Hitchin Pairs for non-compact real Lie groups

by Peter B. Gothen[1]

Abstract

Hitchin pairs on Riemann surfaces are generalizations of Higgs bundles, allowing the Higgs field to be twisted by an arbitrary line bundle. We consider this generalization in the context of G-Higgs bundles for a real reductive Lie group G. We outline the basic theory and review some selected results, including recent results by Nozad and the author [32] on Hitchin pairs for the unitary group of indefinite signature $\mathrm{U}(p,q)$.

1. Introduction

Let X be a closed Riemann surface with holomorphic cotangent bundle $K = \Omega^1_X$. A rank n *Higgs bundle* on X is a pair (E,φ), where $E \to X$ is a rank n holomorphic vector bundle and $\varphi\colon E \to E \otimes K$ is an endomorphism valued holomorphic 1-form on X. Higgs bundles are fundamental objects in the non-abelian Hodge theorem [20, 22, 37, 60]. In the simplest (abelian) case of $n = 1$ this can be expressed as the isomorphism

$$\mathrm{Hom}(\pi_1 X, \mathbb{C}^*) \simeq T^* \mathrm{Jac}(X),$$

whose infinitesimal version gives the Hodge decomposition $H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X)$. Thus, for $n = 1$, a flat line bundle on X corresponds to a pair (E,φ) consisting of a holomorphic line bundle $E \to X$ and a holomorphic 1-form φ on X. For general n, non-abelian Hodge theory produces an isomorphism

$$\mathrm{Hom}(\pi_1 X, \mathrm{GL}(n,\mathbb{C}))/\!\!/\mathrm{GL}(n,\mathbb{C}) \simeq \mathcal{M}(\mathrm{GL}(n,\mathbb{C})).$$

Here the space on the right hand side is the moduli space of isomorphism classes of Higgs bundles (of degree 0) and the space on the left hand side is the space of representations of $\pi_1 X$ modulo the action of $\mathrm{GL}(n,\mathbb{C})$ by overall conjugation. Note that, viewed in this way, the non-abelian Hodge theorem generalizes the Narasimhan–Seshadri theorem [54] to non-compact groups.

For many purposes, rather than considering φ as a 1-form, one might as well consider pairs (E,φ), where $\varphi\colon E \to E \otimes L$ is twisted by an arbitrary line bundle

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$L \to X$. Such a pair is known as a *Hitchin pair* or a *twisted Higgs bundle*. This point of view was probably first explored systematically by Nitsure [55]. The non-abelian Hodge theorem generalizes to this context and involves, on the one side, meromorphic Higgs bundles and on the other side meromorphic connections. This generalization has been carried out by Simpson [61], for Higgs fields with simple poles, and Biquard–Boalch [5], for more general polar parts (see Boalch [8] for a survey).

Another generalization of the non-abelian Hodge theorem has to do with representations of $\pi_1 X$ in groups G other than the general linear group. This already goes back to Hitchin's seminal papers [37, 38, 39] and indeed was also treated by Simpson [62]. Here we shall focus on the theory for real G, which has quite a different flavour from the theory for complex G. A systematic approach to non-abelian Hodge theory for real reductive groups G and applications to the study of character varieties has been explored in a number of papers; see, for example, [30, 12, 24, 25]. The focus of the present paper are the objects which are obtained by allowing for an arbitrary twisting line bundle L in G-Higgs bundles rather than just the canonical bundle K. These objects are known as *G-Hitchin pairs*.

There are many other important aspects of Higgs bundle theory and without any pretense of completenes, we mention here a few. One of the important features of the Higgs bundle moduli space for complex G is that it is an algebraically completely integrable Hamiltonian system (see Hitchin[38]), known as the Hitchin system. This is closely related to the fact that this moduli space is a holomorphic symplectic manifold admitting a hyper-Kähler metric. This aspect of the theory can be generalized to Hitchin pairs using Poisson geometry, as pioneered by Bottacin [9] and Markman [47]; see Biquard–Boalch [5] for the existence of hyper-Kähler metrics on the symplectic leaves. Closely related is the theory of parabolic Higgs bundles (see, for example, Konno [45] and Yokogawa [67]). Parabolic G-Higgs bundles for real G have been considered by, among others, Logares [46], García-Prada–Logares–Muñoz [28] and Biquard–García-Prada–Mundet [6]. Higgs bundles also play an important role in mirror symmetry (see, for example, Hausel–Thaddeus [34]) and in the geometric Langlands correspondence (see, for example, Kapustin–Witten [42]). Also a number of results on G-Higgs bundles for real groups can be obtained via the study of the Hitchin fibration; for this we refer the reader to Baraglia–Schaposnik [4], García-Prada–Peón-Nieto–Ramanan [29], Hitchin–Schaposnik [40], Peón-Nieto [57] and Schaposnik [58], as well as further references found therein.

