REPRESENTATIONS OF SURFACE GROUPS IN THE PROJECTIVE GENERAL LINEAR GROUP

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ABSTRACT. Given a closed, oriented surface X of genus $g \geqslant 2$, and a semisimple Lie group G, let $\mathcal{R}_G$ be the moduli space of reductive representations of $\pi_1 X$ in G. We determine the number of connected components of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$, for $n \geqslant 4$ even. In order to have a first division of connected components, we first classify real projective bundles over such a surface. Then we achieve our goal, using holomorphic methods through the theory of Higgs bundles over compact Riemann surfaces.

We also show that the complement of the Hitchin component in $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$ is homotopically equivalent to $\mathcal{R}_{\mathrm{SO}(3)}$.

1. INTRODUCTION

Let X be a closed, oriented surface of genus $g \geqslant 2$ and $\pi_1 X$ be its fundamental group. Let

$$\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})} = \mathrm{Hom}^{\mathrm{red}}(\pi_1 X, \mathrm{PGL}(n,\mathbb{R}))/\mathrm{PGL}(n,\mathbb{R})$$

be the quotient space of reductive representations of $\pi_1 X$ in the projective general linear group $\mathrm{PGL}(n,\mathbb{R}) = \mathrm{GL}(n,\mathbb{R})/\mathbb{R}^*$, where $\mathrm{PGL}(n,\mathbb{R})$ acts by conjugation. In this paper we determine the number of connected components of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$, for $n \geqslant 4$ even, applying the general theory of G-Higgs bundles to the $\mathrm{PGL}(n,\mathbb{R})$ case.

For a semisimple Lie group G, this general theory, created among others by Hitchin [14], Simpson [29, 30, 31], Corlette [8] and Donaldson [9], supplies a strong relation between different subjects such as topology, holomorphic and differential geometry and analysis. On the one hand, we have the moduli space $\mathcal{R}_G$ of reductive representations of $\pi_1 X$ in G, also known as a character variety. An element in $\mathcal{R}_G$ is topologically classified by certain invariants of the isomorphism class of the associated flat principal G-bundle over X. If c is a topological class of principal G-bundles, we denote by $\mathcal{R}_G(c)$ the subspace of $\mathcal{R}_G$ consisting of classes of representations which belong to the class c. On the other hand we fix a complex structure on X turning it into a Riemann surface, and consider G-Higgs bundles over it. A G-Higgs bundle is a pair consisting of a holomorphic bundle, whose structure group depends on G, and a section of a certain associated bundle (see below for precise definitions). Topologically, a G-Higgs bundle is also classified by invariants taking values in the same set as the representations in $\mathcal{R}_G$. Again, if c is one topological class, we denote by $\mathcal{M}_G(c)$ the moduli space of polystable G-Higgs bundles in the class c.

Now, the above mentioned authors have proved that the spaces $\mathcal{R}_G(c)$ and $\mathcal{M}_G(c)$ are homeomorphic (see Theorem 2.8). More generally, for a reductive Lie group G, there is

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a correspondence similar to the previous one, but replacing $\pi_1 X$ by its universal central extension Γ, defined in (2.2) below. We denote the space of such representations, with fixed topological class c, by $\mathcal{R}_{\Gamma,G}(c)$. Related to these two moduli spaces, and essential in the proof of the existence of the homeomorphism, is a third moduli space: the moduli space of solutions to the so-called Hitchin's equations on a fixed C^∞ principal G-bundle over X.

For G compact and connected, the spaces $\mathcal{R}_G$ and $\mathcal{M}_G$ have been studied in the seminal papers of Narasimhan and Seshadri [18] and of Ramanathan [21] from an algebraic viewpoint, and by Atiyah and Bott [1] from a gauge theoretic point of view. In this case, the answer about the number of components is known: for each topological type c, each subspace of $\mathcal{R}_G(c)$ is connected. Since then much has been done to study the geometry and topology of these spaces. When G is complex, connected and reductive, the answer to the problem of counting connected components is the same as in the compact case by the works of Hitchin [14], Donaldson [9], Corlette [8] and Simpson [29, 30, 31]. When G is a non-compact real form of a complex semisimple Lie group, the study of the topology of $\mathcal{R}_G$ started with the seminal papers of Goldman [12] and Hitchin [15] and, although much work has been done since then by several people (see, in particular, the paper [13] of Gothen, the works [2, 3, 4] of Bradlow, García-Prada and Gothen and also [10] by García-Prada, Gothen and Mundet i Riera), it is still far from finished.

In this paper we are interested in studying the components of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$, for $n \geqslant 4$ even (when $n \geqslant 3$ is odd, $\mathrm{PGL}(n,\mathbb{R}) \cong \mathrm{SL}(n,\mathbb{R})$ hence the components of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ are known to be 3 by the work of Hitchin in [15]; the $n = 2$ case was studied by Xia in [33]). Following the ideas of Hitchin [14, 15], the main tool to reach our goal should be the L^2-norm of the Higgs field in $\mathcal{M}_{\mathrm{PGL}(n,\mathbb{R})}(c)$, but in our case another group naturally appears. We will work with the space $\mathcal{M}_{\mathrm{EGL}(n,\mathbb{R})}$ of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles ($\mathrm{EGL}(n,\mathbb{R}) = \mathrm{GL}(n,\mathbb{R}) \times \mathrm{U}(1)/\sim$, where $(A,z) \sim (-A,-z)$). This is done mainly for two related reasons. One is that with this new group we can work with holomorphic vector bundles, rather than just principal or projective bundles. The other is that we can realize space of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles as closed subspace of $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})} \times \mathrm{Jac}^d(X)$, where $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}$ is the moduli space of Higgs bundles (see [14]). In general, when $\mathcal{M}_G(c)$ is smooth, the function f given by the L^2-norm of the Higgs field is a non-degenerate Morse-Bott function which is also a proper map and, in some cases, the critical submanifolds are well enough understood to allow the extraction of topological information such as the Poincaré polynomial. However, even when $\mathcal{M}_G(c)$ has singularities, the properness of f allows us to draw conclusions about the connected components, although one cannot directly apply Morse theory.

The study of the local minima of f is sufficient to obtain the number of connected components of the space of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles and thus of $\mathcal{R}_{\Gamma,\mathrm{EGL}(n,\mathbb{R})}$, for $n \geqslant 4$. There is a projection from this space to $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ and using this we compute the components of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$, obtaining the first of our two main results (see Theorem 10.2):

Theorem 1.1. *Let $n \geqslant 4$ be even. Then the space $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ has $2^{2g+1} + 2$ connected components.*

Essential in the count of components of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ is the topological classification of real projective bundles over X. This is done in the first part of the paper, where we have found explicit discrete invariants which classify continuous principal $\mathrm{PGL}(n,\mathbb{R})$-principal bundles over any closed oriented surface. This classification shows for example that, in contrast to the complex case, there are real projective bundles which are not projectivization of real vector bundles. It shows also that, in most cases, there is a collapse of the second Stiefel-Whitney class, when we pass from real vector bundles to projective bundles.

Combining the results of Xia [33] for $n = 2$ and of Hitchin [15] for $n \geqslant 3$ odd with our Theorem 1.1, we have the number of connected components of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$, for arbitrary n, as follows:

Theorem 1.2. *The number of connected components of* $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ *is:*

- $2^{2g+1} + 4g - 5$ *if* $n = 2$*;*
- 3 *if* $n \geqslant 3$ *is odd;*
- $2^{2g+1} + 2$ *if* $n \geqslant 4$ *is even.*

Using the results of Hitchin in [15], we are able to obtain more topological information of $\mathcal{R}_{\mathrm{PGL}(3,\mathbb{R})}$ (observe that this is the same as $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$ since $\mathrm{PGL}(3,\mathbb{R}) \cong \mathrm{SL}(3,\mathbb{R})$), because in this case there are no critical submanifolds of f besides the local minima and these are of a very special type. The result we obtain is the following (see Theorem 11.1):

Theorem 1.3. *The space* $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$ *has one contractible component and the space consisting of the other two components is homotopically equivalent to* $\mathcal{R}_{\mathrm{SO}(3)}$*.*

Actually, using a computation of the Poincaré polynomials of $\mathcal{R}_{\mathrm{SO}(3)}$ recently done by Ho and Liu in [16], this theorem gives the Poincaré polynomials of $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$ almost for free.

2. Representations of $\pi_1 X$ in G and G-Higgs bundles

2.1. **Representations of $\pi_1 X$ in G.** Let X be a closed oriented surface of genus $g \geqslant 2$ and let G be a semisimple Lie group.

Consider the space $\mathrm{Hom}(\pi_1 X, G)$ of all homomorphisms from the fundamental group of X to G. Such a homomorphism $\rho : \pi_1 X \to G$ is also called a *representation* of $\pi_1 X$ in G. Considering the presentation of $\pi_1 X$ given by the usual $2g$ generators

$$\pi_1 X = \Big\langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \Big\rangle \tag{2.1}$$

one sees that a representation $\rho \in \mathrm{Hom}(\pi_1 X, G)$ is determined by its values on the set of generators $a_1, b_1, \ldots, a_g, b_g$. The set $\mathrm{Hom}(\pi_1 X, G)$ can thus be embedded in G^{2g} via $\rho \mapsto (\rho(a_1), \ldots, \rho(b_g))$, becoming the subset of $2g$-tuples $(A_1, B_1, \ldots, A_g, B_g)$ of G^{2g} satisfying the algebraic equation $\prod_{i=1}^{g}[A_i, B_i] = 1$, and we consider the induced topology on $\mathrm{Hom}(\pi_1 X, G)$.

Letting G act on $\mathrm{Hom}(\pi_1 X, G)$ by conjugation

$$g \cdot \rho = g \rho g^{-1}$$

we obtain the quotient space

$$\mathrm{Hom}(\pi_1 X, G)/G.$$

This space may not be Hausdorff because there may exist different orbits with non-disjoint closures, so we consider only *reductive* representations of $\pi_1 X$ in G, meaning the ones that, when composed with the adjoint representation of G on its Lie algebra, become a sum of irreducible representations. Denote the space of such representations by $\mathrm{Hom}^{\mathrm{red}}(\pi_1 X, G)$. The corresponding quotient is the space we are interested in:

Definition 2.1. *The* moduli space of representations of $\pi_1 X$ in G *is the quotient space*

$$\mathcal{R}_G = \mathrm{Hom}^{\mathrm{red}}(\pi_1 X, G)/G.$$

The space $\mathcal{R}_G$ is also known as the *G-character variety* of X.

If G acts on G^{2g} through the diagonal adjoint action, the inclusion $j : \mathrm{Hom}(\pi_1 X, G) \hookrightarrow G^{2g}$ becomes G-equivariant and, from Theorem 11.4 in [24], a representation $\rho \in \mathrm{Hom}(\pi_1 X, G)$ is reductive if and only if the orbit of $j(\rho)$ in G^{2g} is closed, hence it follows that $\mathcal{R}_G$ is indeed Hausdorff.

If we allow G to be reductive and not just semisimple, then we consider a universal central extension Γ of $\pi_1 X$ given by the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma \longrightarrow \pi_1 X \longrightarrow 0.$$

It is generated by $2g$ generators $a_1, \dots, b_g$ (which are mapped to the corresponding ones of $\pi_1 X$) and by a central element J, subject to the relation $\prod_{i=1}^g [a_i, b_i] = J$:

$$\Gamma = \Big\langle a_1, b_1, \dots, a_g, b_g, J \mid \prod_{i=1}^{g} [a_i, b_i] = J,\ J \in Z(\Gamma) \Big\rangle. \tag{2.2}$$

Let $H \subseteq G$ be a maximal compact subgroup of G. Analogously to the case of $\pi_1 X$, let us consider the reductive representations ρ of Γ in G such that $\rho(J) \in (Z(G) \cap H)_0$, the identity component of the centre of G intersected with H:

$$\mathrm{Hom}^{\mathrm{red}}_{\rho(J) \in (Z(G) \cap H)_0}(\Gamma, G) = \{\rho : \Gamma \longrightarrow G \mid \rho \text{ is reductive and } \rho(J) \in (Z(G) \cap H)_0\}. \tag{2.3}$$

This definition does not depend on the choice of H. Indeed, any other maximal compact subgroup H' of G is conjugate to H, so $(Z(G) \cap H)_0 = (Z(G) \cap H')_0$.

Definition 2.2. *The* moduli space of representations of Γ in G *is the quotient space*

$$\mathcal{R}_{\Gamma, G} = \mathrm{Hom}^{\mathrm{red}}_{\rho(J) \in (Z(G) \cap H)_0}(\Gamma, G)/G.$$

To give a representation $\rho \in \mathrm{Hom}^{\mathrm{red}}_{\rho(J) \in (Z(G) \cap H)_0}(\Gamma, G)$ is equivalent to give a representation of $\pi_1(X \setminus \{x_0\})$, the fundamental group of the punctured surface, in G such that the image of the homotopy class of the loop around the puncture is $\rho(J) \in (Z(G) \cap H)_0$.

Of course, if G is semisimple, $\mathcal{R}_{\Gamma, G} = \mathcal{R}_G$. The main result of this paper is the computation of the number of connected components of $\mathcal{R}_{\mathrm{PGL}(n, \mathbb{R})}$, for $n \geqslant 4$ even.

As is well-known, there is a bijection between isomorphism classes of representations of $\pi_1 X$ in G and isomorphism classes of flat G-bundles over X. There is as well a one-to-one correspondence between isomorphism classes of representations of Γ in G and isomorphism classes of projectively flat G-bundles over X, i.e., G-bundles equipped with connections with constant central curvature in $Z(\mathfrak{g}) = \mathrm{Lie}(Z(G)_0)$. Taking these correspondences into account, we make the following definition:

Definition 2.3. *Let ρ be a representation of $\pi_1 X$ in G. A* topological invariant *of ρ is a topological invariant of the associated flat G-bundle.*

Let ρ be a representation of Γ in G. A topological invariant *of ρ is a topological invariant of the associated projectively flat G-bundle.*

If two representations $\rho_1, \rho_2 \in \mathrm{Hom}^{\mathrm{red}}(\pi_1 X, G)$ are equivalent, then the associated principal flat G-bundles E_{ρ_1} and E_{ρ_2} are isomorphic and vice-versa. Hence the topological invariants of ρ_1 and of ρ_2 are the same. Thus it makes sense to define a topological invariant of an equivalence class of representations. Given a topological class c of G-bundles over X, denote by

$$\mathcal{R}_G(c)$$

the subspace of $\mathcal{R}_G$ whose representations belong to the class c. Analogously, define $\mathcal{R}_{\Gamma,G}(c)$.

2.2. **G-Higgs bundles.** In this section we introduce the main objects which we shall work with. These are called *Higgs bundles* and roughly are pairs consisting of a holomorphic bundle and a section of an associated bundle (see Definition 2.4 below). Higgs bundles were introduced by Hitchin [14] on compact Riemann surfaces and by Simpson [29] on any compact Kähler manifold.

Let $H \subseteq G$ be a maximal compact subgroup of G and $H^{\mathbb{C}} \subseteq G^{\mathbb{C}}$ their complexifications. There is a Cartan decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

where $\mathfrak{m}$ is the complement of $\mathfrak{h}$ with respect to the non-degenerate $\mathrm{Ad}(G)$-invariant bilinear B form on $\mathfrak{g}$. If $\theta : \mathfrak{g} \to \mathfrak{g}$ is the corresponding Cartan involution then $\mathfrak{h}$ and $\mathfrak{m}$ are its $+1$-eigenspace and -1-eigenspace, respectively. Complexifying, we have the decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$$

and $\mathfrak{m}^{\mathbb{C}}$ is a representation of $H^{\mathbb{C}}$ through the so-called *isotropy representation*

$$\mathrm{Ad}\,|_{H^{\mathbb{C}}} : H^{\mathbb{C}} \longrightarrow \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}}) \tag{2.4}$$

which is induced by the adjoint representation of $G^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$. If $E_{H^{\mathbb{C}}}$ is a principal $H^{\mathbb{C}}$-bundle over X, we denote by $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) = E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}}$ the vector bundle, with fibre $\mathfrak{m}^{\mathbb{C}}$, associated to the isotropy representation.

Let $K = T^*X^{1,0}$ be the canonical line bundle of X.

Definition 2.4. *A G-*Higgs bundle *over a Riemann surface X is a pair $(E_{H^{\mathbb{C}}}, \Phi)$ where $E_{H^{\mathbb{C}}}$ is a principal holomorphic $H^{\mathbb{C}}$-bundle over X and Φ is a global holomorphic section of $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K$, called the* Higgs field.

Any continuous G-bundle has certain discrete invariants which distinguish bundles which are not isomorphic as continuous (or equivalently C^∞) G-bundles. On Riemann surfaces, if G is connected, these invariants take values in $\pi_1 G$. For example, complex vector bundles of rank n are classified by their degree $d \in \mathbb{Z} = \pi_1 \mathrm{U}(n)$. For G not necessarily connected, these topological invariants may take values in more complicated sets which depend only on the homotopy type of G. If H is a maximal compact subgroup of G, then the inclusion $H \subset G$ is a homotopy equivalence so the classification of G-bundles is equivalent to that of H-bundles. Now, a G-Higgs bundle $(E_{H^{\mathbb{C}}}, \Phi)$ is topologically classified by the topological invariant of the corresponding $H^{\mathbb{C}}$-bundle $E_{H^{\mathbb{C}}}$ and, as the maximal compact subgroup of $H^{\mathbb{C}}$ is H, the topological classification of G-Higgs bundles is the same as the one of H-principal bundles.

Now we consider the moduli space of G-Higgs bundles. The notion of (poly)stability, for general G is subtle (see [6, 10, 25, 27]) but for $\mathrm{GL}(n,\mathbb{C})$ it is easy. Consider a ($\mathrm{GL}(n,\mathbb{C})$-)Higgs bundle (V, Φ) and let

$$\mu(V) = \frac{\deg(V)}{\mathrm{rk}(V)}$$

be the slope of V. A subbundle $W \subseteq V$ is said Φ-*invariant* if $\Phi(W) \subset W \otimes K$.

Definition 2.5. *A Higgs bundle (V, Φ) is:*

- stable *if* $\mu(W) < \mu(V)$ *for all* Φ*-invariant proper subbundle* $W \subset V$*;*
- semistable *if* $\mu(W) \leqslant \mu(V)$ *for all* Φ*-invariant proper subbundle* $W \subset V$*;*
- polystable *if* $V = W_1 \oplus \cdots \oplus W_k$ *and* $\Phi = \Phi_1 \oplus \cdots \oplus \Phi_k$ *where, for each* i, $\Phi_i : W_i \to W_i \otimes K$ *and* (W_i, Φ_i) *is stable with* $\mu(W_i) = \mu(V)$.

Definition 2.6. *Two* G*-Higgs bundles* $(E_{H^{\mathbb{C}}}, \Phi)$ *and* $(E'_{H^{\mathbb{C}}}, \Phi')$ *over* X *are* isomorphic *if there is an holomorphic isomorphism* $f : E_{H^{\mathbb{C}}} \to E'_{H^{\mathbb{C}}}$ *such that* $\Phi' = \tilde{f}(\Phi)$*, where* $\tilde{f} \otimes 1_K : E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K \to E'_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K$ *is the map induced from* f *and from the isotropy representation* $H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$.

In order to construct moduli spaces, we need to consider S-equivalence classes of semistable G-Higgs bundles (cf. [27]). For a stable G-Higgs bundle, its S-equivalence class coincides with its isomorphism class and for a strictly semistable G-Higgs bundle, its S-equivalence contains precisely one (up to isomorphism) representative which is polystable so this class can be thought as the isomorphism class of the unique polystable G-Higgs bundle which is S-equivalent to the given strictly semistable one.

These moduli spaces have been constructed by Schmitt in [25, 26, 27], using methods of Geometric Invariant Theory, showing that they carry a natural structure of algebraic/complex variety.

Definition 2.7. *For a reductive Lie group* G*, the* moduli space of G-Higgs bundles over a Riemann surface X *is the algebraic/complex variety of isomorphism classes of polystable* G*-Higgs bundles. We denote it by* $\mathcal{M}_G$*:*

$$\mathcal{M}_G = \{\text{Polystable } G\text{-Higgs bundles on } X\}/\sim .$$

For a fixed topological class c *of* G*-Higgs bundles, denote by* $\mathcal{M}_G(c)$ *the moduli space of* G*-Higgs bundles which belong to the class* c.

The relation between G-Higgs bundles over X and representations $\pi_1 X \to G$ is given by the following fundamental theorem.

Theorem 2.8. *Let* G *be a semisimple Lie group. A* G*-Higgs bundle is polystable if and only if it arises from a reductive representation of* $\pi_1 X$ *in* G*. Moreover, this correspondence induces a homeomorphism between the spaces* $\mathcal{R}_G(c)$ *and* $\mathcal{M}_G(c)$.

If G *is reductive, there is a similar correspondence which induces a homeomorphism between the spaces* $\mathcal{R}_{\Gamma,G}(c)$ *and* $\mathcal{M}_G(c)$.

Strictly speaking, this theorem has been proved for $G = \mathrm{GL}(n, \mathbb{C})$ and $G = \mathrm{SL}(n, \mathbb{C})$ by Hitchin in [14] and Simpson in [29] (see also the papers [8] of Corlette and [9] of Donaldson). The general definition of polystability and the proof of the Hitchin-Kobayashi correspondence for arbitrary G-Higgs bundles appears in the preprint [10] of García-Prada, Gothen and Mundet i Riera.

3. Topological invariants for $\mathrm{PGL}(n, \mathbb{R})$-bundles over closed oriented surfaces

In this section we obtain a topological classification of continuous principal $\mathrm{PGL}(n, \mathbb{R})$-bundles over X, with $n \geqslant 4$ even. We shall, however, start by obtaining a general topological classification for any principal G-bundles, with $\pi_0 G$ abelian.

3.1. **The case of any topological group G with $\pi_0 G$ abelian.** Let G be a topological group. Denote by $\mathcal{C}(G)$ the sheaf of continuous G-valued functions on X and by G_0 the identity component of G. We have the short exact sequence of groups

$$0 \longrightarrow G_0 \longrightarrow G \xrightarrow{p_1} \pi_0 G \longrightarrow 0$$

and, associated to the corresponding short exact sequence of sheaves of continuous functions with values in the corresponding groups, we have the sequence of cohomology sets:

$$H^1(X, \mathcal{C}(G_0)) \longrightarrow H^1(X, \mathcal{C}(G)) \xrightarrow{p_{1,*}} H^1(X, \pi_0 G) \ .$$

Recall that the cohomology set $H^1(X, \mathcal{C}(G))$ is in natural bijection with the set of isomorphism classes of continuous G-principal bundles over X. So, from the previous sequence, we define the first topological invariant of a continuous G-bundle E.

Definition 3.1. *The topological invariant μ_1 of E is defined by*

$$\mu_1(E) = p_{1,*}(E) \in H^1(X, \pi_0 G).$$

Of course, this invariant yields the obstruction to reducing the structure group of E to G_0. Notice that, if $\pi_0 G$ is abelian, $H^1(X, \pi_0 G) \cong \mathrm{Hom}(\pi_1 X, \pi_0 G) \cong \pi_0 G^{2g}$. From now on we assume that we are on this case: $\pi_0 G$ is an abelian group.

