PSEUDOVARIETIES OF SEMIGROUPS

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ABSTRACT. The most developed aspect of the theory of finite semigroups is their classification in pseudovarieties. The main motivation for investigating such entities comes from their connection with the classification of regular languages via Eilenberg's correspondence. This connection prompted the study of various natural operators on pseudovarieties and led to several important questions, both algebraic and algorithmic. The most important of these questions is decidability: given a finite semigroup is there an algorithm that tests whether it belongs to the pseudovariety? Since the most relevant operators on pseudovarieties do not preserve decidability, one often seeks to establish stronger properties. A key role is played by relatively free profinite semigroups, which is the counterpart of free algebras in universal algebra. The purpose of this paper is to give a brief survey of the state of the art, highlighting some of the main developments and problems.

1. Why should we be interested in pseudovarieties of semigroups?

In a general algebraic framework, meaning that the operations considered are fixed, *pseudovarieties* are classes of finite algebras which are closed under taking homomorphic images, subalgebras and finite direct products. Although this is a natural analog for Birkhoff varieties when one is interested in finite algebras, it lacked external motivation to be investigated until the seminal work of Eilenberg showing how algebraic structures play a key in the theory of automata and formal languages [85,86]. Earlier results had shown that regular (word) languages over finite alphabets have certain combinatorial properties if and only if their syntactic semigroups have corresponding algebraic properties: star-free versus aperiodic semigroups (meaning that all subgroups are trivial) [147], locally testable versus local semilattices [71, 126], piecewise testable versus \mathcal{J} -trivial semigroups [148]. Eilenberg [86] showed how all three results fit in a general bijective correspondence between classes of regular languages with suitable properties and pseudovarieties of semigroups: the class of languages corresponding to a given pseudovariety of semigroups V simply consists of all regular languages over finite alphabets whose syntactic semigroups belong to V.

The above three earlier instances of Eilenberg's correspondence are particularly important because the pseudovarieties in question are *decidable* in the sense that, for each of them, there is an algorithm that, given a finite semigroup S (say, by its multiplication table), determines whether or not S

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belongs to the pseudovariety. Since the syntactic semigroup of a regular language (say, given by a regular expression or a finite automaton recognizing it) is effectively computable, we see that a pseudovariety is decidable if and only if so is the corresponding class of regular languages.

Thus, not only pseudovarieties serve as classifying algebraic invariants for natural classes of regular languages but they also translate to algebraic problems combinatorial problems on regular languages. This provided strong external motivation for the study of pseudovarieties of semigroups and led to several books dedicated to the subject [112,130,9,143]. Various extensions of Eilenberg's correspondence have been found, such as: replacing semigroups by arbitrary algebraic structures and languages by subsets of free algebras in varieties [7,9], in particular tree languages (sets of terms) [89,156]; enriching the algebraic structures with a partial order and/or retaining the information about the alphabet as generating set for the syntactic algebra (see [160]); or even putting the correspondence in a general categorical setting [165, 1]. In this latter work, duality plays a key role. The importance of Stone duality had already been pointed out in [9, Section 3.6] and is behind the success of the profinite approach, but it has only been systematically explored more recently (see [91]). One important direction that the research in this area has been exploring is to extend the Eilenberg theory to deal with classes of languages that are not necessarily regular. The aim is to separate classes of languages. Such classes may, for instance, be defined by the complexity of the membership problem for its members, such as the famous classes \mathbf{P} and NP [88]. Various recognition devices have been considered, from Stone topological algebras [40], in particular minimal compact automata [155], to typed monoids [90].

2. The beginnings

The remainder of the paper deals mostly with pseudovarieties of semigroups, although sometimes it is convenient to deal with an enriched algebraic structure. We start with a brief history of the subject that predates the motivation arising from the connections with computer science framed by Eilenberg's correspondence.

Fix an algebraic signature, by which we mean a set graded by the natural numbers $\Sigma = \biguplus_{n \ge 0} \Sigma_n$, where the elements of Σ_n are called the *n*-ary operation symbols. A Σ -algebra is a nonempty set A together with an evaluation mapping $E_n : \Sigma_n \times A^n \to A$ for each $n \ge 0$. Usually, we write $o_A(a_1, \ldots, a_n)$ instead of $E_n(o, a_1, \ldots, a_n)$.

In case A and each Σ_n is further endowed with a topology, then we say that A is a topological Σ -algebra if each mapping E_n is continuous.

By a variety we mean a class of $(\Sigma$ -)algebras closed under the operators H (adding all homomorphic images), S (adding all subalgebras), and P (adding all direct products). We say the algebra A divides the algebra B, if $A \in HS\{B\}$. An *identity* is a formal equality u = v between members of an absolutely free algebra $F_{\Sigma}X$; we say that it is valid in the algebra A or that A satisfies u = v if, for every homomorphism $\varphi : F_{\Sigma}X \to A$, the equality $\varphi(u) = \varphi(v)$ holds. For a set I of identities, we say that the class [I] consisting of all algebras A that satisfy all identities from I, written $A \models I$, is

defined by I; such classes of algebras are also known as equational classes. In 1935, Birkhoff [67] proved that varieties are precisely the equational classes.

Except for the variety of singleton algebras, varieties contain infinite algebras and many interesting classes of finite algebras (such as, in the language with just one binary operation: groups, nilpotent or solvable groups; nilpotent, completely regular, or aperiodic semigroups) are not the subclasses of all finite members of some variety of algebras, that is they are not equational as classes of finite algebras. The natural notion that emerged is that of a *pseudovariety*, which is a class of finite (Σ -)algebras closed under the operators H, S, and P_{fin} (adding all finite direct products).

In 1976, Baldwin and Berman [64] showed that, in case Σ is finite, pseudovarieties are the classes of finite algebras in the union of a countable (increasing) chain of varieties. In 1985, Ash [57] further showed that the finiteness assumption on Σ is superfluous provided chains are replaced by upper directed unions. Meanwhile, also in 1976, Eilenberg and Schützenberger [87] proved that, in case Σ is finite, pseudovarieties are *ultimately defined* by sequences $(u_n = v_n)_{n \geq 0}$ of identities, namely as

$$\bigcup_{n \ge 0} \bigcap_{m \ge n} \llbracket u_n = v_n \rrbracket,$$

where, more generally, for a set I of identities, $\llbracket I \rrbracket$ denotes the class of all finite algebras that satisfy all identities from I.

In 1981, Reiterman [140] came up with what proved to be a rather more fruitful alternative description, which we proceed to describe. In case Σ and X are finite, let $\hat{F}_{\Sigma}X$ be the completion of $F_{\Sigma}X$ with respect to the following (ultra)metric

$$d(u, v) = 2^{-r(u, v)} \text{ where}$$

$$r(u, v) = \min\{|A| : A \text{ finite } \Sigma \text{-algebra such that } A \not\models u = v\}.$$

A formal equality u = v with $u, v \in \widehat{F}_{\Sigma}X$ is called a *pseudoidentity*. For a finite algebra $A, A \models u = v$ means that, for every homomorphism φ : $F_{\Sigma}X \to A$, the equality $\hat{\varphi}(u) = \hat{\varphi}(v)$ holds for the unique extension of φ to a continuous mapping $\hat{\varphi} : \widehat{F}_{\Sigma}X \to A$, where A is endowed with the discrete topology. Reiterman proved that a class of finite algebras is a pseudovariety if and only if it is defined by a set P of pseudoidentities:

$$\llbracket P \rrbracket = \{ A \text{ finite } \Sigma \text{-algebra} : A \models P \}.$$

Banaschewski [65] dropped the finiteness assumption on Σ by considering instead of a metric structure on $F_{\Sigma}X$ a uniform structure, namely that given by taking as a fundamental system of entourages the set of all congruences of finite index. The completion considered by Banaschewski may be realized as the closed subalgebra of the direct product of all quotients $F_{\Sigma}X/\theta$ by congruences of finite index consisting of all elements $(u_{\theta})_{\theta}$ such that, if $\theta_1 \subseteq \theta_2$ and $\varphi : F_{\Sigma}X/\theta_1 \to F_{\Sigma}X/\theta_2$ is the natural homomorphism, then $\varphi(u_{\theta_1}) = u_{\theta_2}$. The topology considered in the product is the product topology, where finite algebras are viewed as discrete topological spaces. Thus, the completion is an inverse limit of finite algebras, that is, a *profinite algebra*. Equivalently, it is a compact topological algebra which is residually finite. For a pseudovariety V, we also say that a profinite algebra is a *pro-V* algebra if it is an inverse limit of algebras from V.

In particular, profinite algebras are topological algebras with underlying Stone spaces (compact zero-dimensional spaces). Following [40], we call such algebras S Stone topological algebras. As recognition devices of subsets of relatively free algebras F, we are interested in the pre-images of clopen subsets of S under continuous homomorphisms $F \to S$. The fact that profinite algebras recognize the same sets as finite algebras, explains the role played by profinite algebras in finite semigroup theory. On the other hand, the fact that all Stone topological semigroups are profinite [129] shows that one needs to get out of the realm of semigroups to cover non-regular languages. Exactly when a Stone topological algebra is residually finite, that is, when it is profinite, is a question that has been recently solved with various alternative characterizations (see [146, 41]). Among such characterizations is the condition that, for every clopen subset, the syntactic congruence is determined by a finite set of linear polynomials. Perhaps to understand what this condition means, it is worth recalling the easy fact that, on an arbitrary monoid M. the linear polynomials sxt determine the syntactic congruence of a subset L in the sense that the elements $m, n \in M$ are syntactically equivalent precisely when, for all $s, t \in M$, smt belongs to L if and only if so does snt. Here, the linear polynomial sxt is viewed as a transformation of M defined by $x \mapsto sxt$, where $s, t \in M$ are fixed and the syntactic congruence σ_L of a subset L of an algebra A is the largest congruence on A saturating L.

