (1830) | 32 crystallographic point groups (local order in crystals) |
| Bravais (1850) | 14 Bravais-lattices (translational order in crystals) |
| Sohncke (1879) | 65 motion groups (regular systems of points) |
| Fedorov (1890), Schoenflies (1891) | 230 space groups (global order in crystals) |
| Fedorov (1891) – 2D after 3D | 17 plane (wallpaper) groups; later rediscovered by Klein, and by Pólya |
| Shubnikov (1951) | 1651 antisymmetry (black-white or two-colored) space groups |
| Shechtman (1982 / 1984) | quasicrystals with "forbidden" symmetries; new approach is needed |

Bolyai's and Lobachevski's non-Euclidean geometry from the 1830s (they did not receive appropriate credit for a long time), then in the 20$^{th}$ century the new developments in science, from quantum mechanics to particle and astrophysics physics, from non-linear mathematics to quantum computing presented many fields where generalized symmetries can be used. Artists also participated in the developments. In the education of basic design, with an eye for industrial design, symmetry studies became an important field. Here we should remember our late honorary president, the architect William Huff. Luckily his work is continued by Van Hoeydonck. We should use our five senses, and not only sight, as was discussed by the architect Pallasmaa. Symmetry may also help in this by comparing visual and musical patterns, presenting symmetric shapes for a tactile survey, and so on.

The power of symmetry principles in some fields is to help to list all of the theoretically possible cases, and thus predict new ones that were not yet discovered. It happened, for example, in the case of crystallographic space groups (see Table 5), chemical isomers (van't Hoff, Fischer), and quarks (Gell-Mann, Ne'eman). There are similar possibilities in art and the humanities in a few special cases. For example, the surviving manuscript of Bach's *Kunst der Fuge* was considered just a set of exercises and not a musical piece for performance, but Graeser, a student of mathematics, found a symmetry principle in the composition and reorganized the mixed-up pages accordingly in 1924. Following this, it became a frequently played work. Wallpaper groups were used to study settlement patterns, as it was suggested by two architects March and Steadman in their book of 1971. It is remarkable that all of the 17 wallpaper groups were used on the walls and floors in the Alhambra, Granada, which was demonstrated by Pérez Gómez during the conference "Alhambra 2000". Various scientific and artistic problems need a concept of "fuzzy symmetries", not a rigid one with "yes/no" questions. In fact, *symmetry measures* were introduced with different motivations (we also dealt with this problem from a crystallographic point of view).

Artists and architects often made experiments with the interplay of symmetry and asymmetry. Earlier we referred to this possibility in the case of the Portuguese late Gothic. It is also important to consider folk (popular or vernacular) architecture, where we may observe the harmony with the local environment and the usage of the optimal shapes and proportions, which were "crystallized" during the ages. This is an important source of inspiration for artists and architects, and also useful to deal with the problems of sustainability, as well as to find "local ways" between the old historic styles and the modern globalized architecture. We may give good examples for surveying folk architecture. In Portugal, the *Inquérito à Habitação Rural* was started by agricultural motivations in the early 1940s, then the comprehensive *Inquérito* was initiated by the architects Keil do Amaral and Távora in 1947 and conducted in the late 1950s describing the varieties of buildings in the six regions of the country. Later the "Porto School", Távora with his students Siza and Souto de Moura, both became winners of the Pritzker-prize, excelled in using the traditions and shaping modern Portuguese architecture. The Spanish also published the result of their survey in five volumes in the 1970s, and "vernacular simplicity" influenced modern Spanish architecture. The case of Hungary is also interesting, where the survey of folk architecture was initiated by Kós and some young architects, then by the Friends of the Hungarian House, but their work was stopped by World War I and World War II, respectively. Still, this tradition survived and led to the birth of the Hungarian Organic Movement in the early1970s. Walking through the *Parque Eduardo VII* in Lisbon, which was designed by Keil do Amaral, or visiting some of the buildings of the mentioned architects is a good possibility to observe the role of symmetry and symmetry breaking in modern architecture.

## 7 CONCLUSION AND FUTURE TASKS

In some sense, the birth of SIS in 1989 signaled a new tendency, which is demonstrated by the foundations of some other organizations with similar goals, the recent popularity of art-math events, and the idea of STEAM (Science, Technology, Engineering, the Arts, and Mathematics) in education, where originally "A" was missing.

The topics related to symmetry may play an important role in interdisciplinary education. It is possible to inspire pupils by presenting the "beauty of science". There are good examples of teaching symmetry groups in mathematics and crystallography by using Escher's graphics with periodic patterns, which were also influenced by those ones in the Alhambra. Even a mistake in the crystallographic tables was discovered by analyzing Escher's drawings and realizing that the list of colored symmetry groups is incomplete. The discovery of quasicrystals was predicted by Islamic artists who created patterns with fivefold symmetry. The theory of tilings was significantly enriched by amateurs. While mathematicians believed that they have the complete list of convex pentagons that generate tilings, later further ones were discovered. We think here about monohedral or "ein-stein" (one-stone) tilings where just one type of pentagon is used. It would be an advantage in education to use those patterns and tilings that are available in the local art and architecture.

In research, symmetry may help to link distant fields, as our congresses gave some examples, like space engineering and origami by Miura, nanotubes and traditional basket weaving by Iijima, AI and ethnomathematics by Goranson, or discrete geometry and *wasan* (traditional Japanese mathematics) by our mini conference.  In 1995, a book was published with the title *The Third Culture, Beyond the Scientific Revolution*; it was written by a large group of authors, including Paul Davis, Richard Dawkins, and Roger Penrose. I still hope that we do need not a "third culture", but we have just one with a growing number of connections, "symmetrically".

[1] Picture credit to Figure 1: Leonardo's drawing is in the public domain; the analysis; the additional lines are from me.

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