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Higgs Bundles and Geometric Structures

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Para a minha avó

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Abstract

In this thesis, we use the theory of Higgs bundles as a tool to understand complex projective structures, i.e. maximal atlas of charts valued in the projective line and whose transition functions are Möbius transformations. The setup uses the celebrated non-abelian Hodge correspondence, mapping a polystable $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle on a compact Riemann surface X of genus $g \geq 2$ to a flat connection, which in some cases, is the holonomy of a branched hyperbolic structure. Together with Gaiotto's conformal limit, which maps the same bundle to a partial oper, i.e., to a connection whose holonomy is that of a branched complex projective structure compatible with X, these are two examples of already known constructions of connections corresponding to projective structures. Our main contribution is to show these are both instances of the same phenomenon: the family of connections appearing in the conformal limit can be understood as a family of complex projective structures, deforming the hyperbolic ones into the ones compatible with X. When the Higgs bundle has zero Toledo invariant, we also show that this deformation is optimal, inducing a geodesic for the Teichmüller (Finsler) metric on Teichmüller space. The proof makes essential use of a Beltrami differential that naturally comes up in the construction and for which we provide an explicit expression. We also give insight on the geometric significance of this differential, by relating it with the pull-back metric in X induced by the harmonic map associated to the Higgs bundle. To carry out the construction, we also give a new proof of the existence of the conformal limit for $\mathrm{SL}(2,\mathbb{C})$, which, contrary to previously known ones, works in the configuration space of polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. The argument used seems simpler than the ones found in the literature, relying only on a symmetry of the Hitchin equation. As further results, we provide a characterization of the Higgs bundles corresponding to elementary representations. On the expository side, we include proofs of some classical results about complex projective structures and the Schwarzian differential equation in contemporary language, and some non-standard descriptions of the geometry of the harmonic maps appearing in the theory of Higgs bundles.

Resumo

Nesta tese utilizamos a teoria de fibrados de Higgs como uma ferramenta para compreender as estruturas complexas projetivas, i.e. atlas maximais de cartas com valores na linha projetiva e cujas funções de transição são transformações de Möbius. Como enquadramento é utilizada a celebrada correspondência de Hodge não abeliana, que faz corresponder um fibrado de Higgs poliestável numa superfície de Riemann X de género $g \geq 2$ a uma conexão plana. Em conjunto com o limite conforme de Gaiotto, que envia o mesmo fibrado num oper parcial, i.e. numa conexão cuja holonomia provém de uma estrutura projetiva ramificada compatível com X, estes são dois exemplos de construções já conhecidas de conexões que correspondem a estruturas projetivas. A principal contribuição é uma demonstração de que estes dois exemplos são instâncias de um mesmo fenómeno: a família que surge no limite conforme pode ser vista como uma família de estruturas projetivas complexas que deforma as hiperbólicas naquelas que são compatíveis com X. Quando o fibrado de Higgs tem invariante de Toledo nulo, também mostramos que esta deformação é ótima, induzindo uma geodésica para a métrica (Finsler) de Teichmüller no espaço com o mesmo nome. A prova utiliza instrumentalmente um diferencial de Beltrami para o qual providenciamos uma expressão explícita. Também exploramos o seu significado geométrico, relacionando-o com a métrica em X induzida pelo mapa harmónico associado ao fibrado de Higgs. Para concretizar a construção, fazemos uso de uma nova prova da existência do limite conforme para $\mathrm{SL}(2,\mathbb{C})$, que incluímos e que, ao contrário das previamente conhecidas, é válida no espaço de configurações de fibrados de Higgs $\mathrm{SL}(2,\mathbb{C})$ poliestáveis. O argumento utilizado parece mais simples do que os presentes na literatura, recorrendo apenas a uma simetria das equações de Hitchin. Como resultados adicionais, apresentamos uma caracterização dos fibrados de Higgs correspondentes a representações elementares. Como parte expositiva, incluímos, em linguagem contemporânea, provas de alguns resultados clássicos sobre estruturas projetivas complexas e a equação diferencial de Schwarz, bem como uma descrição adaptada da geometria do mapa harmónico que surge na teoria de fibrados de Higgs.

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List of symbols

Roman Symbols

$\mathrm{Aff}(\mathbb{C})$ Complex affine group

$\mathbb{C}^*$ Non-zero complex numbers

$\mathbb{CP}^1$ Riemann sphere

End_0 Traceless endomorphisms

$\hbar$ Constant in $\mathbb{C}^*$

$\mathbb{H}^2$ Hyperbolic plane

$\mathbb{H}^3$ Hyperbolic space

$\mathrm{H}^0(E)$ Holomorphic sections of the bundle E

Hom^{cr} Completely reducible representations

$\mathrm{D}_\infty(\mathbb{CP}^1)$ Dihedral group

$\mathcal{A}$ Projective atlas

$\mathcal{B}(S)$ Branched projective structures on S

$\mathcal{B}(X)$ Branched structures compatible with X

$\mathcal{D}$ Positive definite hermitian matrices of unit determinant

$\mathcal{H}$ Horizontal foliation

$\mathcal{P}(S)$ Projective structures on S

$\mathcal{P}(X)$ Projective structures compatible with X

$\mathfrak{X}(\mathcal{D})$ Vector fields on $\mathcal{D}$

$\mathfrak{T}(S)$ Teichmüller space of S

$\mathrm{PSL}(2,\mathbb{C})$ Projective special linear group

$\mathrm{PSU}(2,\mathbb{C})$ Projective special unitary group

$\mathrm{S}(f)$ Schwarzian derivative of f

$\mathrm{SL}(n,\mathbb{C})$ Complex special linear group

$E \to X$ Vector bundle

H Hermitian metric or matrix

K Canonical bundle

$L \subset E$ Line subbundle

S Surface

$T\mathcal{D}$ Complexified tangent space

$W[u_1,u_2]$ Wronskian determinant

X Riemann surface

Greek Symbols

β_L Second fundamental form of L

$\Omega^{(j,k)}(E)$ Smooth (j,k)-forms on E

Γ Group of Möbius transformations

$\mathcal{X}^{\mathrm{SL}(2,\mathbb{C})}$ Character variety

μ Beltrami differential

Φ Higgs field

ϕ Endomorphism part of Higgs field

Φ^* Adjoint Higgs field

$\pi : \widetilde{S} \to S$ Universal cover

Ψ Real Higgs field

$\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ Representation

$\Omega^0(E)$ Smooth sections of the bundle E

Other Symbols

$\overline{\partial}_E$ Holomorphic structure on E

$\lrcorner Y$ Contraction with Y

∇ Flat connection

Introduction

The overall goal of this thesis is the study of geometric structures and their relation with Higgs bundles. More concretely, our objective is to explore how gauge theoretic techniques can be used to describe complex projective structures on closed surfaces.

The subject of complex projective structures is an old one. These geometric entities have been studied for the last 150 years, in several guises, reappearing in the literature every now and then, if only to be described with different languages. The starting point can be traced back to the classical theory of linear differential equations and to the work of Riemann, Schwarz, Poincaré, Picard, and other mathematicians at the end of the 19th century. Throughout the 20th century, a plethora of modern techniques were used with success to solve several central questions in the theory, providing a deeper and wider understanding of the mathematical landscape surrounding these objects. These techniques came from a variety of different fields, from algebraic geometry and the theory of connections to the theory of Kleinian groups and hyperbolic geometry. By the end of the century, most of the results were being obtained through topological cut-and-paste techniques and the analytical theory had been gradually abandoned. For example, this is what happens with the breakthrough result in (Gallo et al., 2000), at the beginning of our century.

Parallel to this development, gauge theory started to find relevant applications in the study of the geometry of manifolds. In the late '80s and throughout the '90s, after a seminal paper by Hitchin, and as a result of the work of several others, like Donaldson, Corlette, Simpson etc., the use of gauge theoretic techniques on the study of representations of the fundamental group of Kähler manifolds, and Riemann surfaces in particular, was widely diffused as the theory of Higgs bundles.

So, at the start of my PhD, and after three decades of successes of this theory, the interest of some mathematical communities in its applications had grown significantly. So it only seemed natural to try applying Higgs bundles to study geometric structures, and the result in (Gallo et al., 2000) provided a bridge to complex projective structures.

Motivation

The research presented here lies at the intersection of two mathematical subjects, the study of complex projective structures, and the theory of Higgs bundles.

A complex projective structure is a way to model a surface S onto the geometry of the complex projective line $\mathbb{CP}^1$. It is a maximal atlas $\mathcal{A}$ of $\mathbb{CP}^1$-valued charts, whose transition functions are Möbius transformations. In particular, it induces a structure of a Riemann surface X which is said to be compatible with $\mathcal{A}$. By analytic continuation of a chart along the surface, one can produce a map out of the universal cover $d : \widetilde{S} \to \mathbb{CP}^1$ called the developing map. This is a local diffeomorphism which is ρ-equivariant, where $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ is a representation of the fundamental group, called the holonomy of the structure. These objects were first studied as solutions of a linear differential equation of Schwarz, the holonomy corresponding to the classical monodromy of the equation. The main questions in the theory concern the geometric properties of the developing map, the classification of what kind of homomorphisms can appear as the holonomy, the moduli and Teichmüller spaces naturally associated, the topological properties of the holonomy mapping to the character variety, the uniqueness of the projective structure inducing a given complex one, and so on.

In the 60s, Gunning reframed the theory in terms of projective connections (Gunning, 1966, §9). This marks the beginning of the use of bundle and sheaf-cohomological techniques to obtain further information about these objects. The author describes a projective structure compatible with X as a holomorphic vector bundle together with a holomorphic subbundle of maximal degree, effectively realizing projective structures as maximally unstable holomorphic bundles (Gunning, 1967b, Theorem 2), a result which is referred in the text as Gunning's Theorem. To build this bundle, we note that the holonomy representation can be used to construct a flat projective bundle P, for which the developing map corresponds to a section s. The fact that it is a local diffeomorphism translates into s being transverse to the horizontal foliation of the bundle. Taking these objects into account, he showed one could actually find adequate lifts, producing a holomorphic flat vector bundle E together with a transverse line bundle L. Nonetheless, the theory progressed by different paths, and its later developments were more closely related to Teichmüller Theory and Kleinian groups. In terms of the characterization of holonomies this culminated with the celebrated work (Gallo et al., 2000), where the authors prove that, for a closed surface of genus $g \geq 2$, a representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ appears as the holonomy of a projective structure if and only if ρ is non-elementary and lifts to to $\mathrm{SL}(2, \mathbb{C})$, a result which had been conjectural for some decades. To prove this, they rely on Kleinian groups, in particular Schottky groups, to build a pants decomposition of the surface as determined by the representation ρ, and effectively building a complex projective structure with holonomy ρ and establishing this previously unsettled direction of the result.

In parallel with these developments, the theory of Higgs bundles, started in the seminal paper (Hitchin, 1987), and further developed by others, made great progress in the study of representations of the fundamental group into Lie groups, through the non-abelian Hodge correspondence. A Higgs bundle is a complex vector bundle E over a closed Riemann surface X of genus g with a holomorphic structure $\overline{\partial}_E$ and a holomorphic $\mathrm{End}(E)$-valued one form Φ called the Higgs field. Under some stability conditions, Hitchin (Hitchin, 1987) proved that a certain dimensional reduction of the Yang-Mills equations depending on Φ, now called

Hitchin equations, admitted a solution, the so-called harmonic metric. This metric can be used to build a flat connection out of the Higgs field. It established the bridge to the theory of representations. The fact that this construction could be reversed, a theorem in (Donaldson, 1987) and generalized in (Corlette, 1988), established the non-abelian Hodge correspondence, in the form of a homeomorphism between the moduli space of polystable Higgs bundles $\mathcal{M}_{Dol}$ and that of completely reducible flat connections $\mathcal{M}_{dR}$, i.e. the character variety of the fundamental group. This was further generalized to compact Kähler manifolds in Simpson (1992).

The theory of Higgs bundles has proven to be quite successful in providing information about these moduli spaces, leading to the discovery of special components which were previously unknown, called Hitchin components. The representations in these components share many interesting properties with the Fuchsian ones appearing in the theory of Riemann surfaces and were the starting point of Higher Teichmüller Theory (Wienhard, 2018).

Higgs bundles were also used by Hitchin to provide a new proof of the uniformization theorem. This proof used the solutions of the gauge theoretic equations to build all possible metrics of negative constant curvature on the closed surface S, (Hitchin, 1987, Theorem 11.2). These metrics came from the family of Higgs bundles with maximal Toledo invariant, parametrized by a quadratic differential $q \in \mathrm{H}^0(K^2)$, allowing to identify the Teichmüller space of the surface $\mathcal{T}(S)$ with the vector space $\mathrm{H}^0(K^2)$. Hitchin's parametrization of $\mathcal{T}(S)$ is indeed the first example of the use of Higgs bundles to study geometric structures.

For some time, the non-explicit nature of the non-abelian Hodge correspondence, made it difficult to generalize this approach to other geometric structures, but, more recently, a lot of work has been done in this direction. In (Biswas et al., 2021) the authors generalize the results of Hitchin to non-maximal Toledo invariant, showing that, under some conditions, these Higgs bundles correspond to branched hyperbolic structures. Several other authors have approached the use of Higgs bundles to build geometric structures such as the works (Baraglia, 2010; Collier, 2017; Collier and Toulisse, 2023; Labourie, 2007). Specifically for complex projective structures see (Alessandrini, 2019; Alessandrini et al., 2021). From a slightly different perspective, geometric structures have also appeared in other constructions related to the non-abelian Hodge correspondence. In relation to physics and the Thermodynamic Bethe Ansatz, Gaiotto defined his conformal limit in (Gaiotto, 2014). This involved taking a $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle in the Hitchin component, and defining a family of flat connections $\nabla_{\hbar,R}$, with fixed $\hbar \in \mathbb{C}^*$ and parametrized by $R \in \mathbb{R}^+$, starting in the flat connection given by the non-abelian Hodge correspondence, and obtained by combining the $\mathbb{C}^*$ action on the moduli space together with the twistor line construction. Gaiotto conjectured that the limit of this family when $R \to 0$ existed and it was an oper. Opers, defined using modern language in (Beilinson and Drinfeld, 2005), were already known to be equivalent to complex projective structures compatible with X when $n = 2$. This conjecture thus established a relation with geometric structures. The statement was proved to be true in (Dumitrescu et al., 2021). After that, the limit was extended to include all stable $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles and its existence was established on the moduli space in (Collier and Wentworth, 2018). In this article, it was also shown that the limit is a partial oper in the sense of (Simpson, 1992). Under some conditions, and for $n = 2$, this again corresponds to geometric structures, namely branched projective ones which are compatible with X. The research presented here aims at offering a tentative understanding of these two seemingly distinct appearances of projective structures, in the hope of using the conformal limit to shed light on the results in (Gallo et al., 2000), using gauge theoretic terms.

The main contribution is the realization of these two instances as a unified phenomenon. In particular, we show that, under some conditions, every connection in the family appearing in the conformal limit is the holonomy of a branched complex projective structure. These structures are compatible with Riemann surface structures X_μ determined by a Beltrami differential μ which we calculate. The main tool in the construction is the theorem of Gunning, used to identify the adequate transverse subbundles that define the structure. To achieve the results, we also provide a simpler and more general proof of the existence of the conformal limit for polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles, which works on the configuration space (and not only on $\mathcal{M}_{Dol}$) and makes a simple use of the symmetries of the equation, avoiding the use of the inverse function theorem for infinite dimensional Banach spaces, which was used in previous proofs of the existence of the limit. We also show that for Higgs bundles with minimal Toledo invariant, the construction has interesting geometric properties, inducing a geodesic in Teichmüller space when it is given the Teichmüller (Finsler) metric. The origin of the Beltrami differential is also explored, and we show that μ comes up naturally as the differential that provides conformal coordinates for the metric induced in X by the harmonic map associated with the Higgs bundle. As further results, we include a characterization of $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles corresponding to elementary representations.

Results and structure of the thesis

We start by reviewing some of the classical theory of projective structures in Chapter 1. This is an expository chapter where we collect and present proofs of some well-known results. We start by quickly reviewing complex projective geometry in $\mathbb{CP}^1$ and then we introduce the definition of complex projective structure as an atlas of $\mathbb{CP}^1$-valued charts with complex-projective transition functions. We give some examples and provide equivalent definitions using the notions of holonomy and development and a fiber bundle description. After that, we introduce certain subclasses of complex projective structures and recall the uniformization theorem to show that every Riemann surface X admits one such structure. Then we study the classical parameterization of complex projective structures compatible with X as solutions of the Schwarz differential equation. Before giving detailed accounts of structures on the torus and the sphere, we show that the representations that can come up as holonomies of projective structures lift to representations in the complex special linear group. We also study some restrictions on the representations that appear in the case of closed surfaces of genus $g \geq 2$. We proceed to a proof of a theorem of Poincaré that states that, precisely on these closed surfaces of genus $g \geq 2$, the holonomy and the compatible Riemann surface structure are enough to determine the projective one. We finally give a more algebraic-geometric equivalent description of complex projective structures using a theorem of Gunning and a holomorphic vector bundle over the surface. Finally, we close the chapter by reviewing branched projective structures and verifying whether the non-branched results do hold for these structures.

In Chapter 2 we provide further background on a diversity of topics related to Higgs bundles and opers. This chapter is expository, but fewer details are provided, given the accessibility of the proofs in the literature, and it sets the stage for the main results in chapter 3. We begin by recalling the non-abelian Hodge correspondence for polystable Higgs bundles. Then we go over the construction of Gaiotto's conformal limit. We proceed to recall the relation between partial opers and the branched projective structures of chapter 1. We finish by stating some classical results in Teichmüller theory, namely how one can use Beltrami differentials to change the complex structure on a Riemann surface.

Chapter 3 is the main chapter of the thesis where our primary contributions are presented. We give a new proof of the existence of the conformal limit in the context of $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles. This proof works in the configuration space and for polystable bundles, which means it generalizes (for rank 2) part of the results in (Collier and Wentworth, 2018; Dumitrescu et al., 2021). It relies simply on a symmetry of the Hitchin equation and dispenses the use of the inverse function theorem on infinite dimensional Banach spaces, as in those papers. Then we use the result of Gunning to show, again in the configuration space, that this conformal limit is a partial oper, i.e. a branched projective structure compatible with a base Riemann surface X. We proceed to the main construction, where, under some conditions, we build branched projective structures associated with the family of connections used to define the conformal limit. The instrumental tool is a Beltrami differential μ that appears naturally associated with the family of connections and which we calculate explicitly. The branched projective structures are then compatible with the complex structure determined by this μ. The special case of minimal Toledo invariant is approached separately, since the proofs are slightly different, even though the result still holds. We then provide an interpretation of the results, showing that the constructed family of projective structures interpolates between branched hyperbolic structures and branched projective structures compatible with X. In the case of minimal Toledo invariant, the projective structures constructed cover a (reparametrization of a) geodesic arc in the Teichmüller space with the Teichmüller (Finsler) metric. We finish the chapter by providing a generalization of our proof of the existence of the conformal limit to all polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. Apart from the last section, this chapter collects the results of the pre-print (Silva and Gothen, 2024).

We finish with Chapter 4, which is a mixed chapter, being both expository and with some new results. It was born out of a tentative understanding of (non)-elementary representations in terms of Higgs bundles and their geometry. So we begin by recalling some of the geometry of the harmonic map associated with $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles. We recall some known results about equivariant maps, the geometry of the associated homogeneous space, and the calculation of the (possibly degenerate) metric induced by the harmonic map on the surface X. The exposition is non-standard. We then provide a new explicit expression for the Beltrami differential that renders this metric conformal in general. We also present an expression for the second fundamental form of the harmonic map in terms of the Higgs bundle. We apply the developed results to the case of $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles showing that in this case, the Beltrami differential is exactly the one that appeared in the construction of Chapter 3. We finish by providing a characterization of the Higgs bundles associated to elementary representations.

References and further bibliography

Since the subject is profoundly vast, any kind of complete bibliography seems out of purpose to be included here, as it seems any historically detailed overview. We thus have tried to use as sources contemporary expositions in the form of survey articles or monographs, which not only try to approach the topics from a modern perspective but also include detailed bibliographies for further information. We cannot understate how the survey (Dumas, 2009), mainly in Chapter 1, the articles (Alessandrini, 2019; Hitchin, 1987), in Chapter 3, and the lecture notes (Li, 2019a), for Chapter 4, have impacted the work presented here.

Chapter 1

Complex projective structures

The original purpose of this chapter was to collect some classical results on complex projective structures since these are needed for the understanding of the main contributions in Chapter 2. The problem is that most of the proofs of these date back to Poincaré, Goursat, Picard and other mathematicians of the end of the nineteenth century, and, to the best of my knowledge, they were not updated to modern terms without a shift in the techniques used. So the goal of the chapter changed, from the simple presentation of the results to the writing up of the theory with some amount of detail. I tried to use techniques similar in spirit to the classical ones which means the solutions of Schwarzian differential equation take the main role as both objects of study and tools of geometric understanding. The literature on this subject is extremely vast, and it would be unreasonable to attempt a complete historical account here. Nonetheless, useful general references for this chapter are the excellent survey (Dumas, 2009, Chapters 1,2,3 and 5) and (De Saint-Gervais, 2016, Chapter IX) for the historical perspective. Since we try to keep the approach close to the theory of differential equations, we did not include here the beautiful topological techniques developed by Thurston, rooted in the previous notion of grafting. For details about this, we point to (Dumas, 2009, Chapter 4) and the references therein.

1.1 Complex projective geometry

The main objects of study in this thesis will be complex projective structures which are a way to transport the geometry of the complex projective line $\mathbb{CP}^1$ to more general surfaces. So we start by recalling some facts about this geometry. We let $\mathbb{CP}^1$ be the set of lines through the origin in $\mathbb{C}^2$ and consider on it the action of $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm Id\}$. This is the action covered by the linear action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathring{\mathbb{C}}^2 = \mathbb{C}^2 - \{(0,0)\}$, and we have an equivariant diagram

$$\begin{tikzcd} \mathrm{SL}(2,\mathbb{C}) \arrow[d, "\mathrm{P}"] & \circlearrowleft & \mathring{\mathbb{C}}^2 \arrow[d, "\mathbb{P}"] \\ \mathrm{PSL}(2,\mathbb{C}) & \circlearrowleft & \mathbb{CP}^1 \end{tikzcd}$$

where $\mathbb{P} : \mathring{\mathbb{C}}^2 \to \mathbb{CP}^1$ sends a point to the line it determines, and $\mathrm{P} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ is the quotient map. Nevertheless, we will mostly see $\mathbb{CP}^1$ as the Riemann sphere, i.e., the extended plane $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. The action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{CP}^1$ reads as the action of the group of Möbius transformations on $\mathbb{C} \cup \{\infty\}$. The group isomorphism sends a matrix $\pm\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to its Möbius representative, the extended complex function $\frac{az+b}{cz+d}$. Note that the arithmetic is extended to infinity as usual, and this function maps ∞ to $\frac{a}{c}$ and attains ∞ when $z = -\frac{d}{c}$. We will make no distinction between these two isomorphic groups and their elements. The action of $\mathrm{PSL}(2,\mathbb{C})$ is sharply 3-transitive, meaning that there is a single Möbius transformation taking any triple of distinct points in $\mathbb{CP}^1$ to any other such triple. In particular, it follows that if two Möbius transformations agree on three points, then they are identical. This is a much stronger condition from which the rigidity of this action follows. That is, Möbius transformations are determined by their action on any open set. The geometry of $\mathbb{CP}^1$ with the action of $\mathrm{PSL}(2,\mathbb{C})$ is an example of a non-metric geometry.

