

# HOPF BIFURCATION WITH TETRAHEDRAL AND OCTAHEDRAL SYMMETRY

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**ABSTRACT.** In the study of the periodic solutions of a  $\Gamma$ -equivariant dynamical system, the  $H \bmod K$  theorem gives all possible periodic solutions, based on the group-theoretical aspects. By contrast, the equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for each  $\mathbf{C}$ -axial subgroup of  $\Gamma \times \mathbb{S}^1$ . In this paper while characterizing the Hopf bifurcation, we identify which periodic solution types, whose existence is guaranteed by the  $H \bmod K$  theorem, are obtainable by Hopf bifurcation from the origin, when the group  $\Gamma$  is either tetrahedral or octahedral. The two groups are isomorphic, but their representations in  $\mathbb{R}^3$  and in  $\mathbb{R}^6$  are not, and this changes the possible symmetries of bifurcating solutions.

**Keywords:** equivariant dynamical system ; tetrahedral symmetry; octahedral symmetry; periodic solutions ; Hopf bifurcation.

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## 1. INTRODUCTION

The formalism of  $\Gamma$ -equivariant differential equations, ie. those equations whose associated vector field commutes with the action of a finite group  $\Gamma$  has been developed by Golubitsky, Stewart and Schaeffer in [4], [7] and [6]. Within this formalism, two methods for obtaining periodic solutions have been described: the  $H \bmod K$  theorem [2, 6, Ch.3] and the equivariant Hopf theorem [6, Ch.4]. The equivariant Hopf theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all  $\mathbf{C}$ -axial subgroups of  $\Gamma \times \mathbb{S}^1$ , under some generic conditions. The  $H \bmod K$  theorem offers the complete set of possible periodic solutions based exclusively on the structure of the group  $\Gamma$  acting on the differential equation. It also guarantees the existence of a model with this symmetry having these periodic solutions, but it is not an existence result for any specific equation.

Steady-state bifurcation problems with octahedral symmetry have been analysed by Melbourne [9] using results from singularity theory. For non-degenerate bifurcation problems equivariant with respect to the standard action on  $\mathbb{R}^3$  of the octahedral group he finds three branches of symmetry-breaking steady-state bifurcations corresponding to the three maximal isotropy subgroups with one-dimensional fixed-point subspaces. Hopf bifurcation with the rotational symmetry of the tetrahedron is studied by Swift and Barany [12], motivated by problems in fluid dynamics. They find evidence of chaotic dynamics, arising from secondary bifurcations from periodic branches created at Hopf bifurcation. Generic Hopf bifurcation with the rotational symmetries of the cube is studied by Ashwin and Podvina [1], also with the motivation of fluid dynamics.

Solutions predicted by the  $H \bmod K$  theorem cannot always be obtained by a generic Hopf bifurcation from the trivial equilibrium. When the group is finite abelian, the

periodic solutions whose existence is allowed by the  $H \bmod K$  theorem that are realizable from the equivariant Hopf theorem are described in [3].

In this article, we pose a more specific question: which periodic solutions predicted by the  $H \bmod K$  theorem are obtainable by Hopf bifurcation from the trivial steady-state when  $\Gamma$  is either the group  $\langle \mathbb{T}, \kappa \rangle$  of symmetries of the tetrahedron or the group  $\mathbb{O}$  of rotational symmetries of the cube? As abstract groups,  $\langle \mathbb{T}, \kappa \rangle$  and  $\mathbb{O}$  are isomorphic, but their standard representations in  $\mathbb{R}^3$  and  $\mathbb{C}^3$  are not. However, the representations of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  and  $\mathbb{O} \times \mathbb{S}^1$  on  $\mathbb{C}^3$  are isomorphic, and this is the relevant action for dealing with equivariant Hopf bifurcation. Thus it is interesting to compare results for these two groups, and we find that indeed our question has different answers for the two groups. In particular, some solutions predicted by the  $H \bmod K$  theorem for  $\langle \mathbb{T}, \kappa \rangle$ -equivariant vector fields can only arise at a resonant Hopf bifurcation, but this is not the case for  $\mathbb{O}$ .

We will answer this question by finding for both groups that not all periodic solutions predicted by the  $H \bmod K$  theorem occur as primary Hopf bifurcations from the trivial equilibrium. For this we analyse bifurcations taking place in 4-dimensional invariant subspaces and giving rise to periodic solutions with very small symmetry groups.

**Framework of the article.** The relevant actions of the groups  $\langle \mathbb{T}, \kappa \rangle$  of symmetries of the tetrahedron,  $\mathbb{O}$  of rotational symmetries of the cube and of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  and  $\mathbb{O} \times \mathbb{S}^1$  on  $\mathbb{R}^6 \sim \mathbb{C}^3$  are described in Section 3, after stating some preliminary results and definitions in Section 2. Hopf bifurcation is treated in Section 4, where we present the results of Ashwin and Podvigina [1] on  $\mathbb{O} \times \mathbb{S}^1$ , together with the formulation of the same results for the isomorphic action of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ . This includes the analysis of Hopf bifurcation inside fixed-point subspaces for submaximal isotropy subgroups, one of which we perform in more detail than in [1], giving a geometric proof of the existence of three branches of periodic solutions, for some values of the parameters in a degree three normal form. Finally, we apply the  $H \bmod K$  theorem in Section 5 where we compare the bifurcations for the two group actions.

## 2. PRELIMINARY RESULTS AND DEFINITIONS

Before stating the theorem we give some definitions from [6]. The reader is referred to this book for results on bifurcation with symmetry.

Let  $\Gamma$  be a compact Lie group. A representation of  $\Gamma$  on a vector space  $W$  is  $\Gamma$ -*simple* if either:

- (a)  $W \sim V \oplus V$  where  $V$  is absolutely irreducible for  $\Gamma$ , or
- (b)  $W$  is non-absolutely irreducible for  $\Gamma$ .

Let  $W$  be a  $\Gamma$ -simple representation and let  $f$  be a  $\Gamma$ -equivariant vector field in  $W$ . Then it follows [7, Ch. XVI, Lemma1.5] that if  $f$  is a  $\Gamma$ -equivariant vector field, and if Jacobian matrix  $(df)_0$  of  $f$  evaluated at the origin has purely imaginary eigenvalues  $\pm\omega i$ , then in suitable coordinates  $(df)_0$  has the form:

$$(df)_0 = \omega J = \omega \begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix}$$

where  $Id$  is the identity matrix. Consider the action of  $\mathbb{S}^1$  on  $W$  given by  $\theta x = e^{i\theta J} x$ . A subgroup  $\Sigma \subseteq \Gamma \times \mathbb{S}^1$  is  $\mathbb{C}$ -*axial* if  $\Sigma$  is an isotropy subgroup and  $\dim \text{Fix}(\Sigma) = 2$ .

