Necessary Conditions Optimality for Impulsive Control Problems *

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Abstract

In this article, we present first and second order necessary conditions of optimality for impulsive control problems recently obtained by the authors. Two kind of results are discussed: first order necessary conditions of optimality for the free time optimal control problem with state and control constraints and second order necessary conditions for fixed time optimal control problems with state endpoint and control constraints. The main feature of these conditions is their nondegeneracy. A discussion of these results in the context of the state-of-the-art providing a historical perspective and insight for future perspectives is also included.

Key words: Optimal Impulsive Control, Necessary Conditions of Optimality, Maximum Principle, Nondegeneracy

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1 Introduction

After some initial advances before the sixties in the last century and significant progresses since then, today the optimal control theory for impulsive systems constitutes a rich body of results.

In this article, results on necessary conditions of optimality for impulsive control problems recently obtained by the authors is surveyed, being some relations with comparable results in the literature discussed having in mind to provide insight for future developments.

We only consider systems generating state trajectories of bounded variation whose dynamics are given by measure differential equations. That is, the components of the trajectory that evolve singularly with respect to the Lebesgue measure (for example, discontinuities or jumps) are obtained by extending the usual class of measurable controls in order to include measures which enter affinely in the dynamics. The state trajectory is obtained by integrating the dynamics in the sense of Lebesgue-Stieltjes. However, since we consider vector fields associated with the singular term of the dynamics depending on the state variable, the fact that the control measure may contain atoms implies that some care is needed in defining the trajectory. This will be presented below and reveals the need to characterize a path joining the endpoints of each trajectory jumps.

We will consider, for now, the following basic problem:

$$\begin{aligned} &(P)\ \text{Minimize } e_0(p) && (1)\\ &\quad\text{subject to } dx(t) = f(t,x(t),u(t))dt + G(t,x(t))d\mu(t),\ t \in [t_0,t_1], && (2)\\ &\qquad e_1(p) \leq 0, \quad e_2(p) = 0, && (3)\\ &\qquad u \in \mathcal{U}, \quad d\mu \in \mathcal{K}. && (4) \end{aligned}$$

Here, $p = (x(t_0), x(t_1), t_0, t_1)$, $t_0 < t_1$ are fixed, and the functions $f : [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $G : [t_0,t_1] \times \mathbb{R}^n \to \mathbb{R}^{n\times q}$, and $e_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{d(e_i)}$, $i = 0, 1, 2$, with $d(e_0) = 1$, are given. By $d(f)$ it is denoted the dimension of the range space of the function f. The function u is the conventional measurable control, being

$$\mathcal{U} := \{u \in L^m_\infty[t_0,t_1] \ : u(t) \in U(t)\ L \text{ a.e.}\}. \quad (5)$$

The impulsive control $d\mu$ is a q-dimensional Borel measure taking values in the cone $\mathcal{K}$, given by

$$\{d\mu \in C^*([t_0,t_1]; R^q) : \forall \phi([t_0,t_1]) \text{ s.t. } \phi(t) \in K^0\ \forall t,$$

$$\int_B \phi(t)d\mu \geq 0 \ \forall \text{ Borel } B \subset [t_0, t_1]\}.$$

Here, K is a given convex, closed, pointed cone from R^q, and K^0 is its dual.

Although, a wide range of different problems have been considered in the literature, the paradigm considered here appears to be the one for which more results have been derived and emerges as a natural extension of the conventional optimal control problem.

One of the key features of our optimality conditions is that these do not degenerate, i.e., they remain informative even for abnormal control processes. While for the first-order conditions a certain controllability assumption is required in order to ensure the nondegeneracy, for second-order conditions no a priori normality assumptions are imposed. Here, the second-order information is used in order to select informative multipliers.

In order to derive the necessary conditions of optimality the following set of basic assumptions will be considered.

B1) The functional e_0, the vector functions e_1, e_2, and the matrix function G are continuously differentiable.

B2) f is continuously differentiable in (x, u) for almost all t, and, together with its partial derivatives in x, is measurable in t for all fixed (x, u).

B3) f and its partial derivatives in x are bounded on any bounded set and continuous in (x, u) uniformly in t.

B4) The set-valued function U is measurable in t, and bounded, being $U(t)$ convex and closed for almost all t.

Since the first-order and the second-order necessary conditions were derived for somewhat different problems and under different assumptions, the additional features and assumptions will be presented in the corresponding sections.

The definition of trajectory associated with a control pair (x_0, u, μ) solution to (2), x, has been presented, for example, in (3). It is a function of bounded variation satisfying, $\forall t \in (t_0, t_1]$:

$$x(t) = x_0 + \int_{t_0}^{t} [f(s, x(s), u(s)) + G(s, x(s))w_{ac}(s)]ds$$
$$+ \int_{t_0}^{t} G(s, x(s))d\mu_{sc}(s) + \sum_{t_i \leq t} \Delta x(t_i)$$

where w_{ac}, $d\mu_{sc}$, and $d\mu_{sa} := \sum_{t_i < t} \mu(\{t_i\})\delta_{t_i}(t)$, are, respectively, the absolutely continuous, the singular continuous, and the atomic components of the control measure, and $\Delta x(t_i) = \xi_i(1) - x(t_i^-)$, being ξ_i solution to

$$\dot{\xi}_i(s) = G(t_i, \xi_i(s))\mu_{sa}(\{t_i\}), \quad [0,1] - \text{a.e.}, \quad \xi_i(0) = x(t_i^-). \tag{6}$$

An admissible process (x_0^*, u^*, μ^*) is a local minimizer of the problem (P) if there exists $\varepsilon > 0$ and, for any finite-dimensional subspace $\mathbf{R} \subset L_\infty^m[t_0, t_1]$, $\varepsilon_{\mathbf{R}} > 0$ such that the process (x_0^*, u^*, μ^*) yields a minimum to the problem (P) with the additional constraints $\|a - a^*\| < \varepsilon$, $\|d\mu - d\mu^*\|_{C^*([t_0,t_1];\mathbb{R}^q)} < \varepsilon$, $u \in \mathcal{U}_\varepsilon(u^*)$ defined as the set of controls u satisfying $\|u - u^*\|_{L_\infty^m[t_0,t_1]} < \varepsilon_{\mathbf{R}}$, $u \in \mathbf{R}$, and $u(t) \in U(t)$.

Besides the technical motivation underlying the extension of the rich body of optimal control theory well established for systems with absolutely continuous trajectories for those with trajectories which are merely of bounded variation, there is a wide range of applications driving the progress in this field:

- Space navigation, (17; 18; 16; 11).
- Finance and resources management, (8; 4; 11; 12)
- Quantum electronics,(12).
- Nonsmooth impact mechanics, (5; 6; 7)

This article is structured as follows. In the next section, we present first-order conditions for the free-time impulsive control problem with state constraints. Here, there are two variants requiring different sets of assumptions: a weakened Maximum Principle, and nondegenerate necessary conditions of optimality. The third section concerns nondegenerate second-order necessary conditions of optimality for the fixed time impulsive control problem without state constraints. These conditions are proved without a priori normal assumptions. Finally, we close the article with a discussion of the presented results in the context of the state-of-the-art in impulsive optimal control.

2 First-Order Optimality Conditions

In this section, we present first-order necessary conditions of optimality for a free time impulsive control problem which, besides state endpoints equality and inequality constraints and control constraints, also includes state constraints of the form

$$\varphi(x, t) \le 0, \quad \forall t \in [t_0, t_1] \tag{7}$$

where $\varphi : \mathbb{R}^n \times [t_0, t_1] \to \mathbb{R}^{d(\varphi)}$ is assumed to be continuously differentiable in all its arguments. A remark concerning this inequality is in order: it has to hold not only for all $t \in [t_0, t_1]$ but also for all points in the trajectory path joining the jump endpoints, i.e., for any atom t_i of μ, we have $\varphi(\xi_i(s), t_i) \leq 0$, $\forall s \in [0,1]$, where ξ_i is solution to (6).

Furthermore, the control measure is scalar, i.e., $q = 1$, and the vector function f in (2) is linear in the conventional control variable u.

The two results presented here - a Maximum Principle and a weakened Maximum Principle - were proved in (2) under different subsets of the following set of assumptions:

F1) f is continuously differentiable in all arguments, and the set-valued map $U(\cdot)$ is constant and convex and compact valued.

F2) The endpoint constraints are *regular*, i.e., for any endpoint vector $p = (x_0, x_1, t_0, t_1)$ satisfying (3):

1) $\{e^j_{2p}(p) : j = 1, .., d(e_2)\}$ is linearly independent.
2) $\exists \bar{p} \in \mathbb{R}^{2n+2}$ s.t. $e_{2p}(p)\bar{p} = 0$, and $\langle e^j_{1p}(p), \bar{p}\rangle > 0$, $\forall j$ s.t. $e^j_1(p) = 0$.