Remark 1.1.1. Indeed, if we consider the stabilizer of a point, say $\infty \in \mathbb{CP}^1$, we find $\mathrm{Stab}(\infty) = \left\{\frac{az+b}{cz+d} \in \mathrm{PSL}(2,\mathbb{C}) \middle| \, c = 0\right\} = \mathrm{Aff}(\mathbb{C})$, where $\mathrm{Aff}(\mathbb{C})$ is the group of $\mathbb{C}$-affine transformations of the form $az + b$. This means $\mathbb{CP}^1$ is a Homogeneous space for $\mathrm{PSL}(2,\mathbb{C})$ identified with $\mathrm{PSL}(2,\mathbb{C})/\mathrm{Aff}(\mathbb{C})$. In particular, $\mathrm{Aff}(\mathbb{C}) = \mathbb{C}^* \ltimes \mathbb{C}$ is non-compact. It is possible to show that there is no invariant Riemannian metric on this homogeneous space.

Even though the geometry is not metric, $\mathbb{CP}^1$ can be given the spherical metric. The Möbius transformations $g = \frac{az+b}{cz+d}$ that preserve this metric are precisely the ones in $\mathrm{PSU}(2) \subset \mathrm{PSL}(2,\mathbb{C})$, i.e the ones with norm $||g|| = 2$, where $||g||^2 = \frac{a^2+b^2+c^2+d^2}{ad-bc}$, see (Beardon, 1983, Theorem 4.2.2).

1.2 Definitions

The idea of a complex projective structure is then to transfer the geometry of the complex projective line to a surface S (Hausdorff second-countable topological 2-manifold).

Definition 1.2.1. Let S be a surface. A *complex projective chart* is a map $\varphi_\alpha : U_\alpha \to \mathbb{CP}^1$ which is a homeomorphism of an open set U_α of S onto its image. A *complex projective atlas* is a collection $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{CP}^1\}_{\alpha \in A}$ of complex projective charts such that the transition functions

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta).$$

are Möbius transformations (one for each connected component of $\varphi_\alpha(U_\alpha \cap U_\beta)$). The set of complex projective atlas is ordered by inclusion, and a *complex projective structure* is a maximal complex projective atlas for this ordering.

Remark 1.2.2. This is an example of a (G,X)-manifold of Thurston (Thurston, 1997, 3.3, page 125) where the model space is the projective line $\mathrm{X} = \mathbb{CP}^1$ and the group of transformations is $\mathrm{G} = \mathrm{PSL}(2, \mathbb{C})$. For this reason, we also call the maximal atlas $\mathcal{A}$ a $\mathbb{CP}^1$*-structure* and S together with it a $\mathbb{CP}^1$*-surface*. The constructions in this chapter that do not depend on specific properties of complex projective geometry can be carried out for general (G,X)-manifolds.

This definition is similar to that of the complex structure on a Riemann surface, i.e. a maximal complex atlas whose transition functions are holomorphic. In fact, given a complex projective structure $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{CP}^1\}_{\alpha \in A}$, we can give S a structure X of a Riemann surface, simply by composing the charts φ_α with the complex-manifold-charts of $\mathbb{CP}^1$ and adequately restricting their domains. The maps so obtained are $\mathbb{C}$-valued charts and, since Möbius transformations are holomorphic, i.e. $PSL(2, \mathbb{C})$ acts holomorphically on $\mathbb{CP}^1$, the collection yields a complex structure X in the usual sense. We write that X is *induced* by $\mathcal{A}$ or that $\mathcal{A}$ is *compatible* with X. We denote by $\mathcal{P}(S)$ the set of complex projective structures on S and by $\mathcal{P}(X)$ the set of the ones compatible with X.

Note that a similar reasoning holds for an induced smooth structure, but since smooth structures are unique on surfaces, this point won't play an important role in what follows, and we always assume S has been given this smooth structure.

One can gather the complex projective structures in a category whose morphisms are the complex projective maps.

Definition 1.2.3. A *complex projective map* between $\mathbb{CP}^1$-surfaces $F : (S, \mathcal{A}) \to (S', \mathcal{A}')$ is a map which is locally in $\mathrm{PSL}(2, \mathbb{C})$, i.e. such that every $x \in S$ has a connected open neighborhood U where the local representation of F coincides with a Möbius transformation $g \in \mathrm{PSL}(2, \mathbb{C})$:

$$\varphi' \circ F|_U \circ \varphi^{-1} = g|_{(\varphi(U))}$$

where φ and φ' are charts in $\mathcal{A}$ and $\mathcal{A}'$. An *isomorphism* is a complex projective map $F : (S, \mathcal{A}) \to (S, \mathcal{A}')$, on the same surface S, whose inverse is also complex projective.

Remark 1.2.4. Note, firstly, that the composition of complex projective maps is still complex projective because the local representations of the composition are just the composition of the local representations. Observe, secondly, that a complex projective map is automatically a local biholomorphism for the induced complex structures since, locally, it looks like a Möbius transformation. Further, an isomorphism of $\mathbb{CP}^1$-structures is a biholomorphism of the induced complex structures. In particular, it is a diffeomorphism.

Before proceeding to concrete examples, let us observe that we can use local homeomorphisms to produce new complex projective structures from old ones, as is typical when working with atlases.

Remark 1.2.5. Given a local homeomorphism $F : S' \to S$ if S has a complex projective structure $\mathcal{A}$ then we can give S' a complex projective structure $F^*\mathcal{A}$ whose charts are the pull-backs of the charts in $\mathcal{A}$ under the restrictions of F to neighborhoods where it is a homeomorphism. The transition functions for $F^*\mathcal{A}$ match the ones in $\mathcal{A}$, and so $F^*\mathcal{A}$ is a complex projective structure, and it is the only one such that F becomes a complex projective map. Given a map $f' : S' \to S'$ which covers $f : S \to S$ with respect to F we have further that f' is complex projective for $F^*\mathcal{A}$ if and only if f is complex projective for $\mathcal{A}$.

Remark 1.2.6. The previous construction has a kind of reverse analog, the push-forward. If instead S' is given a $\mathbb{CP}^1$-structure $\mathcal{A}'$ and S is a quotient of S' by a (discrete and fixed-point free) group of complex projective isomorphisms Γ whose quotient map is $F : S' \to S'/\Gamma = S$ then we can give S a complex projective structure $F_*\mathcal{A}'$. This atlas is built as before by pull-back but now using the inverses of restrictions of F to neighborhoods where it is a homeomorphism. It is a complex projective atlas because the transition functions in $F_*\mathcal{A}'$ are either the ones in $\mathcal{A}'$ or new ones built from composing these with local representations of elements of Γ which are Möbius transformations. With this structure, F becomes a complex projective map. We also have the equality of structures $F^*F_*\mathcal{A}' = \mathcal{A}'$.

Example 1.2.7. The simplest example of a $\mathbb{CP}^1$-surface is of course $\mathbb{CP}^1$. An atlas for this is $\{id : \mathbb{CP}^1 \to \mathbb{CP}^1\}$, which is trivially a complex projective atlas. Another possible atlas is made of the non-homogeneous coordinate charts of the affine patches of $\mathbb{CP}^1$. The usual one has two charts $\varphi_1([z : w]) = \frac{w}{z}$ and $\varphi_2([z : w]) = \frac{z}{w}$, and it is a complex projective atlas because the transition function is the Möbius transformation $z \mapsto \frac{1}{z}$. All of these define the same maximal atlas $\mathcal{A}$ called the *standard complex projective structure* on $\mathbb{CP}^1$.

Example 1.2.8. The complex plane $\mathbb{C}$, and in general any open subset U of $\mathbb{CP}^1$, have complex projective structures whose atlas has a single chart, the inclusion $\iota : U \hookrightarrow \mathbb{CP}^1$.

Example 1.2.9. A complex torus T is a quotient of $\mathbb{C}$ by a lattice $u\mathbb{Z} \oplus v\mathbb{Z}$, with $u, v \in \mathbb{C}$ linearly independent over $\mathbb{R}$. When u and v become linearly dependent over R, but non-zero, the lattice degenerates and the quotient becomes a complex cylinder C. Since the action of $u\mathbb{Z} \oplus v\mathbb{Z}$ is made by translations which are Möbius transformations, and thus complex projective isomorphisms for the standard $\mathbb{CP}^1$-structure on $\mathbb{C}$, we are in conditions of Remark 1.2.6, and the quotient manifolds T and C have induce projective structures.

Example 1.2.10. Another construction of complex projective structures on the Torus comes from considering the quotient T^H_λ of $\mathbb{C}^*$ by the action of a single loxodromic Möbius transformation of the form $z \mapsto \lambda z, \lambda \in \mathbb{C}^*$. Since this a complex projective isomorphism of $\mathbb{C}^*$, the quotient has a $\mathbb{CP}^1$-structure. To see this is a torus note that the transformation maps the circle of radius 1 to the circle of radius $|\lambda|$, by expanding or contracting and twisting. The closed annulus between the two circles is a fundamental domain of the action, whose boundaries are thus glued into a torus.

Example 1.2.11. Another construction using circles is that of Schottky $\mathbb{CP}^1$-structures. To build them we begin with g pairs of circles $\{(C_j, C'_j)\}_{j=1,\ldots,g}$ inside $\mathbb{C} \subset \mathbb{CP}^1$ and all disjoint. Then we find g Möbius transformations $\{T_j\}_{j=1,\ldots,g}$ each of which maps the exterior of circle C_j to the interior of C'_j. One can see that the outside of all circles in $\mathbb{CP}^1_1$ gives a fundamental domain for the action of group Γ generated by $\{T_j\}_{j=1,\ldots,g}$. The quotient of this fundamental domain by Γ_S gives a compact surface that carries a complex projective structure by the quotient construction, the *Schottky* $\mathbb{CP}^1$*-structures*. The identifications work in such a way that the circle C_j gets identified with C'_j. This means that when passing to the quotient g handles form and these are in fact structures on closed surfaces of genus g. See Figure 1.2.11.

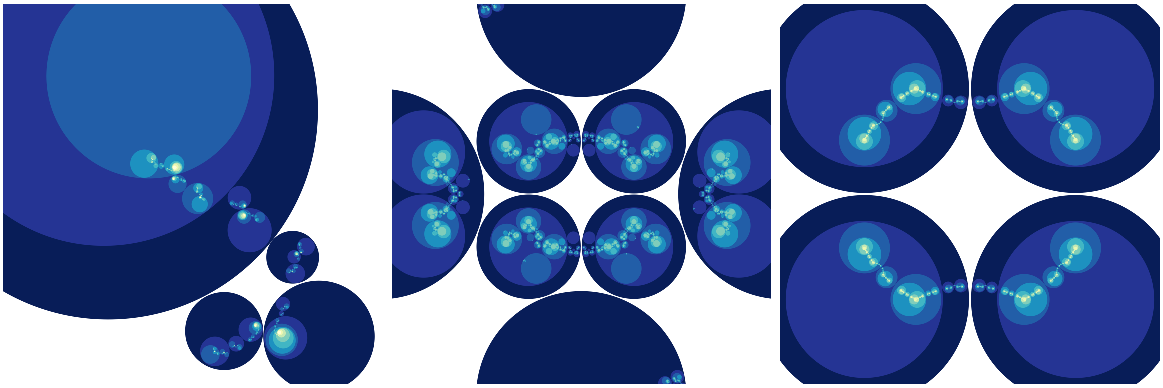

Fig. 1.1 A Schottky group: Representation of the fundamental domain of some Schottky groups. This is the white complement of the large circles. These circles are swapped under the action of the group and correspond to its generators. So both the left and right frames depict a Schottky group with two generators, and the middle one, a group with three generators. The successive images of the circles are represented by lighter and lighter circles. The bright Cantor dust is the limit set. These images were produced using the algorithms in the beautiful book (Mumford et al., 2002, Chapter 5). We include the generators of the group in Appendix A.2.

Example 1.2.12. The theorem of uniformization provides an identification of a compact Riemann surface of genus $g \geq 2$ with a quotient of $\mathbb{H}^2$ by a *Fuchsian group* Γ_F, i.e. a discrete and fixed point free group of $\mathrm{PSL}(2,\mathbb{R})$. This identification in fact yields a quotient complex projective structure $\mathbb{H}^2/\Gamma_F$.

Example 1.2.13. Deforming slightly the generators of a Fuchsian group one can obtain a group Γ_{QF} whose limit set is a Jordan curve. These are the *Quasi-Fuchsian* complex projective structures defined on a closed surface as the quotient structure of a fundamental domain by Γ_{QF}. See Figure 1.2.

Example 1.2.14. All these are examples of the general quotient construction U/Γ where a group of Möbius transformations Γ acts freely and properly discontinuously on an open subset $U \subset \mathbb{CP}^1$ without fixed points.

There are several equivalent descriptions of complex projective structures which we study in the next sections.

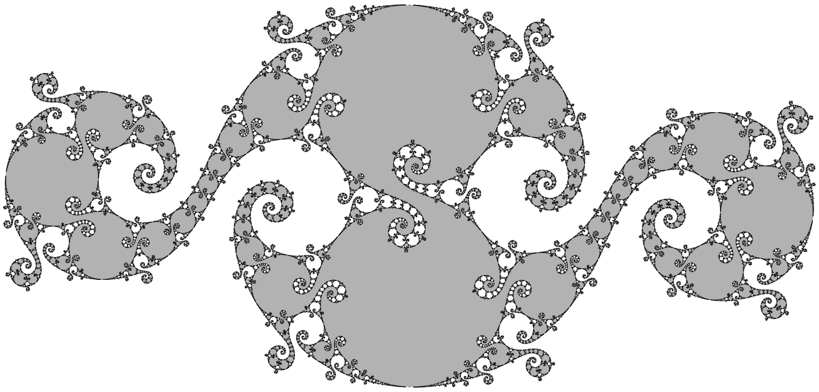

Fig. 1.2 A Quasi-Fuchsian group: Graphical depiction of the fundamental domain of a Quasi-Fuchsian group (in gray). The limit set (in black) is a Jordan curve with fractal-like features. This image was produced using the algorithms in (Mumford et al., 2002, Chapter 8), and we give the generators of the group in Appendix A.2.

1.3 Holonomy and development

Given a complex projective structure, one can produce, via analytic continuation of a single chart, a map defined on the universal cover $\widetilde{S}$. This map is called the *developing map* and it carries all the information of the structure. The construction is detailed in this section. We begin by noting the following which is a result of the rigidity of the action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{CP}^1$.

Lemma 1.3.1. *Let S be a simply connected $\mathbb{CP}^1$-surface. Then there is a complex projective map $d : S \to \mathbb{CP}^1$ and it is unique up to the action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{CP}^1$, i.e., given any other complex projective map d' we have $d' = g \circ d$ for some $g \in \mathrm{PSL}(2,\mathbb{C})$.*

Sketch of Proof. The map d is simply the analytic continuation of any chart of the projective atlas. Explicitly, we consider a point $x \in U \subset S$ and a chart $\varphi : U \to \mathbb{CP}^1$ around it. For any other point $y \in S$ we consider a path γ_y starting in x and ending in y. We cover γ_y by a finite ordered chain of simply connected open sets $U_j, j = 0, 1, 2, ...n$, which are domains of charts φ_j and such that they intersect in pairs, that is $U_{j-1,j} = U_{j-1} \cap U_j \neq \emptyset$, as in the picture. We

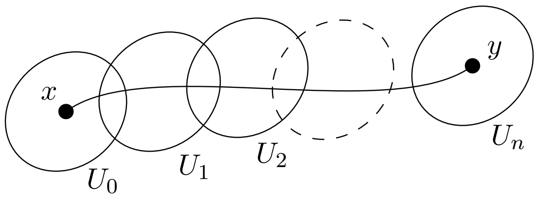

Fig. 1.3 An ordered chain covering the path γ_y from $x \in U = U_0$ to $y \in U_n$.

consider $U_0 = U$ and $\varphi_0 = \varphi$ the starting chart to be the first element of the chain and the last element U_n to contain y. Let us denote by $g_{j-1,j} = \varphi_{j-1} \circ \varphi_j^{-1}$ the transition functions defined in $U_{j-1,j}$ from the coordinates φ_j to coordinates φ_{j-1}. We define the map d as

$$d(y) = g_{0,1} \circ g_{1,2} \circ \cdots \circ g_{n-2n-1} \circ g_{n-1,n} \circ \varphi_n(y).$$

This is basically transporting the coordinate of y rigidly to the same $\mathbb{CP}^1$ as the one in the codomain of the starting chart φ. One can show (Ratcliffe, 2019, §8.4), using the fact that Möbius transformations are defined by their effect on an open set, that this map depends only on the chosen initial chart and not on the hotomopy class of γ. Since the surface S is simply connected this shows that the map is well-defined. Further it is directly a complex projective map, since around each point y the local representation is simply the composition $g_{0,1} \circ g_{1,2} \circ \cdots \circ g_{n-2n-1} \circ g_{n-1,n}$, which is a Möbius transformation. This shows how to build the map d. Of course, changing the starting chart for another one in the atlas would change $g_{0,1}$ by a composition with a Möbius transformation g, and thus d would change to $g \circ d$. Now for the uniqueness. Take any two complex projective maps $d_1 : S \to \mathbb{CP}^1$ and $d_2 : S \to \mathbb{CP}^1$. Since they are locally in $\mathrm{PSL}(2, \mathbb{C})$, there is a simply connected open set $U \neq \emptyset$ where both are given in coordinates by Möbius transformations, i.e. there is a chart $\varphi : U \to \mathbb{CP}^1$ such that $d_1 \circ \varphi^{-1} = g_1$ and $d_2 \circ \varphi^{-1} = g_2$, setting $g = g_2 g_1^{-1}$, this implies that $d_2 \circ \varphi^{-1} = g \circ d_1 \circ \varphi^{-1}$, and since φ is a homeomorphism, $d_2 = g \circ d_1$. Now we consider the set S' of points of S where this equality holds. This is non-empty since $U \subset S'$ and closed since it is defined by a closed condition. (Just take any sequence converging to x in the closure $\overline{S'}$ and take the limit of the equality to check it still holds on $\overline{S'}$, since all functions involved are continuous). It is also true that S' is open for we can reverse the thought process above. Indeed, if the equality holds in a point x we can pass to coordinates in a connected open set U around it, and, since d_1 and d_2 are complex projective maps, they look like (constant) Möbius g_1 and g_2 in U. Since $d_2 = g \circ d_1$ at x, we conclude that $g_1 = g g_2$ at x, but also at U, for g_1 and g_2 are constant there. Since S is connected $S' = S$ and the uniqueness statement follows. □

Definition 1.3.2. We call any such map d a *developing map* of the simply connected $\mathbb{CP}^1$-surface S.

Remark 1.3.3. Note that it follows from the proof we can find around each point x of S a simply connected open neighborhood U where a developing map d agrees with a chart of the atlas. This can also be seen as a corollary since both the restriction of d to U and a chart with domain U are complex projective maps to $\mathbb{CP}^1$ out of a simply connected $\mathbb{CP}^1$-surface, and thus unique up to Möbius transformation.

Remark 1.3.4. Note further that the construction can be carried out using a non-maximal atlas. In particular, if the atlas on S has transition functions valued in some subgroup $\Gamma \subset \mathrm{PSL}(2, \mathbb{C})$, then the uniqueness can be stated up to the action of this subgroup Γ.

So to a simply connected $\mathbb{CP}^1$-surface S we can associate a map from S to $\mathbb{CP}^1$ and this map is essentially unique. If S is not simply connected we can consider its universal covering map $\pi : \widetilde{S} \to S$. The universal cover $\widetilde{S}$ has a unique $\mathbb{CP}^1$-structure which makes π into a complex projective map (Remark 1.2.5). We can consider then a developing map d of this simply connected $\mathbb{CP}^1$-surface. This map preserves the information about the original surface S by behaving adequately with respect to the action of the fundamental group $\pi_1(S)$ seen as a covering group of $\pi : \widetilde{S} \to S$. In fact, given $\gamma \in \pi_1(S)$, since $\gamma : \widetilde{S} \to \widetilde{S}$ covers the identity of S which is a complex projective map, we conclude that γ is also a projective map (cf. Remark 1.2.5). This means that $d \circ \gamma$ is also a projective map to $\mathbb{CP}^1$. By the uniqueness of developing maps in Lemma 1.3.1, this map must be $g_\gamma \circ d$ for some $g_\gamma \in \mathrm{PSL}(2,\mathbb{C})$. So $d \circ \gamma = g_\gamma \circ d$, and for each $\gamma \in \pi_1(S)$, we get an element $g_\gamma \in \mathrm{PSL}(2,\mathbb{C})$, and thus a map $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ given by $\gamma \mapsto g_\gamma$. This map is a homomorphism of groups since $d \circ \gamma_1 \circ \gamma_2 = g_{\gamma_1} \circ d \circ \gamma_2 = g_{\gamma_1} \circ g_{\gamma_2} \circ d$ and so $g_{\gamma_1 \circ \gamma_2} = g_{\gamma_1} \circ g_{\gamma_2}$. We call ρ the *holonomy representation for* d . If instead of d we had picked a different developing map to begin with, it would be of the form $g \circ d$ for some $g \in \mathrm{PSL}(2,\mathbb{C})$ and it would satisfy $g \circ d \circ \gamma = g \circ g_\gamma \circ d = g \circ g_\gamma \circ g^{-1} \circ g \circ d$. This means the holonomy representation for $g \circ d$ is simply the same as for d but left-conjugated by g. The conclusion is that, as the developing map is only defined up to the action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{CP}^1$, the holonomy representation is only defined up to the action of $\mathrm{PSL}(2,\mathbb{C})$ by conjugation.

Definition 1.3.5. Let S be a $\mathbb{CP}^1$-surface and give its universal cover $\pi : \widetilde{S} \to S$ the pull-back projective structure. The complex projective map defined above $d : \widetilde{S} \to \mathbb{CP}^1$, unique up to the action of $\mathrm{PSL}(2,\mathbb{C})$, is called the *developing map* of the $\mathbb{CP}^1$-surface S. It is equivariant with respect to a representation $\rho : \pi_1(\widetilde{S}) \to \mathbb{CP}^1$, unique up to conjugation by G, called the *holonomy* of the $\mathbb{CP}^1$-surface S. Equivariance means that d and ρ satisfy

$$d \circ \gamma = \rho(\gamma) \circ d, \tag{1.3.1}$$

for every $\gamma \in \pi_1(S)$. The pair (d, ρ) is called the *development pair* of the $\mathbb{CP}^1$-surface.

$$\begin{tikzcd}
\widetilde{S} \arrow[loop left, "\gamma\in\pi_1(S)"] \arrow[r] \arrow[d] & \mathbb{CP}^1 \arrow[loop right, "\rho(\gamma)\in\mathrm{PSL}(2,\mathbb{C})"] \\
S &
\end{tikzcd}$$

Remark 1.3.6. Using Remarks 1.3.3 and 1.3.4 we note that, since each developing map d built agrees with some chart of the structure, any two such maps differ by some g which is a transition function of the atlas. This implies, in particular, that the elements g_γ used to build the holonomy homomorphism are also in the group of transition functions of the atlas. We conclude that, if there is an atlas with transition functions in a subgroup $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ then the holonomy can be chosen as a representation $\rho : \pi_1(S) \to \Gamma$. The converse is not true, namely it is possible for the representation ρ to have image in Γ without one being able to find an atlas where all transition functions are in Γ. This is intuitive since the representation only sees some of the transformations in the atlas corresponding to global deck transformations $\gamma \in \pi_1(S)$ and not all the local ones.