As argued in Section 1, the typical goal in the application of the theory of pseudovarieties of semigroups in computer science is to find, if possible, effective characterizations of the pseudovariety associated by Eilenberg's correspondence to a suitable class of regular languages. To be more precise, such classes of languages are called *varieties of languages* and are correspondences \mathcal{V} associating with each finite alphabet A a Boolean subalgebra $\mathcal{V}(A)$ of the powerset Boolean algebra $\mathcal{P}(A^+)$, which consists only of regular languages, such that:

- if $L \in \mathcal{V}(A)$ and $a \in A$, then $a^{-1}L, La^{-1} \in \mathcal{V}(A)$;
- if $\varphi : A^+ \to B^+$ is a homomorphism and $L \in \mathcal{V}(B)$ then $\varphi^{-1}(L) \in \mathcal{V}(A)$.

The associated pseudovariety V according to Eilenberg's correspondence is the pseudovariety of semigroups generated by all syntactic semigroups A^+/σ_L of languages $L \in \mathcal{V}(A)$ with arbitrary finite alphabets A. The inverse correspondence can be described by saying that $\mathcal{V}(A)$ consists of all languages $L \subseteq A^+$ such that A^+/σ_L belongs to V. If one is interested in languages that may contain the empty word, the role of the free semigroup A^+ is then played by the free monoid A^* , and instead of semigroups one works with monoids. It is easy to see that a pseudovariety of monoids consists of the monoids in the pseudovariety of semigroups it generates.

Now, there are too many pseudovarieties of semigroups for all of them to be decidable, as there are only countably many algorithms according to Church's thesis. Indeed the set of all pseudovarieties of semigroups has at most the power of the continuum as, up to isomorphism, there are only countably many finite semigroups. But, even the set of pseudovarieties consisting of finite Abelian groups has the power of the continuum: for any set P of prime integers, the cyclic groups of order a member of P generate a pseudovariety (of semigroups) for which the set of orders of its cyclic groups of prime order is precisely P; there are a continuum of such sets of primes.

One of the fruitful extensions of the Eilenberg correspondence involves pseudovarieties of finite ordered semigroups, and positive varieties of languages [132]. By an ordered semigroup we man a semigroup with a partial order compatible with the semigroup operation; homomorphisms between such structures are required to preserve the operation and the order. *Positive varieties of languages* generalize varieties of languages by replacing Boolean subalgebras of $\mathcal{P}(A^*)$ by 0,1-sublattices. Pseudovarieties of ordered semigroups may also be viewed as generalizations of pseudovarieties of semigroups as the mapping sending the latter to the pseudovariety of ordered semigroups generated by V is injective.

3. Operators on pseudovarieties

Several natural algebraic constructions lead to operators on pseudovarieties. There are basically two kinds of such operators:

- those that give an explicit, constructive, necessary and sufficient condition for a finite semigroup to belong to the resulting pseudovariety; and
- those that define the resulting pseudovariety in terms of generators.

Some authors prefer to use a different type of letter to indicate operators but it is most common to use the same type of letter as for pseudovarieties. Examples of operators of the first kind:

- the operator L: LV is the class of all finite semigroups S such that the monoid eSe belongs to V for every idempotent $e \in S$;
- the operator D: DV is the class of all finite semigroups whose regular \mathcal{D} -classes are subsemigroups from V;
- the operator E: EV is the class of all finite semigroups whose idempotents generate a subsemigroup which belongs to V;
- the operator "bar": for a pseudovariety of groups H, \overline{H} is the class of all finite semigroups whose subgroups belong to H;
- the operator B: BV is the class of all finite semigroups whose *blocks* are semigroups from V.

For the last example, a *block* of a finite semigroup S is defined as follows. Consider the smallest equivalence relation ρ on the union U of the subgroups of S that contains the restriction of both of Green's relations \mathcal{L} and \mathcal{R} to U. A block of S is obtained by taking the Rees quotient T/I of the subsemigroup T of S generated by a ρ -class by the ideal I consisting of the elements of Tthat do not lie in the \mathcal{D} -class of the generators.

Note that all the operators in the preceding paragraph preserve decidability as all the constructions involved can be effectively carried out in a given finite semigroup.

Examples of operators of the second kind:

- the *join* operator: $V \vee W$ is the pseudovariety generated by the union $V \cup W$; this consists of all divisors of direct products $S \times T$ with $S \in V$ and $T \in W$;
- the semidirect product operator: V * W is the pseudovariety generated by all semidirect products S * T with $S \in V$ and $T \in W$ and consists of all divisors of such semidirect products; this is an associative operator [86];
- the Mal'cev product operator: V m W is the pseudovariety generated by the finite semigroups S for which there exists a homomorphism $\varphi: S \to T$ such that $T \in W$ and $\varphi^{-1}(e) \in V$ for every idempotent $e \in S$;
- the power operator P: PV is the pseudovariety generated by all semigroups $\mathcal{P}(S)$ of subsets of S, under the usual subset multiplication $AB = \{ab : a \in A, b \in B\}$, with $S \in V$;
- the operator M: MV is the pseudovariety generated by all monoids S^1 with $S \in V$, obtained from S by adding an identity element if S is not a monoid and taking S itself otherwise; the pseudovarieties in the image of M are called *monoidal*.

It is known that the first four of these operators do not preserve decidability [2, 61]. It seems to be an open problem whether the much less studied operator M (see [9, Chapter 7]) preserves decidability.

The join, semidirect product, Mal'cev product, and power operators are all known to correspond to natural operators on varieties of languages. Thus, it is of interest to be able to relate concrete values of these operators with decidable operators. There are many important results of this kind. Such results involve some natural examples of pseudovarieties which we proceed to list:

- I: singleton semigroups
- S: finite semigroups
- N: finite nilpotent semigroups
- D: finite semigroups in which idempotents are right zeros
- SI: finite semilattices
- B: finite bands
- Ab: finite Abelian groups
- G: finite groups
- G_p: finite p-groups
- $G_{p'}$: finite p'-groups (no elements of order p)
- G_{nil} : finite nilpotent groups
- G_{sol}: finite solvable groups
- A: finite aperiodic semigroups
- J: finite \mathcal{J} -trivial semigroups
- R: finite *R*-trivial semigroups
- Com: finite commutative semigroups
- CS: finite (completely) simple semigroups
- CR: finite completely regular semigroups
- $C_n = (A * G)^n * A$.

In the last example, the *n*-th power is taken with respect to the semidirect product, and C_0 is interpreted to be A. The pseudovariety C_n is the class

of all finite semigroups with Krohn-Rhodes complexity at most n (see [143, Chapter 4]). Thus, Krohn-Rhodes complexity of a finite semigroup S is the minimum n such that $S \in C_n$; it is computable if and only if all the pseudovarieties C_n are decidable.

4. Locality

One of the historical instances of Eilenberg's correspondence is that the pseudovariety corresponding to the variety of all piecewise testable languages is precisely LSI. Here, a language over a finite alphabet A is said to be *piecewise testable* if membership of a word w in it can be tested by determining the factors of length at most some integer n of the word w#, where the symbols \$ and # do not belong to A. The proof that LSI is the right pseudovariety envolves two steps: to show that the right pseudovariety is SI * D and that LSI = SI * D. Both steps have been extensively generalized. For the first step, see [159], and [9, Chapter 10]. The pseudovariety equation LV = V * D was already considered by Eilenberg, who called a pseudovariety V *local* if it is a solution. Examples of local pseudovarieties:

- R, R * G [157]
- every nontrivial pseudovariety of groups [159]
- CR [105]
- $[x^{n+1} = x]$ [104]
- DS [103]
- DA [10]
- DG [106]

The following are examples of non-local pseudovarieties:

- J [107, 108]
- Com [162]
- DAb [106]
- $C_n (n > 0)$ [142]

In case V is not local, the question remains as to what V * D may be. Straubing [159] showed that, if V is a decidable monoidal pseudovariety containing the Brandt semigroup B_2 (which is the syntactic semigroup of the language $(ab)^+$ over the alphabet $\{a, b\}$), then V * D is also decidable. This restricts the problem to the subpseudovarieties of DS as this is the largest pseudovariety that does not contain B_2 [120]. Tilson [164] gave a non-effective characterization of local (monoidal) pseudovarieties in terms of *pseudovarieties* of categories which is used in the proof of some of the above results.

It seems to be an open problem whether locality is decidable for pseudovarieties given say by a finite basis of computable pseudoidentities. Probably, the answer is negative, as proofs of locality/non-locality are in general quite hard.

The interest of locality goes beyond the equation V * D = LV. For instance: if the pseudovariety V is local and decidable and the pseudovariety W has a strong form of decidability to be discussed in Section 8, then V * W is decidable. This follows from results of [53] and [45].

5. Some notable equations

It is easy to see that, for a finite group G, the blocks of the power semigroup $\mathcal{P}(G)$ are divisors of G. In particular, the inclusion $\mathsf{PH} \subseteq \mathsf{BH}$ holds for every pseudovariety of groups H . Margolis and Pin [122] proved more precisely that, for such H ,

(1)
$$\mathsf{PH} \subseteq \mathsf{J} * \mathsf{H} \subseteq \mathsf{J} \textcircled{m} \mathsf{H} \subseteq \mathsf{BH}.$$

They also proved that, in the case where H = G, the first and third inclusions are actually equalities. On the other hand, if the pseudovariety V is local, then the equality

(2)
$$\mathsf{V} * \mathsf{G} = \mathsf{V} \widehat{} \mathfrak{m} \mathsf{G}$$

holds, see [97, Theorem 3.1]. The equality (2) also holds for some non-local pseudovarieties such as J: this was proved by Henckell and Rhodes [98] based on results of Knast [108] and Ash [59], the latter is a very deep and seminal paper that will be mentioned further below. That completed the proof of the equality PG = BG, thereby showing that PG is decidable; see [131] for more on the history of the proof of this equality.

By a profinite graph we mean a topological graph which is an inverse limit of finite graphs. A profinite graph is profinitely connected if all its finite continuous homomorphic images are connected. Let G be a profinite group and X be a subset generating a dense subgroup. The profinite Cayley graph $\Gamma_X(G)$ has G as vertex space and $G \times X$ as edge space, where (g, x) is an edge from g to gx. In a series of deep papers, Auinger and Steinberg characterized the pseudovarieties of groups H for which equality holds in each of the first two inclusions in (1):

- PH = J * H if and only if H is a *Hall* pseudovariety, a property which has many equivalent formulations perhaps the simplest of which is that every profinitely connected subgraph of the profinite Cayley graph of each finitely generated free pro-H group containing the ends of an edge must contain the edge itself [63];
- J * H = J m H if and only if H is *arboreous*, meaning that the profinite Cayley graphs of all finitely generated pro-H groups are *tree-like* in the sense that, given any two vertices, there is a unique minimal profinitely connected subgraph containing them [62].

