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Optimal Control of Infinite-Horizon Growth Models — a direct approach*

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Abstract

We propose a framework to solve dynamic nonlinear infinite-horizon models like those found in the standard economic growth literature. We employ a direct method to solve the underlying optimal control problem, something novel in the economic literature. Instead of deriving the necessary optimality conditions and solving the originated ordinary differential equations, this method first discretizes and then optimizes, in effect transforming the problem into a nonlinear programming problem to be optimized at each sampling instant. We incorporate the work of Fontes (2001) in order to transform the infinite-horizon problem into an equivalent finite-horizon representation of the model. This framework presents several advantages in comparison to the available alternatives that use indirect methods. First, no linearization is required, which sometimes can be erroneous. The problem can be studied in its nonlinear form. Secondly, it enables the simulation of a shock when the economy is not at its steady state, a broad assumption required by all available numerical methods. Thirdly, it allows for the easy study of anticipated shocks. It also allows for the analysis of multiple, sequential shocks. Finally, it is extremely robust and easy to use. We illustrate the application of the framework by solving the standard Ramsey-Cass-Koopsman exogenous growth model and the Uzawa-Lucas endogenous two-sector growth model.

JEL Code: C61, C63, O40.

Keywords: optimal control, direct methods, transitional dynamics, economic growth, non-steady state shocks, sequential shocks.

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1 Introduction

Optimal control theory has been extensively applied to the solution of economic problems since the pioneering article of Arrow (1968). Its main application in economics is usually found within dynamic macroeconomic theory, more specifically in the repertoire of both exogenous and endogenous economic growth models, like in Ramsey (1928); Uzawa (1965); Lucas (1988); Romer (1994), to name just a few. At their core, such models typically incorporate microeconomic foundations and involve multiple distinct and decentralized optimization problems over an infinite-horizon. This formulation gives rise to a system of nonlinear differential equations describing the economy. Nonlinearity usually arises from the diminishing marginal utility of consumption and from the diminishing marginal productivity of the factors of production, but can also be the result of incorporating R&D (Williams and Jones, 1995) or government spending (Barro, 1990).

The study of these dynamic growth models follows a standard procedure, which consists in applying Pontryagin's Maximum Principle and obtaining the necessary optimality conditions (NOC), along with the transversality condition. If we define initial values for the state variables we have a complete description of the system, enough to expound the economy around its steady state equilibrium. But nonlinearity in the production and utility functions and saddle-path stability introduced by the forward-looking assumption of agents, while not posing much of a problem in the characterization of the static short-run, can indeed affect the analysis of the transitional dynamics following a structural change or a policy shock, as Atolia et al. (2010) points out. According to the author, one way of overcoming this issue is to linearize the dynamic system around its (post-shock) steady state and then to study the properties of this linearized, thus simplified, version of the dynamic system as a proxy to the original nonlinear system.

Linearization can be extremely misleading, though. In fact, this mathematical stunt may yield specious predictions and lead to erroneous qualitative assessments, even when probed near the steady state. Wolman and Couper (2003) had already identified the problem. They point out three main pitfalls of excess of reliance in the linearization process: (i) *spurious nonexistence*, that is, results suggesting that there is no nonexplosive solution when global analysis shows that one in fact exists; (ii) *spurious existence*, i.e., a unique equilibrium found in the linearized version of the model while no equilibria indeed exists for a wide range of initial conditions; and (iii) *spurious uniqueness*, when linearization gives origin to a unique nonexplosive equilibrium when in fact there are multiple nonexplosive equilibria. One particular model clearly drives this point home. Futagami et al. (2008) devised a model to study the long-run growth effect of borrowing for public investment. If the public debt target is defined as a ratio to private capital ($\bar{b} \equiv B/K$), as they originally do, then the model exhibits a multiplicity of balanced growth paths (BGP) in the long run and a possible indeterminacy of the transition path to the high-growth BGP. If, on the other hand, one follows Minea and Villieu (2012) and opts to define the public debt target as a ratio to output ($\bar{b} \equiv B/Y$) then the model exhibits a unique BGP and a unique adjustment path to equilibrium. This contrast in results highlights the fallacious influence linearization can have on the interpretation of a model.

Atolia et al. (2010) conducted an extensive analysis on this precise subject. Expectedly, they show that the further away the economy is from a steady-state equilibrium, the larger are the errors generated by linearization, both qualitatively and quantitatively. Furthermore, in models where government expenditure is introduced in the form of stock of public capital, the linearized model over-predicts consumption and welfare gains from an increase in public investment, providing an extremely erroneous signal to the policy maker. More worryingly, the errors can be of a qualitative nature. The authors give the example of when the linear proxy predicts a short-run increase in consumption and the original, nonlinear model solution shows a decline. Their conclusion is remarkably important — linearization is potentially quite misleading. Notwithstanding, linearization is still the dominant practice adopted in the macro-growth literature.

In fact, up until some time ago and even with linearization at our disposal, the transitional dynamics of models with stiff differential equations or giving origin to a center manifold had hardly been investigated, as Trimborn et al. (2004) duly noted. They confer added importance to the transition process in growth models, insofar as the positive and normative implications might differ dramatically depending on whether an economy converges towards a BGP or grows along it, as we have already noted. Moreover, conducting welfare comparisons between different policy regimes or instruments depends on studying this process. Consequently, some numerical procedures started emerging to overcome this problem. The "projection method" (Judd, 1992), the "discretization method" (Mercenier and Michel, 1994), the "shooting method" (Judd, 1998), the "time elimination method" (Mulligan and Sala-i Martin, 1991), the "backward integration" procedure (Brunner and Strulik, 2002) and the "relaxation procedure" (Trimborn et al., 2004), this last one now widely used in the economic growth literature.

Although more reliable than linearization, the aforementioned procedures rely on *indirect* methods to solve the intrinsic optimal control problem of the growth model. Indirect methods consist of calculus of variations, which function by first applying Pontryagin's Maximum Principle and introducing an adjoint variable for each state variable in order to obtain the NOCs from the first order conditions of the Hamiltonian, along with the transversality condition. According to Betts (2010, Chapter 4.3), indirect methods might suffer from the following issues: a) the NOCs have to be computed explicitly, and an explicit expression for $u^*(t)$ (the Euler-Lagrange equation) might be hard to determine, especially in systems with singular arcs; this is also far from flexible, since a new derivation is required each time the problem is changed; b) problems with path inequalities require an estimation of the constrained-arc sequence, which can be a considerably cumbersome task; c) the basic method is not robust, requiring an initial guess for the adjoint variables, which if done properly can still lead to ill-conditioned adjoint equations. Economic models are usually conceived with this in mind and sometimes simplicity is forced on to them so that an analytical solution can be determined, which might not always be the case. But most importantly, the transitional dynamics are always studied as a departure from a steady state. Real-world economies have yet to reach the theoretical boundary set by a steady state[1], and the adjustment trajectories differ sharply for a shock that hits an economy that is at a non-steady state, as we will see in Section 4.

Given this, we propose a framework that transforms the infinite-horizon problem into an equivalent finite-horizon representation so that a *direct* method (control discretization) can be employed to solve the underlying infinite-horizon optimal control problem. The procedure incorporates the seminal work of Fontes (2001). It is worth stressing the main advantages of this approach: (i) this procedure solves a nonlinear programming problem (NLP) and not its linear representation; (ii) it is capable of solving complex problems where the NOCs are hard to determine; (iii) allows for a non-steady state analysis of the transitional dynamics; (iv) anticipated shocks can be studied without introducing discontinuities or reformulating the original problem; (v) it allows for the study of multiple, sequential shocks (either expected or not); (vi) it is extremely easy to use, as no knowledge of the analytical trajectories is required *a priori* and everything is handled numerically; (vii) it is extremely robust, as the optimal control problem is solved using well-established and tested NLP solvers.