Remark 1.3.7. The uniqueness statement translates into uniqueness up to the action of $\mathrm{PSL}(2, \mathbb{C})$ on pairs given by

$$g \cdot (d, \rho) = (g \circ d, g\rho g^{-1}). \tag{1.3.2}$$

So the development pair is properly defined as an equivalence class of pairs under this action. We will always assume a representative has been picked and thus refer to *the* development pair. Further, the equivariance guarantees that the image-group $\rho(\pi_1(S))$ always preserves the image of d.

Example 1.3.8. When the complex structure is a quotient $S \cong U/\Gamma$, where $U \subset \mathbb{CP}^1$ is a simply connected open set and Γ a group of complex projective isomorphisms, the development map is identified with the inclusion $U \iota \mathbb{CP}^1$ and the holonomy representation is simply the identification $\pi_1(S) \cong \Gamma$. Even when U is not simply connected, it's still true that the developing map can be identified with the inclusion, possibly precomposed with a covering. This covering can make elements of $\pi_1(S)$ act trivially, and the representation is no longer faithful. This is what happens for Schottky groups, where half of the generators of the group act trivially. It is still true that the image of the developing map can be identified with the maximal domain of discontinuity, i.e. the complement of the limit set, which is the light Cantor dust in Figure 1.2.11.

The development pair retains all the information about the projective structure.

Theorem 1.3.9. *Let S be a surface and $\pi : \widetilde{S} \to S$ its universal cover. If $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C}))$ is a group-homomorphism and $d : \widetilde{S} \to \mathbb{CP}^1$ a local homeomorphism which is ρ-equivariant, i.e. $d \circ \gamma = \rho(\gamma) \circ d$, then there is a complex projective structure $\mathcal{A}$ on S whose development pair is (d, ρ).*

Proof. Define an atlas on the universal cover $\widetilde{\mathcal{A}}$ by pull-back by d of the standard structure on $\mathbb{CP}^1$. Since d is equivariant, the local representation of any $\gamma \in \pi_1(S)$ using this atlas is simply $\rho(\gamma)$ (recall that the pull-back charts are local restrictions of d). So every γ is a complex projective map. Thus the discrete and fixed-point free action of $\pi_1(S)$ on $\widetilde{S}$ is made by projective maps and thus descends to the quotient (Remark 1.2.6). The developing map of this structure can be taken to be d by the uniqueness of the developing map. The holonomy must then be ρ, for if d is ρ'-equivariant there are small open sets where d is invertible and thus $\rho(\gamma) = d \circ \gamma \circ d^{-1} = \rho'(\gamma)$ for every γ. □

This result shows that complex projective structures admit a more algebraic description as the set of equivalence classes of holonomy pairs.

Remark 1.3.10. Observe also, that if S is given a complex structure X, then the complex projective structure determined by the developing pair (ρ, d) is compatible with X if and only if d is a holomorphic map. (Where $\widetilde{S}$ is given the unique induced complex structure $\widetilde{X}$.)

1.4 Fiber bundle description

In this section, we introduce a third equivalent description of a complex projective structure $\mathcal{A}$ as a smooth $\mathbb{CP}^1$-bundle over S together with a transverse section. The construction makes use of the representation ρ in the developing pair (ρ, d) for $\mathcal{A}$. Taking this homomorphism one can build an action of $\pi_1(S)$ on $\widetilde{S} \times \mathbb{CP}^1$ as

$$\pi_1(S) \circlearrowleft \widetilde{S} \times \mathbb{CP}^1 \qquad\qquad \gamma \cdot (\widetilde{x}, u) = (\gamma(\widetilde{x}), \rho(\gamma)(u))\,.$$

This action is properly discontinuous and the quotient $P = \widetilde{S} \times_\rho \mathbb{CP}^1 = \frac{\widetilde{S}\times\mathbb{CP}^1}{\pi_1(S)}$ has a structure of a fiber bundle over S, with fiber $\mathbb{CP}^1$. Another way to see this is to consider the universal cover $\widetilde{S} \to S$ as principal $\pi_1(S)$-bundle. Then P is just the associated bundle with fiber $\mathbb{CP}^1$ where $\pi_1(S)$ acts using ρ. The projection on the first factor $\mathrm{proj}_1 : \widetilde{S} \times \mathbb{CP}^1 \to \widetilde{S} \xrightarrow{\pi} S$ descends to the quotient yielding the projection map $p : P \to S$ of the bundle. For details see for instance (Tu, 2017, §31.1). This bundle is in fact a flat fiber bundle, in the sense that each horizontal leaf $\widetilde{S} \times \{u\}$ glues in the quotient, producing a leaf of a foliation $\mathcal{H}$ of the total space P, which is complementary to the vertical foliation $\mathcal{V}$, whose leaves are $p^{-1}(x)$. The developing map then corresponds to a section $s : S \to P$ defined by its graph $s(x) = [(\widetilde{x}, d(\widetilde{x}))]$, where $\widetilde{x}$ is any lift of $x \in S$ to the universal cover. The map d is a local diffeomorphism precisely when s is everywhere transverse to the horizontal foliation $\mathcal{H}$. We call P the *flat* $\mathbb{CP}^1$*-bundle associated* to the structure and s its *transverse section*. The pair (P, s) is called the graph of the complex projective structure. Note that if we had picked a different representative $g \circ d$ for the developing map and $g\rho g^{-1}$ for the holonomy, we would have obtained another flat $\mathbb{CP}^1$-bundle, namely $P' = \widetilde{S} \times_{g\rho g^{-1}} \mathbb{CP}^1$, and another section, given by $s'(x) = [(\widetilde{x}, g \circ d(\widetilde{x}))]$. In this case, the element g determines an isomorphism of bundles $\mathsf{F} : P \to P'$ given by $[(\widetilde{x}, u)] \mapsto [(\widetilde{x}, g(u))]$. This isomorphism maps the horizontal foliation of P to the one of P' and the transverse section s to s'. When such an isomorphism exists we say that the two graphs (P, s) and (P', s') are *equivalent*. In conclusion, as the development pair of a structure is only uniquely determined up to the action of $\mathrm{PSL}(2, \mathbb{C})$, the graph of the structure is only uniquely determined up to equivalence.

The graph retains the complete information about the structure.

Theorem 1.4.1. *Let S be a surface. If P is a flat $\mathbb{CP}^1$-bundle over S with horizontal foliation $\mathcal{H}$, and s a section everywhere transverse to $\mathcal{H}$ then there is a complex projective structure $\mathcal{A}$ on S whose graph is (P, s).*

Proof. To prove this we recall that the parallel transport establishes an isomorphism F between any $\mathbb{CP}^1$-flat bundle P with holonomy ρ and the bundle of the form $\widetilde{S} \times_\rho \mathbb{CP}^1$, with its natural horizontal foliation. To this see for example (Kolář et al., 1993, Chapter III - 10. and 10.12). This means that a section s is of mapped to one of the form $x \mapsto [\widetilde{x}, \widetilde{s}(\widetilde{x})]$, where $\widetilde{s} : \widetilde{S} \to \mathbb{CP}^1$ is a ρ-equivariant map. The section is transverse if and only if $\widetilde{s}$ is a local diffeomorphism. Thus (ρ, d) is a developing pair of a complex projective structure. Its graph is precisely $\widetilde{S} \times_\rho \mathbb{CP}^1$, together with the section $x \mapsto [\widetilde{x}, \widetilde{s}(\widetilde{x})]$, which under the isomorphism F^{-1} yields the equivalent graph (P, s). $\square$

This means that complex projective structures have a third interpretation as flat $\mathbb{CP}^1$-bundles together with a transverse section.

Remark 1.4.2. Observe that again, if S is given a complex structure X, then the flat bundle P has an induced complex structure. The projective structure induced by the transverse section s is compatible with X if and only if s is holomorphic with respect to these complex structures.

1.5 Affine, dihedral and hyperbolic $\mathbb{CP}^1$-surfaces

Armed with the previous equivalent definitions of $\mathbb{CP}^1$-surfaces we are in position to study their geometry. We begin by introducing special types of complex projective structures distinguished by the geometry of $\mathbb{CP}^1$. Recall that the special unitary group $\mathrm{SU}(2)$ injects in $\mathrm{SL}(2, \mathbb{C})$ as the matrices of norm 2. The embeddings $\mathbb{C} \hookrightarrow \mathbb{CP}^1$ and $\mathbb{H}^2 \to \mathbb{CP}^1$ determine special subgroups of $\mathrm{PSL}(2, \mathbb{C})$, namely the set of those elements that fix the image.

Definition 1.5.1. We define the following subgroups of $\mathrm{PSL}(2, \mathbb{C})$.

(I) $\mathrm{PSU}(2, \mathbb{C})$ – the *projective special unitary* group – the image of $SU(2)$ in $\mathrm{PSL}(2, \mathbb{C})$ or, equivalently, Möbius transformations of norm equal to 2;

(II) $\mathrm{Aff}(\mathbb{C})$ – the *affine* group – elements of $\mathrm{PSL}(2, \mathbb{C})$ that preserve $\mathbb{C} \hookrightarrow \mathbb{CP}^1$ or, equivalently, Möbius transformations of the form $z \mapsto az + b$, with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$;

(III) $\mathrm{D}_\infty(\mathbb{CP}^1)$ – the *dihedral* group – complex numbers $a \in \mathbb{C}^*$ embedded as diagonal or anti-diagonal elements of $\mathrm{PSL}(2, \mathbb{C})$ or, equivalentely, Möbius transformations of the form $z \mapsto az$ or $z \mapsto \frac{a}{z}$ with $a \in \mathbb{C}^*$.

(IV) $\mathrm{PSL}(2, \mathbb{R})$ – the hyperbolic group – elements of $\mathrm{PSL}(2, \mathbb{C})$ that preserve $\mathbb{H}^2 \hookrightarrow \mathbb{CP}^1$ or equivalentely Möbius transformation with real coefficients.

Groups conjugated to subgroups of any of the first three cases are called *elementary* of type (I), (II), or (III). We say that a complex *projective structure has unitary* (resp. *affine, dihedral or hyperbolic*) *holonomy* if it has a holonomy with image in $\mathrm{PSU}(2, \mathbb{C})$ (resp. $\mathrm{Aff}(\mathbb{C}), \mathrm{D}_\infty(\mathbb{CP}^1)$ or $\mathrm{PSL}(2, \mathbb{R})$).

The restriction on the images of the holonomies of complex projective structures is of a representation-theoretic nature and does not consider the underlying geometry of the developing map. For this we need the further restriction.

Definition 1.5.2. A complex projective structure is *affine* (resp. *hyperbolic*) if it has a developing map with image in $\mathbb{C} \hookrightarrow \mathbb{CP}^1$ (resp. $\mathbb{H}^2 \hookrightarrow \mathbb{CP}^1$).

Lemma 1.5.3. *A complex projective structure is* affine *(resp.* hyperbolic*) if and only if there is a compatible atlas whose charts have image in* $\mathbb{C} \hookrightarrow \mathbb{CP}^1$ *(resp.* $\mathbb{H}^2 \hookrightarrow \mathbb{CP}^1$*) and all the transition functions are in* $\mathrm{Aff}(\mathbb{C})$ *(resp.* $\mathrm{PSL}(2, \mathbb{R})$*).*

Proof. Denote by $\iota : U \to \mathbb{CP}^1$ the embbedding, where $U = \mathbb{C}$ or $U = \mathbb{H}^2$, and by $\Gamma = \mathrm{Aff}(\mathbb{C})$ or $\Gamma = \mathrm{PSL}(2, \mathbb{R})$, respectively. Note that a Möbius transformation $\gamma \in \mathrm{PSL}(2, \mathbb{C})$ is in Γ if and only if it preserves $\iota(U)$. Since the developing map is built using the transition functions as

$$d(y) = g_{0,1} \circ g_{1,2} \circ \cdots \circ g_{n-2n-1} \circ g_{n-1,n} \circ \varphi_n(y), \quad y \in \widetilde{S}$$

we see that if the charts φ have image in $\iota(U)$ and the transition functions $g_{j,j+1}$ preserve this image, then for every $y \in \widetilde{S}$, the point $d(y)$ is still in $\iota(U)$. For the converse assume $d(\widetilde{S}) \subset \iota(U)$. We have already seen in Remark 1.3.3 that a non-maximal maximal can be chosen in such a way that d agrees locally with its charts. Thus, in particular, the charts have image in $\iota(U)$. The transition functions then map subsets of $\iota(U)$ to subsets of $\iota(U)$. As such they preserve $\iota(U)$ and are in Γ. $\square$

Remark 1.5.4. Since the image of the holonomy representation preserves the image of the developing map d (Remark 1.3.7), we have in particular that affine and hyperbolic $\mathbb{CP}^1$-surfaces have affine and hyperbolic holonomies. The converse is not true, and there are examples of exotic projective structures that have hyperbolic holonomy but are not hyperbolic. For more information about these we refer to (Dumas, 2009, 5.4) and the references therein.

We are now in conditions of using the uniformization theorem to prove that, in fact, every Riemann surface X can be given a *compatible* complex projective structure, with its type being determined by the universal cover.

Theorem 1.5.5. *Every Riemann surface X is induced by a complex projective structure. Equivalently, the space $\mathcal{P}(X)$ is non-empty for every X.*

Proof. The uniformization theorem identifies the universal cover $\widetilde{X}$ of X with one of $\mathbb{CP}^1$, $\mathbb{C}$ or $\mathbb{H}$. This realizes X as a quotient of either $\mathbb{CP}^1$, $\mathbb{C}$ or $\mathbb{H}$ by a group of biholomorphisms. Since the biholomorphism groups are $\mathrm{Aut}(\mathbb{CP}^1) \cong \mathrm{PSL}(2, \mathbb{C})$, $\mathrm{Aut}(\mathbb{C}) \cong \mathrm{Aff}(\mathbb{C})$ and $\mathrm{Aut}(\mathbb{C}) \cong \mathrm{PSL}(2, \mathbb{R})$, the surface σ has a complex projective structure induced by the quotient map. $\square$

Remark 1.5.6. Note that we actually see that if the universal cover is $\mathbb{C}$ then X has an affine structure and if it is $\mathbb{H}^2$, X has a hyperbolic structure.

Observe that the classical uniformization can be replaced by other kinds of uniformization, for example, Schottky uniformization.

1.6 Schwarz's equation and parameterization of structures

1.6.1 Schwarzian derivative

At this point we have seen that a $\mathbb{CP}^1$-surface has naturally induced Riemann surface structure, and, conversely, every Riemann surface X is induced by a $\mathbb{CP}^1$-surface. We can now ask what is the size of the (non-empty) space $\mathcal{P}(X)$ for each X, or, in other words, how many complex projective structures induce the same Riemann surface structure. The purpose of this section is to prove that $\mathcal{P}(X)$ is an affine space modeled on the vector space of holomorphic quadratic differentials $\mathrm{H}^0(K^2)$. We start by reviewing a classical construction related to ordinary differential equations and the work of Schwarz. This approach marked the starting point of the investigations into the subject during the late 19th century, from which the modern definition used here can be traced back. Only in this chapter, we will use a prime to denote complex differentiation.

Definition 1.6.1. Let $U \subset \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ a holomorphic function. Assume its complex derivative f' is nonzero in U. We define the *Schwarzian derivative* of f as

$$\mathrm{S}(f) := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{1.6.1}$$

This is a third-order differential operator satisfying the following properties:

(i) If $g \in \mathrm{PSL}(2, \mathbb{C})$ is a Möbius transformation, $\mathrm{S}(g) = 0$;

(ii) If f_1 and f_2 are composable functions then $\mathrm{S}(f_1 \circ f_2) = (\mathrm{S}(f_1) \circ f_2)\, f_2'^2 + \mathrm{S}(f_2)$.

The proofs of the properties follow by direct calculation. In fact, one can even go further and check that the Schwarzian derivative of a function f determines it up to Möbius transformation. The treatment we present here follows (Lehto, 1987, chapter II, 1.2).

Theorem 1.6.2. *Let $\vartheta : U \to \mathbb{C}$ be a holomorphic function on a simply connected open set $U \subset \mathbb{C}$. Then there is a meromorphic function f such that*

$$\mathrm{S}(f) = \vartheta$$

which is unique up to post-composition with an arbitrary Möbius transformation.

Proof. The function f is built using the intimately related Schwarz differential equation:

$$u'' + \frac{1}{2}\vartheta u = 0. \tag{1.6.2}$$

So we consider this linear equation whose solution space on the simply connected $U \subset \mathbb{C}$ is a vector space generated by two linear independent holomorphic solutions, say u_1 and u_2. Define $f = \frac{u_1}{u_2}$. We will show f satisfies $\mathrm{S}(f) = \vartheta$. To begin with, f is meromorphic since it is the quotient of holomorphic functions. Now, consider the Wronskian $W[u_1, u_2] = u_1' u_2 - u_1 u_2'$ of the two solutions. Since u_1 and u_2 are linearly independent there is a point $z_0 \in U$ where $W[u_1, u_2](z_0) \neq 0$. But observe that, using Equation 1.6.2, $W[u_1, u_2]$ satisfies

$$W[u_1, u_2]' = u_1''u_2 + u_1'u_2' - u_1'u_2' - u_1u_2'' = -\frac{1}{2}\vartheta u_1 u_2 + \frac{1}{2}\vartheta u_1 u_2 = 0.$$

Thus $W[u_1, u_2]$ is constant and non-zero since $W[u_1, u_2](z_0) \neq 0$. Write $W[u_1, u_2] = W \in \mathbb{C}^*$. Now note that, $f' = \frac{u_1'u_2 - u_1u_2'}{u_2^2} = \frac{W}{u_2^2}$. This means that, outside of the poles, $f' \neq 0$. Now, $f'' = -2\frac{W}{u_2^3}u_2'$. Thus $\frac{f''}{f'} = -2\frac{u_2'}{u_2}$ and $\left(\frac{f''}{f'}\right)' = -2\frac{u_2''}{u_2} + 2\frac{(u_2')^2}{u_2^2}$ this shows, using the linear Equation 1.6.2, that

$$\mathrm{S}(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = -2\frac{u_2''}{u_2} + 2\frac{(u_2')^2}{u_2^2} - 2\left(\frac{u_2'}{u_2}\right)^2 = \frac{u_2\vartheta}{u_2} = \vartheta.$$

Now for the uniqueness. Start by noting that if $\mathrm{S}(g) = 0$ on a simply connected open set then g is a Möbius transformation. Indeed $\mathrm{S}(g) = 0$ if and only if $y = g''/g'$ satisfies the Ricatti equation $y' - \frac{1}{2}y^2 = 0$. Integrating gives $y^{-1} = -\frac{1}{2}(z + c_1)$ i.e $y = -\frac{2}{z+c_1}$ for some constant c_1. Then $g''/g' = y = -\frac{2}{z+c_1}$ can be integrated once to give $\log g' = -2\log(z + c_1) + \log c_2$, i.e. $g' = (z+c_1)^{-2}c_2$ for some constant c_2. Integrating a third time yields $g = -(z+c_1)^{-1}c_2 + c_3$ for some constant c_3. This is a Möbius transformation $g = \frac{c_3 z + c_3 c_1 - c_2}{z + c_1}$. Now consider the solution f we built and any other function h that inverts the Schwarzian derivative $\mathrm{S}(h) = \vartheta$. Consider a simply connected open set V where f is injective (it exists since $f' \neq 0$). On V,

$$\vartheta = \mathrm{S}(h) = \mathrm{S}(h \circ f^{-1} \circ f) = (\mathrm{S}(h \circ f^{-1}) \circ f)f'^2 + \mathrm{S}(f) = (\mathrm{S}(h \circ f^{-1}) \circ f)f'^2 + \vartheta$$

by (ii). This implies $\mathrm{S}(h \circ f^{-1}) = 0$ since $f' \neq 0$, which then means $h \circ f^{-1} = g$ for some Möbius transformation $g \in \mathrm{PSL}(2, \mathbb{C})$. We conclude that $h = g \circ f$ on V. Now varying V, we conclude that there is a Möbius transformation by which f and g differ around every point of U. Since U is connected these transformations must be the same everywhere and thus $h = g \circ f$ on U . □

Remark 1.6.3. The uniqueness statement actually strengthens (i) after Definition 1.6.1. In particular, we have shown in the proof that, on a simply connected open set, $\mathrm{S}(f) = 0$ if and only if f is a Möbius transformation.

We take note that, in fact, we have shown something stronger.

Corollary 1.6.4. *Every solution f of the equation $\mathrm{S}(f) = \vartheta$ on a simply connected domain is actually of the form $f = u_1/u_2$, where u_1 and u_2 are linearly independent solutions of the Schwarz linear differential equation*

$$u'' + \frac{1}{2}\vartheta u = 0,$$

with Wronskian $W[u_1, u_2] = 1$.

