UNRAMIFIED COVERS AND BRANES ON THE HITCHIN SYSTEM

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ABSTRACT. We study the locus of the moduli space of $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles on a curve given by those Higgs bundles obtained by pushforward under a connected unramified cover. We equip these loci with a hyperholomorphic bundle so that they can be viewed as BBB-branes, and we introduce corresponding BAA-branes which can be described via Hecke modifications. We then show how these branes are naturally dual via explicit Fourier–Mukai transform (recall that $\mathrm{GL}(n, \mathbb{C})$ is Langlands self dual). It is noteworthy that these branes lie over the singular locus of the Hitchin fibration.

As a particular case, our construction describes the behavior under mirror symmetry of the fixed loci for the action of tensorization by a line bundle of order n. These loci play a key role in the work of Hausel and Thaddeus on topological mirror symmetry for Higgs moduli spaces.

CONTENTS

1. Introduction

1.1. **In a nutshell.** Among the many fundamental contributions of Narasimhan and Ramanan to the study of moduli of vector bundles on curves are the Hecke correspondence [NR1, NR2] and the study of generalized Prym varieties as fixed points [NR3]. In this paper we use these ideas to explore mirror symmetry for the moduli space of Higgs bundles in the spirit of the seminal work of A. Kapustin and E. Witten [KW]. More precisely, we exhibit pairs of dual branes for the Langlands self dual group $\mathrm{GL}(n, \mathbb{C})$. The interest of our construction relies on two aspects: firstly the branes we consider are sheaves (rather than just submanifolds) and the duality is realized via an explicit Fourier–Mukai transform; secondly, we are making progress in the understanding of mirror symmetry in the singular locus of the Hitchin system, since the branes lie entirely over this locus. As far as we know, this is the first example of dual branes lying over the singular locus, where mirror symmetry is explicitly realized by a Fourier–Mukai transform. Finally, it is also important to note that among the branes we construct are the fixed loci under tensorization by an order n line bundle, central in the work of Hausel and Thaddeus [HT] on topological mirror symmetry, so our construction ought to be important in a deeper understanding of the topological mirror symmetry phenomenon. In the remainder of this section, we explain our constructions and results in more detail.

1.2. **Context.** N. Hitchin introduced in [Hi1] Higgs bundles over a smooth projective complex curve X of genus $g \geqslant 2$ as solutions to certain equations obtained by dimensional reduction of the self-dual equations on a 4-manifold. These are pairs (E, φ), where E is a holomorphic vector bundle over X and φ is a holomorphic one-form with values in $\mathrm{End}(E)$. The moduli space $\mathbf{M}_X(n, d)$ of Higgs bundles of rank n and degree d is a holomorphic symplectic manifold carrying a hyperkähler metric. Moreover, it admits the structure of an algebraically completely integrable system given by the Hitchin map $h_{X,n} \colon \mathbf{M}_X(n, d) \longrightarrow B_{X,n}$. Here the Hitchin base $B_{X,n}$ is an affine space whose dimension is half that of $\mathbf{M}_X(n, d)$, and the components of $h_{X,n}$ are the coefficients of the characteristic polynomial of φ. The fiber of $h_{X,n}$ over a generic point of the Hitchin base is a torsor for an abelian variety, namely the Jacobian of an associated spectral curve.

The concept of a G-Higgs bundle can be defined for any complex (and even real) reductive Lie group G. In these terms, the above definition becomes that of a $\mathrm{GL}(n, \mathbb{C})$-Higgs bundle. The Hitchin map can also be defined in this generality, and it has been shown that it is an algebraically completely integrable system for any complex reductive Lie group G [Hi2, Fa, Sco, DG].

A new development arose with the discovery by T. Hausel and M. Thaddeus [HT] of a close relation between Higgs bundles, mirror symmetry and the Langlands correspondence. They proved that the moduli spaces of Higgs bundles for the group $\mathrm{SL}(n, \mathbb{C})$ and its Langlands dual group $\mathrm{PSL}(n, \mathbb{C})$ form a pair of SYZ-mirror partners [SYZ], in the sense that the respective Hitchin maps have naturally isomorphic bases and their fibers over corresponding points are, generically, half-dimensional torsors for a pair of dual abelian varieties. This was subsequently generalized by N. Hitchin [Hi4] for the self-dual group G_2 and then by R. Donagi and T. Pantev [DP] for any pair $(G, {}^L G)$ of Langlands dual groups. The duality is reflected by a Fourier–Mukai transform between the moduli spaces interchanging fibers of the Hitchin map over corresponding points in the base. These dualities were obtained over the locus of the Hitchin base where the corresponding spectral curves are smooth.

As mentioned above, the moduli space $\mathbf{M}_X(n, d)$ is hyperkähler. This means that it carries three natural complex structures I_1, I_2 and I_3 verifying the quaternionic relations and a metric which is Kähler with respect to all three holomorphic structures. In the present case, I_1 is the natural complex structure on the moduli space of Higgs bundles $\mathbf{M}_X(n, d)$, while the complex structures I_2 and $I_3 = I_1 I_2$ arise via the non-abelian Hodge Theorem, which identifies $\mathbf{M}_X(n, d)$ with the moduli space of projectively flat $\mathrm{GL}(n, \mathbb{C})$-connections (see [Hi1, Si1]).

A. Kapustin and E. Witten considered in [KW] certain special subvarieties of $\mathbf{M}_X(n, d)$, equipped with special sheaves. The pair composed by such a subvariety and the corresponding sheaf is called a brane. For each of the complex structures on $\mathbf{M}_X(n, d)$ a brane is classified

as follows: it is of type A if it is a Lagrangian subvariety with respect to the corresponding Kähler form and the sheaf over it is equipped with a flat connection, and it is of type B if it is a holomorphic subvariety and the sheaf over it is also holomorphic. Thus, for instance, a (BBB)-brane is a subvariety equipped with a sheaf, holomorphic with respect to all three complex structures I_1, I_2 and I_3; in other words, it is a hyperholomorphic subvariety equipped with a hyperholomorphic sheaf (this is a sheaf with a connection whose curvature is of type $(1,1)$ with respect to all complex structures). A (BAA)-brane is a subvariety which is holomorphic with respect to I_1, and Lagrangian with respect to the Kähler forms ω_2 and ω_3 associated to I_2 and I_3 (hence complex Lagrangian for $\Omega_1 = \omega_2 + \mathrm{i}\,\omega_3$), and which in addition supports a flat vector bundle. There are only two other possible types of branes on $\mathbf{M}_X(n,d)$, namely (ABA)- and (AAB)-branes. Again all this holds for any complex Lie group and not just $\mathrm{GL}(n,\mathbb{C})$.

According to [KW], mirror symmetry conjecturally interchanges (BBB)-branes and (BAA)-branes, and mathematically this duality should again be realised via a Fourier–Mukai transform (in complex structure I_1). The support of the (BAA)-brane should depend not only on the support of the dual (BBB)-brane but also on the hyperholomophic sheaf over it (and vice-versa). A similar story holds for pairs of (ABA)-branes and also for pairs of (AAB)-branes.

Since Kapustin and Witten's paper—and because of it—an intense study of several kinds of branes on Higgs bundle moduli spaces has been carried out. Some examples may be found in [Hi5, BS1, BGP, HeS, BCFG, Hi6, Ga, FJ, BS2, FP, B, HMP] (see also [AFES] for a survey on this subject). Most of these works mainly focus either on the smooth locus of the Hitchin system (exceptions are [BS2, FP, B]) or only deal with the support of the branes and not with the sheaves on it (exceptions are [Hi5, Hi6, Ga, FJ, FP]).

1.3. Our construction.

Starting from a connected unramified cover $p : C \longrightarrow X$ of degree n and Galois group Γ, we introduce in this paper new types of (BBB)-branes and (BAA)-branes on $\mathbf{M}_X(n,d)$, the moduli space for the self-dual group $\mathrm{GL}(n,\mathbb{C})$. As required in the general picture, our (BBB)-branes come equipped with flat, hence hyperholomorphic, bundles. We explicitly prove (when $d = 0$) that their (fiberwise) Fourier–Mukai transform generically yields a sheaf supported exactly over the support of our (BAA)-brane. As expected, the support of the (BAA)-brane depends on the hyperholomorphic bundle over the (BBB)-brane.

These branes are supported on a subspace $B^p \subset B_{X,n}$ of the *singular locus* of the Hitchin system. For a dense open subset $B^p_{\mathrm{ni}} \subset B^p$ of *nodal* and *integral* spectral curves, the normalization of these curves is C itself. Since $p : C \longrightarrow X$ is unramified, B^p_{ni} is, by definition, contained in the so-called *endoscopic locus* of $h_{X,n}$ (cf. [HPa, Ngo]). So our construction (more precisely, its analogue for the Langlands dual groups $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{PSL}(n,\mathbb{C})$) may eventually be relevant in the context of geometric endoscopy, introduced by E. Frenkel and E. Witten in [FW].

In the following we outline our construction in more detail, starting with the (BBB)-branes. Fix the rank n to coincide with the degree of p and set $\mathbf{M}_X(n,d)^p$ to be the locus of Higgs bundles obtained as a pushforward under p of Higgs bundles in $\mathbf{M}_C(1,d) \cong T^* \mathrm{Jac}^d(C)$. Let B^p be the image of $\mathbf{M}_X(n,d)^p$ under the Hitchin map $h_{X,n} \colon \mathbf{M}_X(n,d) \longrightarrow B_{X,n}$. As a direct consequence of non-abelian Hodge theory, one concludes that $\mathbf{M}_X(n,d)^p$ is a hyperholomorphic subvariety. The pushforward by p yields an isomorphism between $\mathbf{M}_X(n,d)^p$ and the quotient of $T^* \mathrm{Jac}^d(C)$ by the Galois group, acting by pullback. From this, one defines a hyperholomorphic line bundle $\mathscr{L}$ over $\mathbf{M}_X(n,d)^p$, naturally associated to a flat line bundle $\mathcal{L}$ on X. We call the pair $(\mathbf{M}_X(n,d)^p, \mathscr{L})$ a *rank 1 Narasimhan–Ramanan* (BBB)-*brane*. We represent it by $(\mathbf{BBB})^p_{\mathcal{L}}$ and write $(\mathbf{BBB})^{p,\mathcal{L}}_{\mathrm{ni}}$ for its restriction to B^p_{ni}. More generally, we can construct a rank n coherent and hyperholomorphic sheaf $\mathscr{F}$ on $\mathbf{M}_X(n,d)^p$ which canonically associated to a flat line bundle $\mathcal{F}$ over C, and we call the pair $(\mathbf{M}_X(n,d)^p, \mathscr{F})$ a *rank n Narasimhan–Ramanan* (BBB)-*brane* and represent it by $(\mathbf{BBB})^p_{\mathcal{F}}$.

Let $p : C \longrightarrow X$ be a Galois $\mathbb{Z}_n$-cover, and let $\xi \in \mathbb{Z}_n$ be the standard generator. Parallel transport of the lifts from X to C provides a line bundle $L_\xi \in \mathrm{Jac}^0(X)$ of order n. In this case, it basically follows from [NR3] that the locus $\mathbf{M}_X(n,d)^p$ coincides with the subvariety $\mathbf{M}_X(n,d)^\xi \subset \mathbf{M}_X(n,d)$ of points (E,φ) fixed by tensorization of by L_ξ, *i.e.* $(E,\varphi) \cong (E \otimes L_\xi, \varphi)$.

The study of $\mathbf{M}_X(n,d)^{\xi}$ was our original motivation. So this justifies the name chosen for the (BBB)-branes appearing in this paper.

If our (BBB)-branes are intimately related to the work of Narasimhan–Ramanan in [NR3], the construction of our (BAA)-branes is closely linked to their work on *Hecke modifications* of vector bundles published in [NR1, NR2]. Hecke modifications in the context of Higgs bundles have previously appeared in several papers; see, for example, [Hi7, HR, Ra, Wi, W]. Before describing the construction, we recall that under certain assumptions on the values of the rank and the degree, there exists a Hitchin section on the moduli space $\mathbf{M}_C(r, d + \delta)$ constructed out of a line bundle $\mathcal{J} \in \mathrm{Jac}^{d+\delta}(C)$. The pushforward under p defines a Hitchin–type section of $\mathbf{M}_X(n, d+\delta)^p \longrightarrow B^p$. We define the subvariety $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}} \subset \mathbf{M}_X(n,d)$ of those Higgs bundles over B_{ni}^p obtained as Hecke modifications of this Hitchin–type section at the divisor of singularities of the corresponding integral and nodal spectral curve (which has length δ) classified by B_{ni}^p. The notation we use for this subvarieties is chosen to recognize the pioneer work Narasimhan and Ramanan on Hecke modifications. We prove next that the subvarieties $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}}$ are complex Lagrangian with respect the holomorphic symplectic form $\Omega_1 = \omega_2 + \mathrm{i}\,\omega_3$ on $\mathbf{M}_X(n,d)$. This shows that this subvariety is the support of a (BAA)-brane on $\mathbf{M}_X(n,d)$, when endowed with a flat bundle.

Our construction of $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}}$ (for $d = 0$ and p of degree n) was aimed at obtaining the support of a (BAA)-brane dual to the rank 1 (BBB)-brane $(\mathbf{BBB})_{\mathcal{L}}^p$, for an appropriate choice of the line bundle $\mathcal{J}$. Towards this goal, we provide an extensive study of the spectral data of the Higgs bundles appearing in $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}}$ and in $\mathbf{M}_X(n,d)^p$, the support of $(\mathbf{BBB})_{\mathcal{L}}^p$. For a given $b \in B_{\mathrm{ni}}^p$, let X_b be the corresponding spectral curve and $\nu_b : C \longrightarrow X_b$ the normalization. Over the Hitchin fiber associated to X_b, the spectral data in $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}}$ are those contained in the closure of the preimage of $\mathcal{J}$ by the pull-back under ν_b. On the other hand, the spectral data contained in $\mathbf{M}_X(n,d)^p$ are those given by pushforward under ν_b. This paves the way for our main result, Theorem 6.5, which is described below.

Theorem. *Let $p : C \longrightarrow X$ be a connected unramified n-cover. Consider the moduli space $\mathbf{M}_X(n,0)$.*

(i) *Let $\mathcal{J} = p^*(\mathcal{L} \otimes K_X^{(n-1)/2})$. The (fiberwisewise) dual of the rank 1 Narasimhan-Ramanan (BBB)-brane $(\mathbf{BBB})_{\mathcal{L}}^p$ (restricted to the locus of nodal and irreducible spectral curves) is the (BAA)-brane supported on $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}}$, and whose flat bundle satisfies (6.28).*

(ii) *Let $\mathcal{J} = \mathcal{F} \otimes p^* K_X^{(n-1)/2}$. The (fiberwise) dual of the rank n Narasimhan-Ramanan (BBB)-brane $(\mathbf{BBB})_{\mathcal{F}}^p$ (restricted to the locus of nodal and irreducible spectral curves) is the (BAA)-brane supported on $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}}$, and whose flat bundle satisfies (6.30).*

It is important to note that this duality is proved by an explicit fiberwise Fourier–Mukai transform, on the fibers over B_{ni}^p, mapping the hyperholomorphic sheaf to a sheaf supported on $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}}$. This Fourier–Mukai transform is carried out using the autoduality of compactified Jacobians of integral curves with planar singularities, from the general results of D. Arinkin [Ar]. It uses a Hitchin section (which embeds B_{ni}^p as a subvariety of $\mathrm{NR}_{\mathrm{ni}}^{p;\mathcal{J}}$) to identify $\overline{\mathrm{Jac}}^{\delta}(X_b)$ with the corresponding $\overline{\mathrm{Jac}}^0(X_b)$, and then apply Arinkin's Fourier–Mukai functor. In order to explicitly do it, we relate this functor with the classical Fourier–Mukai functor of $\mathrm{Jac}^{\delta}(C)$, via the pullback and the pushforward maps induced by the normalization morphism $\nu_b : C \longrightarrow X_b$.

It is worth noticing in this case that $(\mathbf{BBB})_{\mathcal{F}}^p$ appears as the pushforward of (BBB)-brane $(\mathscr{F}, \nabla_{\mathscr{F}}) \longrightarrow \mathbf{M}_C(1,0)$ supported over the whole moduli space, where $\mathscr{F}$ is the pullback under $\mathbf{M}_C(1,0) \longrightarrow \mathrm{Jac}^0(X)$ of the flat line bundle over $\mathrm{Jac}^0(X)$ associated to $\mathcal{F} \longrightarrow X$. Mirror symmetry conjectures that $(\mathscr{F}, \nabla_{\mathscr{F}}) \longrightarrow \mathbf{M}_C(1,0)$ is dual to the (BAA)-brane given by the Hitchin section associated to $\mathcal{F}$. As we said before, $(\mathbf{BAA})_{\mathrm{ni}}^{p;\mathcal{F}}$ can be interpreted in terms of Hecke modifications of the pushforward of this Hitchin section. This suggests a deep relation between duality of branes in $\mathbf{M}_X(n,0)$, duality in $\mathbf{M}_C(1,0)$ and the Hecke operators appearing in geometric Langlands conjecture (see [DP]). For d non-multiple of n a similar result should

hold, but the duality should require a gerbe to work out properly. We also note that the results in this paper provide evidence for the dualities suggested in [FP].

Remark 1.1. We actually construct the support of the (BBB)-brane (and describe its spectral data) in a wider generality, namely in the case where the unramified cover $p : C \longrightarrow X$ is of degree m not necessarily equal to the rank n. In such a case, one must consider polystable Higgs bundles over C of rank r, such that $n = mr$. It is however unclear how to endow such (BBB)-branes with hyperholomorphic bundles.

Similarly, we construct $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ in the more general setup of a degree m cover $p : C \longrightarrow X$. In the absence of a Hitchin section on $\mathbf{M}_C(r, d)$ we make use of very stable bundles on C, which define natural complex Lagrangian multisections of the Hitchin fibration. We explore this in Section 7.

As mentioned above, when the Galois group is cyclic, the support of our (BBB)-branes is $\mathbf{M}_X(n, d)^\xi$. It is interesting to notice that $\mathbf{M}_X(n, d)^\xi$ plays a central role in the proof by T. Hausel and M. Thaddeus [HT] of topological mirror symmetry for the moduli spaces of Higgs bundles for the Langlands dual groups $\mathrm{SL}(n, \mathbb{C})$ and $\mathrm{PSL}(n, \mathbb{C})$ for $n = 2, 3$ (the general case has recently been proved by M. Groechenig, D. Wyss and P. Ziegler [GWZ], and, more recently, by D. Maulik and J. Shen [MS]). One might thus hope that further study of our dual branes in this setting would provide a better geometric understanding of the calculation by Hausel and Thaddeus. We hope to come back to this question in a future article.

1.4. **Organization of the paper.** Here is a brief description of the organization of the paper. In Section 2 we recall some background material on the Hitchin system. In Section 3 we study the locus $\mathbf{M}_X(n, d)^p$, including the corresponding spectral data, for p an unramified cover of degree m, with m dividing n. Section 4 deals with the construction and description of the Narasimhan–Ramanan (BBB)-branes. In Section 5 we construct the complex Lagrangian subvarieties $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$, which support (BAA)-branes. In Section 6, after recalling some background facts on the Fourier–Mukai transform for compactified Jacobians of integral curves and describing in Section 6.2 the role of the normalization of the curve in the transform, we prove our main duality result, namely Theorem 6.5. Finally, in Section 7, we generalize parts of the previous study to the case where $p : C \longrightarrow X$ has degree strictly less than n and no Hitchin section exists on $\mathbf{M}_C(r, d)$.

2. Higgs bundles and the Hitchin system

The purpose of this section is to recall the basics on Higgs bundle moduli spaces which will be used in the remaining part of the paper.

2.1. **Higgs bundles and their moduli space.** Let X be a smooth projective curve over $\mathbb{C}$, of genus $g \geqslant 2$. A *Higgs bundle* over X is a pair (E, φ) given by a holomorphic vector bundle $E \longrightarrow X$, and $\varphi \in H^0(X, \mathrm{End}(E) \otimes K_X)$, where K_X is the canonical bundle. The section φ is called the *Higgs field*. The *rank* and *degree* of a Higgs bundle are those of the underlying vector bundle E. Such a rank n Higgs bundle is also said to be a $\mathrm{GL}(n, \mathbb{C})$-*Higgs bundle*. Occasionally, we shall refer to $\mathrm{SL}(n, \mathbb{C})$-*Higgs bundles*, in which E is required to have a fixed given determinant bundle and φ to be traceless.

Let $\mathbf{M}_X(n, d)$ denote the moduli space os S-equivalence classes of semistable rank n and degree d Higgs bundles on X. Its points are represented by the unique polystable representative of the corresponding S-equivalence class. It is a quasi-projective variety of complex dimension

$$(2.1) \qquad \dim \mathbf{M}_X(n, d) = 2n^2(g - 1) + 2.$$

The closely related *de Rham moduli space* $\mathbf{M}_X^{\mathrm{dR}}(n, d)$ is the moduli space of connections with constant central curvature on a fixed C^∞ vector bundle over X of rank n and degree d. Non-abelian Hodge theory [Hi1, Si2, Si3, Do, Co] establishes the existence of a homeomorphism

between these spaces, $\mathbf{M}_X(n, d) \cong \mathbf{M}_X^{\mathrm{dR}}(n, d)$. This homeomorphism restricts to a diffeomorphism on the smooth locus of $\mathbf{M}_X(n, d)$, whose underlying manifold is a hyperkähler manifold [Hi1] with complex structures

$$I_1, \quad I_2 \quad \text{and} \quad I_3 = I_1 I_2.$$

Here I_1 is the complex structure coming from $\mathbf{M}_X(n, d)$ and I_2 is the one coming from $\mathbf{M}_X^{\mathrm{dR}}(n, d)$. Let also ω_j be the Kähler form associated to I_j and $\Omega_{X,j} = \omega_{j+1} + \mathrm{i}\omega_{j-1}$ the corresponding holomorphic symplectic form.

2.2. **The Hitchin system.** We recall here the spectral construction given in [Hi2, BNR]. Let $(P_1, \ldots, P_n)$ be a basis of $\mathrm{GL}(n, \mathbb{C})$-invariant polynomials with $\deg(P_i) = i$; for instance, we could take $P_i(x) = (-1)^i \mathrm{tr}(\wedge^i x)$. The *Hitchin map* or *Hitchin fibration* is

$$\begin{array}{cccc} (2.2) \qquad h_{X,n}: & \mathbf{M}_X(n, d) & \longrightarrow & B_{X,n} := \bigoplus_{i=1}^n H^0(X, K_X^i) \\ & (E, \varphi) & \longmapsto & (P_1(\varphi), \ldots, P_n(\varphi)). \end{array}$$

Note that $\dim(B_{X,n}) = n^2(g - 1) + 1 = \frac{1}{2}\dim(\mathbf{M}_X(n, d))$.

Consider the total space $|K_X|$ of the canonical bundle and the surjective morphism $\pi: |K_X| \longrightarrow X$. The pullback bundle $\pi^* K_X \longrightarrow |K_X|$ has a tautological section λ. Given an element $b \in B_{X,n}$, with $b = (b_1, \ldots, b_n)$, the *spectral curve* $X_b \subset |K_X|$ is the vanishing locus of the section of $\pi^* K_X^n$ given by

$$(2.3) \qquad \lambda^n + \pi^* b_1 \lambda^{n-1} + \cdots + \pi^* b_{n-1} \lambda + \pi^* b_n \in H^0(|K_X|, \pi^* K_X^n).$$

The restriction of $\pi: |K_X| \longrightarrow X$ to X_b yields a ramified degree n cover denoted by

$$(2.4) \qquad \pi_b: X_b \longrightarrow X.$$

For generic b, the spectral curve X_b is smooth. For any b, the (arithmetic) genus of X_b is [Hi2]

$$g(X_b) = n^2(g - 1) + 1.$$

Additionally, $\pi_{b,*}\mathcal{O}_{X_b} \cong \mathcal{O}_X \oplus K_X^{-1} \oplus \cdots \oplus K_X^{1-n}$, thus

$$\deg(\pi_{b,*}\mathcal{O}_{X_b}) = -n(n - 1)(g - 1).$$

Notation 2.1. For the remainder of the paper, let us denote the degree of the ramification divisor of the spectral curve $X_b \longrightarrow X$ in $B_{X,n}$, by

$$(2.5) \qquad \delta := n(n - 1)(g - 1).$$

Given a rank 1 torsion-free sheaf $\mathcal{F}$ over X_b of degree $d + \delta$, we have that

$$(2.6) \qquad E_{\mathcal{F}} := \pi_* \mathcal{F}$$

is a vector bundle on X of rank n and degree d. Tensorization by the tautological section yields

$$\mu_{\mathcal{F}} : \mathcal{F} \xrightarrow{\otimes \lambda} \mathcal{F} \otimes \pi_b^* K_X.$$

Since π is an affine morphism, $\mu_{\mathcal{F}}$ corresponds to a Higgs field

$$(2.7) \qquad \varphi_{\mathcal{F}} := \pi_{b,*}\mu_{\mathcal{F}} : E_{\mathcal{F}} \longrightarrow E_{\mathcal{F}} \otimes K_X$$

on $E_{\mathcal{F}}$ with characteristic polynomial determined by $b \in B_{X,n}$ [BNR, Sch, Si3]. The pair $(X_b, \mathcal{F})$ is said to be the *spectral datum* of the Higgs bundle $(E_{\mathcal{F}}, \varphi_{\mathcal{F}})$. This establishes a one-to-one correspondence, sometimes called *spectral correspondence*, between the *Hitchin fiber* $h_{X,n}^{-1}(b)$ and the moduli space of rank 1 torsion-free sheaves on X_b of degree $d + \delta$ and linearization naturally induced from the base [Si3, Corollary 6.9], and so

$$(2.8) \qquad h_{X,n}^{-1}(b) \cong \overline{\mathrm{Jac}}^{d+\delta}(X_b).$$

The construction of $\overline{\mathrm{Jac}}^{d+\delta}(X_b)$ follows from [Si2, Theorem 1.21] and it is a compactification of the Jacobian $\mathrm{Jac}^{d+\delta}(X_b)$ of degree $d + \delta$ line bundles on X_b, hence we refer to it as the *compactified Jacobian*. Denote by $\mathscr{X} \longrightarrow B_{X,n}$ the family of n-to-1 spectral curves inside $|K_X|$ and endow it with a linearization induced from a linearization on X. Thanks again to [Si2, Theorem 1.21] one can consider the relative compactified Jacobian $\overline{\mathrm{Jac}}_{B_{X,n}}^{d+\delta}(\mathscr{X})$. The

spectral correspondence promotes to the whole moduli space [Si3, Section 6], giving rise to the isomorphism

$$(2.9) \qquad S_{X,n}: \quad \overline{\mathrm{Jac}}_{B_{X,n}}^{d+\delta}(\mathscr{X}) \quad \longrightarrow \quad \mathbf{M}_X(n,d)$$
$$\mathcal{F} \longrightarrow X_b \quad \longmapsto \quad (E_{\mathcal{F}}, \varphi_{\mathcal{F}}).$$

When the degree is a multiple of the rank, $d = nd'$, the Hitchin fibration $h_{X,n}: \mathbf{M}_X(n,nd') \longrightarrow B_{X,n}$ admits a so-called *Hitchin section* associated to any line bundle $\mathcal{L} \in \mathrm{Jac}^{d'+\delta/n}(X)$. This section is constructed by assigning to each $b \in B$ the Higgs bundle whose spectral datum is given by the line bundle $\pi_b^* \mathcal{L} \longrightarrow X_b$. In other words, we have a morphism

$$(2.10) \qquad \sigma_{X,\mathcal{L}}: \quad B_{X,n} \quad \longrightarrow \quad \mathbf{M}_X(n,nd')$$
$$b \quad \longmapsto \quad (\pi_{b,*}\pi_b^*\mathcal{L}, \pi_{b,*}\mu_{\pi_b^*\mathcal{L}}).$$

Hitchin [Hi3] considered such sections for $\mathcal{L} = K_X^{(n-1)/2}$. In this case we omit the reference to the line bundle in our notation and we simply denote the corresponding Hitchin section by σ_X.

