Optimal Control of Infinite-Horizon Growth Models - a direct approach

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Abstract

We propose a framework for solving nonlinear infinite-horizon economic growth models using an NLP direct method. This approach has several benefits in comparison to the indirect analytical methods found in the literature:

- It can solve a model in its nonlinear form
- It allows for the study of anticipated and multiple, sequential shocks
- It does no require the dynamic system to part from a steady-state equilibrium
- Opens the way for more complex models that are analytically intractable

Introduction

Optimal control theory has been extensively applied to the solution of economic problems since the pioneering work of [1]. The study of dynamic growth models follows a standard procedure, which consists in applying the Maximum Principle and obtaining the NOCs along with the transversality condition. This approach has served the economics profession well but lacks the flexibility and robustness of state-of-the-art numerical tools.

Building on the work of [2], we propose a numerical framework that transforms the original infinite-horizon problem into a finite-horizon equivalent form and solves it using interior point optimizers. This new procedure is capable of dealing with complex models, once deemed intractable when using analytical tools.

We exemplify the usage of this framework by numerically solving the Uzawa-Lucas endogenous growth model and analyzing how it copes with anticipated shocks, but more recent models like [3] are also eligible.

Procedure

The procedure consists in first discretizing the problem and then optimizing, and works as follows:

1. Transform the original infinite-horizon optimal control problem P_∞ into an equivalent finite-horizon problem P_T by using Theorem 1
2. Use a numerical framework to input the O.C. problem (e.g. ICLOCS) and use a solver for the underlying NLP problem (e.g. IPOPT).

Theorem 1

Given a generic optimal control problem

$$P_\infty: \ \min \int_0^\infty L(x(t), u(t)) \cdot dt$$

for which we assume there is a finite solution. Assume additionally that after some time T, the state is within some invariant set S (that is, there is a control u such that $x(t) \in S$, $S \subset \Gamma(t)$, for all $t \geq T$) for which the problem still has a finite solution. Then, there exists a terminal cost function W, such that the problem is equivalent to the finite horizon problem

$$P_T: \ \min \int_0^T L(x(t), u(t)) \cdot dt + W(x(T))$$

Application

We exemplify the usage of the framework solving the Uzawa-Lucas endogenous growth model.

$$\max U = \int_0^\infty \frac{c^{1-\theta}}{1-\theta} e^{-\rho t} dt, \quad s.t. \tag{1}$$

$$\begin{aligned}
&c > 0, && 0 \leq \mu \leq 1 \\
&\dot{k} = Ak^\alpha(\mu h)^{1-\alpha} - c - \delta_k k && \\
&\dot{h} = B(1-\mu)h && \\
&k(0) = k_0 && k \geq 0, \ \forall t > 0 \\
&h(0) = h_0 && h \geq 0, \ \forall t > 0
\end{aligned} \tag{2}$$

with the social planner or household choosing an allocation $(c, \mu)_{t=0}^\infty$ that maximizes U.