F3) The state constraints are *regular*: $\forall (x,t)$ s.t. (7), $\exists q = q(x,t) \in \mathbb{R}^n$ s.t.

$$\langle \varphi^j_x(x,t), q\rangle > 0 \text{ for all } j \text{ s.t. } \varphi^j(x,t) = 0.$$

F4) The state constraints are *compatible* with endpoint constraints at p^* satisfying (3) and $\varphi(x^*(t^*_k), t^*_k) \leq 0$, $k = 0, 1$, i.e., $\exists \varepsilon > 0$ such that

$$\{p \in \mathbb{R}^{2n+2} : |p - p^*| \leq \varepsilon, e_1(p) \leq 0, e_2(p) = 0\} \subseteq \{p : \varphi(x_k, t_k) \leq 0, k = 0, 1\}.$$

F5) The reference trajectory $x^*(\cdot)$ is *controllable* at the end points with regard to state constraints: $\exists u_k \in U$ and $m_k \in [0, +\infty)$, $k = 0, 1$ such that, $\forall j$ with $\varphi^j(x^*(t^*_k), t^*_k) = 0$,

$$(-1)^k\langle f(x^*(t^*_k), u_k, t^*_k) + G(x^*(t^*_k), t^*_k)m_k, \varphi^j_x(x^*(t^*_k), t^*_k)\rangle$$
$$+\varphi^j_t(x^*(t^*_k), t^*_k) < 0.$$

F6) Let $x^*(\cdot)$ be the reference optimal trajectory on $[t^*_0, t^*_1]$, $E^*_0 = (x^*(t^{*+}_0), t^*_0)$, $E^*_1 = (x^*(t^{*-}_1), t^*_1)$. The function φ is twice continuously differentiable and satisfies: $\exists \delta > 0$ s.t. $\forall E \in \mathbb{R}^n \times \mathbb{R}$ with $|E - E^*_k| \leq \delta$ $\forall j, k$ and $\varphi^j(E^*_k) = W^j(E^*_k) = 0$,

$$\dot{W}^j(E) \geq 0$$

where, for $j = 1, .., d(\varphi)$, $W^j(x,t) = \langle \varphi^j_x(x,t), G(x,t)\rangle$, and $\dot{W}^j = \langle G_x G, \varphi^j_x\rangle + \langle G, \varphi^j_{xx} G\rangle$ is the evolution derivative of W^j along the arc joining the jump endpoints.

While the assumptions $F1)$ - $F5)$ are needed in order to ensure the nondegeneracy of the necessary conditions of optimality, $F6)$ together with $F4)$ is used to obtain time transversality conditions for problems with state constraints and dynamics measurable in t.

In order to present our results, we need to introduce some notation. For some $\lambda = (\lambda_0, \lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}^{d(e_1)} \times \mathbb{R}^{d(e_2)}$, let $H = H_0 + H_1 v$, $H_0(x,u,\psi,t) = \langle f(x,u,t), \psi\rangle$, $H_1(x,\psi,t) = \langle G(x,t), \psi\rangle$, and $l(p,\lambda) = \sum_{j=0}^{2}\langle e_j(p), \lambda_j\rangle$.

Furthermore, in order to facilitate following the results, we adopt a short notation as follows: $H_0(t,u) = H_0(x^*(t), u, \psi(t), t)$, $H_0(t) = H_0(t, u^*(t))$, $H_1(t) = H_1(x^*(t), \psi(t), t)$, $H_{0x}(t) = H_{0x}(x^*(t), u^*(t), \psi(t), t)$, …, etc. In other words, if H_0, H_1 or their partial derivatives miss some of arguments x, ψ, u, then it is understood that the values $x^*(t)$, $\psi(t)$, and $u^*(t)$ are considered in their place. Let also, $T^* = [t_0^*, t_1^*]$, and $\Delta_k^* = \mu^*(\{t_k^*\})$, $k = 0, 1$.

Theorem 3.1 (Weakened Maximum Principle). Let (p^*, u^*, μ^*), with $p^* = (x^*(t_0^*), x^*(t_1^*), t_0^*, t_1^*)$, be the solution to the optimal control problem (1)-(4) and (7) whose data satisfies the basic assumptions $B1$)-$B4$) and $F4$).

Then, there exist

- number $\lambda_0 \geq 0$, vectors $\lambda_1 \in \mathbb{R}^{d(e_1)}$, $\lambda_1 \geq 0$, $\lambda_2 \in \mathbb{R}^{d(e_2)}$,
- vector function $\psi \in BV^n(T^*)$,
- vector measure $\eta = (\eta^1, .., \eta^{d(\varphi)})$, $\eta^j \in C_+^*(T^*)$, s.t. $\mathrm{Ds}(\mu^*) \cap \mathrm{Ds}(\eta^j) = \emptyset$ $\forall j$, and
- for every atom $r \in \mathrm{Ds}(\mu^*)$,
  - · vector function $\sigma_r \in BV^n([0,1])$,
  - · vector measure $\eta_r = (\eta_r^1, .., \eta_r^{d(\varphi)})$, $\eta_r^j \in C_+^*([0,1])$, $j = 1, .., d(\varphi)$,

satisfying:

$$\psi(t) = \psi_0 - \int_{t_0^*}^{t} H_{0x}(s)ds - \int_{[t_0^*,t]} H_{1x}(s)d\mu_c^* + \int_{[t_0^*,t]} \varphi_x^\top(x^*, s)d\eta + \sum_{r\in \mathrm{Ds}(\mu^*),\, r\leq t} [\sigma_r(1) - \psi(r^-)], \quad (8)$$

$$\begin{cases} \dot{\alpha}_r^*(s) &= g(\alpha_r^*(s), r)\Delta_r^*, \;\; s \in [0,1], \\ d\sigma_r(s) &= -G_x^\top(\alpha_r^*(s), r)\sigma_r(s)\Delta_r^* ds + \varphi_x^\top(\alpha_r^*(s), r)d\eta_r, \;\; s \in [0,1], \\ \alpha_r^*(0) &= x^*(r^-), \quad \sigma_r(0) = \psi(r^-), \quad \Delta_r^* = \mu^*(\{r\}), \end{cases} \quad (9)$$

$$\psi_0 = \frac{\partial l}{\partial x_0}(p^*, \lambda), \quad \psi_1 = -\frac{\partial l}{\partial x_1}(p^*, \lambda), \quad (10)$$

$$\langle G(\alpha_r^*(s), r), \sigma_r(s)\rangle = 0 \quad \forall s \in [0,1], \quad \forall r \in \mathrm{Ds}(\mu^*) \quad (11)$$

$$\text{supp } (\eta_r^j) \subseteq \{s \in [0,1] : \varphi^j(\alpha_r^*(s), r) = W^j(\alpha_r^*(s), r) = 0\} \quad \forall j, \quad (12)$$

$$\langle \lambda_1, e_1(p^*)\rangle = 0,$$

$$\varphi^j(x^*(t), t) = 0 \;\; \eta^j\text{-a.e.}, \quad \forall j \quad (13)$$

$$\max_{u\in U(t)} H_0(u,t) = H_0(t) \quad \text{a.e.}, \quad (14)$$

$$H_1(t) \leq 0 \;\; \forall t, \qquad H_1(t) = 0 \quad \mu^*\text{-a.e.}, \quad (15)$$

$$\operatorname{ess}\liminf_{t\to t_k^*} \max_{u\in U(t)} H_0(x^*(t_k^*),u,\psi(t_k^*),t) + (-1)^k \frac{\partial l}{\partial t_k}(p^*,\lambda) \le 0,\ k=0,1,\quad (16)$$
$$|\lambda| + |\eta| + \sum_{r\in \mathrm{Ds}(\mu^*)} |\eta_r| = 1.$$

Furthermore, when assumption $F6$) holds, then, for the case where $\Delta_k^* = 0$, we can obtain, instead of (16),

$$\begin{cases} \operatorname{ess}\limsup_{t\to t_k^*} \max_{u\in U(t)} H_0(x^*(t_k^*),u,\psi(t_k^*),t) + (-1)^k \frac{\partial l}{\partial t_k}(p^*,\lambda) \ge 0, \\ \operatorname{ess}\liminf_{t\to t_k^*} \max_{u\in U(t)} H_0(x^*(t_k^*),u,\psi(t_k^*),t) + (-1)^k \frac{\partial l}{\partial t_k}(p^*,\lambda) \le 0, \end{cases} \quad k=0,1.$$

■

Here is our main result of this section, the nondegenerate necessary conditions of optimality for the free time impulsive control problem in its full generality, i.e., with endpoint state constraints and state constraints.

Theorem 3.2. (Nondegenerate Maximum Principle). Let (p^*,u^*,μ^*), with $p^* = (x^*(t_0^*), x^*(t_1^*), t_0^*, t_1^*)$, be a solution to problem (1)-(4) and (7) which is assumed to satisfy the basic hypotheses $B1$)-$B4$), and $F1$)- $F5$).