Let  $\dot{x} = f(x)$  be a  $\Gamma$ -equivariant differential equation with a  $T$ -periodic solution  $x(t)$ . We call  $(\gamma, \theta) \in \Gamma \times \mathbb{S}^1$  a spatio-temporal symmetry of the solution  $x(t)$  if  $\gamma \cdot x(t + \theta) = x(t)$ .

A spatio-temporal symmetry of the solution  $x(t)$  for which  $\theta = 0$  is called a spatial symmetry, since it fixes the point  $x(t)$  at every moment of time.

The main tool here will be the following theorem.

**Theorem 1** (Equivariant Hopf Theorem [6]). *Let a compact Lie group  $\Gamma$  act  $\Gamma$ -simply, orthogonally and nontrivially on  $\mathbb{R}^n$ . Assume that*

- (a)  $f : \mathbb{R}^{2m} \times \mathbb{R} \rightarrow \mathbb{R}^{2m}$  is  $\Gamma$ -equivariant. Then  $f(0, \lambda) = 0$  and  $(df)_{0,\lambda}$  has eigenvalues  $\sigma(\lambda) \pm i\rho(\lambda)$  each of multiplicity  $m$ ;
- (b)  $\sigma(0) = 0$  and  $\rho(0) = 1$ ;
- (c)  $\sigma'(0) \neq 0$  the eigenvalue crossing condition;
- (d)  $\Sigma \subseteq \Gamma \times \mathbb{S}^1$  is a  $\mathbb{C}$ -axial subgroup.

*Then there exists a unique branch of periodic solutions with period  $\approx 2\pi$  emanating from the origin, with spatio-temporal symmetries  $\Sigma$ .*

The group of all spatio-temporal symmetries of  $x(t)$  is denoted  $\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1$ . The symmetry group  $\Sigma_{x(t)}$  can be identified with a pair of subgroups  $H$  and  $K$  of  $\Gamma$  and a homomorphism  $\Phi : H \rightarrow \mathbb{S}^1$  with kernel  $K$ . We define

$$H = \{\gamma \in \Gamma : \gamma\{x(t)\} = \{x(t)\}\} \quad K = \{\gamma \in \Gamma : \gamma x(t) = x(t) \ \forall t\}$$

where  $K \subseteq \Sigma_{x(t)}$  is the subgroup of spatial symmetries of  $x(t)$  and the subgroup  $H$  of  $\Gamma$  consists of symmetries preserving the trajectory  $x(t)$  but not necessarily the points in the trajectory. We abuse notation saying that  $H$  is a group of spatio-temporal symmetries of  $x(t)$ . This makes sense because the groups  $H \subseteq \Gamma$  and  $\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1$  are isomorphic; the isomorphism being the restriction to  $\Sigma_{x(t)}$  of the projection of  $\Gamma \times \mathbb{S}^1$  onto  $\Gamma$ .

Given an isotropy subgroup  $\Sigma \subset \Gamma$ , denote by  $N(\Sigma)$  the normaliser of  $\Sigma$  in  $\Gamma$ , satisfying  $N(\Sigma) = \{\gamma \in \Gamma : \gamma\Sigma = \Sigma\gamma\}$ , and by  $L_\Sigma$  the variety  $L_\Sigma = \bigcup_{\gamma \notin \Sigma} \text{Fix}(\gamma) \cap \text{Fix}(\Sigma)$ .

The second important tool in this article is the following result.

**Theorem 2.** (*H mod K Theorem* [2, 6]) *Let  $\Gamma$  be a finite group acting on  $\mathbb{R}^n$ . There is a periodic solution to some  $\Gamma$ -equivariant system of ODEs on  $\mathbb{R}^n$  with spatial symmetries  $K$  and spatio-temporal symmetries  $H$  if and only if the following conditions hold:*

- (a)  $H/K$  is cyclic;
- (b)  $K$  is an isotropy subgroup;
- (c)  $\dim \text{Fix}(K) \geq 2$ . If  $\dim \text{Fix}(K) = 2$ , then either  $H = K$  or  $H = N(K)$ ;
- (d)  $H$  fixes a connected component of  $\text{Fix}(K) \setminus L_K$ .

*Moreover, if (a) – (d) hold, the system can be chosen so that the periodic solution is stable.*

When  $H/K \sim \mathbb{Z}_m$ , the periodic solution  $x(t)$  is called either a standing wave or (usually for  $m \geq 3$ ) a discrete rotating wave; and when  $H/K \sim \mathbb{S}^1$  it is called a rotating wave [6, page 64]. Here all rotating waves are discrete.

### 3. GROUP ACTIONS

Our aim in this article is to compare the bifurcation of periodic solutions for generic differential equations equivariant under two different representations of the same group. In this section we describe the two representations.

**3.1. Symmetries of the tetrahedron.** The group  $\mathbb{T}$  of rotational symmetries of the tetrahedron [12] has order 12. Its action on  $\mathbb{R}^3$  is generated by two rotations  $R$  and  $C$  of orders 2 and 3, respectively, and given by

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Next we want to augment the group  $\mathbb{T}$  with a reflection, given by

$$\kappa = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

to form  $\langle \mathbb{T}, \kappa \rangle$ , the full group of symmetries of the tetrahedron, that has order 24. We obtain an action on  $\mathbb{R}^6$  by identifying  $\mathbb{R}^6 \equiv \mathbb{C}^3$  and taking the same matrices as generators.

The representation of  $\langle \mathbb{T}, \kappa \rangle$  on  $\mathbb{C}^3$  is  $\langle \mathbb{T}, \kappa \rangle$ -simple. The isotropy lattice of  $\langle \mathbb{T}, \kappa \rangle$  is shown in Figure 1.

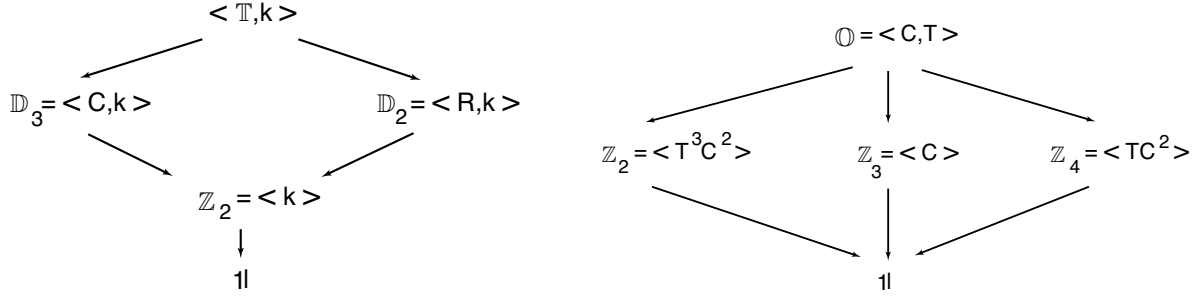


FIGURE 1. Isotropy lattices for the groups  $\langle \mathbb{T}, \kappa \rangle$  of symmetries of the tetrahedron (left) and  $\mathbb{O}$  of rotational symmetries of the cube (right).

