ON SPECIAL LIFTING MODULES

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Abstract

Let R be a ring with identity and M a right R–module. M is called a *special lifting module*, if M is lifting and $M = \oplus_{i\in I} M_i$, where M_i is a module with local endomorphism ring, for all $i \in I$. The purpose of this paper is to develop some properties of these modules. Semiperfect rings such that any direct sum of a simple and a projective local module is lifting are characterized as semiprimary rings with Jacobson radical square-zero.

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1 Introduction and Preliminaries

Throughout this paper all rings will have an identity and all modules will be unital right R–modules. $\mathrm{Rad}(M)$, $\mathrm{Soc}(M)$ and $\mathrm{Jac}(R)$ will indicate Jacobson radical of M, socle of M and the radical of R, respectively. We will use the notation ACC (DCC) to indicate that any module M satisfies the ascending chain condition (descending chain condition).

Let M be a module and $S \leq M$. S is called *small* in M (notation $S \ll M$) if $M \neq S + T$ for any proper submodule T of M. Dually, a submodule E of M is called *essential* in M (notation $E \leq_e M$) if $E \cap A \neq 0$ for any non-zero submodule A of M. A module M is called *hollow* if every proper submodule of M is small in M and it is called *local* if it is hollow and has a unique maximal submodule. The module M is called *uniform* if every non-zero submodule of M is essential in M.

Let M be a module. M is called a *lifting* module (or *satisfies* (D_1)) if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$, equivalently, for every submodule N of M there exist submodules K and K' of M such that $M = K \oplus K'$, $K \leq N$ and $N \cap K' \ll K'$. Dually, M is called an *extending* module (or *satisfies* (C_1)) if for any submodule N of M there exists a direct summand K of M such that N is essential in K.

Let M be a module. M is called *discrete* if M is lifting and has the following condition:

(D_2) If N is a submodule of M such that M/N is isomorphic to a direct summand of M, then N is a direct summand of M.

The module M is called *quasi-discrete* if M is lifting and has the following condition:

(D_3) For every direct summands K and L of M with $M = K + L$, $K \cap L$ is a direct summand of M.

Santa-Clara and Smith introduced the notion of *special extending modules* in [11]:
Let M be a module. M is called a *special extending module* if M is extending and $M = \oplus_{i\in I} M_i$, where M_i is a module with local endomorphism ring, for all $i \in I$. Following this idea, in this paper, we introduce the *special lifting*

modules as a dual notion to the special extending modules: Any module M is called *special lifting* if M is lifting and M is a direct sum of modules M_i with local endomorphism rings for all $i \in I$. Let M be a lifting right R-module. In this paper we prove that M is a special lifting module if one of the following cases is satisfied:

(1) M has ACC on small submodules;

(2) $\mathrm{Rad}(M)$ has ACC on direct summands and R is right artinian;

(3) $\mathrm{Rad}(M)$ has ACC on direct summands and R is a commutative max ring;

(4) M is finitely generated and R is commutative.

In this paper we also characterize rings for which every direct sum of a (finitely generated) special lifting module and a semisimple module is lifting.

Let M_1 and M_2 be modules. The module M_1 is *small M_2-projective* if every homomorphism $f : M_1 \longrightarrow M_2/A$, where A is a submodule of M_2 and $\mathrm{Im}f \ll M_2/A$, can be lifted to a homomorphism $\varphi : M_1 \longrightarrow M_2$. Let M and N be modules. If for all modules F, epimorphisms $f : M \longrightarrow F$ and homomorphisms $h : N \longrightarrow F$ there exists eithere $\varphi : N \longrightarrow M$ with $f\varphi = h$ or a non-zero direct summand M_1 of M and $\varphi : M_1 \longrightarrow N$ with $h\varphi = f_{|M_1}$, then N is called *almost M-projective.*

Let M be a module. M is said to have *finite Goldie dimension* if M contains a finite independent set of uniform submodules $\{N_1, \cdots, N_n\}$ such that $\oplus_{i=1}^{n} N_i \leq_e M$. In this case n is the Goldie dimension of M and we denote n by $dim(M)$. The module M is said to have *finite dual Goldie dimension* if there exists an epimorphism from M to a finite direct sum of n hollow factor modules with small kernel. In this case n is the dual Goldie dimension of M and we denote n by $codim(M)$. Call a function $d : R\text{–Mod} \longrightarrow \mathbb{N} \cup \{0, \infty\}$ a *rank function on R–Mod* if for all $M, N \in R$–Mod: (R0) $d(M) = 0 \Leftrightarrow M = 0$ and (R1) $d(M \oplus N) = d(M) + d(N)$ holds. Note that if d is a rank function and M a module with $d(M) = 1$, then M is indecomposable. Clearly, $dim(M)$ and $codim(M)$ are rank functions.