Then, besides λ_0, λ_1, λ_2, ψ, $\eta = (\eta^1,..,\eta^{d(\varphi)})$, and a pair $(\sigma_r \in BV^n([0,1])$, $\eta_r = (\eta_r^1,..,\eta_r^{d(\varphi)}))$ for every atom $r \in \mathrm{Ds}(\mu^*)$, as in Theorem 3.1, there exist

- a scalar function $\phi \in V(T^*)$, and,
- a scalar function $\theta_r \in V([0,1])$ for every atom $r \in \mathrm{Ds}(\mu^*)$,

satisfying (8)-(10),

$$\phi(t) = \phi_0 + \int_{t_0^*}^{t} H_{0t}(s)ds + \int_{[t_0^*,t]} H_{1t}(s)d\mu_c^* - \int_{[t_0^*,t]} \varphi_t^\top(x^*,s)d\eta + \sum_{r\in \mathrm{Ds}(\mu^*),\, r\le t} [\theta_r(1) - \phi(r^-)],$$
$$d\theta_r(s) = \langle G_t(\alpha_r^*(s),r),\sigma_r(s)\rangle \Delta_r^* ds - \varphi_t^\top(\alpha_r^*(s),r)d\eta_r,\ \theta_r(0) = \phi(r^-),$$
$$\phi_0 = -\frac{\partial l}{\partial t_0}(p^*,\lambda), \quad \phi_1 = \frac{\partial l}{\partial t_1}(p^*,\lambda),$$

(11)-(15), and

$$\max_{u\in U} H_0(u,t) = \phi(t) \quad \forall t \in (t_0^*, t_1^*),$$
$$\lambda_0 + L\Big(\{t : |\psi(t)| > 0\}\Big) + \sum_{r\in \mathrm{Ds}(\mu^*)} L\Big(\{s : |\sigma_r(s)| > 0\}\Big)\Delta_r^* = 1.$$

■

3 Second-Order Optimality Conditions

In this section, we present second order conditions of optimality for a fixed time impulsive control problems with control and endpoint state trajectory constraints. The result presented here can be regarded as an extension of the result proved in (3) in that, now, we consider also control constraints. However, this is not an essential feature of the overall result.

Besides dropping the state constraints and considering a fixed time problem, and therefore $p = (x(t_0), x(t_1))$, now, the set $U(t)$ in the constraint on the conventional control in (5) is defined by $U(t) := \{u \in R^m : M(t,u) = 0\}$ for some function $M : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^{d(M)}$ satisfying a regularity assumption.

Let us consider the basic problem with the following assumptions. This essentially consist in higher smoothness of the data as well as in the Frobenius condition to be satisfied by the singular vector field, that is, $B1$)-$B4$) are now replaced by:

S1) Functions e_0, e_1, e_2 and G are C^2.

S2) The function f is twice continuously differentiable w.r.t. (x,u) for all $t \in [t_0, t_1]$.

S3) Functions f and G and their first and second order derivatives are bounded on any bounded subset and measurable w.r.t. t.

S4) The $n \times q$ matrix G satisfies the so called Frobenius condition, i.e.,

$$G^i_x(t,x)G^j(t,x) - G^j_x(t,x)G^i(t,x) \equiv 0, \quad i,j = 1\ldots q,\ i \neq j, \qquad (19)$$

where G^i is i^{th} column of G.

S5) The function M satisfies the following regularity condition: For any set $V \subset \mathbb{R}^m$, $\exists \epsilon > 0$ such that, for a.a. $t \in [t_0, t_1]$,

$$\det(M_u(t,u)M_u^T(t,u)) \geq \epsilon$$

for all $u \in V$ s.t. $|M(t,u)| \leq \epsilon$.

Notice that, given $S2$), the uniqueness is guaranteed by the Frobenius condition, $S5$). This ensures the robustness of the dynamic system (2) with respect to the approximation of generalized control $d\mu$ by conventional controls $v(\cdot) \in L^q_\infty([t_0,t_1];K)$ (see (11; 20; 21? ; 35)).

The key feature of this result consists in the fact that second-order conditions are used to select an adequate subset from the set of all multipliers that satisfy the local maximum principle. This subset of multipliers is such that the optimality conditions remain informative for optimal abnormal control processes in the absence of a priori assumptions.

Let us now restate the first-order optimality conditions discussed in the previous section in the context of this section, i.e., absence of state constraints, specific structure of the conventional control constraints, and fixed time interval.

We say that a process (p^*, u^*, μ^*) satisfies the local maximum principle if there exists $\lambda = (\lambda_0, \lambda_1, \lambda_2) \neq 0$, such that $\lambda_0 \geq 0$, $\lambda_1 \geq 0$, $\langle \lambda_1, e_1(p^*) \rangle = 0$ a vector function $\psi \in BV^n([t_0, t_1])$, solution (in the integral sense of the definition introduced in the first section) to the adjoint system

$$\begin{cases} -d\psi(t) = H_{0x}(t)dt + H_{x_v}(t)d\mu^*(t), \\ -\psi(t_1) = l_{x_1}(p^*, \lambda), \end{cases} \tag{20}$$

and a vector function $m \in L_\infty^{d(M)}[t_0, t_1]$ which satisfy the following conditions:

$$\begin{aligned} \psi(t_0) &= l_{x_0}(p^*, \lambda), \\ H_u(t) - M_u(t)m(t) &= 0 \quad L\text{-a.e.}, \\ \langle H_v(t), v \rangle &\leq 0 \quad \forall (t, v) \in [t_0, t_1] \times K, \\ \langle H_v(t), \omega^*(t) \rangle &= 0 \quad d\mu^*\text{-a.e.}, \end{aligned} \tag{21}$$

where $\omega^*(t) = \frac{d\mu^*(t)}{d|\mu^*(t)|}$ is the Radon Nicodym derivative of the measure $d\mu^*$ with respect to its total variation measure. Remark that any adjoint trajectory $\psi(t)$ and the function $H(t)$ depend on λ due to the transversality condition (21). The jump of ψ at the atom $r \in \text{Ds}(\mu^*)$ is now given by $\psi(r) = \sigma^r(1)$ being $\sigma^r(\cdot)$ solution to the adjoint limiting system

$$-\dot{\sigma}^r(s) = H_{1x}(r, \alpha^r(s), \sigma^r(s))\Delta_r^*, \quad \sigma^r(0) = \psi(r^-)$$

with $\alpha^r(\cdot)$ and Δ_r^* as in (9).

Let us denote by $\Lambda(p^*, u^*, \mu^*)$ the set of all normalized (i.e., $\|\lambda\| = 1$) Lagrange multipliers λ satisfying the local maximum principle. The following necessary condition of optimality is well known (see (1)) : if (p^*, u^*, μ^*) is a local minimizer for (1)-(3), then $\Lambda(p^*, u^*, \mu^*)$ is nonempty.

We preserve the short notation adopted in the previous section and also not that the dot over the function label means the total derivative with respect to time, and that an argument variable appearing in sub index means that a partial derivative is being considered, e.g., $H_{x_v}(t) = \dfrac{\partial^2 H}{\partial v \partial x}(t)$.

Given the complexity of the formulae that follows, we will assume that the control measure has no singular continuous components, i.e., it takes the form

$d\mu(t) = v(t)dt + \sum_{r \in \mathrm{Ds}} \mu(\{r\})\delta_r(t)$. Furthermore, in order to ensure the compactness of the statement of the second order conditions, we shall use the total derivative w.r.t. time along the solution to the following ordinary differential system

$$\begin{cases} \dot{x} = f(t,x,u) + G(t,x)v \\ -\dot{\psi} = H_x(t,x,\psi,u,v) \\ \dot{w} = v, \qquad v(t) \in K \end{cases}$$

where w is the right continuous function of bounded variation associated with μ. Under the Frobenius condition, this derivative does not depend on v.

$\mathcal{K}_{cr}$ - Cone of critical variations. A variation $(\delta x_0, \delta u, \delta w) \in \mathbb{R}^n \times L_\infty^m[t_0, t_1] \times BV^q$ is critical if the corresponding state trajectory variation, $\delta x \in BV^n$, satisfies the following conditions:

$$\begin{aligned}
&\langle e_{0p}(p^*), \delta p\rangle + \langle e_{0x_1}(p^*), G(t_1)\delta w_1\rangle \le 0,\\
&\langle e_{1p}(p^*), \delta a\rangle + \langle e_{1x_1}(p^*), G(t_1)\delta w_1\rangle \le 0,\\
&\langle e_{2p}(p^*), \delta p\rangle + \langle e_{2x_1}(p^*), G(t_1)\delta w_1\rangle = 0,\\
&\delta p = (\delta x(t_0), \delta x(t_1)), \qquad \delta w_1 = \delta w(t_1)\\
&\dot{\delta x} = (f_x + G_x v)(t)\delta x + f_u(t)\delta u - (\dot{H}_v)_\psi^T(t)\delta w, \quad t \notin \mathrm{Ds}(\mu^*)\\
&M_u(t)\delta u(t) = 0 \quad L-\text{a.e.}\\
&d(\delta w) \in K + \mathrm{Lin}\{d\mu^*\},\ \delta w(t_0) = 0 && (22)\\
&\delta x(r) = \delta q(1; r, \mu^*(\{r\})), \quad \forall r \in \mathrm{Ds}(\mu^*) && (23)
\end{aligned}$$

where $\delta q(s; r, \mu^*(\{r\})) := \delta q^r(s)$ is a solution to

$$\dot{\delta q}^r(s) = [H_{1\psi}(r, \alpha^r(s), \sigma^r(s))\mu^*(\{r\})]_x \delta q^r(s), \tag{24}$$

with $\delta q^r(0) = \delta x(r^-)$ for $r > t_0$ and $\delta q^{t_0}(0) = \delta x_0$, being $\alpha^r(\cdot)$ and $\sigma^r(\cdot)$ as defined above.