In this paper we describe the basics of the theory of G-Hitchin pairs and give a few examples (Section 2). We explain the Hitchin–Kobayashi correspondence which relates the (parameter dependent) stability condition for G-Hitchin pairs to solutions to Hitchin's gauge theoretic equations (Section 3). We then describe recent work of Nozad and the author [32] on $\mathrm{U}(p,q)$-Hitchin pairs (introduced in Section 4), the Milnor–Wood inequality for such pairs (Section 5) and how wall-crossing arguments can be used to study their moduli (Section 6).

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2. Hitchin pairs for real groups

Let G be a connected real reductive Lie group. Following Knapp [44], we shall take this to mean that the following data has been fixed:

- a maximal compact subgroup $H \subset G$;
- a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$;
- a non-degenerate $\mathrm{Ad}(G)$-invariant quadratic form, negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{m}$, which restricts to the Killing form on the semisimple part $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$.

Note that the above data complexify (with the possible exception of G) and that there is an isotropy representation

$$\iota\colon H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$$

coming from restricting and complexifying the adjoint representation of G.

Let X be Riemann surface and let $K = \Omega^1_X$ be its holomorphic cotangent bundle. Fix a line bundle $L \to X$. For a principal $H^{\mathbb{C}}$-bundle $E \to X$ and a representation $\rho\colon H^{\mathbb{C}} \to \mathrm{GL}(V)$ of $H^{\mathbb{C}}$, we denote the associated vector bundle by $E(V) = E \times_\rho V$.

Definition 2.1. A *G-Hitchin pair* (twisted by L) on X is a pair (E,φ), where $E \to X$ is a holomorphic principal $H^{\mathbb{C}}$-bundle and $\varphi \in H^0(X, L \otimes E(\mathfrak{m}^{\mathbb{C}}))$ is a holomorphic 1-form with values in the vector bundle defined by the isotropy representation of $H^{\mathbb{C}}$. If $L = K$, the pair (E,φ) is called a *G-Higgs bundle*.

Example 2.2. If G is compact, a G-Hitchin pair is nothing but a holomorphic principal $G^{\mathbb{C}}$-bundle.

Example 2.3. If $G = \mathrm{GL}(n,\mathbb{C})$, a G-Hitchin pair is a pair (E,φ), where $E \to X$ is a rank n holomorphic vector bundle and $\varphi \in H^0(X, L \otimes \mathrm{End}(E))$ is an L-twisted endomorphism of E. A $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle is given by the same data, with the additional requirements that $\det(E) = \mathcal{O}_X$ and $\varphi \in H^0(X, L \otimes \mathrm{End}_0(E))$, where $\mathrm{End}_0(E) \subset \mathrm{End}(E)$ is the subbundle of φ with $\varphi = 0$.

Example 2.4. Let $G = \mathrm{SL}(n,\mathbb{R})$. A maximal compact subgroup is $\mathrm{SO}(n)$ defined by the standard inner product $\langle x, y\rangle = \sum x_i y_i$ and the isotropy representation is the subspace of $A \in \mathfrak{sl}(n,R)$ which are symmetric with respect to the inner product:

$$\langle Ax, y\rangle = \langle x, Ay\rangle.$$

Hence a $\mathrm{SL}(n,\mathbb{R})$-Hitchin pair can be viewed as a pair $((U,Q),\varphi)$, where (U,Q) is a holomorphic orthogonal bundle, i.e., $U \to X$ is a rank n vector bundle with a non-degenerate holomorphic quadratic form Q, and $\varphi \in H^0(X, L \otimes S^2_Q U)$. Here $S^2_Q U \subset \mathrm{End}(U)$ denotes the subbundle of endomorphism of U, which are symmetric with respect to Q.

Example 2.5. Let $G = \mathrm{U}(p,q)$, the group of linear transformations of $\mathbb{C}^{p+q}$ which preserves an indefinite hermitian form of signature (p,q) on $\mathbb{C}^{p+q} = \mathbb{C}^p \times \mathbb{C}^q$. Taking the obvious $\mathrm{U}(p) \times \mathrm{U}(q)$ as the maximal compact subgroup, we have $H^{\mathbb{C}} = \mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$ and the isotropy representation is

$$\mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C}) \to \mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^q)$$

acting by restricting the adjoint representation of $\mathrm{GL}(p+q,\mathbb{C})$. Hence a $\mathrm{U}(p,q)$-Hitchin pair can be identified with a quadruple (V,W,β,γ), where

$$\beta \in H^0(L \otimes \mathrm{Hom}(W,V)) \quad \text{and} \quad \gamma \in H^0(L \otimes \mathrm{Hom}(V,W)).$$

The $\mathrm{GL}(p+q,\mathbb{C})$-Hitchin pair associated via the inclusion $\mathrm{U}(p,q) \subset \mathrm{GL}(p+q,\mathbb{C})$ is (E,φ), where $E = V \oplus W$ and $\varphi = \left(\begin{smallmatrix} 0 & \beta \\ \gamma & 0 \end{smallmatrix}\right)$. Of course a $\mathrm{SU}(p,q)$-Hitchin pair is given by the same data, with the additional requirement that $\det(V) \otimes \det(W) = \mathcal{O}_X$.