3. UNRAMIFIED COVERS AND HIGGS BUNDLES

3.1. Unramified covers and hyperholomorphic subvarieties in the moduli space.
Let $p: C \longrightarrow X$ be a connected unramified cover of degree m and Galois group Γ. In this section, we study the subvarieties that arise in the moduli space of Higgs bundles out of this geometrical setting. Some of the following results have been already obtained in [HPa].

Let K_C be the canonical bundle of C, and let

$$\eta: |K_C| \longrightarrow C$$

be the corresponding projection. As p is unramified

$$(3.1) \qquad K_C \cong p^* K_X$$

and

$$|K_C| \cong |K_X| \times_X C,$$

hence we have a Cartesian diagram

$$(3.2) \qquad \begin{array}{ccc} |K_C| & \xrightarrow{\eta} & C \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ |K_X| & \xrightarrow{\pi} & X, \end{array}$$

q being the obvious projection. In particular, q is an unramified Γ-cover and $\eta: |K_C| \longrightarrow C$ is Γ-equivariant. Note that the automorphism $\gamma: C \longrightarrow C$, associated to any element of the Galois group, gives rise to an automorphism $\gamma: |K_C| \longrightarrow |K_C|$ that we still denote by γ by abuse of notation.

By (3.1), the pullback under $p: C \longrightarrow X$ of a Higgs bundle is again a Higgs bundle. Moreover, polystability is preserved (e.g. because it sends solutions to the Hitchin equations on X to solutions to the Hitchin equations on C, cf. [Hi1]). So we have a morphism

$$(3.3) \qquad \hat{p}: \quad \mathbf{M}_X(n,d) \quad \longrightarrow \quad \mathbf{M}_C(n,md)$$
$$(E,\varphi) \quad \longmapsto \quad (p^*E, p^*\varphi)$$

between the moduli spaces.

Remark 3.1. The image of $\hat{p}$ lies in $\mathbf{M}_C(n,md)^\Gamma$, the fixed point locus under the induced Galois group action on $\mathbf{M}_C(n,md)$ by pullback.

By the projection formula and (3.1), if (F,ϕ) is a Higgs bundle over C of rank r, then $(p_*F, p_*\phi)$ is a rank $n = mr$ Higgs bundle over X. Since p is unramified and X and C are proper, p is finite, so by [NR3, Lemma 2.1 (ii)] we have

$$(3.4) \qquad p^*p_*(F,\phi) = \bigoplus_{\gamma \in \Gamma} \gamma^*(F,\phi).$$

Consider the moduli space $\mathbf{M}_C(r, d)$ of rank r and degree d Higgs bundles over C.

Proposition 3.2. *Let $p : C \longrightarrow X$ be a connected unramified m-cover with Galois group Γ and let $n = mr$. The pushforward under p,*

$$(3.5) \qquad \begin{array}{rccc} \check{p} : & \mathbf{M}_C(r, d) & \longrightarrow & \mathbf{M}_X(n, d) \\ & (F, \phi) & \longmapsto & (p_* F, p_* \phi), \end{array}$$

is a hyperholomorphic finite morphism. Moreover, two rank r Higgs bundles over C have the same image if and only if they are in the same orbit under the Γ-action by pullback, so

$$\mathbf{M}_X(n, d)^p := \mathrm{Im}(\check{p}) \cong \mathbf{M}_C(r, d)/\Gamma.$$

Proof. It follows from [NR3, Proposition 3.1] that $\check{p}$ has image contained in the semistable locus and so it is well defined. Moreover, it is hyperholomorphic because it corresponds to pushforward of projectively flat bundles under the Non-abelian Hodge Theorem.

Since p is unramified, it is obvious that two rank r Higgs bundles over C in the same orbit under the Γ-action by pullback will give the same image under $\check{p}$. Thanks to (3.4), we see that they have the same image only if they lie in the same Γ-orbit, so $\mathbf{M}_X(n, d)^p \cong \mathbf{M}_C(r, d)/\Gamma$. Note that $\mathbf{M}_C(r, d)/\Gamma$ is naturally a geometric quotient since $\mathbf{M}_C(r, d)$ is quasi-projective, hence $\mathbf{M}_C(r, d) \longrightarrow \mathbf{M}_C(r, d)/\Gamma$ is finite. Thus $\check{p}$ is a finite morphism as it commutes with the composition of the isomorphism $\mathbf{M}_X(n, d)^p \cong \mathbf{M}_C(r, d)/\Gamma$ with the finite quotient, which is a finite morphism. $\qquad \square$

3.2. **The Hitchin map and unramified covers.** Fix a connected unramified cover $p : C \longrightarrow X$ of degree m, with Galois group Γ. In this section we study the restriction of the Hitchin map to $\mathbf{M}_X(n, d)^p$, with n a multiple of m. Let $B^p := h_{X,n}(\mathbf{M}_X(n, d)^p) \subset B_{X,n}$ to be the image under the Hitchin map of the image of $\check{p}$.

Notation 3.3. Let $r = n/m$. We shall employ the same notation for the Hitchin system in $\mathbf{M}_C(r, d)$ as the one used in Section 2.2. So let

$$h_{C,r} : \mathbf{M}_C(r, d) \longrightarrow B_{C,r} = \bigoplus_{i=1}^{r} H^0(C, K_C^i)$$

be the Hitchin map. For any given $a = (a_1, \dots, a_r) \in B_{C,r}$, denote by C_a the corresponding spectral curve in $|K_C|$, with projection map

$$(3.6) \qquad \eta_a = \eta|_{C_a} : C_a \longrightarrow C,$$

where η is defined in (3.2). The curve C_a is defined by the equation

$$(3.7) \qquad \hat{\lambda}^r + \eta^* a_1 \hat{\lambda}^{r-1} + \cdots + \eta^* a_r = 0,$$

with $\hat{\lambda}$ the tautological section of $\eta^* K_C$. Set $\gamma(a) := \gamma^* a$ for every element $\gamma \in \Gamma$, and write $\gamma(a) = (\gamma(a)_1, \dots, \gamma(a)_r)$ where $\gamma(a)_i \in H^0(C, K_C^i)$.

The next proposition establishes the behavior of the Hitchin map under pullback by the Galois group.

Proposition 3.4. *The Hitchin map $h_{C,r} : \mathbf{M}_C(r, d) \longrightarrow B_{C,r}$ is equivariant for the action of the Galois group Γ of $p : C \longrightarrow X$ by pullback. Furthermore, for any $a \in B_{C,r}$, one has the Cartesian diagram*

$$(3.8) \qquad \begin{array}{ccc} C_{\gamma(a)} & \xrightarrow{\;\gamma\;} & C_a \\ {\scriptstyle \eta_{\gamma(a)}} \downarrow & & \downarrow {\scriptstyle \eta_a} \\ C & \xrightarrow{\;\gamma\;} & C. \end{array}$$

In particular, for any $\gamma \in \Gamma$, the spectral curves $C_{\gamma(a)}$ and C_a are isomorphic.

Proof. Let (F, ϕ) be a Higgs bundle in $\mathbf{M}_C(r, d)$ such that $h_{C,r}(F, \phi) = a \in B_{C,r}$. Let P_i be an invariant polynomial of degree i, and observe that $P_i(\gamma^*\phi) = \gamma^* P_i(\phi)$. It then follows that $h_{C,r}(\gamma^*(F, \phi)) = \gamma^* h_{C,r}(F, \phi)$ and the first part follows.

Since the spectral curve $C_a \subset |K_C|$ is given by the vanishing of (3.7), then $C_{\gamma(a)}$ is given by the vanishing of the pullback of (3.7) under γ. Note that $(\gamma(a))_i = \gamma^* a_i$ and that the embedding $C_a \hookrightarrow |K_C|$ is Γ-equivariant (so that $\eta^* \gamma^* = \gamma^* \eta^*$). Therefore, given $y \in C_{\gamma(a)}$, by definition of pullback, one has that

$$\left(\hat{\lambda}^r + \eta^*(\gamma^* a)_1 \hat{\lambda}^{r-1} + \cdots + \eta^*(\gamma^* a)_r \right)(y) = 0$$

is equivalent to

$$\left(\hat{\lambda}^r + (\eta^* a_1)\hat{\lambda}^{r-1} + \cdots + \eta^* a_r \right)(\gamma(y)) = 0$$

because $\hat{\lambda}$ is Γ-invariant, since $\hat{\lambda} = q^*\lambda$, where $q : |K_C| \longrightarrow |K_X|$ and λ is the tautological section of $\pi^* K_X$. Thus $\gamma(y) \in C_a$ and the commutativity of (3.8) holds. The rest of the proposition follows from this. $\qquad\square$

Consider the moduli space of rank n and degree d Higgs bundles on C and its associated Hitchin map

$$h_{C,n} : \mathbf{M}_C(n, d) \longrightarrow B_{C,n} = \bigoplus_{i=1}^{n} H^0(C, K_C^i).$$

By (3.1), it follows that p induces

(3.9) $$p^* : B_{X,n} \longrightarrow B_{C,n}.$$

Lemma 3.5. *The induced map $p^* : B_{X,n} \longrightarrow B_{C,n}$ is injective and the following diagram commutes, where $\hat{p}$ is defined in (3.3):*

$$\begin{array}{ccc}
\mathbf{M}_X(n, d) & \xrightarrow{\hat{p}} & \mathbf{M}_C(n, md) \\
{\scriptstyle h_{X,n}} \downarrow & & \downarrow {\scriptstyle h_{C,n}} \\
B_{X,n} & \xhookrightarrow{p^*} & B_{C,n}.
\end{array}$$

Proof. Since p is a local isomorphism, $p^* : H^0(X, K_X^i) \longrightarrow H^0(C, K_C^i)$ is injective for every i, so $p^* : B_{X,n} \longrightarrow B_{C,n}$ is injective as well. The commutativity of the diagram is immediate from functoriality of pullback. $\qquad\square$

Proposition 3.6. *Let $b \in B_{X,n}$ and $\tilde{b} = p^*b \in B_{C,n}$. Let $X_b \subset |K_X|$ and $C_{\tilde{b}} \subset |K_C|$ be the corresponding spectral curves. Then*

$$C_{\tilde{b}} \cong X_b \times_X C$$

and there is a Cartesian diagram

(3.10)
$$\begin{array}{ccc}
C_{\tilde{b}} & \xrightarrow{\eta_{\tilde{b}}} & C \\
{\scriptstyle q_{\tilde{b}}} \downarrow & & \downarrow {\scriptstyle p} \\
X_b & \xrightarrow{\pi_b} & X,
\end{array}$$

where $q_{\tilde{b}}$, $\eta_{\tilde{b}}$ and π_b are the restrictions of the maps from (3.2). In particular,

  (i) *X_b is reduced if and only if $C_{\tilde{b}}$ is reduced,*
  (ii) *$q_{\tilde{b}} : C_{\tilde{b}} \longrightarrow X_b$ is a connected unramified Γ-cover, and*
 (iii) *$\eta_{\tilde{b}}$ is a Γ-equivariant ramified degree n cover, whose ramification locus is the pullback of that of π_b.*

Proof. View the curve X_b as a divisor in $|K_X|$. First we prove that the pullback by $q : |K_C| \longrightarrow |K_X|$ of this divisor is $C_{\tilde{b}}$. Write $b = (P_1(\varphi), \ldots, P_n(\varphi))$ for some Higgs bundle (E, φ) in the Hitchin fiber of b. Then $X_b \subset |K_X|$ is defined by

$$\lambda^n + \pi^* P_1(\varphi)\lambda^{n-1} + \cdots + \pi^* P_n(\varphi) = 0$$

where we recall that $\lambda \in H^0(|K_X|, \pi^* K_X)$ is the tautological section. Thanks to Lemma 3.5, $C_{\tilde{b}}$ is defined by

$$(3.11) \qquad \hat{\lambda}^n + \eta^* p^* P_1(\varphi) \hat{\lambda}^{n-1} + \cdots + \eta^* p^* P_n(\varphi) = 0,$$

where $\hat{\lambda} \in H^0(|K_C|, \eta^* K_C)$ is the tautological section. Clearly $\hat{\lambda} = q^* \lambda$ and so, in view of (3.2), the equation defining $C_{\tilde{b}}$ is

$$q^* \lambda^n + q^* \pi^* P_1(\varphi) q^* \lambda^{n-1} + \cdots + q^* \pi^* P_n(\varphi) = 0.$$

This shows that $C_{\tilde{b}} = q^* X_b$ as desired.

Viewing $X_b \times_X C$ inside $|K_X| \times_X C \cong |K_C|$ one readily sees that it satisfies (3.11) and, therefore, by the universal property of the fiber product, it is isomorphic to $q^* X_b$. The rest of the lemma follows from this observation. $\qquad \square$

We now study the relation of $\hat{p} \circ \check{p} : \mathbf{M}_C(r, d) \longrightarrow \mathbf{M}_C(n, md)$ with the corresponding Hitchin maps (recall that $n = mr$).

Proposition 3.7. *Let (F, ϕ) be a Higgs bundle of rank r over C and consider $\hat{p} \circ \check{p}(F, \phi)$. Let $a \in B_{C,r}$ and $\tilde{b} \in B_{C,n}$ be the image under the Hitchin map of (F, ϕ) and $\hat{p} \circ \check{p}(F, \phi)$ respectively. Then, the spectral curve $C_{\tilde{b}} \subset |K_C|$ is given by the vanishing of the section*

$$\prod_{\gamma \in \Gamma} \left(\hat{\lambda}^r + \eta^* \gamma(a)_1 \hat{\lambda}^{r-1} + \cdots + \eta^* \gamma(a)_r \right) \in H^0(|K_C|, \eta^* K_C^n).$$

In particular $C_{\tilde{b}}$ is reduced if and only if C_a is reduced and $\gamma(a) \neq \gamma'(a)$ for all distinct $\gamma, \gamma' \in \Gamma$ (i.e., if $a \in B_{C,r}$ is not fixed by any non-trivial element of the Galois group). In that case $C_{\tilde{b}}$ is reducible and

$$(3.12) \qquad C_{\tilde{b}} = \bigcup_{\gamma \in \Gamma} C_{\gamma(a)}.$$

Proof. From (3.4), one has that $C_{\tilde{b}}$ is given by the vanishing of

$$\hat{\lambda}^n + \eta^* P_1 \left(\bigoplus_{\gamma \in \Gamma} \gamma^* \phi \right) \hat{\lambda}^{n-1} + \cdots + \eta^* P_n \left(\bigoplus_{\gamma \in \Gamma} \gamma^* \phi \right)$$

i.e., of

$$\prod_{\gamma \in \Gamma} \left(\hat{\lambda}^r + \eta^* P_1(\gamma^* \phi) \hat{\lambda}^{r-1} + \cdots + \eta^* P_r(\gamma^* \phi) \right).$$

The rest of the proposition follows immediately. $\qquad \square$

We can now describe the subspace $B^p = h_{X,n}(\mathbf{M}_X(n, d)^p) \subset B_{X,n}$.

Proposition 3.8. *There exists a map,*

$$\zeta : B_{C,r} \longrightarrow B^p,$$

making the Γ-equivariant diagram

$$(3.13) \qquad \begin{array}{ccc} \mathbf{M}_C(r, d) & \xrightarrow{\ \check{p}\ } & \mathbf{M}_X(n, d)^p \\ {\scriptstyle h_{C,r}} \downarrow & & \downarrow {\scriptstyle h_{X,n}} \\ B_{C,r} & \xrightarrow[\ \zeta\ ]{} & B^p \end{array}$$

commutative, with Γ acting by pullback on $\mathbf{M}_C(r, d)$ and $B_{C,r}$, and trivially on $\mathbf{M}_X(n, d)^p$ and B^p. The map ζ induces an isomorphism

$$(3.14) \qquad B^p \cong B_{C,r} / \Gamma.$$

Hence,

$$(3.15) \qquad \dim(B^p) = rn(g - 1) + 1.$$

Proof. By Proposition 3.4, $\hat{p} \circ \check{p}$ induces the morphism $B_{C,r} \longrightarrow B_{C,n}$ defined by

$$(P_1(\phi), \dots, P_r(\phi)) \mapsto \left(P_1\Big(\bigoplus_{\gamma \in \Gamma} \gamma^* \phi \Big), \dots, P_n\Big(\bigoplus_{\gamma \in \Gamma} \gamma^* \phi \Big) \right).$$

The image of this morphism is clearly contained in $p^*(B^p)$, and by Lemma 3.5, p^* is injective, so the previous map defines the map $\zeta : B_{C,r} \longrightarrow B^p$. Explicitly,

$$\zeta(P_1(\phi), \dots, P_r(\phi)) = (P_1(p_*\phi), \dots, P_1(p_*\phi)).$$

The commutativity of (3.13) is immediate from construction, and so is (3.14).

Since B^p is isomorphic to the finite quotient $B_{C,r}/\Gamma$, its dimension is

$$\dim(B^p) = \dim(B_{C,r}) = r^2(g(C) - 1) + 1.$$

As $C \longrightarrow X$ is an unramified m-cover, the genus of C equals $g(C) = m(g-1) + 1$ and (3.15) holds. $\qquad\square$

Definition 3.9. Let $B_{C,r}^{\mathrm{sm}}$ be the dense open subset of $B_{C,r}$ given by those points $a \in B_{C,r}$ whose associated spectral curve C_a is smooth. Let us also consider the Γ-invariant subset $B_{C,r}^{\mathrm{it}}$ to be the open subset of $B_{C,r}^{\mathrm{sm}}$ given by those elements where Γ acts freely. Next, define $B_{C,r}^{\mathrm{ni}} \subset B_{C,r}^{\mathrm{it}}$ as the subset given by those curves C_a whose intersection with $C_{\gamma(a)}$ has nodal singularities, for every $\gamma \in \Gamma$, and such that $C_a \cap C_{\gamma(a)} \cap C_{\gamma'(a)} = \varnothing$ for all $\gamma' \in \Gamma \backslash \{1, \gamma\}$. Set as well $B_{\mathrm{it}}^p := B_{C,r}^{\mathrm{it}}/\Gamma$ and $B_{\mathrm{ni}}^p := B_{C,r}^{\mathrm{ni}}/\Gamma$. Finally, define

$$\mathbf{M}_X(n,d)_{\mathrm{ni}}^p := \mathbf{M}_X(n,d)^p \times_{B^p} B_{\mathrm{ni}}^p$$

and

$$\mathbf{M}_C(r,d)_{\mathrm{ni}} := \mathbf{M}_C(r,d) \times_{B_{C,r}} B_{C,r}^{\mathrm{ni}}.$$

Remark 3.10. Note that B_{it}^p parametrizes reduced curves by Propositions 3.7 and 3.6 (i). By Theorem 3.11 (v) below, such spectral curves are *integral*, explaining the notation. The notation used for B_{ni}^p stands for *nodal and integral*.

Recall that we defined δ in (2.5) as the degree of the ramification divisor of the spectral curves in $B_{X,n}$. Accordingly, we define ρ to be the degree of the ramification divisor of the spectral curves $C_a \longrightarrow C$ in $B_{C,r}$,

$$(3.16) \qquad \rho := r(r-1)(g(C)-1) = n(r-1)(g-1).$$

Next, we describe the geometry of the spectral curves X_b when b lies in B_{it}^p. For the statement of the following theorem, recall the notation of the Cartesian diagrams (3.10) and (3.13).

Theorem 3.11. *Let $a \in B_{C,r}^{\mathrm{it}}$ and consider $b = \zeta(a) \in B_{\mathrm{it}}^p$ and $\tilde{b} = p^*b \in B_{C,n}$.*

(i) *The spectral curve $C_{\tilde{b}}$ is reduced, connected, with $m = |\Gamma|$ irreducible components $\{C_{\gamma(a)}\}_{\gamma \in \Gamma}$, all isomorphic to each other. The singular divisor $\mathrm{sing}(C_{\tilde{b}})$ of $C_{\tilde{b}}$ is given by the intersections of distinct components, and has degree $n^2(m-1)(g-1)$. The Galois group Γ of $q_{\tilde{b}}$ permutes the components of $C_{\tilde{b}}$.*

(ii) *The spectral curve X_b is reduced but singular. Its singular divisor $\mathrm{sing}(X_b)$ satisfies $q_{\tilde{b}}^* \mathrm{sing}(X_b) = \mathrm{sing}(C_{\tilde{b}})$. In particular*

$$(3.17) \qquad \deg(\mathrm{sing}(X_b)) = \delta - \rho = n(n-r)(g-1).$$

If $x \in X_b$ is a singularity, and $y \in C_{\gamma'(a)} \cap C_{\gamma''(a)}$ (with $\gamma' \neq \gamma''$) is a singular point on $C_{\tilde{b}}$ mapping to x, then the m singularities of $C_{\tilde{b}}$ mapping to x are precisely the ones of the form $\gamma(y) \in C_{\gamma\gamma'(a)} \cap C_{\gamma\gamma''(a)}$, for all $\gamma \in \Gamma$.

(iii) *Let $C_a \subset C_{\tilde{b}}$ be an irreducible component. Let $q : |K_C| \longrightarrow |K_X|$ be as in (3.2). Then $q(C_a) = q_a(C_a) = X_b$ and*

$$(3.18) \qquad \nu_a := q_a : C_a \longrightarrow X_b$$

is a normalization fitting in the commutative diagram

(3.19)
$$\begin{array}{ccc} C_a & \xrightarrow{\eta_a} & C \\ \nu_a \downarrow & & \downarrow p \\ X_b & \xrightarrow{\pi_b} & X. \end{array}$$

In addition,

(3.20)
$$\nu_{\gamma(a)} = q_{\gamma(a)} = \nu_a \circ \gamma.$$

(iv) *The (disconnected) curve in $|K_C| \times B_{C,r}$ given by*

$$\widetilde{C}_{\tilde{b}} = \bigsqcup_{\gamma \in \Gamma} C_{\gamma(a)} \times \{\gamma(a)\}$$

is the normalization of $C_{\tilde{b}}$, where the normalization morphism $\tilde{\nu}_{\tilde{b}} : \widetilde{C}_{\tilde{b}} \longrightarrow C_{\tilde{b}}$ is given by projecting onto the first factor. Furthermore, the diagram

(3.21)
$$\begin{array}{ccc} \widetilde{C}_{\tilde{b}} & \xrightarrow{\tilde{\nu}_{\tilde{b}}} & C_{\tilde{b}} \\ \tilde{q}_a \downarrow & & \downarrow q_{\tilde{b}} \\ C_a & \xrightarrow{\nu_a} & X_b \end{array}$$

is Cartesian, where $\tilde{q}_a$ is the unramified Γ-cover given by $\tilde{q}_a(z, \gamma(a)) = \gamma^{-1}(z)$.

(v) *The spectral curve X_b is integral.*

Proof. Since a is taken in $B_{C,r}^{\mathrm{it}}$, $C_{\tilde{b}}$ satisfies the hypothesis of the second part of Proposition 3.7. Then, $C_{\tilde{b}}$ is reduced and reducible, decomposing as described in (3.12). Again by hypothesis, C_a is smooth and therefore irreducible. Furthermore, $C_a \cong C_{\gamma(a)}$ by Proposition 3.4. Therefore, (3.12) is, in fact, a decomposition of $C_{\tilde{b}}$ into its irreducible components and the singularities of $C_{\tilde{b}}$ are the intersections of the distinct components,

$$\mathrm{sing}(C_{\tilde{b}}) = \bigcup_{\gamma \neq \gamma'} C_{\gamma(a)} \cap C_{\gamma'(a)}.$$

Note that all the $C_{\gamma(a)}$ lie in the linear system rC inside $|K_C|$. This has two important consequences. The first one is that we can perform the count of the intersection divisor taking a generic element of B_{it}^p without triple intersections of the components $C_{\gamma(a)}$. The second one is that two distinct components, $C_{\gamma(a)}$ and $C_{\gamma'(a)}$, lie in the linear system of rC inside $|K_C|$, so their intersection is

(3.22)
$$r^2 \cdot (C)^2 = r^2(g(C) - 1) = 2nr(g - 1).$$

Hence,

$$\deg(\mathrm{sing}(C_{\tilde{b}})) = \binom{m}{2} 2nr(g - 1) = n^2(m - 1)(g - 1).$$

It follows from Proposition 3.4 that the Galois group permutes the $C_{\gamma(a)}$. This completes the proof of (i).

For (ii), note that by (i) and Proposition 3.6 (i), X_b is reduced. The rest follows from the fact that $q_{\tilde{b}} : C_{\tilde{b}} \longrightarrow X_b$ is an unramified Γ-cover, as we have shown in Proposition 3.6 (ii).

To see (iii), we start by observing that (3.20) follows from (3.8). Then, the maps $\{q_{\gamma(a)} = \nu_{\gamma(a)}\}_{\gamma \in \Gamma}$, have all the same image. From this, in view of (i) and the fact that $q_{\tilde{b}} : C_{\tilde{b}} \longrightarrow X_b$ is surjective, we conclude that $q_{\gamma(a)} = \nu_{\gamma(a)}$ is surjective for each $\gamma \in \Gamma$. Since C_a is smooth, in order to prove that the maps $\nu_{\gamma(a)}$ are normalization morphisms, it now suffices to show that one of them (say ν_a) is generically injective. If $x \in X_b$ does not belong to the ramification locus $\mathrm{sing}(X_b)$ then, by (i) and (ii), each point inside the fiber $q_{\tilde{b}}^{-1}(x)$ lies in a different irreducible component of $C_{\tilde{b}}$. So ν_a is injective over the smooth locus of X_b. The commutativity of (3.10) concludes the proof of (iii).

Consider now (iv). The map $\tilde{\nu}_{\tilde{b}} : \widetilde{C}_{\tilde{b}} \longrightarrow C_{\tilde{b}}$ is a normalization morphism by the description of $C_{\tilde{b}}$ and its singularities given in (i). To see that (3.21) is Cartesian, note that by the universal property of fibered products there is a morphism

$$\widetilde{C}_{\tilde{b}} \longrightarrow C_a \times_{X_b} C_{\tilde{b}} \, .$$

Since any morphism of principal bundles is an isomorphism, the statement follows.

Finally, for (v), we already know that X_b is reduced. Since C_a is smooth, it is irreducible, thus X_b is irreducible by (iii). $\qquad \square$

Remark 3.12. For every $b \in B_{\mathrm{it}}^p$ the corresponding spectral curve X_b is normalized by a smooth spectral curve C_a in $B_{C,r}^{\mathrm{sm}}$.

We introduce some notation to describe the fibers of the Hitchin map restricted to the subvariety $\mathbf{M}_X(n,d)^p$. Recall that ρ is defined in (3.16) as the degree of the ramification divisor of the spectral curves $C_a \longrightarrow C$. Consider the pushforward under the normalization morphism $\nu_a : C_a \longrightarrow X_b$ defined in (3.18),

$$(3.23) \qquad \begin{array}{cccc} \check{\nu}_a : & \mathrm{Jac}^{d+\rho}(C_a) & \longrightarrow & \overline{\mathrm{Jac}}^{d+\delta}(X_b) \\ & L & \longmapsto & \nu_{a,*}L, \end{array}$$

where $\delta - \rho = \deg(\mathrm{sing}(X_b))$ by (ii) of Theorem 3.11. Any other $\gamma(a)$ also projects to b under the map $\zeta : B_{C,r} \longrightarrow B^p$, defined in Proposition 3.8, and for such $\gamma(a)$, a similar map $\check{\nu}_{\gamma(a)}$ exists as well. Furthermore, (3.20) implies that all these morphisms share the same image,

$$\mathrm{Im}(\check{\nu}_a) = \mathrm{Im}(\check{\nu}_{\gamma(a)}).$$

Recall the morphism $\check{p}$ in (3.5). The following proposition describes the fibers of the Hitchin map over B_{it}^p restricted to $\mathbf{M}_X(n,d)^p$.