Our contribution is a significant alternative to either the linearization approach and to the above mentioned numerical procedures that suffer from the perils common to the indirect methods. It opens the way for the study of the transitional dynamics for models in a non-steady state, while keeping intact all the properties inherent to a nonlinear model. Also, it makes it easier for the analysis of anticipated shocks like policy measures which are usually announced well before being enforced. Since the stylized analysis of unexpected shocks does not take into account possible anticipatory actions conducted by the agents, this is a viable and straightforward way of examining the possible behavior.

The remainder of this paper is structured as follows. Section 2 introduces the models that will be used as a testbed for the framework, namely the neoclassical Ramsey growth model (Ramsey, 1928; Cass, 1965; Koopmans, 1963, – henceforth, RCK) and the endogenous two-sector growth model of (Uzawa, 1965; Lucas, 1988, – henceforth, UL). Section 3 exposes the framework and its theoretical background, along with the numerical solution of the selected growth models. Section 4 focus in the transitional dynamics of the models and shows the impulse responses upon expected shocks and multiple, sequential shocks. Also, it compares the effect of a shock when the economy is *not* at a steady state. Finally, a brief overview of our findings can be found in Section 5.

2 Models of economic growth

In order to illustrate the use of the framework we will employ the RCK growth model, a model exhibiting exogenous growth, and the UL endogenous growth model. The neoclassical growth

[1] It is interesting to ask whether infinite growth is possible on a world with finite resources. If it is not, economies will eventually reach a steady-state of no further growth, with capital growing just at the rate required to compensate for depreciations and to equip new borns.

model exhibits saddle-point stability, for which a closed-form solution exists for a particular choice of parameters. This will allow us to compare the accuracy of the numerical results obtained with the analytical solution of the system of differential equations. The second model exhibits multi-dimensional stable manifold and is considerably more complex. In fact, the transition process of this growth model has hardly been investigated due to its intrinsic complexity. These models are a powerful workhorse for studying some of the mechanisms of growth.

2.1 Neoclassical growth model

We will consider the version of the model as defined by Barro and Sala-i Martin (2003). Our population grows according to $L(t) = L(0) \cdot e^{nt}$, normalized to unity at $t = 0$. The households wish to maximize their overall utility U by the means of consumption, C. Also, we consider a current-value formulation with a discount factor ρ. All variables are time-dependent so we will omit the subscripts. This can be summarized as follows

$$U = \int_0^{\infty} u(C) \cdot e^{(n-\rho)t} \cdot dt, \quad C \in [0, +\infty) \tag{1}$$

Families hold assets b and obtain capital gains from assets, rb, and wages from working, w. Labor supply is inelastic and no unemployment exists. The budget constraint, in per capita terms, is then represented by the following equation:

$$\dot{b} = (r - n)b + w - c \tag{2}$$

Also, utility is given by a constant inter-temporal elasticity of substitution (CIES) function form

$$u(c) = \frac{C^{(1-\theta)} - 1}{1 - \theta} \tag{3}$$

The -1 term implies that $\lim_{\theta \to \infty} u(C) = ln(C)$. To avoid numerical errors from potential divisions by zero we can replace the CIES function with the equivalent

$$u(C) = \begin{cases} \frac{C^{1-\theta}}{1-\theta} & \theta \neq 1 \\ ln(C) & \theta = 1 \end{cases} \tag{4}$$

We will be working with per effective worker ratios, so we need to transform the variable consumption C. Henceforth, we consider $\hat{L} \equiv L \cdot X$ to be "effective labor", the product of raw labor and the level of technology. Let us denote c as consumption per unit of effective labor such that $c \equiv C/\hat{L}$. The technological progress grows at rate x, such that $X = X(0) \cdot e^{xt}$. We normalize $X(0)$ to unity and so we obtain:

$$\frac{C^{1-\theta}}{1-\theta} = \frac{e^{(1-\theta)xt} \cdot c^{1-\theta}}{1-\theta} \tag{5}$$

This results in the following objective function, rewritten in consumption per effective worker quantities:

$$U = \int_0^{\infty} \frac{c^{1-\theta}}{1-\theta} \cdot e^{(n+(1-\theta)x-\rho)t} dt \tag{6}$$

The goods to be consumed are produced by firms by employing labor and capital. We assume a standard Cobb-Douglas production function $Y \equiv F(K, \hat{L}) = AK^{\alpha}\hat{L}^{1-\alpha}$ with $0 < \alpha < 1$ and A the level of technology. We include labor-augmenting technological progress at a constant rate, which we already know that grows at rate x. Like with consumption, we will express all our variables in quantities per unit of effective labor, $y \equiv Y/\hat{L}$ and $k \equiv K/\hat{L}$. The output of the economy can then be expressed in the intensive form

$$y = f(k) = Ak^{\alpha}, \;\; f(0) = 0 \tag{7}$$

Goods and labor markets clear. From this assumption we know that supply and demand quantities meet. This implies that the supply of loans b is met by the demand of capital, k. Knowing this we can get the resource constraint for the overall economy, expressed in units of effective labor:

$$\dot{k} = f(k) - c - (x + n + \delta)k, \quad k(0) = k_0, \;\; k \geq 0 \tag{8}$$

Equations (6), (7) and (8) sum up the interactions between agents in the Ramsey growth model.

2.2 Uzawa-Lucas endogenous growth model

The neoclassical growth model falls short of explaining the engine of long-term growth in income per capita observed in developed countries. The introduction of technological progress causes such phenomenon to occur but provides no explanation on its origin. Endogenous growth models were conceived as an attempt to overcome such theoretical fragility and to give a consistent account to what causes economies to keep on growing. One prominent endogenous growth model was developed by Uzawa (1965) and later used by Lucas (1988). We will follow the formulation laid by Barro and Sala-i Martin (2003).

The Uzawa-Lucas model introduces human capital h, another productive input of the economy that is produced by a different technology than that of physical capital K. Also, labor L can either be employed in final output production, μ, with the remaining share $1 - \mu$ dedicated to formal education. This model provides for a very comprehensive assessment of the capabilities of our framework. For one, it exhibits steady-state growth, meaning that consumption and capital (physical and human) is unbounded. Secondly, the introduction of human capital and specialized labour adds another state and control variable to our system, respectively, which results in increased complexity. In fact, the transition process of this model is still unclear, since the indirect methods for solving the underlying optimal control problem employed by researchers give origin to stiff ordinary differential equations, again adding an extra burden to the task of the analyst, as Trimborn et al. (2004) notes.

This model uses the Ramsey consumption-optimizer framework specified in Section 2.1. As before, households try to maximize their utility by consuming according to a standard CIES function $u(C)$:

$$U = \int_0^\infty u(C) \cdot e^{-\rho t} dt \tag{9}$$

Goods are produced according to the following production function:

$$Y = AK^\alpha (\mu h L)^{1-\alpha} \tag{10}$$

Physical capital K and human capital H growth follow the laws of motion:

$$\begin{aligned} \dot{K} &= Y - C - \delta_K K \\ \dot{h} &= B(1-\mu)h - \delta_H h \end{aligned} \tag{11}$$

where $B > 0$ is a constant reflecting productivity of quality adjusted effort in education and δ_H $(0 \leq \delta_H < B)$ is the rate of depreciation of human capital, which will be set to $\delta_h = 0$.

Assuming per-capita variables $k \equiv K/L$, $y \equiv Y/L$ and $c \equiv C/L$ and no population growth $(n = 0)$ we end up with the following system describing our economy

$$\max U = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} dt, \quad s.t. \tag{12}$$

$$\begin{array}{lcll} c & > & 0, & 0 \leq \mu \leq 1 \\ \dot{k} & = & Ak^\alpha (\mu h)^{1-\alpha} - c - \delta_k k & \\ \dot{h} & = & B(1-\mu)h & \\ k(0) & = & k_0 & k \geq 0, \ \forall t > 0 \\ h(0) & = & h_0 & h \geq 0, \ \forall t > 0 \end{array} \tag{13}$$

with the social planner or household choosing an allocation $(c, \mu)_{t=0}^\infty$ that maximizes U.