Our initial classification of G-bundles with μ_1 fixed was much more complicated and was splitted into two assymetric parts: $\mu_1 = 0$ and $\mu_1 \neq 0$. I am gretly indebted to an anonymous referee for providing a much simpler argument for the case $\mu_1 \neq 0$ and which allows to study both cases $\mu_1 = 0$ and $\mu_1 \neq 0$ simultaneously. The argument is as follows.

The surface X is homeomorphic to the result of identifying (using orientation reversing homeomorphisms) the sides of a regular $4g$-gon P according to the rule

$$A_1 B_1 A_1^{-1} B_1^{-1} A_2 B_2 A_2^{-1} B_2^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1}.$$

Let $\pi : P \to X$ be the natural projection. Let c be the centre of P, $B(c, \epsilon) \subset P$ be a small disc centred at c of radius ϵ, disjoint from the boundary of P, and let

$$U = \pi(P \setminus \{c\}) \ \text{ and } \ V = \pi(B(c, \epsilon)).$$

A G-principal bundle E on X can be described by its restrictions

$$E_U = E|_U \ \text{ and } \ E_V = E|_V$$

and by the gluing data

$$\rho : E_U|_{U \cap V} \xrightarrow{\cong} E_V|_{U \cap V}.$$

As V is contractible, E_V is isomorphic to the trivial bundle. On the other hand, the fact that the bundle E_U can be extended to X implies that $\pi^* E_U \to P \setminus \{c\}$ can be trivialized. The invariant $\mu_1(E)$ describes the isomorphism type of E_U and can be thought of as specifying, up to homotopy, how to glue the restrictions of $\pi^* E_U \to P \setminus \{c\}$ to the sides of P. Choosing a trivialization of $\pi^* E_U$, this is the same as associating, for each j, connected components of G to A_j and to B_j. This is how μ_1 can be seen as a homomorphism

$$\mu_1 : \pi_1 X \longrightarrow \pi_0 G.$$

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Let $\mathcal{G}(E_U)$ and $\mathcal{G}(E_V)$ be the gauge groups of E_U and E_V. Then the relevant gluing information to recover the bundle E, up to isomorphism, is the class of $\rho : E_U|_{U \cap V} \to E_V|_{U \cap V}$ in the set of connected components of the double quotient $\mathcal{G}(E_V)\backslash\mathrm{Isom}(E_U|_{U\cap V}, E_V|_{U\cap V})/\mathcal{G}(E_U)$, that is, in

$$\pi_0(\mathcal{G}(E_V))\backslash\pi_0(\mathrm{Isom}(E_U|_{U\cap V}, E_V|_{U\cap V}))/\pi_0(\mathcal{G}(E_U)). \tag{3.1}$$

Choosing adequate trivializations of $\pi^*(E_U|_{U\cap V})$ and of $E_V|_{U\cap V}$, the map ρ is given by a map

$$\rho_0 : U\cap V \longrightarrow G_0$$

(note that $U\cap V \cong P\setminus\{c\}\cap B(c,\epsilon)$). Since $U\cap V\sim S^1$ and $\pi_1(G_0)$ is abelian, we can identify ρ_0 with an element, still denoted by ρ_0, of $\pi_1(G_0)=\pi_1 G$ (we define the fundamental group of a topological group as the fundamental group of its identity component):

$$\rho_0\in\pi_1 G.$$

Now, recall that $\pi_0 G$ acts on $\pi_1 G$ via the adjoint action (hence $[\alpha_1]=[\alpha_2]$ in $\pi_1 G/\pi_0 G$ if and only if there is $a\in G$ such that α_1 and $a\alpha_2 a^{-1}$ are homotopic). Since $\pi_1 G$ is an abelian group, we denote the group structure additively. Given $\mu_1:\pi_1 X\to\pi_0 G$, define $\Gamma_{\mu_1}\subset\pi_1 G$ as the subgroup of $\pi_1 G$ generated by the elements of the form $\gamma_2-\gamma_1\cdot\gamma_2$, where $\gamma_2\in\pi_1 G$ and γ_1 lies in the image of μ_1:

$$\Gamma_{\mu_1}=\langle\gamma_2-\gamma_1\cdot\gamma_2 \mid \gamma_2\in\pi_1 G,\, \gamma_1\in\mathrm{Im}(\mu_1)\subseteq\pi_0 G\rangle\,. \tag{3.2}$$

Then, since $\pi_0 G$ is abelian, the action of $\pi_0 G$ on $\pi_1 G$ descends to the quotient $\pi_1 G/\Gamma_{\mu_1}$.

Definition 3.2. *The topological invariant μ_2 of E is defined as the class of $\rho_0\in\pi_1 G$ in $(\pi_1 G/\Gamma_{\mu_1})/\pi_0 G$:*

$$\mu_2(E)=[\rho_0]\in(\pi_1 G/\Gamma_{\mu_1})/\pi_0 G.$$

It should be noticed that the values which the invariant μ_2 can take depend on the invariant μ_1.

Similar arguments to the ones used in Proposition 5.1 of [21] show that the pair (μ_1,μ_2) is well-defined and that uniquely characterizes the bundle E. In terms of (3.1) this can understood as follows:

- $\pi_1 G$ corresponds to $\pi_0(\mathrm{Isom}(E_U|_{U\cap V}, E_V|_{U\cap V}))$.
- the action of Γ_{μ_1} on $\pi_1 G$ corresponds to the action of $\pi_0(\mathcal{G}(E_U))$ on $\pi_0(\mathrm{Isom}(E_U|_{U\cap V}, E_V|_{U\cap V}))$.
- the action of $\pi_0 G$ on $\pi_1 G$ corresponds to the action of $\pi_0(\mathcal{G}(E_V))$ on $\pi_0(\mathrm{Isom}(E_U|_{U\cap V}, E_V|_{U\cap V}))$.

We have therefore the following topological classification of G-principal bundles over closed oriented surfaces.

Proposition 3.3. *Let X be a closed, oriented surface and let G be a topological group such that $\pi_0 G$ is abelian. Given $\mu_1\in H^1(X,\pi_0 G)$, there is a bijection between the set of isomorphism classes of continuous G-principal bundles E over X, with $\mu_1(E)=\mu_1$, and $(\pi_1 G/\Gamma_{\mu_1})/\pi_0 G$.*

Remark 3.4. In case G is connected, this classification coincides with the well-known topological classification of G bundles over X, given $\pi_1 G$ (cf. [21]).

Remark 3.5. For the case of the sphere S^2 (in fact for S^n) this was already known (cf. [32], Section 18).

Remark 3.6. The same result is valid not only for closed, oriented surfaces, but also for any 2-dimensional connected CW-complex. A proof of this fact, using different methods, can be found in [20].

3.2. **The case of** $\mathrm{PGL}(n, \mathbb{R})$**.** Now we shall apply the result obtained in the previous section to obtain invariants which classify continuous $\mathrm{PGL}(n, \mathbb{R})$-principal bundles over our surface X.

As $\mathrm{PGL}(n, \mathbb{R})$ is homotopically equivalent to $\mathrm{PO}(n, \mathbb{R}) = \mathrm{O}(n, \mathbb{R})/\mathbb{Z}_2$, its maximal compact subgroup, this is equivalent to classify $\mathrm{PO}(n, \mathbb{R})$-bundles. From now on, we will write $\mathrm{PO}(n)$ instead of $\mathrm{PO}(n, \mathbb{R})$ for the real projective orthogonal group, as well as $\mathrm{O}(n)$ instead of $\mathrm{O}(n, \mathbb{R})$ for the real orthogonal group.

For $\mathrm{PO}(n)$, we have that

$$\mu_1 \in H^1(X, \pi_0\mathrm{PO}(n)) \cong (\mathbb{Z}_2)^{2g}. \tag{3.3}$$

This class is the obstruction to reduce the structure group to $\mathrm{PSO}(n)$.

For $n \geqslant 4$ even,

$$\pi_1\mathrm{PO}(n) = \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{if } n = 0 \bmod 4 \\ \mathbb{Z}_4 & \text{if } n = 2 \bmod 4. \end{cases}$$

More precisely, the universal cover of $\mathrm{PO}(n)$ is $\mathrm{Pin}(n)$ and, if $p : \mathrm{Pin}(n) \to \mathrm{PO}(n)$ is the covering projection, then, as a set, $\ker(p) = \{1, -1, \omega_n, -\omega_n\}$ where $\omega_n = e_1 \cdots e_n$ is the oriented volume element of $\mathrm{Pin}(n)$ in the standard construction of this group via the Clifford algebra $\mathrm{Cl}(n)$ (see, for example, [17]).

Notation 3.7. From now on we shall use the additive notation for $\{1, -1, \omega_n, -\omega_n\}$. Hence, under this notation, $\{1, -1, \omega_n, -\omega_n\} = \{0, 1, \omega_n, -\omega_n\}$ (so 1 becomes 0 and -1 becomes 1). This is done because we will identify $\{0, 1, \omega_n, -\omega_n\}$ with $\pi_1\mathrm{PO}(n)$ which is an abelian group.

Recall that $\mathrm{Pin}(n)$ is a group with two connected components, $\mathrm{Pin}(n)^-$ and $\mathrm{Spin}(n)$, where $\mathrm{Pin}(n)^-$ denotes the component which does not contain the identity. We have $\pm\omega_n \notin \mathrm{Z}(\mathrm{Pin}(n)) = \{0, 1\}$, so the action of $\pi_0\mathrm{PO}(n)$ on $\pi_1\mathrm{PO}(n)$ is not trivial. In fact, ω_n commutes with elements in $\mathrm{Spin}(n)$ and anti-commutes with elements in $\mathrm{Pin}(n)^-$, so

$$\pi_1\mathrm{PO}(n)/\pi_0\mathrm{PO}(n) = \{0, 1, \omega_n\}$$

where we also write ω_n for the class of $\omega_n \in \pi_1\mathrm{PO}(n)$ in $\pi_1\mathrm{PO}(n)/\pi_0\mathrm{PO}(n)$, which consists by $\pm\omega$.

For $\mathrm{PO}(n)$-bundles with $\mu_1 = 0$, we have $\Gamma_0 = 0$, where Γ_0 is the subgroup of $\pi_1\mathrm{PO}(n)$ defined in the general setting in (3.2).

For $\mathrm{PO}(n)$-bundles with $\mu_1 \neq 0$, then it is easy to see that $\Gamma_{\mu_1} = \{0, 1\} \cong \mathbb{Z}_2$, therefore

$$\pi_1\mathrm{PO}(n)/\Gamma_{\mu_1} = \{0, \omega_n\} \cong \mathbb{Z}_2,$$

and $\pi_0\mathrm{PO}(n)$ acts trivially on this quotient:

$$(\pi_1\mathrm{PO}(n)/\Gamma_{\mu_1})/\pi_0\mathrm{PO}(n) = \{0, \omega_n\} \cong \mathbb{Z}_2.$$

Hence, we have the invariant μ_2 defined in general in Definition 3.2, which, for $\mathrm{PO}(n)$-principal bundles over X, is such that:

$$\mu_2 \in (\pi_1\mathrm{PO}(n)/\Gamma_{\mu_1})/\pi_0\mathrm{PO}(n) = \begin{cases} \{0, 1, \omega_n\} & \text{if } \mu_1 = 0 \\ \{0, \omega_n\} & \text{if } \mu_1 \neq 0 \end{cases}. \tag{3.4}$$

Remark 3.8. When $\mu_1 \neq 0$, we also write the possible elements of $\mu_2 \in \mathbb{Z}_2$ by 0 and by ω_n, instead of $[0]$ and $[\omega_n]$. This requires a little attention because, for example, $\mu_2 = 0$ has different meanings whenever $\mu_1 = 0$ or $\mu_1 \neq 0$. However, it should always be clear in which situation we are.

Remark 3.9. When $\mu_1 = 0$, we are reduced to the topological classification of $\mathrm{PSO}(n)$-bundles over X which, for $\mathrm{PSO}(n)$-equivalence, is given by the elements in $\{0, 1, \omega_n, -\omega_n\} = \pi_1\mathrm{PSO}(n)$. However, since we are interested in $\mathrm{PO}(n)$-equivalence, the bundles with invariants ω_n and $-\omega_n$ become identified.

The next proposition gives the interpretation of the class μ_2 in terms of obstructions.

Proposition 3.10. *Let $n \geqslant 4$ be even.*

(i) *Let E be a continuous $\mathrm{PO}(n)$-bundle over X with $\mu_1(E) = 0$. Then:*
- *E lifts to a continuous $\mathrm{SO}(n)$-bundle if and only if $\mu_2(E) \in \{0, 1\}$;*
- *E lifts to a continuous $\mathrm{Spin}(n)$-bundle if and only if $\mu_2(E) = 0$.*

(ii) *Let E be a continuous $\mathrm{PO}(n)$-bundle over X with $\mu_1(E) \neq 0$. Then E lifts to a continuous $\mathrm{Pin}(n)$-bundle if and only if $\mu_2(E) = 0$.*

Proof. Suppose $\mu_1(E) = 0$, so that E is in fact a $\mathrm{PSO}(n)$-bundle. From the construction of μ_2 in the previous subsection, we have

$$\mu_2(E) = [g] \in \pi_1\mathrm{PO}(n)/\pi_0\mathrm{PO}(n),$$

where $g : (S^1, y_0) \to (\mathrm{PO}(n), [I_n])$. Let $p : \mathrm{O}(n) \to \mathrm{PO}(n)$ be the projection. There is a lift $g' : (S^1, y_0) \to (\mathrm{O}(n), I_n)$ if and only if $g_*(\pi_1 S^1) \subseteq p_*(\pi_1\mathrm{O}(n))$, which happens if and only if $[g] \in \{0, 1\}$. The case for the lift to $\mathrm{Pin}(n)$ is completely analogous.

The case of $\mu_1(E) \neq 0$ is proved in a similar way, noticing also that over the 1-skeleton X_1 of X there are no obstructions to lifting the bundle because there the bundle is trivialized on contractible open sets. □

Remark 3.11. Notice that, when $\mu_1 \neq 0$, a $\mathrm{PO}(n)$-bundle lifts to an $\mathrm{O}(n)$-bundle if and only if ot lifts to a $\mathrm{Pin}(n)$-bundle. This is clear since, when $\mu_1 \neq 0$, the 0 in $\{0, \omega_n\}$ is the class of 0 and 1 in the quotient $(\pi_1 PO(n)/\Gamma_{\mu_1})/\pi_0\mathrm{PO}(n)$ (cf. Remark 3.8).

Another way to see that a $\mathrm{PO}(n)$-bundle lifts to a $\mathrm{Pin}(n)$-bundle if it lifts to an $\mathrm{O}(n)$-bundle is as follows. Suppose that E is a real projective bundle, with $\mu_1(E) \neq 0$, and which is the projectivization of a real vector bundle W. Since the projection from $\mathrm{O}(n)$ onto $\mathrm{PO}(n)$ preserves components of the groups (because n is even), $w_1(W) = \mu_1(E) \neq 0$ where $w_1(W)$ is the first Stiefel-Whitney class of W. So the first Stiefel-Whitney class of all lifts of E to $\mathrm{O}(n)$ is the same (another way to see this is to note that $w_1(W \otimes F) = w_1(W)$, for any real line bundle F, whenever $\mathrm{rk}(W)$ is even). Nevertheless, different lifts of E can have different second Stiefel-Whitney class because their first Stiefel-Whitney class is non-zero. In fact, given a real vector bundle W of rank n on X with $w_1(W) \neq 0$, it is easy to see that there exists a real line bundle F such that $w_2(W) \neq w_2(W \otimes F)$ (note that $w_2(W \otimes F) = w_2(W) + w_1(W)w_1(F)$). This is the reason why the second Stiefel-Whitney class "disappears" on projective bundles with $\mu_1 \neq 0$. Hence either W or $W \otimes F$ has $w_2 = 0$ and therefore lifts to a $\mathrm{Pin}(n)$-bundle. Choosing this lift of E, we see that E lifts to a $\mathrm{Pin}(n)$-bundle.

Remark 3.12. If E is a real projective bundle with $\mu_1(E) = 0$ and $\mu_2(E) \in \{0, 1\}$, then also the second Stiefel-Whitney class of the lifts is well defined and is equal to μ_2.

From Proposition 3.3, we obtain a full topological classification of real projective bundles over X.

Theorem 3.13. *Let $n \geqslant 4$ be even, and let X be a closed oriented surface of genus $g \geqslant 2$. Then continuous $\mathrm{PO}(n)$-bundles over X are classified by*

$$(\mu_1, \mu_2) \in (\{0\} \times \{0, 1, \omega_n\}) \cup \left(\left((\mathbb{Z}_2)^{2g} \setminus \{0\}\right) \times \mathbb{Z}_2\right).$$

4. Representations and topological classification

4.1. **Representations of $\pi_1 X$ in $\mathrm{PGL}(n,\mathbb{R})$.** In this section we begin our analysis of the space $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$. The first thing to do is to define a topological invariant of a representation $\rho : \pi_1 X \to \mathrm{PGL}(n,\mathbb{R})$. From Definition 2.3, we already know that is done via the correspondence between representations and flat bundles.

Definition 4.1. *Let ρ be a representation $\pi_1 X \to \mathrm{PGL}(n,\mathbb{R})$ and let $E_\rho = \widetilde{X} \times_\rho \mathrm{PGL}(n,\mathbb{R})$, the principal flat $\mathrm{PGL}(n,\mathbb{R})$-bundle over X associated to ρ, viewed as a continuous bundle. The* topological invariants $\mu_1(\rho)$ and $\mu_2(\rho)$ of ρ *are defined by $\mu_1(\rho) = \mu_1(E_\rho)$ and $\mu_2(\rho) = \mu_2(E_\rho)$ where $\mu_1(E_\rho)$ and $\mu_2(E_\rho)$ are the invariants defined in (3.3) and (3.4). Thus*

$$\mu_1(\rho) = \mu_1(E_\rho) \in \mathbb{Z}_2^{2g} \text{ and } \mu_2(\rho) = \mu_2(E_\rho) = \begin{cases} \{0,1,\omega_n\} & \text{if } \mu_1(\rho) = 0 \\ \{0,\omega_n\} & \text{if } \mu_1(\rho) \neq 0. \end{cases}$$

Recall that our goal is to determine the number of connected components of

$$\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})} = \mathrm{Hom}^{\mathrm{red}}(\pi_1 X, \mathrm{PGL}(n,\mathbb{R}))/\mathrm{PGL}(n,\mathbb{R})$$

for $n \geqslant 4$ even.

For fixed topological invariants,

$$(\mu_1,\mu_2) \in (\{0\} \times \{0,1,\omega_n\}) \cup \left(\left((\mathbb{Z}_2)^{2g} \setminus \{0\}\right) \times \mathbb{Z}_2\right),$$

we define the subspace $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1,\mu_2)$ of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ as

$$\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1,\mu_2) = \{\rho \mid \mu_i(\rho) = \mu_i, i = 1,2\}.$$

4.2. **Non-emptiness of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1,\mu_2)$.** For fixed invariants (μ_1,μ_2), we will now study the non-emptiness of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1,\mu_2)$. To do so, we will see how to detect the classes μ_1 and μ_2 of a flat $\mathrm{PGL}(n,\mathbb{R})$-bundle, using only the corresponding representation of $\pi_1 X$ in $\mathrm{PGL}(n,\mathbb{R})$.

Let $\mathrm{PGL}(n,\mathbb{R})_0$ denote the identity component of $\mathrm{PGL}(n,\mathbb{R})$ and let $\mathrm{PGL}(n,\mathbb{R})^-$ denote the component of $\mathrm{PGL}(n,\mathbb{R})$ which does not contain the identity.

Definition 4.2. *Given a representation $\rho : \pi_1 X \to \mathrm{PGL}(n,\mathbb{R})$, let $A_1, B_1, \ldots, B_g \in \mathrm{PGL}(n,\mathbb{R})$ be the images of the generators of $\pi_1 X$ by ρ. The* invariant δ_1 *of ρ is defined as*

$$\delta_1(\rho) \in (\mathbb{Z}_2)^{2g}$$

to be such that:

- *the $(2i-1)$-th coordinate of $\delta_1(\rho)$ is 0 if $A_i \in \mathrm{PGL}(n,\mathbb{R})_0$;*
- *the $(2i-1)$-th coordinate of $\delta_1(\rho)$ is 1 if $A_i \in \mathrm{PGL}(n,\mathbb{R})^-$;*
- *the $2i$-th coordinate of $\delta_1(\rho)$ is 0 if $B_i \in \mathrm{PGL}(n,\mathbb{R})_0$;*
- *the $2i$-th coordinate of $\delta_1(\rho)$ is 1 if $B_i \in \mathrm{PGL}(n,\mathbb{R})^-$.*

Obviously, $\delta_1(\rho)$ is the obstruction to reducing the representation to $\mathrm{PGL}(n,\mathbb{R})_0$. So we have

$$\delta_1(\rho) = \mu_1(\rho). \tag{4.1}$$

In order to obtain something similar for the invariant μ_2, we will consider representations in the maximal compact $\mathrm{PO}(n)$. In terms of topological invariants, there is no loss of generality in doing this and has the advantage that these representations are automatically reductive due to the compactness of $\mathrm{PO}(n)$.

Let $p' : \mathrm{O}(n) \to \mathrm{PO}(n)$ be the projection. Choose $A_i' \in p'^{-1}(A_i)$ and $B_i' \in p'^{-1}(B_i)$ in $\mathrm{O}(n)$, and consider the product

$$\prod_{i=1}^{g}[A_i', B_i'].$$

Since $\ker(p') \subseteq Z(\mathrm{O}(n))$, the value of this product does not depend on the choice of the lifts A_i', B_i' and it is the obstruction to lifting $\rho : \pi_1 X \to \mathrm{PO}(n)$ to a representation $\rho' : \pi_1 X \to \mathrm{O}(n)$.

Definition 4.3. *Let $\rho : \pi_1 X \to \mathrm{PO}(n)$ be a representation and let $A_1, B_1, \dots, B_g \in \mathrm{PO}(n)$ be the images of the generators of $\pi_1 X$ by ρ. The* invariant δ_2 *of ρ is defined as*

$$\delta_2(\rho) = \prod_{i=1}^{g}[A_i', B_i'] \in \{\pm I_n\}$$

where A_i' and B_i' are lifts of A_i and B_i, respectively, to $\mathrm{O}(n)$.

Remark 4.4. In this remark (and only here) we will *not* use the additive notation of Notation 3.7, since here we are going to work on the $\mathrm{Pin}(n)$ and $\mathrm{Spin}(n)$ group (which are not abelian). If $\delta_2(\rho) = I_n$, one can ask whether $\rho' : \pi_1 X \to \mathrm{O}(n)$ lifts to a representation $\rho'' : \pi_1 X \to \mathrm{Pin}(n)$ under the projection $p'' : \mathrm{Pin}(n) \to \mathrm{O}(n)$ and the way to measure the obstruction to the existence of this lift is exactly the same as in the previous case: choose lifts $\tilde{A}_i \in p''^{-1}(A_i')$ and $\tilde{B}_i \in p''^{-1}(B_i')$, for all $i \in \{1, \dots, g\}$, and consider the value

$$\prod_{i=1}^{g}[\tilde{A}_i, \tilde{B}_i] \in \{\pm 1\} = p''^{-1}(I_n). \tag{4.2}$$

Again this is well-defined because $\ker(p'') \subseteq Z(\mathrm{Pin}(n))$ and it is the obstruction to lifting ρ' to a representation $\rho'' : \pi_1 X \to \mathrm{Pin}(n)$.