The equality J m H = BH turns out to be exceptional as it holds only when H = G. This follows from [101], where it is shown that G is the only pseudovariety of groups H such that $A \cap ESI \subseteq DA \textcircled{m} H$. This was already observed in [152], an explicit proof being given in [38, Corollary 3.5].

The computation of the Mal'cev product V m G has deserved a lot of attention. By a *relational morphism* of monoids we mean a relation $\mu \subseteq M \times$ N with domain S which is also a submonoid of $M \times N$. For a pseudovariety of groups H, the H-*kernel* of a finite monoid M is the intersection $K_{\mathsf{H}}(M)$ of all $\mu^{-1}(1)$ when μ ranges over all relational morphisms μ from M to groups in H. A finite semigroup S belongs to $V \textcircled{m} \mathsf{H}$ if and only if $K_{\mathsf{H}}(S^1) \cap S$ belongs to V . The Rhodes Type-II Conjecture states that $K_{\mathsf{G}}(M)$ is the smallest submonoid of M containing all idempotents that is closed under the transformations $m \mapsto amb$ with aba = a or bab = b (known as weak conjugation). The conjecture was proved independently by Ash [59] and Ribes and Zalesskiĭ [144] (in an equivalent group-theoretic formulation previously obtained by Pin and Reutenauer [135]. Algorithms for the H-kernel of finite monoids are known for other pseudovarieties H such as G_p [145], G_{nil} [119], and Ab [83]. Following the approach of Pin and Reutenauer, all these results involve the computation of the closure of finitely generated subgroups in the pro-H topology of a relatively free group (see [119]). The case of G_{sol} remains open although there has been some recent progress [123, 124].

Is is easy to see that BG = EJ [121]. Thus, the pseudovariety J is a solution of each of the following equations:

- (3) $\mathsf{V} * \mathsf{G} = \mathsf{V} \textcircled{m} \mathsf{G}$
- (4) V m G = EV
- (5) $\mathsf{V} * \mathsf{G} = \mathsf{E}\mathsf{V}.$

6. IRREDUCIBILITY

Some pseudovarieties are decomposable in terms of others using for instance one of the binary operators \lor , \ast , m. In a vague sense, a decomposition of a pseudovariety V reduces the study of its members to the study of the members of the pseudovarieties used in the decomposition which, for simplicity, we call *factors*. Also, even though the operator used in the decomposition may not preserve decidability, it may be possible to explore stronger properties of the factors, cf. Section 8.

This prompted the investigation of those pseudovarieties that are indecomposable. For instance, a pseudovariety V is

- strictly finite join irreducible (sfji) if $V = U \lor W$ implies V = U or V = W;
- finite join irreducible (fji) if $V \subseteq U \lor W$ implies $V \subseteq U$ or $V \subseteq W$.

Similar notions may be considered for the operators * and m. Note that fji implies sfji. The pseudovariety N is sfji but not fji, see [143, Corollary 7.3.30]. The pseudovariety J is not sfji [8].

Margolis, Sapir and Weil [118] showed that, if H is an extension-closed pseudovariety of groups, then \overline{H} is irreducible in the stronger sense for \lor , *, m. Klíma and the author showed that the extension-closure assumption on the pseudovariety of groups H is superfluous [37] and that the pseudovariety A, and for a nontrivial pseudovariety of groups H, the pseudovarieties

$$\overline{\mathsf{H}}, \ \mathsf{CR} \cap \overline{\mathsf{H}}, \ \mathsf{DS} \cap \overline{\mathsf{H}}, \ \mathsf{C}_n \cap \overline{\mathsf{H}}$$

are all fji even in the lattice of pseudovarieties of ordered semigroups [39].

The proofs of irreducibility of \overline{H} in [118] and [37] depend on encoding results that started in [109]. Recall that a code is a subset C of a free semigroup A^+ such that the induced homomorphism of free semigroups $C^+ \to A^+$ is injective. The theory of such (variable-length) codes is a well-developed area of combinatorics on words, see [66]. Similarly, if V is a pseudovariety, Ais finite set, and C is a subset of the relatively free pro-V semigroup $\overline{\Omega}_A V$ generated by A, then we say that C is a V-code if the unique continuous homomorphism $\overline{\Omega}_C \mathsf{V} \to \overline{\Omega}_A \mathsf{V}$ extending the inclusion mapping $C \hookrightarrow \overline{\Omega}_A \mathsf{V}$ is injective. A closed (even clopen) subsemigroup of $\overline{\Omega}_A V$ need not be a free pro-V semigroup. However, clopen subgroups of free profinite groups are finitely generated free profinite groups. In case V satisfies no nontrivial identities, which is the case of any pseudovariety that contains N or G_p for some prime p, the subsemigroup of $\overline{\Omega}_A V$ generated by A is a free semigroup on A, which we identify with A^+ . Steinberg and the author [47] proved that, if H is any extension closed pseudovariety of groups, then the clopen free pro- $\overline{\mathsf{H}}$ subsemigroups of $\overline{\Omega}_A \overline{\mathsf{H}}$ are precisely the closures of the subsemigroups generated by a regular code in A^+ . Both to investigate irreducibility problems and for the role of relatively free profinite semigroups in symbolic dynamics [27], it would be worthwhile to characterize all regular V-codes more generally when the pseudovariety V satisfies no nontrivial identities.

The idea for the irreducibility results in [39] can be traced back to the work of Rhodes (see [143, Section 4.6]) considering the double partial action of a semigroup on a \mathcal{J} -class given by left and right multiplication. As compact semigroups always have a minimum ideal, on which both left and right actions are total, the question that arises is when is it faithful. A weaker property is actually used in [39] to establish irreducibility results, namely that both actions are faithful outside the minimum ideal and the double action is also faithful. This property is established for free pro-V semigroups on more than one generator for several pseudovarieties V.

One may also ask when a pseudovariety V(S) generated by a semigroup S is irreducible for operators such as \vee and *. Note that the stronger irreducibility property holds for V(S) if and only if whenever S divides a product $T \times U$ of finite semigroups, respectively a semidirect product T * U, S must divide a direct power of at least one of the factors T and U. In the case of the join, finite semigroups with this property are said to be *fji*. Lee, Rhodes, and Steinberg have started a systematic program of determining (up to isomorphism) all finite fji semigroups. They have done so for all semigroups of order up to five [113] and for all \mathcal{J} -trivial semigroups of order up to six [114].

Further irreducibility results on pseudovarieties of semigroups may be found in [143, Section 7.3].

7. Two key problems

7.1. Krohn-Rhodes complexity. A long outstanding problem in finite semigroup theory is the computation of the Krohn-Rhodes complexity of a finite semigroup. Its formulation in terms of pseudovarieties was already given at the end of Section 3: for a given finite semigroup, to determine the smallest natural number n such that $S \in C_n$. That there is such an n follows from a theorem of Krohn and Rhodes [110] showing that every finite semigroup divides a wreath product of finite simple groups dividing it and as many copies as needed of a 3-element \mathcal{L} -trivial band monoid. The computation of the Krohn-Rhodes complexity was started in [111] and has motivated many developments in the theory of finite semigroups. In the book [117] one finds many examples arguing how the Krohn-Rhodes complexity measures complexity of various natural phenomena.

The computation of the Krohn-Rhodes complexity was given a prominent role in [86], thus being already recognized by Eilenberg as a key problem motivating the theory of pseudovarieties of semigroups. The books [56] and [143] give two pictures, separated by four decades, of the developments aiming toward the solution of the problem. Recently, a complete, positive, solution of the problem has been announced by Margolis, Rhodes, and Schilling [117], thereby culminating almost six decades of deep research in which Rhodes has been the main driving force. Once confirmed, this is a truly remarkable achievement and may open the door to further applications of finite semigroup theory.

7.2. **Dot-depth.** A second key problem in the theory of pseudovarieties of semigroups is related with Schützenberger's characterization of star-free languages [147]. Star-free languages (possibly including the empty word) are languages that can be expressed in terms of single letter languages, the language reduced to the empty word, and the empty language, using only binary union, complementation, and concatenation. In essence, the Kleene star is replaced by complementation in the definition of regular expression.

Cohen and Brzozowski [72] proposed a hierarchy of star-free languages defined roughly by how many nested levels of concatenation are needed in a star-free expression for a language. The levels of the hierarchy define varieties of languages, hence a corresponding decomposition of the pseudovariety A as the union of a chain $(B_n)_n$ of pseudovarieties. Brzozowski and Knast [70] proved that the hierarchy is strict. Thérien [161] and Straubing [159] proposed a variant of the hierarchy consisting of monoidal pseudovarieties which is known as the *Straubing-Thérien hierarchy* and which has been the main target of investigation.

On the pseudovariety side, all levels beyond zero of the (Cohen-)Brzozowski hierarchy $(B_n)_n$ are obtained from the Straubing-Thérien hierarchy $(V_n)_n$ via the the equation $V_n * D = B_n$. In view of Simon's [148] ($V_1 = J$, whence it is decidable), Knast's [107] ($V_1 * D$ is decidable), and Straubing's [159] already mentioned results, since B_2 belongs to V_2 , we know that V_n is decidable if and only if and only if B_n is decidable.

More generally, given a positive variety of languages \mathcal{V} , define Pol \mathcal{V} to be the class of languages defined by letting Pol $\mathcal{V}(A)$ to be the closure under binary union of the set of languages of the form $L_0a_1L_1 \dots a_nL_n$ with the $a_i \in A$ and the $L_i \in \mathcal{V}(A)$. Then Pol \mathcal{V} is a positive variety of languages [133]. Adding complementation in A^* to the closure operator, one gets a variety of languages BPol \mathcal{V} . The refinement of the Straubing-Thérien starts with $\mathcal{V}_0(A) = \{\emptyset, A^*\}$ and takes $\mathcal{V}_{n+1/2} = \text{Pol }\mathcal{V}$ and $\mathcal{V}_{n+1} = \text{BPol }\mathcal{V}$ for each natural number n.