2 Special Lifting Modules

Let M be a module. M is called a *special lifting module*, if M is lifting and $M = \oplus_{i \in I} M_i$, where M_i is a module with local endomorphism ring, for all $i \in I$(see [5]). By [10, Lemma 4.7 and Corollary 4.9], any special lifting module has the decomposition $M = \oplus_{i \in I} H_i$, where H_i is hollow, for all $i \in I$.

It is well-known from [10, Corollary 5.5] that any indecomposable discrete module has local endomorphism ring. From this fact, [10, Theorem 4.15 and Lemma 4.7], discrete modules are also special lifting modules. The Prüfer p-group $\mathbb{Z}(p^\infty)$ is an example of a special lifting $\mathbb{Z}$–module that is not discrete. Note that the endomorphism ring of $\mathbb{Z}(p^\infty)$ is local by [8, Theorem 7.2.8] and $\mathbb{Z}(p^\infty)$ is not discrete because every nonzero factor module of $\mathbb{Z}(p^\infty)$ is isomorphic to $\mathbb{Z}(p^\infty)$.

The following lemma is well-known from [10, Lemma 5.1].

Lemma 2.1 *A quasi-discrete module M is discrete if and only if every epimorphism $M \longrightarrow M$ with small kernel is an isomorphism.*

Recall that a module M is called *hopfian* if every surjective endomorphism of M is an isomorphism. Examples of hopfian modules are noetherian modules (see [2, 11.6]) and finitely generated modules over commutative rings (this is due to Vasconcelos [13]). By the above lemma we get:

Corollary 2.2 *Any hopfian quasi-discrete module is discrete. In particular a hollow module is discrete if and only if it is hopfian.*

Lemma 2.3 *Let M be a lifting right R-module. Then the following are equivalent.*

(a) M is a finite direct sum of hollow modules.

(b) M satisfies the ACC on direct summands.

(c) M satisfies the DCC on direct summands.

(d) There exists a rank function d such that $d(M)$ is finite.

(e) M has finite dual Goldie dimension.

Proof. The equivalences $(b) \Leftrightarrow (c) \Leftrightarrow (e)$ follow from [12, Proposition 4.11] and the fact that supplements in lifting modules are direct summands (see also [10, Proposition 4.8]).
$(a) \Rightarrow (e) \Rightarrow (d)$ are obvious.
We will show $(d) \Rightarrow (a)$ by induction. Let M be a lifting module such that there exists a rank function d and $d(M) = 1$. Then M is indecomposable and hence hollow. Assume now $n \geq 1$ and assume that for every lifting module N such that there exists a rank function d with $d(N) < n$, N is a direct sum of hollow modules. Let M be a lifting module and d a rank function with $d(M) = n$. If M is indecomposable, then it is hollow and we are done. Otherwise M has a decomposition $M = N \oplus L$ with N and L non-zero. Thus by (R1), we have $n = d(N) + d(L)$ and hence $d(N)$ and $d(L)$ are less than n and as N and L are lifting modules as well, by hypothesis they are finite direct sums of hollows and so is M. $\square$

Before we state a general decomposition theorem for lifting modules we need the following lemma.