**3.2. Symmetries of the cube.** The action on  $\mathbb{R}^3$  of the group  $\mathbb{O}$  of rotational symmetries of the cube is generated by the rotation  $C$  of order 3, with the matrix above, and by the rotation  $T$  of order 4

$$T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As in the case of the symmetries of the tetrahedron, we obtain an action of  $\mathbb{O}$  on  $\mathbb{R}^6$  by identifying  $\mathbb{R}^6 \equiv \mathbb{C}^3$  and using the same matrices.

As abstract groups,  $\langle \mathbb{T}, \kappa \rangle$  and  $\mathbb{O}$  are isomorphic, the isomorphism maps  $C$  into itself and the rotation  $T$  of order 4 in  $\mathbb{O}$  into the rotation-reflection  $C^2 R \kappa$  in  $\langle \mathbb{T}, \kappa \rangle$ . However, the two representations are not isomorphic, as can be seen comparing their isotropy lattices in Figure 1.

**3.3. Adding  $\mathbb{S}^1$ .** The corresponding actions of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  and of  $\mathbb{O} \times \mathbb{S}^1$  on  $\mathbb{C}^3$  are obtained by adding the elements  $e^{i\theta} \cdot \text{Id}$ ,  $\theta \in (0, 2\pi)$  to the group. Note that with this action the elements of  $\mathbb{S}^1$  commute with those of  $\langle \mathbb{T}, \kappa \rangle$  and of  $\mathbb{O}$ . The representations  $\mathbb{O} \times \mathbb{S}^1$  and  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  are isomorphic, the isomorphism being given by:

$$C \mapsto C \quad T \mapsto e^{i\pi} C^2 R \kappa \quad e^{i\theta} \mapsto e^{i\theta}.$$

## 4. HOPF BIFURCATION

The first step in studying  $\Gamma$ -equivariant Hopf bifurcation is to obtain the  $\mathbb{C}$ -axial subgroups of  $\Gamma \times \mathbb{S}^1$ . Isotropy subgroups of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1 \sim \mathbb{O} \times \mathbb{S}^1$  are listed in Table 1.

TABLE 1. Isotropy subgroups and corresponding types of solutions, fixed-point subspaces for the action of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1 \sim \mathbb{O} \times \mathbb{S}^1$  on  $\mathbb{C}^3$ , where  $\omega = e^{2\pi i/3}$ .

Index	Name	Isotropy subgroup	Generators in $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$	Generators in $\mathbb{O} \times \mathbb{S}^1$	Fixed-point subspace	dim
(a)	Origin	$\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$	$\{C, R, \kappa, e^{i\theta}\}$	$\{C, T, e^{i\theta}\}$	$\{(0, 0, 0)\}$	0
(b)	Pure mode	$\mathbb{D}_4$	$\{e^{\pi i} C^2 R C, \kappa\}$	$\{TC^2, e^{\pi i} T^2\}$	$\{(z, 0, 0)\}$	2
(c)	Standing wave	$\mathbb{D}_3$	$\{C, \kappa\}$	$\{C, e^{\pi i} T^2 C^2 T\}$	$\{(z, z, z)\}$	2
(d)	Rotating wave	$\mathbb{Z}_3$	$\{\bar{\omega} C\}$	$\{\bar{\omega} C\}$	$\{(z, \omega z, \omega^2 z)\}$	2
(e)	Standing wave	$\mathbb{D}_2$	$\{e^{\pi i} R, \kappa\}$	$\{e^{\pi i} TC^2 TC^2, T^3 C^2\}$	$\{(0, z, z)\}$	2
(f)	Rotating wave	$\mathbb{Z}_4$	$\{e^{\pi i/2} C^2 R \kappa\}$	$\{e^{-\pi i/2} T\}$	$\{(z, iz, 0)\}$	2
(g)	2-Sphere solutions	$\mathbb{Z}_2$	$\{e^{\pi i} R\}$	$\{e^{\pi i} TC^2 TC^2\}$	$\{(0, z_1, z_2)\}$	4
(h)	2-Sphere solutions	$\mathbb{Z}_2$	$\{\kappa\}$	$\{e^{\pi i} T^2 C^2 TC\}$	$\{(z_1, z_2, z_2)\}$	4
(i)	General solutions	$\mathbb{1}$	$\{Id\}$	$\{Id\}$	$\{(z_1, z_2, z_3)\}$	6

The normal form for a  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ -equivariant vector field truncated to the cubic order is

$$(1) \quad \begin{cases} \dot{z}_1 = z_1 (\lambda + \gamma (|z_1|^2 + |z_2|^2 + |z_3|^2) + \alpha (|z_2|^2 + |z_3|^2)) + \bar{z}_1 \beta (z_2^2 + z_3^2) = P(z_1, z_2, z_3) \\ \dot{z}_2 = z_2 (\lambda + \gamma (|z_1|^2 + |z_2|^2 + |z_3|^2) + \alpha (|z_1|^2 + |z_3|^2)) + \bar{z}_2 \beta (z_1^2 + z_3^2) = Q(z_1, z_2, z_3) \\ \dot{z}_3 = z_3 (\lambda + \gamma (|z_1|^2 + |z_2|^2 + |z_3|^2) + \alpha (|z_1|^2 + |z_2|^2)) + \bar{z}_3 \beta (z_1^2 + z_2^2) = R(z_1, z_2, z_3) \end{cases}$$

where  $\alpha, \beta, \gamma, \lambda$  are all complex coefficients. This normal form is the same used in [12] for the  $\mathbb{T} \times \mathbb{S}^1$  action, except that the extra symmetry  $\kappa$  forces some of the coefficients to be equal. The normal form (1) is slightly different, but equivalent, to the one given in [1]. As in [1, 12], the origin is always an equilibrium of (1) and it undergoes a Hopf bifurcation when  $\lambda$  crosses the imaginary axis. By the Equivariant Hopf Theorem, this generates several branches of periodic solutions, corresponding to the  $\mathbb{C}$ -axial subgroups of  $\mathbb{T} \times \mathbb{S}^1$ .