If $\tilde{p} : \mathrm{Pin}(n) \to \mathrm{PO}(n)$ is the universal cover ($\tilde{p} = p' \circ p''$) then, in the case $\delta_1(\rho) \neq 0$, we could not use the same procedure as in the previous cases to measure directly the obstruction to lifting ρ to a representation $\tilde{\rho} : \pi_1 X \to \mathrm{Pin}(n)$ because $\ker(\tilde{p}) = \{\pm 1, \pm\omega_n\} \not\subset Z(\mathrm{Pin}(n)) = \{\pm 1\}$. In principle, the above procedure only gives partial information about the possible lifts of ρ to $\mathrm{Pin}(n)$: if $\delta_2(\rho) = -I_n$ then clearly ρ does not lift to $\mathrm{Pin}(n)$; if $\delta_2(\rho) = I_n$ and the lift ρ' of ρ to $\mathrm{O}(n)$ lifts to $\mathrm{Pin}(n)$ then ρ lifts to $\mathrm{Pin}(n)$; if $\delta_2(\rho) = I_n$ but the lift ρ' of ρ to $\mathrm{O}(n)$ does not lift to $\mathrm{Pin}(n)$, we cannot conclude that ρ does not lift to $\mathrm{Pin}(n)$ because if we change the lift of ρ to $\mathrm{O}(n)$ (or, equivalently, if we change the lifts of some of the generators A_i and B_i) then this new representation of $\pi_1 X$ on $\mathrm{O}(n)$ might lift to $\mathrm{Pin}(n)$. In fact, this is always possible, if $\mu_1(\rho) \neq 0$ (i.e., if $\delta_1(\rho) \neq 0$). To see this, suppose ρ is such that $\delta_1(\rho) \neq 0$ and $\delta_2(\rho) = I_n$. Then $\prod_{i=1}^{g}[A_i', B_i'] = I_n$ for any lifts of A_i and of B_i. On the other hand, there is some $A_{i_0} \in \mathrm{PO}(n)^-$, so $A_{i_0}' \in \mathrm{O}(n)^-$. If $\prod_{i=1}^{g}[\tilde{A}_i, \tilde{B}_i] = -1$, then $-B_{i_0}' \in \mathrm{O}(n)$ is other lift of B_{i_0} and, choosing it, we have a new lift of ρ to $\mathrm{O}(n)$. Lifting $-B_{i_0}'$ to $\mathrm{Pin}(n)$ we obtain $\pm\omega_n \tilde{B}_{i_0}$ and now, since $\tilde{A}_{i_0}^{-1} \in \mathrm{Pin}(n)^-$ and since $\tilde{B}_{i_0}$ and $\tilde{B}_{i_0}^{-1}$ belong to the same component of $\mathrm{Pin}(n)$, we have

$$\tilde{A}_1\tilde{B}_1\tilde{A}_1^{-1}\tilde{B}_1^{-1}\cdots\tilde{A}_{i_0}\omega_n\tilde{B}_{i_0}\tilde{A}_{i_0}^{-1}\tilde{B}_{i_0}^{-1}\omega_n^{-1}\cdots\tilde{A}_g\tilde{B}_g\tilde{A}_g^{-1}\tilde{B}_g^{-1} = -\prod_{i=1}^{g}[\tilde{A}_i, \tilde{B}_i] = 1.$$

Thus, if $\delta_1(\rho) \neq 0$, the value of (4.2) does not give any new information.

If $\delta_1(\rho) = 0$, ρ reduces to a representation in $\mathrm{PSO}(n)$ and, as $\ker(\tilde{p}) \subseteq Z(\mathrm{Spin}(n))$ where $\tilde{p} : \mathrm{Spin}(n) \to \mathrm{PSO}(n)$, we have a well defined obstruction $\tilde{\delta}(\rho)$ to lifting ρ to $\mathrm{Spin}(n)$, defined as follows:

Definition 4.5. *Let $n \geqslant 4$ be even. Let $\rho : \pi_1 X \to \mathrm{PO}(n)$ be a representation with $\delta_1(\rho) = 0$ and let $A_1, B_1, \ldots, B_g \in \mathrm{PO}(n)$ be the images of the generators of $\pi_1 X$ by ρ. The* invariant $\tilde{\delta}$ *of ρ is defined as*

$$\tilde{\delta}(\rho) = \prod_{i=1}^{g} [\tilde{A}_i, \tilde{B}_i] \in \{0, 1, \omega_n\}$$

where $\tilde{A}_i$ and $\tilde{B}_i$ are lifts of A_i and B_i, respectively, to $\mathrm{Spin}(n)$.

Again, in $\{0, 1, \omega_n\}$ of this definition we have identified ω_n and $-\omega_n$ due to the $\mathrm{PO}(n)$-equivalence.

Recall Definition 4.1. From Proposition 3.10 and from what we have seen, we have the following lemma:

Lemma 4.6. *Let $n \geqslant 4$ be even. The following equivalences hold:*

$$\delta_2(\rho) = -I_n \Longleftrightarrow \mu_2(\rho) = \omega_n$$

and, if $\delta_1(\rho) \neq 0$,

$$\delta_2(\rho) = I_n \Longleftrightarrow \mu_2(\rho) = 0.$$

If $\delta_1(\rho) = 0$, we have

$$\tilde{\delta}(\rho) = \mu_2(\rho) \in \{0, 1, \omega_n\}.$$

Proposition 4.7. *Let $n \geqslant 4$ even be given. Then, the space $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1, \mu_2)$ is non-empty, for each pair $(\mu_1, \mu_2) \in (\{0\} \times \{0, 1, \omega_n\}) \cup \big(((\mathbb{Z}_2)^{2g} \setminus \{0\}) \times \mathbb{Z}_2\big)$.*

Proof. Let us start by seeing that $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1, \omega_n)$ is non-empty for each $\mu_1 \in (\mathbb{Z}_2)^{2g}$. To do so we will find an explicit representation of $\pi_1 X$ in $\mathrm{PO}(n)$ (hence in $\mathrm{PGL}(n, \mathbb{R})$) with these invariants. From (4.1) and Lemma 4.6, in order to show that $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1, \omega_n)$ is non-empty we only need to find a reductive representation $\rho : \pi_1 X \to \mathrm{PO}(n) \subset \mathrm{PGL}(n, \mathbb{R})$ with

$$\delta_1(\rho) = \mu_1$$

and

$$\delta_2(\rho) = -I_n.$$

In other words, from the definition of $\delta_1(\rho)$, we need to find $n \times n$ invertible matrices A_i' and B_i' such that

- $A_i' \in \mathrm{SO}(n)$ if and only if the $(2i-1)$-th coordinate of $\delta_1(\rho)$ is 0;
- $A_i' \in \mathrm{O}(n)^-$ if and only if the $(2i-1)$-th coordinate of $\delta_1(\rho)$ is 1;
- $B_i' \in \mathrm{SO}(n)$ if and only if the $2i$-th coordinate of $\delta_1(\rho)$ is 0;
- $B_i' \in \mathrm{O}(n)^-$ if and only if the $2i$-th coordinate of $\delta_1(\rho)$ is 1.

and, from the definition of $\delta_2(\rho)$, which satisfy the equality

$$\prod_{i=1}^{g} [A_i', B_i'] = -I_n.$$

As we are using the compact group $\mathrm{PO}(n)$, the reductiveness condition on the representation is automatically satisfied.

Let us start with the following orthogonal matrices:

$$X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ X_2' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ Y_2 = X_2',\ Y_2' = -X_2' \text{ and } Z_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that X_2, X_2' and Z_2 are pairwise anti-commuting and that Y_2 and Y_2' commute. For $n \geqslant 4$ even, define

$$X_n = \begin{pmatrix} X_2 & 0 \\ 0 & X_{n-2} \end{pmatrix},\ X_n' = \begin{pmatrix} X_2' & 0 \\ 0 & X_{n-2}' \end{pmatrix},\ Y_n = \begin{pmatrix} Y_2 & 0 \\ 0 & I_{n-2} \end{pmatrix},\ Y_n' = \begin{pmatrix} Y_2' & 0 \\ 0 & I_{n-2} \end{pmatrix}$$

$$Z_n = \begin{pmatrix} Z_2 & 0 \\ 0 & X_{n-2}' \end{pmatrix},\ W_n = \begin{pmatrix} X_2 & 0 \\ 0 & Z_{n-2} \end{pmatrix} \text{ and } W_n' = \begin{pmatrix} Z_2 & 0 \\ 0 & X_{n-2} \end{pmatrix}.$$

We have the following facts:

- $X_n, X_n' \in \mathrm{SO}(n) \iff n = 0 \bmod 4$;
- $X_n, X_n' \in \mathrm{O}(n)^- \iff n = 2 \bmod 4$;
- $Y_n, Y_n' \in \mathrm{O}(n)^-$ for all n even;
- $Z_n \in \mathrm{O}(n)^- \iff n = 0 \bmod 4$;
- $Z_n \in \mathrm{SO}(n) \iff n = 2 \bmod 4$;
- $W_n, W_n' \in \mathrm{SO}(n) \iff n = 2 \bmod 4$;
- $W_n, W_n' \in \mathrm{O}(n)^- \iff n = 0 \bmod 4$;
- X_n and X_n' anti-commute for all n even;
- Y_n and Y_n' commute for all n even;
- Z_n anti-commutes with X_n for all n even;
- W_n and W_n' anti-commute for all $n \geqslant 4$ even.

Using these orthogonal matrices and the identity I_n it is possible to construct the required representation. The important thing to note is that for each n we always have a pair of commuting and anti-commuting matrices both in $\mathrm{SO}(n)$ or both in $\mathrm{O}(n)^-$ or one in $\mathrm{SO}(n)$ and the other in $\mathrm{O}(n)^-$.

The case of commuting matrices is easy: if one of the matrices is to be in $\mathrm{SO}(n)$, use the identity I_n; if both must be in $\mathrm{O}(n)^-$, use Y_n and Y_n':

Commuting matrices	$\mathrm{SO}(n)$, $\mathrm{SO}(n)$	$\mathrm{SO}(n)$, $\mathrm{O}(n)^-$	$\mathrm{O}(n)^-$, $\mathrm{O}(n)^-$
n even	I_n, any	I_n, any	Y_n, Y_n'

The case of anti-commuting matrices is also easy:

Anti-commuting matrices	$\mathrm{SO}(n)$, $\mathrm{SO}(n)$	$\mathrm{SO}(n)$, $\mathrm{O}(n)^-$	$\mathrm{O}(n)^-$, $\mathrm{O}(n)^-$
$n = 0 \bmod 4$	X_n, X_n'	X_n, Z_n	W_n, W_n'
$n = 2 \bmod 4$	W_n, W_n'	Z_n, X_n	X_n, X_n'

This shows that given $\mu_1 = \delta_1 = (x_1, x_2, \ldots, x_{2g}) \in (\mathbb{Z}_2)^{2g}$, we can choose A_1' and B_1' in $\mathrm{SO}(n)$ or in $\mathrm{O}(n)^-$ depending on x_1 and on x_2 and such that $[A_1', B_1'] = -I_n$. Then, for $i \geqslant 2$, we can also choose A_i' and B_i' accordingly to x_{2i-1} and to x_{2i} respectively, and such that $[A'_i, B'_i] = I_n$. Hence $\prod_{i=1}^{g}[A'_i, B'_i] = -I_n$ as wanted. Putting $\rho(a_i) = p_2(A'_i)$ and $\rho(b_i) = p_2(B'_i)$ gives a representation $\rho : \pi_1 X \to \mathrm{PGL}(n,\mathbb{R})$ with the given μ_1 and $\mu_2(\rho) = \omega_n$.

For the other cases, the proof is similar but easier. For $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1, 0)$ with $\mu_1 \neq 0$, we have, from (4.1) and Lemma 4.6, to find a representation ρ with $\delta_1(\rho) = \mu_1$ and $\delta_2(\rho) = I_n$ and this is done in same way as above, using the first table.

The cases $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(0, \mu_2)$ with $\mu_2 = 0, 1$ should be dealt with similarly, but now we would need to consider the invariant $\tilde{\delta}$ of Definition 4.5 and, hence, elements on the $\mathrm{Pin}(n)$ group. Instead, note that, since $\mu_1 = 0$, we are looking for representations on the connected group $\mathrm{PGL}(n,\mathbb{R})_0$. Hence, the non-emptiness of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(0, \mu_2)$ follows from Proposition 7.7 of [21]. □

The map in $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ which takes a class ρ to $(\mu_1(\rho), \mu_2(\rho))$ is continuous hence, if classes lie in the same connected component of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$, they must have the same topological invariants. From this and from Theorem 3.13 and Proposition 4.7, we conclude that, for $n \geqslant 4$ even, $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ has at least $2^{2g+1} + 1$ connected components this being the number of topological invariants.

It remains to see whether, for each pair (μ_1, μ_2), $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1, \mu_2)$ is connected or not, and it is now that the theory of Higgs bundles comes into play.

5. $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundles and $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles

In this section we begin the study of $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundles and explain why and how one wants to work with another group instead of $\mathrm{PGL}(n,\mathbb{R})$.

We begin by defining $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundles, using Definition 2.4. Recall that $\mathrm{PO}(n)^{\mathbb{C}} = \mathrm{PO}(n,\mathbb{C}) = \mathrm{O}(n,\mathbb{C})/\mathbb{Z}_2$.

Definition 5.1. *A* $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundle *over X is a pair (E, Φ), where E is a holomorphic principal* $\mathrm{PO}(n,\mathbb{C})$*-bundle and $\Phi \in H^0(X, E \times_{\mathrm{PO}(n,\mathbb{C})} \mathfrak{so}(n,\mathbb{C})^{\perp} \otimes K)$ where $\mathfrak{so}(n,\mathbb{C})^{\perp}$ is the vector space of $n \times n$ symmetric and traceless complex matrices.*

We would like to work naturally with holomorphic vector bundles associated to the corresponding $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundles. However, this will not be done directly because $\mathrm{PO}(n,\mathbb{C})$ does not have a standard action on $\mathbb{C}^n$, and to fix this we use a standard procedure as follows. Enlarge the complex orthogonal group $\mathrm{O}(n,\mathbb{C})$ so that it still has a canonical action on $\mathbb{C}^n$ and such that it has a non-discrete centre, and consider the sheaf of holomorphic functions with values in this centre. Then, the second cohomology of X of this sheaf vanishes, so there is no obstruction to lifting a holomorphic $\mathrm{PO}(n,\mathbb{C})$-bundle to a holomorphic bundle with this new structure group, and hence to do the same to Higgs bundles with the corresponding groups.

Let us then consider the group $\mathrm{GL}(n,\mathbb{R}) \times \mathrm{U}(1)$, the normal subgroup $\{(I_n, 1), (-I_n, -1)\} \cong \mathbb{Z}_2 \lhd \mathrm{GL}(n,\mathbb{R}) \times \mathrm{U}(1)$ and the corresponding quotient group

$$\mathrm{GL}(n,\mathbb{R}) \times_{\mathbb{Z}_2} \mathrm{U}(1) = (\mathrm{GL}(n,\mathbb{R}) \times \mathrm{U}(1))/\mathbb{Z}_2.$$

Its maximal compact is $\mathrm{O}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$, whose complexification is $\mathrm{O}(n,\mathbb{C}) \times_{\mathbb{Z}_2} \mathbb{C}^*$.

Notation 5.2. From now on, we shall write

$$\mathrm{EGL}(n,\mathbb{R}) = \mathrm{GL}(n,\mathbb{R}) \times_{\mathbb{Z}_2} \mathrm{U}(1)$$

and

$$\mathrm{EO}(n) = \mathrm{O}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$$

as well as

$$\mathrm{EO}(n,\mathbb{C}) = \mathrm{O}(n,\mathbb{C}) \times_{\mathbb{Z}_2} \mathbb{C}^*.$$

The "E" stands for enhanced or extended.

Applying again Definition 2.4, we give now a concrete definition of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle. Notice that, if $\overline{G} = \mathrm{EGL}(n,\mathbb{R})$, then a maximal compact subgroup of $\overline{G}$ is $\overline{H} = \mathrm{EO}(n)$, so $\overline{H}^{\mathbb{C}} = \mathrm{EO}(n,\mathbb{C})$. Also, $\overline{\mathfrak{g}}^{\mathbb{C}} = \overline{\mathfrak{h}}^{\mathbb{C}} \oplus \overline{\mathfrak{m}}^{\mathbb{C}}$ where $\overline{\mathfrak{g}}^{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C}) \oplus \mathbb{C}$, $\overline{\mathfrak{h}}^{\mathbb{C}} = \mathfrak{o}(n,\mathbb{C}) \oplus \mathbb{C}$ and $\overline{\mathfrak{m}}^{\mathbb{C}} = \{(A,0) \in \overline{\mathfrak{g}}^{\mathbb{C}} \mid A = A^T\}$ is naturally isomorphic to the space of symmetric matrices.

Definition 5.3. *A* $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle *over X is a pair (E,Φ), where E is a holomorphic principal* $\mathrm{EO}(n,\mathbb{C})$*-bundle and* $\Phi \in H^0(X, E\times_{\mathrm{EO}(n,\mathbb{C})} \overline{\mathfrak{m}}^{\mathbb{C}} \otimes K)$*, where* $\overline{\mathfrak{m}}^{\mathbb{C}} = \{(A,0) \in \overline{\mathfrak{g}}^{\mathbb{C}} \mid A = A^T\}$.

Proposition 5.4. *Every* $\mathrm{PGL}(n,\mathbb{R})$*-Higgs bundle* (E,Φ) *on X lifts to a* $\mathrm{EGL}(n,\mathbb{R})$*-Higgs bundle* $(\overline{E},\overline{\Phi})$.

Proof. We have the following short exact sequence of groups

$$0 \longrightarrow \mathbb{C}^* \xrightarrow{\ i\ } \mathrm{EO}(n,\mathbb{C}) \xrightarrow{\ p\ } \mathrm{PO}(n,\mathbb{C}) \longrightarrow 0$$

where $i(\lambda) = [(I_n,\lambda)]$ and $p([(w,\lambda)]) = [w]$.

Consider the sheaf $\mathrm{EO}(n,\mathcal{O})$ of holomorphic functions on X with values in $\mathrm{EO}(n,\mathbb{C})$. The above short exact sequence induces the following exact sequence

$$H^1(X,\mathcal{O}^*) \longrightarrow H^1(X,\mathrm{EO}(n,\mathcal{O})) \xrightarrow{\ p_*\ } H^1(X,\mathrm{PO}(n,\mathcal{O})) \longrightarrow 0 \tag{5.1}$$

hence, we see that there is no obstruction to lifting E to a principal $\mathrm{EO}(n,\mathbb{C})$-bundle

$$\overline{E} \in H^1(X,\mathrm{EO}(n,\mathcal{O})).$$

Write $G = \mathrm{PGL}(n,\mathbb{R})$, $H = \mathrm{PO}(n)$, $\overline{G} = \mathrm{EGL}(n,\mathbb{R})$ and $\overline{H} = \mathrm{EO}(n)$. We have $\overline{\mathfrak{g}}^{\mathbb{C}} = \overline{\mathfrak{h}}^{\mathbb{C}} \oplus \overline{\mathfrak{m}}^{\mathbb{C}}$, where

$$\overline{\mathfrak{g}}^{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C}) \oplus \mathbb{C} \supset \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C} = \mathfrak{g}^{\mathbb{C}} \oplus \mathbb{C}$$
$$\overline{\mathfrak{h}}^{\mathbb{C}} = \mathfrak{o}(n,\mathbb{C}) \oplus \mathbb{C} \supset \mathfrak{so}(n,\mathbb{C}) \oplus \mathbb{C} = \mathfrak{h}^{\mathbb{C}} \oplus \mathbb{C}$$

and

$$\overline{\mathfrak{m}}^{\mathbb{C}} = \{(A,0) \in \overline{\mathfrak{g}}^{\mathbb{C}} \mid A = A^T\}.$$

If we identify $\overline{\mathfrak{m}}^{\mathbb{C}}$ with $\{A \in \mathfrak{gl}(n,\mathbb{C}) \mid A = A^T\}$, then $\mathfrak{m}^{\mathbb{C}} = \mathfrak{so}(n,\mathbb{C})^{\perp}$ is the subspace of matrices in $\overline{\mathfrak{m}}^{\mathbb{C}}$ with trace equal to zero.

Now, the isotropy action of $H^{\mathbb{C}} = \mathrm{PO}(n,\mathbb{C})$ in $\mathfrak{m}^{\mathbb{C}}$ is given by (where $[A] \in \mathrm{PO}(n,\mathbb{C})$ and $B \in \mathfrak{m}^{\mathbb{C}}$)

$$\mathrm{Ad}([A])(B) = ABA^{-1} = ABA^T \tag{5.2}$$

and the isotropy action of $\overline{H}^{\mathbb{C}}$ in $\overline{\mathfrak{m}}^{\mathbb{C}}$ is given by (where $[(A,\lambda)] \in \overline{H}^{\mathbb{C}}$ and $(B,0) \in \overline{\mathfrak{m}}^{\mathbb{C}}$)

$$\overline{\mathrm{Ad}}([(A,\lambda)])(B,0) = (ABA^{-1},0) = (ABA^T,0). \tag{5.3}$$

We have the bundle $\overline{E}$, which is a lift of E and we have a map $\pi : \overline{E} \to E$ induced by the projection $\overline{H}^{\mathbb{C}} \to H^{\mathbb{C}}$. Now, $\Phi \in H^0(X, E\times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}} \otimes K)$ can be thought as a $H^{\mathbb{C}} \times \mathbb{C}^*$-equivariant map $E \times_X E_K \to \mathfrak{m}^{\mathbb{C}}$ where E_K is the $\mathbb{C}^*$-principal bundle associated to K (i.e., the frame bundle associated to K), and $E \times_X E_K$ is the fibred product over X of E and E_K. Let $\overline{\Phi} : \overline{E} \times_X E_K \to \overline{\mathfrak{m}}^{\mathbb{C}}$ be defined by

$$\overline{\Phi} = (i \otimes 1_K)\Phi(\pi \times_X 1_{E_K}) \tag{5.4}$$

where $i : \mathfrak{m}^{\mathbb{C}} \hookrightarrow \overline{\mathfrak{m}}^{\mathbb{C}}$ is the inclusion. So, we have the commutative diagram

$$\begin{CD}
\overline{E} \times_X E_K @>{\overline{\Phi}}>> \overline{\mathfrak{m}}^{\mathbb{C}} \\
@V{\pi \times_X 1_{E_K}}VV @AA{i}A \\
E \times_X E_K @>>{\Phi}> \mathfrak{m}^{\mathbb{C}}.
\end{CD}$$

From (5.2) and (5.3) follows that $\overline{\Phi}$ is $\overline{H}^{\mathbb{C}} \times \mathbb{C}^*$-equivariant. Hence $\Phi \in H^0(X, E \times_{H^{\mathbb{C}}} \mathfrak{m}^{\mathbb{C}} \otimes K)$ induces, in a natural way, a Higgs field $\overline{\Phi} \in H^0(X, \overline{E} \times_{\overline{H}^{\mathbb{C}}} \overline{\mathfrak{m}}^{\mathbb{C}} \otimes K)$ given by (5.4).