Lemma 2.4 *[9, Lemma 4.2.1] Let M be an R-module with essential radical. For every direct summands $D_1 \subseteq D_2$ of M we have $Rad(D_1) = Rad(D_2)$ if and only if $D_1 = D_2$. In particular if $Rad(M)$ has ACC or DCC on direct summands, then so has M.*

Proof. Let $M = D_1 \oplus D_1'$. Then $D_2 = D_1 \oplus (D_2 \cap D_1')$ and $\mathrm{Rad}(D_2) = \mathrm{Rad}(D_1) \oplus \mathrm{Rad}(D_2 \cap D_1')$. If $\mathrm{Rad}(D_1) = \mathrm{Rad}(D_2)$, then $0 = \mathrm{Rad}(D_2 \cap D_1') = \mathrm{Rad}(M) \cap D_2 \cap D_1'$. This implies $D_2 \cap D_1' = 0$ since $\mathrm{Rad}(M) \leq_e M$ and hence $D_1 = D_2$. $\square$

Theorem 2.5 *Any lifting module M has a decompositon $M = M_1 \oplus M_2 \oplus M_3$ such that*

- *M_1 is semisimple;*
- *M_2 is a lifting module with $Rad(M_2) \ll M_2$ and $Rad(M_2) \leq_e M_2$;*
- *M_3 is a lifting module with $Rad(M_3) = M_3$.*

If $Rad(M)$ satisfies the ACC or DCC on direct summands, then M_2 and M_3 are finite direct sums of hollow modules.

Proof. Let M be any lifting module. By [4, Proposition 2.9], we have a decomposition $M = M_1 \oplus N$ with M_1 semisimple and N lifting with $\mathrm{Rad}(M) = \mathrm{Rad}(N)$ essential in N. As N is lifting, $\mathrm{Rad}(M)$ contains a direct summand M_3 such that $N = M_2 \oplus M_3$ and $\mathrm{Rad}(M) \cap M_2 = \mathrm{Rad}(M_2) \ll M_2$. Clearly, $\mathrm{Rad}(M_2) \leq_e M_2$. As $M_3 \subseteq \mathrm{Rad}(M)$ and M_3 a direct summand of M, we have $\mathrm{Rad}(M_3) = M_3$. Assume now that $\mathrm{Rad}(M)$ has ACC or DCC on direct summands. By Lemma 2.4, M_2 and M_3 also satisfy those chain conditions and by Lemma 2.3 they are finite direct sums of hollow modules. □

Recall that a ring R is called a right *max ring* if every right R-module has a maximal submodule.

Corollary 2.6 *Let M be a lifting right R-module. Then M is special lifting in each of the following cases:*

1. *M has ACC on small submodules;*
2. *$Rad(M)$ has ACC on direct summands and R is right artinian;*
3. *$Rad(M)$ has ACC on direct summands and R is a commutative max ring;*
4. *M is finitely generated and R is commutative.*

Proof. The first case follows from [1, Proposition 2], that states that $\mathrm{Rad}(M)$ is noetherian if M has ACC on small submodules, and from [7, Proposition 3.1] which gives a decomposition of M into a semisimple and a noetherian module. The second case follows from Theorem 2.5 and the fact that hollow modules over artinian rings are noetherian and therefore hopfian. The third case follows from Theorem 2.5 and the fact that every hollow module H over a max ring R is local and since R is commutative, H is also hopfian. The last case follows from Theorem 2.5 and the fact that finitely generated modules over commutative rings are hopfian. □

Theorem 2.7 *Let R be a right artinian ring or a commutative max-ring. Let $M = M_1 \oplus \cdots \oplus M_n$ be a direct sum of relatively projective quasi-discrete right R–modules M_i $(1 \leq i \leq n)$. Then M is special lifting.*

Proof. M is quasi-discrete by [6, Theorem 2.13]. Then we can write $M = \oplus_{i\in I} H_i$, where H_i is hollow for all $i \in I$ by [10, Theorem 4.15]. Thus by the above, M is special lifting. □

3 A Class of Lifting Modules Satisfying (*)

The following property was considered in [6]:

> $M_1 \oplus M_2$ is a lifting right R-module whenever M_1 is a simple right R-module and M_2 is a lifting right R-module.

Let us say that a class $\mathcal{C}$ of lifting modules satisfies (*) if any direct sum of a module in $\mathcal{C}$ with a semisimple module is again a lifting module.

A family of modules $\{M_i\}_{i\in I}$ is called *locally semi-T-nilpotent (lsTn)* if for any infinitely countable set of non-isomorphisms $\{f_n : M_{i_n} \to M_{i_{n+1}}\}$ with all i_n distinct in I, and for any $x \in M_{i_1}$, there exists a positive integer k (depending on x) such that $f_k \dots f_1(x) = 0$.