Under additional conditions on the parameters in (1) there may be other periodic solution branches arising through Hopf bifurcation outside the fixed-point subspaces for  $\mathbb{C}$ -axial subgroups. These have been analysed in [1], we proceed to describe them briefly, with some additional information from [10].

**4.1. Submaximal branches in  $\{(z_1, z_2, 0)\}$ .** As a fixed-point subspace for the  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  action, this subspace is conjugated to  $\text{Fix}(\mathbb{Z}_2(e^{\pi i} R))$ , (see Table 1) the conjugacy being realised by  $C^2$ . The same subspace appears as  $\text{Fix}(\mathbb{Z}_2(e^{\pi i} T^2))$ , under the  $\mathbb{O} \times \mathbb{S}^1$  action, and is conjugated to  $\text{Fix}(\mathbb{Z}_2(e^{\pi i} TC^2 TC^2))$  in Table 1. It contains the fixed-point subspaces  $\{(z, 0, 0)\}$ ,  $\{(z, z, 0)\}$ ,  $\{(iz, z, 0)\}$ , corresponding to  $\mathbb{C}$ -axial subgroups, as well as a conjugate copy of each one of them.

Solution branches of (1) in the fixed-point subspace  $\{(z_1, z_2, 0)\}$  have been analysed in [10, 1]. Restricting the normal form (1) to this subspace and eliminating solutions

that lie in the two-dimensional fixed-point subspaces, one finds that for  $\beta \neq 0$  there is a solution branch with no additional symmetries, if and only if both  $|\alpha/\beta| > 1$  and  $-1 < \mathcal{R}e(\alpha/\beta) < 1$  hold. These solutions lie in the subspaces  $\{(\xi z, z, 0)\}$ , with  $\xi = re^{i\phi}$ , where

$$(2) \quad \cos(2\phi) = -\mathcal{R}e(\alpha/\beta) \quad \sin(2\phi) = \pm \sqrt{1 - (\mathcal{R}e(\alpha/\beta))^2} \quad r^2 = \frac{\mathcal{I}m(\alpha/\beta) + \sin(2\phi)}{\mathcal{I}m(\alpha/\beta) - \sin(2\phi)}.$$

The submaximal branch of periodic solutions connects all the maximal branches that lie in the subspace  $\{(z_1, z_2, 0)\}$ . This can be deduced from the expressions (2), as we proceed to explain, and as illustrated in Figure 2.

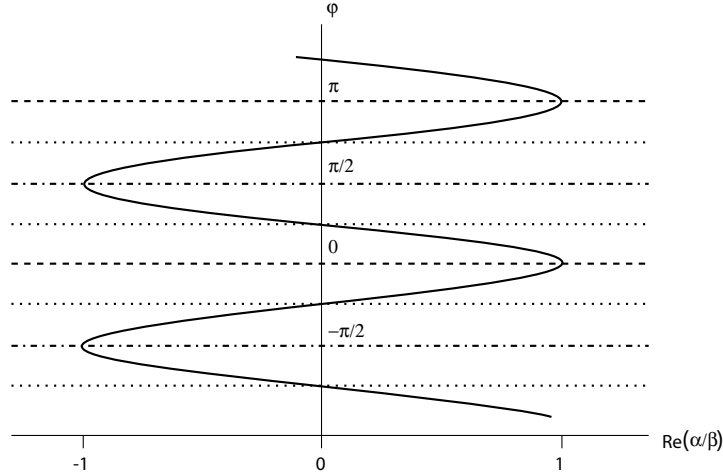


FIGURE 2. Submaximal branch in the subspace  $\{(z_1, z_2, 0)\}$  and its connection to the maximal branches. Dashed lines stand for the branches in  $\{(\pm z, z, 0)\}$  and dot-dash for the branches in  $\{(\pm iz, z, 0)\}$ . Bifurcation into the branches that lie in  $\{(0, z, 0)\}$  and  $\{(z, 0, 0)\}$ , represented by the dotted lines, only occur when both  $|\mathcal{R}e(\alpha/\beta)| = 0$  and  $|\mathcal{I}m(\alpha/\beta)| = 1$ .

When  $\mathcal{R}e(\alpha/\beta) = +1$  we get  $\xi = \pm i$  and the submaximal branches lie in the subspaces  $\{(\pm iz, z, 0)\}$ . Since the submaximal branches only exist for  $\mathcal{R}e(\alpha/\beta) \leq 1$ , then, as  $\mathcal{R}e(\alpha/\beta)$  increases to  $+1$ , pairs of submaximal branches coalesce into the subspaces  $\{(\pm iz, z, 0)\}$ , as in Figure 2.

Similarly, for  $\mathcal{R}e(\alpha/\beta) = -1$  we have  $\xi = \pm 1$ . As  $\mathcal{R}e(\alpha/\beta)$  decreases to  $-1$ , the submaximal branches coalesce into the subspace  $\{(\pm z, z, 0)\}$ , at a pitchfork.

To see what happens at  $|\alpha/\beta| = 1$ , we start with  $|\alpha/\beta| > 1$ . The submaximal branch exists when  $|\mathcal{R}e(\alpha/\beta)| < 1$  and hence  $|\mathcal{I}m(\alpha/\beta)| > 1$ . As  $|\alpha/\beta|$  decreases, when we reach the value 1 it must be with  $|\mathcal{I}m(\alpha/\beta)| = 1$  and therefore  $0 = |\mathcal{R}e(\alpha/\beta)| = -\cos(2\phi)$ , hence  $\sin(2\phi) = \pm 1$ . The expression for  $r^2$  in (2) shows that in this case either  $r$  tends to 0 or  $r$  tends to  $\infty$  and pairs of submaximal branches come together at the subspaces  $\{(0, z, 0)\}$  and  $\{(z, 0, 0)\}$ . Figure 2 is misleading for this bifurcation, because  $0 = |\mathcal{R}e(\alpha/\beta)| = -\cos(2\phi)$  does not in itself imply that the solution lies in one of the subspaces  $\{(0, z, 0)\}$ ,  $\{(z, 0, 0)\}$ . Indeed, it is possible to have  $\cos(2\phi) = 0$  and  $\sin(2\phi) = \pm 1$  with  $|\alpha/\beta| > 1$ . In this case, the right hand side of the expression for  $r^2$  in (2) is positive, and the solution branches are of the form  $(re^{i\phi} z, z, 0)$ , for some  $r > 0$  and with  $\phi = (2k + 1)\pi/4$ ,  $k \in \{1, 2, 3, 4\}$ .