Proposition 3.13. *Let $b \in B_{\mathrm{it}}^p$, and pick $a \in B_{C,r}^{\mathrm{it}}$ such that $\zeta(a) = b$. Then,*

(i) the first line of diagram (3.13) restricts to

$$(3.24) \qquad \bigsqcup_{\gamma \in \Gamma} \mathrm{Jac}^{d+\rho}(C_{\gamma(a)}) \xrightarrow{\check{p}_b} h_{X,n}^{-1}(b) \cap \mathbf{M}_X(n,d)^p,$$

and $\check{p}_b$ is an unramified cover, with the Galois group Γ acting by pullback, hence permuting the connected components of the domain.

(ii) the intersection of $\mathbf{M}_X(n,d)^p$ with the Hitchin fiber is

$$(3.25) \qquad h_{X,n}^{-1}(b) \cap \mathbf{M}_X(n,d)^p \cong \bigsqcup_{\gamma \in \Gamma} \mathrm{Jac}^{d+\rho}(C_{\gamma(a)})/\Gamma \cong \mathrm{Jac}^{d+\rho}(C_a) \cong \mathrm{Im}(\check{\nu}_a) \subset \overline{\mathrm{Jac}}^{d+\delta}(X_b).$$

where the inverse of the second isomorphism is defined by assigning to $L \in \mathrm{Jac}^d(C_a)$ its Γ-orbit, which can be naturally identified with an element of $\bigsqcup_{\gamma \in \Gamma} \mathrm{Jac}^{d+\rho}(C_{\gamma(a)})/\Gamma$.

Proof. The first statement in (i) is clear by Proposition 3.8. The union is disjoint since a is not fixed by any element of Γ, by definition of $B_{C,r}^{\mathrm{it}}$. This also shows that $\check{p}_b$ is an unramified cover with Galois group Γ acting by pullback.

To see that (3.25) holds, by *(i)* we have

$$h_{X,n}^{-1}(b) \cap \mathbf{M}_X(n,d)^p \cong \left(\bigsqcup_{\gamma \in \Gamma} \mathrm{Jac}^{\rho+d}(C_{\gamma(a)}) \right)/\Gamma,$$

hence, recalling that $C_{\gamma(a)} \cong C_a$ by Proposition 3.4, we can choose a representative and

$$h_{X,n}^{-1}(b) \cap \mathbf{M}_X(n,d)^p \cong \mathrm{Jac}^{\rho+d}(C_a).$$

Since the restriction of $\check{p}_b$ to $\mathrm{Jac}^{\rho+d}(C_a)$ coincides with $\check{\nu}_a$, (3.25) follows. $\qquad \square$

We finish the section with an observation that will be useful in Sections 5.3 and 7.

Remark 3.14. It follows from (3.4) that the image of the map $\hat{p} \circ \check{p}$ is contained in the Levi subgroup $\mathrm{L}_{(r,m)} = \mathrm{GL}(r,\mathbb{C})^{\times m}$ associated to the parabolic subgroup $\mathrm{P}_{(r,m)} = N_{\mathrm{GL}(n,\mathbb{C})}(\mathrm{L}_{(r,m)})$. Denote the unipotent radical of $\mathrm{P}_{(r,m)}$ by $\mathrm{U}_{(r,m)}$ and recall that $\mathrm{P}_{(r,m)} = \mathrm{L}_{(r,m)} \ltimes \mathrm{U}_{(r,m)}$. Note that $\mathrm{L}_{(1,n)}$ is the Cartan subgroup of $\mathrm{GL}(n,\mathbb{C})$ while $\mathrm{P}_{(1,n)}$ is the Borel subgroup.

Before stating the following corollary, we need to briefly introduce some notation from [FP]. Denote by $\mathbf{M}_{(r,m)} \subset \mathbf{M}_C(n, md')$ the image of the moduli space of $\mathrm{L}_{(r,m)}$-Higgs bundles of multidegree $(d', \ldots, d')$. When $r = 1$ and $m = n$, $\mathrm{L}_{(1,n)}$ is the Cartan subgroup and $\mathbf{M}_{(1,n)}$ is called in [FP] the *Cartan locus* of $\mathbf{M}_C(n, nd')$ (and, as proved in loc. cit., it supports a (BBB)-brane). Let $V_{(r,m)} \subset B_{C,n}$ be the image of $\mathbf{M}_{(r,m)}$ under the Hitchin map $h_{C,n}$.

Corollary 3.15. *The commutative diagram of Lemma 3.5 restricts to*

$$
\begin{array}{ccc}
\mathbf{M}_X(n, nd')^p & \xrightarrow{\ \hat{p}\ } & \mathbf{M}_{(r,m)} \\
{\scriptstyle h_{X,n}}\big\downarrow & & \big\downarrow{\scriptstyle h_{C,n}} \\
B^p & \overset{p^*}{\hookrightarrow} & V_{(r,m)}.
\end{array}
$$

Proof. It is enough to prove that $\hat{p}$ maps $\mathbf{M}_X(n, d)^p$ to $\mathbf{M}_{(r,m)}$ and this follows from (3.4). $\square$

3.3. **Cyclic covers and tensorization by torsion line bundles.** We study in this section the fixed point subvariety of $\mathbf{M}_X(n, d)$ under tensorization by a fixed line bundle of order n. For the moduli of vector bundles, Narasimhan and Ramanan [NR3] proved that bundles on X which are in the image of the pushforward map from C are fixed by tensorization with all line bundles associated to characters of the Galois group Γ. When Γ satisfies a certain technical condition (which is the case, for example, of cyclic covers, see [NR3, Lemma 2.5]), the converse is also true for simple bundles. Nasser [Na] proved that when $\Gamma \cong \mathbb{Z}_n$ is cyclic, the converse holds also for non simple bundles. The study of these fixed points in the moduli of vector bundles with fixed determinant, has also been carried out in [GR2] and, in detail in rank $n = 2$, in [GR1].

Let $\mathrm{Jac}(X)[n]$ be the subgroup of n-torsion points of the Jacobian $\mathrm{Jac}(X) = \mathrm{Jac}^0(X)$ of X,

$$
\mathrm{Jac}(X)[n] := \{L \in \mathrm{Jac}(X) \mid L^n \cong \mathcal{O}_X\} \cong \mathbb{Z}_n^{2g}.
$$

For convenience of notation, we shall use different symbols for an element $\xi \in \mathrm{Jac}(X)[n]$ as an abstract group and for the corresponding order n line bundle $L_\xi \in \mathrm{Jac}(X)[n]$ on X.

This group acts on $\mathbf{M}_X(n, d)$ by tensorization,

$$
\begin{array}{cccc}
\xi : & \mathbf{M}_X(n, d) & \longrightarrow & \mathbf{M}_X(n, d) \\
& (E, \varphi) & \longmapsto & (E \otimes L_\xi, \varphi).
\end{array}
$$

Denote by $\mathbf{M}_X(n, d)^\xi$ the subvariety of points fixed by $\xi \in \mathrm{Jac}(X)[n]$. It is a hyperholomorphic subvariety since tensorization by a flat line bundle is holomorphic in the three complex structures of $\mathbf{M}_X(n, d)$ (see [GR2] for a proof in the case of $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles, which also applies to the case of $\mathrm{GL}(n,\mathbb{C})$).

Notation 3.16. Let m be the order of L_ξ in $\mathrm{Jac}(X)[n]$, and set $r = n/m$.

Then $L_\xi \in \mathrm{Jac}(X)[m]$ is a primitive element. Consider the projection $p_\xi : |L_\xi| \longrightarrow X$ and let $\lambda_\xi : |L_\xi| \longrightarrow p_\xi^* L_\xi$ be the tautological section. Define C_ξ to be the curve in the total space $|L_\xi|$ given by the zero locus of the section $\lambda_\xi^m - p_\xi^* \mathbf{1} \in H^0(|L_\xi|, \mathcal{O}_{|L_\xi|})$. Denote the restriction to C_ξ of the projection morphism by the same symbol,

(3.26)
$$
p_\xi : C_\xi \longrightarrow X.
$$

Then, (3.26) is a connected unramified regular cover of X with Galois group $\mathbb{Z}_m$.

Remark 3.17. Reciprocally, a connected unramified regular $\mathbb{Z}_m$-cover $p : C \longrightarrow X$ defines a line bundle $L_p \longrightarrow X$ of order m by setting the holonomy of L_p to be given by parallel transport of the lifts from X to C.

We next describe all the points fixed by any $\xi \in \mathrm{Jac}(X)[n]$.

Proposition 3.18. *Let $\xi \in \mathrm{Jac}(X)[n]$ be of order m and let $r = n/m$. A semistable Higgs bundle $(E, \varphi) \in \mathbf{M}_X(n, d)$ is fixed under ξ if and only if it is the pushforward of an element in $\mathbf{M}_{C_\xi}(r, d)$.*

Proof. Since semistability and degree are preserved under pushforward by p_ξ (cf. [NR3, Lemma 3.1]) and since points in the image of the pushforward are fixed (this is a direct consequence of the projection formula), all we need to prove is that all fixed points are in such image.

Let then $(E, \varphi) \in \mathbf{M}_X(n, d)^\xi$. Then there exists an isomorphism

$$f : E \xrightarrow{\cong} E \otimes L_\xi$$

such that

$$\varphi \otimes \mathrm{Id}_{L_\xi} = f \otimes \mathrm{Id}_{L_\xi} \circ \varphi \circ f^{-1}.$$

We can then treat the pair (E, f) as an L_ξ-twisted Higgs bundle of rank n (i.e. the Higgs field f is twisted by L_ξ instead of K_X). The spectral correspondence described in section 2.2 also holds for L_ξ-twisted Higgs bundles [BNR, Sch], hence establishing a one-to-one correspondence between isomorphism classes of pairs (E, f) and their spectral datum, $\mathcal{L} \longrightarrow C'$, where $\mathcal{L}$ is a rank one torsion free sheaf on the corresponding spectral curve $C' \subset |L_\xi|$, given by the vanishing of the section $\sum_{i=1}^n p_\xi^* s_i \lambda_\xi^{n-i}$ of $p_\xi^* L_\xi^n \cong \mathcal{O}_{|L_\xi|} \longrightarrow |L_\xi|$, where $s_i \in H^0(X, L_\xi^i)$. Since L_ξ has no global sections unless it is trivial, in which case all the global sections are constant, we find that C' is given by the vanishing of

$$(3.27) \qquad \lambda_\xi^{m \cdot r} + p_\xi^* s_m \lambda_\xi^{m \cdot (r-1)} + \cdots + p_\xi^* s_{mr},$$

where $s_i \in H^0(X, L_\xi^{m(r-i)}) = \mathbb{C}$. We can recover $f : E \xrightarrow{\cong} E \otimes L_\xi$ as the pushforward of the tensorization morphism $\mu_{\lambda_\xi} : \mathcal{L} \xrightarrow{\otimes \lambda_\xi} \mathcal{L} \otimes p^* L_\xi$.

Consider now the curve $C_0' \subset |L_\xi^m| \cong |\mathcal{O}_X|$ given by

$$(3.28) \qquad \lambda_0^r + p_0^* s_1 \lambda_0^{r-1} + \cdots + p_0^* s_r,$$

where $p_0 : |\mathcal{O}_X| \longrightarrow X$ and $\lambda_0 \in H^0(|\mathcal{O}_X|, \mathcal{O}_{|\mathcal{O}_X|})$ is the tautological section. Note that C_0' is naturally identified with the characteristic polynomial of $f^m : E \longrightarrow E$.

Since $L_\xi^m \cong \mathcal{O}_X$, there is a morphism $|L_\xi| \longrightarrow |L_\xi^m|$ given by tensorization with itself m-times. Under this morphism, C' is sent to C_0'. Fixing a trivialization $\mathcal{O}_X \cong X \times \mathbb{C}$, since the sections s_i are constant, one has $C_0' \cong X \times D$, where $D = \sum_i \ell_i d_i$ with $\deg D = r$ is the divisor of the points in $\mathbb{C}$ defined by the zeros of the polynomial associated to (3.28). It follows that (3.27) can be rewritten as

$$\prod_{j=1}^t (\lambda_\xi^m - p^* d_j)^{\ell_j}.$$

Since f is an isomorphism it follows that $\det(f) = \prod_{j=1}^t d_j^{\ell_j} \neq 0$, so all the d_j are non-zero. Then, the vanishing locus of $\lambda_\xi^m - p^* d_j$ is naturally isomorphic to C_ξ after scaling. It then follows that $C' = \bigsqcup_{j=1}^t C_\xi^{l_j}$, where $C_\xi^{l_j}$ is the possibly non reduced curve given by the vanishing of $(\lambda_\xi^m - p^* d_j)^{l_j}$. As a consequence C' projects onto C_ξ and the following diagram commutes

$$
\begin{array}{ccc}
C' & \xrightarrow{\;u\;} & C_\xi \\
& \searrow{\scriptstyle p'} & \downarrow{\scriptstyle p_\xi} \\
& & X.
\end{array}
$$

Then, setting $E' := u_* \mathcal{L}$ one has that

$$p_{\xi,*} E' \cong E.$$

Taking $f' : E' \xrightarrow{\cong} E' \otimes p_\xi^* L_\xi$ to be the pushforward under u of $\mathcal{L} \xrightarrow{\otimes \lambda_\xi} \mathcal{L} \otimes (p_\xi \circ t)^* L_\xi$, one also has

$$p_{\xi,*} f' \cong f.$$

Since p_ξ is unramified, one has that $p_\xi^* E$ is given by m copies of E' although we denote them by

$$(3.29) \qquad p_\xi^* E \cong \bigoplus_{i=0}^{m-1} E' \otimes p_\xi^* L_\xi^i,$$

where we recall that $p_\xi^* L_\xi$ is isomorphic to $\mathcal{O}_{C_\xi}$. Note also that $p_\xi^* f$ is given by copies of f', permuting cyclically the factors of $p_\xi^* E$,

$$f' : E' \otimes p_\xi^* L_\xi^i \xrightarrow{\ \cong\ } E' \otimes p_\xi^* L_\xi^{i+1}.$$

After (3.29), the pullback $p_\xi^* \varphi$ can be described in terms of a $m \times m$ matrix $p_\xi^* \varphi = \left(\varphi'_{ij} \right)$, where $\varphi'_{ij} : E' \otimes p_\xi^* L_\xi^i \longrightarrow E' \otimes p_\xi^* L_\xi^i \otimes K_C$. Taking a Jordan-Hölder filtration of $(p_\xi^* E, p_\xi^* \varphi)$ one can always write $p_\xi^* \varphi$ in a upper diagonal form, i.e. with $\varphi'_{ij} = 0$ if $i < j$. Observe that $p_\xi^* f$ corresponds with the permutation of the factors of $p_\xi^* E$ by $i \mapsto i+1$. Since φ and f commute, so do $p_\xi^* \varphi$ and $p_\xi^* f$, and therefore $\varphi'_{ij} = 0$ for any $i \neq j$ and $\varphi'_{ii} = \varphi'_{jj}$. Picking $\phi = \varphi'_{ii}$, one has that $\varphi = p_{\xi,*} \phi$. $\square$

The following is the fundamental result describing the fixed point subvariety $\mathbf{M}_X(n,d)^\xi$ for any $\xi \in \mathrm{Jac}(X)[n]$. This generalizes the description given in [HT] for r and d coprime.

Theorem 3.19. *Let $\xi \in \mathrm{Jac}(X)[m]$ be of order m with $n = mr$. Then pushforward under $p_\xi : C_\xi \longrightarrow X$ induces a hyperholomorphic isomorphism*

$$\mathbf{M}_X(n,d)^\xi \cong \mathbf{M}_X(n,d)^{p_\xi} \cong \mathbf{M}_{C_\xi}(r,d)/\mathbb{Z}_m,$$

with the Galois group $\mathbb{Z}_m$ acting by pullback.

Proof. Straightforward from Propositions 3.2 and 3.18. $\square$

4. NARASIMHAN–RAMANAN (BBB)-BRANES FOR COVERS OF MAXIMAL DEGREE

By definition (cf. [KW]), a (BBB)-*brane* on a hyperkähler manifold $\mathbf{M}$ is a pair

$$(\mathrm{N}, (\mathscr{F}, \nabla_{\mathscr{F}})),$$

where:

- $\mathrm{N} \subset \mathbf{M}$ is a *hyperholomorphic subvariety*, i.e. a subvariety which is holomorphic with respect to the three complex structures I_1, I_2 and I_3.
- $(\mathscr{F}, \nabla_{\mathscr{F}})$ is a *hyperholomorphic sheaf* supported on N, i.e. a locally free sheaf $\mathscr{F}$ of finite rank over the ring of C^∞-functions on N equipped with a connection $\nabla_{\mathscr{F}}$ whose curvature is of type $(1,1)$ in the complex structures I_1, I_2 and I_3.

In this section we construct natural (BBB)-branes on the moduli space of Higgs bundles supported on the image under $\check{p}$. We shall construct two different hyperholomorphic sheaves on this subvariety. Our constructions depend on the choice of a flat line bundle either on X (yielding a *rank 1* (BBB)-brane, in Gukov's [Gu] terminology) or on C (yielding a *rank n* (BBB)-brane).

Assumption 1. From now on, until the end of this section, we will be assuming that the rank n coincides with the order of the unramified cover $p : C \longrightarrow X$. So, in the notation of Proposition 3.2, $m = n$ and $r = 1$. In this case we say that the cover p is of *maximal degree* (note that for fixed rank n, the degree of $p : C \longrightarrow X$ yielding Narasimhan-Ramanan branes in $\mathbf{M}_X(n,d)$ is bounded by n).

Under this assumption $p \colon C \longrightarrow X$ is a connected unramified n-cover of X, with genus

$$(4.1) \qquad\qquad g(C) = n(g-1) + 1$$

and

$$\check{p} : \mathbf{M}_C(1,d) \longrightarrow \mathbf{M}_X(n,d)^p$$

is the finite morphism (3.5).

Since the moduli space of rank 1 Higgs bundles is the cotangent bundle of the Jacobian of C, we have the natural projection

$$\beta : \mathbf{M}_C(1, d) \cong T^*\mathrm{Jac}^d(C) \longrightarrow \mathrm{Jac}^d(C), \tag{4.2}$$

which is Γ-equivariant.

Remark 4.1. The support of our (BBB)-branes will be $\mathrm{N} = \mathbf{M}_X(n, d)^p \cong \mathbf{M}_C(1, d)/\Gamma$. Over the generic locus, it is smooth, so it makes sense to speak of a hyperholomorphic sheaf on a dense open subset of N. Moreover, it will be clear from our construction that the branes extend to coherent sheaves (in complex structure I_1) on $\mathbf{M}_X(n, d)$, supported on N.

Let

$$(\mathcal{F}, \nabla_\mathcal{F}) \longrightarrow C$$

be a flat line bundle on C. Since $\pi_1(\mathrm{Jac}^d(C))$ is the abelianization of $\pi_1(C)$, there is a unique flat line bundle

$$(\check{\mathcal{F}}, \check{\nabla}_\mathcal{F}) \longrightarrow \mathrm{Jac}^d(C) \tag{4.3}$$

which restricts to $(\mathcal{F}, \nabla_\mathcal{F})$ on $C \subset \mathrm{Jac}^d(C)$, viewed as subspace under the Abel–Jacobi map. Define

$$(\mathscr{F}, \nabla_\mathscr{F}) := \check{p}_* \beta^*(\check{\mathcal{F}}, \check{\nabla}_\mathcal{F}). \tag{4.4}$$

Then $(\mathscr{F}, \nabla_\mathscr{F})$ is a rank n coherent sheaf over $\mathbf{M}_X(n, d)^p$. Since $\check{p}$ is hyperholomorphic and $\beta^*(\check{\mathcal{F}}, \check{\nabla}_\mathcal{F})$ is flat (thus with curvature trivially of type $(1, 1)$ in any complex structure), it is also hyperholomorphic, and then so is $(\mathscr{F}, \nabla_\mathscr{F})$. Hence the pair $(\mathbf{M}_X(n, d)^p, (\mathscr{F}, \nabla_\mathscr{F}))$ is a (BBB)-brane.

Definition 4.2. Let $p : C \longrightarrow X$ be a connected unramified n-cover and let $(\mathcal{F}, \nabla_\mathcal{F})$ be a flat line bundle on C. The *rank n Narasimhan–Ramanan* (BBB)-*brane* is

$$(\mathbf{BBB})^p_\mathcal{F} := \left(\mathbf{M}_X(n, d)^p, (\mathscr{F}, \nabla_\mathscr{F})\right).$$

We will omit the cover from the name when it is clear from the context.

The brane just constructed is a rank n brane in Gukov's language [Gu]. Next we construct a rank 1 brane arising from a flat line bundle on the base curve X. Since β in (4.2) is Γ-equivariant and the norm map $\mathrm{Nm} : \mathrm{Jac}^d(C) \longrightarrow \mathrm{Jac}^d(X)$ of p is Γ-invariant, their composition is also Γ-invariant, and we have the following quotient maps (denoted by a bar).

$$\alpha : \mathbf{M}_X(n, d)^p \cong \mathbf{M}_C(1, d)/\Gamma \xrightarrow{\overline{\beta}} \mathrm{Jac}^d(C)/\Gamma \xrightarrow{\overline{\mathrm{Nm}}} \mathrm{Jac}^d(X), \tag{4.5}$$

where the isomorphism is given by Theorem 3.19.

Remark 4.3. The map $\alpha : \mathbf{M}_X(n, d)^p \longrightarrow \mathrm{Jac}^d(X)$ can be interpreted as a twisted determinant map. Indeed, if $(E, \varphi) = p_*(F, \phi)$ with $F \in \mathrm{Jac}^d(C)$, then $\det(E) = \det(p_*F) = \mathrm{Nm}(F) \det(p_*\mathcal{O}_C)$, so

$$\alpha(E, \varphi) = \det(E) \det(p_*\mathcal{O}_C)^{-1}.$$

Let now $(\mathcal{L}, \nabla_\mathcal{L}) \longrightarrow X$ be any flat line bundle on the base curve X and let

$$(\check{\mathcal{L}}, \check{\nabla}_\mathcal{L}) \longrightarrow \mathrm{Jac}^d(X) \tag{4.6}$$

be the associated flat line bundle as above. Define the flat line bundle on $\mathbf{M}_X(n, d)^p$ by

$$(\mathscr{L}, \nabla_\mathscr{L}) := \alpha^*(\check{\mathcal{L}}, \check{\nabla}_\mathcal{L}). \tag{4.7}$$

Definition 4.4. Let $p : C \longrightarrow X$ be a connected unramified n-cover and let $(\mathcal{L}, \nabla_\mathcal{L})$ be a flat line bundle on X. The associated *rank 1 Narasimhan–Ramanan* (BBB)-*brane* is

$$(\mathbf{BBB})^p_\mathcal{L} := \left(\mathbf{M}_X(n, d)^p, (\mathscr{L}, \nabla_\mathscr{L})\right).$$

In the remaining part of this section we shall study the restriction of the branes $(\mathbf{BBB})^p_{\mathcal{L}}$ and $(\mathbf{BBB})^p_{\mathcal{F}}$ to a certain Hitchin fiber. Recall that $m = n$ so $r = 1$ in our case. Then, $B^{\mathrm{red}}_{C,1} = B^{\mathrm{ni}}_{C,1} = B^{\mathrm{sm}}_{C,1} = B_{C,1} = H^0(C, K_C)$. Therefore $B^p_{\mathrm{it}} = B^p_{\mathrm{ni}}$ by Proposition 3.7. Also, note that $B^{\mathrm{it}}_{C,1}$ is just the subset of elements with free Γ-orbits,

$$(4.8) \qquad H^0(C, K_C)^{\mathrm{free}} := \{\phi \in H^0(C, K_C) \text{ such that } \gamma(\phi) \neq \phi \text{ for any } \gamma \in \Gamma\}.$$

Proposition 3.8 and Theorem 3.11 imply that, in this particular case, $B^p_{\mathrm{ni}} = H^0(C, K_C)^{\mathrm{free}}/\Gamma$. Moreover, for $b \in B^p_{\mathrm{ni}}$ such that $b = \zeta(\phi) \in B^p_{\mathrm{ni}}$, for $\phi \in H^0(C, K_C)^{\mathrm{free}}$, the spectral curve X_b is singular and integral, with $\deg(\mathrm{sing}(X_b)) = \delta$. Furthermore, using the notation of (3.10),

$$(4.9) \qquad \nu_\phi := q_\phi \colon C \longrightarrow X_b$$

is a normalization morphism and the following diagram commutes:

$$
\begin{array}{ccc}
C & & \\
 & \searrow^{\nu_\phi} & \\
p \downarrow & & X_b \\
 & \swarrow_{\pi_b} & \\
X. & &
\end{array}
$$

In addition, the same statement holds for $\nu_{\gamma(\phi)} = q_{\gamma(\phi)} = \gamma \circ \nu_\phi \colon C \longrightarrow X_b$, for every $\gamma \in \Gamma$.

Remark 4.5. Notice that in this case, namely when the cover $p \colon C \longrightarrow X$ has degree n, the normalization of the spectral curve X_b with b in B^p_{ni} is C_ϕ, the spectral curve for the moduli of rank 1 Higgs bundles over C, associated to ϕ. But C_ϕ is isomorphic to C via the section $\phi \colon C \xrightarrow{\cong} C_\phi \subset |K_C|$, so all spectral curves share (up to isomorphism) the same normalization C. This justifies the slightly different notation for the normalization morphism in (4.9), when compared with (3.18). We have implicitly used the identification $C_\phi \cong C$ in (4.9), so that, strictly speaking, $\nu_\phi = q_\phi \circ \phi$.

In this case, we denote the pushforward morphism (3.23) by

$$(4.10) \qquad \check{\nu}_\phi \colon \mathrm{Jac}^d(C) \hookrightarrow \overline{\mathrm{Jac}}^{d+\delta}(X_b),$$

and Proposition 3.13 (ii) reads as follows

$$(4.11) \quad h^{-1}_{X,n}(b) \cap \mathbf{M}_X(n,d)^p \cong \left(\bigsqcup_{\gamma \in \Gamma} (\mathrm{Jac}^d(C) \times \{\gamma(\phi)\}) \right)/\Gamma \cong \mathrm{Jac}^d(C) \cong \mathrm{Im}(\check{\nu}_\phi) \subset \overline{\mathrm{Jac}}^{d+\delta}(X_b),$$

with the inverse of the second isomorphism defined by choosing one representative ϕ in the Γ-orbit given by b, and taking

$$(4.12) \qquad f_\phi(L) = \Gamma \cdot (L, \phi)$$

for $L \in \mathrm{Jac}^d(C)$.

For each $\gamma \in \Gamma$, let $\hat{\gamma} \colon \mathrm{Jac}^d(C) \longrightarrow \mathrm{Jac}^d(C)$ denote the pullback map associated to the covering automorphism of C determined by γ. In the following proposition we study how $\mathcal{F}$ and $\mathcal{L}$, defining the Narasimhan-Ramanan (BBB)-branes, restrict to a Hitchin fiber in B^p_{ni}.