We have Baba and Harada's result:

Theorem 3.1 (Baba and Harada, [3, Theorem 2]) *Let $M = \bigoplus_{i\in I} M_i$ be an R-module where the M_i's are local modules with local endomorphism rings. Then the following are equivalent.*

1. *M is lifting.*
2. *M_i is almost M_j-projective for all $i \neq j$ and $\{M_i\}_{i\in I}$ is lsTn.*

We first need a technical Lemma regarding locally semi-T-nilpotent families of modules.

Lemma 3.2 *Let $\{M_i\}_{i\in I}$ be a family of R-modules and $\{S_j\}_{j\in J}$ a family of simple R-modules. If $\{M_i\}_{i\in I}$ is lsTn, then also the disjoint union $\{M_i\}_{i\in I} \cup \{S_j\}_{j\in J}$ is lsTn.*

Proof. Let us denote $\{L_k\}_{k\in K} := \{M_i\}_{i\in I} \cup \{S_j\}_{j\in J}$ the disjoint union of the two families, where K is the disjoint union of I and J. Let $\{f_n : L_{i_n} \to L_{i_{n+1}}\}$ be a countably infinite family of non-isomorphisms with all i_n pairwise distinct. We may also assume that all f_n are non-zero. Consider the chain of homomorphisms at a position $k > 1$:

$$\cdots L_{i_{k-1}} \xrightarrow{f_{k-1}} L_{i_k} \xrightarrow{f_k} L_{i_{k+1}} \cdots$$

If $L_{i_k} \in \{S_j\}_{j\in J}$ then neither $L_{i_{k-1}}$ nor $L_{i_{k+1}}$ can be simple since otherwise the map f_{k-1} respectively f_{k+1} is zero or an isomorphism. Hence $L_{i_{k-1}}, L_{i_{k+1}} \in \{M_i\}_{i\in I}$. Moreover the composition $f_k f_{k-1}$ is a non-isomorphism. We refer to this observation by (†).
Let $\Lambda := \{n \in \mathbb{N} : L_{i_n} \in \{S_j\}_{j\in J}\}$. By (†) we know that $\mathbb{N} \setminus \Lambda$ is a (countably) infinite set. Denote the characteristic function of Λ by $\mathbf{1}_\Lambda$, that is $\mathbf{1}_\Lambda(n) = 1$ if $n \in \Lambda$ and otherwise $\mathbf{1}_\Lambda(n) = 0$. Define a bijection $\varphi : \mathbb{N} \to \mathbb{N} \setminus \Lambda$ recursively by setting

$$\varphi(1) := 1 + \mathbf{1}_\Lambda(1) \text{ and } \varphi(n+1) := (\varphi(n) + 1) + \mathbf{1}_\Lambda(\varphi(n) + 1) \text{ for all } n \geq 1.$$

(†) guarantees that φ defines a bijection, since the successor of each $k \in \Lambda$ lies in $\mathbb{N} \setminus \Lambda$.

$$\hat{f}_n := \begin{cases} f_{\varphi(n)} & \text{if } \varphi(n) + 1 \notin \Lambda \\ f_{\varphi(n)+1} f_{\varphi(n)} & \text{otherwise} \end{cases}$$

By the remark above, property (†) guarantees that we always "jump" correctly over the modules in $\{S_j\}_{j\in J}$, and that all $\hat{f}_n$ are non-isomorphisms. By construction we get that $\{\hat{f}_n : L_{i_{\varphi(n)}} \to L_{i_{\varphi(n+1)}}\}$ is a countably infinite family of non-isomorphisms of modules in $\{M_i\}_{i\in I}$. Since $\{M_i\}_{i\in I}$ is lsTn, for every element $x \in L_{i_{\varphi(1)}}$ there exists an $k \geq 1$ such that $\hat{f}_k \cdots \hat{f}_1(x) = 0$. Hence for all $x \in L_{i_1}$ we also get an $k \geq 1$ such that $f_k \cdots f_1(x) = 0$. Thus $\{L_k\}_{k\in K}$ is lsTn. □

With the last Lemma and Baba and Harada's Theorem at hand we easily deduce now the following criteria for a special lifting module to satisfy (*):

Proposition 3.3 *Let $M = \bigoplus_{i\in I} M_i$ be a special lifting module with all M_i's local and let $S := \bigoplus_{j\in J} S_j$ be a semisimple module. Then the following are equivalent.*

(a) $M \oplus S$ is lifting.