There is an additional motivation to study the Straubing-Thérien hierarchy coming from the specification of languages by a model-theoretic description, where words are viewed as linear relational models, with a unary predicate P_a for each letter a in the alphabet such that $P_a(i)$ holds if the letter a is at position i in the word. Additionally, one considers numerical predicates, for instance < to say that position i comes before position j, and \leq to add the possibility of the two positions coinciding. For instance, the language $(ab)^+$ is defined over the alphabet $\{a, b\}$ by the following first-order sentence:

$$\exists x \left((P_a(x) \lor P_b(x)) \land \left((\forall y (x \leqslant y)) \Rightarrow P_a(x) \right) \land \left((\forall y (y \leqslant x)) \Rightarrow P_b(x) \right) \right) \\ \land \forall x, y \left(\left(x < y \land \forall z (z \leqslant x \lor y \leqslant z) \right) \Rightarrow \left(P_a(x) \Leftrightarrow P_b(y) \right) \right)$$

McNaughton and Papert [127] showed that the languages definable by first order sentences are precisely those whose minimal automata have aperiodic transition monoids which, combined with [147] means those are the star-free languages. Thomas [163] proved that the dot-depth hierarchy corresponds to the quantifier-alternation hierarchy in terms of optimal description of starfree languages.

Various attempts have been made to prove decidability of all pseudovarieties V_n and there is a long history of partial results. See [134] for a relatively recent account and various extensions of the core problem discussed here. In passing, let us just mention that Pin and Straubing [136] gave an interesting algebraic description of V_2 as the pseudovariety generated by all monoids of upper-triangular Boolean matrices, and also as PJ. The author [5] further showed that the smallest pseudovariety V such that $PV = V_2$ is generated by the (four-element) syntactic semigroup of the language a^*bc^* over the alphabet $\{a, b, c\}$. Until recently, these, as many other attempts to prove decidability of V_2 failed. The best results so far, due to Place and Zeitoun, give a strong form of decidability for V_2 [138] which allows them to establish decidability for V_3 [139]. The proofs of these results are deep and difficult, which seems to indicate that, with such an approach, it is going to be very hard to go further up in the hierarchy as, at least so far, going one step up it has not been possible to deduce the same strong decidability property as at the previous step, which precludes a general induction argument.

Starting with a given variety of languages \mathcal{V} instead of the trivial variety, one can similarly build on top of it a concatenation hierarchy whose union is the least concatenation closed variety of languages containing \mathcal{V} . Moreover, if one restricts the products $L = L_0 a_1 L_1 \dots a_n L_n$ considered in the definition of the operator Pol to be unambiguous, in the sense that every word in L has a unique factorization $w_0 a_1 w_1 \dots a_n w_n$ with the $w_i \in L_i$, then one

gets corresponding operators UPol and BUPol and unambiguous concatenation hierarchies. The study of such generalizations has been extensively developed and has been intricately linked with progress on understanding the original dot-depth hierarchy. On the pseudovariety side, it is also worth mentioning that the closure under concatenation of V, meaning the pseudovariety corresponding to the closure under concatenation of the variety of languages associated with V is precisely A m V [158], while the unambiguous concatenation analog is given by LI m V.

8. TAMENESS

An approach that has led to many decidability results is to try to find a basis of pseudoidentities for a given pseudovariety for which it can be effectively checked whether a given finite semigroup satisfies it. When the pseudovariety is obtained by applying an operator to other pseudovarieties, this is sometimes achieved provided the "factor" pseudovarieties satisfy suitable hypotheses. The general notion of tame pseudovariety was conceived to explore this idea for the semidirect product.

To explain what is involved in this approach, we need to say a bit more about relatively fee profinite semigroups first. Let V be a pseudovariety of semigroups and recall that we denote by $\Omega_A V$ the free pro-V semigroup on A, which is endowed with a function $\iota_{A,V}: A \to \overline{\Omega}_A V$ such that, for every function $\varphi: A \to S$ into a pro-V semigroup S, there is a unique continuous homomorphism $\hat{\varphi}: \overline{\Omega}_A \mathsf{V} \to S$ such that $\hat{\varphi} \circ \iota_{A,\mathsf{V}} = \varphi$. We may then interpret each $w \in \overline{\Omega}_A V$ as a natural operation on a pro-V semigroup S, namely as the function $w_S: S^A \to S$ given by $w_S(\varphi) = \hat{\varphi}(w)$. Such operations are known as implicit operations. As $\overline{\Omega}_A V$ is pro-V, the interpretations of $w \in \overline{\Omega}_A V$ in semigroups from V completely determine w. Among such operations are those defined by elements of the subsemigroup of some $\overline{\Omega}_A V$ generated by A. which are called *explcit operations*; in particular, when $|A| \ge 2$ and a and b are distinct letters from A, the explicit operation defined by ab is just the semigroup multiplication. When V = S, a set of implicit operations containing multiplication is known as an *implicit signature* and thus gives a structural enrichment of profinite semigroups. Two commonly considered implicit signatures are:

- ω , consisting of multiplication and the ω -power, a unary operation whose interpretation in a finite semigroup S sends each element s to its unique idempotent power s^{ω} ;
- κ , consisting of multiplication and the ω -1-power, a unary operation whose interpretation in a finite semigroup S sends each element s to the inverse of ss^{ω} in the maximal subgroup containing the idempotent s^{ω} .

In general, for an implicit signature σ , the subalgebra of $\overline{\Omega}_A \mathsf{V}$ generated by the image of $\iota_{A,\mathsf{V}}$ is the free σ -algebra in the variety generated by V and it is denoted $\Omega_A^{\sigma}\mathsf{V}$. For instance, $\Omega_A^{\kappa}\mathsf{G}$ is the free group on A.

One natural and classical problem on a pseudovariety V is whether the *word problem* for $\Omega_A^{\sigma} V$ is decidable, that is, whether there is an algorithm that, given two σ -terms u and v, determines whether or not $V \models u = v$. If the word problem for $\Omega_A^{\sigma} V$ is decidable for every finite set A, then we say that the

 σ -word problem for V is decidable. Some notable examples of decidability of κ -word problems are those for the pseudovarieties J [6,9] (which is intimately related with Simon's characterization of piecewise testable languages [148]), A [125,31], R [55], DA [128], D(D \vee G) [68], LG [81], A \cap ESI [69], DAb [43]. Several other examples would be worth considering such as DS and DG as well as preservation of decidability of the κ -word problem under natural operators, like D, L, _ * D, _ * G, LI \widehat{m} _, and A \widehat{m} _.

A rather different kind of problem that intervenes in the definition of tameness has to do with the V-solution of systems of σ -equations. Given σ -identities $u_i = v_i$ $(i \in I)$ over a given finite alphabet X, a V-solution (over the alphabet A) is an evaluation of the variables $\varphi : X \to \overline{\Omega}_A S$ such that: $\mathsf{V} \models \hat{\varphi}(u_i) = \hat{\varphi}(v_i)$ for every $i \in I$, where $\hat{\varphi}$ is the unique extension of φ to a σ -algebra homomorphism $F_{\sigma}X \to \overline{\Omega}_A S$. Such a solution φ is a σ -solution if takes its values in $\Omega_A^{\sigma}\mathsf{V}$. The pseudovariety V is said to be σ -reducible for the system $u_i = v_i$ $(i \in I)$ if the set of all σ -solutions over each finite alphabet A is dense in the subspace of the product space $(\overline{\Omega}_A \mathsf{S})^X$ consisting of all V-solutions. The pseudovariety V is σ -reducible for a set E of systems of σ -equations if it is σ -reducible for all systems in E. Finally, for a recursively enumerable implicit signature σ consisting of computable implicit operations, a pseudovariety V is σ -tame for E if the σ -word problem for V is decidable and V is σ -reducible for E. In particular, the following terminology is used instead of saying that V is σ -tame for E:

- when E consists of all finite systems of σ -equations, we say that V is completely σ -tame;
- when E consists of all finite systems of equations associated with finite directed graphs, in which the variables are the vertices and the edges and there is an equation xy = z for each edge $x \xrightarrow{y} z$, we say that V is graph σ -tame;
- when E consists of the systems $x_1 = x_2 = \cdots = x_n$ $(n \ge 2)$, we say that V is *pointlike* σ -tame,
- when E consists of the systems $x_1 = x_2 = \cdots = x_n = x_n^2$ $(n \ge 2)$, we say that V is *idempotent pointlike* σ -tame.

More generally, such notions can be considered when inequalities are taken instead of equations and V is a pseudovariety of ordered semigroups or monoids.

The interest in tameness comes from the observation, proved in [45, 46] that if V is σ -tame for a finite system of equations over a finite set of variables X and a clopen constraint $K_x \subseteq \overline{\Omega}_A S$ is given (since K_x is the topological closure of a regular language $L_x \subseteq A^+$, it may be described by finite data) for each $x \in X$, then it is decidable whether the system has a V-solution over A. This decidability condition had previously been introduced in [11] under the name of *hyperdecidability* as an approach to proving decidability of semidirect products.