Proof. We have shown that every solution differs from $f = \frac{u_1}{u_2}$ by a Möbius transformation $g = \frac{az+b}{cz+e} \in \mathrm{PSL}(2,\mathbb{C})$, i.e every solution is of the form

$$g \circ f = \frac{a\frac{u_1}{u_2} + b}{c\frac{u_1}{u_2} + e} = \frac{au_1 + bu_2}{cu_1 + eu_2}.$$

But $au_1 + bu_2$ and $cu_1 + eu_2$ are still linearly independent solutions of the Schwarz linear differential equation, since $\det\left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right) = 1$. So every solution f is a quotient of linearly independent solutions. Suppose now that $f = \frac{u_1}{u_2}$. We have shown in the proof that $W[u_1, u_2] = W \in \mathbb{C}^*$ is constant. So $\hat{u}_1 = \frac{u_1}{\sqrt{W}}$ and $\hat{u}_2 = \frac{u_1}{\sqrt{W}}$ are still solutions of the linear equation where

$$\frac{\hat{u}_1}{\hat{u}_2} = \frac{u_1/\sqrt{W}}{u_2/\sqrt{W}} = \frac{u_1}{u_2} = f \qquad \text{with}$$

$$W[\hat{u}_1, \hat{u}_2] = \begin{vmatrix} \hat{u}_1 & \hat{u}_2 \\ \hat{u}_1' & \hat{u}_2' \end{vmatrix} = \begin{vmatrix} u_1/\sqrt{W} & u_2/\sqrt{W} \\ u_1'/\sqrt{W} & u_2'/\sqrt{W} \end{vmatrix} = \frac{1}{W} W[u_1, u_2] = 1.$$

$\square$

1.6.2 Parameterization by quadratic differentials

The Schwarzian derivative can be used to measure the difference between compatible complex projective structures. So suppose $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{CP}^1\}_{\alpha \in A}$ and $\mathcal{A}' = \{\varphi'_{\alpha'} : U_{\alpha'} \to \mathbb{CP}^1\}_{\alpha' \in A'}$ are two structures of $\mathbb{CP}^1$-surface on S, and suppose further they are compatible with the Riemann surface structure X. From now on, we will always assume that the charts are $\mathbb{C}$-valued, which can be achieved by performing a Möbius transformation on the original chart if needed. The compatibility condition is then equivalent to saying that φ_α (and $\varphi'_{\alpha'}$) are (local) holomorphic functions. We will construct a complex quantity $\mathcal{A}' - \mathcal{A}$ which we call the *difference* of the projective structures. To do this at each point $x \in S$ we simply use the atlas $\mathcal{A}$ as complex coordiantes, $z_\alpha = \varphi_\alpha(x)$, and write the local expressions of the charts $\varphi'_{\alpha'}$ of $\mathcal{A}'$ as functions of z_α, i.e. $\varphi'_{\alpha'} \circ \varphi_\alpha^{-1}(z_\alpha)$. These are transition functions from the charts of one of the atlases to the other. To measure how much these combined transition functions differ from a Möbius transformation we will take the Schwarzian derivative with respect to the complex coordinate z_α:

$$\left(\mathcal{A}' - \mathcal{A}\right)(z_\alpha) = \mathrm{S}(\varphi'_{\alpha'} \circ \varphi_\alpha^{-1}(z_\alpha)). \tag{1.6.3}$$

To check that this is well defined we need to see what happens when we change the chosen charts in both the atlases $\mathcal{A}$ and $\mathcal{A}'$. We can note firstly that choosing another projective chart of $\mathcal{A}'$ changes the function $\varphi'_{\alpha'} \circ \varphi_\alpha^{-1}$ by post-composition with a Möbius transformation g' which have $\mathrm{S}(g') = 0$, by (i) after Definition 1.6.1. Using the transformation law (ii) one checks that this leaves the Schwarzian derivative invariant, and thus the definition does not depend on the chart of $\mathcal{A}'$ chosen. The same thought process can be applied to charts in $\mathcal{A}$, but this time the function $\varphi'_{\alpha'} \circ \varphi_\alpha^{-1}$ changes by pre-composition with a $g \in \mathrm{PSL}(2,\mathbb{C})$. This means, according to (ii), that the Schwarzian derivative is not invariant but it changes by multiplication by the square of the derivative of g. To remedy this, the difference $\mathcal{A}' - \mathcal{A}$ must be defined as a *quadratic differential*.

Definition 1.6.5. Let $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to \mathbb{CP}^1\}_{\alpha\in A}$ and $\mathcal{A}' = \{\varphi'_{\alpha'} : U_{\alpha'} \to \mathbb{CP}^1\}_{\alpha'\in A'}$ be complex projective structures compatible with a Riemann surface structure X. The *difference* $\mathcal{A}' - \mathcal{A}$ is the holomorphic quadratic differential expressed in $\mathcal{A}$-local coordinates $z_\alpha = \varphi_\alpha(x)$ as

$$(\mathcal{A}' - \mathcal{A})(z_\alpha) = \mathrm{S}(\varphi'_{\alpha'} \circ \varphi_\alpha^{-1}(z_\alpha))\, dz_\alpha^2. \tag{1.6.4}$$

Remark 1.6.6. Note that since the structures are compatible with X, the function $\varphi'_{\alpha'} \circ \varphi_\alpha^{-1}$ is holomorphic. Further, since charts are local homeomorphisms, the function is a local biholomorphism and thus it has non-zero complex derivative. So its Schwarzian derivative is locally well-defined. By the previous discussion, the factor dz_α^2 was chosen to cancel the one coming from the transformation law of the Schwarzian derivative, and thus the difference is a well-defined quadratic differential.

It is clear that $\mathcal{A}'$ and $\mathcal{A}$ are projectively related if and only if their difference is the zero quadratic differential. This follows because the comparing transition functions from $\mathcal{A}'$ to $\mathcal{A}$ which appear in Definition 1.6.1 are Möbius transformations if and only if they have zero Schwarzian derivative (Remark 1.6.3). (Of course in that case $\mathcal{A}'$ and $\mathcal{A}$ actually define the same maximal atlas and we really have $\mathcal{A} = \mathcal{A}'$). Observe further that if $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ are complex projective structures compatible with X we have that the sum of quadratic differentials $(\mathcal{A}_1 - \mathcal{A}_2) + (\mathcal{A}_2 - \mathcal{A}_3) = \mathcal{A}_1 - \mathcal{A}_3$ since, if we write $z_2 = \varphi_2 \circ \varphi_3(z_3)$ for the relations between the coordinates z_2 of $\mathcal{A}_2$ and z_3 of $\mathcal{A}_3$, we have

$$\begin{aligned}
(\mathcal{A}_1 - \mathcal{A}_2) + (\mathcal{A}_2 - \mathcal{A}_3) &= \mathrm{S}(\varphi_1 \circ \varphi_2^{-1}(z_2))dz_2^2 + \mathrm{S}(\varphi_2 \circ \varphi_3^{-1}(z_3))dz_3^2 \\
&= \mathrm{S}(\varphi_1 \circ \varphi_2^{-1}(z_2)) \left(\frac{d(\varphi_2 \circ \varphi_3)}{dz_3}(z_3)\right)^2 dz_3^2 + \mathrm{S}(\varphi_2 \circ \varphi_3^{-1}(z_3))dz_3^2 \\
&\overset{(ii)}{=} \Big(\mathrm{S}(\varphi_1 \circ \varphi_2^{-1} \circ \varphi_2 \circ \varphi_3)\Big)\, dz_3^2 = \mathcal{A}_1 - \mathcal{A}_3.
\end{aligned}$$

In particular, $(\mathcal{A}_1 - \mathcal{A}_2) + (\mathcal{A}_2 - \mathcal{A}_1) = \mathcal{A}_1 - \mathcal{A}_1 = 0$. We will now show that the construction can be reversed, in the sense that given a holomorphic quadratic differential $q \in \mathrm{H}^0(K^2)$ and a compatible complex projective structure $\mathcal{A}$ we can find another one $\mathcal{A}'$ such that $\mathcal{A}' - \mathcal{A} = q$. Together with the previous identity, these are the axioms for an affine space modeled on $\mathrm{H}^0(K^2)$.

Theorem 1.6.7. *Let $\mathcal{A}$ be a complex projective structure compatible with the Riemann surface X and $q \in \mathrm{H}^0(K_X^2)$ a holomorphic quadratic differential. Then there exists a unique $\mathcal{A}'$ such that $\mathcal{A}' - \mathcal{A} = q$.*

Proof. Locally on a projective coordinate $z = \varphi(x)$ of $\mathcal{A}$, defined around $x \in U$, U a simply connected open set, the quadratic differential has a representation $q = q(z)dz^2$. Using Theorem 1.6.2 one can find a local meromorphic function f_φ such that $\mathrm{S}(f_\varphi(z)) = q(z)$ (in particular

$\frac{df_\varphi}{dz} \neq 0$ since f_φ has a well-defined Schwarzian derivative). This function is unique up to Möbius transformation. Now, for each chart $\varphi \in \mathcal{A}$ with simply connected domain, we define new charts $f_\varphi \circ \varphi$ for each possible solution f_φ. We collect all of them together in a new atlas $\mathcal{A}'$. To check that $\mathcal{A}'$ is projective we only need to verify that the transition functions are Möbius transformations in coordinates. We first write $z_1 = \varphi_1(x)$ and $z_2 = \varphi_2(x)$ for the coordinates in $\mathcal{A}$. As this is a projective atlas we have $z_2 = g_{21}(z_1)$ for some Möbius transformation $g_{21} \in \mathrm{PSL}(2, \mathbb{C})$. Then we write $w_1 = f_{\varphi_1} \circ \varphi_1(x) = f_{\varphi_1}(z_1)$ and $w_2 = f_{\varphi_2} \circ \varphi_2(x) = f_{\varphi_2}(z_2)$ for the coordinates in $\mathcal{A}'$. The transition functions are then $\tau_{21}(w_1) = f_{\varphi_2} \circ \varphi_2 \circ (f_{\varphi_1} \circ \varphi_1)^{-1}(w_1)$. We will show they are Möbius transformations by calculating the Schwarzian derivative $\mathrm{S}(\tau_{21}(w_1))$ and showing it is zero (cf. Remark 1.6.3). Note that the transition functions satisfy the equality

$$\tau_{21} \circ f_{\varphi_1}(z_1) = f_{\varphi_2} \circ g_{21}(z_1).$$

Applying to this identity the Schwarzian derivative with respect to z_1 and successively using the transformation law (ii), we get

$$\begin{aligned}
\mathrm{S}(\tau_{21}) \circ f_{\varphi_1}(z_1) \left(\frac{df_{\varphi_1}}{dz_1}(z_1)\right)^2 + \mathrm{S}(f_{\varphi_1}(z_1)) &= \mathrm{S}(f_{\varphi_2}) \circ g_{21}(z_1) \left(\frac{dg_{21}}{dz_1}(z_1)\right)^2 + 0, && \text{by (i)}\\
\mathrm{S}(\tau_{21}(w_1)) \left(\frac{df_{\varphi_1}}{dz_1}(z_1)\right)^2 + \mathrm{S}(f_{\varphi_1}(z_1)) &= \mathrm{S}(f_{\varphi_2}) \circ g_{21}(z_1) \left(\frac{dg_{21}}{dz_1}(z_1)\right)^2, && w_1 = f_{\varphi_1}(z_1)\\
\mathrm{S}(\tau_{21}(w_1)) \left(\frac{df_{\varphi_1}}{dz_1}(z_1)\right)^2 + q(z_1) &= q(g_{21}(z_1)) \left(\frac{dg_{21}}{dz_1}(z_1)\right)^2, && q(z_j) = f_{\varphi_j}(z_j)\\
\mathrm{S}(\tau_{21}(w_1)) \left(\frac{df_{\varphi_1}}{dz_1}(z_1)\right)^2 + q(z_1) &= q(z_1), && q \in \mathrm{H}^0(K^2)\\
\mathrm{S}(\tau_{21}(w_1)) &= 0, && \frac{df_{\varphi_1}}{dz_1}(z_1) \neq 0.
\end{aligned}$$

Thus the transition functions are Möbius transformations. To show uniqueness just note that any other atlas $\mathcal{A}''$ such that $\mathcal{A}'' - \mathcal{A} = q$ will have $\mathcal{A}'' - \mathcal{A}' = (\mathcal{A}'' - \mathcal{A}) + (\mathcal{A} - \mathcal{A}') = q + (-q) = 0$. □

Corollary 1.6.8. *The space $\mathcal{P}(X)$ of compatible projective structures is an affine space modeled on the space of holomorphic quadratic differentials $\mathrm{H}^0(K_X^2)$.*

Of course, we could have also expressed the difference in terms of the developing maps, since they locally agree with projective charts on the universal cover (Remark 1.3.3). So if $\mathcal{A}_1$ and $\mathcal{A}_2$ are $\mathbb{CP}^1$-structures, take $\widetilde{\mathcal{A}_1}$ and $\widetilde{\mathcal{A}_2}$ to be the ones induced in the universal cover $\widetilde{S}$. If we use coordinates $z_1 = \varphi_1(x)$ for $\widetilde{\mathcal{A}_1}$, the developing map d_2 of $\mathcal{A}_2$ locally agrees with some $\varphi_2 \in \widetilde{\mathcal{A}_2}$, and thus the definition of difference reads

$$\widetilde{\mathcal{A}_2} - \widetilde{\mathcal{A}_1} = \mathrm{S}(d_2 \circ \varphi_1^{-1}(z_1)) dz_1^2, \tag{1.6.5}$$

i.e. we just take the Schwarzian derivative of d_2 in projective coordinates of $\mathcal{A}_1$. The same line of arguments as before can be used to show that this is a $\pi_1(S)$-invariant quadratic differential on $\widetilde{S}$. Indeed each $\gamma \in \pi_1(S)$ is a projective map for the induced structure $\widetilde{\mathcal{A}_1}$ and thus acts in coordinates as $\varphi \circ \gamma \circ \varphi_1^{-1} = g \in \mathrm{PSL}(2,\mathbb{C})$. But the coordinates of the quadratic differential $\widetilde{\mathcal{A}_2} - \widetilde{\mathcal{A}_1}$ pulled-back under γ are just

$$\begin{aligned} \mathrm{S}(d_2 \circ \varphi^{-1}(gz_1)) \left(\frac{dg}{dz_1}\right)^2 &\overset{(ii)}{=} \mathrm{S}(d_2 \circ \varphi^{-1} \circ g(z_1)) = \mathrm{S}(d_2 \circ \gamma \circ \varphi_1^{-1}(z_1)) \\ &= \mathrm{S}(\rho(\gamma) \circ d_2 \circ \varphi_1^{-1}(z_1)) \overset{(ii)}{=} \mathrm{S}(d_2 \circ \varphi_1^{-1}(z_1)). \end{aligned}$$

Throughout the rest of the text, we will suppress the explicit dependence on the charts and will denote by z a complex coordinate (sometimes projective) both on S and on the universal cover $\widetilde{S}$. This means, for example, that if z is the projective coordinate of $\mathcal{A}_1$ and $d_2(z)$ the coordinate representation of a developing map of $\mathcal{A}_2$, then

$$\widetilde{\mathcal{A}}_2 - \widetilde{\mathcal{A}}_1 = \mathrm{S}(d_2(z))dz^2.$$

Note that if d_2 reaches infinity we simply calculate $\mathrm{S}(d_2(z)) = \mathrm{S}(1/d_2(z))$, since $w \mapsto 1/w$ is a Möbius transformation for which S is invariant.

1.7 Holonomies lift

We will see how there is more information available on what kind of homomorphisms do come up as holonomies of projective structures. So let S a surface and $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ a homomorphism. Denote by $\mathrm{P} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$ the quotient map represented in matrix form as $\left[\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right] \mapsto \pm\left[\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right]$ or as a Möbius transformation $\left[\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right] \mapsto \frac{az+b}{cz+d}$.

Definition 1.7.1. A homomorphism $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$ *lifts to* $\mathrm{SL}(2,\mathbb{C})$ if there is another one $\widetilde{\rho} : \pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$ that covers it relatively to the quotient map $\mathrm{P} : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{PSL}(2,\mathbb{C})$, i.e. $\rho = \mathrm{P} \circ \widetilde{\rho}$. We get the commutative diagram.

$$\begin{tikzcd} & \mathrm{SL}(2,\mathbb{C}) \arrow[d, "\mathrm{P}"] \\ \pi_1(M) \arrow[ur, dashed, "\widetilde{\rho}"] \arrow[r, "\rho"] & \mathrm{PSL}(2,\mathbb{C}) \end{tikzcd}$$

We will see now that all homomorphisms that come up as holonomies of $\mathbb{CP}^1$-structures are in fact of this kind.

Theorem 1.7.2. *Let S be a $\mathbb{CP}^1$-surface with holonomy $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{C})$. Then ρ lifts to* $\mathrm{SL}(2,\mathbb{C})$.

Remark 1.7.3. The lift is not necessarily unique and the images of $\widetilde{\rho}$ and of ρ are not necessarily isomorphic.

Proof. Recall that the only Riemann surface that has $\mathbb{CP}^1$ as universal cover is $\mathbb{CP}^1$ itself (Farkas and Kra, 1992, §IV.6.3. Theorem). This is because $\pi_1(S)$ acts without fixed points and a Möbius transformation necessarily has a fixed point on the sphere. So in the case of the Riemann sphere $\mathbb{CP}^1$, the holonomy is trivial, and thus the trivial homomorphism to $\mathrm{SL}(2,\mathbb{C})$ lifts it.

Now let (ρ, d) be the developing pair of a projective structure on S. Take S to be uniformized by the coordinate z, and thus having its universal cover identified with either $\widetilde{S} = \mathbb{C}$ or $\widetilde{S} = \mathbb{H}^2$. Using the Schwarzian parameterization (Theorem 1.6.7), $d : \widetilde{S} \to \mathbb{CP}^1$ corresponds to the $\pi_1(S)$-invariant quadratic differential

$$\vartheta = \mathrm{S}(d(z))dz^2.$$

This means by Corollary 1.6.4 that d is locally a quotient of linearly independent solutions of the linear equation $u'' + \frac{1}{2}\vartheta u = 0$. But, using the uniformizing coordinate, this linear equation is actually global, and thus $d = \frac{u_1}{u_2}$ for some global functions u_1 and u_2 with Wronskian $W[u_1, u_2] = 1$. Given that the surface is uniformized, an element $\gamma \in \pi_1(S)$ acts as $\gamma z = \frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma}$, and $\gamma'(z) = (c_\gamma z + d_\gamma)^{-2}$. Since ϑ is $\pi_1(S)$-invariant we will show in a computation (that we carry out the lemma below) that $u_1(\gamma z)(c_\gamma z + d_\gamma)$ is another solution of the linear equation. Note that c_γ and b_γ are not uniquely determined by γ, since

$$\gamma z = \frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma} = \frac{-a_\gamma z - b_\gamma}{-c_\gamma z - d_\gamma}.$$

To express this ambiguity we write $\pm\hat{u}_1(z) := \pm u_1(\gamma z)(c_\gamma z + d_\gamma)$ for the new solution. Since the Schwarz differential equation is linear and $\widetilde{S}$ is simply connected, the solution space is a vector space, and thus $\pm\hat{u}_1 = \alpha_\gamma\, u_1 + \beta_\gamma\, u_2$ for $\alpha_\gamma, \beta_\gamma \in \mathbb{C}$. Analogously $\pm\hat{u}_2(z) := \pm u_2(\gamma z)(c_\gamma z + d_\gamma) = \delta_\gamma\, u_1(z) + \epsilon_\gamma\, u_2(z)$ for $\delta_\gamma, \epsilon_\gamma \in \mathbb{C}$. Both equalities read in matrix form

$$\begin{bmatrix} \pm\hat{u}_1(z) \\ \pm\hat{u}_2(z) \end{bmatrix} = A_\gamma \begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix}, \qquad \text{with } A_\gamma = \begin{bmatrix} \alpha_\gamma & \beta_\gamma \\ \delta_\gamma & \epsilon_\gamma \end{bmatrix}.$$

We will check in another computation below that $W[\hat{u}_1, \hat{u}_2] = W[u_1, u_2]$. Since we have taken u_1 and u_2 with $W[u_1, u_2] = 1$, the matrix A_γ is in $\mathrm{SL}(2,\mathbb{C})$.

We now want to make the correspondence $\gamma \mapsto A_\gamma$ into the lift $\widetilde{\rho} : \pi_1(S) \to \mathrm{SL}(2,\mathbb{C})$. However, there is a problem, since the matrix A_γ has a sign ambiguity coming from $\pm\hat{u}_j(z)$, $j = 1, 2$. So, for each γ, we need to make an adequate choice of $\pm\hat{u}_j$ that reflects on the corresponding A_γ. This choice needs to be done in a way such that $\gamma \mapsto A_\gamma$ is a homomorphism $\widetilde{\rho}$. If we manage to do this the proof is finished since $\widetilde{\rho}$ will cover ρ. This is seen because the fact that d is ρ-equivariant reads

$$\begin{aligned}\rho(\gamma) \circ d(z) = d(\gamma\, z) = \frac{u_1(\gamma\, z)}{u_2(\gamma\, z)} = \frac{u_1(\gamma\, z)(c_\gamma z + d_\gamma)}{u_2(\gamma\, z)(c_\gamma z + d_\gamma)} = \\ = \frac{\alpha_\gamma u_1(z) + \beta_\gamma u_2(z)}{\delta_\gamma u_1(z) + \epsilon_\gamma u_2(z)} = \frac{\alpha_\gamma \frac{u_1(z)}{u_2(z)} + \beta_\gamma}{\delta_\gamma \frac{u_1(z)}{u_2(z)} + \epsilon_\gamma} = \mathrm{P}(A_\gamma) \circ d(z).\end{aligned}$$

and thus $\widetilde{\rho}$ covers ρ with respect to P. To make a consistent choice of signs we will use a square root of the canonical bundle. So choose a $K^{1/2}$ with a holomorphic section f lifted to the universal cover. The fact that f is a square root reads as $f(\gamma z)\sqrt{\gamma'} = f(z)$ for a defined and consistent choice of sign of root which we have denoted by $\sqrt{\gamma'}$. Noting that $\gamma' = (c_\gamma z + d_\gamma)^{-2}$ we see that

$$(c_\gamma z + d_\gamma)\sqrt{\gamma'} = \pm 1.$$

So we chose the signs of $\hat{u}_j(z), j = 1, 2$ in a way that the above quantity is always 1. This means that now $\gamma \mapsto A_\gamma$ is a well-defined correspondence. Let us show it is a homomorphism. If $\gamma_1 = \frac{a_1 z + b_1}{c_1 z + d_1}$ and $\gamma_2 = \frac{a_2 z + b_2}{c_2 z + d_2}$ we have that $(\gamma_1\gamma_2)' = ((c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2)^{-2}$. Since the choices of the square root are determined by f, f is such that

$$f(\gamma_1\gamma_2 z)\sqrt{(\gamma_1\gamma_2(z))'} = f(\gamma_1\gamma_2 z)\sqrt{\gamma_1'(\gamma_2(z))}\sqrt{\gamma_2'(z)} = f(\gamma_2 z)\sqrt{\gamma_2'(z)} = f(z),$$

with choices included. We conclude that the choices are compatible, in the sense that if $(c_1 z + d_1)\sqrt{\gamma_1'} = 1$ and $(c_2 z + d_2)\sqrt{\gamma_2'} = 1$, we have

$$\begin{aligned}
((c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2)\sqrt{(\gamma_1\gamma_2(z))'} &= \\
&= (c_1(a_2 z + b_2) + d_1(c_2 z + d_2))\sqrt{\gamma_1'(\gamma_2(z))}\sqrt{\gamma_2'(z)} = \\
&= \left(c_1\frac{a_2 z + b_2}{c_2 z + d_2} + d_1\right)(c_2 z + d_2)\sqrt{\gamma_1'(\gamma_2(z))}\sqrt{\gamma_2'(z)} = 1.
\end{aligned}$$

This means that with the given choices

$$\begin{aligned}
A_{\gamma_1\gamma_2}\begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix} &= \begin{bmatrix} u_1(\gamma_1\gamma_2 z) \\ u_2(\gamma_1\gamma_2 z) \end{bmatrix}((c_1 a_2 + d_1 c_2)z + c_1 b_2 + d_1 d_2) = \\
&= \begin{bmatrix} u_1(\gamma_1\gamma_2 z) \\ u_2(\gamma_1\gamma_2 z) \end{bmatrix}\left(c_1\frac{a_2 z + b_2}{c_2 z + d_2} + d_1\right)(c_2 z + d_2) = \\
&= A_{\gamma_1}\begin{bmatrix} u_1(\gamma_2 z) \\ u_2(\gamma_2 z) \end{bmatrix}(c_2 z + d_2) = A_{\gamma_1}A_{\gamma_2}\begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix},
\end{aligned}$$

and $\gamma \mapsto A_\gamma$ indeed defines a homomorphism. We are only left with the mentioned computations. □

Lemma 1.7.4. *Let $\gamma z = \frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma}$. If $u(z)$ is a solution of the Schwarz linear equation so is $\pm u(\gamma z)(c_\gamma z + d_\gamma)$.*

Proof. We suppress the index γ. Noting that $\gamma' = (cz + d)^{-2}$ we compute

$$(u(\gamma z)(cz + d))' = u'(\gamma z)\gamma'(z)(cz + d) + u(\gamma z)c = u'(\gamma z)(cz + d)^{-1} + u(\gamma z)c$$

and

$$
\begin{aligned}
(u(\gamma z)(cz+d))'' &= u''(\gamma z)\gamma'(cz+d)^{-1} - u'(\gamma z)(cz+d)^{-2}c + u'(\gamma z)\gamma' c \\
&= u''(\gamma z)(cz+d)^{-3}.
\end{aligned} \tag{1.7.1}
$$

Now, since $u(z)$ is a solution we have $u''(\gamma z) = -\frac{1}{2}\vartheta(\gamma z)u(\gamma z)$, and the $\pi_1(S)$-invariance reads $\vartheta(z) = \vartheta(\gamma z)(\gamma')^2 = \vartheta(\gamma z)(cz+d)^{-4}$. Combining this with the calculation we obtain

$$
(u(\gamma z)(cz+d))'' = u''(\gamma z)(cz+d)^{-3} = -\frac{1}{2}\vartheta(\gamma z)u(\gamma z)(cz+d)^{-3} = -\frac{1}{2}\vartheta(z)u(\gamma z)(cz+d).
$$

□

Lemma 1.7.5. *The Wronskians satisfy* $W[\hat{u}_1, \hat{u}_2] = W[u_1, u_2]$.