(b) S_j is almost M_i-projective for all $j \in J$ and $i \in I$.

Proof. $(a) \Rightarrow (b)$ follows from Baba and Harada's Theorem 3.1.
$(b) \Rightarrow (a)$ Note that any module M is almost S-projective for all simple modules S (this is trivial). Since M was lifting, by Theorem 3.1 we have that M_i is almost M_j-projective for all $i \neq j$. By hypothesis all S_j are almost M_i-projective, hence the family $\{M_i\} \cup \{S_j\}$ is relatively almost-projective. Lemma 3.2 assures that this family is also lsTn since M was lifting and by Theorem 3.1 the family of the M_i's is lsTn. Thus applying once again Baba and Harada's Theorem we get that $M \oplus S$ is lifting. □

As an immediate consequence we can state:

Corollary 3.4 *Let $M = \bigoplus_{i \in I} M_i$ be a special lifting module with all M_i's local. Then the following statements are equivalent.*

(a) M satisfies (*).

(b) $M \bigoplus S$ is lifting for any simple module S.

(c) Any simple module S is almost M_i-projective for all $i \in I$.

Before we characterize when (*) holds for all special lifting modules of the above type, we show that small-M-projectivity and almost M-projectivity for some local module M coincide for simple modules:

Lemma 3.5 *Let M be a local right R-module and S a simple right R-module. Then S is small M-projective if and only if S is almost small M-projective.*

Proof. Assume S is almost M-projective. Let $K \subseteq M$ and $f : S \to M/K$ be a non-zero homomorphism such that $f(S)$ is small in M/K. Hence f is monomorph, but not epimorph. By hypthesis, there exists either a map $g : S \to M$ such that $f = \pi_K g$ or a map $g : M \to S$ such that $fg = \pi_K$ (since M is indecomposable, M is the only non-zero direct summand). We will rule out the second possibility. Assume there exists an $g : M \to S$ such that $fg = \pi_K$, then g is epimorph factors through $M/\mathrm{Rad}(M)$, that is there exists an isomorphism $\bar{g} : M/\mathrm{Rad}(M) \to S$ such that $g = \bar{g}\pi$ (where $\pi := \pi_{\mathrm{Rad}(M)}$). Hence we get the following commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\pi} & M/\mathrm{Rad}(M) & \longrightarrow & 0 \\ {\scriptstyle id_M}\downarrow & & \downarrow{\scriptstyle f\bar{g}} & & \\ M & \xrightarrow{\pi_K} & M/K & \longrightarrow & 0 \end{array}$$

Thus $\pi_K(\mathrm{Rad}(M)) = f\bar{g}\pi(\mathrm{Rad}(M)) = 0$ and as $\mathrm{Rad}(M)$ is maximal, $K = \mathrm{Rad}(M)$. But as $M/\mathrm{Rad}(M)$ is simple, $f\bar{g}$ and hence f must be isomorphisms contradicting that $f(S)$ was small in M/K. Therefore we can not have this possibility. Thus S is small M-projective.
Now assume that S is small M-projective and assume we have the following diagram:

$$\begin{array}{ccccc} & & S & & \\ & & \downarrow{\scriptstyle f} & & \\ M & \xrightarrow{p} & F & \longrightarrow & 0 \end{array}$$

where p is an epimorphism and f is non-zero. If f is epimorph, then f is an isomorphism and we have $g := f^{-1}p : M \to S$ a non-zero map from a direct summand of M to S such that $fg = p$. If f is not epimorph, then $f(S)$ must be small since F is local. As p factors through $M/\mathrm{Ker}p$, there exists an isomorphism $\bar{p} : M/\mathrm{Ker}p \to F$ such that $p = \bar{p}\pi$ (where $\pi := \pi_{\mathrm{Ker}p}$). Hence $\bar{p}^{-1}f$ is a map from S to $M/\mathrm{Ker}p$ with small image and by hypothesis there exists a map $g : S \to M$ such that $\pi g = \bar{p}^{-1}f$. Thus $pg = \bar{p}\pi g = \bar{p}\bar{p}^{-1}f = f$ shows that g is our desired map. Hence S is almost M-projective. $\square$

Analyzing the proof of [6, Proposition 4.1] we also have:

Proposition 3.6 *Let M be a lifting right R-module with local endomorphism ring such that $M \oplus S$ is lifting for all simple subfactors S of M, then M is simple or local with $Rad(M) = Soc(M)$.*

Now we can characterize when (*) holds for all special lifting modules that a direct sums of locals.