For instance, if the system is reduced to the equation x = y, then being able to determine whether such a solution exists implies that V is decidable. For, given a finite semigroup S and a continuous homomorphism $\varphi : \overline{\Omega}_A S \to$ S, we may take as constraints the sets $K_x = \varphi^{-1}(s)$ and $K_y = \varphi^{-1}(t)$ for a pair of distinct points $s, t \in S$. The existence of a V-solution of x = y satisfying those constraints means that there is a pseudoidentity u = v, with $u, v \in \overline{\Omega}_A S$, which is satisfied by V and fails in S, so that $S \notin V$. On the other hand, the non-existence of a V-solution satisfying those constraints means that the closed sets $p_V(K_x)$ and $p_V(K_y)$ are disjoint, where $p_V: \overline{\Omega}_A S \to \overline{\Omega}_A V$ is the unique continuous homomorphism such that $p_{V} \circ \iota_{A,S} = \iota_{A,V}$; in a Stone space, this means that there is a clopen set separating the two closed sets, which in turn means that there is a language $L\subseteq A^+$ recognized by a semigroup from V that contains the language $K_x \cap A^+$ and is disjoint from $K_y \cap A^+$. Thus, it is decidable whether the equation x = y has a V-solution satisfying given clopen constraints if and only if it is decidable whether given disjoint regular languages over a finite alphabet A may be separated by a V-recognizable language. This was first observed in [12]. Note finally that, if the languages $\varphi^{-1}(r) \cap A^+$ with $r \in S$ may all be pairwise separated by V-recognizable languages then, as they partition A^+ , they are themselves V-recognizable, which entails that $S \in V$. In particular, if V is tame for the equation x = y, then V is decidable.

What may be considered striking is that difficult decidability problems may be settled by solving a classical word problem plus proving a nonalgorithmic topological property. Although none of these ingredients may be easy, achieving them usually means reaching a deep understanding of the pseudovariety in question, which is why the combined property is called *tameness*. But, of course, there are various degrees of tameness depending on for what kind of systems we are able to prove reducibility and how complicated a signature needs to be considered. Before proceeding with a survey of tameness results, we give the main motivation that led to the notion of tameness.

8.1. Tameness and semidirect product. Exploring Tilson's seminal results using pseudovarieties of categories to describe semidirect products of pseudovarieties of semigroups or monoids, namely through his Derived Category Theorem [164], Weil and the author [53] attempted to describe bases of pseudoidentities for such semidirect products V * W. We proceed to describe briefly how such bases are obtained.

There is an analog of Reiterman's Theorem for pseudovarieties of categories. Categories are viewed as generalizations of monoids, which in turn are viewed as categories on a single virtual vertex whose edges are the elements of the monoid. The role of free profinite monoids $\overline{\Omega}_A S$ on a set Ais played by free profinite categories $\overline{\Omega}_{\Gamma} Cat$ on a directed graph Γ [102, 53]. However, extra care needs to be taken (which was already present in [102] but not in [53]) when the graph Γ has an infinite vertex set and in fact it was shown in [20] that Γ may not generate a dense subcategory of $\overline{\Omega}_{\Gamma} Cat$, which was previously taken for granted in several papers. In fact, symbolic dynamics is used in [20] to show that, starting from the graph Γ , it may require an arbitrarily large countable ordinal number of alternations of taking algebraic generation and topological closure before $\overline{\Omega}_{\Gamma} Cat$ is reached. Yet, pseudovarieties of categories can be defined by formal equalities ($u = v; \Gamma$) of elements of free profinite categories $\overline{\Omega}_{\Gamma} Cat$ over finite directed graphs Γ starting and ending at the same vertices [102, 53]. Extra care is needed when semigroupoids (which are like categories but with no requirement for local identities) are considered, see Problem 6 in [143].

The Basis Theorem (5.3) of [53] states that, given pseudovarieties of monoids V and semigroups W, if $\{(u_i = v_i; \Gamma_i) : i \in I\}$ is a basis of pseudoidentities for the pseudovariety of categories gV generated by V, then the set of all the following semigroup pseudoidentities is a basis of pseudoidentities for V * W: $\delta(p)\varepsilon(u_i) = \delta(p)\varepsilon(v_i)$, where p is the common initial vertex of u_i and v_i , δ is a mapping from the vertex set of Γ_i to $(\overline{\Omega}_A S)^1$, and ε is a continuous mapping from the edge set of $\overline{\Omega}_{\Gamma_i} \text{Cat}$ to $\overline{\Omega}_A S$ respecting multiplication such that, for every edge $q \xrightarrow{x} r$, $W \models \delta(q)\varepsilon(x) = \delta(r)$.

Unfortunately, besides sloppiness in handling graphs with infinite vertex sets and pseudovarieties of semigroupoids, there is a serious gap in the proof of a key step (Proposition 3.6) in the proof of the Basis Theorem where an unjustified exchange of quantifiers is implicitly made. While no counterexample has ever been produced, it seems rather unlikely that this key ingredient holds in its full generality. Thus, the Basis Theorem can for now only be used under one of the following two extra finiteness assumptions:

- (1) W is generated by a finite semigroup;
- (2) V has finite vertex rank, meaning that the pseudovariety of categories gV admits a basis of semigroupoid identities over graphs with a bounded number of vertices.

Noting that V has vertex rank one if and only if V is local, local pseudovarieties are specially amenable to this approach. Yet, there are many pseudovarieties of interest with infinite vertex rank, for instance,

- letting $\operatorname{Com}_{m,\alpha}$ be the pseudovariety of commutative monoids satisfying the pseudoidentity $x^{m+\alpha} = x^m$, where *m* is a positive integer and α is either a positive integer or ω , when $m \ge 2$ there is no pseudovariety of finite vertex rank in the (uncountable) interval $[\operatorname{Com}_{m,1}, \operatorname{Com}_{m,\omega}]$ [17];
- several other intervals of pseudovarieties of infinite vertex rank are given in [154] and in the review by Auinger of this paper in Math-SciNet, MR2025914, including the interval between the pseudovariety generated by B_2^1 and DA * H when H is a proper nontrivial pseudovariety of groups; among pseudovarieties of infinite vertex rank covered by the paper and the review, one finds pseudovarieties such as $\overline{H} * G$ when $H \subsetneq G$ (in particular, A * G) and SI * H when $I \neq H \subsetneq G$.

In view of the above discussion, using the tameness approach, the best that can be stated at present is that if V is a decidable pseudovariety of monoids of finite vertex rank and W is graph tame, then V * W is decidable [45].

8.2. Tameness and Mal'cev product. There is also a Basis Theorem for Mal'cev products V m W [137]. The theorem states that, if the set $\{u_i(x_1, \ldots, x_{n_i}) = v_i(x_1, \ldots, x_{n_i}) : i \in I\}$ is a basis of pseudoidentities for V, then the following is a basis for V m W:

$$u_i(w_1, \dots, w_{n_i}) = v_i(w_1, \dots, w_{n_i})$$
 whenever $\mathsf{W} \models w_1^2 = w_1 = \dots = w_{n_i} \ (i \in I).$

As a corollary, one gets that if V is decidable and W is idempotent pointlike tame, then V m W is decidable.

Thus, the Mal'cev product turns out to be much easier to handle than the semidirect product by the tameness approach. In contrast, there is a representation theorem for $\overline{\Omega}_A(V * W)$ [51] but no such representation is known for $\overline{\Omega}_A(V \oplus W)$.

8.3. **Tameness results.** The following is a summary of known tameness results so far.

- In seminal work of Ash [59] it was proved a property that turns out to be equivalent to G being graph κ -tame (see [45]). Yet, it follows from [82] that G is not completely κ -tame, which leads to the question as to whether G is completely tame for some signature. As observed in [33, 34], Ash's result turns out to have an interesting formulation in model theory, where it was, in that sense, rediscovered by Herwig and Lascar [100].
- The pseudovariety J is completely ω-tame [15]. Proving that J is hyperdecidable without going through tameness turns out to be much more complicated [54]. But, in fairness, it should be mentioned that the algorithm that comes from the tameness approach is totally impractical as it involves generating in parallel all favorable and unfavorable cases until the one of interest is produced.
- If W is such that finitely generated free pro-W semigroups are finite and computable and V is graph hyperdecidable then so is V * W [44]. This should be improvable to tameness but does not appear to have been done so far.
- The pseudovariety CR is κ -tame for graph systems of equations. This follows from [48] together an observation of K. Auinger that the required supposedly improved tameness of G is actually granted by Ash's result. There is also a potential problem with the proof because of the usage of free profinite categories over infinite-vertex graphs, which are assumed to generate dense subcategories. Yet, it was shown in [23] that the required property does hold in the case in question.
- The pseudovariety G_p is not graph κ -tame but it is graph σ -tame for a certain infinite implicit signature σ constructed using ideas from symbolic dynamics [13]. This depends on results of Steinberg [151], who previously proved a weak form of hyperdecidability for G_p .
- The pseudovariety Ab is completely κ -tame [35].
- The pseudovariety R is completely ω-tame [29]. The idea of the proof of complete reducibility is to adapt that of Makanin's algorithm to solve equations in free semigroups [115, 116], even though in our case there is no algorithm involved.
- The following was established in [84]:
 - A monoidal pseudovariety of commutative semigroups is completely κ -hyperdecidable if and only if it is decidable.
 - If a proper pseudovariety of Abelian groups is κ -reducible for systems of graph equations, then it is locally finite.
- The pseudovariety LSI is completely ω -tame [79].

- The pseudovarieties A and DA are pointlike ω -tame [32]. In the case of A, the proof uses ideas of Henckell [96,99] giving an algorithm to determine the semigroup of all A-pointlike subsets of a finite semigroup.
- The pseudovariety LG is graph κ -tame [81]. The proof uses the more general result that, if V is graph κ -reducible, then so is V * D [80].
- The pseudovariety of groups H is completely κ-tame if and only if so is D(D ∨ H) [19]. The method is similar to that used for R, described above, but is more complicated because groups are involved.
- The pseudovariety G_{nil} is graph tame [3].
- That the pseudovariety DAb is completely κ -tame has recently been announced by Kufleitner, Wächter and the author but at the moment only a preprint is available for the κ -word problem [43].

The tameness approach has also been explored to compute joins. For instance, $J \lor G$ was independently shown by Steinberg [150] and Azevedo, Zeitoun and the author [18]. The former work fits in a more comprehensive approach to joins, giving hyperdecidability results, and was part of the author's Ph.D. thesis [149]. There are many other papers dealing with joins of pseudovarieties, often using the profinite, but not necessarily tameness, aproach. [4, 16, 28, 50, 77, 60, 78, 153]

It is also worth mentioning that the tameness approach has been proposed to establish decidability of the Straubing-Thérien hierarchy. This was started by Klíma, Kunc, and the author [42] by showing that the ω -inequality problem, meaning solving the inequality $x \leq y$, is decidable over all pseudovarieties V_n and $V_{n+1/2}$ (*n* non-negative integer) in the refined hierarchy. This reduces the decidability of the hierarchy to establishing the purely topological property of ω -reducibility for the inequality $x \leq y$ of all levels. As evidence that such a property may hold, Volaříková [167] has shown that V_2 is defined by ω -identities: if the topological property holds then all levels of the hierarchy would be defined by ω -inequalities.