3 A framework for infinite-horizon models

The framework we present derives from the work of Fontes (2001), originally conceived to tackle industrial optimal control problems of similar characteristics. The procedure is as follows: we transcribe the infinite horizon problem into a finite dimensional, NLP representation of the initial problem, which we prove to be an equivalent representation of the original. We then use a state-of-the-art NLP solver to find the optimal trajectories. We are in fact first discretizing and then optimizing, inverting the process used by indirect methods.

This approach overcomes the problems already identified in Section 1 while introducing several degrees of freedom. The optimal trajectories can be numerically determined without requiring the linearization of the differential equations. This, in itself, avoids the problems posed by a change of base or any other supposedly neutral manipulations. Also, it allows for the study of the transition process without having the system depart from or be at a steady state. Actually, the economy can depart from any given state. Furthermore, it is a powerful tool to study complex phenomena like anticipated or multiple, sequential shocks.

Indeed, one does not need to derive or even know any NOCs, which can be especially useful for problems whose adjoint functions are hard to determine. In fact, this is the current method of choice in the field of optimal control for engineering applications due to its easy applicability and robustness. Oddly so, direct methods are still novelty in growth theory.

We start by showing the theorem and the proof and then we do a numerical implementation and solution of two mainstream economic growth models to serve as an example.

Theorem 1. *Consider the following generic optimal control problem.*

$$P_\infty: \ \min \int_0^\infty L(x(t), u(t)) \cdot dt, \tag{14}$$

subject to:

$\dot{x} = f(x, u) \quad a.e.$

$x(0) = x_0,$

$x(t) \in \Gamma(t),$

$u(t) \in \Omega(t)$

for which we assume there is a finite solution. Assume additionally that after some time T, the state is within some invariant set S (that is, $x(t) \in S, \ S \subset \Gamma(t)$, for all $t \geq T$) for which the problem still has a finite solution. Then, there exists a terminal cost function W, such that the problem is equivalent to the finite horizon problem

$$P_T: \ \min \int_0^T L(x(t), u(t)) \cdot dt + W(x(T)), \tag{15}$$

subject to:

$\dot{x} = f(x, u) \quad a.e.$

$x(0) = x_0,$

$x(t) \in \Gamma(t),$

$u(t) \in \Omega(t),$

$x(T) \in S.$

Proof. Consider problem P_∞ with the additional assumption, i.e., redefine $\Gamma(t) = S$ for all $t \geq T$. The value for P_∞ is

$$V(0, x_0) = \min \left\{ \int_0^T L(x, u(t)) \cdot dt + V(T, x(T)) \right\}$$

We can now define

$$W(x(T)) = V(T, x(T)) = \min_{x \in S} \left\{ \int_T^{+\infty} L(x, u) \cdot dt \right\}$$

and rewrite the problem as P_T. □

3.1 Application

The above theorem is useful only for the case that the set S is such that a characterization of the solution to the problem

$$W(x(T)) = \min_{x \in S} \left\{ \int_T^{+\infty} L(x,u) \cdot dt \right\}$$

is possible (via NCO or other means) and therefore we can explicitly compute $W(x)$ for any $x \in S$.

One such example is when economies are at a BGP, either because the characteristics of the problem always lead to it, or because we assume (or impose) that to be the case. We will exemplify how to use the framework on two of such models, previously described in Section 2.1 and in Section 2.2.

The procedure is as follows:

1. Transcribe the infinite-horizon problem into an equivalent finite-horizon problem by applying Theorem 1;
2. Add the necessary boundary conditions that ensure that the set S is invariant (and so $W(x)$ exists for any $x \in S$);
3. Use a NLP solver to determine the trajectories of the control and state variables.

3.1.1 Neoclassical growth model

Consider the Ramsey-Cass-Koopmans growth problem described in Section 2.1. If we impose that $\rho > n + (1-\theta)x$ so that the model converges to a BGP, with $k(t) = k^*$ constant for $t \geq T$, then define

$$\begin{aligned} &S = \{k : \ \dot{k} = 0\} \Leftrightarrow \\ &S = \{k : \ A(k^*)^\alpha - c^* - (\delta + n + x)k^* = 0\}, \quad k(T) \in S \end{aligned} \tag{16}$$

where k^* and c^* are given by the values obtained at instant T, i.e., $k^* = k(T)$ and $c^* = c(T)$. Equation (16) is the invariant set S required by Theorem 1.

We are now in a position to compute W, the *boundary cost.* From Theorem 1 we know that

$$W(k^*) = \min \int_T^{+\infty} L(k^*, c^*) dt$$

where $L \equiv U$. Since we are maximizing, and max $= -$ min, it becomes

$$W = -\max \int_T^{+\infty} U = U(\cdot, \infty) - U(\cdot, T) \tag{17}$$

We apply a discount factor ρ and ensure that $\rho > n + (1-\theta)x$ must hold true for all t, so $U(\cdot, \infty)$ is 0. Hence, W will be equal to $-U(\cdot, T)$.

Since U is defined by equation 6, it is trivial to integrate equation (17) and obtain the following value for the boundary cost

$$W = -\frac{e^{(n+(1-\theta)x-\rho)t}}{\rho - n - (1-\theta)x} \cdot \frac{c^{(1-\theta)}}{1-\theta} \tag{18}$$

and the boundary condition

$$k(T) \in S \Leftrightarrow \dot{k} = 0 \Leftrightarrow Ak(T)^{\alpha} - c(T) - (n + x + \delta)k(T) = 0 \tag{19}$$

Numerical solution The next step is the implementation of the transcribed finite-horizon problem as described in the previous section on a NLP solver. In order to numerically solve this problem we will make use of the Imperial College London Optimal Control Software[2] (ICLOCS). As a general guideline, we also show how to specify the model as required by ICLOCS. Of course, another interface to an NLP solver could be used and these instructions apply solely to this case.

The stage cost `L` is given by equation (3). The ordinary differential equation `f` is the resource constraint of the economy in units of effective labor, defined in equation (8). The boundary cost `E` is defined in equation (18). Finally, the boundary condition `BC` is defined in equation (19).

To run a simulation we will use the following parameterization. We will consider a fixed 300 year timespan (`tmin=300`, `tmax=300`), enough for the model to grow and converge, since it exhibits saddle-point stability. Without loss of generality, we will define the initial stock of private capital k to be ten percent of its steady state value ($k^* \simeq (\frac{\alpha A}{\delta+\rho+\theta x})^{1/(1-\alpha)},\ k_0 = 0.1k^*$). State bounds are also fairly loose. k can assume any non-negative value since there is no economic definition of negative capital stock ($0 \leq k < \infty$) and the corner solution ($k = 0$) is non-optimal. In the end, capital should approach its steady state value ($k(T) \simeq k^*$). As for the input boundaries, we will just say that consumption per unit of effective labor c is limited between zero and infinity ($0 \leq c < \infty$). Again, a negative consumption has no meaning in economic terms. In the end, we will compare consumption to its steady state value ($c^* \simeq A(k^*)^{\alpha} - (n+x+\delta)k^*,\ c(T) \simeq c^*$). Likewise, the same will be done with the output of the economy, which should converge to its steady state value ($y^* \simeq A(k^*)^{\alpha}$).

For the parameterization of the model we will strictly follow the benchmark set by Barro and Sala-i Martin (2003), reported in Table 1. These are standard values.

Parameter	α	θ	ρ	n	x	δ
Value	0.3/0.6	3	0.02	0.02	0.02	0.02

Table 1: Calibrated parameters for the numerical simulation of the RCK model. Values taken from (Barro and Sala-i Martin, 2003).