Example 2.6. Let $G = \mathrm{Sp}(2n,\mathbb{R})$, the real symplectic group in dimension $2n$, defined as the subgroup of $\mathrm{SL}(2n,\mathbb{R})$ of transformations of $\mathbb{R}^{2n}$ preserving the standard symplectic form, which can be written in coordinates $(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$ as

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n.$$

Then a $\mathrm{Sp}(2n,\mathbb{R})$-Hitchin pair can identified with a triple (V,β,γ), where $V \to X$ is a rank n vector bundle and

$$\beta \in H^0(L \otimes S^2 V) \quad \text{and} \quad \gamma \in H^0(L \otimes S^2 V^*).$$

Note how the inclusions $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{SL}(2n,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{R}) \subset \mathrm{SU}(n,n)$ are reflected in the associated vector bundle data. In the former case, the rank $2n$ orthogonal bundle (U,Q) is given by $U = V \oplus V^*$ with the quadratic form $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

3. The Hitchin–Kobayashi correspondence

We now move on to the central notion of stability for G-Hitchin pairs. The stability condition depends on a parameter $c \in i\mathfrak{z}$, where $\mathfrak{z}$ denotes the centre of $\mathfrak{h}$.

From the point of view of construction of moduli spaces, stability allows for a GIT construction of the moduli space $\mathcal{M}^c_d(X,G)$ of c-semistable G-Higgs bundles for a fixed topological invariant $d \in \pi_1(H)$; this construction has been carried out by Schmitt (see [59]).

On the other hand, there is a Hitchin–Kobayashi correspondence for G-Higgs bundles, which gives necessary and sufficient conditions in terms of stability for the existence of solutions to the so-called Hitchin's equations. To state these equations, we need some notation. By a hermitian metric on the $H^{\mathbb{C}}$-bundle

E we mean a reduction of structure group to $H \subset H^{\mathbb{C}}$, i.e., a smooth section $h\colon X \to E(H^{\mathbb{C}}/H)$. We denote the corresponding principal H-bundle by E_h. Note that h defines a compact real structure, denoted by σ_h, on the bundle of Lie algebras $E(\mathfrak{g}^{\mathbb{C}})$, compatible with the decomposition $E(\mathfrak{g}^{\mathbb{C}}) = E(\mathfrak{h}^{\mathbb{C}}) \oplus E(\mathfrak{m}^{\mathbb{C}})$. If we combine σ_h with the conjugation on complex 1-forms on X, we obtain a complex antilinear involution $A^1(E(\mathfrak{g}^{\mathbb{C}})) \to A^1(E(\mathfrak{g}^{\mathbb{C}}))$. This restricts to an antilinear map which, by a slight abuse of notation, we denote by the same symbol:

$$\sigma_h\colon A^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to A^{0,1}(E(\mathfrak{m}^{\mathbb{C}})).$$

Fix a hermitian metric h_L on L and let ω_X denote the Kähler form of a metric on X compatible with its complex structure, normalized so that $\int_X \omega_X = 2\pi$. Then, for $c \in i\mathfrak{z}$, Hitchin's equation for a metric h on E is the following

$$F(A_h) + [\varphi, \sigma_h(\varphi)]\omega_X = -ic\omega_X. \tag{3.1}$$

Here A_h denotes the Chern connection on E_h (i.e., the unique H-connection compatible with the holomorphic structure on E) and $F(A_h)$ its curvature. Moreover, the bracket $[\varphi, \sigma_h(\varphi)]$ is defined by combining the Lie bracket on $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$ with the contraction $L \otimes \overline{L} \to \mathcal{O}_X$ given by the metric h_L. Note also that in the case when $L = K$, the second term on the left hand side can be written simply as $[\varphi, \sigma_h(\varphi)]$ where the bracket on the Lie algebra is now combined with the wedge product on forms.

In order to state the Hitchin–Kobayashi correspondence for G-Hitchin pairs, giving necessary and sufficient conditions for the existence of solutions to the Hitchin equation, one needs an appropriate stability condition. The general condition needed can be found in [24] (based, in turn, on Bradlow–García-Prada-Mundet [14] and Mundet [41]). It is fairly involved to state in general, so we shall refer the reader to loc. cit. for the full statement and here just give a couple of examples which cover our present needs. Note that, just as the Hitchin equation, the stability condition will depend on a parameter $c \in i\mathfrak{z}$.