It follows that $(\overline{E}, \overline{\Phi})$ is an $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundle. $\square$

Consider the actions of $\mathrm{EO}(n, \mathbb{C})$ on $\mathbb{C}^n$ and on $\mathbb{C}$ induced, respectively, by the group homomorphisms

$$\mathrm{EO}(n, \mathbb{C}) \longrightarrow \mathrm{GL}(n, \mathbb{C}), \quad [(w, \lambda)] \mapsto \lambda w \tag{5.5}$$

and

$$\mathrm{EO}(n, \mathbb{C}) \longrightarrow \mathbb{C}^*, \quad [(w, \lambda)] \mapsto \lambda^2. \tag{5.6}$$

and by the corresponding standard actions of $\mathrm{GL}(n, \mathbb{C})$ and $\mathbb{C}^*$.

Proposition 5.5. *Let $(\overline{E}, \overline{\Phi})$ be an $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundle on X. Through the actions (5.5) and (5.6) of $\mathrm{EO}(n, \mathbb{C})$ on $\mathbb{C}^n$ and on $\mathbb{C}$, associated to $(\overline{E}, \overline{\Phi})$ there is a quadruple $(V, L, Q, \overline{\Phi})$ on X, where V is a holomorphic rank n vector bundle, L is a holomorphic line bundle, Q is a nowhere degenerate quadratic form on V with values in L and $\overline{\Phi} \in H^0(X, S^2_Q V \otimes K)$ where $S^2_Q V$ denotes the bundle of endomorphisms of V which are symmetric with respect to Q.*

Proof. Keeping the notation of the proof of Proposition 5.4, let $\overline{H}^{\mathbb{C}} = \mathrm{EO}(n, \mathbb{C})$. From the actions (5.5) and (5.6) we define, respectively, the vector bundle $V = \overline{E} \times_{\overline{H}^{\mathbb{C}}} \mathbb{C}^n$ and the line bundle $L = \overline{E} \times_{\overline{H}^{\mathbb{C}}} \mathbb{C}$.

With these two bundles we have a $\overline{H}^{\mathbb{C}}$-equivariant map

$$Q : \overline{E} \times_{\overline{H}^{\mathbb{C}}} (\mathbb{C}^n \otimes \mathbb{C}^n) \longrightarrow \overline{E} \times_{\overline{H}^{\mathbb{C}}} \mathbb{C}$$

given fibrewise by

$$v \otimes u \mapsto \sum v_i u_i = \langle v, u \rangle$$

where $\overline{H}^{\mathbb{C}}$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n$ by $[(w, \lambda)] \mapsto \lambda w \otimes \lambda w$ and on $\mathbb{C}$ as above. In other words

$$Q : V \otimes V \longrightarrow L$$

is a nowhere degenerate quadratic form on V with values in L.

Since $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{o}(n, \mathbb{C}) \oplus \overline{\mathfrak{m}}^{\mathbb{C}}$, we have $\overline{E}(\overline{\mathfrak{m}}^{\mathbb{C}}) = \overline{E} \times_{\overline{H}^{\mathbb{C}}} \overline{\mathfrak{m}}^{\mathbb{C}} \subset \overline{E} \times_{\overline{H}^{\mathbb{C}}} \mathfrak{gl}(n, \mathbb{C}) = \mathrm{End}(V)$ and, indeed, $\overline{E}(\overline{\mathfrak{m}}^{\mathbb{C}}) = S^2_Q V$. Thus $\overline{\Phi} \in H^0(X, S^2_Q V \otimes K)$ hence $\overline{\Phi} = \overline{\Phi}^*$ where $\overline{\Phi}^* : V \to V \otimes K$ is such that $Q(\overline{\Phi} u, v) = Q(u, \overline{\Phi}^* v) \in LK$. This means that the diagram

$$\begin{CD}
V @>{q}>> V^* \otimes L \\
@V{\overline{\Phi}}VV @VV{\overline{\Phi}^t \otimes 1_K \otimes 1_L}V \\
V \otimes K @>{q \otimes 1_K}>> V^* \otimes LK
\end{CD}$$

commutes where $q : V \to V^* \otimes L$ is the isomorphism associated to Q, such that $q^t = q \otimes 1_L$. $\square$

The outcome of these results is that one can work with $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles instead of $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundles with the advantage that in the former case we work with the objects (V,L,Q,Φ), involving holomorphic vector bundles. That is what we will do from now on.

We shall also call $\mathrm{EGL}(n,\mathbb{R})$-*Higgs bundles* the quadruples (V,L,Q,Φ) mentioned in the previous proposition.

Given an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle (V,L,Q,Φ), we associate a $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundle (E,Φ_0), where E is given by the projection in sequence (5.1) and Φ_0 is obtained by projecting Φ to its traceless part.

Proposition 5.6. *Given a* $\mathrm{PGL}(n,\mathbb{R})$*-Higgs bundle* (E,Φ_0)*, it is possible to choose a lift of* (E,Φ) *to an* $\mathrm{EGL}(n,\mathbb{R})$*-Higgs bundle* (V,L,Q,Φ) *such that* L *is trivial or* $\deg(L)=1$.

Proof. From (5.1) and from the actions (5.5) and (5.6) defining V and L, two $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles (V,L,Q,Φ) and (V',L',Q',Φ') give rise to the same $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundle if and only if $V'=V\otimes F$ and $L'=L\otimes F^2$ where F is a holomorphic line bundle and $\Phi'_0=\Phi_0$.

Suppose we have an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle (V,L,Q,Φ) associated to (E,Φ_0). Since V and $V^*\otimes L$ are isomorphic, we have

$$\deg(V)=n\deg(L)/2.$$

If $\deg(L)$ is even we can choose a square root F of L^{-1} and, from above, $(V\otimes F,\mathcal{O},Q',\Phi\otimes 1_F)$ also projects to (E,Φ_0).

If $\deg(L)$ is odd then there is no such line bundle F. Anyway, we can take F such that $\deg(F)=(1-\deg(L))/2$ and $(V\otimes F,L\otimes F^2,Q',\Phi\otimes 1_F)$ is also a lift of (E,Φ_0) and the degree of the line bundle $L\otimes F^2$ is 1. □

From [3], a $\mathrm{GL}(n,\mathbb{R})$-Higgs bundle is a triple (V,Q,Φ) where V is a rank n holomorphic vector bundle, equipped with a nowhere degenerate quadratic form, and Φ is symmetric endomorphism of V.

Corollary 5.7. *Let* (E,Φ_0) *be a* $\mathrm{PGL}(n,\mathbb{R})$*-Higgs bundle. Let* (V,L,Q,Φ) *be an* $\mathrm{EGL}(n,\mathbb{R})$*-Higgs bundle which is a lift of* (E,Φ_0)*. Then* (E,Φ_0) *lifts to a* $\mathrm{GL}(n,\mathbb{R})$*-Higgs bundle if and only if* $\deg(L)$ *is even.*

Proof. This follows directly from the proof of the above proposition: $\deg(L)$ is even if and only if we can change the lift (V,L,Q,Φ) to $(V\otimes F,\mathcal{O},Q',\Phi\otimes 1_F)$ (where $F^2=L^{-1}$) and this corresponds to a $\mathrm{GL}(n,\mathbb{R})$-Higgs bundle. □

Definition 5.8. *Two* $\mathrm{EGL}(n,\mathbb{R})$*-Higgs bundles* (V,L,Q,Φ) *and* (V',L',Q',Φ') *are* isomorphic *if there is a pair* (f,g) *of isomorphisms* $f:V\to V'$ *and* $g:L\to L'$ *such that the diagrams*

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi'} \\ V\otimes K & \xrightarrow{f\otimes 1_K} & V'\otimes K \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} V & \xrightarrow{f} & V' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} \\ V^*\otimes L & \xrightarrow{(f^t)^{-1}\otimes g} & V'^*\otimes L' \end{array}$$

commute, where q *and* q' *are the isomorphisms associated to* Q *and* Q'*, respectively.*

Now we consider *twisted orthogonal bundles*, i.e., triples (V,L,Q) where V is a holomorphic rank n vector bundle equipped with a nowhere degenerate L-valued quadratic form Q. Of course, two twisted quadratic pairs (V,L,Q) and (V',L',Q') are isomorphic if there is a pair (f,g) of isomorphisms $f:V\to V'$ and $g:L\to L'$ such that $((f^t)^{-1}\otimes g)q=q'f$.

Let E and E' be two principal $\mathrm{EO}(n,\mathbb{C})$-bundles over X and let (V,L,Q) and (V',L',Q') be the corresponding twisted orthogonal bundles through the actions (5.5) and (5.6). It is easy to see that E and E' are isomorphic if and only if (V,L,Q) and (V',L',Q') are isomorphic. Now, consider the notion of isomorphism between two $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles, by applying Definition 2.6. Consider also Definition 5.8 of isomorphism of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles written in terms of vector bundles, through Proposition 5.5. We have then that, when applied to $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles, the isomorphism notion of Definition 2.6 is equivalent to the one of Definition 5.8, and that Proposition 5.5 gives a bijection between isomorphism classes of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles and of the objects (V,L,Q,Φ) which, because of this bijection, we also called $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles.

Furthermore, the construction of Proposition 5.5 can be naturally applied to families of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles parametrized by varieties. Hence, the bijection between isomorphism classes of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles and of the objects (V,L,Q,Φ), naturally extends to families.

6. Moduli space of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles

6.1. **Moduli space of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles.** Recall from Subsection 2.2 the definition of (poly,semi)stability of ($\mathrm{GL}(n,\mathbb{C})$-)Higgs bundles, and let

$$\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(d)$$

denote the moduli space of isomorphism classes of polystable Higgs bundles of rank n and degree d. $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(d)$ is [19] a quasi-projective variety of complex dimension $2n^2(g-1)+2$ which is smooth at the stable locus.

Given an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle (V,L,Q,Φ), we have a natural way of associating to it a Higgs bundle (V,Φ) by simply forgetting the line bundle L and the quadratic form Q.

Definition 6.1. *We say that an* $\mathrm{EGL}(n,\mathbb{R})$*-Higgs bundle* (V,L,Q,Φ) *is* polystable *if the corresponding Higgs bundle* (V,Φ) *is polystable.*

The comparison of stability and strict polystability of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles and that of the associated Higgs bundles is a delicate question, since the conditions might not correspond. Nevertheless, this problem does not occur for polystability due to the correspondence with the solutions to Hitchin's equations.

For a Lie group G, and a maximal compact subgroup H, the Hitchin's equations are equations for a pair (A,Φ) where A is a H-connection on a C^∞ $H^{\mathbb{C}}$-principal bundle $E_{H^{\mathbb{C}}}$ and $\Phi\in\Omega^{1,0}(X,E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$. The equations are

$$(6.1)\qquad \begin{cases} F(A)-[\Phi,\tau(\Phi)]=\lambda\,\omega \\ \bar\partial_A\Phi=0 \end{cases}$$

where $F(A)\in\Omega^2(E_H,\mathfrak{h})$ is the curvature of A, τ is the involution on G which defines H, $\lambda\in Z(\mathfrak{h})$ and ω is the normalized volume form on X so that $\mathrm{vol}(X)=2\pi$. Furthermore, $\bar\partial_A$ is the unique $\bar\partial$ operator on $E_{H^{\mathbb{C}}}$, corresponding to the H-connection A ($\bar\partial_A$ is then the unique holomorphic structure on $E_{H^{\mathbb{C}}}$ induced from the $(0,1)$-form $A^{0,1}$) and the second equation on (6.1) says that Φ is holomorphic with respect to this holomorphic structure. The details of this theory can be found in [14, 11, 6].

Now, given a G-Higgs bundle, $(E_{H^{\mathbb{C}}},\Phi)$ one associates to it a pair (A,Φ) given by a H-connection A on the C^∞ $H^{\mathbb{C}}$-principal bundle $E_{H^{\mathbb{C}}}$ and $\Phi\in\Omega^{1,0}(X,E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}))$ (see, for instance, [14, 11]). The Hitchin-Kobayashi correspondence says that a G-Higgs bundle (E,Φ) is polystable if and only if the associated pair (A,Φ) is a solution to the G-Hitchin's equations (see [11] for the details).

Let us check that, the homomorphism

$$j:\mathrm{EGL}(n,\mathbb{R})\longrightarrow\mathrm{GL}(n,\mathbb{C}) \tag{6.2}$$

given by $j([(M,\alpha)])=\alpha M$ (which precisely corresponds to forgetting L and Q in (V,L,Q,Φ)) induces a correspondence from solutions to the $\mathrm{EGL}(n,\mathbb{R})$-Hitchin's equations to solutions to the $\mathrm{GL}(n,\mathbb{C})$-Hitchin's equations. Now, for $G=\mathrm{EGL}(n,\mathbb{R})$ we have $H=\mathrm{EO}(n)$, hence $\mathfrak{h}=\mathfrak{o}(n)\oplus\mathfrak{u}(1)$ (so $Z(\mathfrak{h})=0\oplus\mathfrak{u}(1)$) and $\mathfrak{m}^{\mathbb{C}}=\mathfrak{o}(n,\mathbb{C})^{\perp}\oplus 0\cong\mathfrak{o}(n,\mathbb{C})^{\perp}$ (the space of symmetric matrices). For $G'=\mathrm{GL}(n,\mathbb{C})$, we have $H'=\mathrm{U}(n)$ and $\mathfrak{h}'=\mathfrak{u}(n)$ (hence $Z(\mathfrak{h})=\mathfrak{u}(1)$) and $(\mathfrak{m}^{\mathbb{C}})'=\mathfrak{gl}(n,\mathbb{C})$. The homomorphism $j:G\to G'$, defined above, restricts to $j:H\to H'$ and therefore yields the map $j_*:\mathfrak{h}\to\mathfrak{h}'$ given by $j_*(M,\alpha)=\alpha+M$ and also $j_*:\mathfrak{m}^{\mathbb{C}}\to(\mathfrak{m}^{\mathbb{C}})'$ given by $j_*(M,0)\mapsto M$. Hence, given a connection $(A,\beta)\in\Omega^1(E_H,\mathfrak{h})$ (where $A\in\Omega^1(E_H,\mathfrak{o}(n))$ and $\beta\in\Omega^1(E_H,\mathfrak{u}(1))$), we obtain the connection $\beta+A\in\Omega^1(E_{H'},\mathfrak{h}')$ in $E_{H'}$.

Proposition 6.2. *Let $(E_{\mathrm{EO}(n,\mathbb{C})},\Phi)$ be an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle and $(E_{\mathrm{GL}(n,\mathbb{C})},\Phi)$ the corresponding $\mathrm{GL}(n,\mathbb{C})$-Higgs bundle obtained from $E_{\mathrm{EO}(n,\mathbb{C})}$ by extending the structure group through the homomorphism j defined in (6.2). Let $((A,\beta),\Phi)$ be the pair associated to $(E_{\mathrm{EO}(n,\mathbb{C})},\Phi)$ and (A',Φ) be the pair associated to $(E_{\mathrm{GL}(n,\mathbb{C})},\Phi)$. Then $((A,\beta),\Phi)$ is a solution to the $\mathrm{EGL}(n,\mathbb{R})$-Hitchin's equations if and only if (A',Φ) is a solution to the $\mathrm{GL}(n,\mathbb{C})$-Hitchin's equations.*

Proof. The previous discussion shows that $j^*A'=\beta+A$ where j is the homomorphism in (6.2). Let $(M,\alpha)\in\Omega^2(E_H,\mathfrak{h})$ the curvature of (A,β). Then, from (6.1), the $\mathrm{EGL}(n,\mathbb{R})$-Hitchin's equations are

$$\begin{cases}(M,\alpha)+[\Phi,\Phi]=(0,\lambda)\,\omega\\ \bar\partial_{(A,\beta)}\Phi=0\end{cases}$$

where $\lambda\in\mathfrak{u}(1)$. Notice that, in this case, $\tau(X)=-X^t$. Moreover $[\Phi,\Phi]\in\Omega^2(E_H,\mathfrak{o}(n))$, and, from the definition of $\mathfrak{m}^{\mathbb{C}}$, we have $\bar\partial_{(A,\beta)}\Phi=\bar\partial_A\Phi$. Hence the above equations are indeed

$$\begin{cases}M+[\Phi,\Phi]=0\\ \alpha=\lambda\,\omega\\ \bar\partial_A\Phi=0\end{cases}.$$

From this we obtain

$$\begin{cases}\alpha+M+[\Phi,\Phi]=\lambda\,\omega\\ \bar\partial_A\Phi=0\end{cases}$$

which are the $\mathrm{GL}(n,\mathbb{C})$-Hitchin's equations. Furthermore, $\alpha+M$ is the curvature of $\beta+A$, and notice again that $\bar\partial_{\beta+A}\Phi=\bar\partial_A\Phi$ due to the definition of $\mathfrak{m}^{\mathbb{C}}$ and to the map $\mathfrak{m}^{\mathbb{C}}\to(\mathfrak{m}^{\mathbb{C}})'$ defined above. So, the equations in $E_{\mathrm{EO}(n,\mathbb{C})}$ and in the $\mathrm{GL}(n,\mathbb{C})$ bundle $E_{\mathrm{GL}(n,\mathbb{C})}$ obtained from $E_{\mathrm{EO}(n,\mathbb{C})}$ by extending the structure group through the homomorphism j, are equivalent, and this proves the result. □

Hence, the previous proposition and the Hitchin-Kobayashi correspondence show that Definition 6.1 is consistent with the notion of polystability for $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles.

Notation 6.3. Write

$$\mathcal{M}_d$$

for the set of isomorphism classes of polystable $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles (V,L,Q,Φ) with $\mathrm{rk}(V)=n$ and $\deg(L)=d$ (hence $\deg(V)=nd/2$).

The group $\mathrm{EGL}(n,\mathbb{R})$ can be seen as a closed subgroup of $\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)$ through

$$[(A,\lambda)]\mapsto(\lambda A,\lambda^2).$$

The moduli space of $\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)$-Higgs bundles is $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(d_1)\times\mathrm{Jac}^{d_2}(X)$, where $\mathrm{Jac}^{d_2}(X)$ is the subspace of the Picard group of the compact Riemann surface X which parametrizes holomorphic line bundles of degree d_2. It is isomorphic to the Jacobian of X.

We shall realize $\mathcal{M}_d$ as a subspace of $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)\times\mathrm{Jac}^d(X)$ and it will be in this subspace that we will work.

Lemma 6.4. *The map*

$$\begin{array}{rccl} i: & \mathcal{M}_d & \longrightarrow & \mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)\times\mathrm{Jac}^d(X) \\ & (V,L,Q,\Phi) & \longmapsto & ((V,\Phi),L). \end{array}$$

is injective.

Proof. First of all we see that we only have to take care with the form Q. Indeed, if $i(V,L,Q,\Phi)=i(V',L',Q',\Phi')$, then there are isomorphisms $f:V\to V'$ such that $\Phi'f=(f\otimes 1_K)\Phi$ and $g:L\to L'$. Therefore (f,g) is an isomorphism between (V,L,Q'',Φ) and (V',L',Q',Φ') where Q'' is given by $q''=(f^t\otimes g^{-1})q'f$.

Consider then the $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles (V,L,Q,Φ) and (V,L,Q',Φ). These are mapped to $((V,\Phi),L)$ and we have to see that (V,L,Q,Φ) and (V,L,Q',Φ) are isomorphic.

Suppose that (V,Φ) is stable. The automorphism $(q')^{-1}q$ of V is Φ-equivariant hence, from stability, $(q')^{-1}q=\lambda\in\mathbb{C}^*$, so $(\sqrt{\lambda},1_L)$ is an isomorphism between (V,L,Q,Φ) and (V,L,Q',Φ).

Suppose now that (V,Φ) is strictly polystable, with

$$(V,\Phi)=(V_1,\Phi_1)\oplus\cdots\oplus(V_k,\Phi_k).$$

Here $\Phi_i:V_i\to V_i\otimes K$ and all the Higgs bundles (V_i,Φ_i) are stable and with the same slope $\mu=\deg(L)/2$.

Consider the decomposition of $q:V\to V^*\otimes L$ as a matrix (q_{ij}) compatible with that of V and of

$$V^*\otimes L=V_1^*\otimes L\oplus\cdots\oplus V_k^*\otimes L.$$

Hence

$$q_{ij}\in\mathrm{Hom}(V_j,V_i^*\otimes L)$$

(note that $q|_{V_j}=q_{1j}\oplus\cdots\oplus q_{kj}$). Suppose that q_{ij} is non-zero, for some i,j. Since

$$(\Phi_i^t\otimes 1_K\otimes 1_L)q_{ij}=(q_{ij}\otimes 1_K)\Phi_j$$

then q_{ij} is a homomorphism between (V_j,Φ_j) and $(V_i^*\otimes L,\Phi_i^t\otimes 1_K\otimes 1_L)$. These are stable Higgs bundles and $\mu(V_j)=\mu(V_i^*\otimes L)$, therefore q_{ij} must be an isomorphism. Hence for each pair i,j, if q_{ij} is non-zero, then it is an isomorphism.

We will consider now three cases.

In the first case we suppose that (V,Φ) is a direct sum of isomorphic copies of the same Higgs bundle (W,Φ_W), with $\Phi_W : W \to W\otimes K$. Let then

$$(V,\Phi) = \underbrace{(W,\Phi_W)\oplus\cdots\oplus(W,\Phi_W)}_{k \text{ summands}}.$$

We have $(W,\Phi_W)\cong(W^*\otimes L,\Phi_W^t\otimes 1_K\otimes 1_L)$ but we can have more than one isomorphism on each column of (q_{ij}).

Choose i_0 and j_0 such that $q_{i_0j_0} : (W,\Phi_W)\to(W^*\otimes L,\Phi_W^t\otimes 1_K\otimes 1_L)$ is non-zero, being therefore an isomorphism. If $q_{ij} : (W,\Phi_W)\to(W^*\otimes L,\Phi_W^t\otimes 1_K\otimes 1_L)$ is any homomorphism, then $(q_{i_0j_0}^{-1})q_{ij}$ is an endomorphism of (W,Φ_W) and, since (W,Φ_W) is stable, $q_{ij} = \alpha_{ij}q_{i_0j_0}$ where $\alpha_{ij}\in\mathbb{C}$ and, moreover, $\alpha_{ij}=0$ if and only if $q_{ij}=0$. Hence the choice of $q_{i_0j_0}$ gives a way to represent (q_{ij}) by a symmetric $k\times k$ matrix where each entry is α_{ij} and which can be diagonalized through a $k\times k$ matrix (λ_{ij}). Define the automorphism g of V given, with respect to the decomposition of V, by a $k\times k$ matrix (g_{ij}) where $g_{ij}=\lambda_{ij} : W\to W$. Thus g is such that

$$(g^t\otimes 1_L)qg = \tilde{q}$$

where $\tilde{q} : V\to V^*\otimes L$ is an isomorphism which is diagonal, by $\mathrm{rk}(W)\times\mathrm{rk}(W)$ blocks, with respect to the given decomposition of V.

Note also that g is Φ-equivariant. Hence, if $\tilde{Q}$ is the quadratic form associated to $\tilde{q}$, $(g,1_L) : (V,L,\tilde{Q},\Phi)\to(V,L,Q,\Phi)$ is an isomorphism. So we can suppose that (q_{ij}) and (q'_{ij}) are diagonal and, an argument analogous to the case where (V,Φ) was stable shows then that (V,L,Q,Φ) and (V,L,Q',Φ) are isomorphic, the isomorphism being $(f,1_L)$ where f is given, according to the decomposition of V, by

$$f = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \sqrt{\lambda_k} \end{pmatrix}.$$

Here, each $\lambda_i\in\mathbb{C}^*$ is such that $q_{ii}=\lambda_i q'_{ii}$.