Numerical Results for an Anticipated Shock

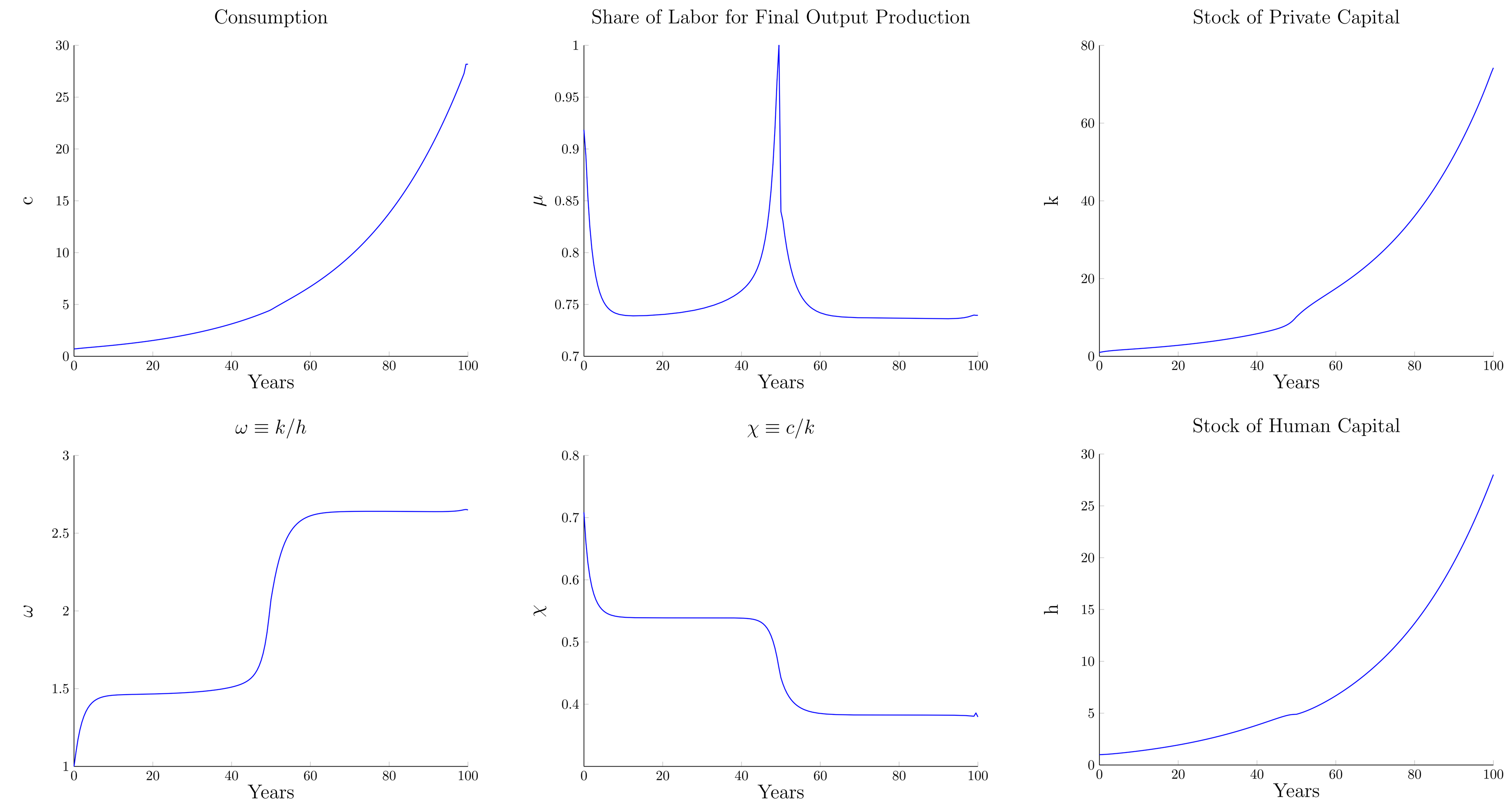

Figure 1: Numerical optimization of the Uzawa-Lucas endogenous growth model when subject to an anticipated capital elasticity increase from $\alpha = 0.3$ to $\alpha = 0.4$ at time $t = 50$.

Conclusion

We provide a new approach for solving exogenous and endogenous economic growth models that is far more powerful than the analytical methods hitherto used, as it is not limited to problems whose NOCs can be derived analytically and does not require the ODEs to be linearized.

In short, this framework opens a whole new realm of possibilities, being able to cope with extremely complex and nonlinear dynamic systems, continuous or discrete, and making it extremely easy to study expected and unexpected shocks, single or multiple. We believe it will be an important asset in the toolkit of a macro-growth researcher.

Further Work

We are extending this framework to run unanticipated shocks by running a similar MPC problem with a moving-horizon. Moreover, we are using the framework to study optimal debt adjustment using time-variant tax rates, something deemed intractable using analytical methods.

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