Quadratic form. For any $\lambda \in \Lambda$, we define the quadratic form

$$\Omega^\lambda(\delta x_0, \delta u, \delta w) = \delta p^T l_{pp}^\lambda(p^*)\delta p + Q_1^\lambda(\delta p, \delta w_1) - \int_{t_0}^{t_1} Q^\lambda(\delta x, \delta u, \delta w)(t)dt$$
$$+ \int_{t_0}^{t_1} \delta u^T(t)\langle M(t), m(t)\rangle_{uu}\delta u(t)dt$$

where Q^λ and Q_1^λ are the following quadratic forms:

$$
\begin{aligned}
Q^\lambda(\delta x, \delta u, \delta w) = {} & \delta u^T H^\lambda_{uu}\delta u + 2\delta x^T H^\lambda_{xu}\delta u - 2\delta w^T(\dot H^\lambda_v)_u \delta u \\
& -\delta w^T (\ddot H^\lambda_v)_v \delta w - 2\delta w^T (\dot H^\lambda_v)_x \delta x + \delta x^T H^\lambda_{xx}\delta x \\
Q_1^\lambda(\delta x(\cdot), \delta w_1) = {} & 2\delta x(t_0)^T l^\lambda_{x_0 x_1}(p^*) G(t_1)\delta w_1 - 2\delta x(t_1)^T H^\lambda_{xv}(t_1)\delta w_1 \\
& +\delta w_1^T G^T(t_1)(l^\lambda_{x_1 x_1}(a^*)G(t_1) - H^\lambda_{xv}(t_1))\delta w_1 \qquad (25) \\
& - \sum_{s\in \mathrm{Ds}(\mu^*)} [\delta x^T(s)\Psi^\lambda(s)\delta x(s) - \delta x^T(s^-)\Psi^\lambda(s^-)\delta x(s^-)].
\end{aligned}
$$

Here, the t dependence in Q^λ is not explicit, and $\delta x(\cdot)$ is the corresponding solution to (22), (23), (24), $\delta x(t_0^-) = \delta x_0$, $w(t_0^-) = 0$. The $n \times n$ matrix $\Psi^\lambda(t) \in BV^{n\times n}$ in formula (25) is given by

$$\Psi^\lambda(t) = -Z^T(1;t) \int_0^1 Z^{-1T}(s,t) H_{1xx}(s;t) Z^{-1}(s;t) ds Z(1;t),$$

where $H_{1xx}(s;t)$ is $H_{1xx}(t, \alpha^*(s;t), \sigma^*(s;t), w^*(t))$ and the $n \times n$ matrix $Z(s;t)$ satisfies the linear differential equation

$$-\frac{dZ(s;t)}{ds} = Z(s;t)[H_{1\psi}(t, \alpha^*(s,t), \sigma^*(s,t))w^*(t)]_x, \qquad Z(0;t) = I,$$

with $\alpha^*(s;t)$, $\sigma^*(s;t)$ being the solutions to the corresponding limiting systems with $\alpha^*(1;t) = x^*(t)$, and $\sigma^*(1;t) = \psi(t)$.

In order to state the main result of this section, it is convenient to consider the following modified variational equation for $t \notin \mathrm{Ds}(\mu^*)$:

$$\dot{\delta x} = (f_x + G_x v)(t)\delta x + f_u(t)\delta u - (\dot H_v)_\psi(t)\pi\delta w, \qquad (26)$$

with jump conditions (23) and (24), with

$$\delta x(0) = \delta x_0 \in \mathbb{R}^n,\ \delta w \in L^k_\infty, \delta u \in L^m_\infty,\ \text{with } \delta u(t) \in \mathrm{Ker} M_u(t) \qquad (27)$$

where π is the matrix of the orthogonal projection from R^q onto the linear subspace $N = K \cap (-K)$.

Define the quadratic form Ω^λ_a on $\mathbb{R}^n \times L^m_\infty \times L^q_\infty \times \mathbb{R}^q$ obtained from Ω^λ by formally replacing δw_1 by h. Put $e = (e_1, e_2)$ and denote by K_π the linear subspace of $\mathbb{R}^n \times L^m_\infty \times L^q_\infty \times \mathbb{R}^q$ of all tuples $(\delta x_0, \delta u, \delta w, h)$ such that the corresponding solution of (26) with (23), (24), satisfies

$$e_p(p^*)\delta p + e_{x_1}(p^*)G(t_1)\pi h = 0, \quad (h \in \mathbb{R}^q).$$

Define the linear operator $\mathcal{A} : \mathcal{K}_\pi \to \mathbb{R}^{d(e)}$ by the formula

$$\mathcal{A}(\delta x(0), \delta u, \delta w, h) = e_{x_0}(p^*)\delta x_0 + e_{x_1}(p^*)\delta x_1 + L_{x_1}(p^*)G(t_1)\pi h,$$

where δx is the corresponding solution to (26), (27), (23), (24). Let $d = \text{codim}\,(Im\mathcal{A})$.

Consider the subset $\Lambda_a(x^*, u^*, w^*)$ (or Λ_a for short) of vectors $\lambda \in \Lambda(x^*, u^*, w^*)$ such that the index of the form Ω_a^λ on the subspace $\mathcal{K}_\pi$ is not greater then d. We recall that the index of a quadratic form q on a given subspace V is the maximum dimension of any subspace of V where the quadratic form is negative definite.

Theorem 4.1. (Necessary conditions of optimality). Let the control process (x^*, u^*, w^*) be a local optimal to the problem (P). Then, $\Lambda_a \neq \emptyset$ and, for any $(\delta x_0, \delta u, \delta w) \in \mathcal{K}_{cr}$, we have

$$\max_{\lambda \in \Lambda_a} \Omega^\lambda(\delta x_0, \delta u, \delta w) \geq 0. \tag{28}$$

The proof of this result is organized into several steps as follows. First, (P) is transformed into an equivalent problem whose impulsive dynamics do not depend on the state variable which can be easily converted into an abstract optimization problem. Then, the optimality conditions proved in (1) are applied, and the stated local maximum principle and second-order conditions are obtained by decoding the abstract conditions in terms of data of the original problem.

4 Discussion

There is a substantial body of literature addressing impulsive optimal control, and, in particular, necessary conditions of optimality. For an excellent survey see (22) and for somewhat complementary comprehensive overviews consult (23; 11; 35; 33).

Degeneracy of the necessary conditions of optimality is an important issue that has an important history for conventional optimal control problems, (1; 9; 14). However, this is not the case in impulsive control. One of the most distinct features of our optimality conditions relatively to comparable results in the impulsive control literature, see for example (13; 29), is the fact that they do not degenerate. While, for first-order conditions, we have to assume some assumptions concerning the controllability of the reference control process relatively the the constraints, as well as their regularity, and compatibility, for second-order conditions, the second-order information is used to restrict the set of multipliers in such a way that the conditions remain informative even for abnormal control processes.

The solution concept adopted in both results in this paper is in the spirit of the works of (5; 6; 23; 19; 26; 31). These have their roots in the reparameterization techniques used in some of the first studies on impulsive control, (27; 24; 34), which were formulated having in mind to ensure the existence of solution to optimal control problems with unbounded controls emerging, for example, in minimum fuel and optimal space navigation problems. The existence of solution is ensured by enlarging the control space in order to include measures and the necessary conditions of optimality were derived by decoding in terms of the original data, the conditions for an equivalent conventional optimal control problem. This is generated by reparameterising time so that, for each atom of the control measure, there corresponds a new time subinterval, being the graph of the trajectory (in the new parameterization) completed according to the singular dynamics.

However, these approaches require that time be treated as an additional component of the state variable, this meaning that the time dependence and the state dependence of the data have to be of the same nature. Like in (26; 32; 26), we derived first-order necessary conditions of optimality in the form of a maximum principle for impulsive control problems for which only the measurable dependence of the absolutely dynamics w.r.t. time is required. Although the optimality conditions derived in (28) is for problems with data also merely measurable on time, this work concerns generalized problems of Bolza with Lagrangians jointly convex in $(x, \dot{x})$ and the duality interpretations of the multipliers as well as properness assumptions are emphasized.

Unlike any of these references, the optimal impulsive control problem that we consider for first order conditions was more complete in the sense that the endpoints of the time interval are free, and state constraints have to be satisfied by any feasible trajectory. These features not only greatly increase the complexity of the optimality conditions but also make the arguments of the proof extremely more involved.

It can be seen that our first-order conditions can be extended for the case of vector valued control measures (quite common in the literature). In fact, this is the case for our result on second-order conditions where we assumed the Frobenius condition to be satisfied by the singular vector field. This property of differential geometric character is present in most results for impulsive control with vector valued measures, (5; 20; 21; 10; 15; 13), in order to ensure the correctness. One of the challenges is to derive the optimality conditions for our optimal control problem without this condition, as in (6; 26).

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NECESSARY OPTIMALITY CONDITIONS FOR IMPULSIVE CONTROL PROBLEMS*

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Abstract: In this article, we present first and second order necessary conditions of optimality for impulsive control problems recently obtained by the authors. Two kind of results are discussed: first order necessary conditions of optimality for the free time optimal control problem with state and control constraints and second order necessary conditions for fixed time optimal control problems with state endpoint and control constraints. The main feature of these conditions is their nondegeneracy. A discussion of these results in the context of the state-of-the-art providing a historical perspective and insight for future perspectives is also included.

Keywords: Optimal Impulsive Control, Necessary Conditions of Optimality, Maximum Principle, Nondegeneracy

1. INTRODUCTION

After some initial advances before the sixties in the last century and significant progresses since then, today the optimal control theory for impulsive systems constitutes a rich body of results.

In this article, results on necessary conditions of optimality for impulsive control problems recently obtained by the authors is surveyed, being some relations with comparable results in the literature discussed having in mind to provide insight for future developments.