Proposition 3.7 *Let R be a ring. Then the following are equivalent.*

(a) Any special lifting right R-module $M = \bigoplus_{i\in I} M_i$ with all M_i's local satisfies (*).

(b) Finitely generated special lifting right R-modules satisfy (*).

(c) Any direct sum of a simple right R-module and a local right R-module with local endomorphism is lifting.

(d) Any local right R-module M with local endomorphism ring is simple or $Rad(M) = Soc(M)$.

(e) Any simple right R-module is small M-projective for every local right R-module M with local endomorphism ring.

(f) Any simple right R-module is almost M-projective for every local right R-module M with local endomorphism ring.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ is trivial; $(c) \Rightarrow (d)$ follows from Proposition 3.6.
$(d) \Rightarrow (e)$ Let M be a local module with local endomorphism ring and S a simple module. Assume that there is a non-zero homomorphism $f : S \to M/K$ with $f(S)$ small in M/K and K a submodule of M. Since $\mathrm{Rad}(M) = \mathrm{Soc}(M)$ is the unique maximal submodule of M, we have $K \subseteq \mathrm{Soc}(M)$. Let L be a direct summand of K in $\mathrm{Soc}(M)$, then $L \cong (L \oplus K)/K = \mathrm{Soc}(M)/K = \mathrm{Rad}(M)/K = \mathrm{Rad}(M/K)$. Denote by $g : \mathrm{Rad}(M/K) \to L$ the natural isomorphism and denote by $\pi_K : M \to M/K$ the canonical projection. Then $\pi_K \circ g = id_{\mathrm{Rad}(M/K)}$ and hence $h := g \circ f : S \to M$ has the property that $\pi_K \circ h = \pi_K \circ g \circ f = id_{\mathrm{Rad}(M/K)} \circ f = f$. Hence S is small-M-projective.
$(e) \Leftrightarrow (f)$ follows from Lemma 3.5. $(f) \Rightarrow (a)$ follows from Corollary 3.4. $\square$

Recall that a ring R is called *semilocal* if $R/\mathrm{Jac}R$ is semisimple. A semilocal ring with nilpotent Jacobson radical is called *semiprimary*. It is well-known that semiprimary rings are right and left perfect.

Theorem 3.8 *Let R be a ring. Then the following are equivalent.*

(a) Every special lifting right R-module satisfies (*) and R is right perfect.

(b) Finitely generated special lifting right R-modules satisfy (*) and R is semiperfect.

(c) Any direct sum of a simple and a projective local right R-module is lifting and R is semiperfect.

(d) $(JacR)^2 = 0$ and R is semilocal.

(e) the left-version of (a), (b) or (c) hold.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ is trivial; $(c) \Rightarrow (d)$ As R is semiperfect $R = \bigoplus_{i=1}^{n} e_i R$ with e_i primitive idempotents. Let J denote the Jacobson radical of R. By hypothesis any direct sum of a simple and $e_i R$ is lifting, hence by Proposition

3.6, e_iR is simple or $\mathrm{Rad}(e_iR) = e_iJ = \mathrm{Soc}(e_iR)$. Hence $J = \bigoplus_{i=1}^{n} e_iJ \subseteq \mathrm{Soc}(R_R) = l.ann(J)$ by [2, Proposition 15.17]. Thus $J^2 = 0$.
$(d) \Rightarrow (a)$ Semiprimary rings are right perfect. Moreover any hollow right R-module over a right perfect ring R is local. So all special lifting modules can be written as a direct sum of locals with local endomorphisms. Let L be a local, non-simple right R-module with local endomorphism ring. As R is semilocal $\mathrm{Rad}(L) = LJ$ and as $J^2 = 0$ it follows that $\mathrm{Rad}(L)J = 0$. Hence $\mathrm{Rad}(L) = \mathrm{Soc}(L)$. By Proposition 3.7, all special lifting right R-modules satisfy (*).
Since (d) is left-right independent, we also get the equivalence of left versions of $(a) - (c)$ with (d). $\square$

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