From the expression for  $r^2$  in (2) it follows that the condition  $r = 1$  at the submaximal branch only occurs when  $\sin(2\phi) = 0$ . This is satisfied at the values of  $\alpha$  and  $\beta$  for which the submaximal branch bifurcates from one of the  $\mathbb{C}$ -axial subspaces, as we have already seen. This implies that the submaximal solution branches have no additional symmetry, because, for any  $\gamma$  in  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  or  $\mathbb{O} \times \mathbb{S}^1$ , the norm of the first coordinate of  $\gamma \cdot (\xi z, z, 0)$  is either  $|z|$ , or  $r|z|$ , or zero. If  $r \neq 1$  then the only possible symmetries are those that fix the subspace  $\{(z_1, z_2, 0)\}$ .

**4.2. Submaximal branches in  $\{(z_1, z_1, z_2)\}$ .** As a fixed-point subspace for the  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  action, this subspace is conjugated to  $\text{Fix}(\mathbb{Z}_2(\kappa))$ , and, under the  $\mathbb{O} \times \mathbb{S}^1$  action, as  $\text{Fix}(\mathbb{Z}_2(e^{\pi i} T^2 C^2 T C))$  (see Table 1) the conjugacy being realised by  $C^2$ . It contains the fixed-point subspaces  $\{(z, 0, 0)\}$ ,  $\{(0, z, z)\}$ ,  $\{(z, z, z)\}$ , corresponding to  $\mathbb{C}$ -axial subgroups, as well as a conjugate copy of each one of them. In what follows, we give a geometric construction for finding solution branches of (1) in the fixed-point subspace  $\{(z_1, z_1, z_2)\}$ , completing the description given in [1]. In particular, we show that the existence of these branches only depends on the parameters  $\alpha$  and  $\beta$  in the normal form, and that there are parameter values for which three submaximal solution branches coexist.

Restricting the normal form (1) to this subspace and eliminating some solutions that lie in the two-dimensional fixed-point subspaces, one finds that for  $\beta \neq 0$  there is a solution branch through the point  $(z, z, \xi z)$  with  $\xi = e^{-i\psi}$ , if and only if

$$(3) \quad \beta - \alpha + r^2 \alpha + \beta (r^2 e^{-2i\psi} - 2e^{2i\psi}) = 0.$$

For  $(x, y) = (\cos 2\psi, \sin 2\psi)$ , this is equivalent to the conditions (4) (5) below:

$$(4) \quad \mathcal{R}(x, y) = (x - x_0)^2 - (y - y_0)^2 + K_r > 0$$

$$K_r = [(3 \operatorname{Im}(\alpha/\beta))^2 - (\operatorname{Re}(\alpha/\beta) - 1)^2 + 12 \operatorname{Re}(\alpha/\beta)] / 16$$

and

$$(5) \quad \mathcal{I}(x, y) = (x - x_0)(y - y_0) + K_i = 0 \quad K_i = \frac{3}{16} \operatorname{Im}(\alpha/\beta) [\operatorname{Re}(\alpha/\beta) + 1]$$

where

$$(6) \quad x_0 = (1 - 3\operatorname{Re}(\alpha/\beta)) / 4 \quad y_0 = \operatorname{Im}(\alpha/\beta) / 4 \quad \text{and} \quad r^2 = \mathcal{R}(x, y) / |\alpha + \beta e^{-2i\psi}|^2.$$

Thus, solution branches will correspond to points  $(x, y)$  where the (possibly degenerate) hyperbola  $\mathcal{I}(x, y) = 0$  intersects the unit circle, subject to the condition  $\mathcal{R}(x, y) > 0$ . There may be up to four intersection points. By inspection we find that  $(x, y) = (1, 0)$  is always an intersection point, where either  $\psi = 0$  or  $\psi = \pi$ . Substituting into (3), it follows that  $r = 1$ . These solutions corresponds to the two-dimensional fixed point subspace  $\{(z, z, z)\}$  and its conjugate  $\{(z, z, -z)\}$ .

Solutions that lie in the fixed-point subspace  $\{(z, z, 0)\}$  correspond to  $r = 0$  and hence to  $\mathcal{R}(x, y) = 0$ . The subspace  $\{(0, 0, z)\}$  corresponds to  $r \rightarrow \infty$  in (3), i.e. to  $e^{2i\psi} = (1 - \alpha/\beta) / 2$ , hence  $\mathcal{I}(x, y) = 0$  if and only if either  $\operatorname{Im}(\alpha/\beta) = 0$  or  $\operatorname{Re}(\alpha/\beta) = 3/2$ .

Generically, the fact that  $(1, 0)$  is an intersection point implies that  $\mathcal{I}(x, y) = 0$  meets the circle on at least one more point. We proceed to describe some of the situations that may arise.

When  $\operatorname{Re}(\alpha/\beta) = -1$ , we have  $K_i = 0$  and hence  $\mathcal{I}(x, y) = 0$  consists of the two lines  $x = 1$  and  $y = y_0$ , shown in Figure 3. The line  $x = 1$  is tangent to the unit circle, hence, when  $|\operatorname{Im}(\alpha/\beta)| \leq 4$ , the intersection of  $\mathcal{I}(x, y) = 0$  and the circle consists of three

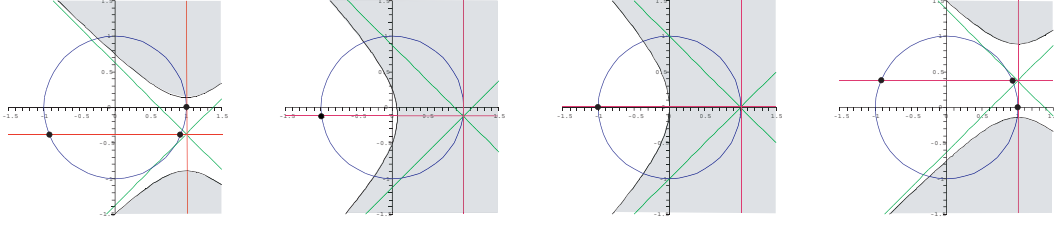


FIGURE 3. For  $\operatorname{Re}(\alpha/\beta) = -1$  and  $|\operatorname{Im}(\alpha/\beta)| \leq 4$ , the set  $\mathcal{I}(x, y) = 0$  consists of two lines (red), meeting the unit circle (blue) at two or three points that correspond to periodic solution branches (black dots) if  $\mathcal{R}(x, y) > 0$  (outside gray area). Green lines are asymptotes to  $\mathcal{R}(x, y) = 0$ . Graphs, from left to right are for  $\operatorname{Im}(\alpha/\beta) = -3/2; -1/2; 0; +3/2$ .

points, except when  $\operatorname{Im}(\alpha/\beta) = 0$  (Figure 3, left) and in the limit cases  $\operatorname{Im}(\alpha/\beta) = \pm 4$ , when two solutions come together at a saddle-node. The other case when  $\mathcal{I}(x, y) = 0$  consists of two lines parallel to the axes occurs when  $\operatorname{Im}(\alpha/\beta) = 0$ , when solutions lie in the subspace  $\{(0, 0, z)\}$ .