Proof. Dropping the index in γ we have $\hat{u}_j(z) = u_j(\gamma z)(cz+d)$. Thus we calculate

$$
\begin{aligned}
&W[\hat{u}_1, \hat{u}_2](z) = \hat{u}_1(z)'\hat{u}_2(z) - \hat{u}_1(z)\hat{u}_2'(z) = \\
&= \left(u_1'(\gamma z)(cz+d)^{-1} + u_1(\gamma z)c\right) u_2(\gamma z)(cz+d) - u_1(\gamma z)(cz+d)\left(u_2'(\gamma z)(cz+d)^{-1} + u_2(\gamma z)c\right) \\
&= u_1'(z)u_2(z) - u_1(z)u_2'(z).
\end{aligned}
$$

□

This proof is modeled on the one of (Kra, 1985, Theorem 3.2), an article where one can find further details about the intricate history of this result and its several proofs. We note some points about the argument. The fact that $u_j(\gamma z)(c_\gamma z + d_\gamma)$ are still solutions of the differential equation already hints at a transformation law related to square roots of the canonical bundle, because $c_\gamma z + d_\gamma = \gamma'^{-1/2}$. It's possible to express the linear equation in a more invariant way, without using projective coordinates to begin with. When this is done, one observes that, in fact, the u_j transform like $-\frac{1}{2}$-differentials. This was first done in (Hawley and Schiffer, 1966, II. 1.), where the full transformation law of the Schwarzian derivative (ii), after Definition 1.6.1, is considered. A contemporary view on this approach reads the Schwarzian linear equation as a differential operator corresponding to a connection $D_{\mathrm{S}} : \Omega^0(K^{-1/2}) \to \Omega^0(K^{3/2})$, and leads to the introduction of *opers*. Even though we will treat opers in a further section, this approach won't be needed and we refer to (Frenkel, 2007, 4 − 4.1) for further details.

It is also possible to provide cohomological proofs of this result, being either of group-theoretic or sheaf-theoretic nature, for which we again refer to the bibliography of (Kra, 1985). In any case, it is clear from our proof that the obstruction to the existence of lift is the same as the obstruction to the existence of a square root of the canonical bundle K. In the case of a Riemann surface X it is always possible to find such a root since the obstruction to find one is the image of the Chern class of K in $H^2(X, \mathbb{Z}_2)$. When X is open $H^2(X, \mathbb{Z}_2) = 0$. When X is closed the degree of K is even. In both cases, the obstruction vanishes.

Remark 1.7.6. It follows from the proof, that each possible square square-root of K determines a lift. This means, in particular, the number of lifts for a closed surface of genus g is 2^{2g}. This can also be seen from the algebraic fact that if we find a lift, we can obtain all others by changing the signs of the generators of the group in a way that the defining relation is still satisfied. In this case, there are $2g$ generators γ_j of the fundamental group $\pi_1(S) = \langle \{a_j, b_j\}_{j=1,\cdots,g} | \prod_{k=1}^{g} [a_k, b_k] = id \rangle$, and the defining relation of the group is satisfied regardless of the signs of the lifts $\widetilde{\rho}(\gamma_j)$, because it only depends on commutators.

For higher dimensional complex projective manifolds M of dimension m, i.e. manifolds locally modeled in $\mathrm{PGL}(m+1, \mathbb{C})$, the obstruction to the lifting turns out to be the image of the Chern class of the canonical bundle $K_M = \det T^*M^{(1,0)}$ inside $H^2(X, \mathbb{Z}_{m+1})$, as proved in (Simha, 1989).

1.8 Characterization of structures on closed surfaces

In this section, we assume S is a closed surface of genus g. Using the parameterization by quadratic differentials we can already obtain classification results of the projective structures in lower genus.

1.8.1 The sphere

The sphere has a unique complex structure $X_{\mathbb{CP}^1}$, and since any $\mathbb{CP}^1$-structure induces a complex one, all such projective structures must be compatible with $X_{\mathbb{CP}^1}$. Using stereographic projection, $X_{\mathbb{CP}^1}$ is the complex structure on $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$, whose charts are the inclusions of $\mathbb{C}$ and $\mathbb{C}^* \cup \{\infty\}$ and the transition function is $z \mapsto 1/z$. This atlas makes it clear that in fact $X_{\mathbb{CP}^1}$ is a complex projective structure since the transition function is a Möbius transformation. Indeed it would be a complex projective structure with dihedral holonomy if it weren't for the fact that, since the sphere is simply connected, the holonomy of the structure is trivial. Given that, by the Riemann-Roch Theorem, there are no quadratic differentials on $\mathbb{CP}^1$, there are no further complex projective structures, i.e., the affine space parameterizing compatible complex projective structures is trivial.

Theorem 1.8.1. *There is only one complex projective structure on the sphere $\mathbb{CP}^1$, i.e. $\mathcal{P}(\mathbb{CP}^1) = \{X_{\mathbb{CP}^1}\}$.*

1.8.2 The torus

For this section we follow (Gunning, 1981, 2.) and (Loray and Marìn, 2009, Remark 1.1). The uniformization theorem describes each complex torus $T_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ as the quotient of the complex plane $\mathbb{C}$ by the integer lattice generated by 1 and $\tau \in \mathbb{H}^2$. For a fixed complex structure T_τ, the Schwarzian parameterization gives a holomorphic quadratic differential ϑ determined by each complex projective structure compatible with T_τ. Since the canonical bundle of a torus is trivial we have that K^2 is also trivial and $\vartheta = c \in \mathbb{C}$ is a constant. This means we can explicitly integrate the linear equation of Schwarz to obtain the developing map. Indeed on the universal cover $\mathbb{C}$ of T_τ we have $u''(z) + \frac{1}{2}cu(z) = 0$. This is a linear equation with constant coefficients. Its characteristic polynomial is $r^2 + \frac{c}{2}$. This has two distinct roots $\pm i\sqrt{\frac{c}{2}}$, if $c \neq 0$, or a double root if $c = 0$. A basis of solutions u_1, u_2 is then

$$(1') \quad u_1(z) = e^{-i\sqrt{\frac{c}{2}}z} \quad u_2(z) = e^{i\sqrt{\frac{c}{2}}z} \quad \text{if } c \neq 0 \qquad (2') \quad u_1(z) = 1 \quad u_2(z) = z \quad \text{if } c = 0.$$

To be consistent, namely for the parameterization to be continuous in c, we will use instead the basis of solutions

$$\begin{aligned} &(1) \quad u_1(z) = i\frac{\sin\left(\sqrt{c/2}\,z\right)}{\sqrt{c/2}} \quad u_2(z) = ie^{i\sqrt{c/2}z} \quad \text{if } c \neq 0 \\ &(2) \quad u_1(z) = iz \quad u_2(z) = i \quad \text{if } c = 0. \end{aligned}$$

where the solutions now satisfy $W[u_1, u_2](z) = 1$. These were chosen in such a way that $\lim_{c\to 0} i\frac{\sin\left(\sqrt{c/2}\,z\right)}{\sqrt{c/2}} = iz$ and $\lim_{c\to 0} ie^{i\sqrt{c/2}z} = i$. The developing map is thus

$$d(z) = \frac{u_1(z)}{u_2(z)} = \frac{i\frac{\sin\left(\sqrt{c/2}\,z\right)}{\sqrt{c/2}}}{ie^{i\sqrt{c/2}z}} = \frac{\left(e^{i\sqrt{c/2}z} - e^{-i\sqrt{c/2}z}\right)e^{-i\sqrt{c/2}z}}{2i\sqrt{c/2}} = \frac{1 - e^{-i\sqrt{2c}z}}{i\sqrt{2c}}$$

for every c, with $\lim_{c\to 0} \frac{1-e^{-i\sqrt{2c}z}}{i\sqrt{2c}} = z$ understood. Note that the developing map never attains ∞, which means that any complex projective structure is actually an affine one. Now the translations $\gamma_1(z) = z + 1$ and $\gamma_\tau(z) = z + \tau$ form a basis of the covering group $\pi_1(T_\tau)$. Thus the equivariance of d reads

$$\begin{aligned} d(z+\tau) &= \frac{1 - e^{-i\sqrt{2c}(z+\tau)}}{i\sqrt{2c}} = \frac{1 - \left(1 - 1 + e^{-i\sqrt{2c}z}\right)e^{-i\sqrt{2c}\tau}}{i\sqrt{2c}} = \\ &= \frac{1}{i\sqrt{2c}} - \left(\frac{1}{i\sqrt{2c}} - d(z)\right)e^{-i\sqrt{2c}\tau} = e^{-i\sqrt{2c}\tau}d(z) + \frac{1 - e^{-i\sqrt{2c}\tau}}{i\sqrt{2c}}. \end{aligned}$$

It means that $d(z+\tau) = \alpha_\tau d(z) + \beta_\tau$ for $\alpha_\tau = e^{-i\sqrt{2c}\tau}$ and $\beta_\tau = \frac{1-e^{-i\sqrt{2c}\tau}}{i\sqrt{2c}}$. Thus the holonomy is affine, as expected. It is given in matrix form as

$$\gamma_\tau \mapsto \pm \begin{bmatrix} e^{-i\sqrt{2c}/2\tau} & \frac{\sin\left(\sqrt{c/2}\,\tau\right)}{\sqrt{c/2}} \\ 0 & e^{i\sqrt{2c}/2\tau} \end{bmatrix} \qquad \gamma_1 \mapsto \pm \begin{bmatrix} e^{-i\sqrt{2c}/2} & \frac{\sin\left(\sqrt{c/2}\right)}{\sqrt{c/2}} \\ 0 & e^{i\sqrt{2c}/2} \end{bmatrix}.$$

Note that for $c = 0$ we get simply the holonomy of the uniformized torus

$$\gamma_\tau \mapsto \pm \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} \cong z + \tau \qquad \gamma_1 \mapsto \pm \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cong z + 1.$$

A few comments come to light by observing the parameterization. The first one is that all developing maps omit $\infty \in \mathbb{CP}^1$. This means that all structures are in fact affine projective structures, which accounts for their affine holonomy. The second one is that the developing map is extremely transcendental. In general, it is known that the Schwarz differential equation will only have algebraic solutions for very special sets of parameters. In fact, in the most general case, the equation can't even be solved by quadrature, that is, its solution cannot be expressed in terms of elementary functions and their integrals. This is because the differential Galois group is not solvable. For more details about this, and further results on differential Galois theory, we refer to (Van Der Put and Singer, 2003, pag. 127), where a variant of Kovacic's algorithm is used to determine when the Schwarzian equation has algebraic solutions. Of course, the fact the equation can't be solved by quadrature means that, for higher genus surfaces, one cannot expect such a complete solution to be obtained solely by the methods of this subsection.

1.8.3 Higher genus surfaces

For closed surfaces with genus $g \geq 2$, the results are rather different from the previous ones, since the quadratic differential in the Schwarz parameterization can be non-constant. In particular, a complete classification in any explicit sense is currently unavailable. We will prove the classical result that restricts the holonomies of complex projective structures in this case since it is difficult to find the result expressed in contemporary language.

Theorem 1.8.2. *Let S be a closed surface with genus $g \geq 2$. Any complex projective structure on S has non-elmentary holonomy $\rho(\pi_1(S))$.*

To prove the result we will exclude each type of elementary group.

Lemma 1.8.3. *For $g \geq 2$, the holonomy $\rho(\pi_1(S))$ is not an elementary group of type (I).*

Proof. If the holonomy is of type (I) it is conjugate to a subgroup of $\mathrm{PSU}(2, \mathbb{C})$. These are isometries of the spherical Riemannian metric $ds^2_{\mathbb{CP}^1}$ on $\mathbb{CP}^1$. This means that the pull-back metric under the developing map is invariant under the action of $\pi_1(S)$, and thus it descends to S. Since the Gaussian curvature of this metric is $k = 1$ we get that

$$\chi(S) = \int_S k \,\mathrm{vol} = \int_S 1 \,\mathrm{vol} > 0$$

by the Gauss-Bonnet theorem. So we conclude that the Euler characteristic is $\chi(S) = 2 - 2g \geq 1$ and thus $g = 0$. $\square$

Lemma 1.8.4. *For $g \geq 2$, the holonomy $\rho(\pi_1(S))$ is not an elementary group of type (II).*

Proof. We give S the Riemann surface structure X induced by the complex projectvie structure. Then we proceed as in the proof of the lifting Theorem 1.7.2, using Corollary 1.6.4 to write the developing map in $\widetilde{S} = \mathbb{H}^2$ as $d = \frac{u_1}{u_2}$, with $u_j(z)$ solutions of the Schwarz linear equation. If the holonomy is elementary of type (II) then the image of every γ under the holonomy is

$\rho(\gamma)(z) = \alpha_\gamma z + \beta_\gamma$. Choosing $\gamma \mapsto A_\gamma = \begin{bmatrix} \sqrt{\alpha_\gamma} & \beta_\gamma \\ 0 & \sqrt{\alpha_\gamma}^{-1} \end{bmatrix}$ a lift of the holonomy using a square root of the canonical bundle $K^{1/2}$ we obtain as before

$$\begin{bmatrix} u_1(\gamma z)(\gamma')^{-1/2} \\ u_2(\gamma z)(\gamma')^{-1/2} \end{bmatrix} = A_\gamma \begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix}. \tag{1.8.1}$$

In particular, $u_2(\gamma z)(\gamma')^{-1/2} = \sqrt{\alpha_\gamma}^{-1} u_2(z)$. Taking the second derivative of both sides, as in Lemma 1.7.4 (Equation 1.7.1), we get that $u_2''(\gamma z)(\gamma')^{3/2} = \sqrt{\alpha_\gamma}^{-1} u_2''(z)$. Suppose now $u_2''(z) \not\equiv 0$. Define $v(z) = \frac{u_2(z)}{u_2''(z)}$. It satisfies

$$v(\gamma z)(\gamma')^{-2} = \frac{u_2(\gamma z)(\gamma')^{-1/2}}{u_2(\gamma z)(\gamma')^{3/2}} = \frac{\sqrt{\alpha_\gamma}^{-1} u_2(z)}{\sqrt{\alpha_\gamma}^{-1} u_2''(z)} = v(z),$$

and as such it determines a meromorphic bivector field on the surface X. It has a pole whenever the non-zero holomorphic function $u_2''(z)$ has a zero of higher order than $u_2(z)$. This is impossible. Thus the bivector field is holomorphic. Since the genus is $g > 1$, Riemann-Roch then implies that $v(z) = 0$, given that the holomorphic bitangent bundle K^{-2} has no nontrivial sections. This means $u_2(z) = 0$, a contradiction to the assumption $u_2''(z) \not\equiv 0$. We conclude that $u_2''(z) \equiv 0$. Integrating this identity we get that globally $u_2(z) = c_1 z + c_2$ for $c_1, c_2 \in \mathbb{C}$ not both zero (because $u_j, j = 1, 2$, are linearly independent solutions). This means $d = \frac{u_1}{u_2}$ has at most one pole at $z_0 = -\frac{c_2}{c_1}$, i.e $d^{-1}(\infty) = \{z_0\}$ or d avoids infinity. The first case is impossible because all elements in the holonomy group $\rho(\gamma) = \alpha_\gamma z + \beta_\gamma$ preserve infinity. Thus the pre-image $d^{-1}(\infty) = \{z_0\}$ is $\pi_1(S)$-invariant and cannot be a single point since $\pi_1(M)$ acts without fixed points. This means d actually avoids infinity, and thus the complex projective structure is an affine one. But affine structures can only exist on the torus. Indeed, by Lemma 1.5.3, we can find an atlas for which all transition functions are elements of $\mathrm{Aff}(C)$, i.e. of the form $w = az + b$. This is impossible on a genus $g > 1$ surface since then $w' = a$ are locally constant transition functions for the canonical bundle K. This means K is flat, and in particular $2g - 2 = \deg(K) = 0$, i.e. $g = 1$. □

Lemma 1.8.5. *For $g \geq 2$, the holonomy $\rho(\pi_1(S))$ is not an elementary group of type (III).*

Proof. If the holonomy is an elementary group of type (III) then it is conjugate to a subgroup of $\mathrm{D}_\infty(\mathbb{CP}^1)$, i.e. Möbius transformations of the form $z \mapsto az$ or $z \mapsto \frac{a}{z}$ for $a \in \mathbb{C}^*$. Now recall that $\mathbb{CP}^1$ can be covered with two charts namely $(U = \mathbb{C}, z)$ and $(V = \mathbb{C}^* \cup (\infty), w)$, where the transition function is $w = \frac{1}{z}$. In particular, $dw = -\dfrac{1}{z^2} dz$ and $\frac{\partial}{\partial w} = -z^2 \frac{\partial}{\partial z}$. This means that the vector field $Z = z\frac{\partial}{\partial z}$ in U, has coordinate representation $z\frac{\partial}{\partial z} = \frac{1}{w}\frac{-1}{z^2}\frac{\partial}{\partial w} = -w\frac{\partial}{\partial w}$ in V, and it is a global holomorphic vector field on $\mathbb{CP}^1$. Consider now the holomorphic 2−tensor $Z \otimes Z = z\frac{\partial}{\partial z} \otimes z\frac{\partial}{\partial z} \in \mathcal{O}_{\widetilde{S}}(K^{-2})$. We will show that the transformations of $\mathrm{D}_\infty(\mathbb{CP}^1)$ actually preserve this tensor. After that, the equivariance of the developing map d means that, in this case, the pullback $d^*(Z \otimes Z)$ will be a $\pi_1(S)$-invariant holomorphic bivector field. It will descend to a holomorphic section of K^{-2}, meaning that $\deg(K^{-2}) \geq 0$, by Riemann-Roch. Since $\deg(K^{-2}) = -2\deg(K) = 4 - 4g$ we get that $g = 0$ or $g = 1$, which finishes the proof. We are thus left to show that $Z \otimes Z$ is an invariant tensor field. In the coordinates of U, any transformation of the holonomy group that is of the form $z \mapsto az$, transforms $\frac{\partial}{\partial z}$ to $\frac{1}{a}\frac{\partial}{\partial z}$, and thus Z is mapped to $az\frac{1}{a}\frac{\partial}{\partial z} = z\frac{\partial}{\partial z} = Z$. For transformations of the form $z \mapsto \frac{a}{z}$, the vector $\frac{\partial}{\partial z}$ is mapped to $\frac{-z^2}{a}\frac{\partial}{\partial z}$, and Z to $\frac{a}{z}\frac{-z^2}{a}\frac{\partial}{\partial z} = -z\frac{\partial}{\partial z} = -Z$. This means that $Z \otimes Z$ is mapped to itself, and as so it is invariant. For the other coordinate patch, we use the transition function to note that the transformation $z \mapsto az$ acts as $w \mapsto z = \frac{1}{w} \mapsto az = \frac{a}{w} \mapsto \frac{w}{a}$ and the transformation $z \mapsto \frac{a}{z}$ acts as $w \mapsto z = \frac{1}{w} \mapsto \frac{a}{z} = aw \mapsto \frac{1}{wa}$. In both cases the calculation is the same as before with a replaced by $1/a$. □

Remark 1.8.6. Of course, these restrictions also yield information about the lifted holonomies (cf. Theorem 1.7.2). For example, any lift $\widetilde{\rho} : \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$ of a holonomy representation $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ on a closed surface of genus $g \geq 2$ is irreducible. For suppose the representation $\widetilde{\rho}$ preserves a line in $\mathbb{C}^2$, then ρ preserves a point in $\mathbb{CP}^1$. Conjugating this point to $\infty \in \mathbb{CP}^1$, we see that the holonomy is of type (II) since Möbius transformations that preserve ∞ are affine. By Lemma 1.8.4 this cannot happen.

This Theorem characterizes the possible representations that can appear as holonomies of complex projective structures. The question that follows, of whether these are the only restrictions, appears naturally and dates back to the original problem. That this was indeed the case was conjectured for years, and proved in (Gallo et al., 2000, Theorem 1.1.1.).

Theorem 1.8.7. *Let S be a closed surface of genus $g \geq 2$ and $\rho : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ a non-elementary representation that lifts to* $\mathrm{SL}(2, \mathbb{C})$*. Then there exists a complex projective structure on S whose holonomy is ρ.*

Remark 1.8.8. For each ρ the projective structure is not unique.

The proof of this theorem has proven to be quite elusive and delicate. For instance, the research announcement (Gallo, 1989, Theorem 1.) claimed a proof, but the details never ended up being published. Then the proof presented in (Kapovich, 1995) turned out to be flawed (there are incorrections or omissions in Lemmas 1 and 4). These problems were precisely the motivation for the work in (Gallo et al., 2000), to which introduction we refer to for further details about the history of the result. We note that a further gap in the proof there is present in paragraph 5.5. This was found and corrected in (Fils, 2019).

This theorem has also played a crucial role in the development of the research presented in this thesis. The starting point and original idea was to use the theory of Higgs bundles to provide another proof of the statement. This has proved quite difficult mainly because there is no control over the Riemann surface structure on S compatible with the complex projective structures produced by the theorem. The Riemann surface structure is the kind of information that remains fixed in the non-abelian Hodge correspondence (see Chapter 2).