Proposition 4.6. *Let $b \in B^p_{\mathrm{ni}}$, and let $\mathcal{L}$ and $\mathcal{F}$ be the hyperholomorphic bundles defined in (4.6) and (4.4) respectively. The restrictions of $\mathcal{L}$ and of $\mathcal{F}$ to $h^{-1}_{X,n}(b) \cap \mathbf{M}_X(n,d)^p$ are identified, under the isomorphism (4.11), with the bundles $\mathrm{Nm}^* \check{\mathcal{L}}$ and $\bigoplus_{\gamma \in \Gamma} \hat{\gamma}^* \check{\mathcal{F}}$ respectively.*

Proof. Pick some $\phi \in H^0(C, K_C)^{\text{free}}$ such that $\zeta(\phi) = b$ and consider the following commutative diagram

$$
\begin{array}{ccccc}
\operatorname{Jac}^d(C) \cong \operatorname{Jac}^d(C) \times \{\phi\} & \overset{\tilde{i}}{\hookrightarrow} & T^* \operatorname{Jac}^d(C) \cong \mathbf{M}_C(1,d) & \overset{\beta}{\longrightarrow} & \operatorname{Jac}^d(C) \\
\downarrow{\scriptstyle f_\phi} & & \downarrow{\scriptstyle \check{p}} & & \downarrow{\scriptstyle \operatorname{Nm}} \\
h_{X,n}^{-1}(b) \cap \mathbf{M}_X(n,d)^p & \overset{i}{\hookrightarrow} & T^* \operatorname{Jac}^d(C)/\Gamma \cong \mathbf{M}_X(n,d)^p & \overset{\alpha}{\longrightarrow} & \operatorname{Jac}^d(X),
\end{array}
$$

where left vertical isomorphism is the one given in (4.11), and $i, \tilde{i}$ are the obvious inclusions.

By definition, $\mathscr{L} = \alpha^* \check{\mathcal{L}}$ hence, by commutativity of the diagram and the fact that $\beta \circ \tilde{i} = \mathbf{1}_{\operatorname{Jac}}$, it follows that $i^* \mathscr{L}$ is identified, via the isomorphism (4.11), with $\tilde{i}^* \beta^* \operatorname{Nm}^* \check{\mathcal{L}} = \operatorname{Nm}^* \check{\mathcal{L}}$, as claimed.

Recall from (4.4) that $\mathscr{F} = \check{p}_* \beta^* \check{\mathcal{F}}$. Now, by Proposition 3.13 we have a commutative diagram

$$
\begin{array}{ccc}
& & \overset{\tilde{i}}{} \\
\operatorname{Jac}^d(C) \cong \operatorname{Jac}^d(C) \times \{\phi\} \overset{j}{\hookrightarrow} \bigsqcup_{\gamma \in \Gamma} \operatorname{Jac}^d(C) \times \{\gamma(\phi)\} \hookrightarrow & & \mathbf{M}_C(1,d) \\
{\scriptstyle \cong} \searrow \qquad \downarrow{\scriptstyle \check{p}_b} & & \downarrow{\scriptstyle \check{p}} \\
h_{X,n}^{-1}(b) \cap \mathbf{M}_X(n,d)^p \overset{i}{\hookrightarrow} & & \mathbf{M}_X(n,d)^p.
\end{array}
$$

where the diagonal isomorphism is the one given by (4.11), the rightmost square is Cartesian and the central downward arrow is the unramified Γ-cover $\check{p}_b : \bigsqcup_{\gamma \in \Gamma} \operatorname{Jac}^d(C) \times \{\gamma(\phi)\} \longrightarrow h_{X,n}^{-1}(b) \cap \mathbf{M}_X(n,d)$, whose action of the Galois group in the bundle factor is given by the pullback $\hat{\gamma}$ associated to each $\gamma \in \Gamma$. Under the isomorphism (4.11), $i^* \mathscr{F}$ is identified with $\tilde{i}^* \check{p}^* \mathscr{F}$ and, since $\beta \circ \tilde{i} = \mathbf{1}_{\operatorname{Jac}}$,

$$
\tilde{i}^* \check{p}^* \mathscr{F} = \tilde{i}^* \check{p}^* \check{p}_* \beta^* \check{\mathcal{F}} = \tilde{i}^* \bigoplus_{\gamma \in \Gamma} \hat{\gamma}^* \beta^* \check{\mathcal{F}} = \tilde{i}^* \beta^* \bigoplus_{\gamma \in \Gamma} \hat{\gamma}^* \check{\mathcal{F}} = \bigoplus_{\gamma \in \Gamma} \hat{\gamma}^* \check{\mathcal{F}},
$$

completing the proof. $\qquad \square$

Remark 4.7. In [FP], the first and fourth authors constructed a brane $\mathbf{Car}(\mathcal{L})$ supported on the Cartan locus $\mathbf{M}_{(1,n)} \subset \mathbf{M}_C(n, nd)$. Recall from Remark 3.14 that $\hat{p}(\mathbf{M}_X(n,d)^p)$ is contained in $\mathbf{M}_{1,n}$. One can check that the hyperholomorphic sheaf in $\mathbf{Car}(\mathcal{L})$ pulls-back under $\hat{p}$ to that of $(\mathbf{BBB})_{\mathcal{L}}^p$. Note that moreover $\mathbf{Car}(\mathcal{L}) =: \mathbf{M}_X(n,d)^p$ for the trivial $\mathbb{Z}_n$-Galois cover $p : \bigsqcup X \longrightarrow X$. In that case however Theorem 3.19 fails.

5. Narasimhan–Ramanan dual (BAA)-branes

We have seen in Section 2.1 that $\mathbf{M}_X(n,d)$ is a hyperkähler variety with Kähler structures $((I_1, \omega_1), (I_2, \omega_2), (I_3, \omega_3))$. Following [KW], a (BAA)-*brane* on $\mathbf{M}_X(n,d)$ is, by definition, a pair $(\Sigma, (W, \nabla_W))$, where:

- Σ is a subvariety of $\mathbf{M}_X(n,d)$, which is a complex Lagrangian for the holomorphic symplectic form $\Omega_1 = \omega_2 + i\omega_3$.
- (W, ∇_W) is a flat bundle supported on Σ.

The purpose of the present section is to construct a natural collection of complex Lagrangian subvarieties, whose image by the Hitchin fibration $h_{X,n} \longrightarrow B_{X,n}$ is B_{ni}^p (cf. Definition 3.9). Each of these subvarieties depends on the unramified cover $p : C \longrightarrow X$ and on a holomorphic line bundle $\mathcal{J}$ on C, and will henceforth be denoted by $\operatorname{NR}_{\text{ni}}^{p,\mathcal{J}}$. We will also see that the Higgs bundles lying in $\operatorname{NR}_{\text{ni}}^{p,\mathcal{J}}$ can be constructed from Hecke modifications of a Hitchin section of $\mathbf{M}_X(n, d+\delta)^p \longrightarrow B^p$ constructed out of $\mathcal{J}$. Up to the choice of a flat bundle on it, $\operatorname{NR}_{\text{ni}}^{p,\mathcal{J}}$ is thus the support of a (BAA)-brane, the *Narasimhan–Ramanan dual* (BAA)-*brane*. In Section 6 will be shown that these branes are the (fiberwise) mirror transform of the Narasimhan–Ramanan (BBB)-branes. This justifies the notation for these subvarieties, as well as their name.

5.1. **Construction of the support.** In this section we construct certain subvarieties that later will be shown to be complex Lagrangian, hence the support of a (BAA)-brane after the choice of a flat bundle on them.

Donagi and Pantev defined in [DP] the abelianized Hecke correspondence. For the general linear group $\mathrm{GL}(n, \mathbb{C})$, this correspondence is given by tensorizing the spectral data by the dual of the ideal of a point on the corresponding spectral curve. In the following lines we generalize this correspondence to 0-dimensional subschemes of length ℓ as it constitutes the first ingredient of our construction.

Recall the family of spectral curves $\mathscr{X} \longrightarrow B_{X,n}$ and consider its restriction to the subset $B^p_{\mathrm{ni}} \subset B_{X,n}$ that parametrizes nodal curves. The existence of the Jacobian $\mathrm{Jac}_{B^p_{\mathrm{ni}}}(\mathscr{X})$ follows by the seminal work of Grothendieck [Gr2, Thm. 3.1]. The existence of the relative compactified Jacobian $\overline{\mathrm{Jac}}_{B^p_{\mathrm{ni}}}(\mathscr{X})$ follows by Altman and Kleiman's work [AK, Theorem 3.1], being a compactification of $\mathrm{Jac}_{B^p_{\mathrm{ni}}}(\mathscr{X})$, inducing the fiberwise compactification of the Jacobian by rank one torsion-free sheaves. We define the tensorization morphism

$$\mathfrak{t}^\ell_p : \quad \overline{\mathrm{Jac}}^{d+\delta}_{B^p_{\mathrm{ni}}}(\mathscr{X}) \times_{B^p_{\mathrm{ni}}} \mathrm{Hilb}^\ell_{B^p_{\mathrm{ni}}}(\mathscr{X}) \quad \longrightarrow \quad \overline{\mathrm{Jac}}^{d+\delta+\ell}_{B^p_{\mathrm{ni}}}(\mathscr{X})$$
$$(\mathcal{F}, \mathcal{I}_Z) \qquad \longmapsto \qquad \mathcal{F} \otimes \mathcal{I}_Z^\vee.$$

Since we are considering nodal curves, hence Cohen–Macaulay, the sheaves $\mathcal{I}_Z$ are Cohen–Macaulay too. In that case the dual sheaves $\mathcal{I}_Z^\vee$ are Cohen–Macaulay as well, so torsion–free as we are working on curves. Furthermore, one can see that the $\mathcal{I}_Z^\vee$ are semistable (hence stable since we our spectral curves are integral). It follows that $\mathcal{F} \otimes \mathcal{I}_Z^\vee$ is semistable and torsion–free provided that $\mathcal{F}$ is, proving that $\mathfrak{t}^\ell_p$ is well defined.

Remark 5.1. Recall the Abel–Jacobi map $\alpha_\ell : \mathrm{Hilb}^\ell(X_b) \longrightarrow \overline{\mathrm{Jac}}^\ell(X_b)$, $\mathcal{I}_Z \mapsto \mathcal{I}_Z^\vee$. The preimage $\alpha_\ell^{-1}(\mathcal{F}')$ is identified with the open set of $\mathbb{P}(H^0(X_b, \mathcal{F}'))$ of injective sections and the identification is done by considering the short exact sequence $0 \longrightarrow \mathcal{O}_{X_b} \overset{s}{\longrightarrow} \mathcal{F}' \longrightarrow \mathcal{O}_Z \longrightarrow 0$. It then follows that the preimage $(\mathfrak{t}^\ell_p)^{-1}(\mathcal{F}'')$ consists on the pairs $(\mathcal{F}, \mathcal{I}_Z)$ fitting in the short exact sequence

$$0 \longrightarrow \mathcal{F} \overset{s \otimes \mathbf{1}_{\mathcal{F}}}{\longrightarrow} \mathcal{F}'' \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

Recall the isomorphism (2.9) provided by the spectral correspondence. One can define the *p-adapted length ℓ abelianized Hecke morphism* by setting the morphism

$$\mathfrak{h}^\ell_p := S_{X,n} \circ \mathfrak{t}^\ell_p \circ (S^{-1}_{X,n} \times \mathbf{1}_{\mathrm{Hilb}}),$$

$$\mathfrak{h}^\ell_p : \mathbf{M}_X(n, d) \times_{B_{X,n}} \mathrm{Hilb}^\ell_{B^p_{\mathrm{ni}}}(\mathscr{X}) \longrightarrow \mathbf{M}_X(n, d+\ell) \times_{B_{X,n}} B^p_{\mathrm{ni}}.$$

The second ingredient of the construction of the Narasimhan–Ramanan dual branes is a section of the structural morphisms of the relative Hilbert scheme $\mathrm{Hilb}^{\delta-\rho}_{B^p_{\mathrm{ni}}}(\mathscr{X}) \longrightarrow B^p_{\mathrm{ni}}$ given by the singularities of the spectral curves. Recall from (ii) Theorem 3.11 that for every $b \in B^p_{\mathrm{ni}}$, associated to the spectral curve $X_b = \mathscr{X}|_b$, one has that the singularity divisor $\mathrm{sing}(X_b)$ is reduced and has length $\delta - \rho$. Hence, one can construct the section

$$\mathrm{sing}^p_{\mathrm{ni}} : \quad B^p_{\mathrm{ni}} \quad \longrightarrow \quad \mathrm{Hilb}^{\delta-\rho}_{B^p_{\mathrm{ni}}}(\mathscr{X})$$
$$b \quad \longmapsto \quad \mathrm{sing}(X_b).$$

We denote the image of this section by

$$\mathrm{Sing}^p_{\mathrm{ni}} := \mathrm{Im}\left(\mathrm{sing}^p_{\mathrm{ni}}\right).$$

One obviously have a natural isomorphism provided by the structural projection

(5.1) $$\mathbf{M}_X(n, d) \times_{B_{X,n}} \mathrm{Sing}^p_{\mathrm{ni}} \cong \mathbf{M}_X(n, d) \times_{B_{X,n}} B^p_{\mathrm{ni}}.$$

The third ingredient of the construction of the Narasimhan–Ramanan dual brane is a Hitchin section on $\mathbf{M}_C(r, d + \delta - \rho)$. Such a section will only exists under certain conditions.

Assumption 2. From now on, until the end of Section 6, we will be assuming that d is a multiple of r, hence $d = rd'$ for some integer d'.

Remark 5.2. Note that Assumption 1 forces $r = 1$, so it implies Assumption 2.

As we have anticipated, the reason behind requiring Assumption 2 is because in this case one can construct a Hitchin section associated to any $\mathcal{J} \in \mathrm{Jac}^{\delta/r+d'}(C)$,

$$\sigma_{C,\mathcal{J}} : B_{C,r} \longrightarrow \mathbf{M}_C(r, rd' + \delta - \rho).$$

Recall (2.10) and (2.7) from Section 2.2, and compose the Hitchin section with the pushforward map $\check{p}$,

$$\check{p} \circ \sigma_{C,\mathcal{J}} : B_{C,r} \longrightarrow \mathbf{M}_X(n, rd' + \delta - \rho)^p.$$

Thanks to Proposition 3.8, $\check{p} \circ \sigma_{C,\mathcal{J}}$ factors through a Hitchin section for $\mathbf{M}_X(n, rd' + \delta - \rho)^p$,

$$\sigma_{p,\mathcal{J}} : B^p \longrightarrow \mathbf{M}_X(n, rd' + \delta - \rho)^p,$$

satisfying $\sigma_{p,\mathcal{J}} \circ \zeta = \check{p} \circ \sigma_{C,\mathcal{J}}$.

As before, we use capital letters to denote their image,

$$\Sigma_{C,\mathcal{J}} := \mathrm{Im}\,(\sigma_{C,\mathcal{J}}) \subset \mathbf{M}_C(r, rd' + \delta - \rho)$$

and

$$\Sigma_{p,\mathcal{J}} := \mathrm{Im}\,(\sigma_{p,\mathcal{J}}) \subset \mathbf{M}_X(n, rd' + \delta - \rho)^p.$$

Note that for every $\gamma \in \Gamma$ one has that $\sigma_{p,\mathcal{J}} = \sigma_{p,\gamma * \mathcal{J}}$, hence

$$\Sigma_{p,\mathcal{J}} = \Sigma_{p,\gamma * \mathcal{J}}$$

only depends on the class of $\mathcal{J}$ under the action of Γ by pull-back.

Definition 5.3. For each $[\mathcal{J}]_\Gamma \in \mathrm{Jac}^{\delta/r+d'}(C)/\Gamma$, define the subvariety $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ of $\mathbf{M}_X(n, rd')$, closed in $\mathbf{M}_X(n, rd') \times_{B_{X,n}} B_{\mathrm{ni}}^p$, as the image under under the isomorphism (5.1) of the restriction to $\mathbf{M}_X(n, d) \times_{B_{\mathrm{ni}}^p} \mathrm{Sing}_{\mathrm{ni}}^p$ of the preimage of $\Sigma_{p,\mathcal{J}} = \check{p}(\Sigma_{C,\mathcal{J}})$ under the p-adapted length $(\delta - \rho)$ abelianized Hecke morphism, *i.e.*

$$\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}} := \left(\left(\mathfrak{h}_p^{(\delta-\rho)} \right)^{-1} (\Sigma_{p,\mathcal{J}}) \right) \cap \left(\mathbf{M}_X(n, rd')_{\mathrm{ni}}^p \times_{B_{X,n}} \mathrm{Sing}_{\mathrm{ni}}^p \right).$$

Remark 5.4. The justification for the notation used for this subvariety will be clear from Theorem 6.5 below, where we show that $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ supports the (BAA)-brane dual to the Narasimhan–Ramanan (BBB)-brane.

We next turn to the description of the spectral data of $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ in terms of Hecke modifications of suitable Higgs bundles. These have been considered in several works, such as in [Hi7, HR, Ra, Wi, W]. Let D be a reduced effective divisor on X, $E \longrightarrow X$ a vector bundle and $\alpha_y \in E_y^*$ for each point y in the support of D. This defines

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{\alpha} \mathcal{O}_D \longrightarrow 0$$

where E' depends on D and on the projective class of α. The bundle E' is said to be a *Hecke modification of E* (along D and associated to α). If E is equipped with a Higgs field ψ, then a *Hecke modification of the Higgs bundle* (E, ψ) is a Higgs bundle (E', φ) where E' is a Hecke modification of E which is compatible with ψ and φ, *i.e.* such that the restriction of ψ to E' equals φ.

The following theorem gives a description of $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ in terms of Hecke modifications of the Higgs bundles parametrized by the section $\sigma_{p,\mathcal{J}}$. Note that $\pi_b(\mathrm{sing}(X_b))$ is a reduced effective divisor of length $\delta - \rho$, whenever $b \in B_{\mathrm{ni}}^p$.

Theorem 5.5. *Let $(E, \varphi) \in \mathbf{M}_X(n, d)$, with $d = rd'$, and let $b \in B_{\mathrm{ni}}^p$. The following conditions are equivalent:*

(1) $(E, \varphi) \in \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}} \cap h_{X,n}^{-1}(b)$;

(2) (E, φ) is a Hecke modification of the Higgs bundle

$$(E_{\mathcal{J},b}, \psi_{\mathcal{J},b}) := \sigma_{p,\mathcal{J}}(b) \in \mathbf{M}_X(n, rd' + \delta - \rho)^p$$

along the divisor $\pi_b(\mathrm{sing}(X_b))$ on X.

Proof. Choose $a \in B_{C,r}^{\mathrm{ni}}$ such that $b = \zeta(a)$. By construction, we have $E_{\mathcal{J},b} \cong p_* \eta_{a,*} \eta_a^* \mathcal{J}$. Recalling the commuting diagram (3.19) (with $\gamma = 1$), one has

$$(5.2) \qquad\qquad E_{\mathcal{J},b} \cong \pi_{b,*} \nu_{a,*} \eta_a^* \mathcal{J}.$$

As for the Higgs field, by definition (cf. (2.7)),

$$\psi_{\mathcal{J},b} \cong p_* \eta_{a,*} \mu_{\eta_a^* \mathcal{J}} : E_{\mathcal{J},b} \longrightarrow E_{\mathcal{J},b} \otimes K_X.$$

Then, $\psi_{\mathcal{J},b} \cong \pi_{b,*} \nu_{a,*} \mu_{\eta_a^* \mathcal{J}}$ thanks to the commutativity of (3.19) (with $\gamma = 1$). Recall also that $\mu_{\eta_a^* \mathcal{J}}$ is defined by the tensorization under the tautological section $\hat{\lambda} : C_a \longrightarrow \eta_a^* K_C$,

$$\mu_{\eta_a^* \mathcal{J}} : \eta_a^* \mathcal{J} \xrightarrow{\otimes \hat{\lambda}} \eta_a^* \mathcal{J} \otimes \eta_a^* K_C.$$

Let $\mathcal{F} \longrightarrow X_b$ be the spectral datum of (E, φ). Recalling Remark 5.1 the first condition is equivalent to $\mathcal{F}$ fitting in the exact sequence

$$(5.3) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \nu_{a,*} \eta_a^* \mathcal{J} \longrightarrow \mathcal{O}_{\mathrm{sing}(X_b)} \longrightarrow 0,$$

for any choice of $a \in B_{C,r}^{\mathrm{ni}}$ such that $b = \zeta(a)$.

By definition, the second condition above is equivalent to saying that E is given by the short exact sequence

$$(5.4) \qquad 0 \longrightarrow E \longrightarrow E_{\mathcal{J},b} \longrightarrow \mathcal{O}_{\pi_b(\mathrm{sing}(X_b))} \longrightarrow 0,$$

and the Higgs field is obtained by the restriction, *i.e.*

$$(5.5) \qquad\qquad \varphi = \psi_{\mathcal{J},b}|_E$$

We can easily see that the first condition implies the second. Since π_b is a finite morphism, pushing forward the exact sequence (5.3) to X yields the exact sequence (5.4) after the identification (5.2), hence a Hecke modification of vector bundles.

Recall that $\hat{\lambda} = q_a^* \lambda$, where λ is the tautological section of $\pi_b^* K_X$, and that $K_C \cong p^* K_X$. Since $p \circ \eta_a = \pi_b \circ q_a$ and since $q_a = \nu_a$ by definition, then

$$\mu_{\eta_a^* \mathcal{J}} : \eta_a^* \mathcal{J} \xrightarrow{\otimes \hat{\lambda}} \eta_a^* \mathcal{J} \otimes \nu_a^* \pi_b^* K_X.$$

Then, by the projection formula, one has that

$$\nu_{a,*} \mu_{\eta_a^* \mathcal{J}} = \mu_{(\nu_{a,*} \eta_a^* \mathcal{J})} : \nu_{a,*} \eta_a^* \mathcal{J} \xrightarrow{\otimes \lambda} \nu_{a,*} \eta_a^* \mathcal{J} \otimes \pi_b^* K_X$$

on X_b, so

$$(5.6) \qquad\qquad \psi_{\mathcal{J},b} = \pi_{b,*} \mu_{(\nu_{a,*} \eta_a^* \mathcal{J})}.$$

Since tensorization by λ restricts to subsheaves and $\mathcal{F}$ is a subsheaf of $\nu_{a,*} \eta_a^* \mathcal{J}$, it is clear that

$$\mu_{\mathcal{F}} = \mu_{(\nu_{a,*} \eta_a^* \mathcal{J})}|_{\mathcal{F}}.$$

Taking the pushforward under π_b yields $\varphi = \psi_{\mathcal{J},b}|_E$ by (5.6), proving (5.5).

For the converse statement, suppose (E, φ) is a Hecke modification of $(E_{\mathcal{J},b}, \psi_{\mathcal{J},b})$, so that we have (5.4) and (5.5). If $\mathcal{F} \longrightarrow X_b$ is again the spectral datum, then (5.4) is the same as

$$0 \longrightarrow \pi_{b,*} \mathcal{F} \longrightarrow \pi_{b,*} \nu_{a,*} \eta_a^* \mathcal{J} \longrightarrow \mathcal{O}_{\pi_b(\mathrm{sing}(X_b))} \longrightarrow 0.$$

Recall from [BNR, Si2] that the $\pi_{b,*} \mathcal{O}_{X_b}$-module structure on E and on $E_{\mathcal{J},b}$ is precisely given by the corresponding Higgs fields, then (5.5) implies that this an exact sequence of $\pi_{b,*} \mathcal{O}_{X_b}$-modules.

Since π_b is a finite morphism, $\pi_{b,*}$ is an exact functor (as the higher direct image sheaves vanish because its fibers are zero dimensional) from the category of $\mathcal{O}_{X_b}$-modules to the category of $\pi_{b,*} \mathcal{O}_{X_b}$-modules. Moreover, since π_b is affine, this is an equivalence of categories (cf. [Ha, ex. 5.17 p.128], so the previous sequence holds if and only if we have that (5.3) holds and hence $(E, \varphi) \in \mathrm{NR}_{\mathrm{ni}}^{p, \mathcal{J}} \cap h_{X,n}^{-1}(b)$. $\qquad\square$

We finish the section showing that $\mathrm{NR}_{\mathrm{ni}}^{p, \mathcal{J}}$ lies in the smooth locus.

Proposition 5.6. *The subvariety* $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ *is contained in the stable locus of* $\mathbf{M}_X(n, rd')$.

Proof. $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ maps to B_{ni}^p under the Hitchin map, hence every Higgs bundle in $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ corresponds to an irreducible (and reduced) spectral curve, by (v) of Theorem 3.11. But a destabilizing Higgs subbundle of a strictly polystable Higgs bundle gives rise to a proper component of the corresponding spectral curve, which hence is not irreducible. $\square$

5.2. **Spectral data and parabolic modules.** We shall now provide the spectral description of the subvarieties $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$. The use of *parabolic modules* (cf. [Re, C1, C2, Bh]) will be crucial along this section, as they provide a convenient way to relate rank one torsion-free sheaves on a singular curve with line bundles over its normalization.

Given a nodal curve X_b, where $b = \zeta(a) \in B_{\mathrm{ni}}^p$ for some $a \in B_{C,r}^{\mathrm{ni}}$, its normalization is C_a with $\nu_a : C_a \longrightarrow X_b$ being the normalization morphism, as described in (3.18). One can consider the pull-back morphism,

$$(5.7) \qquad \begin{array}{rccc} \hat{\nu}_a : & \mathrm{Jac}^{d+\delta}(X_b) & \longrightarrow & \mathrm{Jac}^{d+\delta}(C_a) \\ & L & \longmapsto & \nu_a^* L. \end{array}$$

Note that $\hat{\nu}_a$ does not extend to $\overline{\mathrm{Jac}}^{d+\delta}(X_b)$. One of the motivations to introduce parabolic modules is that their moduli space is a compactification of $\mathrm{Jac}^{d+\delta}(X_b)$ (different to $\overline{\mathrm{Jac}}^{d+\delta}(X_b)$) where $\hat{\nu}_a$ extends naturally.

Consider

$$\widetilde{D}_a := \nu_a^{-1}(\mathrm{sing}(X_b)).$$

Decompose the singular divisor as $\mathrm{sing}(X_b) = D_b^1 + \cdots + D_b^s$, in such a way that each subdivisor D_b^i is supported on the reduced point $x_i \in \mathrm{sing}(X_b)$, with $x_i \neq x_j$. This induces the decomposition $\widetilde{D}_a = \widetilde{D}_a^1 + \cdots + \widetilde{D}_a^s$, where each subdivisor is $\widetilde{D}_a^i := \nu_a^{-1}(D_a^i)$.

Definition 5.7. A (rank 1) *parabolic module* over C_a associated to $\widetilde{D}_a$, of degree $d+\delta$ and type $\ell = (\ell_1, \ldots, \ell_s)$, is a pair (M, V) where $M \in \mathrm{Jac}^{d+\delta}(C_a)$ and V is a vector subspace of $M \otimes \mathcal{O}_{\widetilde{D}_a}$ such that:

(1) V is $\bigoplus_{i=1}^s V^i$ with $V^i \subset (M \otimes \mathcal{O}_{\widetilde{D}_a^i})$;
(2) for every i, the vector space V^i has dimension $\ell_i > 0$;
(3) V^i is an $\mathcal{O}_{x_i}$-submodule of $M \otimes \mathcal{O}_{\widetilde{D}_a^i}$ via pushforward under ν_a *i.e.* via the inclusion $\mathcal{O}_{X_b} \hookrightarrow \nu_{a,*}\mathcal{O}_{C_a}$.

See [C1, C2] for more details.

Write $\mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a)$ for the moduli space of parabolic modules over C_a associated to $\widetilde{D}_a$, of degree $d + \delta$ and type ℓ. It is an integral and projective variety (see [C1]).

Recall that we have defined $B_{C,r}^{\mathrm{ni}}$ as the open subset of $B_{C,r}^{\mathrm{it}}$ such that the intersection of C_a with any $C_{\gamma(a)}$ as only nodal singularities.