In the second case we consider

$$V = \underbrace{(W,\Phi_W)\oplus\cdots\oplus(W,\Phi_W)}_{l \text{ summands}}\oplus\underbrace{(W^*\otimes L,\Phi_W^t\otimes 1_K\otimes 1_L)\oplus\cdots\oplus(W^*\otimes L,\Phi_W^t\otimes 1_K\otimes 1_L)}_{l \text{ summands}}$$

with $W\not\cong W^*\otimes L$. In this case $q_{ij}=0$ if $-l\leqslant i-j\leqslant l$. So (q_{ij}) splits into four $l\times l$ blocks in the following way

$$(q_{ij}) = \begin{pmatrix} 0 & q_2 \\ q_1 & 0 \end{pmatrix}$$

where q_1 represents

$$q|_{W\oplus\cdots\oplus W} : W\oplus\cdots\oplus W \longrightarrow W\oplus\cdots\oplus W$$

and q_2 represents

$$q|_{W^*\otimes L\oplus\cdots\oplus W^*\otimes L} : W^*\otimes L\oplus\cdots\oplus W^*\otimes L \longrightarrow W^*\otimes L\oplus\cdots\oplus W^*\otimes L.$$

Again, using the stability of (W,Φ_W) and of $(W^*\otimes L,\Phi_W^t\otimes 1_K\otimes 1_L)$ and the fact that Φ is symmetric with respect to q, we see that each entry of q_1 is given by a scalar. The same happens with q_2. Hence we can write

$$(q_{ij}) = \begin{pmatrix} 0 & A^t \\ A & 0 \end{pmatrix}$$

where A is a non-singular $l \times l$ matrix. Now, if we write in the same way,

$$(q'_{ij}) = \begin{pmatrix} 0 & B^t \\ B & 0 \end{pmatrix}$$

then consider the isomorphism of V given by

$$f = \begin{pmatrix} B^{-1}A & 0 \\ 0 & I_l \end{pmatrix}$$

where we mean by this that each entry of $B^{-1}A$ represents a scalar automorphism of (W, Φ_W) and f is the identity over $W^* \otimes L \oplus \cdots \oplus W^* \otimes L$. So $(f, 1_L)$ is an isomorphism between (V, L, Q, Φ) and (V, L, Q', Φ).

The last case is the generic one, where we consider a combination of the previous cases. We can always write

$$V = (V_1 \oplus \cdots \oplus V_i) \oplus (V_{i+1} \oplus \cdots \oplus V_j) \oplus \cdots \oplus (V_{l+1} \oplus \cdots \oplus V_k) \tag{6.3}$$

and

$$\Phi = (\Phi_1 \oplus \cdots \oplus \Phi_i) \oplus (\Phi_{i+1} \oplus \cdots \oplus \Phi_j) \oplus \cdots \oplus (\Phi_{l+1} \oplus \cdots \oplus \Phi_k)$$

where (V_a, Φ_a) and (V_b, Φ_b) are inside the same parenthesis in (6.3) if and only if are isomorphic or $(V_b, \Phi_b) \cong (V_a^* \otimes L, \Phi_a^t \otimes 1_K \otimes 1_L)$. If V_a and V_b are not inside the same parenthesis in (6.3) then $q_{ab} = 0 = q'_{ab}$. Hence we have an isomorphism f between (V, L, Q, Φ) and (V, L, Q', Φ) where f is diagonal by blocks (not all of the same size), each corresponding to an isomorphism of one of the previous cases. □

We identify $\mathcal{M}_d$ with its image by the map i and therefore consider $\mathcal{M}_d$ as a subspace of $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2) \times \mathrm{Jac}^d(X)$.

Note that $\mathrm{EGL}(n,\mathbb{R})$ is a reductive (not semisimple) Lie group. Therefore, in view of Theorem 2.8, $\mathcal{M}(c)$ is homeomorphic to $\mathcal{R}_{\Gamma,\mathrm{EGL}(n,\mathbb{R})}(c)$, for each topological class c.

Notation 6.5. From now on, we shall write $\mathcal{R}$ instead of $\mathcal{R}_{\Gamma,\mathrm{EGL}(n,\mathbb{R})}$.

Remark 6.6. Let $n \geqslant 4$ be even. Notice that a representation

$$\rho \in \mathrm{Hom}^{\mathrm{red}}_{\rho(J)\in(Z(\mathrm{PGL}(n,\mathbb{R}))\cap\mathrm{PO}(n))_0}(\Gamma, \mathrm{PGL}(n,\mathbb{R}))$$

(cf. (2.3)), which is the same as $\rho \in \mathrm{Hom}^{\mathrm{red}}(\pi_1 X, \mathrm{PGL}(n,\mathbb{R}))$, does not always lift to a representation $\rho' \in \mathrm{Hom}^{\mathrm{red}}(\Gamma, \mathrm{GL}(n,\mathbb{R}))$ with the condition $\rho'(J) \in (Z(\mathrm{GL}(n,\mathbb{R})) \cap \mathrm{O}(n))_0 = I_n$, (i.e., to a representation $\rho' \in \mathrm{Hom}^{\mathrm{red}}(\pi_1 X, \mathrm{GL}(n,\mathbb{R}))$) as we have seen in Proposition 4.7. But it always lifts to a representation of Γ in $\mathrm{EGL}(n,\mathbb{R})$ such that $\rho'(J) \in (Z(\mathrm{EGL}(n,\mathbb{R})) \cap \mathrm{EO}(n))_0 \cong \mathrm{U}(1)$, since $Z(\mathrm{EGL}(n,\mathbb{R})) \cap \mathrm{EO}(n)$ is connected. This is an instance of the fact that a $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundle lifts to an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle, but not to a $\mathrm{GL}(n,\mathbb{R})$-Higgs bundle.

The following lemma will be useful in Section 7.

Lemma 6.7. *$\mathcal{M}_d$ is a closed subspace of $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2) \times \mathrm{Jac}^d(X)$.*

Proof. Let $\mathcal{R}_d$ be the subspace of $\mathcal{R}$ which corresponds to $\mathcal{M}_d$, under Theorem 2.8.

Consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{R}_{\Gamma,\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)}(nd/2,d) & \xrightarrow{\simeq} & \mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)\times\mathrm{Jac}^d(X) \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ \mathcal{R}_d & \xrightarrow{\simeq} & \mathcal{M}_d \end{array}$$

where $\mathcal{R}_{\Gamma,\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)}(nd/2,d)$ is the subspace of $\mathcal{R}_{\Gamma,\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)}$ which corresponds, via again Theorem 2.8, to $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)\times\mathrm{Jac}^d(X)$. So, the top and bottom maps are the homeomorphisms given by Theorem 2.8. The map j is induced by the injective map $[(A,\lambda)]\mapsto(\lambda A,\lambda^2)$ of $\mathrm{EGL}(n,\mathbb{R})$ into $\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)$. The diagram commutes due to the fact that the actions of $\mathrm{EO}(n,\mathbb{C})$ on $\mathbb{C}^n$ and on $\mathbb{C}$ defining an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle are given by $[(w,\lambda)]\mapsto\lambda w$ and $[(w,\lambda)]\mapsto\lambda^2$, which are then compatible with the inclusion of $\mathrm{EGL}(n,\mathbb{R})$ into $\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)$ and therefore with the induced action of $\mathrm{EGL}(n,\mathbb{R})$ on $\mathbb{C}^n\times\mathbb{C}$. Now, since, from Lemma 6.4, i is injective, it follows that j is also injective, and, as we identify $\mathcal{M}_d$ with $i(\mathcal{M}_d)$, we also identify $\mathcal{R}_d$ with $j(\mathcal{R}_d)$. So $\mathcal{R}_d$ can be seen as the space of reductive homomorphisms of Γ in $\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)$ which have their image in $\mathrm{EGL}(n,\mathbb{R})$, modulo $\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)$-equivalence.

Now, since $\mathrm{EGL}(n,\mathbb{R})$ is a closed subgroup of $\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)$, it follows that $\mathcal{R}_d$ is closed in $\mathcal{R}_{\Gamma,\mathrm{GL}(n,\mathbb{C})\times\mathrm{U}(1)}(nd/2,d)$, hence $\mathcal{M}_d$ is closed in $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)\times\mathrm{Jac}^d(X)$. □

6.2. **Deformation theory of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles.** In this section, we briefly recall the description of Biswas and Ramanan [7] (see also [19]) of the deformation theory of G-Higgs bundles and, in particular, the identification of the tangent space of $\mathcal{M}_G$ at the smooth points, and then apply it to the case of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles.

The spaces $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$ in the Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$ verify the relation

$$[\mathfrak{h}^{\mathbb{C}},\mathfrak{m}^{\mathbb{C}}]\subset\mathfrak{m}^{\mathbb{C}}$$

hence, given $v\in\mathfrak{m}^{\mathbb{C}}$, there is an induced map $\mathrm{ad}(v)|_{\mathfrak{h}^{\mathbb{C}}}:\mathfrak{h}^{\mathbb{C}}\to\mathfrak{m}^{\mathbb{C}}$. Applying this to a G-Higgs bundle $(E_{H^{\mathbb{C}}},\Phi)$, we obtain the following complex of sheaves on X:

$$C^\bullet(E_{H^{\mathbb{C}}},\Phi):\mathcal{O}(E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}))\xrightarrow{\mathrm{ad}(\Phi)}\mathcal{O}(E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})\otimes K).$$

Proposition 6.8 (**Biswas, Ramanan** [7]). *Let $(E_{H^{\mathbb{C}}},\Phi)$ represent a G-Higgs bundle over the compact Riemann surface X.*

(i) *The infinitesimal deformation space of $(E_{H^{\mathbb{C}}},\Phi)$ is isomorphic to the first hypercohomology group $\mathbb{H}^1(X,C^\bullet(E_{H^{\mathbb{C}}},\Phi))$ of the complex $C^\bullet(E_{H^{\mathbb{C}}},\Phi)$;*

(ii) *There is a long exact sequence*

$$\begin{aligned} 0&\longrightarrow\mathbb{H}^0(X,C^\bullet(E_{H^{\mathbb{C}}},\Phi))\longrightarrow H^0(X,E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}))\longrightarrow H^0(X,E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})\otimes K)\longrightarrow\\ &\longrightarrow\mathbb{H}^1(X,C^\bullet(E_{H^{\mathbb{C}}},\Phi))\longrightarrow H^1(X,E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}))\longrightarrow H^1(X,E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})\otimes K)\longrightarrow\\ &\longrightarrow\mathbb{H}^2(X,C^\bullet(E_{H^{\mathbb{C}}},\Phi))\longrightarrow 0\end{aligned}$$

where the maps $H^i(X,E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}))\to H^i(X,E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})\otimes K)$ are induced by $\mathrm{ad}(\Phi)$.

Proposition 6.8 applied to the case of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles, yields:

Proposition 6.9. *Let (V,L,Q,Φ) be an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle over X. There is a complex of sheaves*

$$C^\bullet(V,L,Q,\Phi):\Lambda^2_QV\oplus\mathcal{O}\xrightarrow{[\Phi,-]}S^2_QV\otimes K$$

and

(i) *the infinitesimal deformation space of* (V, L, Q, Φ) *is isomorphic to the first hypercohomology group* $\mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi))$ *of* $C^\bullet(V, L, Q, \Phi)$*. In particular, if* (V, L, Q, Φ) *represents a smooth point of* $\mathcal{M}_d$*, then*

$$T_{(V,L,Q,\Phi)}\mathcal{M} \simeq \mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi));$$

(ii) *there is an exact sequence*

$$\begin{aligned} 0 &\longrightarrow \mathbb{H}^0(X, C^\bullet(V, L, Q, \Phi)) \longrightarrow H^0(X, \Lambda^2_Q V \oplus \mathcal{O}) \longrightarrow H^0(X, S^2_Q V \otimes K) \longrightarrow \\ &\longrightarrow \mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi)) \longrightarrow H^1(X, \Lambda^2_Q V \oplus \mathcal{O}) \longrightarrow H^1(X, S^2_Q V \otimes K) \longrightarrow \\ &\longrightarrow \mathbb{H}^2(X, C^\bullet(V, L, Q, \Phi)) \longrightarrow 0 \end{aligned}$$

where the maps $H^i(X, \Lambda^2_Q V \oplus \mathcal{O}) \to H^i(X, S^2_Q V \otimes K)$ *are induced by the map* $[\Phi, -]$*.*

6.3. **Topological classification of** $\mathrm{EGL}(n, \mathbb{R})$**-Higgs bundles.** Our calculations will be performed on $\mathcal{M}_d$ so we will also need the topological invariants of $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundles.

We will define these discrete invariants using the relation between $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundles and $\mathrm{PGL}(n, \mathbb{R})$-Higgs bundles. In fact, we already know from Theorem 3.13 that the invariants μ_1 and μ_2 completely classify $\mathrm{PGL}(n, \mathbb{R})$-bundles over X, and also know from Proposition 5.6 that if two $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundles project to the same $\mathrm{PGL}(n, \mathbb{R})$-Higgs bundle, then the degree of the corresponding line bundle L is equal modulo 2.

As we are dealing with topological classification of bundles, we can forget the Higgs field and consider only twisted orthogonal bundles (V, L, Q) which correspond to elements of the set $H^1(X, \mathcal{C}(\mathrm{EO}(n)))$, and $\mathrm{PO}(n)$-bundles E which are in bijection with $H^1(X, \mathcal{C}(\mathrm{PO}(n)))$. There is then a relation between (V, L, Q) and E which is similar to the one between $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundles and $\mathrm{PGL}(n, \mathbb{R})$-Higgs bundles, but now forgetting the Higgs field.

Consider then the following commutative diagram:

$$\begin{array}{ccccccc} & & H^1(X, \mathbb{Z}_2) & \xrightarrow{\cong} & H^1(X, \mathbb{Z}_2) & & \\ & & \downarrow & & \downarrow & & \\ H^1(X, \mathcal{C}(\mathrm{U}(1))) & \longrightarrow & H^1(X, \mathcal{C}(\mathrm{O}(n) \times \mathrm{U}(1))) & \longrightarrow & H^1(X, \mathcal{C}(\mathrm{O}(n))) & \longrightarrow & 0 \\ \cong\downarrow & & \downarrow p_2' & & \downarrow p_2 & & \\ H^1(X, \mathcal{C}(\mathrm{U}(1))) & \longrightarrow & H^1(X, \mathcal{C}(\mathrm{EO}(n))) & \longrightarrow & H^1(X, \mathcal{C}(\mathrm{PO}(n))) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & H^2(X, \mathbb{Z}_2) & \xrightarrow{\cong} & H^2(X, \mathbb{Z}_2). & & \end{array} \tag{6.4}$$

The map

$$p_2' : H^1(X, \mathcal{C}(\mathrm{O}(n) \times \mathrm{U}(1))) \longrightarrow H^1(X, \mathcal{C}(\mathrm{EO}(n)))$$

is induced from the projection $\mathrm{O}(n) \times \mathrm{U}(1) \to \mathrm{EO}(n)$ and hence defined, in terms of vector bundles, by

$$p_2'((W, Q_W), M) = (W \otimes M, M^2, Q_W \otimes 1_{M^2}).$$

Once again we see that (V, L, Q) is in the image of p_2' if and only if $\deg(L)$ is even. Moreover, if this is the case, the pre-image of (V, L, Q) under p_2' is the following set of 2^{2g} pairs:

$$p_2'^{-1}(V, L, Q) = \{((V \otimes L^{-1/2} F, Q \otimes Q_F \otimes 1_{L^{-1}}), L^{1/2} F) \mid F^2 \cong \mathcal{O}\} \tag{6.5}$$

where $L^{1/2}$ is a fixed square root of L (notice that saying that F is a 2-torsion point of the Jacobian is equivalent to say that F has a nowhere degenerate quadratic form $Q_F : F \otimes F \to \mathcal{O}$).

The map p_2 is the one induced from the projection $\mathrm{O}(n) \to \mathrm{PO}(n)$.

Definition 6.10. *Given a twisted orthogonal bundle (V, L, Q), let E be the corresponding $\mathrm{PO}(n)$-bundle under the map $H^1(X, \mathcal{C}(\mathrm{EO}(n))) \to H^1(X, \mathcal{C}(\mathrm{PO}(n)))$. Define the* first invariant $\overline{\mu}_1$ *of (V, L, Q) as*

$$\overline{\mu}_1(V, L, Q) = \mu_1(E) \in (\mathbb{Z}_2)^{2g}.$$

$\overline{\mu}_1(V, L, Q)$ is then the obstruction to reducing the structure group of (V, L, Q) to $\mathrm{ESO}(n) = \mathrm{SO}(n) \times_{\mathbb{Z}_2} \mathrm{U}(1)$. Hence, this happens if and only if E reduces to a $\mathrm{PSO}(n)$-bundle.

If $\deg(L)$ is even, then

$$\overline{\mu}_1(V, L, Q) = w_1(V \otimes L^{-1/2}, Q \otimes 1_{L^{-1}}) \tag{6.6}$$

the first Stiefel-Whitney class of the real orthogonal bundle $V \otimes L^{-1/2}$ (the value of w_1 is independent of the choice of the square root of L because n is even - notice that $w_1(W \otimes F) = w_1(W) + \mathrm{rk}(W)_2 w_1(F)$ where $\mathrm{rk}(W)_2 = \mathrm{rk}(W) \bmod 2$).

Definition 6.11. *Let (V, L, Q) be a twisted orthogonal bundle with* $\mathrm{rk}(V) = n \geqslant 4$. *Define the* second invariant $\overline{\mu}_2$ *of (V, L, Q) as follows:*

(i) *If $\overline{\mu}_1(V, L, Q) = 0$,*

$$\overline{\mu}_2(V, L, Q) = \begin{cases} (w_2(V \otimes L^{-1/2}), \deg(L)) \in \mathbb{Z}_2 \times 2\mathbb{Z} & \text{if } \deg(L) \text{ even} \\ \deg(L) \in 2\mathbb{Z} + 1 & \text{if } \deg(L) \text{ odd} \end{cases}$$

where $w_2(V \otimes L^{-1/2})$ is the second Stiefel-Whitney class of $V \otimes L^{-1/2}$ and $2\mathbb{Z}$ represents the set of even integers and $2\mathbb{Z} + 1$ the set of odd integers.

(ii) *If $\overline{\mu}_1(V, L, Q) \neq 0$,*

$$\overline{\mu}_2(V, L, Q) = \deg(L) \in \mathbb{Z}.$$

Notice that on the first and third items, $w_2(V \otimes L^{-1/2})$ does not depend on the choice of the square root of L due to the vanishing of $\overline{\mu}_1(V, L, Q)$ (cf. Remark 3.12).

Let E be a $\mathrm{PO}(n)$-bundle and (V, L, Q) be a twisted orthogonal bundle which maps to E. From (6.4), E lifts to a $\mathrm{O}(n)$-bundle if and only if (V, L, Q) lifts to a $\mathrm{O}(n) \times \mathrm{U}(1)$-bundle and this occurs if and only if $\deg(L)$ is even (recall also Corollary 5.7). Hence, using Proposition 3.10 and the fact that by definition $\overline{\mu}_1(V, L, Q) = \mu_1(E)$, the following proposition is immediate:

Proposition 6.12. *Let $n \geqslant 4$ be even. Let E be a $\mathrm{PO}(n)$-bundle and (V, L, Q) be a twisted orthogonal bundle which maps to E.*

(i) *If $\overline{\mu}_1(V, L, Q) = 0$, then:*
- $\overline{\mu}_2(V, L, Q) = (0, \deg(L))$ *with* $\deg(L)$ *even* $\iff \mu_2(E) = 0$*;*
- $\overline{\mu}_2(V, L, Q) = (1, \deg(L))$ *with* $\deg(L)$ *even* $\iff \mu_2(E) = 1$*;*
- $\overline{\mu}_2(V, L, Q) = \deg(L)$ *with* $\deg(L)$ *odd* $\iff \mu_2(E) = \omega_n$*.*

(ii) *If $\overline{\mu}_1(V, L, Q) \neq 0$, then:*
- $\overline{\mu}_2(V, L, Q) = \deg(L)$ *even* $\iff \mu_2(E) = 0$*;*
- $\overline{\mu}_2(V, L, Q) = \deg(L)$ *odd* $\iff \mu_2(E) = \omega_n$*.*

From Theorem 3.13, we know that μ_1 and μ_2 completely classify $\mathrm{PO}(n)$-bundles. Moreover, since we also know that the difference between two (V, L, Q) and (V', L', Q') mapping to the same $\mathrm{PO}(n)$-bundle lies in the degree of L, we have then the following:

Theorem 6.13. *Let X be a closed oriented surface of genus $g \geqslant 2$ and let $n \geqslant 4$ be even. Then twisted orthogonal bundles over X are topologically classified by the invariants*

$$(\overline{\mu}_1, \overline{\mu}_2) \in (\{0\} \times ((\mathbb{Z}_2 \times 2\mathbb{Z}) \cup (2\mathbb{Z}+1))) \cup \left(\left(\mathbb{Z}_2^{2g} \setminus \{0\}\right) \times \mathbb{Z}\right).$$

Now, returning to our principal objects - $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles - we see that Theorem 6.13 also gives the topological classification of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles.

Notation 6.14. Let

$$\mathcal{M}(\overline{\mu}_1, \overline{\mu}_2)$$

denote the subspace of the space of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles in which the $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles have invariants $(\overline{\mu}_1, \overline{\mu}_2)$.

Remark 6.15. If $\overline{\mu}_1 = 0$ and if d_1 and d_2 are even, then

$$\mathcal{M}(0, (w_2, d_1)) \simeq \mathcal{M}(0, (w_2, d_2))$$

and, if d_1 and d_2 are odd,

$$\mathcal{M}(0, d_1) \simeq \mathcal{M}(0, d_2).$$

If $\overline{\mu}_1 \neq 0$ and $d_1 = d_2 \bmod 2$, then

$$\mathcal{M}(\overline{\mu}_1, d_1) \simeq \mathcal{M}(\overline{\mu}_1, d_2).$$

In all cases the bijection is given by $(V, L, Q, \Phi) \mapsto (V \otimes F, L \otimes F^2, Q \otimes 1_{F^2}, \Phi \otimes 1_F)$, where F is a holomorphic line bundle of suitable degree.

Again, we define the same invariants for the space of representations $\mathcal{R}$ (recall Notation 6.5). $\mathcal{R}(\overline{\mu}_1, (w_2, d))$ corresponds to $\mathcal{M}(\overline{\mu}_1, (w_2, d))$ if d is even, and $\mathcal{R}(\overline{\mu}_1, d)$ corresponds to $\mathcal{M}(\overline{\mu}_1, d)$ if d is odd.

From Proposition 5.4, the surjective map taking an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle to the corresponding $\mathrm{PGL}(n,\mathbb{R})$-Higgs bundle induces a surjective continuous map $p : \mathcal{R} \to \mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$. Using Propositions 5.6 and 6.12, the following is immediate.