1.9 The theorem of Poincaré

Propelled by the arguments above where we consider the developing map a global quotient of solutions of the linear differential equation, we note a simple consequence, already known to Poincaré (Poincaré, 1884, §4, page 220-221) or (Appell and Goursat, 1930, Chapitre XV. 117, page 310).

Theorem 1.9.1. *Let X be a closed Riemann surface of genus $g \geq 2$. If two complex projective structures $\mathcal{A}_1$ and $\mathcal{A}_2$ are compatible with X and have the same holonomy then they are identical.*

Proof. Let $\mathcal{A}_\infty$ have developing map d_1 and $\mathcal{A}_2$ have developing map d_2. Since both structures are compatible with X we can write, using the parameterization by quadratic differentials and the Schwarz equation, $d_1 = \frac{u_1}{u_2}$ for $u_j, j = 1, 2$, solutions of $u'' + \frac{1}{2}\vartheta_1 u = 0$ and $d_2 = \frac{v_1}{v_2}$ with $v_j, j = 1, 2$, solutions of $v'' + \frac{1}{2}\vartheta_2 v = 0$, where ϑ_j are $\pi_1(S)$-invariant quadratic differentials. If $\rho_1 : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$ and $\rho_2 : \pi_1(S) \to \mathrm{PSL}(2, \mathbb{C})$, the holonomies of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, are the same, this means that $\rho_1 = g\rho_2 g^{-1}$, for $g \in \mathrm{PSL}(2, \mathbb{C})$. Changing d_2 by composition with g, we can assume that, in fact, $\rho_1 = \rho_2$. Now this means that both u_j and v_j satisfy

$$\begin{bmatrix} u_1(\gamma z)(\gamma')^{-1/2} \\ u_2(\gamma z)(\gamma')^{-1/2} \end{bmatrix} = \pm A_\gamma \begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix} \qquad \begin{bmatrix} v_1(\gamma z)(\gamma')^{-1/2} \\ v_2(\gamma z)(\gamma')^{-1/2} \end{bmatrix} = \pm A_\gamma \begin{bmatrix} v_1(z) \\ v_2(z) \end{bmatrix},$$

for the same homomorphism $\gamma \mapsto \pm A_\gamma = \left(\begin{smallmatrix} \alpha_\gamma & \beta_\gamma \\ \delta_\gamma & \epsilon_\gamma \end{smallmatrix}\right)$. Define $w = u_1 v_2 - v_1 u_2$. We will show in a computation that w transforms under $\pi_1(X)$ as a vector field, and since it is holomorphic, we have by Riemann-Roch that $w = 0$, from where we conclude $d_1 = \frac{u_1}{u_2} = \frac{v_1}{v_2} = d_2$. The computation goes as follows, suppressing the argument in z after the first equality,

$$\begin{aligned} &(u_1(\gamma z)v_2(\gamma z) - v_1(\gamma z)u_2(\gamma z))(\gamma')^{-1} = (\alpha_\gamma u_1 + \beta_\gamma u_2)(\delta_\gamma v_1 + \epsilon_\gamma v_2) - (\alpha_\gamma v_1 + \beta_\gamma v_2)(\delta_\gamma u_1 + \epsilon_\gamma u_2) = \\ &= \alpha_\gamma u_1 \delta_\gamma v_1 + \alpha_\gamma u_1 \epsilon_\gamma v_2 + \beta_\gamma u_2 \delta_\gamma v_1 + \beta_\gamma u_2 \epsilon_\gamma v_2 - \alpha_\gamma v_1 \delta_\gamma u_1 - \alpha_\gamma v_1 \epsilon_\gamma u_2 - \beta_\gamma v_2 \delta_\gamma u_1 - \beta_\gamma v_2 \epsilon_\gamma u_2 \\ &= (\alpha_\gamma \epsilon_\gamma - \beta_\gamma \delta_\gamma)u_1 v_2 + (\beta_\gamma \delta_\gamma - \alpha_\gamma \epsilon_\gamma)v_1 u_2 = u_1 v_2 - v_1 u_2. \end{aligned}$$

□

To be clear, the theorem of Poincaré Theorem 1.9.1 shows that if the complex structure is fixed to be X, the holonomy ρ completely determines the projective structure compatible with X, which is unique if it exists. But this does not exclude that different $\mathbb{CP}^1$-strucutres have the same holonomy. When this happens, these projective structures are surely not compatible with the same Riemann surface structure X. We remark that this indeed happens, and, in fact, infinitely often, as proved, for example, in (Dumas, 2009, 5.4).

Note also that this Theorem 1.9.1 and Theorem 1.8.7 together don't imply that, for a given X and a given ρ, there will be a projective structure compatible with X and with holonomy ρ. This is false, and it will only happen for specific pairs (X, ρ) whose description still seems impossible in full generality (contrary to what happens in genus 1).

1.10 The linear bundle

We now observe a consequence of the fact that holonomies lift, related to the bundle construction of Section 1.4. The point here is that if the holonomy representation $\rho : \pi_1(S) \mapsto \mathrm{PSL}(2,\mathbb{C})$ lifts to $\widetilde{\rho} : \pi_1(S) \mapsto \mathrm{SL}(2,\mathbb{C})$, one can carry out the same construction but using the lifted version $\widetilde{\rho}$ using as fiber the vector space $\mathbb{C}^2$. This means that the construction produces a flat complex vector bundle $E = \widetilde{S} \times_{\widetilde{\rho}} \mathbb{C}^2$ with flat connection ∇. Denote by $\mathring{E}$ the slit bundle, i.e. the bundle E with its zero section removed. This bundle is the deprojectivization of the bundle $P = \widetilde{S} \times_\rho \mathbb{CP}^1$ of $\mathbb{CP}^1$ produced in the original construction. This means that there is a map $\mathring{E} \to P$ that on each fiber just maps the line in $\mathbb{C}^2$ to the point it generates in $\mathbb{CP}^1$:

$$\mathbb{P} : \mathring{E} = \widetilde{S} \times_{\widetilde{\rho}} \mathbb{C}^2\text{-}\{[\begin{smallmatrix}0\\0\end{smallmatrix}]\} \to P = \widetilde{S} \times_\rho \mathbb{CP}^1$$
$$\left(\widetilde{x}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) \mapsto (\widetilde{x}, [v_1 : v_2])\,.$$

For this reason, we also write $\mathbb{P}(E)$ for P. The horizontal foliation determined by $\widetilde{\rho}$ is simply the one determined by the connection ∇ and it is mapped to the horizontal foliation of $\mathbb{P}(E)$ defined by $\widetilde{\rho}$. Note that the transverse section s, coming from the developing map of the structure, defines a slit line subbundle $\mathring{L} \subset \mathring{E}$ simply as $\mathring{L}_x = \mathbb{P}^{-1}(s(x))$. By adding back the zero section, we get a line subbundle L uniquely determined by s and the lift $\widetilde{\rho}$. The transversality condition on the section s means that the subbundle L is everywhere transverse to the horizontal foliation $\mathcal{H}E$ determined by ∇. Since both the line bundle and the horizontal foliation have complex dimension one, and lie inside a two-dimensional space, L is transverse to $\mathcal{H}E$ if L_x is not contained in H_xE (the tangent space of $\mathcal{H}E$ inside E). In conclusion, a complex projective structure determines a flat $\mathrm{SL}(2,\mathbb{C})$-vector bundle E together with a transverse line subbundle L. Of course that, conversely, any flat $\mathrm{SL}(2,\mathbb{C})$-vector bundle E together with such a line subbundle L will determine a complex projective structure, which we denote by the tuple (E, ∇, L). This is because the projectivization $(\mathbb{P}(E), s = \mathbb{P}(L))$ is a flat $\mathbb{CP}^1$-bundle with a transverse section, and this is the graph of a complex projective structure (cf. Theorem 1.4.1). Now to express the space of projective structures as a space of flat $\mathrm{SL}(2,\mathbb{C})$-vector bundles together with transverse line subbundles we need to introduce an adequate equivalence relation, as we did when defining the graph. This equivalence relation must deal with the extra ambiguity introduced by the choice of lifting. As we have seen in Remark 1.7.6, these choices are determined by the possible signs in the lift of the generators γ_j of $\pi_1(S)$. So suppose $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ are two lifts to $\mathrm{SL}(2,\mathbb{C})$ of ρ. Then define

$$\delta(\gamma) = \begin{cases} 1 & \text{if } \widetilde{\rho}_1(\gamma) = \widetilde{\rho}_2(\gamma) \\ -1 & \text{if } \widetilde{\rho}_1(\gamma) = -\widetilde{\rho}_2(\gamma) \end{cases}$$

which compares the choice signs of both lifts. Let us identify $\{\pm 1\} = \mathbb{Z}_2 \cong \{id, -id\} \subset \mathbb{C}^* \subset \mathrm{SL}(2,\mathbb{C})$, where id is the 2×2 identity matrix and $\mathbb{C}^*$ is diagonally embedded in $\mathrm{SL}(2,\mathbb{C})$ as

$\lambda \mapsto \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right]$. Then this comparison map is simply the homomorphism $\delta : \pi_1(S) \to \mathbb{Z}_2$ such that $\gamma \mapsto \widetilde{\rho}_1(\gamma)\left(\widetilde{\rho}_2(\gamma)\right)^{-1}$. We see that the identity $\widetilde{\rho}_1 = \delta \otimes \widetilde{\rho}_2$ is satisfied. Further $\delta \otimes \delta = Id$, the trivial homomorphism. This kind of homomorphism accounts for all the possible ambiguity in the lifting, in the sense that any two lifts define such a map and, further, that given any such map δ with trivial $\otimes$-square, we can obtain any other lift by starting from a fixed one and tensoring it with δ. To take into consideration this at the level of the flat connections and bundles we define two tuples (E_1, ∇_1, L_1) and (E_2, ∇_2, L_2) of flat $\mathrm{SL}(2,\mathbb{C})$-bundles E_j with flat connection ∇_j and transverse line subbundle L_j to be *equivalent* if there is a (smooth) vector bundle isomorphism $\mathsf{F} : E_1 \to E_2 \otimes S$ such that $\mathsf{F}(L_1) \subset L_2 \otimes S$ and $\nabla_2 = \mathsf{F}i^*(\nabla_1) \otimes \nabla_\delta$, for some line bundle S with flat connection ∇_δ of order 2. (This is the flat connection whose holonomy is δ.) The equivalence class of the tuple is then uniquely determined by the projective structure. We call any of the possible flat bundles the *associated vector bundle* and the corresponding line subbundle the *transverse* subbundle of the $\mathbb{CP}^1$-structure.

1.10.1 Gunning's transversality criterion

We will specify this construction for $\mathbb{CP}^1$-surfaces compatible with a fixed Riemann surface X. Recall that in this case, the developing map of the structure is a local biholomorphism, or, equivalently, its graph is a complex manifold with a holomorphic section. From the point of view of the associated $\mathrm{SL}(2,\mathbb{C})$-vector bundle (E, ∇) and corresponding transverse line subbundle L, this means that, when E is given the holomorphic structure $\nabla^{(0,1)}$ determined by the complex structure X, i.e.

$$\nabla^{(0,1)} : \Omega^0(E) \mapsto \Omega^{0,1}(E) \quad v(z) \mapsto \nabla(v(z))^{(0,1)},$$

which is trivially integrable, since there are no two-forms on surfaces, then L is a *holomorphic* line subbundle $\nabla^{(0,1)}(\Omega^0(L)) \subset \Omega^{(0,1)}(L)$. The transversality condition of L can now be described in holomorphic terms. Recall that L is transverse if for every $z \in X$ the line L_z is not contained in the horizontal subspace H_zE determined by the flat connection ∇. This happens if and only if $\nabla(L) \neq 0$, i.e $\nabla(s) \neq 0$ for every non-zero local section of L or, equivalently, the sheaf map $\nabla : \Omega^0(L) \to \Omega^1(E)$ is non-zero. Since L is holomorphic we can consider instead the action of this map on holomorphic sections $\mathrm{H}^0(L)$. It restricts then as $\nabla|_{\mathrm{H}^0(L)} : \mathrm{H}^0(L) \to \mathrm{H}^0(E \otimes K)$, because its $(0,1)$-part is zero on such sections. Since L has a local basis of holomorphic sections, $\nabla(s) \neq 0$ for every local section if and only if $\nabla|_{\mathrm{H}^0(L)}(s) \neq 0$ for every local holomorphic section. Composing the restriction with the quotient map $q : E \to E/L$ we obtain

$$\beta_L = q \circ \nabla|_{\mathrm{H}^0(L)} : \mathrm{H}^0(L) \to \mathrm{H}^0(E/L \otimes K). \tag{1.10.1}$$

which is an $\mathcal{O}$-linear map. This means it determines a holomorphic homomorphism $\beta_L : L \to E/L \otimes K$ called the *holomorphic second fundamental form* of L with respect to ∇. The transversality condition $\nabla|_{\mathrm{H}^0(L)}(s) \neq 0$ for every local holomorphic section is equivalent to β_L being nowhere zero, i.e. an isomorphism, since E/L is a line bundle. Note that since an $\mathrm{SL}(2,\mathbb{C})$-bundle has a trivialized determinant, there is a natural isomorphism $E/L \cong L^{-1}$. This means that the second fundamental form can also be seen as a holomorphic section of $L^{-2}K$, and L is transverse to ∇ if and only if $\beta_L \in \mathrm{H}^0(L^{-2}K)$ is nowhere zero. We collect this in the following theorem due to Gunning (Gunning, 1967b, Theorem 2).

Theorem 1.10.1. *Let $E \to X$ be an $\mathrm{SL}(2,\mathbb{C})$-vector bundle on a Riemann surface X with holomorphic structure $\nabla^{(0,1)}$ given by a flat connection ∇. Then any holomorphic line subbundle $L \subset E$ with nowhere-zero holomorphic second fundamental form $\beta_L \in \mathrm{H}^0(L^{-2}K)$, determines a projective structure compatible with X. Moreover any other $\mathrm{SL}(2,\mathbb{C})$-bundle E' with flat connection ∇' and subbundle L' determines an equivalent projective structure if and only if there is an isomorphism $\mathsf{F} : E \otimes S \to E'$ such that $L' = \mathsf{F}(L \otimes S)$ and $\nabla \otimes \nabla_\delta = \mathsf{F}^*(\nabla')$ for some holomorphic line bundle S with flat connection ∇_δ of order 2.*

Remark 1.10.2. The condition of $\beta_L \in \mathrm{H}^0(L^{-2}K)$ being nowhere zero is equivalent to $L^{-2}K$ being trivialized $L^{-2}K \cong \mathcal{O}$ via β_L, or also to $\beta_L : L \to L^{-1}K$ being an isomorphism, as was noted before.

For closed surfaces of genus g the theorem can be combined with the theorem of Riemann--Roch to provide further information.

Lemma 1.10.3. *Let $E \to X$ be an $\mathrm{SL}(2,\mathbb{C})$-vector bundle on a closed Riemann surface X of genus $g \geq 2$ with holomorphic structure $\nabla^{(0,1)}$ given by a flat connection ∇. The second fundamental form $\beta_L \in \mathrm{H}^0(L^{-2}K)$ of the holomorphic line bundle L is nowhere zero if and only if $\deg(L) = g-1$.*

Proof. We have shown that β_L is nowhere zero if and only if $\beta_L : L \to L^{-1}K$ is an isomorphism. This implies that $L^{-2}K \cong \mathcal{O}$, i.e. $L \cong K^{1/2}$ and thus $\deg(L) = g-1$ by Riemann-Roch. Conversely, suppose $\deg(L) = g-1$. The second fundamental form β_L yields a section in $\mathrm{H}^0(L^2K^{-1})$. Since $\deg(L^2K^{-1}) = 2\deg(L) - (2g-2) = 0$, Riemann-Roch guarantees that either this section is the trivial one or it is everywhere non-zero. In this second case, $L^2K^{-1} \cong \mathcal{O}$. So we only need to exclude the first possibility. Suppose then that $\beta_L = 0$ is the trivial section. This means that L is preserved by the connection (recall that $\beta_L = q \circ \nabla|_{\mathrm{H}^0(L)}$, $q : E \to L$). This contradicts the irreducibility of the flat connection (cf. Remark 1.8.6). $\square$

The conclusion is that a complex projective structure compatible with a closed Riemann surface X of genus $g \geq 2$ determines an $\mathrm{SL}(2,\mathbb{C})$-vector bundle $E \to X$ with flat connection ∇ and a $\nabla^{(0,1)}$-holomorphic line subbundle L of degree $g-1$. This uniquely determines the $\mathbb{CP}^1$-structure up to the equivalence relation in Theorem 1.10.1.

This transversality criterion also provides a simple proof of Theorem 1.9.1 of Poincaré. Indeed, if the holonomies of two complex projective structures compatible with X are equal, then one can choose the same lift to $\mathrm{SL}(2,\mathbb{C})$. This means that the associated vector bundle is the same (E,∇) and we have two transverse line subbundles L_1 and L_2. These are holomorphic subbundles of $(E,\nabla^{(0,1)})$ both of degree $g-1$ by the Lemma. Note that even the existence of a single one of such bundles already guarantees that the bundle $(E,\nabla^{(0,1)})$ is unstable. Since

$\deg(L_1) > 0$, this means that L_1 is the maximal destabilizing subbundle of E, which is unique (Gunning, 1967a, §5 (c) Lemma 15) and thus we have $L_1 = L_2$.

Remark 1.10.4. In other words, from the algebraic geometric point of view, the flat vector bundles that support complex projective structures compatible with a closed Riemann surface X of genus $g \geq 2$ are the ones for which the holomorphic bundle $(E, \nabla^{(0,1)})$ is *maximally unstable*, i.e they have a line subbundle of maximal possible degree. (Recall that the degree of an unstable subbundle is bounded by $g-1$.) We will explore this perspective further after this chapter.

1.11 Space of marked projective structures

At this point, we should make a remark about the spaces of projective structures that are usually defined, even though we won't use them in what will follow. We note that, the space $\mathcal{P}(S)$ we have defined has the three following incarnations:

$$\mathcal{P}(S) = \Big\{ \mathcal{A} \,|\, \mathcal{A} \text{ is a maximal } \mathbb{CP}^1\text{-atlas} \Big\} \leftrightarrow$$

$$\leftrightarrow \frac{\{(\rho, d) \ \ |\rho \in \operatorname{Hom}(\pi_1(S), \operatorname{PSL}(2,\mathbb{C})) \text{ and } d : \widetilde{S} \to \mathbb{CP}^1 \text{ is a } \rho \text{ equivariant local diffeomorphism}\}}{\operatorname{PSL}(2,\mathbb{C})}$$

$$\leftrightarrow \frac{\{(P, s) \,|P \text{ is a flat } \mathbb{CP}^1\text{-bundle and } s \text{ is a transverse section}\}}{\sim_{\text{equivalence}}}.$$

These are infinite-dimensional spaces. For example, any diffeomorphism of S can be used to produce another atlas from a given one, which, in general, won't be the same. The usual way to deal with this space $\mathcal{P}(S)$ is to quotient it by an adequate equivalence relation. For atlases, this is the equivalence $\approx_{iso}$ under isomorphism of $\mathbb{CP}^1$-surfaces, which should be translated suitably to the other incarnations of the space. Now, although the quotient $\mathcal{P}(S)$ is finite-dimensional it is not Hausdorff. So to deal with this further question, one proceeds as in Teichmüller theory. We define two complex projective structures on S to be *marked isomorphic* if there is a complex projective isomorphism between them which is isotopic to the identity map in S. Using this stronger notion of equivalence $\approx_{iso}^{mark}$ the space produced is actually a complex manifold $\boldsymbol{\mathcal{P}}(S) = \mathcal{P}(S)/ \approx_{iso}^{mark}$. Further the map sending a projective structure to its induced Riemann surface structure factors as a holomorphic vector bundle map $\boldsymbol{\mathcal{P}}(S) \to \mathcal{T}(S)$ to the Teichmüller space of S, see, for example, (Dumas, 2009, §3.3). An important result we won't need here is that, for a closed surface S of genus $g \geq 2$, the holonomy map

$$\operatorname{hol} : \boldsymbol{\mathcal{P}}(S) \to \mathcal{X}^{\operatorname{SL}(2,\mathbb{C})} = \operatorname{Hom}^{cr}(\pi_1(S), \operatorname{SL}(2,\mathbb{C}))/\operatorname{SL}(2,\mathbb{C}),$$

sending the marked class of a projective structure to its holonomy inside the character variety is a local biholomorphism. Here Hom^{cr} denotes the completely reducible representations. We refer to (Dumas, 2009, §5.2) and the bibliography therein. It was also shown in (Hejhal, 1975, Theorem 8.) that hol is not a covering map.

1.12 Branched projective structures

In this section we will review the branched analog of projective structures introduced and studied in (Mandelbaum, 1972, 1973). The constructions are similar to the ones in previous sections, so we only provide details where the arguments are different. As usual, the idea of branching involves considering geometric structures on $S - B$, where B is a discrete set of points on S. Furthermore, one imposes regularity conditions around these points, so that the structures obtained are still controlled by the geometry of S.

A *branched projective structure* is then the same as a complex projective structure $\mathcal{A} = \{(U_\alpha, \varphi_\alpha : U_\alpha \to \mathbb{CP}^1)\}$, but now the complex projective charts are not required to be homeomorphisms onto their image, but are instead required to be topological (at most) singly-branched r-coverings from topological disks U_α of S onto open sets of $\mathbb{CP}^1$. This is to say that the charts φ_α are coverings with r sheets, $r \geq 1$ an integer, except possibly at a single point of their domain $p_\alpha \in U_\alpha$. This point is imposed to be the only one on its pre-image under φ_α. The points p_α are called the *branching points* of $\mathcal{A}$ and the set $B_\mathcal{A} = \{p_\alpha\}_{\alpha \in A}$, which is well defined and discrete, is called its *branching set*. The transition functions of the atlas $\mathcal{A}$ are still required to be Möbius transformations. One can show that such an atlas also induces a complex structure X, called, as before, the *induced* complex structure. Again we also use the terminology that $\mathcal{A}$ and X are *compatible*. Using holomorphic coordinates for this complex structure, the charts φ_α look like $z \mapsto z^r$, for integer $r > 1$ only at the branching points. The value of $r - 1$ at each p_α is called the *order of* $\mathcal{A}$ *at* p_α, and it is denoted by $\mathrm{ord}_\mathcal{A}(p_\alpha)$. We can collect this information into a divisor supported at B called the *branching divisor* $D_\mathcal{A}$:

$$D_\mathcal{A} = \sum_\alpha \mathrm{ord}_\mathcal{A}(p_\alpha) \cdot p_\alpha.$$

This is an effective divisor that is finite if S is compact since B is finite in that case. When we want to stress the difference between branched projective structures and usual complex projective structures (with $B = \emptyset$) we call these last ones *unbranched*. We denote the space of branched projective structures on a surface S as $\mathcal{B}(S)$ and the set of those compatible with the Riemmann surface structure X as $\mathcal{B}(X)$.

Example 1.12.1. The most common example of a branched projective structure is the pull-back of a (not necessarily branched) projective structure under a branched cover of Riemann surfaces. This means we can take such a cover $\varpi : Y \to X$ and any of the unbranched examples we have seen to produce an assortment of branched ones.