Proposition 5.8. *Take* $a \in B_{C,r}^{\mathrm{ni}}$, *then* $\deg(\widetilde{D}_a) = 2\deg(\mathrm{sing}(X_b))$ *and* $\widetilde{D}_a \longrightarrow \mathrm{sing}(X_b)$ *is a* $2:1$ *cover.*

Proof. Since a is chosen in $B_{C,r}^{\mathrm{ni}}$, it follows that $C_{\tilde{b}}$ has only nodal singularities given by the intersection of two irreducible components. Since $q_{\tilde{b}}$ is a connected unramified cover by Proposition 3.6 (ii) and the normalization ν_a is the restriction of $q_{\tilde{b}}$ to the irreducible component C_a of $C_{\tilde{b}}$, one has that $\widetilde{D}_a = \nu_a^{-1}(\mathrm{sing}(X_b))$ can be described as

$$\widetilde{D}_a = C_a \cap \mathrm{sing}(C_{\tilde{b}})$$

and so we have that $\widetilde{D}_a$ is the union of the nodal intersections $C_a \cap C_{\gamma(a)}$, for all $\gamma \in \Gamma$. Then, it follows from (3.22) that

$$\deg(\widetilde{D}_a) = 2nr(m-1)(g-1),$$

which is twice $\deg(\mathrm{sing}(X_b))$. Consider now a singular point x of X_b. If $y \in C_a \cap C_{\gamma(a)}$ maps under $q_{\tilde{b}}$ to x so does $\gamma^{-1}(y) \in C_{\gamma^{-1}(a)} \cap C_a$. Since the action of Γ is free in C, we conclude that

$y \neq \gamma^{-1}(y)$. If there is another $\gamma' \neq \gamma$ in Γ such that $(\gamma')^{-1}(y) \in C_a$ then $y \in C_{\gamma'(a)}$ as well, so $y \in C_a \cap C_{\gamma(a)} \cap C_{\gamma'(a)}$ is a triple intersection which it is excluded as $a \in B_{C,r}^{\mathrm{ni}}$. We conclude that among the nodal singularities of $C_{\tilde{b}}$ mapping to x under $q_{\tilde{b}}$, there are exactly two which lie in C_a, so the projection $\widetilde{D}_a \longrightarrow \mathrm{sing}(X_b)$ is a 2-cover. $\qquad \square$

For the rest of the section we take $a \in B_{C,r}^{\mathrm{ni}}$ and $b = \zeta(a) \in B_{\mathrm{ni}}^p$, so X_b is irreducible with nodal singularities by Theorem 3.11. Observe that in this case $\deg(D_a^i) = 1$, $\deg(\widetilde{D}_a^i) = 2$ and $s = \deg(\mathrm{sing}(X_b))$. Set also $\ell = (1, \ldots, 1)$. We are under the assumptions considered in [C2], so, by Theorem 1 in loc. cit., there is a finite morphism

$$
(5.8) \qquad \begin{array}{rccc} \tau : & \mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a) & \longrightarrow & \overline{\mathrm{Jac}}^{d+\delta}(X_b), \\ & (M, V) & \longmapsto & \mathcal{F} \end{array}
$$

where $\mathcal{F}$ is defined by the short exact sequence

$$
(5.9) \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \nu_{a,*} M \longrightarrow \nu_{a,*}\big(M \otimes \mathcal{O}_{\widetilde{D}_a}/V\big) \longrightarrow 0,
$$

and the second map is the composition $\nu_{a,*} M \longrightarrow \nu_{a,*}\big(M \otimes \mathcal{O}_{\widetilde{D}_a}\big) \longrightarrow \nu_{a,*}\big(M \otimes \mathcal{O}_{\widetilde{D}_a}/V\big)$. Note that $\deg(\nu_{a,*}(M \otimes \mathcal{O}_{\widetilde{D}_a^i})/V^i) = 1$, so

$$
(5.10) \qquad \nu_{a,*}\big(M \otimes \mathcal{O}_{\widetilde{D}_a}/V\big) \cong \mathcal{O}_{\mathrm{sing}(X_b)}.
$$

By definition of V, the quotient $M \otimes \mathcal{O}_{\widetilde{D}_a}/V$ is an $\mathcal{O}_{X_b}$-module, so $\mathcal{F}$ inherits an $\mathcal{O}_{X_b}$-module structure as well. In addition, $\deg(\nu_{a,*} M) = d + \delta + \deg(\mathrm{sing}(X_b))$ and $\deg\big(\nu_{a,*}\big(M \otimes \mathcal{O}_{\widetilde{D}_a}/V\big)\big) = \deg(\mathrm{sing}(X_b))$, thus indeed $\deg(\mathcal{F}) = d + \delta$.

Let τ_0 denote the restriction of τ to $\tau^{-1}(\mathrm{Jac}^{d+\delta}(X_b))$. From [C2, Theorem 1] we know that it is an isomorphism

$$
(5.11) \qquad \tau_0 : \tau^{-1}(\mathrm{Jac}^{d+\delta}(X_b)) \overset{\cong}{\longrightarrow} \mathrm{Jac}^{d+\delta}(X_b),
$$

so $\mathrm{Jac}^{d+\delta}(X_b)$ can be seen as a dense open subspace of $\mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a)$ via τ_0^{-1}. In other words, $\mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a)$ is a compactification of $\mathrm{Jac}^{d+\delta}(X_b)$, which is different from $\overline{\mathrm{Jac}}^{d+\delta}(X_b)$.

Lemma 5.9. *The morphism $\tau : \mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a) \longrightarrow \overline{\mathrm{Jac}}^{d+\delta}(X_b)$ is surjective.*

Proof. It follows from [AIK] that $\overline{\mathrm{Jac}}^{d+\delta}(X_b)$ is irreducible. The restriction τ_0 is surjective, hence the lemma is an immediate consequence of the compactness of $\mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a)$ and irreducibility of $\overline{\mathrm{Jac}}^{d+\delta}(X_b)$. $\qquad \square$

Consider the projection onto the first factor,

$$
(5.12) \qquad \begin{array}{rccc} \dot{\nu}_a : & \mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a) & \longrightarrow & \mathrm{Jac}^{d+\delta}(C_a) \\ & (M, V) & \longmapsto & M. \end{array}
$$

This is a fiber bundle, with projective fibers given by products of closed subschemes of Grassmanians [C1]. The restriction of this morphism to $\mathrm{Jac}^{d+\delta}(X_b) \subset \mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a)$ coincides with $\hat{\nu}_a$ from (5.7), *i.e.* the diagram

$$
(5.13) \qquad \begin{array}{c} \mathrm{Jac}^{d+\delta}(X_b) \\ {\scriptstyle \tau_0^{-1}} \big\uparrow \qquad \searrow {\scriptstyle \hat{\nu}_a} \\ \mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a) \xrightarrow{\ \dot{\nu}_a\ } \mathrm{Jac}^{d+\delta}(C_a) \end{array}
$$

commutes. So $\mathrm{PMod}_\ell^{d+\delta}(C_a, \widetilde{D}_a)$ is a compactification of $\mathrm{Jac}^{d+\delta}(X_b)$ to which the pullback map $\hat{\nu}_a$ extends, in contrast to what happens with $\overline{\mathrm{Jac}}^{d+\delta}(X_b)$. The next lemma relates the closure of the fiber of $\hat{\nu}_a$ with the fiber of $\dot{\nu}_a$.

Lemma 5.10. *Let $M \in \mathrm{Jac}^{d+\delta}(C_a)$. Then*

$$\overline{\hat{\nu}_a^{-1}(M)} = \tau(\dot{\nu}_a^{-1}(M)), \tag{5.14}$$

and it classifies those $\mathcal{F} \in \overline{\mathrm{Jac}}^{d+\delta}(X_b)$ fitting in the short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \nu_{a,*}M \longrightarrow \mathcal{O}_{\mathrm{sing}(X_b)} \longrightarrow 0. \tag{5.15}$$

Furthermore, the restriction of τ to $\dot{\nu}_a^{-1}(M)$ is a closed embedding.

Proof. First of all, the image under τ of the fiber of $\dot{\nu}_a$ over M is given by $\mathcal{F}$ fitting in (5.9). Then, it follows from (5.10) that $\tau(\dot{\nu}_a^{-1}(M))$ is given by those $\mathcal{F}$ fitting in (5.15).

We now address (5.14). The commutative diagram (5.13) can be completed as

$$
\begin{array}{ccc}
\hat{\nu}_a^{-1}(M) & \lhook\joinrel\longrightarrow & \mathrm{Jac}^{d+\delta}(X_b). \\
{\scriptstyle \tau_0^{-1}} \big\uparrow & & {\scriptstyle \tau_0^{-1}} \big\uparrow \quad\searrow^{\hat{\nu}_a} \\
\dot{\nu}_a^{-1}(M) & \lhook\joinrel\longrightarrow & \mathrm{PMod}_\ell^{d+\delta}(C_a,\widetilde{D}_a) \xrightarrow{\ \dot{\nu}_a\ } \mathrm{Jac}^{d+\delta}(C_a) \\
& & {\scriptstyle \tau}\big\downarrow \\
\overline{\hat{\nu}_a^{-1}(M)} & \lhook\joinrel\longrightarrow & \overline{\mathrm{Jac}}^{d+\delta}(X_b)
\end{array}
$$

Of course, $\tau\tau_0^{-1}$ is the identity on $\mathrm{Jac}^{d+\delta}(X_b)$. Now, since τ is a closed morphism, $\tau(\dot{\nu}_a^{-1}(M))$ is closed in $\overline{\mathrm{Jac}}^{d+\delta}(X_b)$; clearly, it contains $\tau(\tau_0^{-1}(\hat{\nu}_a^{-1}(M))) = \hat{\nu}_a^{-1}(M)$, so $\overline{\hat{\nu}_a^{-1}(M)} \subset \tau(\dot{\nu}_a^{-1}(M))$.

Conversely, given any parabolic module $(M,V) \in \dot{\nu}_a^{-1}(M)$, we know that there is a sequence of elements $(p_j)_j$ in $\mathrm{Jac}^{d+\delta}(X_b)$ such that its image under τ_0^{-1} converges to (M,V). We can find such a sequence in $\tau_0^{-1}(\hat{\nu}_a^{-1}(M))$ as follows (so that $\mathrm{PMod}_\ell^{d+\delta}(C_a,\widetilde{D}_a)$ is actually fiberwise compactification of $\mathrm{Jac}^{d+\delta}(X_b)$). The pullback map $\hat{\nu}_b : \mathrm{Jac}^{d+\delta}(X_b) \longrightarrow \mathrm{Jac}^{d+\delta}(C_a)$ is a locally trivial fibration, with the fiber F being isomorphic to products of powers of $\mathbb{C}^*$ and of $\mathbb{C}$. Trivialize this fibration on an open set U around $M \in \mathrm{Jac}^{d+\delta}(C_a)$, hence becoming a product $U \times F$. On this product, project the sequence $(p_j)_j$ onto a sequence $(q_j)_j$ on $\{M\} \times F$ (*i.e.* via the map $U \times F \longrightarrow \{M\} \times F$, $(L,z) \mapsto (M,z)$). So $(q_j)_j$ is a sequence on $\hat{\nu}_b^{-1}(M)$ and, since (5.13) commutes, $(\tau_0^{-1}(q_j))_j$ is a sequence in $\dot{\nu}_a^{-1}(M)$, which converges to the same point as $(\tau_0^{-1}(p_j))_j$, namely (M,V). Now, taking the image under the closed morphism τ, we see that that $\tau(M,V) = \lim q_j$, thus $\tau(M,V) \in \overline{\hat{\nu}_a^{-1}(M)}$, and therefore $\overline{\hat{\nu}_a^{-1}(M)} \supset \tau(\dot{\nu}_a^{-1}(M))$.

It remains to prove the last claim *i.e.* to prove that τ restricted to $\dot{\nu}_a^{-1}(M)$ is injective. Take (M,V) and (M,V') in $\dot{\nu}_a^{-1}(M)$, both mapping to $\mathcal{F}$ under τ. It follows that both V and V' must be isomorphic to $\mathcal{F} \otimes \mathcal{O}_{\mathrm{sing}(X_b)}$, hence $(M,V) \cong (M,V')$. $\qquad\square$

Remark 5.11. The previous lemma shows that, for a fixed M, we can identify $\overline{\hat{\nu}_a^{-1}(M)}$ and $\dot{\nu}_a^{-1}(M)$ via τ. However, the map τ is not generally injective, as there are $M \not\cong M'$ such that $\tau(\dot{\nu}_a^{-1}(M)) \cap \tau(\dot{\nu}_a^{-1}(M'))$ is non-empty, that is, the *closures* of $\hat{\nu}_a^{-1}(M)$ and of $\hat{\nu}_a^{-1}(M')$ will intersect; cf. [GO1, Example 5.4] for an example of this phenomenon for A-type singularities.

We finish this section describing the restriction of the Hitchin map to $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$.

Theorem 5.12. *Let $b \in B_{\mathrm{ni}}^p$ and $a \in B_{C,r}^{\mathrm{ni}}$ such that $b = \zeta(a)$, then*

$$h_{X,n}^{-1}(b) \cap \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}} = \bigcup_{\gamma^*\mathcal{J} \in \Gamma(\mathcal{J})} S_{X,n}\left(\overline{\hat{\nu}_a^{-1}(\eta_a^*(\gamma^*\mathcal{J}))}\right), \tag{5.16}$$

where $\Gamma(\mathcal{J})$ denotes the Γ-orbit of $\mathcal{J}$. Furthermore,

$$\dim\left(h_{X,n}^{-1}(b) \cap \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}\right) = \delta - \rho = n(n-r)(g-1). \tag{5.17}$$

Proof. As we have seen in the proof of Theorem 5.5, the points $h_{X,n}^{-1}(b) \cap \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ are those Higgs bundles whose spectral data $\mathcal{F}$ fit in the exact sequence (5.3) for any choice of $a \in B_{C,r}^{\mathrm{ni}}$

such that $b = \zeta(a)$. Observing that $\eta^*_{\gamma(a)}\mathcal{J} = \eta^*_a(\gamma^*\mathcal{J})$, if we fix $a \in B^{\mathrm{ni}}_{C,r}$, we can understand $h^{-1}_{X,n}(b) \cap \mathrm{NR}^{p,\mathcal{J}}_{\mathrm{ni}}$ as the set of $\mathcal{F}$ fitting in

$$0 \longrightarrow \mathcal{F} \longrightarrow \nu_{a,*}\eta^*_a\gamma^*\mathcal{J} \longrightarrow \mathcal{O}_{\mathrm{sing}(X_b)} \longrightarrow 0,$$

for every $\gamma \in \Gamma(\mathcal{J})$. Then, the first statement follows from Lemma 5.10. It then follows that

$$\dim\left(h^{-1}_{X,n}(b) \cap \mathrm{NR}^{p,\mathcal{J}}_{\mathrm{it}}\right) = \dim\left(\widehat{\hat{\nu}^{-1}_a(\eta^*_a\mathcal{J})}\right) = \dim\left(\hat{\nu}^{-1}_a(\eta^*_a\mathcal{J})\right).$$

We can find in [Gr1] a description of the fibers of $\hat{\nu}_a$, being a torsor for $H^0(X_b, \mathcal{O}^*_{\mathrm{sing}(X_b)})$. Since,

$$\dim\left(H^0(X_b, \mathcal{O}_{\mathrm{sing}(X_b)})\right) = \deg(\mathrm{sing}(X_b)),$$

the second statement follows from (ii) of Theorem 3.11. $\square$

Remark 5.13. In particular, if $\mathcal{J}$ is pulled back from a line bundle over X then $\Gamma(\mathcal{J}) = \{\mathcal{J}\}$, thus (5.16) just becomes $S_{X,n}\left(\hat{\nu}^{-1}_a(\mathcal{J})\right)$. Notice that this condition on $\mathcal{J} \in \mathrm{Jac}^{\delta/r+d'}(C)$ is only possible if d' is a multiple of m.

5.3. A complex Lagrangian subvariety.

Keeping $\mathcal{J} \in \mathrm{Jac}^{\delta/r+d'}(C)$ fixed, we now study some properties of $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{ni}}$. Particularly relevant is the proof that $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{ni}}$ is a complex Lagrangian subvariety of $\mathbf{M}_X(n, rd')$.

Recall that B denotes the Borel subgroup of $\mathrm{GL}(n, \mathbb{C})$ and U its unipotent subgroup. Fix a square root $K^{1/2}_C$ of K_C. We now study the relation of our subvariety $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{ni}}$ with the *unipotent locus* in $\mathbf{M}_C(n, nd')$, as defined in [FP, Section 4] out of a line bundle $\mathcal{L} \longrightarrow C$ of degree d',

$$(5.18) \qquad \mathrm{Uni}_C(\mathcal{L}) = \left\{ (E, \varphi) \in \mathbf{M}_C(n, nd') \;\middle|\; \begin{array}{l} \exists\, \sigma \in H^0(X, E/\mathrm{B}): \\ \varphi \in H^0(X, E_\sigma(\mathfrak{b}) \otimes K_X); \\ E_\sigma/\mathrm{U} \cong \bigoplus_{i=1}^n \mathcal{L}K_C^{(n+1-2i)/2}. \end{array} \right\},$$

The relevant line bundles for us are those of the orbit $\Gamma(\mathcal{J}) = \{\gamma^*\mathcal{J} \text{ for } \gamma \in \Gamma\}$, and for each of them the corresponding unipotent locus will be denoted by $\mathrm{Uni}_C(\gamma^*\mathcal{J}) \subset \mathbf{M}_C(n, nd')$. Recall the Cartan locus $\mathbf{M}_{(1,n)}$ of $\mathbf{M}_C(n, nd')$, introduced before Corollary 3.15, and let V be $h_{C,n}(\mathbf{M}_{(1,n)})$. Then $h_{C,n}(\mathrm{Uni}(\gamma^{i,*}\mathcal{J})) = V$, for every i (cf. [FP]).

Recall the pullback map $\hat{p}: \mathbf{M}_X(n, rd') \longrightarrow \mathbf{M}_C(n, nd')$ from (3.3).

Proposition 5.14. *Consider the open subvariety* $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{Jac}}$ *of* $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{ni}}$ *defined as the intersection of* $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{ni}}$ *with the open subset* $\mathrm{Jac}^{d+\delta}(X_b)$ *of every Hitchin fiber* $h^{-1}_{X,n}(b)$. *Then* $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{Jac}}$ *is mapped under the pullback map* (3.3) *to the union of* $\mathrm{Uni}_C(\gamma^*\mathcal{J})$, *for* $\gamma \in \Gamma$, *i.e.*

$$\hat{p}\left(\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{Jac}}\right) \subset \bigsqcup_{\gamma^*\mathcal{J} \in \Gamma(\mathcal{J})} \mathrm{Uni}_C(\gamma^*\mathcal{J}).$$

Proof. By the spectral correspondence, the proposition can be proved by showing that the pullback of the spectral data of any Higgs bundle in $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{Jac}}$ gives the spectral data of a Higgs bundle in $\bigsqcup_{\gamma^*\mathcal{J} \in \Gamma(\mathcal{J})} \mathrm{Uni}_C(\gamma^{i,*}\mathcal{J})$.

Take $b \in B^p_{\mathrm{ni}} \subset B^p$ and let $\tilde{b} = p^*b$. Then $C_{\tilde{b}}$ is integral and has nodal singularities. Furthermore, by Corollary 3.15, $\tilde{b} \in V$. So, if (E, φ) represents a point in $\mathrm{NR}^{p,\mathcal{J}}_{\mathrm{Jac}}$ mapping to b, then $\hat{p}(E, \varphi)$ maps to $\tilde{b} \in V$. By Theorem 5.12, the spectral datum of (E, φ) is given by a line bundle $L \in \mathrm{Jac}^{d+\delta}(X_b)$ such that $\nu^*_a L \cong \gamma^*\mathcal{J}$ for some $\gamma \in \Gamma$. Using the commutative diagram of Theorem 3.11 (iv), we see that $\tilde{\nu}^*_{\tilde{b}} q^*_{\tilde{b}} L \cong \tilde{q}^*_a \gamma^*\mathcal{J}$ is the exterior product of all the elements in the orbit $\Gamma(\mathcal{J})$. Recall that $C_{\tilde{b}}$ is a disconnected curve, whose connected components are all isomorphic to C. We see that the spectral datum of $\hat{p}((E, \varphi))$ is a line bundle on $C_{\tilde{b}}$ (namely $q^*_{\tilde{b}} L$) which pulls back under the normalization map $\tilde{\nu}_{\tilde{b}}: \tilde{C}_{\tilde{b}} \longrightarrow C_{\tilde{b}}$ to the exterior product of all the elements in the orbit $\Gamma(\mathcal{J})$. By Proposition 4.5 (4.14) of [FP], this is the spectral datum of an element of $\mathrm{Uni}_C(\gamma^*\mathcal{J})$, proving that $\hat{p}(E, \varphi) \in \mathrm{Uni}_C(\gamma^*\mathcal{J})$. $\square$

Given a Higgs bundle (E, φ), the associated *deformation complex* is defined by

$$(5.19) \qquad C_{(E,\varphi)}^\bullet : \operatorname{End}(E) \xrightarrow{[-,\varphi]} \operatorname{End}(E) \otimes K.$$

Its hypercohomology $\mathbb{H}^*(C_{(E,\varphi)}^\bullet)$ fits in the long exact sequence

(5.20)

$$0 \longrightarrow \mathbb{H}^0(C_{(E,\varphi)}^\bullet) \longrightarrow H^0(X, \operatorname{End}(E)) \longrightarrow H^0(X, \operatorname{End}(E) \otimes K_X) \longrightarrow \mathbb{H}^1(C_{(E,\varphi)}^\bullet) \longrightarrow$$
$$\longrightarrow H^1(X, \operatorname{End}(E)) \longrightarrow H^1(X, \operatorname{End}(E) \otimes K_X) \longrightarrow \mathbb{H}^2(C_{(E,\varphi)}^\bullet) \longrightarrow 0.$$

If (E, φ) is a stable Higgs bundle, then it represents a smooth point of the moduli space $\mathbf{M}_X(n, d)$, with tangent space isomorphic to $\mathbb{H}^1(C_{(E,\varphi)}^\bullet)$.

Recall from Section 2.1 the holomorphic symplectic form $\Omega_{X,1} = \omega_2 + \mathrm{i}\omega_3$ on $\mathbf{M}_X(n, rd')$ associated to the complex structure I_1. We can now prove the main result of this section.

Theorem 5.15. $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ *is a Lagrangian subvariety of* $\mathbf{M}_X(n, rd')$ *with respect to* $\Omega_{X,1}$.

Proof. First we prove that $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ is isotropic. Recall the open subset $\mathrm{NR}_{\mathrm{Jac}}^{p,\mathcal{J}} \subset \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ defined in the previous proposition. It is enough to show that the symplectic form $\Omega_{X,1}$ vanishes on $\mathrm{NR}_{\mathrm{Jac}}^{p,\mathcal{J}}$.

Let $(E, \varphi) \in \mathrm{NR}_{\mathrm{Jac}}^{p,\mathcal{J}}$. It is a smooth point of $\mathbf{M}_X(n, d)$, by Proposition 5.6. Consider the polystable Higgs bundle $(\tilde{E}, \tilde{\varphi}) := \hat{p}(E, \varphi) = (p^*E, p^*\varphi)$ over C. By Proposition 5.14, $(\tilde{E}, \tilde{\varphi}) \in \mathrm{Uni}_C(\gamma^*\mathcal{J}) \subset \mathbf{M}_C(n, nd')$ for some $\gamma^*\mathcal{J} \in \Gamma(\mathcal{J})$. In addition, by [FP, Proposition 4.5], $(\tilde{E}, \tilde{\varphi})$ is stable, thus also represents a smooth point of $\mathbf{M}_C(n, nd')$.

As (E, φ) represents a smooth point of the moduli space, the corresponding tangent space is isomorphic to $\mathbb{H}^1(C_{(E,\varphi)}^\bullet)$, where $C_{(E,\varphi)}^\bullet$ is the complex given by (5.19). Since $\mathbb{H}^1(C_{(E,\varphi)}^\bullet)$ fits in (5.20), any tangent vector in $T_{(E,\varphi)}\mathbf{M}_X(n, rd') \cong \mathbb{H}^1(C_{(E,\varphi)}^\bullet)$ is determined by an element in $H^0(X, \operatorname{End}(E) \otimes K_X) \cong H^{1,0}(X, \operatorname{End}(E))$ (providing the deformation of the Higgs field) and by an element in $H^1(X, \operatorname{End}(E)) \cong H^{0,1}(X, \operatorname{End}(E))$ (providing the deformation of the holomorphic structure of E). Thus, every $v, w \in T_{(E,\varphi)}\mathbf{M}_X(n, rd')$, may be represented as $v = (\alpha_1, \alpha_2)$ and $w = (\beta_1, \beta_2)$, with $\alpha_1, \beta_1 \in \Omega^{1,0}(X, \operatorname{End}(E))$ and $\alpha_2, \beta_2 \in \Omega^{0,1}(X, \operatorname{End}(E))$. Then [Hi1],

$$(5.21) \qquad \Omega_{X,1}(v, w) = \int_X \operatorname{tr}(\alpha_1 \wedge \beta_2 - \alpha_2 \wedge \beta_1) \in \mathbb{C}.$$

Pick the holomorphic symplectic form $\Omega_{C,1}$ on $\mathbf{M}_C(n, nd')$. Analogous statements hold for any pair of tangent vectors $\tilde{v}, \tilde{w} \in T_{(\tilde{E}, \tilde{\varphi})}\mathbf{M}_C(n, nd') \cong \mathbb{H}^1(C_{(\tilde{E}, \tilde{\varphi})}^\bullet)$, where $C_{(\tilde{E}, \tilde{\varphi})}^\bullet$ is the deformation complex of $(\tilde{E}, \tilde{\varphi})$, defined in (5.19). Thus,

$$(5.22) \qquad \Omega_{C,1}(\tilde{v}, \tilde{w}) = \int_C (\tilde{\alpha}_1 \wedge \tilde{\beta}_2 - \tilde{\alpha}_2 \wedge \tilde{\beta}_1) \in \mathbb{C}.$$

If $\tilde{v} = d\hat{p}(v)$ and $\tilde{w} = d\hat{p}(w)$, then $\tilde{\alpha}_i = p^*\alpha_i$ and $\tilde{\beta}_i = p^*\beta_i$, for $i = 1, 2$, hence (5.21) and (5.22) imply that

$$(5.23) \qquad \hat{p}^* \Omega_{C,1} = d\hat{p}^t \Omega_{C,1} = m\Omega_{X,1}$$

because $p : C \longrightarrow X$ is a degree m map.

Assume now that v, w are tangent vectors to $\mathrm{NR}_{\mathrm{Jac}}^{p,\mathcal{J}}$. By Proposition 5.14, we have that $d\hat{p}(v)$ and $d\hat{p}(w)$ are tangent vectors to the unipotent locus of $\mathbf{M}_C(n, nd')$, which is isotropic, by Proposition 4.2 of [FP]. Hence $\Omega_{C,1}(d\hat{p}(v), d\hat{p}(w)) = 0$, so $\Omega_{X,1}(v, w) = 0$ by (5.23), proving that $\mathrm{NR}_{\mathrm{Jac}}^{p,\mathcal{J}}$, thus $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$, is isotropic.

It remains to prove that $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ is a mid-dimensional subvariety of $\mathbf{M}_X(n, rd')$, that is, $\dim(\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}) = n^2(g - 1) + 1$, after (2.1). Since $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ lies in the smooth locus of $\mathbf{M}_X(n, rd')$ after Proposition 5.6, its dimension can be computed through the Hitchin map restricted to $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$, namely by adding the dimension of B_{ni}^p to the dimension of any fiber $h_{X,n}^{-1}(b) \cap \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$.