We only consider systems generating state trajectories of bounded variation whose dynamics are given by measure differential equations. That is, the components of the trajectory that evolve singularly with respect to the Lebesgue measure (for example, discontinuities or jumps) are obtained by extending the usual class of measurable controls in order to include measures which enter affinely in the dynamics. The state trajectory is obtained by integrating the dynamics in the sense of Lebesgue-Stieltjes. However, since we consider vector fields associated with the singular term of the dynamics depending on the state variable, the fact that the control measure may contain atoms implies that some care is needed in defining the trajectory. This will be presented below and reveals the need to characterize a path joining the endpoints of each trajectory jumps.

We will consider, for now, the following basic problem:

$$
\begin{aligned}
(P)\ \text{Minimize}\quad & e_0(p) && (1)\\
\text{subject to}\quad & dx(t) = f(t,x(t),u(t))dt \\
& \quad + G(t,x(t))d\mu(t), \\
& t \in [t_0,t_1], && (2)\\
& e_1(p) \le 0,\ e_2(p) = 0, && (3)\\
& u \in \mathcal{U},\ d\mu \in \mathcal{K}. && (4)
\end{aligned}
$$

Here, $p = (x(t_0), x(t_1), t_0, t_1)$, $t_0 < t_1$ are fixed, and the functions $f : [t_0,t_1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $G : [t_0,t_1] \times \mathbb{R}^n \to \mathbb{R}^{n\times q}$, and $e_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{d(e_i)}$, $i = 0, 1, 2$, with $d(e_0) = 1$, are given. By $d(f)$ it is denoted the dimension of the range space of the function f. The function u is the conventional measurable control, being

$$\mathcal{U} := \{u \in L^m_\infty[t_0,t_1] : u(t) \in U(t)\ \mathcal{L}\ \text{a.e.}\}. \quad (5)$$

The impulsive control $d\mu$ is a q-dimensional Borel measure taking values in the cone K, given by

$$
\begin{aligned}
\{d\mu \in C^*([t_0,t_1]; R^q) :& \\
\forall \phi \in ([t_0,t_1])\ \text{s.t.}\ & \phi(t) \in K^0\ \forall t, \\
\int_B \phi(t)d\mu \ge 0\ &\forall\ \text{Borel}\ B \subset [t_0,t_1]\}.
\end{aligned}
$$

Here, K is a given convex, closed, pointed cone from R^q, and K^0 is its dual.

Although, a wide range of different problems have been considered in the literature, the paradigm considered here appears to be the one for which more results have been derived and emerges as a natural extension of the conventional optimal control problem.

One of the key features of our optimality conditions is that these do not degenerate, i.e., they remain informative even for abnormal control processes. While for the first-order conditions a certain controllability assumption is required in order to ensure the nondegeneracy, for second-order conditions no a priori normality assumptions are imposed. Here, the second-order information is used in order to select informative multipliers.

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In order to derive the necessary conditions of optimality the following set of basic assumptions will be considered.

B1) The functional e_0, the vector functions e_1, e_2, and the matrix function G are continuously differentiable.

B2) f is continuously differentiable in (x,u) for almost all t, and, together with its partial derivatives in x, is measurable in t for all fixed (x,u).

B3) f and its partial derivatives in x are bounded on any bounded set and continuous in (x,u) uniformly in t.

B4) The set-valued function U is measurable in t, and bounded, being $U(t)$ convex and closed for almost all t.

Since the first-order and the second-order necessary conditions were derived for somewhat different problems and under different assumptions, the additional features and assumptions will be presented in the corresponding sections.

The definition of trajectory associated with a control pair (x_0,u,μ) solution to (2), x, has been presented, for example, in [3]. It is a function of bounded variation satisfying, $\forall t \in (t_0,t_1]$:

$$x(t) = x_0 + \int_{t_0}^{t} [f(s,x(s),u(s)) + G(s,x(s))w_{ac}(s)]ds + \int_{t_0}^{t} G(s,x(s))d\mu_{sc}(s) + \sum_{t_i \leq t} \Delta x(t_i)$$

where w_{ac}, $d\mu_{sc}$, and

$$d\mu_{sa} := \sum_{t_i<t} \mu(\{t_i\})\delta_{t_i}(t),$$

are, respectively, the absolutely continuous, the singular continuous, and the atomic components of the control measure, and $\Delta x(t_i) = \xi_i(1) - x(t_i^-)$, being ξ_i solution to

$$\dot{\xi}_i(s) = G(t_i,\xi_i(s))\mu_{sa}(\{t_i\}), \quad [0,1]\text{−a.e.}, \qquad \xi_i(0) = x(t_i^-). \quad (6)$$

An admissible process (x_0^*,u^*,μ^*) is a local minimizer of the problem (P) if there exists $\varepsilon > 0$ and, for any finite-dimensional subspace $\mathbf{R} \subset L^m_\infty[t_0,t_1]$, $\epsilon_{\mathbf{R}} > 0$ such that the process (x_0^*,u^*,μ^*) yields a minimum to the problem (P) with the additional constraints $\|a - a^*\| < \varepsilon$, $\|d\mu - d\mu^*\|_{C^*([t_0,t_1];\mathbb{R}^q)} < \varepsilon$, $u \in \mathcal{U}_\varepsilon(u^*)$ defined as the set of controls u satisfying $\|u - u^*\|_{L^m_\infty[t_0,t_1]} < \epsilon_{\mathbf{R}}$, $u \in \mathbf{R}$, and $u(t) \in U(t)$.

Besides the technical motivation underlying the extension of the rich body of optimal control theory well established for systems with absolutely continuous trajectories for those with trajectories which are merely of bounded variation, there is a wide range of applications driving the progress in this field:

- Space navigation, [17, 18, 16, 11].
- Finance and resources management, [8, 4, 11, 12]
- Quantum electronics, [12].
- Nonsmooth impact mechanics, [5, 6, 7]

This article is structured as follows. In the next section, we present first-order conditions for the free-time impulsive control problem with state constraints. Here, there are two variants requiring different sets of assumptions: a weakened Maximum Principle, and nondegenerate necessary conditions of optimality. The third section concerns nondegenerate second-order necessary conditions of optimality for the fixed time impulsive control problem without state constraints. These conditions are proved without a priori normal assumptions. Finally, we close the article with a discussion of the presented results in the context of the state-of-the-art in impulsive optimal control.

2. FIRST-ORDER OPTIMALITY CONDITIONS

In this section, we present first-order necessary conditions of optimality for a free time impulsive control problem which, besides state endpoints equality and inequality constraints and control constraints, also includes state constraints of the form

$$\varphi(x,t) \leq 0, \quad \forall t \in [t_0,t_1] \qquad (7)$$

where $\varphi : \mathbb{R}^n \times [t_0,t_1] \to \mathbb{R}^{d(\varphi)}$ is assumed to be continuously differentiable in all its arguments. A remark concerning this inequality is in order: it has to hold not only for all $t \in [t_0,t_1]$ but also for all points in the trajectory path joining the jump endpoints, i.e., for any atom t_i of μ, we have $\varphi(\xi_i(s),t_i) \leq 0$, $\forall s \in [0,1]$, where ξ_i is solution to (6).

Furthermore, the control measure is scalar, i.e., $q = 1$, and the vector function f in (2) is linear in the conventional control variable u.

The two results presented here - a Maximum Principle and a weakened Maximum Principle - were proved in [3] under different subsets of the following set of assumptions:

F1) f is continuously differentiable in all arguments, and the set-valued map $U(\cdot)$ is constant and convex and compact valued.

F2) The endpoint constraints are *regular*, i.e., for any endpoint vector $p = (x_0,x_1,t_0,t_1)$ satisfying (3):
1) $\{e^j_{2p}(p) : j = 1,..,d(e_2)\}$ is linearly independent.
2) $\exists \bar{p} \in \mathbb{R}^{2n+2}$ s.t. $e_{2p}(p)\bar{p} = 0$, and $\langle e^j_{1p}(p),\bar{p}\rangle > 0$,

$$\forall j \text{ s.t. } e^j_1(p) = 0.$$

F3) The state constraints are *regular*: $\forall(x,t)$ s.t. (7), $\exists q = q(x,t) \in \mathbb{R}^n$ s.t.

$$\langle \varphi^j_x(x,t), q\rangle > 0 \text{ for all } j \quad \text{s.t. } \varphi^j(x,t) = 0.$$

F4) The state constraints are *compatible* with endpoint constraints at p^* satisfying (3) and $\varphi(x^*(t^*_k),t^*_k) \leq 0$, $k = 0,1$, i.e., $\exists\varepsilon > 0$ such that

$$\{p \in \mathbb{R}^{2n+2} : |p - p^*| \leq \varepsilon, \; e_1(p) \leq 0, e_2(p) = 0\} \subseteq \{p : \varphi(x_k,t_k) \leq 0, k = 0,1\}.$$

F5) The reference trajectory $x^*(\cdot)$ is *controllable* at the end points with regard to state constraints: $\exists u_k \in U$ and $m_k \in [0,+\infty)$, $k = 0,1$ such that, $\forall j$ with $\varphi^j(x^*(t^*_k),t^*_k) = 0$,

$$(-1)^k \langle f(x^*(t^*_k),u_k,t^*_k) + G(x^*(t^*_k),t^*_k)m_k, \varphi^j_x(x^*(t^*_k),t^*_k)\rangle + \varphi^j_t(x^*(t^*_k),t^*_k) < 0.$$

F6) Let $x^*(\cdot)$ be the reference optimal trajectory on $[t^*_0,t^*_1]$, $E^*_0 = (x^*(t^{*+}_0),t^*_0)$, $E^*_1 = (x^*(t^{*-}_1),t^*_1)$. The function φ is twice continuously differentiable and satisfies: $\exists\delta > 0$ s.t. $\forall E \in \mathbb{R}^n \times \mathbb{R}$ with $|E - E^*_k| \leq \delta$ $\forall j,k$ and $\varphi^j(E^*_k) = W^j(E^*_k) = 0$,

$$\dot{W}^j(E) \geq 0$$

where, for $j = 1,..,d(\varphi)$,

$$W^j(x,t) = \langle \varphi^j_x(x,t), G(x,t)\rangle,$$

and

$$\dot{W}^j = \langle G_x G, \varphi^j_x\rangle + \langle G, \varphi^j_{xx} G\rangle$$

is the evolution derivative of W^j along the arc joining the jump endpoints.