Example 3.1. (Cf. Hitchin[37], Simpson [60, 62].) Consider $\mathrm{GL}(n, \mathbb{C})$-Hitchin pairs (E, φ), where $E \to X$ is a rank n vector bundle and $\varphi \in H^0(X, L \otimes \mathrm{End}(E))$. Recall that the *slope* of a vector bundle E on X is the ratio between its degree and its rank: $\mu(E) = \deg(E)/\operatorname{rk}(E)$. A $\mathrm{GL}(n, \mathbb{C})$-Hitchin pair (E, φ) is *semistable* if

$$\mu(F) \leqslant \mu(E) \tag{3.2}$$

for all non-zero subbundles $F \subset E$ which are preserved by φ, i.e., such that $\varphi(F) \subset F \otimes L$. Moreover, (E, φ) is *stable* if additionally strict inequality holds in (3.2) whenever $F \neq E$. Finally, (E, φ) is polystable if it is the direct sum of stable Higgs bundles, all of the same slope. In this case $i\mathfrak{z} \simeq \mathbb{R}$ and the stability parameter is fixed to be the real constant $c = \mu(E)$. Note that this constraint is of a topological nature and can be obtained from Chern–Weil theory by integrating the trace of the Hitchin equation, which in this case is:

$$F(A_h) + [\varphi, \varphi^{*_h}]\omega_X = -ic \operatorname{Id} \omega_X.$$

Example 3.2. (Cf. [10].) Consider $\mathrm{U}(p,q)$-Hitchin pairs (V,W,β,γ). In this case, $i\mathfrak{z} \simeq \mathbb{R}\times\mathbb{R}$ and the Hitchin equation becomes

$$\begin{aligned} F(A_h(V)) + (\beta\beta^{*_h} - \gamma^{*_h}\gamma)\omega_X &= -ic_1\,\mathrm{Id}_V\,\omega_X,\\ F(A_h(W)) + (\gamma\gamma^{*_h} - \beta^{*_h}\beta)\omega_X &= -ic_2\,\mathrm{Id}_W\,\omega_X. \end{aligned} \tag{3.3}$$

Here $A_h(V)$ and $A_h(W)$ denote the Chern connections on V and W, respectively, and the parameter $(c_1,c_2)\in\mathbb{R}\times\mathbb{R}$ is constrained by Chern–Weil theory by

$$\frac{p}{p+q}c_1 + \frac{q}{p+q}c_2 = \mu(V\oplus W).$$

The stability condition is most conveniently described by introducing the *α-slope* of (V,W,β,γ) by

$$\mu_\alpha(V,W,\beta,\gamma) = \mu(V\oplus W) + \alpha\frac{p}{p+q}$$

for a real parameter α, related to (c_1,c_2) by $\alpha = c_2 - c_1$. The α-stability conditions are completely analogous to the ones of Example 3.1, but applied to $\mathrm{U}(p',q')$-subbundles, defined in the obvious way by $V'\subset V$ and $W'\subset W$ such that $\beta(W')\subset V'\otimes L$ and $\gamma(V')\subset W'\otimes L$.

The Hitchin–Kobayashi correspondence for G-Hitchin pairs [37, 62, 14, 24] can now be stated as follows.

Theorem 3.3. *Let (E,φ) be a G-Hitchin pair. There exists a hermitian metric h in E solving Hitchin's equation (3.1) if and only if (E,φ) is c-polystable. Moreover, the solution h is unique up to H-gauge transformations of E_h.*

Next we explain how to give an interpretation in terms of moduli spaces. Fix a C^∞ principal H-bundle $\mathcal{E}$ of topological class $d\in\pi_1 H$ and consider the configuration space of G-Higgs pairs on $\mathcal{E}$:

$$\mathcal{C}(\mathcal{E}) = \{(\bar\partial_A,\varphi) \;\mid\; \bar\partial_A\varphi = 0\}.$$

Here $\bar\partial_A$ is a $\bar\partial$-operator on $\mathcal{E}$ defining a structure of holomorphic principal $H^{\mathbb{C}}$-bundle $E_A\to X$ and the C^∞-Higgs field $\varphi\in A^{1,0}(\mathcal{E}(\mathfrak{m}^{\mathbb{C}}))$. Let $\mathcal{C}^{c-\mathrm{ps}}(\mathcal{E})\subset\mathcal{C}(\mathcal{E})$ be the subset of c-polystable G-Higgs pairs. The *complex gauge group* $\mathcal{G}^{\mathbb{C}}$ is the group of C^∞ automorphisms of the principal $H^{\mathbb{C}}$-bundle $\mathcal{E}_{\mathbb{C}}$ obtained by extending the structure group to the complexification $H^{\mathbb{C}}$ of H. It acts on $\mathcal{C}^{c-\mathrm{ps}}(\mathcal{E})$ and we can identify, as sets[2],

$$\mathcal{M}^c_d(X,G) = \mathcal{C}^{c-\mathrm{ps}}(\mathcal{E})/\mathcal{G}^{\mathbb{C}}.$$