From (3.15) we have that $\dim B_{\mathrm{ni}}^p = \dim B^p = rn(g-1) + 1$. On the other hand, by (5.17), one has that $\dim(h_{X,n}^{-1}(b) \cap \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}) = n(n-r)(g-1)$. Then, their sum equals

$$\dim B^p + \dim(h_{X,n}^{-1}(b) \cap \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}) = rn(g-1) + 1 + n(n-r)(g-1).$$

Adding up yields $\dim(\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}) = n^2(g-1) + 1$ as claimed. $\qquad\square$

6. Mirror symmetry and branes

In two preceding sections we constructed the Narasimhan–Ramanan (BBB)-branes $(\mathbf{BBB})_{\mathcal{F}}^p$ and $(\mathbf{BBB})_{\mathcal{L}}^p$ (cf. Definitions 4.2 and 4.4) and the complex Lagrangian subvarieties $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ (cf. Definition 5.3). Recall that the Definitions 4.2 and 4.4 required Assumption 1, thus $r = 1$. Note as well that, by construction, both branes $(\mathbf{BBB})_{\mathcal{L}}^p$ and $(\mathbf{BBB})_{\mathcal{F}}^p$ determine coherent sheaves on $\mathbf{M}_X(n,d)$ (with respect to the complex structure I_1), and their support fiber (via the Hitchin map) over the locus B^p. On the other hand $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ is only constructed over the open dense subspace $B_{\mathrm{ni}}^p \subset B^p$ parametrizing integral and nodal spectral curves there.

Mirror symmetry predicts that a (BBB)-brane ought to be dual to a (BAA)-brane. It is expected that such duality is realized via a Fourier–Mukai transform relative to a Lagrangian fibration associated to the complex structure I_1. In this section we prove that (the restriction to B_{ni}^p of) the coherent sheaf on $\mathbf{M}_X(n,d)$ given by the Narasimhan–Ramanan (BBB)-brane $(\mathbf{BBB})_{\mathcal{L}}^p$ is Fourier–Mukai transformed into a sheaf supported over $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$, for an explicit choice of $\mathcal{J}$ depending on $\mathcal{L}$. We obtain an explicit relation on this transformed sheaf which is enough to find its support, but we do not have a full description of it. The only missing piece to produce the complete (fiberwise) mirror symmetry is a global description of the corresponding flat bundle over $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$; we have partial information on it, but not a global one. Starting with $(\mathbf{BBB})_{\mathcal{F}}^p$ instead, we obtain a similar duality statement.

We address only the case of degree $d = 0$ (thus the case of d multiple of n also follows), because in this case we have the Hitchin sections [Hi3] as global Lagrangian sections of the Hitchin fibration $h_{X,n} : \mathbf{M}_X(n,0) \longrightarrow B_{X,n}$. This allow us to perform the fiberwise Fourier–Mukai transform without using a gerbe (or using a trivial one). For d non-multiple of n, then $h_{X,n}$ has no such global Lagrangian section, hence a gerbe is required to properly perform the relative Fourier–Mukai (cf. [HT]). On the other hand, we expect that all the analysis in the preceding sections generalizes to the setting of parabolic Higgs bundles, and there, under mild assumptions, the Fourier–Mukai duality can be performed for any degree d without the need for a gerbe. This is because, for an appropriate choice of parabolic weights, the Hitchin fibration always admits a Hitchin section (cf. [GO2]).

6.1. **Review of autoduality of compactified Jacobians of integral curves.** In this section we review autoduality of compactified Jacobians of integral curves with planar singularities and the associated Fourier–Mukai transform given by Arinkin in [Ar]. Since spectral curves are contained in the surface $|K_X|$, his construction applies to any integral spectral curve X_b, in particular to all curves for $b \in B_{\mathrm{ni}}^p$ by Theorem 3.11 (v). In this context, Arinkin's autoduality statement becomes becomes the autoduality of the corresponding Hitchin fibers $h_{X,n}^{-1}(b)$.

Take an integral curve with planar singularities X_b and consider an integer δ. Then every semistable rank 1 torsion-free sheaf on X_b is indeed stable, and $\overline{\mathrm{Jac}}^\delta(X_b)$ is therefore a fine moduli space with universal family $\mathcal{U}_b \longrightarrow X_b \times \overline{\mathrm{Jac}}^\delta(X_b)$. Denote by $\mathcal{U}_b^0$ its restriction to $X_b \times \mathrm{Jac}^\delta(X_b)$. Before constructing the Poincaré sheaf, we first construct the Poincaré bundle using $\mathcal{U}_b$ and $\mathcal{U}_b^0$. Choose a point $\sigma_X(b)$ in $\overline{\mathrm{Jac}}^\delta(X_b)$ and let $\mathcal{U}_b$ and $\mathcal{U}_b^0$ be normalized with respect to $\sigma_X(b)$.

Given a flat morphism $f : Y \longrightarrow S$ whose geometric fibers are curves, we can define the determinant of cohomology (see [KM] and [Es, Section 6.1]) as follows. If $\mathcal{E}$ is an S-flat sheaf on Y, the determinant of cohomology $\mathcal{D}_f(\mathcal{E})$ is an invertible sheaf on S, constructed locally as the determinant of complexes of free sheaves, which is locally quasi-isomorphic to $Rf_*\mathcal{E}$. Consider the triple product $X_b \times \overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b)$ and the projection $f_{23} : X_b \times \overline{\mathrm{Jac}}^\delta(X_b) \times$

$\mathrm{Jac}^\delta(X_b) \longrightarrow \overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b)$, which is flat and whose fibers are curves. Consider as well the corresponding projections f_{12} and f_{13}. The Poincaré line bundle $\mathcal{P}_b \longrightarrow \overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b)$ is the invertible sheaf

$$(6.1) \qquad \mathcal{P}_b := \mathcal{D}_{f_{23}} \left(f_{12}^* \mathcal{U}_b \otimes f_{13}^* \mathcal{U}_b^0 \right)^{-1} \otimes \mathcal{D}_{f_{23}} \left(f_{13}^* \mathcal{U}_b^0 \right) \otimes \mathcal{D}_{f_{23}} \left(f_{12}^* \mathcal{U}_b \right).$$

The restriction of the Poincaré bundle $\mathcal{P}_b$ to the point associated to $M \in \mathrm{Jac}^\delta(X_b)$, that is, $\mathcal{P}_{b,M} := \mathcal{P}_b|_{\overline{\mathrm{Jac}}^\delta(X_b) \times \{M\}}$, is the line bundle over $\overline{\mathrm{Jac}}^\delta(X_b)$ given by

$$(6.2) \qquad \mathcal{P}_{b,M} = \mathcal{D}_{f_2}(\mathcal{U}_b \otimes f_1^* M)^{-1} \otimes \mathcal{D}_{f_2}(f_1^* M) \otimes \mathcal{D}_{f_2}(\mathcal{U}_b),$$

where we have considered the projections $f_1 : X_b \times \overline{\mathrm{Jac}}^\delta(X_b) \longrightarrow X_b$ and $f_2 : X_b \times \overline{\mathrm{Jac}}^\delta(X_b) \longrightarrow \overline{\mathrm{Jac}}^\delta(X_b)$.

Our Poincaré bundle is constructed over $\overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b)$. A similar construction can be performed over $\mathrm{Jac}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b)$, which coincides with $\mathcal{P}_b$ after restricting both to $\mathrm{Jac}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b)$. Gluing both line bundles over $\mathrm{Jac}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b)$, one can define the line bundle

$$(6.3) \qquad \mathcal{P}_b^\sharp \longrightarrow \left(\overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b) \right)^\sharp,$$

where

$$(6.4) \qquad \left(\overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b) \right)^\sharp := \left(\mathrm{Jac}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b) \right) \cup \left(\overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{Jac}^\delta(X_b) \right).$$

Arinkin [Ar] extended this construction to the compactified Jacobian, obtaining the *Poincaré sheaf*

$$\overline{\mathcal{P}}_b \longrightarrow \overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b),$$

showing as well that it is a Cohen-Macaulay sheaf. Therefore, considering the injection

$$(6.5) \qquad j : \left(\overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b) \right)^\sharp \hookrightarrow \overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b),$$

one has that the Poincaré sheaf satisfies [Ar, Lemma 6.1 (2)]

$$(6.6) \qquad \overline{\mathcal{P}}_b \cong j_* \mathcal{P}_b^\sharp.$$

Taking the projections π_1, π_2 onto the first and second factors

$$(6.7)$$

$$\overline{\mathrm{Jac}}^\delta(X_b) \times \overline{\mathrm{Jac}}^\delta(X_b)$$

$$\overset{\pi_1}{\swarrow} \qquad\qquad \overset{\pi_2}{\searrow}$$

$$\overline{\mathrm{Jac}}^\delta(X_b) \qquad\qquad\qquad \overline{\mathrm{Jac}}^\delta(X_b),$$

and using $\overline{\mathcal{P}}_b$ as a kernel, we consider the Fourier–Mukai functor on the bounded derived category of coherent sheaves on $\overline{\mathrm{Jac}}^\delta(X_b)$,

$$(6.8) \qquad \begin{aligned} \Theta_b : \quad D^b \left(\overline{\mathrm{Jac}}^\delta(X_b) \right) &\longrightarrow \quad D^b \left(\overline{\mathrm{Jac}}^\delta(X_b) \right) \\ \mathcal{F}^\bullet &\longmapsto \quad R\pi_{2,*}(\pi_1^* \mathcal{F}^\bullet \otimes \overline{\mathcal{P}}_b). \end{aligned}$$

The following is due to Arinkin in the case of X_b integral.

Theorem 6.1 ([Ar]). *Let X_b be an integral curve with planar singularities and δ an integer. The moduli space of rank 1 torsion-free sheaves over $\overline{\mathrm{Jac}}^\delta(X_b)$ is $\overline{\mathrm{Jac}}^\delta(X_b)$ itself. Furthermore the Fourier–Mukai functor Θ_b is a derived equivalence.*

6.2. **Fourier–Mukai and normalization.** Let us consider the normalization $\nu_a : C_a \longrightarrow X_b$ of an integral curve X_b with nodal singularities, and denote by $\delta - \rho$ the degree of the singular divisor $\text{sing}(X_b)$, where δ and ρ are integers. In this section we study the interplay of ν_a with the Fourier–Mukai transform constructed by Arinkin[1].

Since C_a is smooth, the Jacobian $\text{Jac}^0(C_a)$ is a smooth abelian variety known to be autodual. Choosing a point in $\text{Jac}^\delta(C_a)$ and $\text{Jac}^\rho(C_a)$ provides an isomorphism from the abelian variety $\text{Jac}^0(C_a)$ to the torsors $\text{Jac}^\delta(C_a)$ and $\text{Jac}^\rho(C_a)$. Using this isomorphism, we naturally obtain a Poincaré line bundle

$$\mathcal{P}_a \longrightarrow \text{Jac}^\rho(C_a) \times \text{Jac}^\delta(C_a)$$

from the Poincaré line bundle over $\text{Jac}^0(C_a) \times \text{Jac}^0(C_a)$. Then $\mathcal{P}_a$ is a universal family of topologically trivial line bundles over $\text{Jac}^\rho(C_a)$ parametrized by $\text{Jac}^\delta(C_a)$.

Consider the pushforward morphism

$$\check{\nu}_a : \text{Jac}^\rho(C_a) \longrightarrow \overline{\text{Jac}}^\delta(X_b)$$

induced from the normalization map $\nu_a : C_a \longrightarrow X_b$, and the pullback map

$$\hat{\nu}_a : \text{Jac}^\delta(X_b) \longrightarrow \text{Jac}^\delta(C_a).$$

Let $\mathbf{1}_{\widetilde{\text{Jac}}}$ and $\mathbf{1}_{\text{Jac}}$ be the identity morphisms in $\text{Jac}^\rho(C_a)$ and $\text{Jac}^\delta(X_b)$ respectively. Recall from (6.6) that the Poincaré sheaf $\overline{\mathcal{P}}_b$ on $\overline{\text{Jac}}^\delta(X_b) \times \overline{\text{Jac}}^\delta(X_b)$ is constructed from the Poincaré line bundle

$$\mathcal{P}_b \longrightarrow \overline{\text{Jac}}^\delta(X_b) \times \text{Jac}^\delta(X_b)$$

given in (6.1). Then both $(\check{\nu}_a \times \mathbf{1}_{\text{Jac}})^* \mathcal{P}_b$ and $(\mathbf{1}_{\widetilde{\text{Jac}}} \times \hat{\nu}_a)^* \mathcal{P}_a$ are bundles over $\text{Jac}^\rho(C_a) \times \text{Jac}^\delta(X_b)$. The next result shows that they differ from a line bundle which is a pullback from a line bundle over $\text{Jac}^\delta(X_b)$.

Consider the projections

$$\text{Jac}^\rho(C_a) \times \text{Jac}^\delta(X_b)$$

$$\pi_1' \swarrow \qquad \qquad \searrow \pi_2'$$

$$\text{Jac}^\rho(C_a) \qquad \qquad \qquad \qquad \text{Jac}^\delta(X_b).$$

Proposition 6.2. *Let $\nu_a : C_a \longrightarrow X_b$ be normalization of an integral curve X_b with planar singularities. Then we have that*

$$(\check{\nu}_a \times \mathbf{1}_{\text{Jac}})^* \mathcal{P}_b \cong (\mathbf{1}_{\widetilde{\text{Jac}}} \times \hat{\nu}_a)^* \mathcal{P}_a \otimes (\pi_2')^* \mathcal{W},$$

for some line bundle $\mathcal{W} \longrightarrow \text{Jac}^\delta(X_b)$.

Proof. After a certain adaptation, the proof is analogous to that of [FP, Lemma 5.2]. We include it here for the sake of clarity. First note that $(\check{\nu}_a \times \mathbf{1}_{\text{Jac}})^* \mathcal{P}_b$ is a family of topologically trivial line bundles over $\text{Jac}^\rho(C_a)$ parametrized by $\text{Jac}^\delta(X_b)$. Since $\mathcal{P}_a \longrightarrow \text{Jac}^\rho(C_a) \times \text{Jac}^\delta(C_a)$ is a universal family for such objects, there exists a morphism

$$g : \text{Jac}^\delta(X_b) \longrightarrow \text{Jac}^\delta(C_a),$$

such that

(6.9) $$(\check{\nu}_a \times \mathbf{1}_{\text{Jac}})^* \mathcal{P}_b \cong (\mathbf{1}_{\widetilde{\text{Jac}}} \times g)^* \mathcal{P}_a \otimes (\pi_2')^* \mathcal{W}_b,$$

for some line bundle $\mathcal{W}_b \longrightarrow \text{Jac}^\delta(X_b)$. We claim that $g = \hat{\nu}_a$. In order to prove it, we shall need several preliminary statements.

Recall the description of $\mathcal{P}_{b,M}$ given in (6.2),

$$\mathcal{P}_{b,M} = \mathcal{D}_{f_2}(\mathcal{U}_b \otimes f_1^* M)^{-1} \otimes \mathcal{D}_{f_2}(f_1^* M) \otimes \mathcal{D}_{f_2}(\mathcal{U}_b),$$

---

[1]The results contained in this section were previously known to D. Arinkin, to whom we are indebted for conversations.

for each $M \in \mathrm{Jac}^{\delta}(X_b)$, where

$$f_1 : X_b \times \overline{\mathrm{Jac}}^{\delta}(X_b) \longrightarrow X_b \quad \text{and} \quad f_2 : X_b \times \overline{\mathrm{Jac}}^{\delta}(X_b) \longrightarrow \overline{\mathrm{Jac}}^{\delta}(X_b).$$

Take also the projections

$$\tilde{f}_1 : C_a \times \mathrm{Jac}^{\rho}(C_a) \longrightarrow C_a, \quad \tilde{f}_2 : C_a \times \mathrm{Jac}^{\rho}(C_a) \longrightarrow \mathrm{Jac}^{\rho}(C_a)$$

and

$$f_1' : X_b \times \mathrm{Jac}^{\rho}(C_a) \longrightarrow X_b \quad f_2' : X_b \times \mathrm{Jac}^{\rho}(C_a) \longrightarrow \mathrm{Jac}^{\rho}(C_a).$$

Obviously

(6.10)
$$X_b \times \mathrm{Jac}^{\rho}(C_a) \xrightarrow{\mathbf{1}_{X_b} \times \check{\nu}_a} X_b \times \overline{\mathrm{Jac}}^{\delta}(X_b)$$

with f_1' and f_1 mapping to X_b,

commutes.

The following diagram is also obviously Cartesian,

$$
\begin{array}{ccc}
X_b \times \mathrm{Jac}^{\rho}(C_a) & \xrightarrow{\mathbf{1}_{X_b} \times \check{\nu}_a} & X_b \times \overline{\mathrm{Jac}}^{\delta}(X_b) \\
\downarrow{\scriptstyle f_2'} & & \downarrow{\scriptstyle f_2} \\
\mathrm{Jac}^{\rho}(C_a) & \xrightarrow{\check{\nu}_a} & \overline{\mathrm{Jac}}^{\delta}(X_b),
\end{array}
$$

thus, since we know from [Es, Proposition 44 (1)] that the determinant of cohomology commutes with base change,

(6.11)
$$\check{\nu}_a^* \mathcal{D}_{f_2} = \mathcal{D}_{f_2'}(\mathbf{1}_{X_b} \times \check{\nu}_a)^*.$$

Let $\mathcal{U}_a \longrightarrow C_a \times \mathrm{Jac}^{\rho}(C_a)$ be the universal bundle of topologically trivial line bundles over C_a. Since C_a is smooth, we may apply (6.2), so that the Poincaré bundle $\mathcal{P}_a$ satisfies

(6.12)
$$\mathcal{P}_{a,N} = \mathcal{D}_{\tilde{f}_2}(\mathcal{U}_a \otimes \tilde{f}_1^* N)^{-1} \otimes \mathcal{D}_{\tilde{f}_2}(\tilde{f}_1^* N) \otimes \mathcal{D}_{\tilde{f}_2}(\mathcal{U}_a),$$

for any $N \in \mathrm{Jac}^{\delta}(C_a)$. Recall that $\mathcal{U}_b \longrightarrow X_b \times \overline{\mathrm{Jac}}^{\delta}(X_b)$ is the universal sheaf of degree δ torsion-free sheaves on X_b and consider the pullback $(\mathbf{1}_{X_b} \times \check{\nu}_a)^* \mathcal{U}_b$ which is a sheaf over $X_b \times \mathrm{Jac}^{\rho}(C_a)$. Observe that $(\nu_a \times \mathbf{1}_{\widetilde{\mathrm{Jac}}})_* \mathcal{U}_a$ is a family of rank 1 torsion-free sheaves of degree δ over X_b. Then, by the universality property, there exists a morphism $h : \mathrm{Jac}^{\rho}(C_a) \longrightarrow \overline{\mathrm{Jac}}^{\delta}(X_b)$ and a line bundle $\mathcal{W}_a \longrightarrow \mathrm{Jac}^{\rho}(C_a)$ such that

$$(\mathbf{1}_{X_b} \times h)^* \mathcal{U}_b \cong (\nu_a \times \mathbf{1}_{\widetilde{\mathrm{Jac}}})_* \mathcal{U}_a \otimes (f_2')^* \mathcal{W}_a.$$

Note that h coincides pointwise with $\check{\nu}_a$, since $\mathrm{Jac}^{\rho}(C_a)$ is smooth it follows that $h = \check{\nu}_a$. Then,

(6.13)
$$(\mathbf{1}_{X_b} \times \check{\nu}_a)^* \mathcal{U}_b \cong (\nu_a \times \mathbf{1}_{\widetilde{\mathrm{Jac}}})_* \mathcal{U}_a \otimes (f_2')^* \mathcal{W}_a.$$

Since the diagram

$$C_a \times \mathrm{Jac}^{\rho}(C_a) \xrightarrow{\nu_a \times \mathbf{1}_{\widetilde{\mathrm{Jac}}}} X_b \times \mathrm{Jac}^{\rho}(C_a)$$

with $\tilde{f}_2$ and f_2' mapping to $\mathrm{Jac}^{\rho}(C_a)$,

commutes, the definition of the determinant of cohomology ensures that

(6.14)
$$\mathcal{D}_{f_2'}(\nu_a \times \mathbf{1}_{\widetilde{\mathrm{Jac}}})_* \cong \mathcal{D}_{\tilde{f}_2}.$$

Finally,

$$C_a \times \operatorname{Jac}^\rho(C_a)$$

with maps $\nu_a \circ \tilde{f}_1$, $(\nu_a \times \mathbf{1}_{\widetilde{\operatorname{Jac}}})$, $\tilde{f}_2$ and

$$X_b \xleftarrow{\ f_1'\ } X_b \times \operatorname{Jac}^\rho(C_a) \xrightarrow{\ f_2'\ } \operatorname{Jac}^\rho(C_a).$$

also commutes, so $f_1'((f_2')^{-1}(U)) = \nu_a \tilde{f}_1(\tilde{f}_2^{-1}(U))$ for every open subset $U \subset \operatorname{Jac}^\rho(C_a)$. It then follows that for each $M \in \operatorname{Jac}^\delta(X_b)$ and each open set $U \subset \operatorname{Jac}^\rho(C_a)$,

$$
\begin{aligned}
((f_2')_* (f_1')^* M)(U) &= \lim_{W \supseteq f_1'((f_2')^{-1}(U))} M(W) \\
&= \lim_{W \supseteq \nu_a \tilde{f}_1(\tilde{f}_2^{-1}(U))} M(W) \\
&= ((\tilde{f}_2)_* (\nu_a \circ \tilde{f}_1)^* M)(U),
\end{aligned}
$$

so $(f_2')_* (f_1')^* = (\tilde{f}_2)_* (\nu_a \circ \tilde{f}_1)^*$. As a consequence of this identification, we have

$$(6.15) \qquad \mathcal{D}_{f_2'}(f_1')^* \cong \mathcal{D}_{\tilde{f}_2} \tilde{f}_1^* \nu_a^*.$$

Using the projection formula and (6.9)–(6.15), we have that, for any $M \in \operatorname{Jac}^\delta(X_b)$,

$$
\begin{aligned}
\mathcal{P}_{a,g(M)} &\cong \check{\nu}_a^* \mathcal{P}_{b,M} \\
&\cong \check{\nu}_a^* \left(\mathcal{D}_{f_2} \left(\mathcal{U}_b \otimes f_1^* M \right)^{-1} \otimes \mathcal{D}_{f_2}(f_1^* M) \otimes \mathcal{D}_{f_2}(\mathcal{U}_b) \right) \\
&\cong \check{\nu}_a^* \mathcal{D}_{f_2} \left(\mathcal{U}_b \otimes f_1^* M \right)^{-1} \otimes \check{\nu}_a^* \mathcal{D}_{f_2}(f_1^* M) \otimes \check{\nu}_a^* \mathcal{D}_{f_2}(\mathcal{U}_b) \\
&\cong \mathcal{D}_{f_2'} \left((\mathbf{1}_{X_b} \times \check{\nu}_a)^* (\mathcal{U}_b \otimes f_1^* M) \right)^{-1} \otimes \mathcal{D}_{f_2'} \left((\mathbf{1}_{X_b} \times \check{\nu}_a)^* (f_1^* M) \right) \otimes \mathcal{D}_{f_2'} \left((\mathbf{1}_{X_b} \times \check{\nu}_a)^* \mathcal{U}_b \right) \\
&\cong \mathcal{D}_{f_2'} \left((\mathbf{1}_{X_b} \times \check{\nu}_a)^* \mathcal{U}_b \otimes (f_1')^* M \right)^{-1} \otimes \mathcal{D}_{f_2'} \left((f_1')^* M \right) \otimes \mathcal{D}_{f_2'} \left((\mathbf{1}_{X_b} \times \check{\nu}_a)^* \mathcal{U}_b \right) \\
&\cong \mathcal{D}_{f_2'} \left((\nu_a \times \mathbf{1}_{\widetilde{\operatorname{Jac}}})_* \mathcal{U}_a \otimes (f_1')^* M \otimes (f_2')^* \mathcal{W}_a \right)^{-1} \otimes \\
&\qquad \otimes \mathcal{D}_{f_2'} \left((f_1')^* M \right) \otimes \mathcal{D}_{f_2'} \left((\nu_a \times \mathbf{1}_{\widetilde{\operatorname{Jac}}})_* \mathcal{U}_a \otimes (f_2')^* \mathcal{W}_a \right) \\
&\cong \mathcal{D}_{f_2'} \left((\nu_a \times \mathbf{1}_{\widetilde{\operatorname{Jac}}})_* \mathcal{U}_a \otimes (f_1')^* M \right)^{-1} \otimes \mathcal{W}_a^{-1} \otimes \\
&\qquad \otimes \mathcal{D}_{f_2'} \left((f_1')^* M \right) \otimes \mathcal{D}_{f_2'} \left((\nu_a \times \mathbf{1}_{\widetilde{\operatorname{Jac}}})_* \mathcal{U}_a \right) \otimes \mathcal{W}_a \\
&\cong \mathcal{D}_{f_2'} \left((\nu_a \times \mathbf{1}_{\widetilde{\operatorname{Jac}}})_* \left(\mathcal{U}_a \otimes \tilde{f}_1^* \nu_a^* M \right) \right)^{-1} \otimes \mathcal{D}_{f_2'} \left((f_1')^* M \right) \otimes \mathcal{D}_{f_2'} \left((\nu_a \times \mathbf{1}_{\widetilde{\operatorname{Jac}}})_* \mathcal{U}_a \right) \\
&\cong \mathcal{D}_{\tilde{f}_2}(\mathcal{U}_a \otimes \tilde{f}_1^* \nu_a^* M)^{-1} \otimes \mathcal{D}_{\tilde{f}_2} \left(\tilde{f}_1^* \nu_a^* M \right) \otimes \mathcal{D}_{\tilde{f}_2}(\mathcal{U}_a) \\
&\cong \mathcal{P}_{\nu_a^* M} \\
&\cong \mathcal{P}_{a,\hat{\nu}_a(M)}.
\end{aligned}
$$

This implies that $g(M) = \hat{\nu}_a(M)$ for any $M \in \operatorname{Jac}^\delta(X_b)$. Since $\operatorname{Jac}^\delta(X_b)$ is smooth, this suffices to state that $g = \hat{\nu}_a$, thus completing the proof. $\qquad \square$

Consider now the projections

$$(6.16) \qquad \operatorname{Jac}^\rho(C_a) \times \operatorname{Jac}^\delta(C_a)$$

with maps η_1, η_2 and

$$\operatorname{Jac}^\rho(C_a) \qquad\qquad\qquad \operatorname{Jac}^\delta(C_a).$$

Using the Poincaré line bundle $\mathcal{P}_a \longrightarrow \operatorname{Jac}^\rho(C_a) \times \operatorname{Jac}^\delta(C_a)$ as kernel, define the Fourier–Mukai functor

$$(6.17) \qquad
\begin{aligned}
\Theta_a : \quad D^b(\operatorname{Jac}^\rho(C_a)) &\longrightarrow D^b(\operatorname{Jac}^\delta(C_a)) \\
\mathcal{E}^\bullet &\longmapsto R\eta_{2,*}(\eta_1^* \mathcal{E}^\bullet \otimes \mathcal{P}_a).
\end{aligned}
$$

Mukai [Mu, Theorem 2.2] proved in this classical setting that the above functor is a derived equivalence. Note also that it is the particular case of Arinkin's Theorem 6.1 for smooth curves.

The next theorem establishes a relation between the Fourier–Mukai functors Θ_b in (6.8) and Θ_a in (6.17), for complexes arising as pushforward via $\check{\nu}_a$. Recall the moduli space of parabolic modules $\mathrm{PMod}_\ell^\delta(C_a, \widetilde{D}_a)$ of degree δ and type $\ell = (1, \ldots, 1)$, and let $\mathcal{O}_{\mathrm{PMod}}$ be the corresponding structure sheaf. Recall also the morphisms τ in (5.8) and $\dot{\nu}_a$ in (5.12).