Of course outside the branching set B a branched projective structure induces an unbranched complex projective one, i.e. $S - B$ is a $\mathbb{CP}^1$-surface. This means that the results of previous sections (valid for not-necessarily-compact surfaces) apply directly to $S - B$. We will see if and how they generalize to branched projective structures.

1.12.1 Holonomy, development and fiber bundles

Let then $\mathcal{A}$ a branched projective structure on S, inducing the Riemann surface structure X, and with branching set $B_{\mathcal{A}} = \{p_\alpha\}_{\alpha \in A}$. Clearly, all the previous constructions, the development map, the holonomy, and the graph, do work directly on $X - B$, since it is a $\mathbb{CP}^1$-surface.

As seen in Remark 1.3.3, the developing map agrees locally with charts. This means that around a branching point p_α the developing map d of $X - B$ agrees with a chart, and it is thus of the form $z \mapsto z^{r_\alpha}$, with $r_\alpha = \mathrm{ord}_{\mathcal{A}}(p_\alpha) + 1$. Thus we can extend the developing map d by continuity to $(X - B) \cup \{p_\alpha\}$. This extension is no longer a local diffeomorphism, but the derivative $d'(z) = rz^{r-1}$ has a zero of order $r - 1 = \mathrm{ord}_{\mathcal{A}}(p_\alpha)$ at the point p_α. We can continue this procedure for every p_α, obtaining the *developing map of the branched projective structure*. This is no longer a local diffeomorphism around every point but there is a discrete set where it fails to be an immersion to order $r - 1$. The branching divisor corresponds to the divisor of zeros of the derivative $d'(z)$.

Now for the representation, we note that the holonomy of $X - B$ is well-defined as a map $\rho_{X-B} : \pi_1(X - B) \to \mathrm{PSL}(2, \mathbb{C}))$. The inclusion $\iota : X - B \to X$ induces the map of fundamental groups $\iota_* : \pi_1(X - B) \to \pi_1(X)$ that maps the generators corresponding to loops γ_α around the punctures p_α to the trivial element. We will see later, when studying the lifting properties, that ρ_{X-B} maps these generators to $\pm Id \in \mathrm{PSL}(2, \mathbb{C})$, that is, the holonomy is trivial around the branching point(cf. (De Saint-Gervais, 2016, Proposition IX.1.2.)). Thus ρ_{X-B} factorizes as a map $\rho : \pi_1(X) \to \mathrm{PSL}(2, \mathbb{C}))$ and this defines the *holonomy of the branched structure*. It is clear that the extended developing map d is ρ-equivariant, since it is ρ_{X-B}-equivariant on $S - B$.

Repeating the construction of the graph is now immediate using the holonomy representation ρ and the developing map d of the branched structures. This means we can build a flat $\mathbb{CP}^1$-fiber bundle $P = \widetilde{S} \times_\rho \mathbb{CP}^1$ which comes together with a section $s(x) = [\widetilde{x}, d(\widetilde{x})]$, where $\widetilde{x}$ is any lift of $x \in S$ to the universal cover, corresponding to d. Outside the branching points, this section is transverse to the horizontal foliation $\mathcal{H}$ with leaves $\mathcal{H}_u = \{[\widetilde{x}, u] | \widetilde{x} \in \widetilde{S}\}$, with fixed $u \in \mathbb{CP}^1$. Using the holomorphic coordinates compatible with the structure we see that around a branching point $s(x) = [z, d(z)] = [z, z^{r_\alpha}]$, and so s is horizontal to order $\mathrm{ord}_{\mathcal{A}}(p_\alpha) = r_\alpha - 1$.

The conclusion is that a branched projective structure determines a flat $\mathbb{CP}^1_1$-bundle together P with a section s which is transverse except at a discrete set of points where it has finite order of contact with the horizontal foliation. Note that the branching divisor of the structure $D_{\mathcal{A}}$ is the same as the order of contact divisor between s and $\mathcal{H}$. These objects are only determined by the structure up to a suitable equivalence relation similar to the non-branched case.

Of course, we can still define subclasses of branched projective structures as we did for the unbranched ones. The branched affine or hyperbolic structures are defined as in Definition 1.5.2 and the holonomy subgroups as in Definition 1.5.1, simply by allowing branching.

1.12.2 Schwarz parameterization

Let us observe what happens to the Schwarzian parameterization in the branched case. So we let S be a surface and $\mathcal{A}_2$ be a branched projective structure compatible with the Riemann surface structure X, and which we suppose uniformized by another projective structure $\mathcal{A}_2$. The construction of the $\pi_1(S)$-quadratic differential using the Schwarzian derivative works exactly as before

$$\vartheta = \widetilde{\mathcal{A}}_2 - \widetilde{\mathcal{A}}_1 = \mathrm{S}(d_2(z))dz^2,$$

but it is now possible for the quadratic differential to be singular at the branch points where $d'(z) = 0$. Indeed, if we calculate the Schwarzian derivative around a branch point p_α where $d(z) = z^{r_\alpha}, r_\alpha \geq 2$ we see that

$$d'(z) = r_\alpha z^{r_\alpha - 1}, \qquad d''(z) = r_\alpha(r_\alpha - 1)z^{r_\alpha - 2}, \qquad \frac{d''(z)}{d'(z)} = \frac{r_\alpha - 1}{z}, \qquad \left(\frac{d''(z)}{d'(z)}\right)' = -\frac{r_\alpha - 1}{z^2},$$

$$\begin{aligned} \mathrm{S}(d) = \left(\frac{d''(z)}{d'(z)}\right)' - \frac{1}{2}\left(\frac{d''(z)}{d'(z)}\right)^2 &= -\frac{r_\alpha - 1}{z^2} - \frac{1}{2}\left(\frac{r_\alpha - 1}{z}\right)^2 \\ &= -\frac{(r_\alpha - 1 + r_\alpha^2/2 - r_\alpha + 1/2)}{z^2} = \frac{1 - r_\alpha^2}{2z^2}. \end{aligned}$$

Quadratic differentials of this form, with double poles at the branching points p_α and with Laurent expansion around p_α with coefficient of the form $r_\alpha^2 - \frac{1}{2}$, for r_α an integer, are thus the only ones that do come up in the parameterization of branched projective structures. Nonetheless, the integers r_α together with the form of the expansion around chosen branch points are not enough to uniquely determine a single quadratic differential, and there is, for each branching divisor D, a class that satisfies these requirements. These are called *integrable* meromorphic quadratic differentials of type D, whose set is denoted by $\mathcal{V}_D^X$. In (Hejhal, 1975, Lemma 1) or in (Mandelbaum, 1972, Theorem 3.) it is shown how these differentials are described. A sketch of the idea goes as follows. The relation between the Schwarzian derivative and the linear equation of Schwarz in Corollary 1.6.4 will still hold in the branched case, and the condition for a quadratic differential of said form to appear is that the linear equation has only non-logarithmic solutions. This is because the developing map is meromorphic if and only if this happens, and this condition is necessary and sufficient for d to be locally of the form $z \mapsto z^{r_\alpha}$. From the classical theory of singular differential equations, the quadratic differentials that make the equation have non-logarithmic solutions are determined by the zeros of a polynomial. These are the so-called *apparent singularities* of the solutions at a regular singular point (Poincaré, 1884, pag. 217). This result can be seen from the Frobenius method for the solution of the linear equation, for example, and we refer to (Haraoka, 2020, Theorem 4.9 or Example 4.2) for a contemporary exposition and further details. The conclusion is that the Schwarz parameterization is still valid for branched projective structures compatible with X and with fixed branching divisor D, but the space of usable quadratic differentials is now $\mathcal{V}_D^X$, the space of integrable ones. The spaces $\mathcal{V}_D^X$ are affine algebraic varieties inside the vector space of meromorphic quadratic differentials with at most double poles. This is to be understood in a general sense when X is not compact since then the spaces might not be finite-dimensional and, given that each branched point determines a polynomial equation, there might be an infinite number of them defining the variety $\mathcal{V}_D^X$. In the case of non-branched structures, $D = 0$ is the empty divisor, and $\mathcal{V}_0^X$ is the space of regular quadratic differentials.

1.12.3 Holonomy lifts

We have seen in Theorem 1.7.2 that the holonomies of complex projective structures lift to $\mathrm{SL}(2, \mathbb{C})$. What happens in the branched case? For this last part, we will work only with *finitely* branched structures, i.e., the ones for which the branching divisor is finite B. This won't be relevant as our main results concern closed surfaces.

Theorem 1.12.2. *Let $\mathcal{A}$ be a branched projective structure on a surface S with finite branching divisor $D = \sum_{k=1}^m \mathrm{ord}_{\mathcal{A}}(p_k) \cdot p_k$. Then the holonomy of $\mathcal{A}$ lifts to $\mathrm{SL}(2, \mathbb{C})$ if and only if $\deg(D) = \sum_{k=1}^m \mathrm{ord}_{\mathcal{A}}(p_k)$ is even.*

Sketch of proof. Let B be the branching set of $\mathcal{A}$, and denote by X its induced Riemann surface structure. We have already shown in the proof of Theorem 1.7.2 that the holonomy of an unbranched projective structure lifts to $\mathrm{SL}(2, \mathbb{C})$. So the holonomy $\rho_{X-B} : \pi_1(S - B) \to \mathrm{PSL}(2, \mathbb{C})$ lifts to $\widetilde{\rho_{X-B}} : \pi_1(S - B) \to \mathrm{PSL}(2, \mathbb{C})$. It is known from the classical theory of linear differential equations that the lifted holonomy maps the elements γ_k corresponding to loops around the punctures to $\widetilde{\rho_{X-B}}(\gamma_k) = (-1)^{r_\alpha} Id$, that is

$$\begin{bmatrix} u_1(\gamma_k z)(\gamma_k')^{-1/2} \\ u_2(\gamma_k z)(\gamma_k')^{-1/2} \end{bmatrix} = \begin{bmatrix} (-1)^{r_\alpha} & 0 \\ 0 & (-1)^{r_\alpha} \end{bmatrix} \begin{bmatrix} u_1(z) \\ u_2(z) \end{bmatrix},$$

where u_j are linearly independent solutions of the Schwarz linear equation. We refer to (Haraoka, 2020, section 4.3) for a proof of the general case of differential equations of order n or to (De Saint-Gervais, 2016, Theorem IX.1.1 and) for a proof of a version of this result specifically for the Schwarz equation. Now let $\pi_1(S) = \langle a_j | R(a_j) \rangle$ be a presentation of the fundamental group of S. Then, the the fundamental group of $S - B$ has a presentation of the form $\pi_1(S - B) = \langle a_j \gamma_k | R(a_j) \prod_{k=1}^m \gamma_k \rangle$, with the γ_k corresponding to the punctures. This means $\widetilde{\rho_{X-B}}$ will define a map $\widetilde{\rho} : \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$ if $\prod_{k=1}^m \widetilde{\rho_{X-B}}(\gamma_k) = Id \in \mathrm{SL}(2, \mathbb{C})$. Since $\widetilde{\rho_{X-B}}(\gamma_k) = (-1)^{r_\alpha} Id$ this happens if and only if $\deg(D) = \sum_{k=1}^m \mathrm{ord}_{\mathcal{A}}(p_k)$ is even. □

Remark 1.12.3. This also shows that the holonomy of the branched structure is well defined since projectively all loops around punctures are mapped trivially, and so ρ_{X-B} always factors through $\iota_* : \pi_1(S - B) \to \pi_1(S)$, regardless of the order of D.

1.12.4 Characterization of holonomy on closed surfaces

The results about the restrictions on the holonomy of closed $\mathbb{CP}^1$-surfaces won't always generalize to branched surfaces with branching divisor D. This is easily seen because the use of the theorem of Riemann-Roch plays a fundamental role in their proofs. Note, however, that upon suitable conditions on the degree of D, one can still make these proof work. For example, it is impossible for a branched structure to have an elementary holonomy group of type (III) if $\deg(D) < 2g - 2$. The same proof of Lemma 1.8.5 works, with the detail that the holomorphic bivector used is now meromorphic with pole divisor $-2D$. Since $\deg(K^{-2}) = 4 - 4g$, the result still follows. But when the degree of D jumps to $2g - 2$, the proof breaks down. In fact, it is possible to have a branched complex projective structure with holonomy ρ in $\mathrm{D}_\infty(\mathbb{C})$, for a suitable D. This naturally gives rise to the question of what is the minimum degree $\delta(\rho)$ the branching divisor must have, for a given representation to be the holonomy of a branched projective structure. Theorem 1.8.7 shows that $\delta(\rho) = 0$ for ρ non-elementary and liftable to $\mathrm{SL}(2, \mathbb{C})$. In the same article, the authors also showed that $\delta(\rho) = 1$ for ρ non-elementary and non-liftable. For all other representations, the function $\delta : \mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2, \mathbb{C}))$ is described in (Fils, 2021, Figure 1.) which we refer to for further details.

1.12.5 Gunning's criterion for branched structures

Gunning's construction and his criterion can still be given for finitely branched projective structures, but not all of them. This is because the construction requires a lift of the holonomy representation to $\mathrm{SL}(2, \mathbb{C})$, and for branched structures, its existence is no longer automatic. So, from now on we will only consider branched projective structures whose holonomy lifts to $\mathrm{SL}(2, \mathbb{C})$. By Theorem 1.12.2 this happens if and only if $\deg(D)$ is even. The construction proceeds as in Section 1.10.1. Using the lifted holonomy representation $\widetilde{\rho}$ of a branched structure $\mathcal{A}$, we build the $\mathrm{SL}(2, \mathbb{C})$-vector bundle $E = \widetilde{S} \times_{\widetilde{\rho}} \mathbb{C}^2$ with flat connection ∇, which projectivizes to the graph of the structure $(\mathbb{P}(E) = P, s)$. The transverse section then defines a line subbundle L which is transverse to ∇ except at the branching points p_α where $L_{p_\alpha} \subset H_{p_\alpha}E$, with HE the tangent bundle of the horizontal foliation determined by ∇. One can again give the complex structure X induced by $\mathcal{A}$ to the surface. This means that $E \to X$ is a complex vector bundle that becomes holomorphic by giving it the $\overline{\partial}$-operator $\nabla^{(0,1)}$. The transverse line subbundle L is holomorphic for this structure and we can define its second fundamental form β_L as before by equation 1.10.1. The transversality condition is weakened since it is only satisfied outside the branching points. At the branching set, β_L will have zeros, since $L_{p_\alpha} \subset H_{p_\alpha}E$. These zeros have precisely the same order as the branching points, and thus the branching divisor is expressed as $D = \mathrm{div}(\beta_L)$. Thus a branched projective structure compatible with X and whose holonomy lifts to $\mathrm{SL}(2, \mathbb{C})$ is the same as a flat $\mathrm{SL}(2, \mathbb{C})$-bundle together with a $\nabla^{(0,1)}$-holomorphic line subbundle, whose second fundamental form β_L is non-zero. This data only defined up to the same equivalence relation as before.

Theorem 1.12.4. *Let $E \to X$ be an $\mathrm{SL}(2, \mathbb{C})$-vector bundle on a Riemann surface X with holomorphic structure $\nabla^{(0,1)}$ given by a flat connection ∇. Then any holomorphic line subbundle $L \subset E$ with non-zero holomorphic second fundamental form β_L, determines a branched projective structure compatible with X, with branching divisor $\mathrm{div}(\beta_L)$. Moreover any other $\mathrm{SL}(2, \mathbb{C})$-bundle E' with flat connection ∇' and subbundle L' determines an equivalent projective structure* *if and only if there is an isomorphism* $\mathsf{F} : E \otimes S \to E'$ *such that* $L' = \mathsf{F}(L \otimes S)$ *and* $\nabla \otimes \delta = \mathsf{F}^*(\nabla')$ *for some holomorphic line bundle* S *with flat connection* δ *of order 2.*

Remark 1.12.5. Note that for closed and branched projective surfaces there is no analog of Lemma 1.10.3. Indeed the theorem of Riemann-Roch no longer restricts the holomorphic subbundle to be $K^{1/2}$. In fact, for $\deg(L) < g - 1$ we have $\deg(L^{-2}K) > 0$, and there are non-constant sections $\beta_L \in \mathrm{H}^0(L^{-2}K)$. Note also that when $\deg(L) \leq 0$, the bundle E is no longer unstable (cf. Remark 1.10.4 and the paragraph above it). This happens when the branching divisor $D = \mathrm{div}(\beta_L)$ has degree $\deg(D) \geq 2g - 2$. This is another change in behavior already reflected by the breaking down of the proofs of the restrictions on the holonomy of closed $\mathbb{CP}^1$-surfaces, as it was noted in Subsection 1.12.4.

This finishes the collection of results about complex projective structures we gather in this chapter. In what follows, we will explore further constructions using the vector bundle perspective, and this last Theorem 1.12.4 will be our main tool.

Chapter 2

Higgs bundles, opers and Riemann surfaces

The purpose of this chapter is to recall a diversity of results that will play a main role in the construction of Chapter 3. In particular, we start by recalling the non-abelian Hodge correspondence, the moduli space of Higgs bundles, and their relation with the geometry of Riemann surfaces, as pioneered by Hitchin and his proof of the uniformization theorem (Hitchin, 1987, Theorem (11.2)). Then we set up Gaiotto's conformal limit (Gaiotto, 2014) and cite known existence results. After that, we recall the notion of oper, introduced first in (Drinfel'd and Sokolov, 1985) but reformulated using modern language in (Beilinson and Drinfeld, 2005), and their partial analogs introduced in (Simpson, 2010). Finally, we include some statements about classical Teichmüller space and the Teichmüller (Finsler) metric. Contrary to the previous chapter, we will only provide details when they are needed to understand the constructions of Chapter 3, as most of the other ones are well-known and easily accessible in the literature.

2.1 Non-abelian Hodge correspondence

The non-abelian Hodge correspondence is now a well-known construction, due to the seminal work of several mathematicians, such as Hitchin, Donaldson, Corlette, and others. For details, history and more bibliography, we refer to the introductory surveys (Guichard, 2017; Li, 2019b; Thomas, 2023; Wentworth, 2016). Briefly, for a fixed closed Riemann surface X of genus $g \geq 2$, this correspondence establishes a homeomorphism between the moduli space of Higgs bundles and the moduli space of flat connections. In what follows we will mainly review the existence of solutions to Hitchin equations, the so-called harmonic metrics, because they will play the main role in our results. This means we mostly work with the correspondence in the Hitchin direction, i.e., starting with a Higgs Bundle and building the corresponding flat connection. From now on X will always be a closed Riemann surface of genus $g \geq 2$. We will also define $\mathrm{SL}(n, \mathbb{C})$-Higgs bundles, even though the correspondence works for more general Lie groups.

Definition 2.1.1. Let X be a closed Riemann surface. A *Higgs bundle* is a vector bundle $E \to X$ with holomorphic structure $\overline{\partial}_E$, together with a *Higgs field*, i.e a holomorphic one-form $\Phi \in \Omega^{(1,0)}(\mathrm{End}(E))$ valued in the endomorphisms of E. We denote it by $(E, \overline{\partial}_E, \Phi)$ or simply by $(\overline{\partial}_E, \Phi)$ when the bundle is understood. It is an $\mathrm{SL}(n,\mathbb{C})$-*Higgs bundle* when $(E, \overline{\partial}_E)$ is a holomorphic $\mathrm{SL}(n,\mathbb{C})$-vector bundle (i.e. it has holomorphically trivialized $\det(E)$) and $\Phi \in \Omega^{(1,0)}(\mathrm{End}_0(E))$ is trace-free.

Note that for $n = 1$ Higgs bundle is simply a line bundle L with a holomorphic structure, i.e. an element of $\mathrm{Pic}^0(X)$, together with a holomorphic one form, since $\mathrm{End}(L)$ is trivial in this case. We note also that we can perform the usual operations with Higgs bundles, such as the direct sum, tensor products, and extensions, considering the induced Higgs fields. The notion of isomorphic Higgs bundles is defined similarly, by imposing the isomorphism preserves the Higgs field, and the trivialization of $\det(E)$ in the $\mathrm{SL}(n,\mathbb{C})$-case. The relevant class of Higgs bundles is composed of polystable ones. Recall that the slope of a vector bundle E is $\mathrm{slope}(E) = \frac{\deg(E)}{\mathrm{rank}(E)}$.

Definition 2.1.2. Let $(E, \overline{\partial}_E, \Phi)$ be a Higgs bundle. It is called (semi)stable if every proper Φ-invariant holomorphic subbundle $V \subset E$ satisfies $\mathrm{slope}(V)(\leq) < \mathrm{slope}(E)$. It is called polystable if it is the direct sum of stable Higgs bundles.

Remark 2.1.3. The definition of stability for a vector bundle E is analogous but one considers all proper subbundles $V \subset E$.

We will be interested in $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. This means that $(E, \overline{\partial}_E, \Phi)$ is strictly polystable (i.e polystable but not stable) if and only if it is a direct sum of the form $E = L \oplus L^{-1}$, with L a line bundle with $\deg(L) = 0$. This is because $E = L \oplus E/L$ and $E/L \cong L^{-1}$, where the isomorphism is induced by the trivialization of $\det(E)$, and only happens when $\mathrm{rank}(E) = 2$. One can also see that E is topologically trivial since it has degree zero. We now move on to the metric.

Definition 2.1.4. Let $(\overline{\partial}_E, \Phi)$ be a $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle. Let H be a Hermitian metric on E and let F_{A_H} be the curvature of the Chern connection A_H determined by H and $\overline{\partial}_E$. Fix $R \in \mathbb{R}^+$. A metric which induces the trivial metric on $\det(E)$ is called *harmonic with parameter* R if it satisfies the R-scaled version of Hitchin's equation:

$$F_{A_H} + R^2 \left[\Phi, \Phi^{*_H}\right] = 0. \tag{2.1.1}$$

Remark 2.1.5. If $R = 1$ we get the usual Hitchin equation for the Higgs bundle $(\overline{\partial}_E, \Phi)$, and a metric which is harmonic with parameter 1 is simply called *harmonic*. Thus a metric is harmonic with parameter R if and only if it is a solution to Hitchin's equation for the Higgs bundle $(\overline{\partial}_E, R\Phi)$.

The main point of the correspondence is that polystability is the necessary and sufficient condition for the existence of such a metric.