Figure 1 depicts the results obtained. As expected, the stock of private capital k, the consumption c and the output of the economy y in units of effective labor all converge to their

[2]http://www.ee.ic.ac.uk/ICLOCS/ — This software solves optimal control problems with general path and boundary constraints and free or fixed final time. It uses another intermediary piece of software called Interior Point Optimizer (IPOPT) to solve the transformed NLP problem.

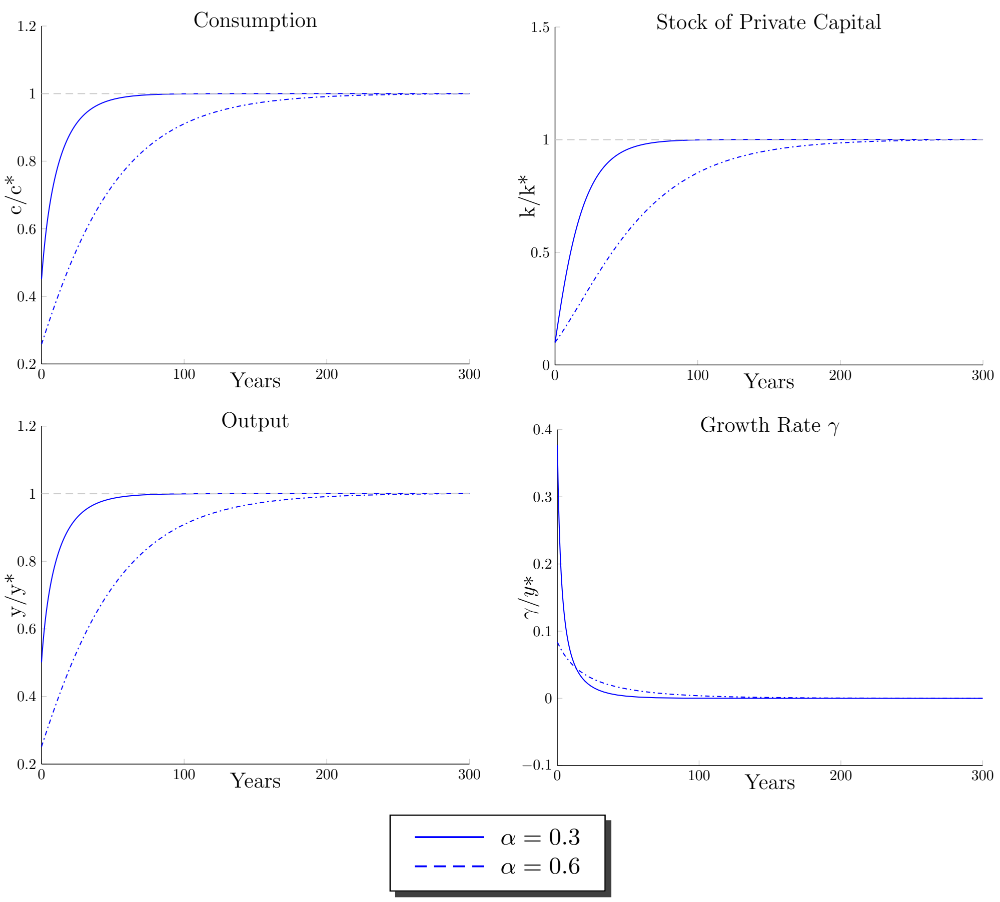

Figure 1: Numerical trajectories for the Ramsey-Cass-Koopmans model, using Barro and Sala-i Martin (2003) benchmark.

steady state values k^*, c^*, y^*. These results are fully in line with the ones obtained by Barro and Sala-i Martin (2003).

3.1.2 Uzawa-Lucas endogenous growth model

As before, consider the Uzawa-Lucas growth model described in Section 2.2 by equations (12) and (13).

We know that in balanced growth the following holds true for our specification of the model

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{h}}{h} = \frac{\dot{y}}{y} = \gamma$$

If we define $\omega \equiv k/h$ and $\chi \equiv c/k$, we know that $\dot{\omega} = 0$ and $\dot{\chi} = 0$ holds true in the steady state, for $t \geq T$[3]. Define the control policy $c(t)$ and $u(t)$ such that

$$\begin{aligned} c(t) &= \chi k(t) \\ k(t) &= \omega h(t) \end{aligned} \tag{20}$$

[3] See Appendix 5B of Barro and Sala-i Martin (2003) for a concise explanation of the mathematical properties of the model.

and $\chi,\ \omega$ are both equal to a given positive constant.

We have that

$$\frac{\dot{k}}{k} = Ak^{\alpha-1}(uh)^{1-\alpha} - (\chi+\delta) = (\frac{uh}{Ak})^{1-\alpha} - (\chi+\delta) = (\frac{u}{A\omega})^{1-\alpha} - (\chi+\delta) = \gamma$$

and

$$\frac{\dot{h}}{h} = \frac{\chi\dot{k}}{\chi k} = \frac{\dot{k}}{k} = \gamma$$

On the other hand,

$$\frac{\dot{c}}{c} = \frac{\chi\dot{k}}{\chi k} = \frac{\dot{k}}{k} = \gamma$$

and

$$\frac{\dot{k}}{k} = \frac{\omega\dot{h}}{\omega h} = B(1-u) = \gamma$$

so (20) is achieved with $u(t) = u$ constant satisfying

$$B(1-u) = (\frac{u}{A\omega})^{1-\alpha} - (\chi+\delta)$$

Moreover we have

$$\frac{\dot{c}}{c} = \frac{\dot{k}}{k} = \frac{\dot{h}}{h} = B(1-u)$$

which implies that $\dot{c} = \gamma c$ in a BGP. At the end time ($t = T$) consumption c will continue to grow at rate γ, according to

$$c(t) = c(T) \cdot e^{\gamma(t-T)},\ t \in [T, +\infty) \tag{21}$$

which enable us to compute W. Utility will be bounded as long as $\rho > \gamma(1-\theta)$, meaning that $U(\cdot, \infty) = 0$. The boundary cost is then given by integrating equation (12) incorporating the definition of $c(t)$ from equation (21). It then becomes

$$W = -\int_T^\infty \frac{(c(T)e^{\gamma(t-T)})^{1-\theta}}{1-\theta} e^{-\rho t} dt$$

which we know to be equal to

$$W = -\frac{e^{\gamma(1-\theta)-\rho}}{\rho - \gamma(1-\theta)} \cdot \frac{[c(T)e^{-\gamma T}]^{1-\theta}}{1-\theta} \tag{22}$$

The boundary condition will then become

$$S = \{(k,h) \in \mathbb{R} :\ \frac{\dot{k}}{k} - \frac{\dot{h}}{h} = 0\} \quad k(T),\ h(T) \in S \tag{23}$$

Numerical solution The system as defined by equations (12) and (13) along with the boundary cost (22) and the boundary condition (23) is all that is required to solve the model numerically. The model was run with the parameters set to those of Table 2.

Parameter	A	B	α	θ	ρ	δ_K	δ_H
Value	1	0.136	0.3	3	0.03	0.05	0

Table 2: Calibrated parameters for the Uzawa-Lucas model.

Figure 2 depicts the optimization results obtained upon running the model from an arbitrary starting point $(k_0,\ h_0)$. As can easily be seen and is expected, $c,\ k,\ h$ exhibit constant growth when the system is in equilibrium. On the contrary, ω and χ converge to a stationary state, as expected. Also, we see that in this economy approximately 2/3 of labor will be employed in producing final goods, while the remaining 1/3 will be developing human capital.