Now consider Hitchin's equation (3.1) as an equation for a pair (A,φ) of a (metric) connection A on $\mathcal{E}$ and a Higgs field $\varphi\in A^{1,0}(\mathcal{E}(\mathfrak{m}^{\mathbb{C}}))$. The complex gauge group $\mathcal{G}^{\mathbb{C}}$ acts transitively on the space of metrics on $\mathcal{E}$ with stabilizer the *unitary gauge group* $\mathcal{G}$, by which we understand the C^∞ automorphism group of the H-bundle $\mathcal{E}$. Thus the Hitchin–Kobayashi correspondence of Theorem 3.3 says that there is a complex gauge transformation taking (A,φ) to a solution to

[2]Indeed a construction of the moduli space using complex analytic methods in the style of Kuranishi should be possible, though we are not aware of the existence of such a construction in the literature.

Hitchin's equation if and only if (E_A, φ) is c-polystable, and this solution is unique up to unitary gauge transformation. In other words, we have a bijection

$$\mathcal{M}_d^c(X, G) \simeq \{(A, \varphi) \mid (A, \varphi) \text{ satisfies (3.1)}\}/\mathcal{G}. \tag{3.4}$$

When G is compact, there is no Higgs field and the Hitchin equation simply says that the Chern connection is (projectively) flat. Hence (3.4) identifies the moduli space of semistable $G^{\mathbb{C}}$-bundles with the moduli space of (projectively) flat G-connections. This latter space can in turn be identified with the *character variety* of representations of (a central extension of) the fundamental group of X in G.

For non-compact G, assume that $L = K$ and that the parameter $c \in iZ(\mathfrak{g})$. Then the Hitchin equation can be interpreted as a (projective) flatness condition for the G-connection B defined by

$$B = A_h + \varphi - \sigma_h(\varphi). \tag{3.5}$$

It is a fundamental theorem of Donaldson [22] and (more generally) Corlette [20] that for any flat reductive[3] connection B on a principal G-bundle $\mathcal{E}_G$, there exists a so-called harmonic metric on $\mathcal{E}_G$. A consequence of harmonicity is that when the metric is used to decompose B as in (3.5), then (A, φ) satisfies the Hitchin equation. Combining this with the Hitchin–Kobayashi correspondence gives the non-abelian Hodge theorem[4]: an identification between the moduli space of G-Higgs bundles and the character variety for representations of (a central extension of) $\pi_1 X$ in G.

Example 3.4. If we want to apply the non-abelian Hodge theorem to $\mathrm{U}(p, q)$-Higgs bundles, we need to fix the parameter in Hitchin's equation to be in the centre of $\mathrm{U}(p, q)$, i.e., in the notation of Example 3.2, we must take $c_1 = c_2 = c = \mu(V \oplus W)$. Of course this corresponds to the value for $\mathrm{GL}(n, \mathbb{C})$-Higgs bundles under the inclusion $\mathrm{U}(p, q) \subset \mathrm{GL}(p + q, \mathbb{C})$ (cf. Examples 2.5 and 3.1).

4. Hitchin pairs for $\mathrm{U}(p, q)$ and quiver bundles

We saw in Example 3.2, that there is a degree of freedom in the choice of stability parameter for $\mathrm{U}(p, q)$-Hitchin pairs. There is another way of viewing this parameter dependence for the stability condition, which is to notice that a $\mathrm{U}(p, q)$-Hitchin pair can be viewed as a *quiver bundle* (see, e.g., King [43], Álvarez-Cónsul–García-Prada [1, 2], and also [31]). To explain this, recall that a quiver Q is an oriented graph (which we shall assume to be finite), given by a set of vertices Q_0, a set of arrows Q_1 and head and tail maps

$$h, t \colon Q_1 \to Q_0.$$

For each $a \in Q_1$, let $M_a \to X$ be a holomorphic vector bundle on X and let $M = \{M_a\}$ be the collection of these *twisting bundles*.

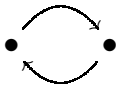
[3]When G is linear this simply means that the holonomy representation is completely reducible.

[4]See the references cited in the Introduction for the generalization to the meromorphic situation.

Definition 4.1. A *Q-bundle twisted by M* on X is a collection of holomorphic vector bundles $E_i \to X$ indexed by the vertices $i \in Q_0$ of Q and a collection of holomorphic maps $\varphi_a \colon M_a \otimes E_{ta} \to E_{ha}$ indexed by the arrows $a \in Q_1$ of Q.