Proposition 6.16. *The map $p : \mathcal{R} \to \mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ satisfies the following identities:*

(i) *If $\overline{\mu}_1 = 0$, then*

$$p(\mathcal{R}(0, (0,0))) = \mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(0,0)$$

$$p(\mathcal{R}(0, (1,0))) = \mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(0,1)$$

and

$$p(\mathcal{R}(0,1)) = \mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(0, \omega_n).$$

(ii) *If $\overline{\mu}_1 \neq 0$, then*

$$p(\mathcal{R}(\overline{\mu}_1, 0)) = \mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1, 0)$$

and

$$p(\mathcal{R}(\overline{\mu}_1, 1)) = \mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1, \omega_n).$$

From this and from Proposition 4.7, we have:

Corollary 6.17. *$\mathcal{R}(\overline{\mu}_1, \overline{\mu}_2)$ is non-empty for any choice of invariants $\overline{\mu}_1$ and $\overline{\mu}_2$.*

7. The Hitchin proper functional

Here we use the method introduced by Hitchin in [14] to study the topology of moduli space $\mathcal{M}_G$ of G-Higgs bundles.

Define

$$f : \mathcal{M}_G(c) \longrightarrow \mathbb{R}$$

by

$$f(E_{H^{\mathbb{C}}}, \Phi) = \|\Phi\|^2_{L^2} = \int_X |\Phi|^2 \mathrm{dvol}. \tag{7.1}$$

This function f is usually called the *Hitchin functional*.

Here we are using the *harmonic metric* (cf. [8, 9]) on $E_{H^{\mathbb{C}}}$ to define $\|\Phi\|_{L^2}$. So we are using the identification between $\mathcal{M}_G(c)$ with the space of gauge-equivalent solutions of Hitchin's equations. We opt to work with $\mathcal{M}_G(c)$, because in this case we have more algebraic tools at our disposal. We shall make use of the tangent space of $\mathcal{M}_G(c)$, and we know from [14] that the above identification induces a diffeomorphism between the corresponding tangent spaces.

We have the following result:

Proposition 7.1 (**Hitchin** [14])**.**

(i) *The function f is proper.*
(ii) *If $\mathcal{M}_G(c)$ is smooth, then f is a non-degenerate perfect Bott-Morse function.*

Since f is proper, it attains a minimum on each connected component of $\mathcal{M}_G(c)$. Moreover, we have the following result from general topology:

Proposition 7.2. *If the subspace of local minima of f is connected, then so is $\mathcal{M}_G(c)$.*

Now, fix $L \in \mathrm{Jac}^d(X)$ and consider the space

$$\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2) \cong \mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2) \times \{L\} \subset \mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2) \times \mathrm{Jac}^d(X).$$

In our case, the Hitchin functional

$$f : \mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2) \longrightarrow \mathbb{R}$$

is given by

$$f(V, \Phi) = \|\Phi\|^2_{L^2} = \frac{\sqrt{-1}}{2} \int_X \mathrm{tr}(\Phi \wedge \Phi^*) \mathrm{dvol}. \tag{7.2}$$

7.1. **Smooth minima.** Away from the singular locus of $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)$, the Hitchin functional f is a moment map for the Hamiltonian S^1-action on $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)$ given by

$$(V, \Phi) \mapsto (V, e^{\sqrt{-1}\theta}\Phi). \tag{7.3}$$

From this it follows immediately that a stable point of $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)$ is a critical point of f if and only if is a fixed point of the S^1-action. Let us then study the fixed point set of the given action (this is analogous to [15] and [3]).

Let (V, Φ) represent a stable fixed point. Then either $\Phi = 0$ or (since the action is on $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)$) there is a one-parameter family of gauge transformations $g(\theta)$ such that $g(\theta) \cdot (V, \Phi) = (V, e^{\sqrt{-1}\theta}\Phi)$. In the latter case, let

$$\psi = \frac{d}{d\theta} g(\theta)|_{\theta=0}$$

be the infinitesimal gauge transformation generating this family. Simpson shows in [29] that this (V, Φ) is what is called a *complex variation of Hodge structure*. This means that

$$V = \bigoplus_j F_j$$

where the F_j's are the eigenbundles of the infinitesimal gauge transformation ψ: over F_j,

$$\psi = \sqrt{-1}j \in \mathbb{C}. \tag{7.4}$$

$\Phi_j = \Phi|_{F_j}$ is a map

$$\Phi_j : F_j \longrightarrow F_{j+1} \otimes K$$

which is non-zero for all j except the maximal one.

Set

$$\mathcal{M}_L = \{(V, L', Q, \Phi) \in \mathcal{M}_d \mid L' \cong L\} \subset \mathcal{M}_d.$$

From Lemma 6.7, we know that $\mathcal{M}_d$ is a closed subspace of $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2) \times \mathrm{Jac}^d(X)$ hence, for each L, $\mathcal{M}_L$ is closed in $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2) \times \{L\} \cong \mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)$. The following proposition is a direct consequence of this and of the properness of the f given in (7.2).

Proposition 7.3. *The restriction of the Hitchin functional f to $\mathcal{M}_L$ is a proper and bounded below function.*

From now on we will consider the restriction of f to $\mathcal{M}_L$. This fact will be important in the counting of components of each $\mathcal{M}_d$, as we shall see in Section 8.

The circle action (7.3) restricts to $\mathcal{M}_L$. So, if (V, L, Q, Φ) is an $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundle such that (V, Φ) is stable and is a fixed point of the S^1-action (i.e., is a critical point of f), then it is a variation of Hodge structure. In this case, $g(\theta) \in H^0(X, \mathrm{EO}(n, \mathcal{O}))$ and, since the Lie algebra of $\mathrm{EO}(n, \mathbb{C})$ is $\mathfrak{o}(n, \mathbb{C}) \oplus \mathbb{C}$, we have $\psi \in H^0(X, \mathfrak{o}(n, \mathcal{O}) \oplus \mathcal{O})$, therefore being skew-symmetric with respect to Q. Thus, using (7.4) we have that, if $v_j \in F_j$ and $v_l \in F_l$,

$$\sqrt{-1}jQ(v_j, v_l) = Q(\psi v_j, v_l) = -Q(v_j, \psi v_l) = -\sqrt{-1}lQ(v_j, v_l).$$

F_j and F_l are therefore orthogonal under Q if $l \neq -j$, and $q : V \to V^* \otimes L$ yields an isomorphism

$$q|_{F_j} : F_j \xrightarrow{\cong} F^*_{-j} \otimes L. \tag{7.5}$$

This means that

$$V = F_{-m} \oplus \cdots \oplus F_m$$

for some m integer or half-integer.

Using these isomorphisms and the fact that Φ is symmetric under Q, we see that

$$(q \otimes 1_K)\Phi_j = (\Phi^t_{-j-1} \otimes 1_K \otimes 1_L)q$$

for $j \in \{-m, \dots, m\}$.

The Cartan decomposition of $\mathfrak{g}^{\mathbb{C}}$ induces a decomposition of vector bundles

$$E_{H^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}}) = E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}) \oplus E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})$$

where $E_{H^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}})$ (resp. $E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}})$) is the adjoint bundle, associated to the adjoint representation of $H^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ (resp. $\mathfrak{h}^{\mathbb{C}}$). For the group $\mathrm{EGL}(n, \mathbb{R})$, we have $E_{H^{\mathbb{C}}}(\mathfrak{g}^{\mathbb{C}}) = \mathrm{End}(V) \oplus \mathcal{O}$ where $\mathcal{O} = \mathrm{End}(L)$ is the trivial line bundle on X and we already know that $E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}}) = \Lambda^2_Q V \oplus \mathcal{O}$ and $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) = S^2_Q V$ where $\Lambda^2_Q V$ is the bundle of skew-symmetric endomorphisms of V with respect to the form Q. The involution in $\mathrm{End}(V)\oplus\mathcal{O}$ defining the above decomposition is $\theta\oplus 1_{\mathcal{O}}$ where θ is the involution on $\mathrm{End}(V)$ defined by

$$\theta(A)=-(qAq^{-1})^t\otimes 1_L. \tag{7.6}$$

Its $+1$-eigenbundle is $\Lambda^2_Q V\oplus\mathcal{O}$ and its -1-eigenbundle is $S^2_Q V$.

We also have a decomposition of this vector bundle as

$$\mathrm{End}(V)\oplus\mathcal{O}=\bigoplus_{k=-2m}^{2m}U_k\oplus\mathcal{O}$$

where

$$U_k=\bigoplus_{i-j=k}\mathrm{Hom}(F_j,F_i).$$

From (7.4), this is the $\sqrt{-1}k$-eigenbundle for the adjoint action $\mathrm{ad}(\psi):\mathrm{End}(V)\oplus\mathcal{O}\to\mathrm{End}(V)\oplus\mathcal{O}$ of ψ. We say that U_k is the subspace of $\mathrm{End}(V)\oplus\mathcal{O}$ with *weight* k.

Write

$$U_{i,j}=\mathrm{Hom}(F_j,F_i).$$

The restriction of the involution θ, defined in (7.6), to $U_{i,j}$ gives an isomorphism

$$\theta:U_{i,j}\to U_{-j,-i} \tag{7.7}$$

so θ restricts to

$$\theta:U_k\longrightarrow U_k.$$

Write

$$U^+=\Lambda^2_Q V\quad\text{and}\quad U^-=S^2_Q V$$

so that

$$E_{H^{\mathbb{C}}}(\mathfrak{h}^{\mathbb{C}})=U^+\oplus\mathcal{O}$$

and

$$E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}})=U^-.$$

Let also

$$U_k^+=U_k\cap U^+$$

and

$$U_k^-=U_k\cap U^-$$

so that $U_k=U_k^+\oplus U_k^-$ is the corresponding eigenbundle decomposition. Hence

$$U^+=\bigoplus_k U_k^+$$

and

$$U^-=\bigoplus_k U_k^-.$$

Observe that $\Phi\in H^0(X,U_1^-\otimes K)$.

The map $\mathrm{ad}(\Phi)=[\Phi,-]$ interchanges U^+ with U^- and therefore maps $U_k^\pm$ to $U_{k+1}^\mp\otimes K$. So, for each k, we have a weight k subcomplex of the complex $C^\bullet(V,L,Q,\Phi)$ defined in Proposition 6.9:

$$C_k^\bullet(V,L,Q,\Phi):U_k^+\oplus\mathcal{O}\xrightarrow{[\Phi,-]}U_{k+1}^-\otimes K.$$

In fact, since $\mathrm{ad}(\psi)|_{\mathcal{O}}=0$, $C_k^\bullet(V,L,Q,\Phi)$ is given by

$$C_0^\bullet(V,L,Q,\Phi):U_0^+\oplus\mathcal{O}\xrightarrow{[\Phi,-]}U_1^-\otimes K$$

and, for $k \neq 0$, by

$$C_k^\bullet(V, L, Q, \Phi) : U_k^+ \xrightarrow{[\Phi,-]} U_{k+1}^- \otimes K.$$

From Proposition 6.9, if an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle (V, L, Q, Φ) is such that (V, Φ) is stable, its infinitesimal deformation space is

$$\mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi)) = \bigoplus_k \mathbb{H}^1(X, C_k^\bullet(V, L, Q, \Phi)).$$

We say that $\mathbb{H}^1(X, C_k^\bullet(V, L, Q, \Phi))$ is the subspace of $\mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi))$ with *weight* k.

By Hitchin's computations in [15], we have the following result which gives us a way to compute the eigenvalues of the Hessian of the Hitchin functional f at a smooth (here we mean smooth in $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)$) critical point.

Proposition 7.4. *Let f be the Hitchin functional. Let (V, L, Q, Φ) be an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle with (V, Φ) stable and which represents a critical point of f. The eigenspace of the Hessian of f corresponding to the eigenvalue k is*

$$\mathbb{H}^1(X, C_{-k}^\bullet(V, L, Q, \Phi)).$$

In particular, (V, L, Q, Φ) is a local minimum of f if and only if $\mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi))$ has no subspaces with positive weight.

For the moment we will only care about the stable points of $\mathcal{M}_L$.

Using Proposition 7.4, one can prove the following result by an argument analogous to the proof of Corollary 4.15 of [2] (see also Remark 4.16 in the same paper and Lemma 3.11 of [5]). It is the fundamental result which makes possible the description of the stable local minima of f.

Theorem 7.5. *Let $(V, L, Q, \Phi) \in \mathcal{M}_L$ be a critical point of f with (V, Φ) stable. Then (V, L, Q, Φ) is a local minimum if and only if either $\Phi = 0$ or*

$$\mathrm{ad}(\Phi)|_{U_k^+} : U_k^+ \longrightarrow U_{k+1}^- \otimes K$$

is an isomorphism for all $k \geqslant 1$.

The following theorem is quite similar to the corresponding one in [15] and in [3] as one would naturally expect. Indeed, the proof of this theorem is inspired in the one of Theorem 4.3 of [3].

Theorem 7.6. *Let the $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle (V, L, Q, Φ) be a a critical point of the Hitchin functional f such that (V, Φ) is stable. Then (V, L, Q, Φ) represents a local minimum if and only if one of the following conditions occurs:*

(i) $\Phi = 0$.
(ii) *For each i, $\mathrm{rk}(F_i) = 1$ and Φ_i is an isomorphism, for $i \neq m$.*

Proof. The proof that a local minimum of f must be of one of the above types is very similar to the one presented in the proof of Theorem 4.3 of [3], so we skip it.

To prove the converse, let (V, L, Q, Φ) represent a point of type (2). Then

$$V = \bigoplus_{i=-m}^{m} F_i \tag{7.8}$$

with $\mathrm{rk}(F_i) = 1$, so $n = 2m+1$, and

$$\Phi = \bigoplus_{i=-m}^{m} \Phi_i \tag{7.9}$$

with $\Phi_i : F_i \to F_{i+1} \otimes K$ isomorphism, if $i \neq m$.

For each $k \in \{1, \ldots, 2m\}$, $\mathrm{rk}(U_k) = 2m - k + 1$ hence

$$\mathrm{rk}(U_k^+) = \frac{2m-k+1}{2} = \mathrm{rk}(U_{k+1}^- \otimes K)$$

if $n = k \bmod 2$, and

$$\mathrm{rk}(U_k^+) = \frac{2m-k}{2} = \mathrm{rk}(U_{k+1}^- \otimes K)$$

if $n \neq k \bmod 2$. Therefore, if we prove that $\mathrm{ad}(\Phi) : U_k^+ \to U_{k+1}^- \otimes K$ is injective, we conclude that it is an isomorphism and, from Theorem 7.5, that (V, L, Q, Φ) represents a local minimum of f.

Let $g \in U_k^+ = U_k \cap U^+ = \bigoplus_{i-j=k} \mathrm{Hom}(F_j, F_i) \cap \Lambda_Q^2 V$. We can write g as

$$g = g_{-m} \oplus g_{-m+1} \oplus \cdots \oplus g_{m-k} \tag{7.10}$$

where $g_j : F_j \to F_{j+k}$ and $g_j = -q^{-1}(g_{-j-k}^t \otimes 1_L)q$. Now,

$$\mathrm{ad}(\Phi)(g) = [\Phi, g] = \Phi g - (g \otimes 1_K)\Phi$$

and, using the decompositions (7.8), (7.9) and (7.10), this yields

$$[\Phi, g] = (\Phi_{-m+k} g_{-m} - (g_{-m+1} \otimes 1_K)\Phi_{-m}) \oplus \cdots \oplus (\Phi_{m-1} g_{m-k-1} - (g_{m-k} \otimes 1_K)\Phi_{m-k-1}).$$

The summands lie in different $U_{i,j}^- \otimes K$, hence $[\Phi, g] = 0$ is equivalent to the following system of equations

$$\begin{cases} \Phi_{-m+k} g_{-m} - (g_{-m+1} \otimes 1_K)\Phi_{-m} = 0 \\ \Phi_{-m+k+1} g_{-m+1} - (g_{-m+2} \otimes 1_K)\Phi_{-m+1} = 0 \\ \vdots \\ \Phi_{m-1} g_{m-k-1} - (g_{m-k} \otimes 1_K)\Phi_{m-k-1} = 0. \end{cases} \tag{7.11}$$

Take any fibre of V and choose suitable basis of V and $V^* \otimes L$ such that, with respect to these basis,

$$\Phi = \begin{pmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ 1 & \ddots & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} 0 & \ldots & \ldots & 0 & 1 \\ \vdots & & & \ldots & 0 \\ \vdots & & 1 & & \vdots \\ 0 & \ldots & & & \vdots \\ 1 & 0 & \ldots & \ldots & 0 \end{pmatrix}$$

and

$$g_j = -g_{-j-k} \tag{7.12}$$

over the corresponding fibre of U_k^+. Then (7.11) implies that, over this fibre, $g_i = g_j$ for all i, j. In particular,

$$g_j = g_{-j-k} \tag{7.13}$$

for all j. From (7.12) and (7.13), we must then have $g = 0$. Since we considered any fibre, the result follows. $\square$

Remark 7.7. Although we are always assuming $\mathrm{rk}(V) \geqslant 4$ even, we will need during the proof of Proposition 7.10 below, to consider $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles of rank 1 and 2 and also of rank bigger or equal than 3 odd. In the first two cases it is straightforward to see that the minima of the Hitchin functional $f(V,\Phi) = \|\Phi\|^2_{L^2}$, with (V,Φ) stable, in the corresponding moduli spaces are the following:

- If $\mathrm{rk}(V) = 1$, (V,L,Q,Φ) is a minimum of f if and only if $\Phi = 0$;
- If $\mathrm{rk}(V) = 2$, (V,L,Q,Φ) is a minimum of f if and only if either $\Phi = 0$ or $V = F \oplus (F^* \otimes L)$ with $\mathrm{rk}(F) = 1$ and

$$\Phi = \begin{pmatrix} 0 & 0 \\ \Phi' & 0 \end{pmatrix}$$

with $\Phi' : F \to F^* \otimes L$ non-zero (not necessarily isomorphism).

For $\mathrm{rk}(V) \geqslant 3$ odd, (V,L,Q,Φ) is a minimum of f if and only if either $\Phi = 0$ or $V = \oplus_{i=-m}^m F_i$ with $\mathrm{rk}(F_i) = 1$ and Φ_i is an isomorphism, for $i \neq m$. This case is completely analogous to the even case considered here. The details can be found in [20].

Let (V,L,Q,Φ) represent a local minimum of f of type (2) of Theorem 7.6. Then,

$$V = F_{-m} \oplus \cdots \oplus F_{-1/2} \oplus F_{1/2} \oplus \cdots \oplus F_m \tag{7.14}$$

where m is an half-integer.

Corollary 7.8. *Let (V,L,Q,Φ) represent a local minimum of f of type (2).*

(i) *Then $F_{-1/2}^2 \cong LK$ and the others F_i are uniquely determined by the choice of this square root of LK as $F_{-1/2+i} \cong F_{-1/2}K^{-i}$.*

(ii) *Then (V,L,Q,Φ) is isomorphic to an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle where*

$$q = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & & \dots & 0 \\ \vdots & & 1 & & \vdots \\ 0 & \dots & & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & \ddots & & \dots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

with respect to the decomposition $V = F_{-m} \oplus \cdots \oplus F_m$.

7.2. **Singular minima.** We must now show that Theorem 7.6 gives us all non-zero minima of the Hitchin proper function f.

Let (V,L,Q,Φ) be an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle such that (V,Φ) is strictly polystable, with $(V,\Phi) = \bigoplus_i (V_i,\Phi_i)$. Suppose moreover that Q also splits accordingly $Q = \bigoplus_i Q_i$ so that we have $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles (V_i,L,Q_i,Φ_i). We have

$$f(V,L,Q,\Phi) = \sum_i f(V_i,L,Q_i,\Phi_i)$$

so, if (V,L,Q,Φ) is a local minimum of f, each of its stable summands is also a local minimum of f in the corresponding lower rank space $\mathcal{M}_L$. Hence each (V_i,L,Q_i,Φ_i) is a fixed point of the circle action and therefore the same happens to (V, L, Q, Φ). So (V, L, Q, Φ) is a complex variation of Hodge structure

$$V = \bigoplus_\alpha W_\alpha$$

where each W_α is the $\sqrt{-1}\alpha$-eigenbundle for an infinitesimal $\mathrm{EO}(n, \mathbb{C})$-gauge transformation ψ and where $\Phi_\alpha : W_\alpha \to W_{\alpha+1} \otimes K$, with the possibility that $\Phi_\alpha = 0$. We can then also write

$$\mathrm{End}(V) \oplus \mathcal{O} = \bigoplus_\lambda U_\lambda \oplus \mathcal{O}$$

where U_λ is the $\sqrt{-1}\lambda$ eigenbundle of $\mathrm{ad}(\psi)$. Let $U_\lambda^\pm = U_\lambda \cap U^\pm$, where $U^+ = \Lambda_Q^2 V$ and $U^- = S_Q^2 V$, and define the following complex of sheaves associated to (V, L, Q, Φ):

$$C^\bullet_{>0}(V, L, Q, \Phi) : \bigoplus_{\lambda>0} U_\lambda^+ \xrightarrow{[\Phi,-]} \bigoplus_{\lambda>1} U_\lambda^- \otimes K. \tag{7.15}$$

Hitchin's computations in [15] for showing that a given fixed point of the circle action is not a local minimum yield the following proposition.

Proposition 7.9. *Let (V, L, Q, Φ) be a fixed point of the S^1-action on $\mathcal{M}_L$. Let (V_t, L, Q_t, Φ_t) be a one-parameter family of polystable $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundles such that $(V_0, L, Q_0, \Phi_0) = (V, L, Q, \Phi)$. If there is a non-trivial tangent vector to the family at 0 which lies in the subspace*

$$\mathbb{H}^1(X, C^\bullet_{>0}(V, L, Q, \Phi))$$

of the infinitesimal deformation space $\mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi))$, then (V, L, Q, Φ) is not a local minimum of f.

In other words, if (V, Φ) is strictly polystable, Hitchin's arguments in [15] are also valid: if there is a non-empty subspace of $\mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi))$ which gives directions in which f decreases and if these directions are integrable into a one-parameter family in $\mathcal{M}_L$, then (V, L, Q, Φ) is not a local minimum of f.

The following result, adapted from [15], shows that there are no more non-zero minima of f besides the ones of Theorem 7.6.

Proposition 7.10. *Let (V, L, Q, Φ) represent a point of $\mathcal{M}_L$ such that (V, Φ) is strictly polystable. If $\Phi \neq 0$, then (V, L, Q, Φ) is not a local minimum of f.*

Proof. Suppose $V = V_1 \oplus V_2$, $\Phi = \Phi_1 \oplus \Phi_2$ and (V, Φ) represents a local minimum of f in $\mathcal{M}_L$, with $\Phi_1 \neq 0 \neq \Phi_2$.

Consider first the case where V_1 and V_2 are not isomorphic and $V_1 \cong V_1^* \otimes L$ and $V_2 \cong V_2^* \otimes L$. Then the quadratic form Q also splits as $Q = Q_1 \oplus Q_2$ with $Q_i : V_i \otimes V_i \to L$, $i = 1, 2$. We have therefore the $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundles (V_1, L, Q_1, Φ_1) and (V_2, L, Q_2, Φ_2) which are local minima of f on the corresponding lower rank moduli space. Let $n_1 = \mathrm{rk}(V_1)$ and $n_2 = \mathrm{rk}(V_2)$ so that $n = n_1 + n_2$ (here, the cases $n_1 = 2$ or $n_2 = 2$ or $n_i \geqslant 3$ odd are included). So we have

$$V_1 = F_{-m} \oplus \cdots \oplus F_m$$

and

$$V_2 = G_{-k} \oplus \cdots \oplus G_k.$$

Consider the complex

$$C^\bullet_{m+k}(V, L, Q, \Phi) : U^+_{m+k} \xrightarrow{[\Phi,-]} U^-_{m+k+1} \otimes K.$$

Since $\Phi \neq 0$, we have $m+k>0$ and $C^\bullet_{m+k}(V,L,Q,\Phi)$ is a subcomplex of the complex $C^\bullet_{>0}(V,L,Q,\Phi)$ defined in (7.15).