9. The structure of relatively free profinite semigroups

In view of the role of relatively free profinite semigroups in the profinite approach, it is worth understanding the structure of such semigroups. This is in general quite hard and has only been achieved in very few cases. An idea that has been extensively explored is that, just as the positions of the letters in finite words are linearly ordered, members of relatively free profinite semigroups, sometimes called *profinite words*, but which the author prefers to call *pseudowords*, should also have some kind of linear structure. Even for the pseudovarieties of groups this is in a sense the case: the profinite Cayley graph of $\overline{\Omega}_A H$ is a profinite H-tree if and only if $(H \cap Ab) * H = H$ [50], which extends results of Gildenhuys and Ribes [92].

The aperiodic case also presents a linear behavior. This had already been observed for J [6], R [52], DA [10], $D(D \lor H)$ [19], but the order types become much more involved for the pseudovariety A [30,25,166]. The latter of these works also brings about interesting connections with model theory, which we have seen in this survey to pop up every so often.

The local structure of relatively free profinite semigroups has also been investigated, particularly, the structure of their regular \mathcal{D} -classes. The author came up with an interesting connection with symbolic dynamics: for every pseudovariety V containing LSI, the regular \mathcal{J} -classes of $\overline{\Omega}_A V$ that are maximal in the partial order of \mathcal{J} -classes are in bijection with the minimal shift spaces $X \subseteq A^{\mathbb{Z}}$ [14]: the \mathcal{J} -class associated with X consists of all non-finite pseudowords in $\overline{L(X)}$, where L(X) consists of all finite words that appear as blocks in elements of X. Recall that a *shift space* over a finite alphabet A is simply a nonempty closed subset of $A^{\mathbb{Z}}$, whose elements are viewed as biinfinite words, which is stable under shifting the origin. We say that X is *sofic* if L(X) is a regular language, *irreducible* if, for all $u, v \in L(X)$ there is w such that $uwv \in L(X)$, *periodic* if L(X) consists of all factors of the powers of a fixed word, and *substitutive* if L(X) consists of all factors of $\varphi^n(a)$ where $a \in A$ and φ is a primitive endomorphism of A^+ .

More generally, irreducible shift spaces also have a unique \mathcal{J} -minimal \mathcal{J} -class intersecting $\overline{L(X)}$, which is denoted $J_V(X)$. Since all maximal subgroups in a \mathcal{J} -class are isomorphic this led to the definition of the *Schützenberger group* of X, denoted $G_V(X)$, to be any of the maximal subgroups of $J_V(X)$. In the case of a minimal shift space, there is a natural geometric interpretation of $G_V(X)$ as an inverse limit of profinite completions of Poincaré groups of certain Rauzy graphs of X [24]. It is also an invariant of topological conjugacy, which is the natural notion of isomorphism between shift spaces [73] (see also [75]). The book [27] gives an introduction to this theory and interesting connections with coding theory which were already explored in [26, 95].

A remarkable result of Costa and Steinberg [74] shows that, whenever H is an extension-closed pseudovariety of groups and X is an irreducible sofic shift space, then $G_{\overline{H}}(X)$ is a free pro- \overline{H} group which is of countable rank unless X is periodic, in which case the group is procyclic [49]. In contrast, it had already been observed in [14] that $G_{\mathsf{S}}(X)$ may not be a free profinite group even for substitutive shift spaces. In the case of substitutive shift spaces, a finite (profinite) presentation can be computed for $G_{\mathsf{S}}(X)$ which entails that it is decidable whether a finite group is a continuous quotient of $G_{\mathsf{S}}(X)$ and allows to prove freeness or non-freeness (even relatively to any pseudovariety, as is the case for the much studied Prouhet-Thue-Morse shift space, which is generated by the substitution $a \mapsto ab, b \mapsto ba$) in many cases [22]. Further relevant results for freeness of Schützenberger groups have also been obtained in [93, 94].

It remains an open problem what kind of profinite group can $G_{V}(X)$ be when X is a minimal shift space. More information provided by \mathcal{J} -classes associated with a shift space has also been explored in [21, 76].

10. CONCLUSION

There are many aspects of the theory of pseudovarieties, which extends for over six decades, that it is impossible to cover in such a brief survey. By no means this is meant to belittle such aspects and the many valuable contributions that many authors have made, but rather reflects the limitations of the author of this survey.

In any case, it is hoped that this work gives a feeling for the richness and depthness of a well-motivated theory whose potential applications have perhaps not yet been fully explored.

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References

- J. Adámek, S. Milius, R. S. R. Myers, and H. Urbat, *Generalized Eilenberg theorem:* varieties of languages in a category, ACM Trans. Comput. Log. 20 (2019), Art. 3, 47.
- [2] D. Albert, R. Baldinger, and J. Rhodes, *The identity problem for finite semigroups* (the undecidability of), J. Symbolic Logic **57** (1992), 179–192.
- [3] K. Alibabaei, The pseudovariety of all nilpotent groups is tame, Int. J. Algebra Comput. 29 (2019), 1019–1034.
- [4] J. Almeida, Some pseudovariety joins involving the pseudovariety of finite groups, Semigroup Forum 37 (1988), 53–57.
- [5] _____, The equation P X = PJ, in Semigroup and its related fields, (M. Yamada and H. Tominaga, eds.), Vol. 1-11, 1990, Matsue University.
- [6] _____, Implicit operations on finite *J*-trivial semigroups and a conjecture of I. Simon, J. Pure Appl. Algebra 69 (1990), 205–218.
- [7] _____, On pseudovarieties, varieties of languages, filters of congruences, pseudoidentities and related topics, Algebra Universalis **27** (1990), 333–350.
- [8] _____, On direct product decompositions of finite |-trivial semigroups, Int. J. Algebra Comput. 1 (1991), 329–337.
- [9] _____. (1994). Finite semigroups and universal algebra, World Scientific, Singapore. English translation.
- [10] _____, A syntactical proof of locality of DA, Int. J. Algebra Comput. 6 (1996), 165–177.
- [11] _____, Hyperdecidable pseudovarieties and the calculation of semidirect products, Int. J. Algebra Comput. 9 (1999), 241–261.
- [12] _____, Some algorithmic problems for pseudovarieties, Publ. Math. Debrecen 54 Suppl. (1999), 531–552.
- [13] _____, Dynamics of implicit operations and tameness of pseudovarieties of groups, Trans. Amer. Math. Soc. 354 (2002), 387–411.
- [14] _____, Profinite groups associated with weakly primitive substitutions, Fundamentalnaya i Prikladnaya Matematika (Fundamental and Applied Mathematics) 11 (2005), 13–48, In Russian. English version in J. Math. Sciences 144, No. 2 (2007) 3881–3903.
- [15] _____, Profinite semigroups and applications, in Structural theory of automata, semigroups and universal algebra, (V. B. Kudryavtsev and I. G. Rosenberg, eds.), 2005, pp. 1–45, Springer.
- [16] J. Almeida and A. Azevedo, The join of the pseudovarieties of *R*-trivial and *L*-trivial monoids, J. Pure Appl. Algebra 60 (1989), 129–137.
- [17] _____, Globals of pseudovarieties of commutative semigroups: the finite basis problem, decidability, and gaps, Proc. Edinburgh Math. Soc. 44 (2001), 27–47.
- [18] J. Almeida, A. Azevedo, and M. Zeitoun, *Pseudovariety joins involving J-trivial semigroups*, Int. J. Algebra Comput. 9 (1999), 99–112.
- [19] J. Almeida and C. Borlido, Complete κ-reducibility of pseudovarieties of the form DRH, Int. J. Algebra Comput. 27 (2017), 189–235.