While the assumptions $F1)$ - $F5)$ are needed in order to ensure the nondegeneracy of the necessary conditions of optimality, $F6)$ together with $F4)$ is used to obtain time transversality conditions for problems with state constraints and dynamics measurable in t.

In order to present our results, we need to introduce some notation. For some $\lambda = (\lambda_0,\lambda_1,\lambda_2) \in \mathbb{R} \times \mathbb{R}^{d(e_1)} \times \mathbb{R}^{d(e_2)}$, let $H = H_0 + H_1 v$, $H_0(x,u,\psi,t) = \langle f(x,u,t),\psi\rangle$, $H_1(x,\psi,t) = \langle G(x,t),\psi\rangle$, and $l(p,\lambda) = \sum_{j=0}^{2}\langle e_j(p),\lambda_j\rangle$.

Furthermore, in order to facilitate following the results, we adopt a short notation as follows:

$$H_0(t,u) = H_0(x^*(t),u,\psi(t),t),$$
$$H_0(t) = H_0(t,u^*(t)),$$
$$H_1(t) = H_1(x^*(t),\psi(t),t),$$
$$H_{0x}(t) = H_{0x}(x^*(t),u^*(t),\psi(t),t),$$
$$\ldots, \text{ etc.}$$

In other words, if H_0, H_1 or their partial derivatives miss some of arguments x, ψ, u, then it is understood that the values

$x^*(t)$, $\psi(t)$, and $u^*(t)$ are considered in their place. Let also, $T^* = [t_0^*, t_1^*]$, and $\Delta_k^* = \mu^*(\{t_k^*\})$, $k = 0, 1$.

Theorem 3.1 (Weakened Maximum Principle). Let (p^*, u^*, μ^*), with

$$p^* = (x^*(t_0^*), x^*(t_1^*), t_0^*, t_1^*),$$

be the solution to the optimal control problem (1)-(4) and (7) whose data satisfies the basic assumptions $B1$)-$B4$) and $F4$).

Then, there exist

- number $\lambda_0 \geq 0$, vectors $\lambda_1 \in \mathbb{R}^{d(e_1)}$, $\lambda_1 \geq 0$, $\lambda_2 \in \mathbb{R}^{d(e_2)}$,
- vector function $\psi \in BV^n(T^*)$,
- vector measure $\eta = (\eta^1, .., \eta^{d(\varphi)})$, $\eta^j \in C_+^*(T^*)$, s.t. $\mathrm{Ds}(\mu^*) \cap \mathrm{Ds}(\eta^j) = \emptyset\ \forall j$, and
- for every atom $r \in \mathrm{Ds}(\mu^*)$,
  - vector function $\sigma_r \in BV^n([0,1])$,
  - vector measure $\eta_r = (\eta_r^1, .., \eta_r^{d(\varphi)})$, $\eta_r^j \in C_+^*([0,1])$, $j = 1, .., d(\varphi)$,

satisfying:

$$\psi(t) = \psi_0 - \int_{t_0^*}^t H_{0x}(s)ds - \int_{[t_0^*,t]} H_{1x}(s)d\mu_c^* + \int_{[t_0^*,t]} \varphi_x^{\mathrm{T}}(x^*, s)d\eta + \sum_{r\in \mathrm{Ds}(\mu^*),\, r\leq t} [\sigma_r(1) - \psi(r^-)], \quad (8)$$

$$\begin{cases} \dot{\alpha}_r^*(s) &= g(\alpha_r^*(s), r)\Delta_r^*,\ s \in [0,1], \\ d\sigma_r(s) &= -G_x^{\mathrm{T}}(\alpha_r^*(s), r)\sigma_r(s)\Delta_r^* ds \\ &\quad +\varphi_x^{\mathrm{T}}(\alpha_r^*(s), r)d\eta_r,\ s \in [0,1], \\ \alpha_r^*(0) &= x^*(r^-), \quad \sigma_r(0) = \psi(r^-), \\ &\quad \Delta_r^* = \mu^*(\{r\}), \end{cases} \quad (9)$$

$$\psi_0 = \frac{\partial l}{\partial x_0}(p^*, \lambda),$$

$$\psi_1 = -\frac{\partial l}{\partial x_1}(p^*, \lambda), \quad (10)$$

$$\langle G(\alpha_r^*(s), r), \sigma_r(s)\rangle = 0\ \forall s \in [0,1],\ \forall r \in \mathrm{Ds}(\mu^*) \quad (11)$$

$$\mathrm{supp}(\eta_r^j) \subseteq \{s \in [0,1] : \varphi^j(\alpha_r^*(s), r) = W^j(\alpha_r^*(s), r) = 0\}\ \forall j, \quad (12)$$

$$\langle \lambda_1, e_1(p^*)\rangle = 0,$$

$$\varphi^j(x^*(t), t) = 0 \ \ \eta^j\text{-a.e., } \forall j \quad (13)$$

$$\max_{u\in U(t)} H_0(u,t) = H_0(t) \ \text{ a.e.,} \quad (14)$$

$$H_1(t) \leq 0\ \forall t,\ \ H_1(t) = 0\ \ \mu^*\text{-a.e.,} \quad (15)$$

$$\operatorname*{ess\,lim\,inf}_{t\to t_k^*} \max_{u\in U(t)} H_0(x^*(t_k^*), u, \psi(t_k^*), t) + (-1)^k \frac{\partial l}{\partial t_k}(p^*, \lambda) \leq 0,\ k = 0, 1, \quad (16)$$

$$|\lambda| + |\eta| + \sum_{r\in \mathrm{Ds}(\mu^*)} |\eta_r| = 1.$$

Furthermore, when assumption $F6$) holds, then, for the case where $\Delta_k^* = 0$, we can obtain, instead of (16),

$$\begin{cases} \operatorname*{ess\,lim\,sup}_{t\to t_k^*} \max_{u\in U(t)} H_0(x^*(t_k^*), u, \psi(t_k^*), t) \\ \quad +(-1)^k \frac{\partial l}{\partial t_k}(p^*, \lambda) \geq 0, \\ \operatorname*{ess\,lim\,inf}_{t\to t_k^*} \max_{u\in U(t)} H_0(x^*(t_k^*), u, \psi(t_k^*), t) \\ \quad +(-1)^k \frac{\partial l}{\partial t_k}(p^*, \lambda) \leq 0, \end{cases}$$

$k = 0, 1$.

Here is our main result of this section, the nondegenerate necessary conditions of optimality for the free time impulsive control problem in its full generality, i.e., with endpoint state constraints and state constraints.

Theorem 3.2. (Nondegenerate Maximum Principle). Let (p^*, u^*, μ^*), with $p^* = (x^*(t_0^*), x^*(t_1^*), t_0^*, t_1^*)$, be a solution to problem (1)-(4) and (7) which is assumed to satisfy the basic hypotheses $B1$)-$B4$), and $F1$)- $F5$).

Then, besides

$$\lambda_0,\ \lambda_1,\ \lambda_2,\ \psi,\ \eta = (\eta^1, .., \eta^{d(\varphi)}),$$

and a pair $(\sigma_r \in BV^n([0,1]),\ \eta_r = (\eta_r^1, .., \eta_r^{d(\varphi)}))$ for every atom $r \in \mathrm{Ds}(\mu^*)$, as in Theorem 3.1, there exist

- a scalar function $\phi \in V(T^*)$, and,
- a scalar function $\theta_r \in V([0,1])$ for every atom $r \in \mathrm{Ds}(\mu^*)$,

satisfying (8)-(10),

$$\phi(t) = \phi_0 + \int_{t_0^*}^t H_{0t}(s)ds + \int_{[t_0^*,t]} H_{1t}(s)d\mu_c^* - \int_{[t_0^*,t]} \varphi_t^{\mathrm{T}}(x^*, s)d\eta + \sum_{r\in \mathrm{Ds}(\mu^*),\, r\leq t} [\theta_r(1) - \phi(r^-)],$$

$$d\theta_r(s) = \langle G_t(\alpha_r^*(s), r), \sigma_r(s)\rangle\Delta_r^* ds - \varphi_t^{\mathrm{T}}(\alpha_r^*(s), r)d\eta_r,\ \theta_r(0) = \phi(r^-),$$

$$\phi_0 = -\frac{\partial l}{\partial t_0}(p^*, \lambda), \quad \phi_1 = \frac{\partial l}{\partial t_1}(p^*, \lambda),$$

(11)-(15), and

$$\max_{u\in U} H_0(u,t) = \phi(t) \quad \forall t \in (t_0^*, t_1^*),$$

$$\lambda_0 + \mathcal{L}\Big(\{t : |\psi(t)| > 0\}\Big) + \sum_{r\in \mathrm{Ds}(\mu^*)} \mathcal{L}\Big(\{s : |\sigma_r(s)| > 0\}\Big)\Delta_r^* = 1.$$

3. SECOND-ORDER OPTIMALITY CONDITIONS

In this section, we present second order conditions of optimality for a fixed time impulsive control problems with control and endpoint state trajectory constraints. The result presented here can be regarded as an extension of the result proved in [3] in that, now, we consider also control constraints. However, this is not an essential feature of the overall result.