In the general case, when  $\operatorname{Re}(\alpha/\beta) \neq -1$ ,  $\operatorname{Im}(\alpha/\beta) \neq 0$  (Figure 4), the curve  $\mathcal{I}(x, y) = 0$  is a hyperbola intersecting the unit circle at  $(x, y) = (1, 0)$  and on one to three other points. Again, these other intersections may correspond to submaximal branches or not, depending on the sign of  $\mathcal{R}(x, y)$ . Examples with one, two and three submaximal branches are shown in Figure 4. These branches bifurcate from the fixed-point subspace  $\{(z, z, 0)\}$ , when the intersection meets the line  $\mathcal{R}(x, y) = 0$ , or from the subspaces  $\{(z, z, \pm z)\}$ . A pair of branches may also terminate at a saddle-node bifurcation.

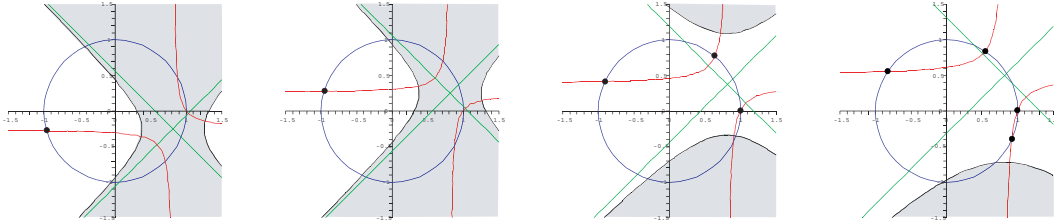


FIGURE 4. Generically the curve  $\mathcal{I}(x, y) = 0$  is a hyperbola (red curves) shown here for  $\operatorname{Re}(\alpha/\beta) = -3/4$  and, from left to right, for  $\operatorname{Im}(\alpha/\beta) = -1; +1; +3/2; +2$ . The hyperbola intersects the unit circle (blue) at up to four points, that will correspond to periodic solution branches (black dots) if  $\mathcal{R}(x, y) > 0$  (outside gray area). Green lines are asymptotes to  $\mathcal{R}(x, y) = 0$ .

## 5. SPATIO-TEMPORAL SYMMETRIES

The  $H$  mod  $K$  theorem [2, 6] states necessary and sufficient conditions for the existence of a  $\Gamma$ -equivariant differential equation having a periodic solution with specified spatial symmetries  $K \subset \Gamma$  and spatio-temporal symmetries  $H \subset \Gamma$ , as explained in Section 2.



TABLE 2. Spatio-temporal symmetries of solutions arising through primary Hopf bifurcation from the trivial equilibrium and number of branches, for the action of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  on  $\mathbb{C}^3$ . The index refers to Table 1. Subgroups of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  below the dividing line are not  $\mathbb{C}$ -axial.

index	subgroup $\Sigma \subset \langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ generators	Spatio-temporal symmetries $H$ generators	Spatial symmetries $K$ generators	number of branches
(b)	$\{e^{\pi i} C^2 R C, \kappa\}$	$\{C^2 R C, \kappa\}$	$\{R, \kappa\}$	3
(c)	$\{C, \kappa\}$	$\{C, \kappa\}$	$\{C, \kappa\}$	4
(d)	$\{e^{-2\pi i/3} C\}$	$\{C\}$	$\mathbb{1}$	8
(e)	$\{e^{\pi i} R, \kappa\}$	$\{R, \kappa\}$	$\{\kappa\}$	6
(f)	$\{e^{\pi i/2} C^2 R \kappa\}$	$\{C^2 R \kappa\}$	$\mathbb{1}$	6
(g)	$\{e^{\pi i} R\}$	$\{R\}$	$\mathbb{1}$	12
(h)	$\{\kappa\}$	$\{\kappa\}$	$\{\kappa\}$	12

TABLE 3. Spatio-temporal symmetries of solutions arising through primary Hopf bifurcation from the trivial equilibrium and number of branches, for the action of  $\mathbb{O} \times \mathbb{S}^1$  on  $\mathbb{C}^3$ . The index refers to Table 1. Subgroups of  $\mathbb{O} \times \mathbb{S}^1$  below the dividing line are not  $\mathbb{C}$ -axial.

index	subgroup $\Sigma \subset \langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ generators	Spatio-temporal symmetries $H$ generators	Spatial symmetries $K$ generators	number of branches
(b)	$\{TC^2, e^{\pi i} T^2\}$	$\{TC^2, T^2\}$	$\{TC^2\}$	3
(c)	$\{C, e^{\pi i} T^2 C^2 T\}$	$\{C, T^2 C^2 T\}$	$\{C\}$	4
(d)	$\{e^{-2\pi i/3} C\}$	$\{C\}$	$\mathbb{1}$	8
(e)	$\{e^{\pi i} TC^2 TC^2, T^3 C^2\}$	$\{TC^2 TC^2, T^3 C^2\}$	$\{T^3 C^2\}$	6
(f)	$\{e^{-\pi i/2} T\}$	$\{T\}$	$\mathbb{1}$	6
(g)	$\{e^{\pi i} TC^2 TC^2\}$	$\{TC^2 TC^2\}$	$\mathbb{1}$	12
(h)	$\{e^{\pi i} T^2 C^2 TC\}$	$\{T^2 C^2 TC\}$	$\mathbb{1}$	12

For a given  $\Gamma$ -equivariant differential equation, the  $H \bmod K$  gives necessary conditions on the symmetries of periodic solutions. Not all these solutions arise by a Hopf bifurcation from the trivial equilibrium — we call this a *primary Hopf bifurcation*. In this session we address the question of determining which periodic solution types, whose existence is guaranteed by the  $H \bmod K$  theorem, are obtainable at primary Hopf bifurcations, when the symmetry group is either  $\langle \mathbb{T}, \kappa \rangle$  or  $\mathbb{O}$ .