- [20] J. Almeida and A. Costa, Infinite-vertex free profinite semigroupoids and symbolic dynamics, J. Pure Appl. Algebra 213 (2009), 605–631.
- [21] _____, On the transition semigroups of centrally labeled Rauzy graphs, Int. J. Algebra Comput. 22 (2012), 25 pages.
- [22] _____, Presentations of Schützenberger groups of minimal subshifts, Israel J. Math. 196 (2013), 1–31.
- [23] _____, A note on pseudovarieties of completely regular semigroups, Bull. Austral. Math. Soc. 92 (2015), 233–237.
- [24] _____, A geometric interpretation of the Schützenberger group of a minimal subshift, Ark. Mat. 54 (2016), 243–275.
- [25] J. Almeida, A. Costa, J. C. Costa, and M. Zeitoun, *The linear nature of pseudowords*, Publ. Mat. **63** (2019), 361–422.
- [26] J. Almeida, A. Costa, R. Kyriakoglou, and D. Perrin, On the group of a rational maximal bifix code, Forum Math. 32 (2020), 553–576.
- [27] _____. (2020). Profinite semigroups and symbolic dynamics, Lect. Notes in Math., vol. 2274, Springer, Cham.
- [28] J. Almeida, J. C. Costa, and M. Zeitoun, Tameness of pseudovariety joins involving R, Monatsh. Math. 146 (2005), 89–111.
- [29] _____, Complete reducibility of systems of equations with respect to R, Portugal. Math. 64 (2007), 445–508.
- [30] _____, Iterated periodicity over finite aperiodic semigroups, European J. Combin. 37 (2014), 115–149.
- [31] _____, McCammond's normal forms for free aperiodic semigroups revisited, LMS J. Comput. Math. 18 (2015), 130–147.
- [32] _____, Reducibility of pointlike problems, Semigroup Forum 94 (2017), 325–335.
- [33] J. Almeida and M. Delgado, Sur certains systèmes d'équations avec contraintes dans un groupe libre, Portugal. Math. 56 (1999), 409–417.
- [34] _____, Sur certains systèmes d'équations avec contraintes dans un groupe libre addenda, Portugal. Math. 58 (2001), 379–387.
- [35] _____, Tameness of the pseudovariety of abelian groups, Int. J. Algebra Comput. 15 (2005), 327–338.
- [36] J. Almeida and A. Escada, On the equation V * G = EV, J. Pure Appl. Algebra **166** (2002), 1–28.
- [37] J. Almeida and O. Klíma, Reducibility vs. definability for pseudovarieties of semigroups, Int. J. Algebra Comput. 26 (2016), 1483–1495.
- [38] _____, On the insertion of n-powers, Discrete Math. & Theor. Comp. Sci. 21 (2019), 18 pp.
- [39] _____, Representations of relatively free profinite semigroups, irreducibility, and order primitivity, Trans. Amer. Math. Soc. 373 (2020), 1941–1981.
- [40] _____, Stone pseudovarieties, Results Math. **79** (2024), Paper No. 252, 43 pp.
- [41] J. Almeida, O. Klíma, and H. Goulet-Ouellet, What makes a Stone topological algebra profinite, Algebra Universalis 84 (2023), article 6.
- [42] J. Almeida, O. Klíma, and M. Kunc, The ω-inequality problem for concatenation hierarchies of star-free languages, Forum Math. 30 (2018), 663–679.
- [43] J. Almeida, M. Kufleitner, and J. Ph. Wächter. (2024). The word problem for $(\omega 1)$ -terms over DAb. https://arxiv.org/abs/2411.08523
- [44] J. Almeida and P. V. Silva, On the hyperdecidability of semidirect products of pseudovarieties, Comm. Algebra 26 (1998), 4065–4077.
- [45] J. Almeida and B. Steinberg, On the decidability of iterated semidirect products and applications to complexity, Proc. London Math. Soc. 80 (2000), 50–74.
- [46] _____, Syntactic and global semigroup theory, a synthesis approach, in Algorithmic problems in groups and semigroups, (J. C. Birget, S. W. Margolis, J. Meakin, and M. V. Sapir, eds.), 2000, pp. 1–23, Birkhäuser.
- [47] _____, Rational codes and free profinite monoids, J. London Math. Soc. (2) 79 (2009), 465–477.
- [48] J. Almeida and P. G. Trotter, The pseudoidentity problem and reducibility for completely regular semigroups, Bull. Austral. Math. Soc. 63 (2001), 407–433.

- [49] J. Almeida and M. V. Volkov, Subword complexity of profinite words and subgroups of free profinite semigroups, Int. J. Algebra Comput. 16 (2006), 221–258.
- [50] J. Almeida and P. Weil, Reduced factorizations in free profinite groups and join decompositions of pseudovarieties, Int. J. Algebra Comput. 4 (1994), 375–403.
- [51] _____, Free profinite semigroups over semidirect products, Russian Math. (Iz. VUZ) 39 (1995), 1–27.
- [52] _____, Free profinite *R*-trivial monoids, Int. J. Algebra Comput. 7 (1997), 625– 671.
- [53] _____, Profinite categories and semidirect products, J. Pure Appl. Algebra 123 (1998), 1–50.
- [54] J. Almeida and M. Zeitoun, The pseudovariety J is hyperdecidable, Theor. Inform. Appl. 31 (1997), 457–482.
- [55] _____, An automata-theoretic approach to the word problem for ω -terms over R, Theor. Comp. Sci. **370** (2007), 131–169.
- [56] M. Arbib. (1968). Algebraic theory of machines, languages and semigroups, Academic Press, New York.
- [57] C. J. Ash, Pseudovarieties, generalized varieties and similarly described classes, J. Algebra 92 (1985), 104–115.
- [58] _____, Finite semigroups with commuting idempotents, J. Austral. Math. Soc., Ser. A 43 (1987), 81–90.
- [59] _____, Inevitable graphs: a proof of the type II conjecture and some related decision procedures, Int. J. Algebra Comput. 1 (1991), 127–146.
- [60] K. Auinger, Join decompositions of pseudovarieties involving semigroups with commuting idempotents, J. Pure Appl. Algebra 170 (2002), 115–129.
- [61] K. Auinger and B. Steinberg, On the extension problem for partial permutations, Proc. Amer. Math. Soc. 131 (2003), 2693–2703.
- [62] _____, The geometry of profinite graphs with applications to free groups and finite monoids, Trans. Amer. Math. Soc. 356 (2004), 805–851.
- [63] _____, On power groups and embedding theorems for relatively free profinite monoids, Math. Proc. Cambridge Phil. Soc. 138 (2005), 211–232.
- [64] J. Baldwin and J. Berman, Varieties and finite closure conditions, Colloq. Math. 35 (1976), 15–20.
- [65] B. Banaschewski, The Birkhoff theorem for varieties of finite algebras, Algebra Universalis 17 (1983), 360–368.
- [66] J. Berstel, D. Perrin, and Ch. Reutenauer. (2010). Codes and automata, Encyclopedia of Mathematics and its Applications, vol. 129, Cambridge University Press, Cambridge.
- [67] G. Birkhoff, On the structure of abstract algebras, Proc. Cambridge Phil. Soc. 31 (1935), 433–454.
- [68] C. Borlido, The κ -word problem over DRH, Theor. Comp. Sci. **702** (2017), 1–22.
- [69] M. J. J. Branco and J. C. Costa, On ω-identities over finite aperiodic semigroups with commuting idempotents, in Semigroups, categories, and partial algebras. icsaa 2019. proceedings of the conference, kochi, india, december 9–12, 2019, 2021, pp. 169–178, Singapore: Springer.
- [70] J. A. Brzozowski and R. Knast, The dot-depth hierarchy of star-free languages is infinite, J. Comput. System Sci. 16 (1978), 37–55.
- [71] J. A. Brzozowski and I. Simon, Characterizations of locally testable events, Discrete Math. 4 (1973), 243–271.
- [72] R. S. Cohen and J. A. Brzozowski, Dot-depth of star-free events, J. Comput. System Sci. 5 (1971), 1–16.
- [73] A. Costa, Conjugacy invariants of subshifts: an approach from profinite semigroup theory, Int. J. Algebra Comput. 16 (2006), 629–655.
- [74] A. Costa and B. Steinberg, Profinite groups associated to sofic shifts are free, Proc. London Math. Soc. 102 (2011), 341–369.
- [75] _____, A categorical invariant of flow equivalence of shifts, Ergodic Theory Dynam. Systems 36 (2016), 470–513.

- [76] _____, The Karoubi envelope of the mirage of a subshift, Comm. Algebra **49** (2021), 4820–4856.
- [77] J. C. Costa, Some pseudovariety joins involving locally trivial semigroups, Semigroup Forum **64** (2002), 12–28.
- [78] _____, Reducibility of joins involving some locally trivial pseudovarieties, Comm. Algebra 32 (2004), 3517–3535.
- [79] J. C. Costa and C. Nogueira, Complete reducibility of the pseudovariety LSl, Int. J. Algebra Comput. 19 (2009), 247–282.
- [80] J. C. Costa, C. Nogueira, and M. L. Teixeira, *Pointlike reducibility of pseudovarieties of the form* V * D, Int. J. Algebra Comput. 26 (2016), 203–216.
- [81] _____, The word problem for κ -terms over the pseudovariety of local groups, Semigroup Forum 103 (2021), 439–468.
- [82] T. Coulbois and A. Khélif, Equations in free groups are not finitely approximable, Proc. Amer. Math. Soc. 127 (1999), 963–965.
- [83] M. Delgado, Abelian pointlikes of a monoid, Semigroup Forum 56 (1998), 127–146.
- [84] M. Delgado, A. Masuda, and B. Steinberg, Solving systems of equations modulo pseudovarieties of abelian groups and hyperdecidability, in Semigroups and formal languages, 2007, pp. 57–65, World Sci. Publ., Hackensack, NJ.
- [85] S. Eilenberg. (1974). Automata, languages and machines, Vol. A, Academic Press, New York.
- [86] _____. (1976). Automata, languages and machines, Vol. B, Academic Press, New York.
- [87] S. Eilenberg and M. P. Schützenberger, On pseudovarieties, Adv. in Math. 19 (1976), 413–418.
- [88] L. Fortnow, The status of the P versus NP problem, Comm. ACM 52 (2009), 78-86.
- [89] F. Gécseg and M. Steinby. (1984). Tree automata, Akadémiai Kiadó, Budapest.
- [90] M. Gehrke and A. Krebs, Stone duality for languages and complexity, ACM SIGLOG News 4 (2017), 29–53.
- [91] M. Gehrke and S. van Gool. (2024). Topological duality for distributive lattices. Theory and applications, Camb. Tracts Theor. Comput. Sci., vol. 61, Cambridge Univ. Press.
- [92] D. Gildenhuys and L. Ribes, Profinite groups and boolean graphs, J. Pure Appl. Algebra 12 (1978), 21–47.
- [93] H. Goulet-Ouellet, Freeness of Schützenberger groups of primitive substitutions, Int. J. Algebra Comput. 32 (2022), 1101–1123.
- [94] _____, Pronilpotent quotients associated with primitive substitutions, J. Algebra 606 (2022), 341–370.
- [95] _____, Suffix-connected languages, Theor. Comp. Sci. 923 (2022), 126–143.
- [96] K. Henckell, Pointlike sets: the finest aperiodic cover of a finite semigroup, J. Pure Appl. Algebra 55 (1988), 85–126.
- [97] K. Henckell, S. Margolis, J.-E. Pin, and J. Rhodes, Ash's type II theorem, profinite topology and Malcev products. Part I, Int. J. Algebra Comput. 1 (1991), 411–436.
- [98] K. Henckell and J. Rhodes, The theorem of Knast, the PG=BG and Type II Conjectures, in Monoids and semigroups with applications, (J. Rhodes, ed.), 1991, pp. 453– 463, World Scientific.
- [99] K. Henckell, J. Rhodes, and B. Steinberg, Aperiodic pointlikes and beyond, Int. J. Algebra Comput. 20 (2010), 287–305.
- [100] B. Herwig and D. Lascar, Extending partial automorphisms and the profinite topology on free groups, Trans. Amer. Math. Soc. 352 (2000), 1985–2021.
- [101] P. M. Higgins and S. W. Margolis, Finite aperiodic semigroups with commuting idempotents and generalizations, Israel J. Math. 116 (2000), 367–380.
- [102] P. R. Jones, Profinite categories, implicit operations and pseudovarieties of categories, J. Pure Appl. Algebra 109 (1996), 61–95.
- [103] P. R. Jones and P. G. Trotter, Locality of ds and associated varieties, J. Pure Appl. Algebra 104 (1995), 275–301.
- [104] P. R. Jones, Monoid varieties defined by $x^{n+1} = x$ are local, Semigroup Forum 47 (1993), 318–326.