3.2 Evaluation

In order to assess the quality of the numerical results obtained we will follow a now standard approach in the literature to measure the accuracy of numerical methods. This is the procedure conducted in other studies like those of Aruoba et al. (2006) or Heer and Maußner (2008). We will calculate the residual against a closed-form linear analytical solution of the RCK model. Since the other numerical methods use an indirect approach, the residual they calculate is for the Euler equation, the ordinary differential equation that describes how consumption evolves over time. The Euler residual provides a (unit-free) measure of the percentage error in the consumption trajectory of the household. In our case, we can directly compare the trajectory for consumption $c(t)$ we obtain numerically against the one determined analytically. This is in fact a more robust comparison, since the Euler equation is mostly concerned with the asymptotic properties of the accuracy of the numerical solutions. As Atolia et al. (2010) duly point out, this might be of interest to the DSGE literature, but not so much to growth theory, since we are concerned with the complete transitional path.

For the closed-form analytical solution of the RCK model we will follow Brunner and Strulik (2002). They show that for the particular case when $(\alpha\delta)/(\delta+\rho) = 1/\theta$ holds true, the consumer will select a constant savings rate of $s = 1/\theta$ and the solution of the model is

$$k(t) = [\frac{s}{\delta} + (k_0^{1-\alpha} - \frac{s}{\delta})e^{-\delta(1-\alpha)t}]^{\frac{1}{1-\alpha}} \tag{24}$$

and $c(t) = (1-s)k(t)^{\alpha}$. For this particular case the authors assume no technological progress and no population growth. Accordingly, the parameters were set as reported in Table 3 and the residuals of the trajectories for the consumption $c(t)$ were calculated for the interval $[0.1k^*, 2k^*]$.

Parameter	α	θ	ρ	n	x	δ
Value	0.3	4.6(7)	0.02	0	0	0.05

Table 3: Calibrated parameters as set byBrunner and Strulik (2002) for the numerical simulation of a particular closed-form solution to the RCK exogenous growth model.

Figure 3 depicts the logarithm of the consumption's trajectory $c(t)$ residual against the stock of private capital, k. Like Aruoba et al. (2006) and Ambler and Pelgrin (2010), we opt for

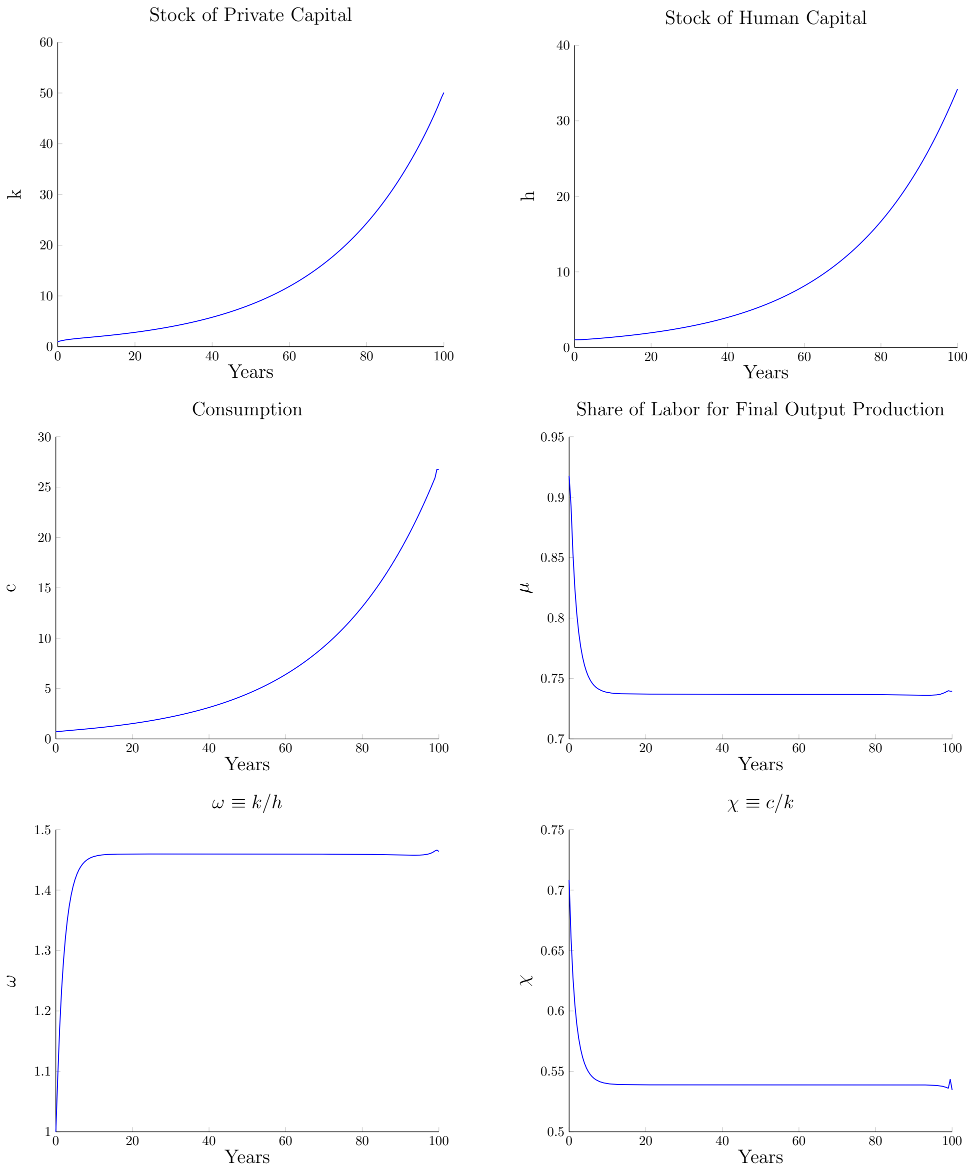

Figure 2: Numerical optimization of the Uzawa-Lucas endogenous growth model.

reporting the absolute errors in base 10 logarithms as it facilitates the economic interpretation. A value of -3 means a \$1 mistake for each \$1000 spent, a value of -4 a \$1 mistake for each \$10 000, and so on. These results go in line with Euler residuals obtained for other numerical procedures, most of them identified in this paper and summarized in Aruoba et al. (2006). It is worth noting that near the steady state ($k^* \simeq 7.96$) the log residual error of -16 is most neglectable.

	$\alpha = 0.3\ \theta = 4.66(7)$			
	Max Abs Error		$(x_A(T) - x_N(T))/x_0$ Error	
	$k_0 = 0.1k^*$	$k_0 = 2k^*$	$k_0 = 0.1k^*$	$k_0 = 2k^*$
c	0.00610	0.00117	$0.88182e^{-4}$	$0.2549e^{-4}$
k	0.01638	0.00271	0.00353	$0.1653e^{-4}$
y	0.00208	$0.1895e^{-3}$	$0.21159e^{-3}$	$0.8056e^{-4}$
	$\alpha = 0.5\ \theta = 2.8$			
	Max Abs Error		$(x_A(T) - x_N(T))/x_0$ Error	
	$k_0 = 0.1k^*$	$k_0 = 2k^*$	$k_0 = 0.1k^*$	$k_0 = 2k^*$
c	0.01097	0.00512	0.00115	$0.37462e^{-4}$
k	0.08491	0.02317	0.00713	$0.16750e^{-3}$
y	0.00828	0.00162	0.00113	$0.11841e^{-3}$

Table 4: Maximum absolute errors and errors as a percentage of the initial pre-shock equilibrium value for two distinct benchmarks, one using the values $\alpha = 0.3,\ \theta = 4.66(7)$ and the other using $\alpha = 0.5,\ \theta = 2.8$.