Besides dropping the state constraints and considering a fixed time problem, and therefore $p = (x(t_0), x(t_1))$, now, the set $U(t)$ in the constraint on the conventional control in (5) is defined by $U(t) := \{u \in R^m : M(t,u) = 0\}$ for some function $M : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^{d(M)}$ satisfying a regularity assumption.

Let us consider the basic problem with the following assumptions. This essentially consist in higher smoothness of the data as well as in the Frobenius condition to be satisfied by the singular vector field, that is, $B1$)-$B4$) are now replaced by:

S1) Functions e_0, e_1, e_2 and G are C^2.

S2) The function f is twice continuously differentiable w.r.t. (x,u) for all $t \in [t_0, t_1]$.

S3) Functions f and G and their first and second order derivatives are bounded on any bounded subset and measurable w.r.t. t.

S4) The $n \times q$ matrix G satisfies the so called Frobenius condition, i.e.,

$$G_x^i(t,x)G^j(t,x) - G_x^j(t,x)G^i(t,x) \equiv 0,\quad i,j = 1\ldots q,\ i \neq j, \quad (17)$$

where G^i is i^{th} column of G.

S5) The function M satisfies the following regularity condition: For any set $V \subset \mathbb{R}^m$, $\exists\epsilon > 0$ such that, for a.a. $t \in [t_0, t_1]$,

$$\det(M_u(t,u)M_u^T(t,u)) \geq \epsilon$$

for all $u \in V$ s.t. $|M(t,u)| \leq \epsilon$.

Notice that, given $S2$), the uniqueness is guaranteed by the Frobenius condition, $S5$). This ensures the robustness of the dynamic system (2) with respect to the approximation of generalized control $d\mu$ by conventional controls $v(\cdot) \in L_\infty^q([t_0, t_1]; K)$ (see [11, 20, 21, 35]).

The key feature of this result consists in the fact that second-order conditions are used to select an adequate subset from the set of all multipliers that satisfy the local maximum principle. This subset of multipliers is such that the optimality conditions remain informative for optimal abnormal control processes in the absence of a priori assumptions.

Let us now restate the first-order optimality conditions discussed in the previous section in the context of this section, i.e., absence of state constraints, specific structure of the conventional control constraints, and fixed time interval.

We say that a process (p^*, u^*, μ^*) satisfies the local maximum principle if there exists $\lambda = (\lambda_0, \lambda_1, \lambda_2) \neq 0$, such that $\lambda_0 \geq 0$, $\lambda_1 \geq 0$, $\langle \lambda_1, e_1(p^*)\rangle = 0$ a vector function $\psi \in BV^n([t_0, t_1])$, solution (in the integral sense of the definition introduced in the first section) to the adjoint system

$$\begin{cases} -d\psi(t) = H_{0x}(t)dt + H_{x_v}(t)d\mu^*(t), \\ -\psi(t_1) = l_{x_1}(p^*, \lambda), \end{cases} \quad (18)$$

and a vector function

$$m \in L_\infty^{d(M)}[t_0, t_1]$$

which satisfy the following conditions:

$$\begin{aligned} \psi(t_0) &= l_{x_0}(p^*, \lambda), \quad (19) \\ H_u(t) - M_u(t)m(t) &= 0 \quad \mathcal{L}\text{-a.e.}, \\ \langle H_v(t), v\rangle &\le 0 \\ &\forall (t, v) \in [t_0, t_1] \times K, \\ \langle H_v(t), \omega^*(t)\rangle &= 0 \quad d\mu^*\text{-a.e.}, \end{aligned}$$

where $\omega^*(t) = \frac{d\mu^*(t)}{d|\mu^*(t)|}$ is the Radon Nicodym derivative of the measure $d\mu^*$ with respect to its total variation measure. Remark that any adjoint trajectory $\psi(t)$ and the function $H(t)$ depend on λ due to the transversality condition (19). The jump of ψ at the atom $r \in \mathrm{Ds}(\mu^*)$ is now given by $\psi(r) = \sigma^r(1)$ being $\sigma^r(\cdot)$ solution to the adjoint limiting system

$$-\dot\sigma^r(s) = H_{1x}(r, \alpha^r(s), \sigma^r(s))\Delta_r^*, \quad \sigma^r(0) = \psi(r^-)$$

with $\alpha^r(\cdot)$ and Δ_r^* as in (9).

Let us denote by $\Lambda(p^*, u^*, \mu^*)$ the set of all normalized (i.e., $\|\lambda\| = 1$) Lagrange multipliers λ satisfying the local maximum principle. The following necessary condition of optimality is well known (see [1]): if (p^*, u^*, μ^*) is a local minimizer for (1)-(3), then $\Lambda(p^*, u^*, \mu^*)$ is nonempty.

We preserve the short notation adopted in the previous section and also not that the dot over the function label means the total derivative with respect to time, and that an argument variable appearing in sub index means that a partial derivative is being considered, e.g., $H_{x_v}(t) = \dfrac{\partial^2 H}{\partial v \partial x}(t)$.

Given the complexity of the formulae that follows, we will assume that the control measure has no singular continuous components, i.e., it takes the form $d\mu(t) = v(t)dt + \sum_{r\in \mathrm{Ds}} \mu(\{r\})\delta_r(t)$. Furthermore, in order to ensure the compactness of the statement of the second order conditions, we shall use the total derivative w.r.t. time along the solution to the following ordinary differential system

$$\begin{cases} \dot x = f(t, x, u) + G(t, x)v \\ -\dot\psi = H_x(t, x, \psi, u, v) \\ \dot w = v, \quad v(t) \in K \end{cases} \quad (20)$$

where w is the right continuous function of bounded variation associated with μ. Under the Frobenius condition, this derivative does not depend on v.

<u>$\mathcal{K}_{cr}$ - Cone of critical variations</u>. A variation $(\delta x_0, \delta u, \delta w) \in \mathbb{R}^n \times L_\infty^m[t_0, t_1] \times BV^q$ is critical if the corresponding state trajectory variation, $\delta x \in BV^n$, satisfies the following conditions:

$$\begin{aligned} &\langle e_{0_p}(p^*), \delta p\rangle + \langle e_{0x_1}(p^*), G(t_1)\delta w_1\rangle \le 0, \\ &\langle e_{1_p}(p^*), \delta a\rangle + \langle e_{1x_1}(p^*), G(t_1)\delta w_1\rangle \le 0, \\ &\langle e_{2_p}(p^*), \delta p\rangle + \langle e_{2x_1}(p^*), G(t_1)\delta w_1\rangle = 0, \\ &\delta p = (\delta x(t_0), \delta x(t_1)), \qquad \delta w_1 = \delta w(t_1) \\ &\dot{\delta x} = (f_x + G_x v)(t)\delta x + f_u(t)\delta u \quad (21) \\ &\qquad -(\dot H_v)_\psi^T(t)\delta w, \quad t \notin \mathrm{Ds}(\mu^*) \\ &M_u(t)\delta u(t) = 0 \quad \mathcal{L} - \text{a.e.} \\ &d(\delta w) \in \mathcal{K} + \mathrm{Lin}\{d\mu^*\}, \ \delta w(t_0) = 0 \quad (22) \\ &\delta x(r) = \delta q(1; r, \mu^*(\{r\})), \\ &\qquad \forall r \in \mathrm{Ds}(\mu^*) \quad (23) \end{aligned}$$

where $\delta q(s; r, \mu^*(\{r\})) := \delta q^r(s)$ is a solution to

$$\dot{\delta q}^r(s) = [H_{1\psi}(r, \alpha^r(s), \sigma^r(s))\mu^*(\{r\})]_x \, \delta q^r(s), \quad (24)$$

with $\delta q^r(0) = \delta x(r^-)$ for $r > t_0$ and $\delta q^{t_0}(0) = \delta x_0$, being $\alpha^r(\cdot)$ and $\sigma^r(\cdot)$ as defined above.