The first step in answering this question is the next lemma:

**Lemma 1.** *Pairs of subgroups  $H, K$  of symmetries of periodic solutions arising through a primary Hopf bifurcation for  $\Gamma = \langle \mathbb{T}, \kappa \rangle$  are given in Table 2 and for  $\Gamma = \mathbb{O}$  in Table 3.*

*Proof.* The symmetries corresponding to the  $\mathbb{C}$ -axial subgroups of  $\Gamma \times \mathbb{S}^1$  provide the first five rows of Tables 2 and 3. The last two rows correspond to the submaximal branches found in 4.1 and 4.2 above.  $\square$

TABLE 4. Possible pairs  $H, K$  for Theorem 2 in the action of  $\langle \mathbb{T}, \kappa \rangle$  on  $\mathbb{C}^3$ .

$K$	Generators of $K$	$H$	Generators of $H$	$\text{Fix}(K)$	$\dim$
$\mathbb{D}_2$	$\{R, \kappa\}$	$K$ $N(K) = \mathbb{D}_4$	$\{R, \kappa\}$ $\{C^2 R C, \kappa\}$	$\{(z, 0, 0)\}$	2
$\mathbb{D}_3$	$\{C, \kappa\}$	$N(K) = K$	$\{C, \kappa\}$	$\{(z, z, z)\}$	2
$\mathbb{Z}_2$	$\{\kappa\}$	$K$ $N(K) = \mathbb{D}_2$	$\{\kappa\}$ $\{R, \kappa\}$	$\{(z_1, z_2, z_2)\}$	4
$\mathbb{1}$	$\{Id\}$	$\mathbb{Z}_4$ $\mathbb{Z}_3$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ $K$	$\{C^2 R \kappa\}$ $\{C\}$ $\{R\}$ $\{\kappa\}$ $\{Id\}$	$\mathbb{C}^3$	6

The next step is to identify the subgroups corresponding to Theorem 2.

**Lemma 2.** *Pairs of subgroups  $H, K$  satisfying conditions (a) – (d) of Theorem 2 are given in Table 4 for  $\Gamma = \langle \mathbb{T}, \kappa \rangle$  and in Table 5 for  $\Gamma = \mathbb{O}$ .*

*Proof.* Conditions (a) – (d) of Theorem 2 are immediate for the isotropy subgroups with two-dimensional fixed-point subspaces. For  $\Gamma = \mathbb{O}$ , these are all the non-trivial subgroups. For  $\Gamma = \langle \mathbb{T}, \kappa \rangle$ , condition (d) has to be verified for  $K = \mathbb{Z}_2$ , that has a four-dimensional fixed-point subspace. In this case  $L_K = \{(z, z, z)\} \cup \{(-z, z, z)\} \cup \{(z, 0, 0)\}$ , and since  $\dim \text{Fix}(\mathbb{Z}_2(\kappa)) = 4$ , and  $L_K$  consists of two-dimensional subspaces, it follows that  $\text{Fix}(K) \setminus L_K$  is connected and condition (d) follows.

For  $K = \mathbb{1}$  we have that  $L_K$  is the union of a finite number of subspaces of dimensions 2 and 4, and again we have condition (d) because  $\text{Fix}(K) \setminus L_K = \mathbb{C}^3 \setminus L_K$  is connected. In this case, condition (a) is the only restriction and  $H$  may be any cyclic subgroup of  $\Gamma$ .  $\square$

Of all the subgroups of  $\langle \mathbb{T}, \kappa \rangle$  of order two that appear as  $H$  in a pair  $H \sim \mathbb{Z}_n, K = \mathbb{1}$  in Table 4, only  $H \sim \mathbb{Z}_3(C)$  is an isotropy subgroup, as can be seen in Figure 1. The two subgroups of  $\langle \mathbb{T}, \kappa \rangle$  of order two,  $H = \mathbb{Z}_2(\kappa)$  and  $H = \mathbb{Z}_2(R)$  are not conjugated, since  $\kappa$  fixes a four dimensional subspace, whereas  $R$  fixes a subspace of dimension two. In contrast, all but one of the cyclic subgroups of  $\mathbb{O}$  are isotropy subgroups, as can be seen comparing Table 5 to Figure 1, and noting that  $\mathbb{Z}_4(T)$  is conjugated to  $\mathbb{Z}_4(TC^2)$  and that  $\mathbb{Z}_2(T^2C^2TC)$  is conjugated to  $\mathbb{Z}_2(T^3C^2)$ . This will have a marked effect on the primary Hopf bifurcations.

**Proposition 1.** *For the representation of the group  $\langle \mathbb{T}, \kappa \rangle$  on  $\mathbb{R}^6 \sim \mathbb{C}^3$ , all pairs of subgroups  $H, K$  satisfying the conditions of the  $H \bmod K$  Theorem, with  $H \neq \mathbb{1}$ , occur as spatio-temporal symmetries of periodic solutions arising through a primary Hopf bifurcation, except for:*

- (1) the pair  $H = K = \mathbb{D}_2$ , generated by  $\{R, \kappa\}$ ;
- (2) the pair  $H = \mathbb{Z}_2(\kappa), K = \mathbb{1}$ .

*Proof.* The result follows by inspection of Tables 2 and 4, we discuss here why these pairs do not arise in a primary Hopf bifurcation. Case (1) refers to a non-trivial isotropy subgroup  $K \subset \langle \mathbb{T}, \kappa \rangle$  for which  $N(K) \neq K$ . For the group  $\langle \mathbb{T}, \kappa \rangle$ , there are two non-trivial

TABLE 5. Possible pairs  $H, K$  for Theorem 2 in the action of  $\mathbb{O}$  on  $\mathbb{C}^3$ .

$K$	Generators of $K$	$H$	Generators of $H$	$\text{Fix}(K)$	$\dim$
$\mathbb{Z}_4$	$\{TC^2\}$	$K$ $N(K) = \mathbb{D}_4$	$\{TC^2\}$ $\{T^2, TC^2\}$	$\{(z, 0, 0)\}$	2
$\mathbb{Z}_3$	$\{C\}$	$K$ $N(K) = \mathbb{D}_3$	$\{C\}$ $\{C, T^2C^2T\}$	$\{(z, z, z)\}$	2
$\mathbb{Z}_2$	$\{T^3C^2\}$	$K$ $N(K) = \mathbb{D}_2$	$\{T^3C^2\}$ $\{TC^2TC^2, T^3C^2\}$	$\{(0, z, z)\}$	2
$\mathbb{1}$	$\{Id\}$	$\mathbb{Z}_4$ $\mathbb{Z}_3$ $\mathbb{Z}_2$ $\mathbb{Z}_2$ $K$	$\{T\}$ $\{C\}$ $\{TC^2TC^2\}$ $\{T^2C^2TC\}$ $\{Id\}$	$\mathbb{C}^3$	6