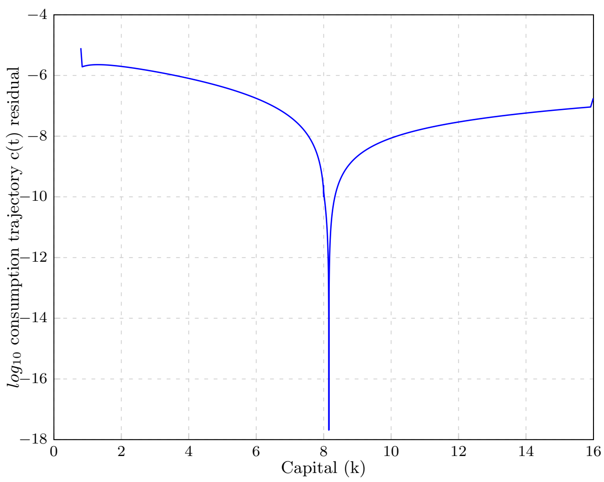

Figure 3: log_{10} of the residual of the consumption trajectory.

Moreover, in Table 4 we present maximum absolute errors for the trajectories of consumption $c(t)$, stock of private capital $k(t)$ and output of the economy $y(t)$. We also show a measure of the errors as a percentage of the initial pre-shock equilibrium value, namely by calculating the ratio $\frac{x_A(T) - x_N(T)}{x_0}$, a procedure also followed by Atolia et al. (2010).

The results obtained, along with the residual for the consumption trajectory, are extremely satisfactory when in comparison to all the other available procedures, according to the results published by Aruoba et al. (2006).

4 Transitional dynamics

The framework we propose is especially useful for the analysis of the transitional dynamics arising from policy or structural shocks. Without any reformulation of the problem, one can easily study expected or unexpected shocks, either departing from a steady state or not. Moreover, it is also extremely easy to study a sequence of multiple shocks. The innovation is that the economy does not have to converge to a new steady-state before a new shock can be applied. Shocks can occur at any given time.

In order to exemplify how to use the framework to study the transition dynamics we will extend the RCK model from Section 2.1 with the introduction of proportional taxes on wage income, τ_w, private asset income, τ_r, and consumption, τ_c. We follow Barro and Sala-i Martin (2003). This time we assume no technological progress, with no loss of generality.

This extension requires a change to equation (2), the budget constraint of the households. The budget constraint will then become

$$\dot{b} = (1-\tau_w)w + (1-\tau_r)rb - (1+\tau_c)c - nb \tag{25}$$

with $r = \alpha k^{\alpha-1} - \delta$ and $w = (1-\alpha)k^{\alpha}$. Since markets clear with $b = k$, equation (25) is also the global constraint of the economy, which assumes the following form

$$\dot{k} = (1-\tau_w)(1-\alpha)k^{\alpha} + (1-\tau_r)\alpha k^{\alpha} - (1+\tau_c)c - (n+\delta)k \tag{26}$$

These modifications allow us to introduce exogenous shocks by manipulating the policy variables $\{\tau_w, \tau_r, \tau_c\}$ and therefore study how the economy copes with a certain expected or unexpected change of policy.

For the parameterization we will consider the values specified in Table 5.

Parameter	α	θ	ρ	n	x	δ	τ_w	τ_r	τ_c
Value	0.3	2	0.02	0.01	0	0.03	0.4	0.3	0.1

Table 5: Calibrated parameters.

4.1 Expected shocks

In this particular case, the agents show perfect foresight, i.e., it is assumed that at the point in time when the maximization occurs, the maximizing agent is aware of the whole set of information. If that holds true, then it is also true that it knows the future time path of variables exogenous to the model, like those of the tax rates. We will first consider a simulation of an expected shock to the RCK model and then to the UL model.

To study such shocks, authors like Trimborn (2007) suggest a reformulation of the optimization problem, namely by decomposing the functional form of the objective function from $f^{(1)}$ to $f^{(2)}$ and the state equations from $g^{(1)}$ to $g^{(2)}$ at time $\tilde{t}$, when the shock occurs. The necessary optimality conditions would have to be augmented with the conditions derived from the interior boundary condition, that for this case are $\psi[\tilde{t}] = \tilde{t} - t_{jump} = 0$. Moreover, the adjoint variable functions introduced by the Maximum Principle have to attend to a continuity requirement, also known as the Weierstrass-Erdmann corner condition. For further details, Bryson and Ho (1975) provide an exhaustive explanation.

We do not require any reformulation of the RCK model. The only step needed is to set a change of τ_c to 20% at time $t = 20$. Again, we will depart from the steady state, with $k_0 = k^*$.

In Figure 4 we present the impulse responses of when an expected policy shock takes place. The results come as no surprise and are in line with the ones obtained by Trimborn (2007).

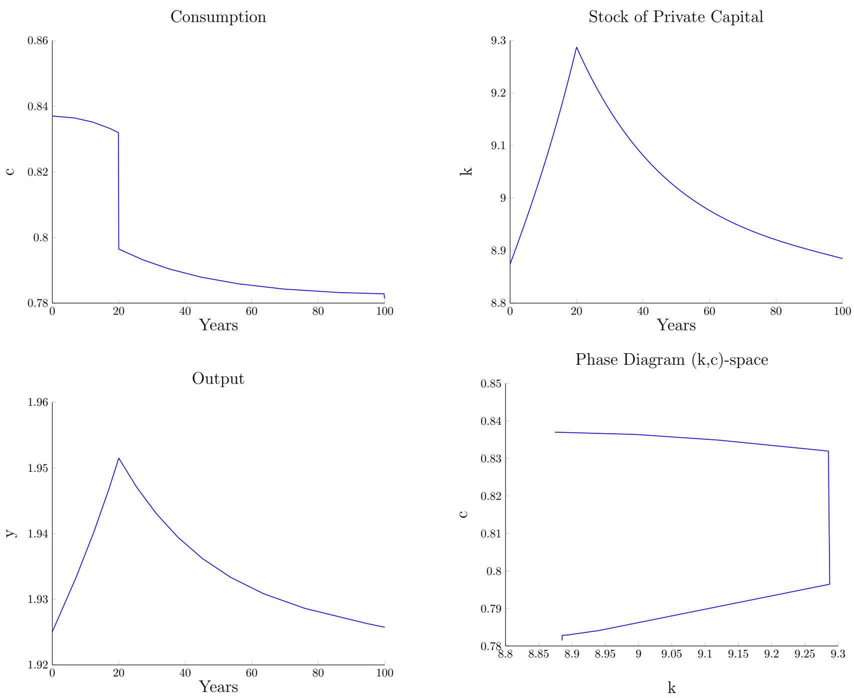

Figure 4: Impulse responses to an anticipated increase in τ_c at $t = 20$.

We also show how the UL model reacts to an anticipated shock. We will consider the scenario of an increase in the elasticity of physical capital from $\alpha = 0.3$ to $\alpha = 0.4$ at time $t = 50$. This means that the marginal productivity of capital increases, so it will become more attractive to work in the production of final goods.

Figure 5 presents the impulse responses of the model to such shock. From a quantitative point of view, we have a welfare increase of 0.132% (welfare with no shock is $U_0 = -9.1678$ and with the aforementioned shock it raised to $U_s = -9.1557$). But this analysis is particularly interesting from a qualitative point of view. Expectedly, we can see a surge in the share of labor dedicated to production in the final goods sector, reaching $\mu = 1$. Since there is no capital attrition, swapping labor between producing final goods and developing human capital comes at no extra cost, but in a more real case scenario it would probably have a greater effect on the evolution of human capital.