Theorem 6.3. *Let* $\nu_a : C_a \longrightarrow X_b$ *be normalization of an integral curve* X_b *with planar singularities and let* $\mathcal{E}^\bullet$ *be a complex in* $D^b(\mathrm{Jac}^\rho(C_a))$. *Then there is an isomorphism*

$$\Theta_b(R\check{\nu}_{a,*}\mathcal{E}^\bullet) \otimes \tau_*\mathcal{O}_{\mathrm{PMod}} \cong R\tau_*\dot{\nu}_a^*\Theta_a(\mathcal{E}^\bullet) \otimes \overline{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \overline{\mathrm{Jac}}}.$$

Proof. Consider the pullback of the Poincaré sheaf to $\overline{\mathrm{Jac}}^\delta(X_b) \times \mathrm{PMod}_\ell^\delta(C_a, \widetilde{D}_a)$,

$$(6.18) \qquad \qquad \dot{\mathcal{P}}_b := (\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau)^*\overline{\mathcal{P}}_b.$$

Applying the projection formula yields

$$(6.19) \qquad \qquad (\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau)_*\dot{\mathcal{P}}_b \cong \overline{\mathcal{P}}_b \otimes (\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau)_*\mathcal{O}_{\overline{\mathrm{Jac}} \times \mathrm{PMod}}.$$

Consider the projections,

$$(6.20)$$

and notice that, since $\overline{\mathrm{Jac}}^\delta(X_b)$ is projective, connected and reduced,

$$\dot{\pi}_{2,*}\mathcal{O}_{\overline{\mathrm{Jac}} \times \mathrm{PMod}} \cong \mathcal{O}_{\mathrm{PMod}}.$$

Note also that

$$\pi_2 \circ (\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau) = \tau \circ \dot{\pi}_2 \quad \text{and} \quad \dot{\pi}_1 = \pi_1 \circ (\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau),$$

where π_1 and π_2 are the projections defined in (6.7). Now, recalling (6.8) and using these relations, the identity (6.19), and the projection formula and the fact that the derived direct image is functorial with respect to compositions, we have that, for $\mathcal{F}^\bullet \in D^b(\overline{\mathrm{Jac}}^\delta(X_b))$,

$$
\begin{aligned}
R\tau_*R\dot{\pi}_{2,*}\big(\dot{\pi}_1^*\mathcal{F}^\bullet \otimes \dot{\mathcal{P}}_b\big) &\cong R\pi_{2,*}R(\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau)_*\big((\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau)^*\pi_1^*\mathcal{F}^\bullet \otimes \dot{\mathcal{P}}_b\big) \\
&\cong R\pi_{2,*}\big(\pi_1^*\mathcal{F}^\bullet \otimes (\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau)_*\dot{\mathcal{P}}_b\big) \\
(6.21) \qquad &\cong R\pi_{2,*}(\pi_1^*\mathcal{F}^\bullet \otimes \overline{\mathcal{P}}_b) \otimes \pi_{2,*}(\mathbf{1}_{\overline{\mathrm{Jac}}} \times \tau)_*\mathcal{O}_{\overline{\mathrm{Jac}} \times \mathrm{PMod}} \\
&\cong R\pi_{2,*}(\pi_1^*\mathcal{F}^\bullet \otimes \overline{\mathcal{P}}_b) \otimes \tau_*\dot{\pi}_{2,*}\mathcal{O}_{\overline{\mathrm{Jac}} \times \mathrm{PMod}} \\
&= \Theta_b(\mathcal{F}^\bullet) \otimes \tau_*\mathcal{O}_{\mathrm{PMod}}.
\end{aligned}
$$

The next step consists of establishing a relation between $\dot{\mathcal{P}}_b$ and $\mathcal{P}_a$. As $\dot{\mathcal{P}}_b$ parametrizes rank 1 torsion-free sheaves over $\overline{\mathrm{Jac}}^\delta(X_b)$, then

$$(\check{\nu}_a \times \mathbf{1}_{\mathrm{PMod}})^*\dot{\mathcal{P}}_b \longrightarrow \mathrm{Jac}^\rho(C_a) \times \mathrm{PMod}_\ell^\delta(C_a, \widetilde{D}_a)$$

is a family of rank 1 torsion-free sheaves over $\mathrm{Jac}^\rho(C_a)$ (*i.e.* line bundles on $\mathrm{Jac}^\rho(C_a)$) parametrized by $\mathrm{PMod}_\ell^\delta(C_a, \widetilde{D}_a)$. Since $\mathcal{P}_a \longrightarrow \mathrm{Jac}^\rho(C_a) \times \mathrm{Jac}^\delta(C_a)$ is a universal bundle, there is a morphism

$$g : \mathrm{PMod}_\ell^\delta(C_a, \widetilde{D}_a) \longrightarrow \mathrm{Jac}^\delta(C_a),$$

such that

$$(\check{\nu}_a \times \mathbf{1}_{\mathrm{PMod}})^*\dot{\mathcal{P}}_b \cong (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times g)^*\mathcal{P}_a \otimes (\pi_2'')^*\mathcal{W}'.$$

where π_2'' is the projection given by

$$(6.22)$$

$$
\begin{array}{ccc}
& \operatorname{Jac}^\rho(C_a) \times \operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a) & \\
{\scriptstyle \pi_1''} \swarrow & & \searrow {\scriptstyle \pi_2''} \\
\operatorname{Jac}^\rho(C_a) & & \operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a),
\end{array}
$$

and $\mathcal{W}'$ is some line bundle on $\operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a)$.

Thanks to (6.6) and to the definition of $\dot{\mathcal{P}}_b$ given in (6.18), we see that that restricting it to $\operatorname{Jac}^\delta(X_b) \hookrightarrow \tau_0^{-1} \operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a)$ gives $\dot{\mathcal{P}}_b|_{\overline{\operatorname{Jac}} \times \operatorname{Jac}} \cong \mathcal{P}_b$. Then, with $|_{\widetilde{\operatorname{Jac}} \times \operatorname{Jac}}$ denoting the restriction to $\operatorname{Jac}^\rho(C_a) \times \operatorname{Jac}^\delta(X_b)$ via τ_0^{-1},

$$
(\mathbf{1}_{\widetilde{\operatorname{Jac}}} \times g)^* \mathcal{P}_a|_{\widetilde{\operatorname{Jac}} \times \operatorname{Jac}} \otimes (\pi_2'')^* \mathcal{W}'|_{\widetilde{\operatorname{Jac}} \times \operatorname{Jac}} \cong (\check{\nu}_a \times \mathbf{1}_{\operatorname{PMod}})^* \dot{\mathcal{P}}_b|_{\widetilde{\operatorname{Jac}} \times \operatorname{Jac}} \cong (\check{\nu}_a \times \mathbf{1}_{\operatorname{Jac}})^* \mathcal{P}_b.
$$

So Proposition 6.2 shows that

$$
(\mathbf{1}_{\widetilde{\operatorname{Jac}}} \times g)^* \mathcal{P}_a|_{\widetilde{\operatorname{Jac}} \times \operatorname{Jac}} \otimes (\pi_2'')^* \mathcal{W}'|_{\widetilde{\operatorname{Jac}} \times \operatorname{Jac}} \cong (\mathbf{1}_{\widetilde{\operatorname{Jac}}} \times \hat{\nu}_a)^* \mathcal{P}_a \otimes (\pi_2')^* \mathcal{W}_b.
$$

Then the diagram

$$
\begin{array}{ccc}
& \operatorname{Jac}^\delta(X_b) & \\
{\scriptstyle \tau_0^{-1}} \downarrow & \searrow {\scriptstyle \hat{\nu}_a} & \\
\operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a) & \xrightarrow{\; g \;} & \operatorname{Jac}^\delta(C_a)
\end{array}
$$

commutes, hence we conclude from (5.13) that $g = \dot{\nu}_a$, as both coincide in the dense open subset $\operatorname{Jac}^\delta(X_b)$. As a consequence, we obtain

$$(6.23) \qquad (\check{\nu}_a \times \mathbf{1}_{\operatorname{PMod}})^* \dot{\mathcal{P}}_b \cong (\mathbf{1}_{\widetilde{\operatorname{Jac}}} \times \dot{\nu}_a)^* \mathcal{P}_a \otimes (\pi_2'')^* \mathcal{W}'.$$

Restricting (6.23) to $\{\mathcal{O}_{C_a}\} \times \operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a)$ yields

$$
\mathcal{W}' \cong ((\check{\nu}_a \times \mathbf{1}_{\operatorname{PMod}})^* \dot{\mathcal{P}}_b)|_{\{\mathcal{O}_{C_a}\} \times \operatorname{PMod}} \cong \dot{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \operatorname{PMod}},
$$

because $\mathcal{P}_a$ is normalized as in (6.27). But $\dot{\mathcal{P}}_b$ is the pullback of $\overline{\mathcal{P}}_b$ under $(\mathbf{1}_{\overline{\operatorname{Jac}}} \times \tau)$, so

$$(6.24) \qquad \mathcal{W}' \cong \tau^*\left(\overline{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \overline{\operatorname{Jac}}}\right).$$

Combining this description with (6.23), we conclude that

$$(6.25) \qquad (\check{\nu}_a \times \mathbf{1}_{\operatorname{PMod}})^* \dot{\mathcal{P}}_b \cong (\mathbf{1}_{\widetilde{\operatorname{Jac}}} \times \dot{\nu}_a)^* \mathcal{P}_a \otimes (\pi_2'')^* \tau^*\left(\overline{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \overline{\operatorname{Jac}}}\right).$$

We now address the last part of the proof. Recall the projections π_1'' and π_2'' from (6.22), $\dot{\pi}_1$ and $\dot{\pi}_2$ from (6.20) and finally η_1 and η_2 from (6.16), and observe that

$$(6.26) \qquad \begin{aligned} \pi_2'' &= \dot{\pi}_2 \circ (\check{\nu}_a \times \mathbf{1}_{\operatorname{PMod}}), \\ \pi_1'' &= \eta_1 \circ (\mathbf{1}_{\widetilde{\operatorname{Jac}}} \times \dot{\nu}_a). \end{aligned}$$

Moreover,

$$
\begin{array}{ccc}
\operatorname{Jac}^\rho(C_a) \times \operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a) & \xrightarrow{\; \pi_1'' \;} & \operatorname{Jac}^\rho(C_a) \\
{\scriptstyle \check{\nu}_a \times \mathbf{1}_{\operatorname{PMod}}} \downarrow & & \downarrow {\scriptstyle \check{\nu}_a} \\
\overline{\operatorname{Jac}}^\delta(X_b) \times \operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a) & \xrightarrow{\; \dot{\pi}_1 \;} & \overline{\operatorname{Jac}}^\delta(X_b),
\end{array}
$$

and

$$
\begin{array}{ccc}
\operatorname{Jac}^\rho(C_a) \times \operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a) & \xrightarrow{\; \mathbf{1}_{\widetilde{\operatorname{Jac}}} \times \dot{\nu}_a \;} & \operatorname{Jac}^\rho(C_a) \times \operatorname{Jac}^\delta(C_a) \\
{\scriptstyle \pi_2''} \downarrow & & \downarrow {\scriptstyle \eta_2} \\
\operatorname{PMod}_\ell^\delta(C_a, \widetilde{D}_a) & \xrightarrow{\; \dot{\nu}_a \;} & \operatorname{Jac}^\delta(C_a)
\end{array}
$$

are Cartesian diagrams.

The derived direct image and pullback are functorial with respect to compositions. Furthermore, the base-change formula applies to the two previous Cartesian diagrams. So, starting from (6.21) and using these facts, together with (6.25) and with the projection formula, finally yields

$$
\begin{aligned}
\Theta_b(R\check{\nu}_{a,*}\mathcal{E}^\bullet) \otimes \tau_*\mathcal{O}_{\mathrm{PMod}}
&\cong R\tau_* R\dot{\pi}_{2,*}\big(\dot{\pi}_1^* R\check{\nu}_{a,*}\mathcal{E}^\bullet \otimes \dot{\mathcal{P}}_b\big) \\
&\cong R\tau_* R\dot{\pi}_{2,*}\big(R(\check{\nu}_a \times \mathbf{1}_{\mathrm{PMod}})_* (\pi_1'')^* \mathcal{E}^\bullet \otimes \dot{\mathcal{P}}_b\big) \\
&\cong R\tau_* R\dot{\pi}_{2,*} R(\check{\nu}_a \times \mathbf{1}_{\mathrm{PMod}})_* \big((\pi_1'')^* \mathcal{E}^\bullet \otimes (\check{\nu}_a \times \mathbf{1}_{\mathrm{PMod}})^* \dot{\mathcal{P}}_b\big) \\
&\cong R\tau_* R\pi_{2,*}'' \big((\pi_1'')^* \mathcal{E}^\bullet \otimes (\check{\nu}_a \times \mathbf{1}_{\mathrm{PMod}})^* \dot{\mathcal{P}}_b\big) \\
&\cong R\tau_* R\pi_{2,*}'' \big((\pi_1'')^* \mathcal{E}^\bullet \otimes (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \dot{\nu}_a)^* \mathcal{P}_a \otimes (\pi_2'')^* \tau^* \big(\overline{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \overline{\mathrm{Jac}}}\big)\big) \\
&\cong R\tau_* R\pi_{2,*}'' \big((\pi_1'')^* \mathcal{E}^\bullet \otimes (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \dot{\nu}_a)^* \mathcal{P}_a\big) \otimes \overline{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \overline{\mathrm{Jac}}} \\
&\cong R\tau_* R\pi_{2,*}'' (\mathbf{1}_{\widetilde{\mathrm{Jac}}} \times \dot{\nu}_a)^* (\eta_1^* \mathcal{E}^\bullet \otimes \mathcal{P}_a) \otimes \overline{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \overline{\mathrm{Jac}}} \\
&\cong R\tau_* \dot{\nu}_a^* R\eta_{2,*}(\eta_1^* \mathcal{E}^\bullet \otimes \mathcal{P}_a) \otimes \overline{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \overline{\mathrm{Jac}}} \\
&\cong R\tau_* \dot{\nu}_a^* \Theta_a(\mathcal{E}^\bullet) \otimes \overline{\mathcal{P}}_b|_{\{\nu_{a,*}\mathcal{O}_{C_a}\} \times \overline{\mathrm{Jac}}},
\end{aligned}
$$

as claimed. $\qquad\square$

6.3. Branes and Fourier–Mukai transform.
We are now at the last step towards the goal of proving the duality statement between the branes we constructed.

Along this section we fix the degree to be trivial, $d = 0$. We require also Assumption 1, so $p : C \longrightarrow X$ has order $m = n$ (hence $r = 1$ and $\rho = 0$) and the spectral data of $\mathbf{M}_X(n,d)^p$ is as described in (4.11). In particular, the normalization of the spectral curves is always C and B_{ni}^p coincides with B_{ni}^p, the subset parametrizing integral and nodal curves.

Let us use the Hitchin section σ_C constructed from a spin structure $K_C^{1/2} = p^* K_X^{1/2}$ to choose a point in $\mathrm{Jac}^\delta(C)$. Consider associated the Poincaré bundle

$$
\mathcal{P} \longrightarrow \mathrm{Jac}^0(C) \times \mathrm{Jac}^\delta(C).
$$

If $N \in \mathrm{Jac}^\delta(C)$, then $\mathcal{P}_N = \mathcal{P}|_{\mathrm{Jac}^0(C) \times \{N\}}$ is the line bundle over $\mathrm{Jac}^0(C)$ corresponding to the point $N \otimes K_C^{-(n-1)/2}$ of $\mathrm{Jac}^0(C)$ under autoduality of $\mathrm{Jac}^0(C)$. We can assume that $\mathcal{P}_a$ is normalized so that

$$
\tag{6.27} \mathcal{P}|_{\{\mathcal{O}_C\} \times \mathrm{Jac}^\delta(C)} \cong \mathcal{O}_{\mathrm{Jac}^\delta(C)}.
$$

Let $b \in B_{\mathrm{ni}}^p$ and $\phi \in H^0(C, K_C)^{\mathrm{free}}$ be a representative of b, and recall from (4.10) the pushforward morphism $\check{\nu}_\phi : \mathrm{Jac}^0(C) \hookrightarrow \overline{\mathrm{Jac}}^\delta(X_b)$. We wish to understand the Fourier–Mukai transform of the sheaf $\check{\nu}_{\phi,*}\mathrm{Nm}^* \check{\mathcal{L}}$ over $\overline{\mathrm{Jac}}^\delta(X_b) \cong h_{X,n}^{-1}(b)$ under the derived equivalence $\Theta_b : D^b\big(\overline{\mathrm{Jac}}^\delta(X_b)\big) \longrightarrow D^b\big(\overline{\mathrm{Jac}}^\delta(X_b)\big)$. Indeed, by Proposition 4.6, $\check{\nu}_{\phi,*}\mathrm{Nm}^* \check{\mathcal{L}}$ is supported on $\mathrm{Im}(\check{\nu}_\phi) \cong \mathrm{Jac}^0(C)$ and is the restriction to the Hitchin fiber over b (intersected with $\mathbf{M}_X(n,0)^p$) of the hyperholomorphic line bundle $\mathscr{L}$ defining the rank 1 Narasimhan-Ramanan (BBB)-brane $(\mathbf{BBB})_{\mathcal{L}}^p$. It is a classical fact that that the Fourier–Mukai of a line bundle over $\mathrm{Jac}^0(C)$ is a complex supported only in one degree (namely the genus of C), so it can be considered as a sheaf as opposed to a complex. Hence, by Theorem 6.3, $\Theta_b(\check{\nu}_{\phi,*}\mathrm{Nm}^* \check{\mathcal{L}})$ is a sheaf over $\overline{\mathrm{Jac}}^\delta(X_b)$, whose support is the intersection of the support of the dual (BAA)-brane with the Hitchin fiber over b.

By considering the rank n Narasimhan-Ramanan (BBB)-brane $(\mathbf{BBB})_{\mathcal{F}}^p$, we conclude by the same token that the support of the sheaf $\Theta_b(\bigoplus_{\gamma \in \Gamma} \check{\nu}_{\phi,*}\hat{\gamma}^* \check{\mathcal{F}})$ determines the support of the dual (BAA)-brane.

Theorem 6.4. *Consider the moduli space $\mathbf{M}_X(n,0)$ and the Narasimhan-Ramanan (BBB)-branes $(\mathbf{BBB})_{\mathcal{L}}^p$ and $(\mathbf{BBB})_{\mathcal{F}}^p$ on it associated to a connected unramified cover $p : C \longrightarrow X$.*

Let $b = \zeta(\phi) \in B^p_{\mathrm{ni}}$ *for some* $\phi \in H^0(C, K_C)^{\mathrm{free}}$. *Let* $\hat{\nu}_\phi : \mathrm{Jac}^\delta(X_b) \longrightarrow \mathrm{Jac}^\delta(C)$ *be the pullback morphism associated to the normalization* ν_ϕ *(thus corresponding to* (5.7)*).*

(i) *Let*

$$\hat{\mathcal{L}} := p^*\mathcal{L} \otimes K_C^{(n-1)/2} \in \mathrm{Jac}^\delta(C).$$

The Fourier–Mukai transform of the hyperholomorphic sheaf $\mathscr{L}|_{h_{X,n}^{-1}(b)} \cong \check{\nu}_{\phi,*}\mathrm{Nm}^*\check{\mathcal{L}}$ *satisfies the relation*

(6.28)
$$\Theta_b(\check{\nu}_{\phi,*}\mathrm{Nm}^*\check{\mathcal{L}}) \otimes \tau_*\mathcal{O}_{\mathrm{PMod}} \cong \overline{\mathcal{P}}_b\big|_{\{\nu_{\phi,*}\mathcal{O}_C\} \times \hat{\nu}_\phi^{-1}(\hat{\mathcal{L}})}$$

and its support is

$$\mathrm{supp}(\Theta_b(\check{\nu}_{\phi,*}\mathrm{Nm}^*\check{\mathcal{L}})) = \overline{\hat{\nu}_\phi^{-1}(\hat{\mathcal{L}})} = \mathrm{NR}_{\mathrm{ni}}^{p,\hat{\mathcal{L}}} \cap h_{X,n}^{-1}(b).$$

(ii) *Analogously, let*

(6.29)
$$\hat{\mathcal{F}} := \mathcal{F} \otimes K_C^{(n-1)/2}$$

and let $\Gamma(\mathcal{F})$ *be the orbit of* $\mathcal{F}$ *by the Galois group* Γ *of* p. *The Fourier–Mukai transform of the hyperholomorphic sheaf* $\mathscr{F}|_{h_{X,n}^{-1}(b)} \cong \bigoplus_{\gamma \in \Gamma} \check{\nu}_{\phi,*}\hat{\gamma}^*\check{\mathcal{F}}$ *satisfies the relation*

(6.30)
$$\Theta_b\left(\bigoplus_{\gamma \in \Gamma} \check{\nu}_{\phi,*}\hat{\gamma}^*\check{\mathcal{F}}\right) \otimes \tau_*\mathcal{O}_{\mathrm{PMod}} \cong \bigoplus_{\gamma \in \Gamma} \tau_*\mathcal{O}_{\hat{\nu}_\phi^{-1}(\gamma^*\hat{\mathcal{F}})} \otimes \overline{\mathcal{P}}_b\big|_{\{\nu_{\phi,*}\mathcal{O}_C\} \times \overline{\mathrm{Jac}}}$$

and its support is

$$\mathrm{supp}\left(\Theta_b\left(\bigoplus_{\gamma \in \Gamma} \check{\nu}_{\phi,*}\hat{\gamma}^*\check{\mathcal{F}}\right)\right) = \bigcup_{\gamma^*\hat{\mathcal{F}} \in \Gamma(\mathcal{F})} \overline{\hat{\nu}_\phi^{-1}(\gamma^*\hat{\mathcal{F}})} = \mathrm{NR}_{\mathrm{ni}}^{p,\hat{\mathcal{F}}} \cap h_{X,n}^{-1}(b).$$

Proof. By Theorem 6.3 we have

(6.31)
$$\Theta_b(\check{\nu}_{\phi,*}\mathrm{Nm}^*\check{\mathcal{L}}) \otimes \tau_*\mathcal{O}_{\mathrm{PMod}} \cong \tau_*\dot{\nu}_\phi^*\Theta(\mathrm{Nm}^*\check{\mathcal{L}}) \otimes \overline{\mathcal{P}}_b\big|_{\{\nu_{\phi,*}\mathcal{O}_C\} \times \overline{\mathrm{Jac}}}$$

so we need to compute $\Theta(\mathrm{Nm}^*\check{\mathcal{L}})$; cf. (6.17), we have removed the index a since now all the C_a are isomorphic to C. This is the classical Fourier–Mukai transform on an abelian variety, the only difference being that $\Theta : D^b(\mathrm{Jac}^0(C)) \longrightarrow D^b(\mathrm{Jac}^\delta(C))$ takes values in the derived category of complexes over the torsor $\mathrm{Jac}^\delta(C)$ and not over the actual abelian variety $\mathrm{Jac}^0(C)$. So we must use the identification we settled $\mathrm{Jac}^0(C) \xrightarrow{\cong} \mathrm{Jac}^\delta(C)$, by tensorization by $K_C^{(n-1)/2}$. As is well-known, $\Theta(\mathrm{Nm}^*\check{\mathcal{L}})$ is the skyscraper sheaf over the point of $\mathrm{Jac}^\delta(C)$ whose corresponding point over $\mathrm{Jac}^0(C)$ corresponds to the line bundle $\mathrm{Nm}^*\check{\mathcal{L}}$ under the autoduality of $\mathrm{Jac}^0(C)$.

Consider the commutative diagram

(6.32)
$$\begin{array}{ccc} C & \xhookrightarrow{A_C} & \mathrm{Jac}^0(C) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Nm}} \\ X & \xhookrightarrow[A_X]{} & \mathrm{Jac}^0(X), \end{array}$$

where the horizontal maps are the Abel–Jacobi maps determined by chosen base points on C_a and X, which correspond under $p : C \longrightarrow X$. Since the isomorphisms yielding the autoduality of $\mathrm{Jac}^0(C)$ and $\mathrm{Jac}^0(X)$ are given by pullback of the Abel–Jacobi maps, this yields the commutative diagram

$$\begin{array}{ccc} \mathrm{Jac}^0(C)^\vee & \xrightarrow[\cong]{A_C^*} \mathrm{Jac}^0(C) \xrightarrow[-\otimes K_C^{(n-1)/2}]{\cong} & \mathrm{Jac}^\delta(C) \\ {\scriptstyle \mathrm{Nm}^*}\big\uparrow & \big\uparrow{\scriptstyle p^*} & \\ \mathrm{Jac}^0(X)^\vee & \xrightarrow[A_X^*]{\cong} \mathrm{Jac}^0(X) & \end{array}$$

(where the maps no longer depend on the choice of base points). Recall that by definition $\check{\mathcal{L}}$ is the line bundle over $\mathrm{Jac}^0(X)$ which corresponds to the flat line bundle $\mathcal{L}$ over X under autoduality, *i.e.* $\mathcal{L} \cong A_X^* \check{\mathcal{L}}$. So we conclude that $\Theta(\mathrm{Nm}^* \check{\mathcal{L}}) \cong \mathcal{O}_{p^* \mathcal{L} \otimes K_C^{(n-1)/2}} = \mathcal{O}_{\hat{\mathcal{L}}}$.

Hence, $\tau_* \dot{\nu}_\phi^* \Theta(\mathrm{Nm}^* \check{\mathcal{L}}) \cong \tau_* \mathcal{O}_{\dot{\nu}_\phi^{-1}(\hat{\mathcal{L}})}$. Since τ is finite morphism, $\tau_* \mathcal{O}_{\dot{\nu}_\phi^{-1}(\hat{\mathcal{L}})}$ is a coherent sheaf, hence (cf. [Ha, ex. 5.5, 5.6, p.124]) its support in $\overline{\mathrm{Jac}}^\delta(X_b)$ is closed and is the closure of the image by τ of the support of $\mathcal{O}_{\dot{\nu}_\phi^{-1}(\hat{\mathcal{L}})}$ which, by Lemma 5.10, is $\dot{\nu}_\phi^{-1}(\hat{\mathcal{L}})$. In addition, by the same lemma, the restriction of τ to $\dot{\nu}_\phi^{-1}(\hat{\mathcal{L}})$ is a closed embedding, thus we actually have $\tau_* \mathcal{O}_{\dot{\nu}_\phi^{-1}(\hat{\mathcal{L}})} \cong \mathcal{O}_{\overline{\hat{\nu}_\phi^{-1}(\hat{\mathcal{L}})}}$.

Considering the transform just as a sheaf (thus ignoring the only degree where the complex is non-zero), we then have, by (6.31),

$$\Theta_b(\check{\nu}_{\phi,*}\mathrm{Nm}^* \check{\mathcal{L}}) \otimes \tau_* \mathcal{O}_{\mathrm{PMod}} \cong \overline{\mathcal{P}}_b\big|_{\{\nu_{\phi,*}\mathcal{O}_C\} \times \overline{\hat{\nu}_\phi^{-1}(\hat{\mathcal{L}})}}.$$

The sheaf $\tau_* \mathcal{O}_{\mathrm{PMod}}$ is supported in $\overline{\mathrm{Jac}}^\delta(X_b)$, thus

$$\mathrm{supp}(\Theta_b(\check{\nu}_{\phi,*}\mathrm{Nm}^* \check{\mathcal{L}})) = \overline{\hat{\nu}_\phi^{-1}(\hat{\mathcal{L}})}$$

as claimed. Note finally that $\overline{\hat{\nu}_\phi^{-1}(\hat{\mathcal{L}})}$ is indeed the spectral data of the intersection $\mathrm{NR}_{\mathrm{ni}}^{p,\hat{\mathcal{L}}} \cap h_{X,n}^{-1}(b)$, by Theorem 5.12 (since $\hat{\mathcal{L}} = p^*(\mathcal{L}^{-1} \otimes K_X^{(n-1)/2})$, then $\Gamma(\hat{\mathcal{L}}) = \{\hat{\mathcal{L}}\}$ in (5.16); cf. Remark 5.13), completing the proof of (i).