isotropy subgroups in this situation, as can be seen in Table 4. The first one,  $K = \mathbb{D}_2$ , is not an isotropy subgroup of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ , so the pair  $H = K = \mathbb{D}_2$  does not occur in Table 2 as a Hopf bifurcation from the trivial solution in a normal form with symmetry  $\langle \mathbb{T}, \kappa \rangle$ . The second subgroup,  $K = \mathbb{Z}_2(\kappa)$  occurs as an isotropy subgroup of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  with four-dimensional fixed-point subspace. As we have seen in 4.2, in this subspace there are periodic solutions with  $K = H = \mathbb{Z}_2(\kappa)$  arising through a Hopf bifurcation with submaximal symmetry. On the other hand, the normaliser of  $\mathbb{Z}_2(\kappa)$  corresponds to a  $\mathbb{C}$ -axial subgroup of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ , so there is a Hopf bifurcation from the trivial solution with  $H = N(K)$ .

Case (2) concerns the situation when  $K = \mathbb{1}$ . All the cyclic subgroups  $H \subset \langle \mathbb{T}, \kappa \rangle$ , with the exception of  $\mathbb{Z}_2(\kappa)$ , are the projection into  $\langle \mathbb{T}, \kappa \rangle$  of cyclic isotropy subgroups of  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$ , so they correspond to primary Hopf bifurcations.  $\square$

Another reason why the pair  $H = K = \mathbb{D}_2$  does not occur at a primary Hopf bifurcation is the following: a non-trivial periodic solution in  $\text{Fix}(\mathbb{D}_2)$  has the form  $X(t) = (z(t), 0, 0)$ . Then  $Y(t) = C^2RCX(t) = -X(t)$  is also a solution contained in the same plane. If the origin is inside  $X(t)$  then the curves  $Y(t)$  and  $X(t)$  must intersect, so they coincide as curves, and this means that  $C^2RC$  is a spatio-temporal symmetry of  $X(t)$ . Hence, if  $H = K = \mathbb{D}_2$ , then  $X(t)$  cannot encircle the origin, and hence it cannot arise from a Hopf bifurcation from the trivial equilibrium. Of course it may originate at a Hopf bifurcation from another equilibrium. The argument does not apply to the other subgroup  $K$  with  $N(K) \neq K$ , because in this case  $\dim \text{Fix}(K) = 4$  and indeed, for some values of the parameters in the normal form, there are primary Hopf bifurcations into solutions with  $H = K = \mathbb{Z}_2(\kappa)$ .

The second case in Proposition 1 is more interesting: if  $X(t) = (z_1(t), z_2(t), z_3(t))$  is a  $2\pi$ -periodic solution with  $H = \mathbb{Z}_2(\kappa)$ ,  $K = \mathbb{1}$ , then for some  $\theta \neq 0 \pmod{2\pi}$  and for all  $t$  we have  $z_1(t + \theta) = z_1(t)$  and  $z_3(t + \theta) = z_2(t) = z_2(t + 2\theta)$ . If  $z_2(t)$  and  $z_3(t)$  are identically zero, then  $X(t) \in \text{Fix}(\mathbb{D}_2)$  and hence  $K = \mathbb{D}_2$ , contradicting our assumption. If  $z_2(t)$  and  $z_3(t)$  are non-zero, then  $\theta = \pi$  and in then  $RX(t) = (z_1(t), -z_2(t), -z_3(t))$  is also a solution whose trajectory may not intersect  $\{X(t)\}$ , since  $R \notin H$ . The possibilities are then of the form  $X(t) = (z_1(t), z_2(t), z_2(t + \pi))$  with a  $\pi$ -periodic  $z_1$ . If  $z_1(t) \equiv 0$ , then

$X(t)$  cannot be obtained from the  $\langle \mathbb{T}, \kappa \rangle \times \mathbb{S}^1$  action, because in this case it would be  $X(t) \in \text{Fix}(\mathbb{Z}_2(e^{\pi i} R))$ , hence  $R \in H$ . The only other possibility is to have all coordinates of  $X(t)$  not zero, with  $z_2(t)$  and  $z_3(t)$  having twice the period of  $z_1(t)$ , so if they do arise at a Hopf bifurcation, it will be at a 2-1 resonance. This last situation does not occur for the group  $\mathbb{O}$ , as can be seen in the next result.

**Proposition 2.** *For the representation of the group  $\mathbb{O}$  on  $\mathbb{R}^6 \sim \mathbb{C}^3$ , all pairs of subgroups  $H, K$  satisfying the conditions of the  $H \bmod K$  Theorem, with  $H \neq 1$ , occur as spatio-temporal symmetries of periodic solutions arising through a primary Hopf bifurcation, except for the following pairs:*

- (1)  $H = K = \mathbb{Z}_4$ , generated by  $\{TC^2\}$ ;
- (2)  $H = K = \mathbb{Z}_3$  generated by  $\{C\}$ ;
- (3)  $H = K = \mathbb{Z}_2$  generated by  $\{T^3C^2\}$ .

*Proof.* As in Proposition 1, the result follows comparing Tables 3 and 5.

All the cyclic subgroups  $H \subset \mathbb{O}$  are the projection into  $\mathbb{O}$  of cyclic isotropy subgroups  $\Sigma$  of  $\mathbb{O} \times \mathbb{S}^1$ , so the pairs  $H, K = \mathbb{Z}_n, 1$  correspond to primary Hopf bifurcations: for  $\mathbb{Z}_4$  and  $\mathbb{Z}_3$  the subgroup  $\Sigma \subset \mathbb{O} \times \mathbb{S}^1$  is  $\mathbb{C}$ -axial, whereas for the subgroups of  $\mathbb{O}$  of order two, the subspace  $\text{Fix}(\Sigma)$  is four-dimensional and has been treated in 4.1 and 4.2 above.

All the non-trivial isotropy subgroups  $K$  of  $\mathbb{O}$  satisfy  $N(K) \neq K$ , so they are candidates for cases where  $H = K$  does not occur. This is indeed the case, since they are not isotropy subgroups of  $\mathbb{O} \times \mathbb{S}^1$ .  $\square$

The pairs  $H, K$  in Proposition 2, that are not symmetries of solutions arising through primary Hopf bifurcation, are of the form  $H = K$ , where there would be only spatial symmetries. Here,  $\dim \text{Fix}(K) = 2$  for all cases. As remarked after the proof of Proposition 1, the origin cannot lie inside a closed trajectory with these symmetries, hence they cannot arise at a primary Hopf bifurcation.

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