Remark 4.2. It is easy to see that Q-bundles on X form a category which can be made into an abelian category by considering coherent Q-sheaves, in a way analogous to what happens for vector bundles.

It should now be clear that L-twisted $\mathrm{U}(p,q)$-Hitchin pairs can be viewed as Q-bundles for the quiver

(4.1) 

where both arrows are twisted by L^*.

There is a natural stability condition for quiver bundles which, just as for Hitchin pairs, gives necessary and sufficient conditions for the existence of solutions to certain natural gauge theoretic equations (cf. King [43] and Álvarez-Cónsul–García-Prada [1, 2]). This condition depends on a parameter vector

$$\alpha = (\alpha_i)_{i \in Q_0} \in \mathbb{R}^{Q_0}$$

and it is defined using the *α-slope* of a Q-bundle E:

$$\mu_\alpha(E) = \frac{\sum_i (\deg(E_i) + \alpha_i \operatorname{rk}(E_i))}{\sum_i \operatorname{rk}(E_i)}.$$

Thus E is α-stable if for any proper non-zero sub-Q-bundle E' of E, we have

$$\mu_\alpha(E') < \mu_\alpha(E),$$

and α-semi- and polystability are defined just as for vector bundles.

Note that the stability condition is unchanged under an overall translation of the stability parameter

$$(\alpha_i) \mapsto (\alpha_i + a)$$

for any constant $a \in \mathbb{R}$. Thus we may as well take $\alpha_0 = 0$ and we see that the number of effective stability parameters is $|Q_0| - 1$. In the case of Q-bundles for the quiver (4.1), i.e., $\mathrm{U}(p,q)$-Hitchin pairs, we then have one real parameter $\alpha = \alpha_1$ and the general Q-bundle stability condition reproduces the stability for $\mathrm{U}(p,q)$-Hitchin pairs of Example 3.2.

5. The Milnor–Wood inequality for $\mathrm{U}(p,q)$-Hitchin pairs

The Milnor–Wood inequality has its origins [48, 66] in the theory of flat bundles. From this point of view there is a long sequence of generalizations and important contributions (see, for example, Dupont [23], Toledo [64], Domic–Toledo [21], Turaev [65], Clerc–Ørsted [19], Burger–Iozzi-Wienhard [17, 18]). Here we shall, however, focus on its Higgs bundle incarnation, again first considered by Hitchin [37]. From this point of view it is a bound on the topological class of a $\mathrm{U}(p,q)$-Hitchin pair. In order to state it we need the following definition.

Definition 5.1. Let $E = (V, W, \beta, \gamma)$ be a $\mathrm{U}(p,q)$-Hitchin pair. The *Toledo invariant* of E is

$$\tau(E) = \frac{2pq}{p+q}\big(\mu(V) - \mu(W)\big).$$

Note that, if we set $a = \deg(V)$ and $b = \deg(W)$, then we can write $\tau(E) = 2(qa - pb)/(p+q)$.

The Milnor–Wood inequality for $\mathrm{U}(p,q)$-Hitchin pairs can now be stated as follows:

Proposition 5.2 (Gothen–Nozad [32, Proposition 3.3])**.** *Let $E = (V, W, \beta, \gamma)$ be an α-semistable $\mathrm{U}(p,q)$-Hitchin pair with twisting line bundle L. Then*

$$-\operatorname{rk}(\beta)\deg(L) + \alpha\big(\operatorname{rk}(\beta) - \frac{2pq}{p+q}\big) \leqslant \tau(E) \leqslant \operatorname{rk}(\gamma)\deg(L) + \alpha\big(\operatorname{rk}(\gamma) - \frac{2pq}{p+q}\big).$$

The proof is analogous to the one for $\mathrm{U}(p,q)$-Higgs bundles in [10]. It applies the α-semistability condition for $\mathrm{U}(p,q)$-Hitchin pairs to certain subobjects defined in a natural way using β and γ. We refer the reader to [32] for details.

Remark 5.3. The Toledo invariant has been defined for G-Higgs bundles for any non-compact simple reductive group G of hermitian type by Biquard–García-Prada–Rubio [7]. These authors also prove a very general Milnor–Wood inequality for such G-Higgs bundles. In the case when $L = K$ their theorem specializes to our Proposition 5.2.

The inequality of Proposition 5.2 has several interesting consequences, for example we get the following bounds on the Toledo invariant (cf. [32, Proposition 3.4]).