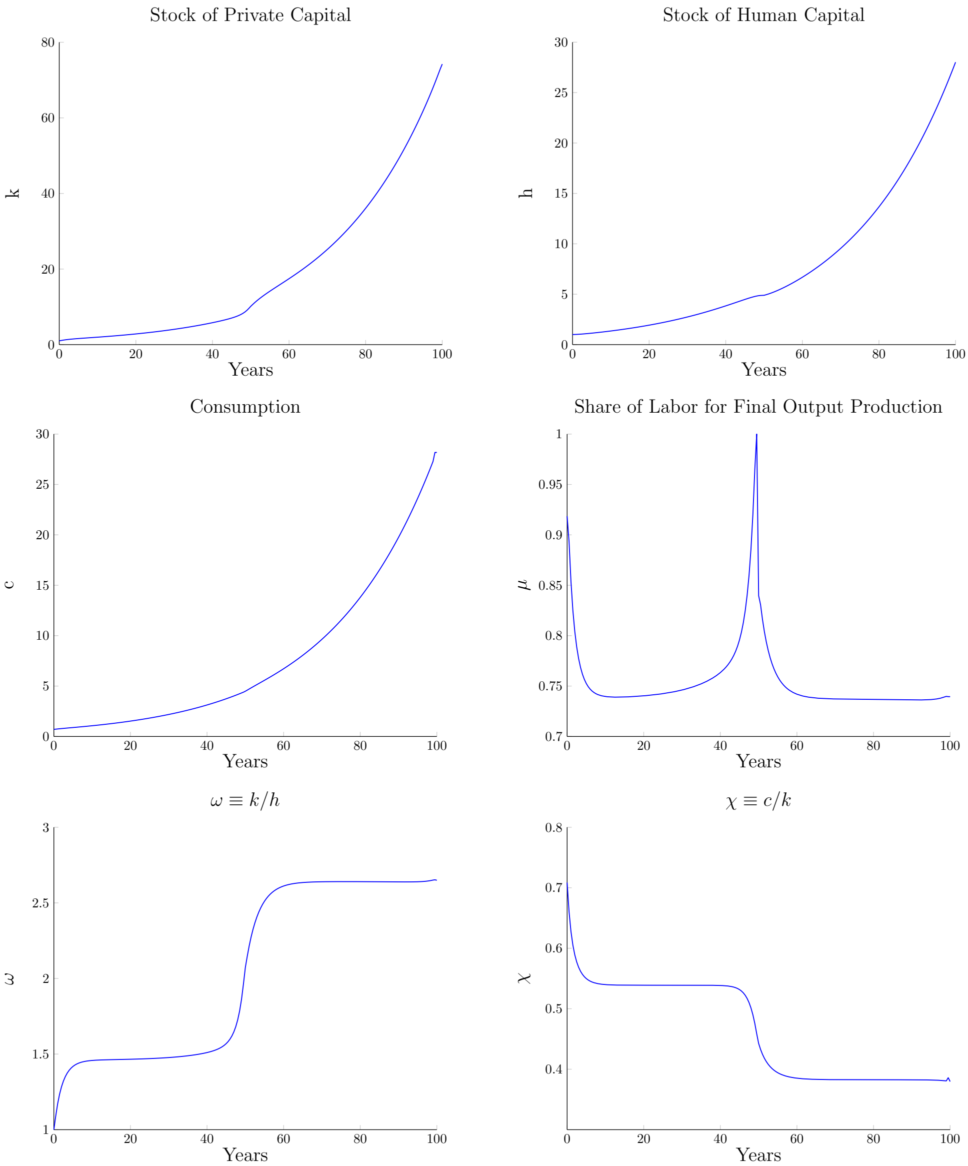

Figure 5: Numerical optimization of the Uzawa-Lucas endogenous growth model when subject to an anticipated capital elasticity increase from $\alpha = 0.3$ to $\alpha = 0.4$ at time $t = 50$.

4.2 Non-steady state shocks

The available numerical approaches assume that the economy departs from a steady state prior to being hit by a shock. Indeed, such information is usually an input of the procedure. To be more precise, some methods (like Trimborn et al., 2004) do not require to start from a steady state, but rather calculate the state of the system at its equilibrium prior to applying a shock, which is conceptually the same.

In our framework there is no requirement to start from a steady state. In fact, a non-steady state analysis is more realistic in the sense that no empirical studies have consistently reported a real world economy to be at its long-term equilibrium state.

Consider the same shock as above, but now taking place place for three different values of

$k_0 = \{0.5k^*, k^*, 1.5k^*\}$ (since the shock takes place at $\tilde{t} = 20$, it is closer to its steady state but still distant enough to serve as a viable example). In the first case, where $k_0 = 0.5k^*$, the initial value for the stock of private capital is set to half of its equilibrium value. In the second case, with $k_0 = k^*$, we are starting from an equilibrium state that goes fully in line with the results already obtained and represented in Figure 4 and also reported by Trimborn, 2007. In the third case, the value for k_0 is set to 50% higher than its steady state, with $k_0 = 1.5k^*$.

As can be seen from Figures 6, the outcome is substantially different. From a qualitative point of view, the trajectory of consumption $c(t)$ manifests a widely different behavior depending on its starting point. In the case where the economy is way over its steady state (red dashed line), there is a sharp drop in the consumption after the shock, as expected, and it continues to converge to its steady state. The adjustment trajectory is quite similar to the case when the economy departs from its steady state (blue straight line). But when the economy departs from a state considerably lower than its equilibrium value (green dashed line) the trajectory is considerably different. Instead of showing a continuous drop in the consumption, it can be seen that after the shock a sudden drop occurs but it is partially mitigated by a subsequent increase up until the new steady-state. The savings rate also exhibits a contrasting effect. Instead of rising (agents will necessarily consume less when their budget decreases), the savings rate will actually *decrease* for the case when the economy departs from a state over its equilibrium value, with a sudden increase at the time the tax policy comes into effect.

Actually, the behavior of consumption is not the only exhibiting such sharp differences in the adjustment trajectory. Also, there is no overshooting of the investment in stock of private capital, as occurs when the economy departs from a steady-state. Same happens with the output of the economy.

From a quantitative point of view, a welfare analysis shows also a difference in costs of adjustment, albeit with a major difference in qualitative and quantitative terms after the shock. Looking at the whole horizon, consumption decreases and so does the welfare. But if we look only to the period after the shock, we can clearly see a welfare decrease for the economy departing from and above the steady state, but a welfare *increase* for the economy departing from below its equilibrium. From a qualitative standpoint, it has a far better acceptance since an increase in consumption even when not exploited to its potential level is still better than a drop.

Table 6 summarizes the welfare analysis for each of the starting initial values of k_0.

Expected Shock ($\tau_0 = 0.1,\ \tau_{\tilde{t}} = 0.2$)			
Initial value (k_0)	Welfare (no shock)	Welfare	Cost
$0.5k^*$	-124.5585	-133.5139	8.9554
k^*	-117.3117	-126.0254	8.7137
$1.5k^*$	-112.6665	-121.1948	8.5283

Table 6: Welfare analysis for three distinct initial values for k_0 upon an expected shock hitting the RCK growth model.

4.3 Multiple, sequential shocks

Another very interesting application of the framework is to simulate multiple sequential shocks, something not seen in the literature due to the complexity of determining the new optimal paths from the original necessary optimality conditions. Also, we allow for the optimization to occur from a non-steady state, like in Section 4.2. Otherwise, if we allowed enough time for the economy to convert back to a new steady state, these sequential shocks could be simulated using traditional methods by connecting the impulse responses to each shock. The deed is even more complex for anticipated shocks as the system becomes increasingly complex.

Again, such analysis is made possible by the fact that the framework transforms the problem into an equivalent problem that can be solved numerically using a direct method, meaning that optimization is done *a posteriori*. The generated NLP problem along with the necessary optimality conditions can then be solved at each instant.