Consider the space $H^1(X,\mathrm{Hom}(G_{-k},F_m)) = H^1(X,F_mG_kL^{-1})$. For $i=1,2$,

$$\deg(V_i) = n_i \deg(L)/2$$

and, since F_m (resp. G_k) is a Φ_1 (resp. Φ_2)-invariant subbundle of V_1 (resp. V_2), we have, from the stability of (V_1,Φ_1) and of (V_2,Φ_2),

$$\deg(F_mG_kL^{-1}) = \deg(F_m)+\deg(G_k)-\deg(L) < 0.$$

It follows, by Riemann-Roch, that $H^1(X,\mathrm{Hom}(G_{-k},F_m))$ is non-zero. Choose then

$$0 \neq h \in H^1(X,\mathrm{Hom}(G_{-k},F_m))$$

and let

$$\sigma = (h,\theta_*(h)) \in H^1(X,\mathrm{Hom}(G_{-k},F_m)\oplus \mathrm{Hom}(F_{-m},G_k)\cap \Lambda^2_Q V) \subset H^1(X,U^+_{m+k}) \tag{7.16}$$

where $\theta_* : H^1(X,\mathrm{End}(V)) \to H^1(X,\mathrm{End}(V))$ is the map induced by the involution θ on $\mathrm{End}(V)$ previously defined. σ is obviously non-zero and, moreover, it is annihilated by

$$\mathrm{ad}(\Phi) = [\Phi,-] : H^1(X,U^+_{m+k}) \longrightarrow H^1(X,U^-_{m+k+1}\otimes K)$$

hence it defines an element in $\mathbb{H}^1(X,C^\bullet_{m+k}(V,L,Q,\Phi))$, which we also denote by σ.

Now, σ defines extensions

$$0 \longrightarrow F_m \xrightarrow{i_\sigma} U_\sigma \xrightarrow{p_\sigma} G_{-k} \longrightarrow 0$$

and

$$0 \longrightarrow G_k \xrightarrow{p^t_\sigma\otimes 1_L} U^*_\sigma \otimes L \xrightarrow{i^t_\sigma\otimes 1_L} F_{-m} \longrightarrow 0.$$

Let

$$V_\sigma = \bigoplus_{i=-m+1}^{m-1} F_i \oplus U_\sigma \oplus \bigoplus_{j=-k+1}^{k-1} G_j \oplus (U^*_\sigma\otimes L) \tag{7.17}$$

and $\Phi_\sigma : V_\sigma \to V_\sigma \otimes K$ given by

$$\begin{aligned} &\Phi_\sigma(v_{-m+1},\dots,v_{m-1},u_\sigma,w_{-k+1},\dots,w_{k-1},u^*_\sigma\otimes l) = \\ &= (\Phi_1 v_{-m+1},\dots,(i_\sigma\otimes 1_K)\Phi_1 v_{m-1},\Phi_2 p_\sigma u_\sigma, \\ &\Phi_2 w_{-k+1},\dots,(p^t_\sigma\otimes 1_L\otimes 1_K)\Phi_2 w_{k-1},\Phi_1(i^t_\sigma\otimes 1_L)(u^*_\sigma\otimes l)). \end{aligned} \tag{7.18}$$

Let us see that (V_σ,Φ_σ) is stable. If W is a proper Φ_σ-invariant subbundle of V_σ then W is one of the following:

- $W = F_m$;
- $W = G_k$;
- $W = \bigoplus_{i=-m+a}^{m-1} F_i \oplus F_m$, with $1 \leqslant a \leqslant 2m-1$;
- $W = \bigoplus_{j=-k+b}^{k-1} G_j \oplus G_k$, with $1 \leqslant b \leqslant 2k-1$;
- $W = \bigoplus_{i=-m+a}^{m-1} F_i \oplus U_\sigma \oplus \bigoplus_{j=-k+1}^{k-1} G_j \oplus G_k$, with $1 \leqslant a \leqslant 2m-1$;
- $W = U_\sigma \oplus \bigoplus_{j=-k+1}^{k-1} G_j \oplus G_k$.

Using the stability of (V_1,Φ_1) or of (V_2,Φ_2) and the fact that $\mu(V_i)=\mu(V)=\mu(V_\sigma)$, $i=1,2$, it follows that $\mu(W)<\mu(V_\sigma)$, (V_σ,Φ_σ) being therefore stable.

The summands $\bigoplus_{i=-m+1}^{m-1} F_i$ and $\bigoplus_{j=-k+1}^{k-1} G_j$ in V_σ have a quadratic form coming from Q, and we also have the canonical L-valued quadratic form on $U_\sigma \oplus (U_\sigma^* \otimes L)$. These give a L-valued quadratic form Q_σ on V_σ.

So we have seen that V_σ defined in (7.17), Φ_σ defined in (7.18) and Q_σ just defined give rise to a stable $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle $(V_\sigma, L, Q_\sigma, \Phi_\sigma)$.

Notice that, if $\sigma = 0$, then $(V_0, L, Q_0, \Phi_0) = (V, L, Q, \Phi)$. Now, consider the family $(V_{t\sigma}, L, Q_{t\sigma}, \Phi_{t\sigma})$ of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles. The induced infinitesimal deformation is given by σ which, from (7.16), lies in a positive weight subspace of $\mathbb{H}^1(X, C^\bullet(V, L, Q, \Phi))$. Taking Proposition 7.9 in account, this proves that (V, L, Q, Φ) is not a local minimum of f.

Suppose now that $V_1 \not\cong V_2$, but the form Q does not decompose. From the stability of (V_1, Φ_1) and of (V_2, Φ_2) we must have

$$q = \begin{pmatrix} 0 & q_{12} \\ q_{21} & 0 \end{pmatrix}$$

where $q_{12} : V_2 \to V_1^* \otimes L$ and $q_{21} : V_1 \to V_2^* \otimes L$ are isomorphisms and $q_{21} = q_{12}^t \otimes 1_L$.

Hence we can write

$$V = V_1 \oplus (V_1^* \otimes L)$$

and

$$\Phi = \Phi_1 \oplus \left(\Phi_1^t \otimes 1_K \otimes 1_L\right).$$

Consider the point in $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)$ represented by (V_1, Φ_1). Since $\Phi_1 \neq 0$, we know from [14] that (V_1, Φ_1) is not a local minimum of f in $\mathcal{M}_{\mathrm{GL}(n,\mathbb{C})}(nd/2)$ (this is because the group $\mathrm{GL}(n,\mathbb{C})$ is complex). Therefore one can find a family $(V_{1,s}, \Phi_{1,s})$ of stable Higgs bundles near (V, Φ) such that $f(V_{1,s}, \Phi_{1,s}) < f(V, \Phi)$ for all s, i.e.,

$$\|\Phi_{1,s}\|_{L^2}^2 < \|\Phi\|_{L^2}^2. \tag{7.19}$$

Consider now the family of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles in $\mathcal{M}_L$ given by

$$\left(V_{1,s} \oplus \left(V_{1,s}^* \otimes L\right) L, L, Q_s, \Phi_{1,s} \oplus \Phi_{1,s}^t \otimes 1_K \otimes 1_L\right)$$

where Q_s is the canonical quadratic form in $V_{1,s} \oplus \left(V_{1,s}^* \otimes L\right) L$. We have

$$\|\Phi_{1,s} \oplus \left(\Phi_{1,s}^t \otimes 1_K \otimes 1_L\right)\|_{L^2}^2 = \|\Phi_{1,s}\|_{L^2}^2 + \|\Phi_{1,s}^t \otimes 1_K \otimes 1_L\|_{L^2}^2 \tag{7.20}$$

where we are using the harmonic metric on $V_{1,s}^*$ and on $V_{1,s} \oplus V_{1,s}^*$ induced by the one on $V_{1,s}$. We have $\mathrm{tr}((\Phi_{1,s}^t \otimes 1_K \otimes 1_L)(\Phi_{1,s}^t \otimes 1_K \otimes 1_L)^*) = \mathrm{tr}(\Phi_{1,s}\Phi_{1,s}^*)$ therefore (7.20) is equivalent to

$$\|\Phi_{1,s} \oplus \Phi_{1,s}^t \otimes 1_K \otimes 1_L\|_{L^2}^2 = 2\|\Phi_{1,s}\|_{L^2}^2$$

and from (7.19) we conclude that

$$\|\Phi_{1,s} \oplus \Phi_{1,s}^t \otimes 1_K \otimes 1_L\|_{L^2}^2 < 2\|\Phi_1\|_{L^2}^2 = \|\Phi\|_{L^2}^2$$

for all s. Hence $(V_1 \oplus V_2, L, Q, \Phi_1 \oplus \Phi_2)$ is not a local minimum of f.

If $V_1 \cong V_2$, then we saw in the proof of Lemma 6.4 that we can decompose $Q = Q_1 \oplus Q_2$ so that we can decompose the $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles (V, L, Q, Φ) as $(V_1, L, Q_1, \Phi_1') \oplus (V_1, L, Q_2, \Phi_2')$. Hence we use the same argument as the first case to prove that (V, L, Q, Φ) is not a minimum of f.

If, for example, $\Phi_1 \neq 0$ and $\Phi_2 = 0$ then, due to the symmetry of Φ relatively to Q, the quadratic form must split into $Q_1 \oplus Q_2$, so that we have (V_1, L, Q_1, Φ_1) and $(V_2, L, Q_2, 0)$ and in a similar manner to the first case considered, we prove that (V, L, Q, Φ) is not a local minimum of f.

For $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles such that (V,Φ) has more than two summands, just consider the first two and use one of the above arguments. □

8. Connected components of the space of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles

In this section we compute the number of components of the subspaces of the moduli space of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles such that the degree of L is 0 and 1. Denote these subspaces by $\mathcal{M}_0$ and $\mathcal{M}_1$, respectively. In other words, using Notation 6.14, we write $\mathcal{M}_0$ as a disjoint union

$$\mathcal{M}_0 = \bigsqcup_{w_2\in\mathbb{Z}_2} \mathcal{M}(0,(w_2,0)) \sqcup \bigsqcup_{\overline{\mu}_1\in\mathbb{Z}_2^{2g}\setminus\{0\}} \mathcal{M}(\overline{\mu}_1,0).$$

On the other hand,

$$\mathcal{M}_1 = \bigsqcup_{\overline{\mu}_1\in\mathbb{Z}_2^{2g}} \mathcal{M}(\overline{\mu}_1,1).$$

Of course, the space $\mathcal{M}$ of isomorphism classes of polystable $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles has an infinite number of components because the invariant given by the degree of L can be any integer. But our computation will also give the number of components of any subspace of $\mathcal{M}$ with the degree of L fixed, due to the identifications given in Remark 6.15.

Before proceeding with the computation, we need some results which will be used. Let $\mathcal{N}_{\mathrm{EO}(n,\mathbb{C})}$ be the moduli space of holomorphic semistable principal $\mathrm{EO}(n,\mathbb{C})$-bundles on X and $\mathcal{N}_{\mathrm{EO}(n,\mathbb{C})}(\overline{\mu}_1,\overline{\mu}_2)$ be the subspace with invariants $(\overline{\mu}_1,\overline{\mu}_2)$.

The following is an adaptation of Proposition 4.2 of [21].

Proposition 8.1. *$\mathcal{N}_{\mathrm{EO}(n,\mathbb{C})}(\overline{\mu}_1,\overline{\mu}_2)$ is connected.*

Proof. Let E' and E'' represent two classes in $\mathcal{N}_{\mathrm{EO}(n,\mathbb{C})}(\overline{\mu}_1,\overline{\mu}_2)$. Let P be the underlying C^∞ principal bundle, and let $\overline{\partial}_{A'}$ and $\overline{\partial}_{A''}$ be the operators on P defining, respectively, E' and E'' and given by unitary connections A' and A''.

Let $\mathbb{D}$ be an open disc in $\mathbb{C}$ containing 0 and 1. Consider the C^∞ principal-$\mathrm{EO}(n,\mathbb{C})$ bundle $\mathbb{E}\to\mathbb{D}\times X$, where $\mathbb{E}=\mathbb{D}\times P$. Define the connection form on $\mathbb{E}$ by

$$A_z(v,w) = zA''(w) + (1-z)A'(w) \in \Omega^1(\mathbb{D}\times P, \mathfrak{o}(n,\mathbb{C})\oplus\mathbb{C})$$

where v is tangent to $\mathbb{D}$ at z and w is tangent to P at some point p. If we consider the holomorphic bundle E_z given by $\mathbb{E}|_{\{z\}\times X}$ with the holomorphic structure given by A_z, then we have that $E_0\cong E'$ and $E_1\cong E''$.

Since semistability is an open condition with respect to the Zariski topology, $\mathbb{D}\setminus D'$ is connected where $D'=\{z\in\mathbb{D} : E_z \text{ is not semistable}\}$. Hence $\{E_z\}_{z\in\mathbb{D}\setminus D'}$ is a connected family of semistable $\mathrm{EO}(n,\mathbb{C})$-principal bundles joining E_0 and E_1. Since $E_0\cong E'$ and $E_1\cong E''$, using the universal property of the coarse moduli space $\mathcal{N}_{\mathrm{GL}(n,\mathbb{C})}$ of $\mathrm{GL}(n,\mathbb{C})$-principal bundles, there is a connected family in $\mathcal{N}_{\mathrm{GL}(n,\mathbb{C})}$ joining E' and E''. But, of course this connected family lies in $\mathcal{N}_{\mathrm{EO}(n,\mathbb{C})}(\overline{\mu}_1,\overline{\mu}_2)$. □

Let

$$\mathcal{M}'_L$$

be the subspace of $\mathcal{M}_L$ consisting of those components of $\mathcal{M}_L$ such that the minimum of the Hitchin function f attained on these components is 0. Hence the local minima on $\mathcal{M}'_L$ are those with $\Phi=0$.

Proposition 7.2 and the previous one yield the following:

Corollary 8.2. *For each* $(\overline{\mu}_1, \overline{\mu}_2)$*, the space* $\mathcal{M}'_L(\overline{\mu}_1, \overline{\mu}_2)$ *is (if non-empty) connected.*

Recall from Corollary 6.17 that $\mathcal{M}'_L(\overline{\mu}_1, \overline{\mu}_2)$ is empty precisely when n and $\deg(L)$ are both odd. Excluding this case, $\mathcal{M}'_L(\overline{\mu}_1, \overline{\mu}_2)$ is hence connected.

All the analysis of the proper function f carried in Section 7 was done over each $\mathcal{M}_L$, hence one can use Proposition 7.2 to compute the number of components of $\mathcal{M}_L$, and then compute the number of components of $\mathcal{M}_0$ and of $\mathcal{M}_1$.

For each L, define

$$\mathcal{M}''_L = \mathcal{M}_L \setminus \mathcal{M}'_L$$

so that we have a disjoint union

$$\mathcal{M}_L = \mathcal{M}'_L \sqcup \mathcal{M}''_L. \tag{8.1}$$

Let us now concentrate attentions on $\mathcal{M}_0$.

For each $L \in \mathrm{Jac}^0(X) = \mathrm{Jac}(X)$, the Jacobian of X, $\mathcal{M}_L$ is a subspace of $\mathcal{M}_0$ and it is the fibre over L of the map

$$\nu_0 : \mathcal{M}_0 \longrightarrow \mathrm{Jac}(X) \tag{8.2}$$

given by

$$\nu_0(V, L, Q, \Phi) = L.$$

To emphasize the fact that now $\mathcal{M}_L \subset \mathcal{M}_0$, we shall write $\mathcal{M}_{L,0}$ instead of $\mathcal{M}_L$. Any two fibres $\mathcal{M}_{L,0}$ and $\mathcal{M}_{L',0}$ of ν_0 are isomorphic through the map

$$(V, L, Q, \Phi) \mapsto (V \otimes L^{-1/2} \otimes L'^{1/2}, L', Q \otimes 1_{L^{-1}} \otimes 1_{L'}, \Phi \otimes 1_{L^{-1/2}} \otimes 1_{L'^{1/2}}).$$

In particular, any fibre is isomorphic to $\mathcal{M}_{\mathcal{O}}$.

More precisely, after lifting to a finite cover, (8.2) becomes a product. This is a similar situation to the one which occurs on the moduli of vector bundles with fixed determinant (cf. [1]). Indeed, we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{O}} \times \mathrm{Jac}(X) & \xrightarrow{\mathrm{pr}_2} & \mathrm{Jac}(X) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ \mathcal{M}_0 & \xrightarrow{\nu_0} & \mathrm{Jac}(X) \end{array} \tag{8.3}$$

where $\pi((W, \mathcal{O}, Q, \Phi), M) = (W \otimes M, M^2, Q \otimes 1_{M^2}, \Phi \otimes 1_M)$ and $\pi'(M) = M^2$. Hence ν_0 is a fibration.

Recall that an $\mathrm{EGL}(n, \mathbb{R})$-Higgs bundle $(V, \mathcal{O}, Q, \Phi)$ is topologically classified by the invariants $(\overline{\mu}_1, \overline{\mu}_2)$ where $\overline{\mu}_1 = w_1(V, Q, \Phi) \in \mathbb{Z}_2^{2g}$ and, if $\overline{\mu}_1 \neq 0$, then $\overline{\mu}_2 = 0 = \deg(\mathcal{O})$, and, if $\overline{\mu}_1 = 0$, then $\overline{\mu}_2 = (w_2(V, Q, \Phi), 0 = \deg(\mathcal{O}))$.

Now, if $\mathcal{M}_{\mathrm{GL}(n,\mathbb{R})}$ denotes the moduli space of $\mathrm{GL}(n, \mathbb{R})$-Higgs bundles [3], which are classified by the first and second Stiefel-Whitney classes, there is a surjective map

$$\mathcal{M}_{\mathrm{GL}(n,\mathbb{R})} \longrightarrow \mathcal{M}_{\mathcal{O}} \tag{8.4}$$

given by $(W, Q, \Phi) \mapsto (W, \mathcal{O}, Q, \Phi)$ and such that:

- $\mathcal{M}_{\mathrm{GL}(n,\mathbb{R})}(0, w_2)$ is mapped onto $\mathcal{M}_{\mathcal{O}}(0, (w_2, 0))$;
- if $w_1 \neq 0$, $\mathcal{M}_{\mathrm{GL}(n,\mathbb{R})}(w_1, w_2)$ is mapped onto $\mathcal{M}_{\mathcal{O}}(w_1, 0)$.

The following result is proved in Proposition 4.6 of [3] and gives a more detailed information about the structure of $\mathcal{M}_{\mathrm{GL}(n,\mathbb{R})}$.

Proposition 8.3. *Let* $(V, \mathcal{O}, Q, \Phi) \in \mathcal{M}_{\mathrm{GL}(n,\mathbb{R})}$ *be a local minimum of* f *with* $\Phi \neq 0$*. Then,*

$$w_1(V, \mathcal{O}, Q, \Phi) = 0$$

and

$$w_2(V, \mathcal{O}, Q, \Phi) = (g-1)n^2/4 \bmod 2.$$

Therefore, using the surjection (8.4) and the fact that any fibre of ν_0 is isomorphic to $\mathcal{M}_{\mathcal{O}}$, we obtain:

Proposition 8.4. *Let* $(V, L, Q, \Phi) \in \mathcal{M}_{L,0}$ *be a local minimum of* f *with* $\Phi \neq 0$*. Then,*

$$\overline{\mu}_1(V, L, Q, \Phi) = 0$$

and

$$\overline{\mu}_2(V, L, Q, \Phi) = ((g-1)n^2/4 \bmod 2, 0).$$

From now on we shall write

$$z_0 = (g-1)\frac{n^2}{4} \bmod 2.$$

From this proposition and from what we saw above follows that,

$$\mathcal{M}_{L,0}(\overline{\mu}_1) = \mathcal{M}'_{L,0}(\overline{\mu}_1) \tag{8.5}$$

if $\overline{\mu}_1 \neq 0$,

$$\mathcal{M}_{L,0}(0, w_2) = \mathcal{M}'_{L,0}(0, w_2) \tag{8.6}$$

if $w_2 \neq z_0$, and

$$\mathcal{M}_{L,0}(0, z_0) = \mathcal{M}'_{L,0}(0, z_0) \sqcup \mathcal{M}''_{L,0}(0, z_0). \tag{8.7}$$

In other words,

$$\mathcal{M}_{L,0} = \bigsqcup_{\overline{\mu}_1 \in (\mathbb{Z}_2)^{2g} \setminus \{0\}} \mathcal{M}'_{L,0}(\overline{\mu}_1) \sqcup \bigsqcup_{w_2 \in \mathbb{Z}_2} \mathcal{M}'_{L,0}(0, w_2) \sqcup \mathcal{M}''_{L,0}(0, z_0). \tag{8.8}$$

Proposition 8.5. *Let* $n \geqslant 4$ *be even and* $L \in \mathrm{Jac}(X)$ *be given. Then* $\mathcal{M}_{L,0}$ *has* $2^{2g+1} + 1$ *connected components. More precisely,*

(i) $\mathcal{M}_{L,0}(\overline{\mu}_1)$ *with* $\overline{\mu}_1 \neq 0$*, is connected;*
(ii) $\mathcal{M}_{L,0}(0, w_2)$ *with* $w_2 \neq z_0$*, is connected;*
(iii) $\mathcal{M}_{L,0}(0, z_0)$ *has* $2^{2g} + 1$ *components.*

This result follows immediately from Theorem 5.2 of [3] and from the existence of the map $\mathcal{M}_{\mathrm{GL}(n,\mathbb{R})} \to \mathcal{M}_{\mathcal{O}}$ described in (8.4). However, for completeness, we are still going to give a proof.

Proof. Let $L \in \mathrm{Jac}(X)$. Fix $\overline{\mu}_1 \neq 0$ and consider the subspace

$$\mathcal{M}_{L,0}(\overline{\mu}_1) \subset \mathcal{M}_{L,0}.$$

This space is connected by (8.5) and by Corollary 8.2. So there are $2^{2g} - 1$ components of $\mathcal{M}_{L,0}$ of this kind.

For the same reason but using (8.6), we see that $\mathcal{M}_{L,0}(0, w_2)$ with $w_2 \neq z_0$, is connected.

For the space $\mathcal{M}_{L,0}(0,z_0)$ we have the decomposition (8.7). The space $\mathcal{M}'_{L,0}(0,z_0)$ is connected from Corollary 8.2. Let us then analyse the space $\mathcal{M}''_{L,0}(0,z_0)$. Consider the non-zero local minima of the Hitchin functional f. From Corollary 7.8, these are such that

$$V = F_{-1/2} \otimes \bigoplus_{i=-r-1}^{r} K^i \tag{8.9}$$

where $r = m - 1/2$ and $F_{-1/2}$ is a square root of LK. There are 2^{2g} different choices for $F_{-1/2}$ thus the space of local minima of this kind consists of 2^{2g} isolated points. Therefore $\mathcal{M}''_{L,0}(0,z_0)$ has 2^{2g} connected components. All these are homeomorphic to a vector space and constitute the so-called *Hitchin* or *Teichmüller components* of $\mathcal{M}_{L,0}$ [15]. So $\mathcal{M}_{L,0}(0,z_0)$ has $2^{2g}+1$ components.