Proposition 5.4. *Let $E = (V, W, \beta, \gamma)$ be an α-semistable $\mathrm{U}(p,q)$-Hitchin pair with twisting line bundle L with $\deg(L) \geqslant 0$. Then the following hold:*

(i) If $\alpha \leqslant -\deg(L)$ then

$$\min\{p,q\}\big(-\alpha\frac{|p-q|}{p+q} - \deg(L)\big) \leqslant \tau(E) \leqslant -\alpha\frac{2pq}{p+q}.$$

(ii) If $-\deg(L) \leqslant \alpha \leqslant \deg(L)$ then

$$\min\{p,q\}\big(-\alpha\frac{|p-q|}{p+q} - \deg(L)\big) \leqslant \tau(E) \leqslant \min\{p,q\}\big(\deg(L) - \alpha\frac{|p-q|}{p+q}\big).$$

(iii) If $\deg(L) \leqslant \alpha$ then

$$-\alpha\frac{2pq}{p+q} \leqslant \tau(E) \leqslant \min\{p,q\}\big(\deg(L) - \alpha\frac{|p-q|}{p+q}\big).$$

Note, in particular, that for $\alpha = 0$ (the value relevant for the non-abelian Hodge theorem) we have by *(ii)* of the proposition that

$$|\tau(E)| \leqslant \min\{p,q\}\deg(L). \tag{5.1}$$

In the case of $\mathrm{U}(p,q)$-Higgs bundles (i.e., $\alpha = 0$ and $L = K$) this is the usual Milnor–Wood inequality (cf. [10]).

The study of properties of Higgs bundles with extremal values for the Toledo invariant is an interesting question. This has been studied for various specific groups G of hermitian type by Hitchin [37] for $\mathrm{PSL}(2,\mathbb{R})$, Gothen [30] for $\mathrm{Sp}(4,\mathbb{R})$, García-Prada–Gothen–Mundet [25] for $\mathrm{Sp}(2n,\mathbb{R})$, Bradlow–García-Prada–Gothen [10, 11, 16] for $\mathrm{SO}^*(2n)$ and $\mathrm{U}(p,q)$. A general study for G-Higgs bundles for non-compact groups of hermitian type was carried out by Biquard–García-Prada–Rubio [7]. From the point of view of representations of surface groups much work has also been done and without being at all exhaustive, we mention here a few works: Toledo [64], Hernández [36] and Burger–Iozzi–Wienhard [17, 18]. From either point of view, one of the key properties of maximal objects (Higgs bundles or representations) is that they exhibit rigidity phenomena, of which we mention but two examples. Firstly, a classical theorem of Toledo, which states that a maximal representation of $\pi_1 X$ in $\mathrm{U}(p,1)$ factors through $\mathrm{U}(1,1)\times\mathrm{U}(p-1)$. Secondly we mention [10, Proposition 3.30], which says that the moduli space of maximal $\mathrm{U}(p,p)$-Higgs bundles is isomorphic to the moduli space of K^2-twisted Hitchin pairs of rank p — so here Hitchin pairs play an important role even in the theory of usual Higgs bundles. Toledo's theorem and its generalizations for surface group representations have clear parallels on the Higgs bundle side of the non-abelian Hodge theory correspondence. On the other hand, the surface group representation parallel of the second kind of rigidity phenomenon is perhaps less clear; see, however, Guichard–Wienhard [33] for the case of representations in $\mathrm{Sp}(2n,\mathbb{R})$.

6. Wall crossing for $\mathrm{U}(p,q)$-Hitchin pairs

We finish this paper by describing an application of wall-crossing techniques to moduli of $\mathrm{U}(p,q)$-Hitchin pairs, following [56, 32]. These techniques have a long history in the subject, going back at least to Thaddeus' proof [63] of the rank 2 Verlinde formula. The main results on connectedness of moduli spaces of $\mathrm{U}(p,q)$-Higgs bundles from [10] were based on the wall-crossing results for *triples* of [11]: triples are Q-bundles for a quiver with two vertices and one arrow between them, so they correspond to $\mathrm{U}(p,q)$-Hitchin pairs with one of the Higgs fields β or γ vanishing. Later some of these results have been generalized to *holomorphic chains*, i.e., Q-bundles for a quiver of type A_n, see Álvarez-Cónsul–García-Prada–Schmitt [3], García-Prada–Heinloth–Schmitt [27], García-Prada–Heinloth [26] and Heinloth [35]. Similar ideas have also been employed by other authors to study various properties of moduli spaces, including their Hodge numbers, such as the works of Bradlow–García-Prada– Muñoz–Newstead [15], Bradlow–García-Prada–Mercat–Muñoz–Newstead [13], Muñoz [49, 50, 51] and Muñoz–Ortega–Vázquez-Gallo [52, 53].