<u>Quadratic form</u>. For any $\lambda \in \Lambda$, we define the quadratic form

$$\begin{aligned} \Omega^\lambda(\delta x_0, \delta u, \delta w) = \\ &\delta p^T l_{pp}^\lambda(p^*)\delta p + Q_1^\lambda(\delta p, \delta w_1) \\ &- \int_{t_0}^{t_1} Q^\lambda(\delta x, \delta u, \delta w)(t)dt \\ &+ \int_{t_0}^{t_1} \delta u^T(t)\langle M(t), m(t)\rangle_{uu}\delta u(t)dt \end{aligned}$$

where Q^λ and Q_1^λ are the following quadratic forms:

$$\begin{aligned} Q^\lambda(\delta x, \delta u, \delta w) &= \delta u^T H_{uu}^\lambda \delta u \\ &+ 2\delta x^T H_{xu}^\lambda \delta u - 2\delta w^T(\dot H_v^\lambda)_u \delta u \\ &- \delta w^T(\ddot H_v^\lambda)_v \delta w - 2\delta w^T(\dot H_v^\lambda)_x \delta x \\ &+ \delta x^T H_{xx}^\lambda \delta x \end{aligned}$$

$$\begin{aligned} Q_1^\lambda(\delta x(\cdot), \delta w_1) = \\ &2\delta x(t_0)^T l_{x_0 x_1}^\lambda(p^*)G(t_1)\delta w_1 \\ &- 2\delta x(t_1)^T H_{xv}^\lambda(t_1)\delta w_1 \\ &+ \delta w_1^T G^T(t_1)(l_{x_1x_1}^\lambda(a^*)G(t_1) - H_{xv}^\lambda(t_1))\delta w_1 \\ &- \sum_{s\in \mathrm{Ds}(\mu^*)} [\delta x^T(s)\Psi^\lambda(s)\delta x(s) \\ &- \delta x^T(s^-)\Psi^\lambda(s^-)\delta x(s^-)]. \quad (25) \end{aligned}$$

Here, the t dependence in Q^λ is not explicit, and $\delta x(\cdot)$ is the corresponding solution to (22), (23), (24), $\delta x(t_0^-) = \delta x_0$, $w(t_0^-) = 0$. The $n \times n$ matrix $\Psi^\lambda(t) \in BV^{n\times n}$ in formula (25) is given by

$$\begin{aligned} \Psi^\lambda(t) = -Z^T(1; t) \\ &\int_0^1 Z^{-1T}(s, t)H_{1xx}(s; t)Z^{-1}(s; t)ds \\ &\qquad Z(1; t), \end{aligned}$$

where $H_{1xx}(s; t)$ is

$$H_{1xx}(t, \alpha^*(s; t), \sigma^*(s; t), w^*(t))$$

and the $n \times n$ matrix $Z(s; t)$ satisfies the linear differential equation

$$\begin{aligned} -\frac{dZ(s; t)}{ds} = Z(s; t) \\ &[H_{1\psi}(t, \alpha^*(s, t), \sigma^*(s, t))w^*(t)]_x, \\ &Z(0; t) = I, \end{aligned}$$

with $\alpha^*(s; t)$, $\sigma^*(s; t)$ being the solutions to the corresponding limiting systems with $\alpha^*(1; t) = x^*(t)$, and $\sigma^*(1; t) = \psi(t)$.

In order to state the main result of this section, it is convenient to consider the following modified variational equation for $t \notin \mathrm{Ds}(\mu^*)$:

$$\dot{\delta x} = (f_x + G_x v)(t)\delta x + f_u(t)\delta u - (\dot H_v)_\psi(t)\pi\delta w, \quad (26)$$

with jump conditions (23) and (24), with

$$\delta x(0) = \delta x_0 \in \mathbb{R}^n, \ \delta w \in L_\infty^k, \delta u \in L_\infty^m, \text{ with } \delta u(t) \in \mathrm{Ker} M_u(t) \quad (27)$$

where π is the matrix of the orthogonal projection from R^q onto the linear subspace $N = K \cap (-K)$.

Define the quadratic form Ω_a^λ on $\mathbb{R}^n \times L_\infty^m \times L_\infty^q \times \mathbb{R}^q$ obtained from Ω^λ by formally replacing δw_1 by h. Put $e = (e_1, e_2)$ and denote by $\mathcal{K}_\pi$ the linear subspace of $\mathbb{R}^n \times L_\infty^m \times L_\infty^q \times \mathbb{R}^q$ of all tuples $(\delta x_0, \delta u, \delta w, h)$ such that the corresponding solution of (26) with (23), (24), satisfies

$$e_p(p^*)\delta p + e_{x_1}(p^*)G(t_1)\pi h = 0, \quad (h \in \mathbb{R}^q).$$

Define the linear operator

$$\mathcal{A} : \mathcal{K}_\pi \to \mathbb{R}^{d(e)}$$

by the formula

$$\begin{aligned} \mathcal{A}(\delta x(0), \delta u, \delta w, h) = e_{x_0}(p^*)\delta x_0 \\ + e_{x_1}(p^*)\delta x_1 + L_{x_1}(p^*)G(t_1)\pi h, \end{aligned}$$

where δx is the corresponding solution to (26), (27), (23), (24). Let

$$d = \mathrm{codim}\,(Im\mathcal{A}).$$

Consider the subset $\Lambda_a(x^*, u^*, w^*)$ (or Λ_a for short) of vectors $\lambda \in \Lambda(x^*, u^*, w^*)$ such that the index of the form Ω_a^λ on the subspace $\mathcal{K}_\pi$ is not greater then d. We recall that the index of a quadratic form q on a given subspace V is the maximum dimension of any subspace of V where the quadratic form is negative definite.

Theorem 4.1. (Necessary conditions of optimality). Let the control process

$$(x^*, u^*, w^*)$$

be a local optimal to the problem (P). Then, $\Lambda_a \neq \emptyset$ and, for any $(\delta x_0, \delta u, \delta w) \in \mathcal{K}_{cr}$, we have

$$\max_{\lambda\in\Lambda_a} \Omega^\lambda(\delta x_0, \delta u, \delta w) \ge 0. \quad (28)$$

The proof of this result is organized into several steps as follows. First, (P) is transformed into an equivalent problem whose impulsive dynamics do not depend on the state variable which can be easily converted into an abstract optimization problem. Then, the optimality conditions proved in [1] are applied, and the stated local maximum principle and second-order conditions are obtained by decoding the abstract conditions in terms of data of the original problem.

4. DISCUSSION

There is a substantial body of literature addressing impulsive optimal control, and, in particular, necessary conditions of optimality. For an excellent survey see [22] and for somewhat complementary comprehensive overviews consult [23, 11, 35, 33].

Degeneracy of the necessary conditions of optimality is an important issue that has an important history for conventional optimal control problems, [1, 9, 14]. However, this is not the case in impulsive control. One of the most distinct features of our optimality conditions relatively to comparable results in the impulsive control literature, see for example [13, 29], is the fact that they do not degenerate. While, for first-order conditions, we have to assume some assumptions concerning the controllability of the reference control process relatively the the constraints, as well as their regularity, and compatibility, for second-order conditions, the second-order information is used to restrict the set of multipliers in such a way that the conditions remain informative even for abnormal control processes.

The solution concept adopted in both results in this paper is in the spirit of the works of [5, 6, 23, 19, 26, 31]. These have their roots in the reparameterization techniques used in some of the first studies on impulsive control, [27, 24, 34], which were formulated having in mind to ensure the existence of solution to optimal control problems with unbounded controls emerging, for example, in minimum fuel and optimal space navigation problems. The existence of solution is ensured by enlarging the control space in order to include measures and the necessary conditions of optimality were derived by decoding in terms of the original data, the conditions for an equivalent conventional optimal control problem. This is generated by reparameterising time so that, for each atom of the control measure, there corresponds a new time subinterval, being the graph of the trajectory (in the new parameterization) completed according to the singular dynamics.

However, these approaches require that time be treated as an additional component of the state variable, this meaning that the time dependence and the state dependence of the data have to be of the same nature. Like in [26, 32], we derived first-order necessary conditions of optimality in the form of a maximum principle for impulsive control problems for which only the measurable dependence of the absolutely dynamics w.r.t. time is required. Although the optimality conditions derived in [28] is for problems with data also merely measurable on time, this work concerns generalized problems of Bolza with Lagrangians jointly convex in $(x, \dot{x})$ and the duality interpretations of the multipliers as well as properness assumptions are emphasized.

Unlike any of these references, the optimal impulsive control problem that we consider for first order conditions was more complete in the sense that the endpoints of the time interval are free, and state constraints have to be satisfied by any feasible trajectory. These features not only greatly increase the complexity of the optimality conditions but also make the arguments of the proof extremely more involved.

It can be seen that our first-order conditions can be extended for the case of vector valued control measures (quite common in the literature). In fact, this is the case for our result on second-order conditions where we assumed the Frobenius condition to be satisfied by the singular vector field. This property of differential geometric character is present in most results for impulsive control with vector valued measures, [5, 20, 21, 10, 15, 13], in order to ensure the correctness. One of the challenges is to derive the optimality conditions for our optimal control problem without this condition, as in [6, 26].

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OPTIMIZATION OF PARAMETERS IN THE SECOND-ORDER IMPROVING ALGORITHMS FOR OPTIMAL CONTROL PROBLEMS

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Abstract: The paper describes modifications of the second-order improving algorithms for optimal control problems. The algorithms presented are based on the localization technique and involve some parameters, which regulate the depth of the functional descent on each iteration. The modifications proposed allow one to organize a search for the best values of parameters on each of the steps.

Keywords: problem of optimal control, optimal control, controlled system of differential equations, algorithms of weak improvement

The paper discusses an algorithms of weak second-order improvement for the problem of optimal control with the free right end.

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Some modifications of the algorithm are proposed. Iteration of the algorithm consists of integration of the initial differential relation and an auxiliary nonlinear vector-matrix system of differential equations dependent on parameters. These parameters are controllers of the step, and their choice substantially influences the efficiency of operation of the algorithm. The modifi-