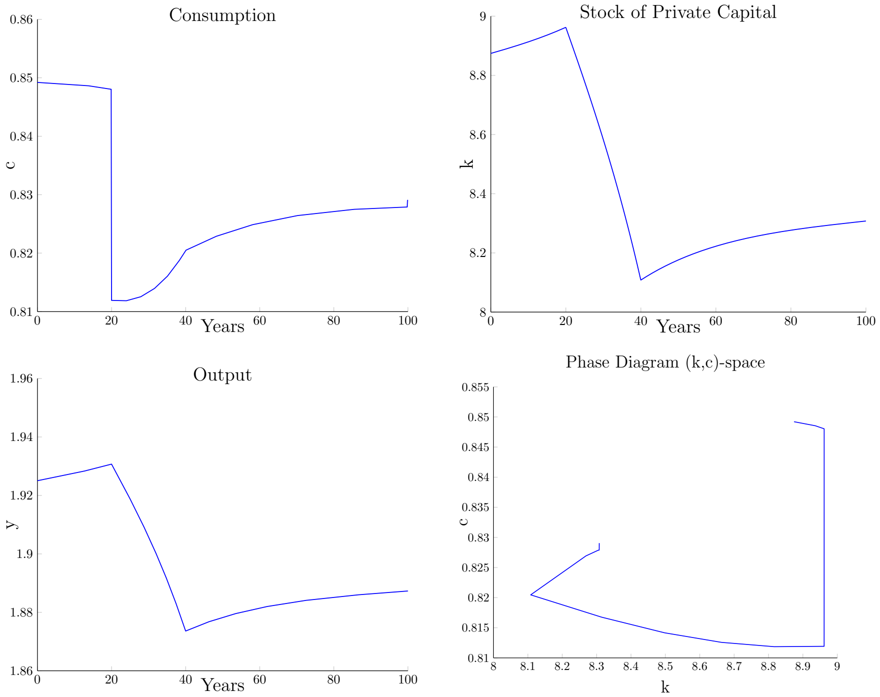

Figure 7: Impulse responses to multiple sequential shocks, both expected. The first is an anticipated increase in τ_c at $t = 20$ and the second is a decrease in τ_k at $t = 40$.

Figure 7 shows the impulse responses of the RCK model augmented with taxes to an anticipated increase in the consumption tax τ_c at time $\tilde{t} = 20$ from $\tau_{c,0} = 0.1$ to $\tau_{c,\tilde{t}} = 0.2$ followed by a decrease in the capital tax τ_k at time $\tilde{t} = 40$ from $\tau_{k,0} = 0.3$ to $\tau_{k,\tilde{t}} = 0.1$. It is interesting to observe that from both a qualitative and a quantitative point of view households will be worse off, even if the tax decrease on capital could potentially increase the long-term output of the economy, therefore making for the levying of the consumption tax. Also, it is interesting to observe the adjustment trajectories for consumption c and for the stock of private capital k.

5 Summary

We have proposed a new framework capable of solving and simulating the transitional dynamics of nonlinear continuous and discrete growth models. This is made possible by the theorem that assures that we can represent an infinite-horizon with a finite-horizon formulation, so that we can solve the underlying optimal control using a direct method. Although widely used in control theory, this approach is still fairly new in the economic growth literature, with most of the relevant numerical procedures making use of indirect methods. The procedure is extremely powerful as it is not limited to problems whose NOCs can be derived analytically. We have already highlighted some of the main advantages when compared to the available procedures, but it is worth emphasizing that this framework allows for the study of the transitional dynamics of models that are not at their steady-state, something that current numerical procedures are not capable of. It also makes it extremely easy to investigate the adjustment trajectories of when multiple shocks hit the economy at different times.

In short, this framework opens a whole new realm of possibilities, being able to cope with extremely complex and nonlinear dynamic systems, continuous or discrete, and making it extremely easy to study expected and unexpected shocks, single or multiple. We believe it will be an important asset in the toolkit of a macro-growth researcher.

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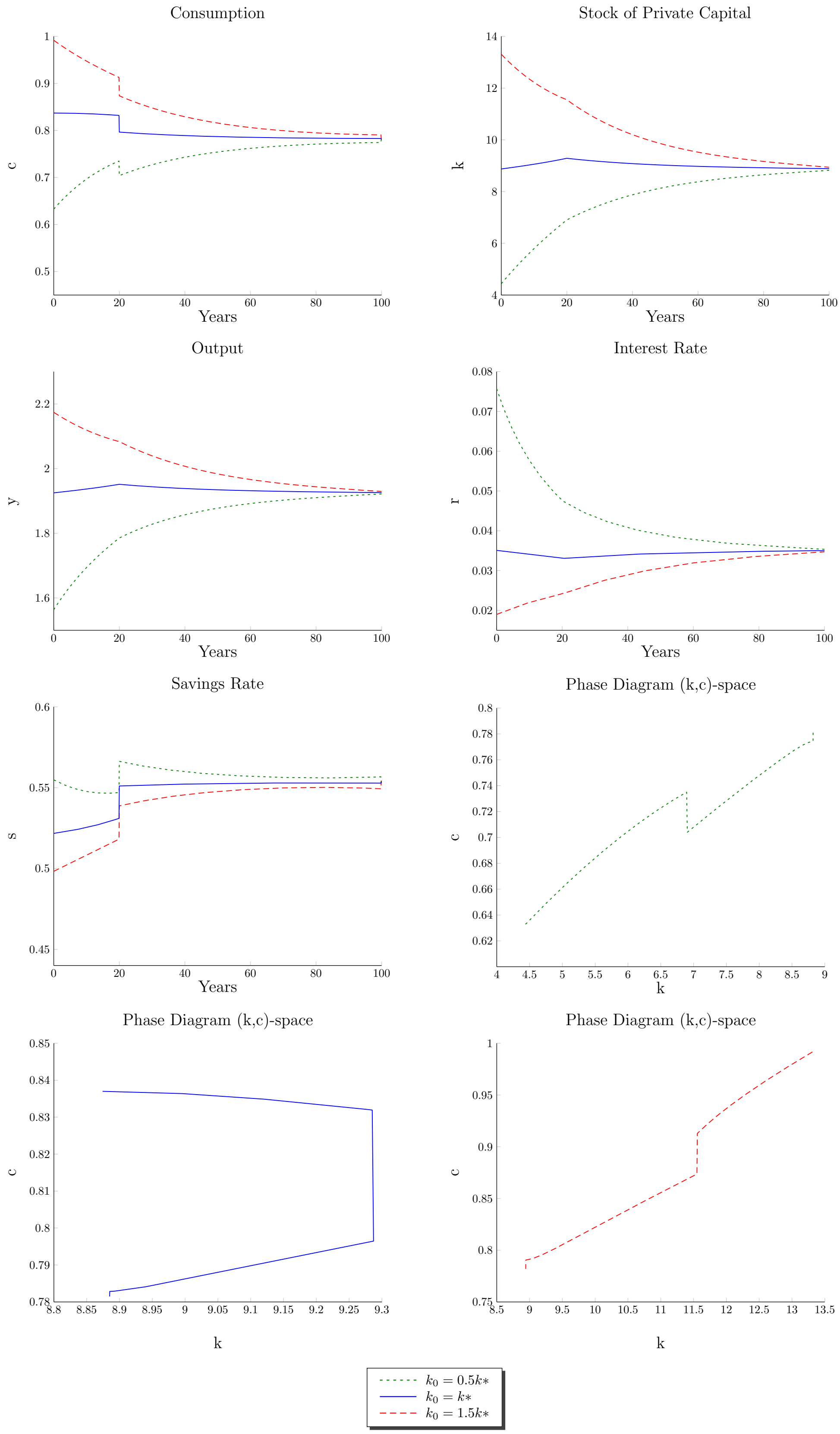

Figure 6: Impulse responses to an anticipated increase in τ_c at $t = 20$ for three distinct initial values for k_0. The straight blue line exhibits the adjustment trajectories for when the economy departs from steady state, $k_0 = k^*$. The red dashed line exhibits the very same trajectories for when the economy departs from over its steady state value, $k_0 = 1.5k^*$. Finally, the green dashed line shows the behavior from a starting point of half its steady state value, $k_0 = 0.5k^*$.

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