One common feature of all these results is that they deal with quivers without oriented cycles, corresponding to nilpotent Higgs fields. It is therefore interesting to investigate to what extend the aforementioned results can be generalized to quivers with oriented cycles. Since we need at least two vertices to have effective

stability parameters, the simplest possible case is that of $\mathrm{U}(p,q)$-Hitchin pairs, corresponding to the quiver (4.1).

It turns out that a direct generalization of the arguments for triples of [11] runs into difficulties. To explain this, we first remark that the stability condition can only change for certain discrete values of the parameter α, called *critical values*. Fix topological invariants $t = (p, q, a, b)$ of $\mathrm{U}(p,q)$-Hitchin pairs, where $a = \deg(V)$ and $b = \deg(W)$. Then α is a critical value of the stability parameter for $\mathrm{U}(p,q)$-Hitchin pairs of type t if it is numerically possible to have a proper subobject $E' \subset E$ of a $\mathrm{U}(p,q)$-Hitchin pair $E = (V, W, \beta, \gamma)$ of type t such that

$$\mu_\alpha(E') = \mu_\alpha(E) \quad \text{and} \quad \frac{p'}{p'+q'} \neq \frac{p}{p+q} \tag{6.1}$$

(Here the type of E' is $t' = (p', q', a', b')$.) This means that α is critical if and only if it is possible for $\mathrm{U}(p,q)$-Hitchin pairs to exist which are α'-stable for $\alpha' < \alpha$ and α'-unstable for $\alpha' > \alpha$ (and vice-versa). Denote by $\mathcal{M}_{\alpha^\pm}$ the moduli space of $\alpha^\pm$-semistable $\mathrm{U}(p,q)$-Hitchin pairs of type t, where $\alpha^\pm = \alpha \pm \epsilon$ for $\epsilon > 0$ small. Then one is led to introduce "flip loci" $\mathcal{S}_{\alpha^\pm} \subset \mathcal{M}_{\alpha^\pm}$ corresponding to $\mathrm{U}(p,q)$-Hitchin pairs which change their stability properties as the critical value α is crossed. If one can estimate appropriately the codimension of these flip loci, it will follow that $\mathcal{M}_{\alpha^\pm}$ are birationally equivalent. The $\mathrm{U}(p,q)$-Hitchin pairs E in the flip loci have descriptions as extensions

$$0 \to E' \to E \to E'' \to 0$$

for α-semistable $\mathrm{U}(p,q)$-Hitchin pairs (of lower rank) E' and E'' satisfying (6.1). Such extensions are controlled by the first hypercohomology of a two-term complex of sheaves $\underline{Hom}^\bullet(E'', E')$ (see [32, Definition 2.14], cf. [31]). Thus, in order to control the number of extensions one needs vanishing results for the zeroth and second hypercohomology groups. This, together with an analysis of the moduli space for large α, was the strategy followed in [10] to prove irreducibility of moduli spaces of holomorphic triples.

The main difficulty in generalizing this approach to $\mathrm{U}(p,q)$-Hitchin pairs is that the vanishing results do not generalize without additional hypotheses (compare, for example, [11, Proposition 3.6] and [32, Proposition 3.22]). However, for a certain range of the parameter α and the Toledo invariant, things can be made to work. Thus we can obtain birationality of moduli spaces of $\mathrm{U}(p,q)$-Hitchin pairs under certain constraints (see [32, Theorem 5.3]). This combined with the results from [10] on connectedness of moduli of $\mathrm{U}(p,q)$-Higgs bundles finally gives the main result:

Theorem 6.1 ([32, Theorem 5.5])**.** *Denote by $\mathcal{M}_\alpha(p,q,a,b)$ the moduli space of semistable K-twisted $\mathrm{U}(p,q)$-Hitchin pairs. Suppose that $\tau = \frac{2pq}{p+q}(a/p - b/q)$ satisfies $|\tau| \leqslant \min\{p,q\}(2g-2)$. Suppose also that either one of the following conditions holds:*

1. *$a/p - b/q > -(2g-2)$, $q \leqslant p$ and $0 \leqslant \alpha < \frac{2pq}{pq-q^2+p+q}\big(b/q - a/p - (2g-2)\big) + 2g - 2$,*

(2) $a/p - b/q < 2g-2$, $p \leqslant q$ *and* $\frac{2pq}{pq-p^2+p+q}(b/q-a/p+2g-2)-(2g-2) < \alpha \leqslant 0$.

Then the closure of the stable locus in the moduli space $\mathcal{M}_\alpha(p,q,a,b)$ *is irreducible. In particular, if* $\gcd(p+q, a+b) = 1$*, then* $\mathcal{M}_\alpha(p,q,a,b)$ *is irreducible.*

Remark 6.2. Unless $p = q$, the conditions on $a/b - b/q$ in the theorem are guaranteed by the hypothesis $|\tau| \leqslant \min\{p,q\}(2g-2)$ (see [32, Remark 5.6]).

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