If follows from (8.8) and from the count above that $\mathcal{M}_{L,0}$ has $2^{2g+1}+1$ connected components. □

We have computed the components of each fibre of ν_0. Let us see that the space $\mathcal{M}_0$ has less components than $\mathcal{M}_{L,0}$.

Theorem 8.6. *The space $\mathcal{M}_0$ has $2^{2g}+2$ components.*

Proof. From Theorem 6.13, there are $2^{2g}+1$ topological invariants of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles in $\mathcal{M}_0$, hence $\mathcal{M}_0$ has at least $2^{2g}+1$ components.

Let $(V,L,Q,\Phi),(V',L',Q',\Phi') \in \mathcal{M}_0(0,z_0)$ such that each $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle is a local minimum of type (2) on the corresponding fibre of ν_0 (see (8.2)). Hence

$$V = F_{-m} \oplus \cdots \oplus F_{-1/2} \oplus F_{1/2} \oplus \cdots \oplus F_m$$

and

$$V' = F'_{-m} \oplus \cdots \oplus F'_{-1/2} \oplus F'_{1/2} \oplus \cdots \oplus F'_m$$

where $F_{-1/2}$ (resp. $F'_{-1/2}$) is a square root of LK (resp. $L'K$). Since $\mathrm{Jac}(X)$ is connected, there is a path L_t in $\mathrm{Jac}(X)$ joining L to L'. Set

$$V_t = F_{-m,t} \oplus \cdots \oplus F_{-1/2,t} \oplus F_{1/2,t} \oplus \cdots \oplus F_{m,t}$$

where $F^2_{-1/2,t} \cong L_tK$ and $F_{-1/2+i,t} \cong F_{-1/2,t}K^{-i}$. With

$$q_t = \begin{pmatrix} 0 & \dots & \dots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & 1 & & \vdots \\ 0 & \dots & & & \vdots \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix} \quad \text{and} \quad \Phi_t = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & \ddots & & \dots & 0 \\ 0 & 1 & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

$(V_t,L_t,Q_t,\Phi_t)_t$ is a path in $\mathcal{M}_0$ joining (V,L,Q,Φ) and (V',L',Q',Φ') and such that, for every t, (V_t,L_t,Q_t,Φ_t) is a minimum of f in $\mathcal{M}_{L_t,0}$ of type (2). Hence we conclude that all the 2^{2g} Hitchin components of all fibres of ν_0 join together to form a unique component of $\mathcal{M}_0$: $\mathcal{M}''_0(0,z_0) = \bigcup_{L\in\mathrm{Jac}(X)} \mathcal{M}''_{L,0}(0,z_0)$. Note that this is not a Hitchin component. Indeed, the group $\mathrm{EGL}(n,\mathbb{R})$ is not a split real form (due to $\mathrm{U}(1)$), so the moduli space of $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundles on X was not expected to have a Hitchin component (cf. [15]).

On the other hand, $\mathcal{M}'_0(\overline{\mu}_1) = \bigcup_{L\in\mathrm{Jac}(X)} \mathcal{M}'_{L,0}(\overline{\mu}_1)$ is connected because $\nu_0|_{\mathcal{M}'_0(\overline{\mu}_1)} : \mathcal{M}'_0(\overline{\mu}_1) \to \mathrm{Jac}(X)$ is surjective and with connected fibre from item (1)(a) of Proposition 8.5 and $\mathrm{Jac}(X)$ is connected. For an analogous reason, we also conclude that $\mathcal{M}_0'(0,w_2) = \bigcup_{L\in \mathrm{Jac}(X)} \mathcal{M}_{L,0}'(0,w_2)$ is connected.

Finally, $\mathcal{M}_0'(0,z_0)$ and $\mathcal{M}_0''(0,z_0)$ are two different connected components of $\mathcal{M}_0(0,z_0)$.

Concluding, we have one component for each $\mathcal{M}_0'(0,z_0)$, $\mathcal{M}_0''(0,z_0)$ and $\mathcal{M}_0'(0,w_2)$ with $w_2 \neq z_0$, and $2^{2g}-1$ components coming from $\mathcal{M}_0'(\overline{\mu}_1)$. These yield the $2^{2g}+2$ components of $\mathcal{M}_0$. □

Let us now deal with the space $\mathcal{M}_1$.

We have again a map $\nu_1 : \mathcal{M}_1 \to \mathrm{Jac}^1(X)$ and, if $\deg(L) = 1$, $\mathcal{M}_{L,1} = \nu_1^{-1}(L)$. In fact, when we fix a line bundle $L_0 \in \mathrm{Jac}^1(X)$, we have a analogous diagram to (8.3):

$$\begin{array}{ccc} \mathcal{M}_{L_0} \times \mathrm{Jac}(X) & \xrightarrow{\ m\ } & \mathrm{Jac}^1(X) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ \mathcal{M}_1 & \xrightarrow{\ \nu_1\ } & \mathrm{Jac}^1(X) \end{array}$$

where $m((W,L_0,Q,\Phi),M) = ML_0$, $\pi((W,L_0,Q,\Phi),M) = (W\otimes M, L_0M^2, Q\otimes 1_{M^2}, \Phi\otimes 1_M)$ and $\pi'(L) = L^2L_0^{-1}$. Hence ν_1 is also a fibration.

If an $\mathrm{EGL}(n,\mathbb{R})$-Higgs bundle (V,L,Q,Φ), with (V,Φ) stable, is a non-zero local minimum of f in $\mathcal{M}_L$ then it follows from Corollary 7.8 that $\deg(L)$ is even. Hence, if $\deg(L) = 1$,

$$\mathcal{M}_{L,1}(\overline{\mu}_1) = \mathcal{M}_{L,1}'(\overline{\mu}_1)$$

thus,

$$\mathcal{M}_{L,1} = \bigsqcup_{\overline{\mu}_1 \in (\mathbb{Z}_2)^{2g}} \mathcal{M}_{L,1}'(\overline{\mu}_1). \tag{8.10}$$

Proposition 8.7. *Let $n \geqslant 4$ be even and let $L \in \mathrm{Jac}^1(X)$. Then $\mathcal{M}_{L,1}$ has 2^{2g} connected components. More precisely, each $\mathcal{M}_{L,1}(\overline{\mu}_1)$ is connected.*

Proof. The result follows from (8.10) and from Corollary 8.2, just like in the proof of Proposition 8.5. □

Now we compute the components of $\mathcal{M}_1$.

Theorem 8.8. *$\mathcal{M}_1$ has 2^{2g} components.*

Proof. $\mathcal{M}_1(\overline{\mu}_1) = \bigcup_{L\in\mathrm{Jac}(X)} \mathcal{M}_{L,1}(\overline{\mu}_1)$ is connected since $\mathrm{Jac}^1(X)$ is connected and $\nu_1|_{\mathcal{M}_1(\overline{\mu}_1)} : \mathcal{M}_1(\overline{\mu}_1) \to \mathrm{Jac}^1(X)$ is a fibration with connected fibre $\mathcal{M}_{L,1}(\overline{\mu}_1)$, from Proposition 8.7. The result follows since $\mathcal{M}_1 = \bigsqcup_{\overline{\mu}_1\in(\mathbb{Z}_2)^{2g}} \mathcal{M}_1(\overline{\mu}_1)$. □

9. Topology of $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}$

In this subsection we shall consider the lower rank case of $\mathrm{SL}(3,\mathbb{R})$-Higgs bundles. In holomorphic terms these are triples (V,Q,Φ) where V is holomorphic vector bundle equipped with a nowhere degenerate quadratic form Q and with trivial determinant, and Φ is a traceless K-twisted endomorphism of V, symmetric with respect to Q.

Let $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}$ be the moduli space of $\mathrm{SL}(3,\mathbb{R})$-Higgs bundles. These objects are classified by the second Stiefel-Whitney class $w_2 \in \{0,1\}$, and let $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ be the subspace of $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}$ whose elements have the given w_2.

The moduli space $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}$ was considered in [15] where the minimum subvarieties of the Hitchin functional were studied. There it was shown that if (V, Q, Φ) represents a fixed point of the circle action (7.3), with $\Phi \neq 0$, then V is of the form

$$V = F_{-m} \oplus \cdots \oplus F_m$$

with $F_{-j} \cong F_j^*$, hence $\mathrm{rk}(F_j) = \mathrm{rk}(F_{-j})$ for all j. From this and since $\mathrm{rk}(V) = 3$, we conclude that fixed points with non-zero Higgs field are precisely those such that

$$V = F_{-1} \oplus \mathcal{O} \oplus F_1$$

with $\mathrm{rk}(F_j) = 1$ and, if $j \neq 1$, $\Phi_j : F_j \to F_{j+1} \otimes K$ is an isomorphism. These are local minima of the Hitchin function f. The corresponding connected component, the Hitchin component, being isomorphic to a vector space, is contractible.

For each $w_2 \in \{0, 1\}$, let

$$\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$$

be the subspace of $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ such that the minima on each of its connected components have $\Phi = 0$. Given $(V, Q, \Phi) \in \mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}(w_2)$, we know from [31], that

$$\lim_{t \to 0} (V, Q, t\Phi)$$

exists on $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ and it is a fixed point of the $\mathbb{C}^*$-action $(V, Q, \Phi) \mapsto (V, Q, t\Phi)$ on $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}(w_2)$, being therefore a minimum of f. Hence, if $(V, Q, \Phi) \in \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$, it follows that $\lim_{t\to 0}(V, Q, t\Phi) = (V', Q', 0) \in \mathcal{N}_{\mathrm{SO}(3,\mathbb{C})}(w_2)$ which is the space of local minima with zero Higgs field. Note that, in principle, it may happen that $\lim_{t\to 0}(V, Q, t\Phi) \neq (V, Q, 0) = (V, Q)$, as (V, Q) may be unstable as an ordinary orthogonal vector bundle.

Let us then consider the map

$$F : \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2) \times [0, 1] \longrightarrow \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$$

given by

$$F((V, Q, \Phi), t) = \begin{cases} (V, Q, t\Phi) & \text{if } t \neq 0 \\ \lim_{t\to 0}(V, Q, t\Phi) & \text{if } t = 0. \end{cases} \tag{9.1}$$

This map, together with the previous discussion, provides the following result (recall that, from [15], we know that $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}$ has 3 components).

Theorem 9.1. *The space $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}$ has one contractible component and the space consisting of the other two components is homotopically equivalent to $\mathcal{N}_{\mathrm{SO}(3,\mathbb{C})}$.*

Proof. The first part has already been discussed. For the second part, we have to see that the map F defined in (9.1) is continuous, providing then a retraction from $\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ into $\mathcal{N}_{\mathrm{SO}(3,\mathbb{C})}(w_2)$, for each value of w_2. When $t \neq 0$, the continuity of F is obvious. We will take care of the case $t = 0$.

The space $\mathcal{M}_{\mathrm{GL}(3,\mathbb{C})}$ is a quasi-projective, algebraic variety and $\mathbb{C}^*$ acts algebraically on it as $(V, \Phi) \mapsto (V, t\Phi)$. Linearise this action with respect to an ample line bundle N (such that N^s is very ample) over $\mathcal{M}_{\mathrm{GL}(3,\mathbb{C})}$. This $\mathbb{C}^*$-action induces one on N^s and, therefore we obtain a $\mathbb{C}^*$-action on $H^0(\mathcal{M}_{\mathrm{GL}(3,\mathbb{C})}, N^s)$ given by

$$(t \cdot s)(V, \Phi) = t \cdot (s(V, t^{-1}\Phi)).$$

One can choose a rank $r+1$, $\mathbb{C}^*$-invariant subspace $W \subseteq H^0(\mathcal{M}_{\mathrm{GL}(3,\mathbb{C})}, N^s)$ and hence $\mathbb{C}^*$ acts on W. From this action we obtain a $\mathbb{C}^*$-action on $\mathbb{P}^r \cong \mathbb{P}(W)$, and there is a $\mathbb{C}^*$-equivariant locally closed embedding

$$\iota : \mathcal{M}_{\mathrm{GL}(3,\mathbb{C})} \hookrightarrow \mathbb{P}^r. \tag{9.2}$$

If we linearise the given $\mathbb{C}^*$-action on $\mathbb{P}^r$ with respect to the very ample $\mathcal{O}_{\mathbb{P}^r}(1)$, then this is compatible with the morphism (9.2) and with the isomorphism $N^s \cong \mathcal{O}_{\mathbb{P}^r}(1)$.

Now, we can decompose W as

$$W = \bigoplus_{i=1}^{k} W_{r_i}$$

where $r_i = \mathrm{rk}(W_{r_i})$ and $\mathbb{C}^*$ acts over each W_{r_i} as $v \mapsto t^{\alpha_i} v$, $t \in \mathbb{C}^*$, $\alpha_i \in \mathbb{Z}$ and $\alpha_i < \alpha_j$ whenever $i < j$. So, for each r_i, we have a subspace of $\mathbb{P}^r$ given by $\mathbb{P}^{r_i-1} = \mathbb{P}(W_{r_i})$. With respect to the above decomposition of W, $\mathbb{C}^*$ acts as

$$(v_1, \dots, v_k) \mapsto (t^{\alpha_1} v_1, \dots, t^{\alpha_k} v_k). \tag{9.3}$$

Then, we also have the induced $\mathbb{C}^*$-action on the closed subspace $\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ and a $\mathbb{C}^*$-equivariant topological embedding

$$\iota|_{\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)} : \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2) \hookrightarrow \mathbb{P}^r \tag{9.4}$$

and we denote the image in $\mathbb{P}^r$ of $\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ through $\iota|_{\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)}$ also by $\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$. So we view $\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ not as a subvariety of $\mathbb{P}^r$, but simply a closed subspace (for the complex topology).

From (9.3), the fixed point set of the $\mathbb{C}^*$-action on $\mathbb{P}^r$ is

$$\mathrm{Fix}_{\mathbb{C}^*}(\mathbb{P}^r) = \bigcup_{i=1}^{k} \mathbb{P}^{r_i-1}$$

so the fixed point set of this action on $\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ is

$$\mathrm{Fix}_{\mathbb{C}^*}(\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)) = \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2) \cap \bigcup_{i=1}^{k} \mathbb{P}^{r_i-1}.$$

But we already know that $\mathrm{Fix}_{\mathbb{C}^*}(\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)) = \mathcal{N}_{\mathrm{SO}(3,\mathbb{C})}(w_2)$ which is an irreducible variety, by Theorem 5.9 of [23]. So we conclude that

$$\mathrm{Fix}_{\mathbb{C}^*}(\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)) = \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2) \cap \mathbb{P}^{r_{i_0}-1} \tag{9.5}$$

for some $i_0 \in \{1, \dots, k\}$.

Actually,

$$i_0 = \min\{i \in \{1, \dots, k\} \mid v_i \neq 0, \text{ for some } (v_1, \dots, v_k) \in \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)\}. \tag{9.6}$$

In fact, and let $j = \min\{i \in \{1, \dots, k\} \mid v_i \neq 0, \text{ for some } (v_1, \dots, v_k) \in \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)\}$ and let $(v_1, \dots, v_k) \in \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)$ so that we can write it as $(0, \dots, 0, v_j, \dots, v_k)$. We have

$$\begin{aligned}
\lim_{t\to 0} t(0, \dots, 0, v_j, \dots, v_k) &= \lim_{t\to 0}(0, \dots, 0, t^{\alpha_j} v_j, \dots, t^{\alpha_k} v_k) \\
&= \lim_{t\to 0}(0, \dots, 0, v_j, t^{\alpha_{j+1}-\alpha_j} v_{j+1}, \dots, t^{\alpha_k-\alpha_j} v_k) \\
&= (0, \dots, v_j, \dots, 0) \in \mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2) \cap \mathbb{P}^{r_j-1}.
\end{aligned}$$

But, since we already know that $\lim_{t\to 0} t(v_1,\dots,v_k)\in \mathrm{Fix}_{\mathbb{C}^*}(\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2))$, we have from (9.5) that $i_0=j$ and this settles (9.6).

If we take the map $\tilde{F}:\mathbb{P}^r\times[0,1]\to\mathbb{P}^r$ given by

$$\tilde{F}((v_1,\dots,v_k),t)=\begin{cases} t(v_1,\dots,v_k)=(t^{\alpha_1}v_1,\dots,t^{\alpha_k}v_k) & \text{if } t\neq 0\\ \lim_{t\to 0} t(v_1,\dots,v_k)=(0,\dots,0,v_{i_0},0,\dots,0) & \text{if } t=0\end{cases}$$

then it is well-defined by the definition of i_0 in (9.6) and it is clearly continuous because i_0 is constant. By the compatibility of the actions, we have that F corresponds, under (9.4), to $\tilde{F}|_{\mathcal{M}'_{\mathrm{SL}(3,\mathbb{R})}(w_2)\times[0,1]}$, so F is also continuous. □

10. Connected components of spaces of representations

Recall that our main goal is to compute the number of components of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ for $n\geqslant 4$ even, but we had to work with the group $\mathrm{EGL}(n,\mathbb{R})$. The work done also gives a way to count the components of the subspace of $\mathcal{R}=\mathcal{R}_{\Gamma,\mathrm{EGL}(n,\mathbb{R})}$ given by the disjoint union $\mathcal{R}_0\sqcup\mathcal{R}_1$. Denote this subspace by $\mathcal{R}_{0,1}$.

Proposition 10.1. *Let $n\geqslant 4$ be even. Then, $\mathcal{R}_{0,1}$ has $2^{2g+1}+2$ connected components. More precisely,*

(i) *$\mathcal{R}_0(\overline{\mu}_1)$ is connected, if $\overline{\mu}_1\neq 0$;*
(ii) *$\mathcal{R}_0(0,w_2)$ is connected, if $w_2\neq z_0$;*
(iii) *$\mathcal{R}_0(0,z_0)$ has 2 components;*
(iv) *$\mathcal{R}_1(\overline{\mu}_1)$ is connected.*

Proof. By Theorem 2.8, $\mathcal{R}_0(\overline{\mu}_1)\cong\mathcal{M}_0(\overline{\mu}_1)$, $\mathcal{R}_0(0,w_2)\cong\mathcal{M}_0(0,w_2)$ and $\mathcal{R}_1(\overline{\mu}_1)\cong\mathcal{M}_1(\overline{\mu}_1)$. The result follows directly from the analysis of the components of $\mathcal{M}_0$ and $\mathcal{M}_1$ in Theorems 8.6 and 8.8. □

Now our main result follows as a corollary.

Theorem 10.2. *Let $n\geqslant 4$ be even, and X a closed oriented surface of genus $g\geqslant 2$. Then the moduli space $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ of reductive representations of π_1X in $\mathrm{PGL}(n,\mathbb{R})$ has $2^{2g+1}+2$ connected components. More precisely,*

(i) *$\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1,0)$ is connected, if $\mu_1\neq 0$;*
(ii) *$\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(0,w_2)$ is connected, if $w_2\neq z_0$;*
(iii) *$\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(0,z_0)$ has 2 components;*
(iv) *$\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1,\omega_n)$ is connected.*

Proof. The result follows immediately from the existence of the surjective continuous map $p:\mathcal{R}\to\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ satisfying the identities of Proposition 6.16 and from the previous proposition. Note that the two components of $\mathcal{R}_0(0,z_0)$ are not mapped into only one in $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(0,z_0)$ because if that were the case, every representation in $\mathrm{PGL}(n,\mathbb{R})$ with $(0,z_0)$ as invariants could deform to a representation into $\mathrm{PO}(n)$, the maximal compact, and then the same would occur for the group $\mathrm{EGL}(n,\mathbb{R})$. We know however that this is not possible because of the analysis of the minima with invariants $(0,z_0)$: the component with minima with $\Phi\neq 0$ corresponds precisely to those representations which do not deform to a representation in $\mathrm{EO}(n)$. On the other hand, $\mathrm{PGL}(n,\mathbb{R})$ is a split real form so by [15] the space $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}$ should have a Hitchin component which in this case corresponds to the representations which do not deform to $\mathrm{PO}(n)$. □

Remark 10.3. For the proof of Theorem 10.2 is not essential to have Proposition 10.1. We could have used Propositions 8.5 and 8.7 and noticed that the vector bundles corresponding to minima of f of type (2) are projectively equivalent. This would give us the number of components of $\mathcal{M}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1,\mu_2)$, therefore of $\mathcal{R}_{\mathrm{PGL}(n,\mathbb{R})}(\mu_1,\mu_2)$ from Theorem 2.8.

Remark 10.4. If $\mu_1 = 0$, then we might expect to get the same components as Hitchin did in [15] but that does not happen. We computed 4 components while Hitchin's result was 6. The difference is that we are considering $\mathrm{PGL}(n,\mathbb{R})$-equivalence (cf. Remark 3.9), while Hitchin considered $\mathrm{PSL}(n,\mathbb{R})$-equivalence.

11. Topology of $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$

We finish with a corollary of Theorem 9.1. When n is odd, $\mathrm{PGL}(n,\mathbb{R}) \cong \mathrm{SL}(n,\mathbb{R})$, so $\mathcal{R}_{\mathrm{PGL}(3,\mathbb{R})} = \mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$. Furthermore, from [15] we know that $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$ has three components.

Theorem 11.1. *Let X be a closed oriented surface of genus $g \geqslant 2$. The moduli space $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$ of reductive representations of $\pi_1 X$ in $\mathrm{SL}(3,\mathbb{R})$ has one contractible component (the Hitchin component) and the space consisting of the other two components is homotopically equivalent to $\mathcal{R}_{\mathrm{SO}(3)}$.*

Proof. The moduli space $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}$ is isomorphic, via Theorem 2.8, to $\mathcal{M}_{\mathrm{SL}(3,\mathbb{R})}$. The result follows from Theorem 9.1. □

Very recently, in [16], Ho and Liu have computed, among other things, the Poincaré polynomials of the spaces $\mathcal{R}_{\mathrm{SO}(2n+1)}(w_2)$, $w_2 = 0, 1$. For $n = 3$, their result is (Theorem 5.5 and Example 5.7 of [16])

$$P_t(\mathcal{R}_{\mathrm{SO}(3)}(0)) = \frac{-(1+t)^{2g}t^{2g+2} + (1+t^3)^{2g}}{(1-t^2)(1-t^4)} \tag{11.1}$$

and

$$P_t(\mathcal{R}_{\mathrm{SO}(3)}(1)) = \frac{-(1+t)^{2g}t^{2g} + (1+t^3)^{2g}}{(1-t^2)(1-t^4)}. \tag{11.2}$$

From this result and from Theorem 11.1, we have:

Theorem 11.2. *The Poincaré polynomials of $\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}(w_2)$, $w_2 = 0, 1$, are given by*

$$P_t(\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}(0)) = \frac{-(1+t)^{2g}t^{2g+2} + (1+t^3)^{2g}}{(1-t^2)(1-t^4)} + 1$$

and

$$P_t(\mathcal{R}_{\mathrm{SL}(3,\mathbb{R})}(1)) = \frac{-(1+t)^{2g}t^{2g} + (1+t^3)^{2g}}{(1-t^2)(1-t^4)}.$$

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