- [105] P. R. Jones and M. B. Szendrei, Local varieties of completely regular monoids, J. Algebra 150 (1992), 1–27.
- [106] J. Kaďourek, On the locality of the pseudovariety DG, J. Inst. Math. Jussieu 7 (2008), 93–180.
- [107] R. Knast, A semigroup characterization of dot-depth one languages, RAIRO Inf. Théor. et Appl. 17 (1983), 321–330.
- [108] _____, Some theorems on graph congruences, RAIRO Inf. Théor. et Appl. 17 (1983), 331–342.
- [109] I. O. Koryakov, Embeddings of pseudofree semigroups, Russian Math. (Iz. VUZ) 39 (1995), 53–59.
- [110] K. Krohn and J. Rhodes, Algebraic theory of machines. I. Prime decomposition theorem for finite semigroups and machines, Trans. Amer. Math. Soc. 116 (1965), 450–464.
- [111] _____, Complexity of finite semigroups, Ann. of Math. (2) 88 (1968), 128–160.
- [112] G. Lallement. (1979). Semigroups and combinatorial applications, Wiley-Interscience, J. Wiley & Sons, Inc., New York.
- [113] E. W. H. Lee, J. Rhodes, and B. Steinberg, *Join irreducible semigroups*, Int. J. Algebra Comput. **29** (2019), 1249–1310.
- [114] _____, On join irreducible J-trivial semigroups, Rend. Semin. Mat. Univ. Padova 147 (2022), 43–78.
- [115] G. S. Makanin, The problem of solvability of equations in a free semigroup, Mat. Sb. (N.S.) 103 (2) (1977), 147–236, In Russian. English translation in: Math. USSR-Sb. 32 (1977) 128-198.
- [116] _____, Equations in a free semigroup, Amer. Math. Soc. Transl. (II Ser.) 117 (1981), 1–6.
- [117] S. Margolis, J. Rhodes, and A. Schilling. (2024). Decidability of Krohn-Rhodes complexity for all finite semigroups and automata. https://arxiv.org/abs/2406.18477
- [118] S. Margolis, M. Sapir, and P. Weil, Irreducibility of certain pseudovarieties, Comm. Algebra 26 (1998), 779–792.
- [119] _____, Closed subgroups in pro-V topologies and the extension problem for inverse automata, Int. J. Algebra Comput. 11 (2001), 405–445.
- [120] S. W. Margolis, On M-varieties generated by power monoids, Semigroup Forum 22 (1981), 339–353.
- [121] S. W. Margolis and J.-E. Pin, Varieties of finite monoids and topology for the free monoid, in Proc. 1984 marquette semigroup conference, 1984, pp. 113–129, Marquette University.
- [122] _____, Products of group languages, in Fct'85, Vol. 199, 1985, pp. 285–299, Springer.
- [123] C. Marion, P. V. Silva, and G. Tracey, The pro-k-solvable topology on a free group, J. Austral. Math. Soc. 116 (2024), 363–383.
- [124] _____, The pro-supersolvable topology on a free group: deciding denseness, J. Algebra 646 (2024), 183–204.
- [125] J. McCammond, Normal forms for free aperiodic semigroups, Int. J. Algebra Comput. 11 (2001), 581–625.
- [126] R. McNaughton, Algebraic decision procedures for local testability, Math. Syst. Theory 8 (1974), 60–76.
- [127] R. McNaughton and S. Papert. (1971). Counter-free automata, MIT Press, Cambridge, MA.
- [128] A. Moura, The word problem for ω -terms over DA, Theor. Comp. Sci. **412** (2011), 6556–6569.
- [129] K. Numakura, Theorems on compact totally disconnetced semigroups and lattices, Proc. Amer. Math. Soc. 8 (1957), 623–626.
- [130] J.-E. Pin. (1986). Varieties of formal languages, Plenum, London. English translation.
- [131] _____, BG = PG: A success story, in Semigroups, formal languages and groups, (J. Fountain, ed.), Vol. 466, 1995, pp. 33–47, Kluwer.

- [132] _____, A variety theorem without complementation, Russian Math. (Iz. VUZ) 39 (1995), 80–90.
- [133] _____, An explicit formula for the intersection of two polynomials of regular languages, in Developments in language theory. 17th international conference, DLT 2013, Marne-la-Vallée, France, June 18–21, 2013. Proceedings, 2013, pp. 31–45, Berlin: Springer.
- [134] _____, The dot-depth hierarchy, 45 years later, in The role of theory in computer science, essays dedicated to Janusz Brzozowski, (S. Konstantinidis et al., eds.), 2017, pp. 177–202, World Scientific.
- [135] J.-E. Pin and C. Reutenauer, A conjecture on the Hall topology for the free group, Bull. London Math. Soc. 23 (1991), 356–362.
- [136] J.-E. Pin and H. Straubing, Monoids of upper triangular matrices, in Semigroups: structure and universal algebraic problems, (G. Pollák, ed.), 1985, pp. 259–272, North-Holland.
- [137] J.-E. Pin and P. Weil, Profinite semigroups, Mal'cev products and identities, J. Algebra 182 (1996), 604–626.
- [138] Th. Place and M. Zeitoun, Separation for dot-depth two, Log. Methods Comput. Sci. 17 (2021), Paper No. 24, 42.
- [139] T. Place and M. Zeitoun, Dot-depth three, return of the *J*-class, in Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science, 2024, pp. 15, ACM, New York.
- [140] J. Reiterman, The Birkhoff theorem for finite algebras, Algebra Universalis 14 (1982), 1–10.
- [141] J. Rhodes, Kernel systems a global study of homomorphisms on finite semigroups, J. Algebra 49 (1977), 1–45.
- [142] J. Rhodes and B. Steinberg, Complexity pseudovarieties are not local; Type II subsemigroups can fall arbitrarily in complexity, Int. J. Algebra Comput. 16 (2006), 739–748.
- [143] _____. (2009). The q-theory of finite semigroups, Springer Monographs in Mathematics, Springer.
- [144] L. Ribes and P. A. Zalesskii, On the profinite topology on a free group, Bull. London Math. Soc. 25 (1993), 37–43.
- [145] _____, The pro-p topology of a free group and algorithmic problems in semigroups, Int. J. Algebra Comput. 4 (1994), 359–374.
- [146] F. M. Schneider and J. Zumbrägel, Profinite algebras and affine boundedness, Adv. in Math. 305 (2017), 661–681.
- [147] M. P. Schützenberger, On finite monoids having only trivial subgroups, Inform. and Control 8 (1965), 190–194.
- [148] I. Simon, Piecewise testable events, in Proc. 2nd GI Conf., Vol. 33, 1975, pp. 214– 222, Springer.
- [149] B. Steinberg. (1998). Decidability and hyperdecidability of joins of pseudovarieties, Ph.D. Thesis.
- [150] _____, On pointlike sets and joins of pseudovarieties, Int. J. Algebra Comput. 8 (1998), 203–231.
- [151] _____, Inevitable graphs and profinite topologies: some solutions to algorithmic problems in monoid and automata theory, stemming from group theory, Int. J. Algebra Comput. 11 (2001), 25–71.
- [152] _____, A note on the equation PH = J * H, Semigroup Forum 63 (2001), 469–474.
- [153] _____, On algorithmic problems for joins of pseudovarieties, Semigroup Forum 62 (2001), 1–40.
- [154] _____, On an assertion of J. Rhodes and the finite basis and finite vertex rank problems for pseudovarieties, J. Pure Appl. Algebra 186 (2004), 91–107.
- [155] _____. (2013). Topological dynamics and recognition of languages. https://doi.org/10.48550/arXiv.1306.1468.
- [156] M. Steinby, A theory of tree languages varieties, Univ. Turku, 1990.
- [157] P. Stiffler, Extension of the fundamental theorem of finite semigroups, Adv. in Math. 11 (1973), 159–209.

- [158] H. Straubing, Aperiodic homomorphisms and the concatenation product of recognizable sets, J. Pure Appl. Algebra 15 (1979), 319–327.
- [159] _____, Finite semigroup varieties of the form V * D, J. Pure Appl. Algebra **36** (1985), 53–94.
- [160] H. Straubing and P. Weil, Varieties, in Handbook of automata theory. Vol. I. Theoretical foundations, 2021, pp. 569–614, EMS Press, Berlin.
- [161] D. Thérien, Classification of finite monoids: the language approach, Theor. Comp. Sci. 14 (1981), 195–208.
- [162] D. Thérien and A. Weiss, Graph congruences and wreath products, J. Pure Appl. Algebra 36 (1985), 205–215.
- [163] W. Thomas, Classifying regular events in symbolic logic, J. Comput. System Sci. 25 (1982), 360–376.
- [164] B. Tilson, Categories as algebra: an essential ingredient in the theory of monoids, J. Pure Appl. Algebra 48 (1987), 83–198.
- [165] H. Urbat, J. Adámek, L.-T. Chen, and S. Milius, Eilenberg theorems for free, in 42nd International Symposium on Mathematical Foundations of Computer Science, Vol. 83, 2017, pp. Art. No. 43, 15, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern.
- [166] S. J. van Gool and B. Steinberg, Pro-aperiodic monoids via saturated models, Israel J. Math. 234 (2019), 451–498.
- [167] J. Volaříková, The omega-reducibility of pseudovarieties of ordered monoids representing low levels of concatenation hierarchies, Int. J. Algebra Comput. 34 (2024), 87–135.
- [168] S. Zhang, An infinite order operator on the lattice of varieties of completely regular semigroups, Algebra Universalis 35 (1996), 485–505.

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