For the proof of (ii), we have, again by Theorem 6.3,

$$\Theta_b\bigg(\bigoplus_{\gamma \in \Gamma} \check{\nu}_{\phi,*}\hat{\gamma}^* \check{\mathcal{F}}\bigg) \otimes \tau_* \mathcal{O}_{\mathrm{PMod}} \cong \bigoplus_{\gamma \in \Gamma}(\Theta_b(\check{\nu}_{\phi,*}\hat{\gamma}^* \check{\mathcal{F}}) \otimes \tau_* \mathcal{O}_{\mathrm{PMod}})$$

$$\cong \bigoplus_{\gamma \in \Gamma} \tau_* \hat{\nu}_\phi^* \Theta(\hat{\gamma}^* \check{\mathcal{F}}) \otimes \overline{\mathcal{P}}_b\big|_{\{\nu_{\phi,*}\mathcal{O}_C\} \times \overline{\mathrm{Jac}}}$$

and a similar argument to the one given above, proves $\Theta(\hat{\gamma}^* \check{\mathcal{F}}) \cong \mathcal{O}_{\gamma * \hat{\mathcal{F}}}$. Thus

$$\Theta_b\bigg(\bigoplus_{\gamma \in \Gamma} \check{\nu}_{\phi,*}\hat{\gamma}^* \check{\mathcal{F}}\bigg) \otimes \tau_* \mathcal{O}_{\mathrm{PMod}} \cong \bigoplus_{\gamma \in \Gamma} \tau_* \mathcal{O}_{\hat{\nu}_\phi^{-1}(\gamma * \hat{\mathcal{F}})} \otimes \overline{\mathcal{P}}_b\big|_{\{\nu_{\phi,*}\mathcal{O}_{C_a}\} \times \overline{\mathrm{Jac}}}.$$

As in the preceding case, $\bigoplus_{\gamma \in \Gamma} \tau_* \mathcal{O}_{\hat{\nu}_\phi^{-1}(\gamma * \hat{\mathcal{F}})}$ is supported in the (not necessarily disjoint) union $\bigcup_{\gamma * \hat{\mathcal{F}} \in \Gamma(\mathcal{F})} \overline{\hat{\nu}_\phi^{-1}(\gamma * \hat{\mathcal{F}})}$, hence so is the right-hand side of the above isomorphism as $\tau_* \mathcal{O}_{\mathrm{PMod}}$ is supported on $\overline{\mathrm{Jac}}^\delta(X_b)$. We conclude that

$$\mathrm{supp}\bigg(\Theta_b\bigg(\bigoplus_{\gamma \in \Gamma} \check{\nu}_{\phi,*}\hat{\gamma}^* \check{\mathcal{F}}\bigg)\bigg) = \bigcup_{\gamma * \hat{\mathcal{F}} \in \Gamma(\mathcal{F})} \overline{\hat{\nu}_\phi^{-1}(\gamma * \hat{\mathcal{F}})}$$

which coincides with $\mathrm{NR}_{\mathrm{ni}}^{p,\hat{\mathcal{F}}} \cap h_{X,n}^{-1}(b)$ by Theorem 5.12. $\qquad\square$

Since for each $b \in B_{\mathrm{ni}}^p$, Θ_b is a derived equivalence by Theorem 6.1, the previous theorem provides the following fiberwise duality statement as an immediate consequence.

Theorem 6.5. *Let $p : C \longrightarrow X$ be a connected unramified n-cover. Consider the moduli space $\mathbf{M}_X(n, 0)$ and let $\hat{\mathcal{L}}$ and $\hat{\mathcal{F}}$ as in Theorem 6.4.*

(i) *The (fiberwise) dual of the rank 1 Narasimhan-Ramanan (BBB)-brane $(\mathbf{BBB})_{\mathcal{L}}^p$ (over B_{ni}^p) is the (BAA)-brane supported on $\mathrm{NR}_{\mathrm{ni}}^{p,\hat{\mathcal{L}}}$, and whose flat bundle satisfies (6.28).*

(ii) *The (fiberwise) dual of the rank n Narasimhan-Ramanan (BBB)-brane $(\mathbf{BBB})_{\mathcal{F}}^p$ (over B_{ni}^p) is the (BAA)-brane supported on $\mathrm{NR}_{\mathrm{ni}}^{p,\hat{\mathcal{F}}}$, and whose flat bundle satisfies (6.30).*

Remark 6.6. In [FP], duality is conjectured between the (BBB)-brane $\mathbf{Car}(\mathcal{L})$ (supported on the Cartan locus $\mathbf{M}_{(1,n)} \subset \mathbf{M}_C(n, nd)$) and the (BAA)-brane $\mathbf{Uni}(\mathcal{L})$ (supported on the unipotent locus $\mathrm{Uni}_C(\mathcal{L}) \subset \mathbf{M}_C(n, nd)$). Recall from Remark 4.7 that $(\mathbf{BBB})_{\mathcal{L}}^p$ pulls-back to $\mathbf{Car}(\mathcal{L})$ under $\hat{p}$ while $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{L}}$ is sent to $\mathrm{Uni}_C(\mathcal{L})$ under $\hat{p}$ as we have seen in Proposition 5.14. Thus, Theorem 6.5 give us some indications that the general principles of this conjecture seem to hold true.

7. Branes in the absence of a Hitchin section

In Section 4 we worked under Assumption 1 to construct a family of (BBB)-branes supported on $\mathbf{M}_X(n, d)^p$. In Section 5 we required Assumption 2, which is weaker than Assumption 1, to define the Lagrangian subvariety $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$ over the locus of Hitchin base B_{ni}^p of those spectral curves whose normalization lives in $B_{C,r}^{\mathrm{sm}}$.

A straight-forward observation is that, when Assumption 1 fails, one can always define a (BBB)-brane on $\mathbf{M}_X(n, d)^p$ by considering the trivial bundle on it.

Without Assumption 2 we face a problem for the construction of our Narasimhan–Ramanan dual (BAA)-branes, namely the lack of a section for the Hitchin fibration $h_{C,r} : \mathbf{M}_C(r, d) \longrightarrow B_{C,r}$. Instead, we pick the Lagrangian multisection of the Hitchin fibration given by a very stable bundle. With this multisection we define a Lagrangian subvariety which we study using the branes associated with parabolic subgroups from [FP, Section 6].

Given a stable bundle $\mathcal{V} \longrightarrow C$, one has that $(\mathcal{V}, \phi)$ is a stable Higgs bundle for every $\phi \in H^0(C, \mathrm{End}(\mathcal{V}) \otimes K_C)$. Then, we have a natural morphism

$$
\begin{array}{ccc}
H^0(C, \mathrm{End}(\mathcal{V}) \otimes K_C) & \longrightarrow & \mathbf{M}_C(r, d) \\
\phi & \longmapsto & (\mathcal{V}, \phi),
\end{array}
$$

which is an embedding as $\mathcal{V}$ is simple. We denote by $\Sigma_{\mathcal{V}}$ the image of this map. It is well-known that this provides a Lagrangian subvariety.

Proposition 7.1. *For every stable bundle $\mathcal{V}$, $\Sigma_{\mathcal{V}}$ is Lagrangian.*

Proof. Since $\mathcal{V}$ is stable it is simple, so one has that

$$
\dim \Sigma_{\mathcal{V}} = \dim H^0(C, \mathrm{End}(\mathcal{V}) \otimes K_C) = \frac{1}{2} \dim \mathbf{M}_C(r, d).
$$

Since the vector bundle is fixed along $\Sigma_{\mathcal{V}}$, note also that the projection

$$
T_{(\mathcal{V}, \phi)} \Sigma_{\mathcal{V}} \longrightarrow H^1(C, \mathrm{End}(\mathcal{V}))
$$

is constantly zero, so $\Omega_{C,1}$ vanishes there. $\qquad \square$

After Laumon [La], a vector bundle $\mathcal{V} \longrightarrow C$ is *very stable* if it has no non-zero nilpotent Higgs field. It can be shown [La] that a very stable bundle is stable (provided $g \geqslant 2$) and that the locus of very stable bundles is a dense open subset of the moduli space of vector bundles. The fourth author and C. Pauly proved the following (see [PP, Theorem 1.1 and Corollary 1.2]).

Theorem 7.2 ([PP]). *Let $\mathcal{V}$ be a stable bundle. Then, $\mathcal{V}$ is very stable if and only if the restriction $h_{C,r}|_{\Sigma_{\mathcal{V}}}$ of the Hitchin fibration $h_{C,r}$ to $\Sigma_{\mathcal{V}}$ is finite and surjective.*

One thus easily deduces the following.

Corollary 7.3. *Under the hypotheses of Theorem 7.2, $\Sigma_{\mathcal{V}} \longrightarrow B_{C,r}$ is a Lagrangian multisection of $h_{C,r}$.*

In view of Corollary 7.3 we provide the following definition analogous to Definition 5.3.

Definition 7.4. For any very stable bundle $\mathcal{V} \longrightarrow C$ of rank r and degree $d - \rho + \delta$, define the subvariety $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}}$ of $\mathbf{M}_X(n, d)$ closed in $\mathbf{M}_X(n, d) \times_{B_{X,n}} B_{\mathrm{ni}}^p$ as

$$
\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}} := \left(\left(\mathfrak{h}_p^{(\delta - \rho)} \right)^{-1} (\Sigma_{\mathcal{V}}) \right) \cap \left(\mathbf{M}_X(n, rd')_{\mathrm{ni}}^p \times_{B_{\mathrm{ni}}^p} \mathrm{Sing}_{\mathrm{ni}}^p \right).
$$

As we did in Theorem 5.12 for $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{J}}$, let us study the spectral datum for $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}}$. Let us denote by $\Sigma_{\mathcal{V},a}$ the fiber $\Sigma_{\mathcal{V}} \longrightarrow B_{C,r}$ over $a \in B_{C,r}$. Set also

$$\Sigma'_{\mathcal{V},a} := S_{C,r}^{-1}(\Sigma_{\mathcal{V},a}),$$

note that, by definition of $S_{C,r}$ and $\Sigma_{\mathcal{V},a}$, for every $\mathcal{J} \in \Sigma'_{\mathcal{V},a} \subset \mathrm{Jac}^{d+\delta}(C_a)$ one has

$$\eta_{a,*}\mathcal{J} \cong \mathcal{V}.$$

Proposition 7.5. *Let $b \in B_{\mathrm{ni}}^p$ and $a \in B_{C,r}^{\mathrm{ni}}$ such that $b = \zeta(a)$, then*

$$(7.1) \qquad h_{X,n}^{-1}(b) \cap \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}} = \bigcup_{\mathcal{J} \in \Sigma'_{\mathcal{V},a}, \gamma^*\mathcal{J} \in \Gamma(\mathcal{J})} S_{X,n}\left(\widehat{\hat{\nu}_a^{-1}(\gamma^*\mathcal{J})}\right),$$

where $\Gamma(\mathcal{J})$ denotes the Γ-orbit of $\mathcal{J}$. Furthermore,

$$(7.2) \qquad \dim\left(h_{X,n}^{-1}(b) \cap \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}}\right) = \delta - \rho = n(n-r)(g-1)$$

and

$$(7.3) \qquad \dim \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}} = n^2(g-1) + 1 = \frac{1}{2}\dim \mathbf{M}_X(n,d).$$

Proof. After a trivial adaptation of the proof of Theorem 5.12 one gets the description given in (7.1). The rest of the proposition follows from this fact, (7.2) is clear after the description of the fibers of $\hat{\nu}_a$ given in [Gr1] and (ii) of Theorem 3.11. The proof of (7.3) follows from (7.2) and (3.15). $\qquad\square$

In order to prove that the above manifold is isotropic we compare it with some complex Lagrangian submanifolds inside $\mathbf{M}_C(n,dm)$ defined in [FP, Section 6].

Let $\mathcal{V} \longrightarrow C$ of rank r and degree $d - \rho + \delta$, and choose an ordering $o \in \mathrm{Ord}(\Gamma)$ of the elements of Γ, $o = (\gamma_{o,1}, \ldots, \gamma_{o,m})$. For each o and each $i = 1, \ldots, m$, define the vector bundle $\mathcal{V}_{o,i} = \gamma_{o,i}^*\mathcal{V} \otimes K_C^{-r(m-i-1)}$ be vector bundles, and consider the variety

$$(7.4) \qquad \mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o) = \left\{ (E,\varphi) \in \mathbf{M}_C(n,dm) \;\middle|\; \begin{array}{l} \exists\, \sigma \in H^0(X, E/\mathrm{P}_{(r,m)}): \\ \varphi \in H^0(X, E_\sigma(\mathfrak{p}_{(r,m)}) \otimes K_X); \\ E_\sigma/\mathrm{U}_{(r,m)} \cong \mathcal{V}_{o,1} \oplus \cdots \oplus \mathcal{V}_{o,m}. \end{array} \right\},$$

where we recall the notation introduced in Remark 3.14. These subvarieties are studied in [FP, Section 6] and it follows from [FP, Theorem 6.7] that $\mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o)$ is Lagrangian (in particular, it is isotropic).

We prove a relation between $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}}$ and $\mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o)$ analogous to that described in Proposition 5.14.

Proposition 7.6. *Let $\hat{p}: \mathbf{M}_X(n,d) \longrightarrow \mathbf{M}_C(n,md)$ be the pullback morphism. Consider the open subvariety $\mathrm{NR}_{\mathrm{Jac}}^{p,\mathcal{V}}$ of $\mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}}$ defined as its intersection with the open subset $\mathrm{Jac}^{d+\delta}(X_b)$ of every Hitchin fiber $h_{X,n}^{-1}(b)$. Then $\mathrm{NR}_{\mathrm{Jac}}^{p,\mathcal{V}}$ is mapped under the pullback map (3.3) to the union of $\mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o)$ for all different $o \in \mathrm{Ord}(\Gamma)$, i.e.*

$$\hat{p}\left(\mathrm{NR}_{\mathrm{Jac}}^{p,\mathcal{V}}\right) \subset \bigsqcup_{o \in \mathrm{Ord}(\Gamma)} \mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o).$$

Proof. We will check that the spectral datum of $p^*(E,\varphi)$ satisfies the conditions of the spectral datum of $\mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o)$, which proves the statement.

Let $L \in \mathrm{Jac}^{d+\delta}(X_b)$ be the spectral datum of $(E,\varphi) \in \mathrm{NR}_{\mathrm{ni}}^{p,\mathcal{V}}$. By Cartesianity of the square in (3.10), we know that $\tilde{L} = q_{\tilde{b}}^*L \in \mathrm{Jac}(X_{\tilde{b}})$ is the spectral datum for $(\tilde{E}, \tilde{\varphi}) := p^*(E,\varphi)$. Also, Cartesianity of (3.21) implies that

$$\tilde{\nu}_{\tilde{b}}^*\tilde{L} = \tilde{q}_a^*\nu_a^*L.$$

By (7.1), $\nu_a^* L$ equals $\gamma^* \mathcal{J}_i$, with $\mathcal{J}_i \in \Sigma'_{\mathcal{V},a}$. So, choosing an ordering $o \in \mathrm{Ord}(\Gamma)$, we write

$$(7.5) \qquad \tilde{\nu}_{\tilde{b}}^* \tilde{L} = \tilde{q}_a^* \gamma^* \mathcal{J}_i = ((\gamma_{o,1}\gamma)^* \mathcal{J}_i, \ldots, (\gamma_{o,m}\gamma)^* \mathcal{J}_i).$$

Since the spectral data $\tilde{L}$ satisfies (7.5), it is then a line over $C_{\tilde{b}} = \bigcup_{\gamma' \in \Gamma} C_{\gamma'(a)}$ whose restriction to any $C_{\gamma'(a)}$ is $(\gamma'\gamma)^* \mathcal{J}_i$. Then, using the order o we can set $Z_j = \bigcup_{i=j}^m$ and $\tilde{L}_j = \ker(\tilde{L} \longrightarrow \tilde{L}|_{Z_j})$. This defines a filtration of $\mathcal{O}_{C_{\tilde{b}}}$

$$0 \subset \tilde{L}_1 \subset \tilde{L}_2 \subset \cdots \subset \tilde{L}_m = \tilde{L}.$$

Taking the pushforward of this filtration to C provides a filtration

$$0 \subset \tilde{E}_1 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_m = \tilde{E}$$

preserved by the Higgs field, and whose graded part $\mathrm{gr}(\tilde{E})$ is precise $\bigoplus_i \mathcal{V}_i$ appearing in the definition of $\mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o)$. Then, $(\tilde{E}, \tilde{\varphi})$ lies in some $\mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o)$ and the proof is completed.

Alternatively, checking that the line bundles in (7.5) satisfy [FP, Assumption 1], we may apply [FP, Proposition 6.6] to conclude. $\qquad \square$

We can finally prove the main theorem of this section, whose proof mimics that of Theorem 5.15 and is thus omitted.

Theorem 7.7. *The manifold* $\mathrm{NR}_{\mathrm{ni}}^{\xi,\mathcal{V}}$ *is Lagrangian.*

Proof. Isotropicity is proved as in Theorem 5.15 making use in the proof of Proposition 7.6 instead of Proposition 5.14, and recalling that the subvarieties $\mathrm{Uni}_C^{(r,m)}(\mathcal{V}, o)$ are isotropic. Then, the proof follows from (7.3). $\qquad \square$

The question of how to produce hyperholomorphic bundles $\mathbf{M}_C(r, d)$ dual to the (BAA)-brane supported on a Lagrangian multisection $\Sigma_\mathcal{V}$ associated to a very stable bundle is being studied by Hausel and Hitchin [HH].

References

[AFES]   L. B. Anderson, L. Fredrickson, M. Esole, L. P. Schaposnik, *Singular geometry and Higgs bundles in string theory.* SIGMA **14** (2018), 037, 27 pages.

[AIK]   A. Altman, A. Iarrobino, and S. Kleiman. *Irreducibility of the compactified Jacobian. In Real and complex singularities* (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976), pages 1–12. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[AK]   A. Altman, S. Kleiman, *Compactifying the Picard scheme. II*, Amer. J. Math. **101** (1979), no. 1, 10–41.

[Ar]   D. Arinkin, *Autoduality of compactified Jacobians for curves with plane singularities*, J. Algebraic Geom. **22** (2013), 363–388.

[BS1]   D. Baraglia, L. P. Schaposnik, *Real structures on moduli spaces of Higgs bundles*, Adv. Theo. Math. Phys., **20** (2016), 525–551.

[BS2]   D. Baraglia, L. P. Schaposnik, *Cayley and Langlands type correspondences for orthogonal Higgs bundles*, Trans. Amer. Math. Soc. **371** (2019), 7451–7492 .

[BNR]   A. Beauville, M. S. Narasimhan, S. Ramanan, *Spectral curves and the generalized theta divisor*, J. Reigne Angew. Math **398** (1989), 169–179.

[Bh]   U. Bhosle, *Generalized parabolic bundles and applications to torsionfree sheaves on nodal curves*, Arkiv für Mat. (2) **30** (1992), 187–215.

[BCFG]   I. Biswas, L. Calvo, E. Franco, O. Garcia-Prada, *Involutions of Higgs moduli spaces over elliptic curves and pseudo-real Higgs bundles.* J. Geom. Phys. **142**, 47–65 (2019).

[BGP]   I. Biswas, O. Garcia-Prada, *Anti-holomorphic involutions of the moduli spaces of Higgs bundles*, J. Éc. polytech. Math. **2** (2015), 35–54.

[B]   L. Branco, *Higgs bundles, Lagrangians and mirror symmetry*, DPhil Thesis, University of Oxford, 2017.

[C1]   P. R. Cook, *Local and Global Aspects of the Module theory of singular curves*, Ph.D. Thesis, University of Liverpool, 1993.

[C2]   P. R. Cook, *Compactified Jacobians and curves with simple singularities*, Algebraic Geometry (Catania, 1993/Barcelona, 1994), 37–47, Lecture Notes in Pure and Appl. Math., **200**, Marcel Dekker, 1998.

[Co]   K. Corlette, *Flat G-bundles with canonical metrics*, J. Diff. Geom., **28**, 3, (1988), 361–382.

[DG]   R. Donagi, D. Gaitsgory, *The gerbe of Higgs bundles.* Transform. Groups, **7** (2002), no. 2, 109–153.

[DP]   R. Donagi, T. Pantev, *Langlands duality for Hitchin systems*, Invent. Math. **189**, no. 3 (2012), 653–735.

[Do]   S. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3), **55** 1 (1987), 127–131.

[Es]   E. Esteves, *Compactifying the relative Jacobian over families of reduced curves.* Trans. Am. Math. Soc., **353** 8 (2001), 3045–3095.

[FJ]   E. Franco, M. Jardim. *Mirror symmetry for Nahm branes.* Preprint `arXiv:1709.01314`.

[FP]   E. Franco, A. Peón-Nieto, *The Borel subgroup and branes on the Hitchin system.* Preprint `arXiv:1709.03549v2`.

[FW]   E. Frenkel, E. Witten, *Geometric Endoscopy and Mirror Symmetry, Commun. Number Theory Phys.* **2** (2008), 113–283.

[Ga]   D. Gaiotto, *S-duality of boundary conditions and the Geometric Langlands program,* Proc. Symp. Pure Math. **98** (2018), 139–180.

[GR1]   O. Garcia-Prada, S. Ramanan, *Involutions of rank 2 Higgs bundle moduli spaces,* Geometry and Physics: Volume II, A Festschrift in honour of Nigel Hitchin, Editors: J.E. Andersen, A. Dancer and O. Garcia-Prada, Oxford University Press, 2018.

[GR2]   O. Garcia-Prada, S. Ramanan, *Involutions and higher order automorphisms of Higgs moduli spaces,* Proc. London Math. Soc. **119** (2019), 681–732.

[GO1]   P. Gothen, A. Oliveira, *The singular fiber of the Hitchin map,* Int. Math. Res. Not., **2013** No. 5 (2013), 1079–1121.

[GO2]   P. Gothen, A. Oliveira, *Topological mirror symmetry for parabolic Higgs bundles,* J. Geom. Phys., **137** (2019), 7–34.

[GWZ]   M. Groechenig, D. Wyss, P. Ziegler, *Mirror symmetry for moduli spaces of Higgs bundles via p-adic integration,* Invent. Math. **221** (2020), 505–596.

[Gr1]   A. Grothendieck, *Éléments de Géométrie Algébrique IV. Étude locale des schémas et des morphismes de schémas,* Publ. Mat. de IHES **32** (1967), 5–361.

[Gr2]   A. Grothendieck, *Technique de descente et théorèmes d'existence en géométrie algébrique. V. Les schémas de Picard: théorèmes d'existence,* Séminaire Bourbaki **7** (1961-1962), Talk no. 232, 143–161.

[Gu]   S. Gukov, *Quantization via mirror symmetry,* Jpn. J. Math. **6** (2011), no. 2, 65–119. MR 2861769

[Fa]   G. Faltings, *Stable G-bundles and projective connections,* J. Algebraic Geom., **2**(3):507–568, 1993.

[Ha]   R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Mathematics **52**, Springer-Verlag, 1977.

[HT]   T. Hausel, M. Thaddeus, *Mirror symmetry, Langlands duality, and the Hitchin system,* Invent. Math. **153** (2003), 197–229.

[HMP]   T. Hausel, A. Mellit, Du Pei, *Mirror symmetry with branes by equivariant Verlinde formulae,* in *Geometry and Physics: Volume I, A Festschrift in honour of Nigel Hitchin.* Oxford Scholarship Online (2018).

[HPa]   T. Hausel, C. Pauly, *Prym varieties of spectral covers,* Geometry & Topology **16** (2012), 1609–1638.

[HeS]   S. Heller, L. P. Schaposnik, *Branes through finite group actions.* J. Geom. Phys. **129** (2018), 279–293.

[Hi1]   N. J. Hitchin, *The self-duality equations on a Riemann surface,* Proc. London Math. Soc. (3), **55**, 1, (1987), 59–126.

[Hi2]   N. J. Hitchin, *Stable bundles and integrable systems,* Duke Math. J. **54**, Number 1 (1987), 91–114.

[Hi3]   N. J. Hitchin, *Lie groups and Teichmüller space,* Topology **31** (1992), 449–473.

[Hi4]   N. J. Hitchin, *Langlands Duality And G_2 Spectral Curves,* Quart. J. Math. **58**, 3, (2007), 319–344.

[Hi5]   N. J. Hitchin, *Higgs bundles and characteristic classes,* Arbeitstagung Bonn 2013, Progr. Math., **319**, Birkhäuser/Springer, Cham, 2016, 247–264.

[Hi6]   N. J. Hitchin, *Spinors, Lagrangians and rank 2 Higgs bundles.* Proc. London Math. Soc. **115** (2017), 33–54.

[Hi7]   N. J. Hitchin, *Critical loci for Higgs bundles.* Comm. Math. Phys. **366** (2019), 841–864.

[HH]   N. J. Hitchin and T. Hausel, work in progress.

[HR]   J.-M. Hwang and S. Ramanan, *Hecke curves and Hitchin discriminant,* Ann. Sci. École Norm. Sup. (4) **37** (2004), 801–817.

[KW]   A. Kapustin, E. Witten, *Electric-magnetic duality and the geometric Langlands program,* Comm. Number Theory Phys. **1** (2007), 1–236.

[KM]   F. Knudsen, D. Mumford, *The projectivity of the moduli space of stable curves I: Preliminaries on "det" and "Div".* Math. Scand. **39** (1976), 19–55.

[La]   G. Laumon, *Un analogue global du cône nilpotent.* Duke Math. J. **57**, 647–671 (1988).

[MS]   D. Maulik and J. Shen, *Endoscopic decomposition and the Hausel–Thaddeus conjecture,* Preprint:arXiv:2008.08520v2 [math.AG].

[Mu]   S. Mukai, *Duality between $\mathcal{D}(X)$ and $\mathcal{D}(\hat{X})$ with its application to Picard sheaves,* Nagoya Math. J., **81** (1981), 153–175.

[Na]   F. Nasser, *Torsion Subgroups of Jacobians Acting on Moduli Spaces of Vector Bundles,* PhD Thesis, University of Aarhus, 2005.

[NR1]   M. S. Narasimhan and S. Ramanan, *Moduli of vector bundles on a compact Riemann surface,* Ann. of Math. (2) **89** (1969), 14–51.

[NR2]   M. S. Narasimhan and S. Ramanan, *Deformations of the moduli space of vector bundles over an algebraic curve,* Ann. Math. (2) **101** (1975), 391–417.

[NR3]   M. S. Narasimhan, S. Ramanan, *Generalized Prym varieties as fixed points*, J. Indian Math. Soc. **39** (1975), 1–19.

[Ngo]   B. C. Ngô, *Fibration de Hitchin et endoscopie*, Invent. Math. **164** (2006), 399–453.

[PP]   C. Pauly and A. Peón-Nieto, *Very stable bundles and properness of the Hitchin map*, Geom. Dedicata **198** (1) (2019) 143–148.

[Ra]   S. Ramanan, *Some aspects of the theory of Higgs pairs*, The many facets of geometry, Oxford Univ. Press, Oxford, 2010, pp. 92–112.

[Re]   C. J. Rego, *The compactified Jacobian*, Ann. Sci. Éc. Norm. Sup. IV, **13** (1980), 211–223.

[Sch]   D. Schaub, *Courbes spectrales et compactifications de Jacobiennes*, Mathematische Zeitschrift **227**, issue 2 (1998), 295–312.

[Sco]   R. Scognamillo, *An elementary approach to the abelianization of the Hitchin system for arbitrary reductive groups.* Compositio Math. **110** (1998), no. 1, 17–37

[Si1]   C. T. Simpson, *Higgs bundles and local systems*, Publ. Math., Inst. Hautes Etudes Sci. **75** (1992), 5–95.

[Si2]   C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*, Publ. Math., Inst. Hautes Etud. Sci. **79** (1994), 47–129.

[Si3]   C. T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety II*, Publ. Math., Inst. Hautes Etud. Sci. **80** (1995), 5–79.

[SYZ]   A. Strominger, S.-T. Yau, E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), 243–259.

[Wi]   G. Wilkin, *The reverse Yang-Mills-Higgs flow in a neighbourhood of a critical point*, J. Differential Geom. **115**, Number 1 (2020), 111–174.

[W]   E. Witten, *More On Gauge Theory And Geometric Langlands*, Adv. Math. **327** (2018), 624–707.

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