Theorem 2.1.6. *An* $\mathrm{SL}(n,\mathbb{C})$*-Higgs bundle* $(\overline{\partial}_E, \Phi)$ *is polystable if and only if, for each* $R \in \mathbb{R}^+$*, there is a harmonic metric with parameter* R*. This metric is unique if* $(\overline{\partial}_E, \Phi)$ *is stable.*

Note that Hitchin's equation for $R = 1$ is equivalent to the flatness of the connection $\nabla = A_H + \Phi + \Phi^{*_H}$. This is the flat connection associated with the Higgs bundle $(\overline{\partial}_E, \Phi)$. Denote by $\mathcal{H}^{ps}(E)$ the configuration space of Higgs bundles on the (topologically) trivial rank 2 bundle E, i.e. $\mathcal{H}^{ps}(E)$ is a subset of the fibered product of the affine space of $\overline{\partial}$-operators on E with the vector space of Higgs fields, and by $\mathcal{C}(E)$ the space of flat connections. We define the *non-abelian Hodge map* as

$$\mathrm{NH} : \mathcal{H}^{ps}(E) \to \mathcal{C}(E) \tag{2.1.2}$$

$$(\overline{\partial}_E, \Phi) \mapsto A_H + \Phi + \Phi^{*_H}, \tag{2.1.3}$$

where H is the harmonic metric, i.e. the solution of Hitchin's equation for $R = 1$. Note that this map is only well defined on the subset of stable Higgs bundles $\mathcal{H}^s$, since only there the metric H is unique. But we can extend it to $\mathcal{H}^{ps}(E)$ by choosing metrics in each strictly polystable case. Since this choice won't be relevant, we write the map in this way, always assuming an adequate choice has been made if needed. This map actually has image inside the set of totally reducible flat connections $\mathcal{C}^{cr}(E)$, and it descends to a map between the moduli spaces $\mathcal{M}_{Db} = \mathcal{H}^{ps}(E)/\sim$ and $\mathcal{M}_{dRh} = \mathcal{C}^{cr}(E)/\mathcal{G}$, where the $\sim$ denotes equivalence by isomorphism and $\mathcal{G} = \Omega^0(\mathrm{Aut}(E))$ the smooth gauge group. We will mainly work on the configuration spaces, and so will not worry about the much-studied geometric properties of the moduli spaces. We note that the moduli space of completely reducible flat connections $\mathcal{M}_{dRh}$ can be identified with the character variety $\mathcal{X}^{\mathrm{SL}(n,\mathbb{C})} = \mathrm{Hom}^{cr}(\pi_1(S), \mathrm{SL}(n,\mathbb{C}))/\mathrm{SL}(n,\mathbb{C})$, with $\mathrm{Hom}^{cr}(\pi_1(S), \mathrm{SL}(n,\mathbb{C}))$ the space of completely reducible representations, where $\mathrm{SL}(n,\mathbb{C})$ acts by conjugation. This identification is made via the holonomy of the flat connection.

Definition 2.1.7. An $\mathrm{SL}(2,\mathbb{R})$-*Higgs bundle* is as an $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle E with a holomorphic decomposition $E = L \oplus L^{-1}$, where L is a holomorphic line bundle and with the Higgs field has the form $\Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$, for this decomposition, where $\alpha \in \mathrm{H}^0(L^2K)$ and $\beta \in \mathrm{H}^0(L^{-2}K)$.

It is so defined because the corresponding connection under the non-abelian Hodge correspondence has holonomy conjugated (in $\mathrm{SL}(2,\mathbb{C})$) to a subgroup of $\mathrm{SL}(2,\mathbb{R})$. For such a Higgs bundle the *Toledo invariant* $\deg(L)$ satisfies a Milnor-Wood type inequality $0 \leq |\deg(L)| \leq g-1$. Note further that the $\mathrm{SL}(2,\mathbb{C})$-gauge transformation $E \to E$ given with respect to the decomposition by $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$ interchanges L and L^{-1}. This means that when picking a representative in the configuration space of a point of the moduli space $\mathcal{M}_{Db}$, we can assume $\deg(L) \geq 0$. The $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle stability condition for the case $\deg(L) > 0$ is simply $\beta \neq 0$. This is because L is the maximal destabilizing subbundle of E, which is not preserved by the Higgs field if and only if $\beta \neq 0$. There are no strictly polystable cases. For the case $\deg(L) = 0$ the condition is more delicate, and it is analyzed in Section 3.4.

Definition 2.1.8. The $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles of the form $E = K^{1/2} \oplus K^{-1/2}$, for $K^{1/2}$ a square root of the canonical bundle, and with Higgs field $\Phi = \left(\begin{smallmatrix} 0 & q \\ 1 & 0 \end{smallmatrix}\right)$, where $1 \in \mathcal{O}$ and $q \in \mathrm{H}^0(K^2)$, are said to be in a *Hitchin component*.

These are the Higgs bundles that, under the non-abelian Hodge correspondence, map to connections whose holonomies lie inside the special components of the real character variety $\mathcal{X}^{\mathrm{SL}(2,\mathbb{R})} \subset \mathcal{X}^{\mathrm{SL}(2,\mathbb{C})}$ called the Hitchin components. So the structure of $\mathcal{X}^{\mathrm{SL}(2,\mathbb{R})}$ is as follows. The subspaces $\mathcal{X}_d^{\mathrm{SL}(2,\mathbb{R})}$ of representations with fixed Toledo invariant $\deg(L) = d$ are connected components, except $\mathcal{X}_{g-1}^{\mathrm{SL}(2,\mathbb{R})}$. The latter has the 2^{2g} connected components: the Hitchin components, which account for the choice of a square root of K. Moreover, they are parameterized by quadratic differentials $q = \alpha \in \mathrm{H}^0(K^2)$. Hitchin in (Hitchin, 1987) proved that a Hitchin component parameterizes all hyperbolic structures on S (Definition 1.5.2) and that under the non-abelian Hodge correspondence a hyperbolic structure is sent to its holonomy. Biswas et al. in (Biswas et al., 2021) generalized this to show that any $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle with $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$ gives rise to a branched hyperbolic structure with branching divisor $\mathrm{div}(\beta)$.

Remark 2.1.9. There are higher rank Hitchin components for $\mathrm{SL}(n,\mathbb{C})$ as described by Hitchin in (Hitchin, 1992), and they share similar properties with the ones inside $\mathrm{SL}(2,\mathbb{C})$, but we won't need them for what follows.

2.2 The conformal limit

The existence of the harmonic metric with parameter allows one to use R to deform the flat connection associated with the Higgs Bundle. This construction was introduced in (Gaiotto, 2014). Given a polystable $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle $(\overline{\partial}_E, \Phi)$ and a fixed $\hbar \in \mathbb{C}^*$ one has the $\mathbb{R}^+$-family of flat connections

$$\nabla_{\hbar,R} = A_{H_R} + \hbar^{-1}\Phi + \hbar R^2 \Phi^{*_{H_R}} \tag{2.2.1}$$

where H_R is the harmonic metric with parameter $R \in \mathbb{R}^+$ for $(\overline{\partial}_E, \Phi)$, A_{H_R} is the Chern connection for $\overline{\partial}_E$ and H_R, and the adjoint $*_{H_R}$ is taken also with respect to the metric H_R.

Definition 2.2.1. The *$\hbar$-conformal limit* of $(\overline{\partial}_E, \Phi)$ is the connection

$$\nabla_{\hbar,0} := \lim_{R \to 0} \nabla_{\hbar,R} \tag{2.2.2}$$

when it exists.

Remark 2.2.2. In the case $\hbar = 1$ note that, if $R = 1$, the connection $\nabla_{1,1}$ is just $\mathrm{NH}(\overline{\partial}_E, \Phi)$ the one given by the usual non-abelian Hodge correspondence.

This limit is usually set up on the moduli space, and so it is understood as the limit of a gauge class of connections. Its existence in this sense was established in (Dumitrescu et al., 2021, Theorem 3.2.) for $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles in the Hitchin components, and in (Collier and

Wentworth, 2018, Proposition 5.1.) for all stable $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles. In both cases, the use of the inverse function theorem in infinite dimensional Banach spaces is instrumental to finding suitable gauge transformations acting on $\nabla_{\hbar,R}$. Here we will work directly with the limit defined on the configuration space, and for $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles we shall give a more direct argument both in the stable case (in Section 3.1) and in the polystable case (in Section 3.4.1)). We give a further proof for $\mathrm{SL}(2,\mathbb{C})$-polystable Higgs bundles in Theorem 3.6.4.

2.3 Partial opers and branched projective structures

Classical opers were introduced in (Drinfel'd and Sokolov, 1985) and the concept was reformulated in modern language in (Beilinson and Drinfeld, 2005). They are defined as a flat bundle together with a full filtration by holomorphic subbundles, whose induced map on quotients satisfies a transversality condition. It turns out that their definition exactly matches the definition of an associated vector bundle to a complex projective structure as stated by Gunning's criterion Theorem 1.10.1. In (Simpson, 2010) the filtration used is not necessarily full, and the transversality condition is replaced by the *gr*-semistability of the associated graded Higgs bundle, giving rise to the notion of *partial oper*. Again, these will be related to branched projective structures by the branched version of Gunning's criterion, Theorem 1.12.4.

Definition 2.3.1. (Simpson, 2010) Let X be a Riemann surface and consider an $\mathrm{SL}(2,\mathbb{C})$-bundle E with flat connection ∇ together with a filtration

$$0 \subset L \subset E$$

by a $\nabla^{(0,1)}$-holomorphic subbundle L. Let $\beta_L : L \to E/L \otimes K$ be the $\mathcal{O}$-linear map induced by ∇. The *associated graded* is the Higgs bundle $(\mathrm{Gr}(E), \theta)$ where

$$\mathrm{Gr}(E) = L \oplus E/L \cong L \oplus L^{-1} \quad \text{and} \quad \theta = \begin{pmatrix} 0 & 0 \\ \beta_L & 0 \end{pmatrix}.$$

If this Higgs bundle is semistable we call the filtration *a partial* $\mathrm{SL}(2,\mathbb{C})$*-oper*. Two partial opers are equivalent if their flat bundles are isomorphic by a gauge transformation that preserves the filtration.

Remark 2.3.2. Note that if $\deg(L) > 0$ then L is the maximal destabilizing subbundle of $\mathrm{Gr}(E)$. This means that $(\mathrm{Gr}(E), \theta)$ is (semi)stable as a Higgs bundle precisely if L is not preserved by θ, and this happens if and only if $\beta_L \not\equiv 0$. In this case, the definitions of associated vector bundle to a branched projective structure compatible with X and of partial oper are the same, by Gunning's criterion Theorem 1.12.4, and they just require β_L to be non-zero. In the case $\deg(L) = 0$ though, the bundle $\mathrm{Gr}(E)$ is semistable (as a holomorphic bundle). Thus $(\mathrm{Gr}(E), \theta)$ is semistable as a Higgs bundle even if β_L is zero. In this situation, the definition of partial oper includes more objects than the branched projective structures. These structures will correspond to the partial opers with non-zero β_L.

Remark 2.3.3. We remark that Simpson's definition in (Simpson, 2010) allows for filtrations which are not full. In particular, $0 \subset E$, with E semistable as a holomorphic bundle, is considered a partial oper structure in that paper, but not here.

Remark 2.3.4. In the case of classical, or *full*, opers is obtained when $\beta_L : L \to L^{-1}K$ is an isomorphism. In this case $L^2 \cong K$ and $\deg(L) = g-1$, and $(\mathrm{Gr}(E), \theta)$ is automatically stable, since $\beta_L \neq 0$. These correspond to unbranched projective structures compatible with X.

For now, we will be interested in the case of $\beta_L \neq 0$. Then, for a Riemann surface X, a branched projective structure compatible with X is the same as such a partial $\mathrm{SL}(2, \mathbb{C})$-oper, with the caveat that the equivalence of projective structures is slightly weaker than that of opers. While for opers the filtration must be preserved by gauge equivalence, in the situation of projective structures the gauge equivalence is allowed to twist the subbundle.

We also recall that the structure of a partial $\mathrm{SL}(2, \mathbb{C})$-oper on E, in particular, realizes E as an extension of the $\nabla^{(0,1)}$-holomorphic line bundles, namely of L by $E/L \cong L^{-1}$:

$$0 \to L \to E \xrightarrow{\beta_L} L^{-1} \to 0.$$

These extensions are classified by an element of the Dolbeault cohomology $\mathrm{H}^1(\mathrm{Hom}(L^{-1}, L)) \cong \mathrm{H}^1(L^2)$, and thus represented by a $(0,1)$-form with values in L^2. In any C^∞ decomposition of E of the form $E = L \oplus L^{-1}$, the holomorphic structure $\nabla^{(0,1)}$ can be written as

$$\nabla^{(0,1)} = \begin{pmatrix} \partial_L & \omega \\ 0 & \partial_{L^{-1}} \end{pmatrix},$$

since L is holomorphic, and the class $[\omega] \in \mathrm{H}^1(L^2)$ is called is the *extension class* of the partial $\mathrm{SL}(2, \mathbb{C})$-oper (E, ∇).

The conformal limit in Definition 2.2.1 of $\mathrm{SL}(2, \mathbb{R})$-Higgs bundles, seen in the moduli space, is known to be a partial oper. For the Hitchin component this is explicit in (Dumitrescu et al., 2021, Theorem 3.2.), and for the other $\mathrm{SL}(2, \mathbb{R})$-components it can be read from (Collier and Wentworth, 2018, Proposition 5.1.). These results, together with the ones of the previous section (cf. Definition 2.1.8 and paragraph after it), show that, for $R = 1$ and $\hbar = 1$, the connections $\nabla_{1,1}$ for $\mathrm{SL}(2, \mathbb{R})$-Higgs bundles with $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$ correspond to branched hyperbolic structures, while the conformal limit connections $\nabla_{\hbar,0}$ correspond to partial opers, and thus to branched projective structures compatible with X. We will see that these are instances of the same phenomenon, as in fact, when $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$, there is a family of branched projective structures continuously deforming the ones with holonomy $\nabla_{\hbar,1}$ into the ones with holonomy $\nabla_\hbar, 0$. This is the main contribution present in Chapter 3. For that, we will need some classical results about Teichmüller theory which we now recall.

2.4 Beltrami differentials and complex structures

There are several approaches to the description of the Teichmüller space $\mathcal{T}(S)$ of a closed surface S. This is the space of marked complex structures on S and it is classically identified with the unit L^1-ball in the space of holomorphic quadratic differentials $\mathrm{H}^0(K^2)$ for some fixed Riemann surface structure via Teichmüller's embedding, see for example (Daskalopoulos and Wentworth, 2007, Theorem 2.9). Its construction relies on the use of Beltrami differentials to change the complex structure.

Definition 2.4.1. Let X be Riemann surface with canonical bundle K. A *Beltrami differential* μ is a (smooth) section of $\overline{K} \otimes K^{-1}$ whose sup-norm is strictly bounded by 1, i.e., an element of the set

$$\mathcal{B}(X) = \left\{ \mu \in \Omega^0(\overline{K} \otimes K^{-1}) = \Omega^{(-1,1)}(X) \,|\, ||\mu||_\infty < 1 \right\}. \tag{2.4.1}$$

Remark 2.4.2. Note that the transformation law for the coordinates of μ is $\mu_2(z_2) = \mu_1(z_1)\frac{dz_2/dz_1}{d\overline{z}_2/d\overline{z}_1}$. As such, the value of $|\mu(z)|$ is independent of the holomorphic coordinate chart and it is a well-defined quantity whose supremum we denote by $||\mu||_\infty$.

A Beltrami differential can be used to build a complex atlas for a different complex structure in the following way. Let z be a coordinate for X and let $\mu = \mu(z)d\overline{z} \otimes \frac{\partial}{\partial z} \in \Omega^0(\overline{K} \otimes K^{-1})$. Consider the local *Beltrami equation* on the contractible open $U \subset X$ for a function $v : U \to \mathbb{C}$

$$\partial_{\overline{z}} v = \mu(z)\partial_z v. \tag{2.4.2}$$

In this situation, $\mu(z)$ is called the *parameter* of the equation. The classical existence result states that such a v exists if $||\mu||_\infty < 1$ on U. Furthermore, such function v is a diffeomorphism onto some open set of $\mathbb{C}$. Now, we can check the condition for existence on each open set U of X, and this happens precisely if μ is a Beltrami differential. If we collect all such local functions v together we can verify they form a complex atlas for a new complex structure denoted by X_μ. A contemporary description of this result can be found for example in (Hubbard, 2006, Theorem 4.8.12).

Theorem 2.4.3. *Let X be a Riemann Surface. Then any Beltrami differential $\mu \in \mathcal{B}(X)$ determines a complex structure X_μ whose local charts are the solutions of the Beltrami equation with parameter $\mu(z)$.*

In conclusion, if X has coordinate z, then X_μ has coordinate v such that $\partial_{\overline{z}} v = \mu(z)\partial_z v$. Writing $\nu = \partial_z v$ we can observe that the bases of $(1,0)$ and $(0,1)$-forms of X_μ are given with respect to X by

$$dv = \nu(dz + \mu(z)d\overline{z}) \tag{2.4.3}$$
$$d\overline{v} = \overline{\nu}(d\overline{z} + \overline{\mu}(z)dz), \tag{2.4.4}$$

with $\nu \neq 0$ since v is a local diffeomorphism.

Remark 2.4.4. It is actually true that any complex structure up to biholomorphism isotopic to the identity arises in this way. (This can be deduced, for example, from the surjectivity statement of (Daskalopoulos and Wentworth, 2007, Theorem 2.9), together with the fact that the map there factors through a map out of $\mathcal{B}(X)$). In particular, there is an identification of $\mathcal{T}(S)$ with the space of Beltrami differentials $\mathcal{B}(X)$ modulo the equivalence relation where $\mu \sim \mu'$ if there is a biholomorphism $X_\mu \to X_{\mu'}$ isotopic to the identity.

2.4.1 Teichmüller geodesics and disks

The Teichmüller space $\mathcal{T}(S)$ has several interesting metrics. One of them is the Teichmüller metric, which is a Finsler metric, and whose distance function is defined using the properties of quasi-conformal mappings. In particular, the distance between $X_\mu, X_{\mu'} \in \mathcal{T}(S)$ is the smallest possible maximal dilation of a quasi-conformal mapping $f : X_\mu \to X_{\mu'}$ in the isotopy class of the identity of S. It is a classical result of Teichmüller that each such class has a unique map that realizes this minimum, the so-called Teichmüller mappings. These maps have complex dilations of the form

$$\mu = \frac{\overline{\partial} f}{\partial f} = c\frac{\overline{q}}{|q|}, \qquad 0 < c < 1,$$

where c is a constant and q a quadratic differential. They allow one to describe the geodesic rays through the origin X_0 in $\mathcal{T}(S)$ (cf. (Lehto, 1987, Section V.7.7.)).

Theorem 2.4.5. *Let $\mu = t\frac{\overline{q}}{|q|}$ be a Beltrami differential with $q \in \mathrm{H}^0(X, K^2)$ a quadratic differential, and $t \in [0, 1)$. Then $t \mapsto X_{\mu(t)}$ is a geodesic ray in the Teichmüller metric, called the* ray associated with q.

Remark 2.4.6. One can even show that if t is allowed to be in the hyperbolic disk $\mathbb{D}$ the map $t \mapsto X_{\mu(t)}$ is an isometry (Lehto, 1987, Theorem V.9.3.). Maps of this form are called complex geodesics or *Teichmüller Disks*.

Chapter 3

Higgs bundles and projective structures

In this chapter we introduce the main contribution of the thesis, to be submitted for publication as (Silva and Gothen, 2024).

It is a construction of branched projective structures associated with the family 2.2.1 that appears in the conformal limit, which can be summarized as follows.

Theorem. *Let* $(E = L \oplus L^{-1}, \Phi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix})$ *be an* $\mathrm{SL}(2,\mathbb{R})$*-Higgs bundle, where* $\alpha \in \mathrm{H}^0(L^2K)$ *and* $\beta \in \mathrm{H}^0(L^{-2}K)$*, with* $0 < \deg(L) \leq g-1$ *and* $\beta \neq 0$*. Assume that* $|\hbar R| \leq 1$ *and* $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$*. Then the following results hold.*

1. *There is a Riemann surface structure* $X_\mu \in \mathcal{T}(S)$ *associated to a Beltrami differential* $\mu = \mu(\hbar, R)$ *on* X *and a branched projective structure* $\mathcal{P}(\hbar, R) \in \mathcal{B}(S)$ *with branching divisor* $\mathrm{div}(\beta)$*, compatible with* X_μ*.*

2. *The family* $\mathcal{P}(\hbar, R)$ *depends continuously on* $(\hbar, R)$ *and interpolates between a branched hyperbolic structure* $\mathcal{P}(\hbar, 1)$ *and the branched projective structure given by the partial oper* $(\nabla_{\hbar,0}, L)$*.*

For $\deg(L) = 0$ *the construction goes through under the same conditions but for the parameters with* $|\hbar R| < 1$ *(strict inequality). Further, the curve* $R \mapsto X_{\mu(\hbar,R)}$ *in* $\mathcal{T}(S)$ *is a geodesic ray in the Teichmüller metric.*

The proof of these results will occupy the rest of the chapter. To establish them, we will also provide a simpler proof of the existence of the conformal limit, working at the level of the configuration spaces of polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles.

3.1 Existence of the $\mathrm{SL}(2,\mathbb{R})$-conformal limit

Recall Definition 2.1.8 where we view an $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle as an $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle E with a holomorphic decomposition $E = L \oplus L^{-1}$, where L is a holomorphic line bundle, and a Higgs field of the form $\Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$ for this decomposition, where $\alpha \in \mathrm{H}^0(L^2K)$ and $\beta \in \mathrm{H}^0(L^{-2}K)$. As we have seen the *Toledo invariant* $\deg(L)$ satisfies a Milnor-Wood type inequality $0 \leq |\deg(L)| \leq g-1$, and we can assume, by duality, that an $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle has $\deg(L) \geq 0$.

Recall that the $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle stability condition for the case $\deg(L) > 0$ is simply $\beta \neq 0$, and that there are no strictly polystable cases. For the case $\deg(L) = 0$ the condition will be analyzed in Section 3.4.

The harmonic metric H in either case is known to diagonalize (Hitchin (1987) or (Alessandrini, 2019, Proposition 5.2), for example) with respect to this decomposition, so $H = \left(\begin{smallmatrix} h & 0 \\ 0 & h^{-1} \end{smallmatrix}\right)$, where h is a metric in the line bundle L.

If we choose a holomorphic frame for L, and the induced holomorphic frame in E, h is locally given by a positive function, still denoted by h, or $h(z)$ if we want to make explicit the dependence on a complex coordinate z in X. The Chern connection for H is given in this frame by $A_H = d + \left(\begin{smallmatrix} \partial \log h & 0 \\ 0 & -\partial \log h \end{smallmatrix}\right) = \left(\begin{smallmatrix} \partial_z \log h \, dz & 0 \\ 0 & -\partial_z \log h \, dz \end{smallmatrix}\right)$. Further α and β are given by 1-forms $\alpha = \alpha(z)dz$ and $\beta = \beta(z)dz$. Recalling that locally $\Phi^{*_H} = H^{-1}\overline{\Phi}^T H$, Hitchin's equations (2.1.1) for this case ($R = 1$) read

$$
\begin{aligned}
&0 = F_{A_H} + [\Phi, \Phi^{*_H}] \Leftrightarrow \\
&0 = \begin{pmatrix} \overline{\partial}\partial \log h & 0 \\ 0 & -\overline{\partial}\partial \log h \end{pmatrix} + \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & \overline{\beta} h^{-2} \\ \overline{\alpha} h^2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \overline{\beta} h^{-2} \\ \overline{\alpha} h^2 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}.
\end{aligned}
$$

This simplifies to the single scalar (and unscaled) vortex equation

$$\partial_{\overline{z}}\partial_z \log h = |\alpha|^2 h^2 - |\beta|^2 h^{-2}.$$

The R-scaled version is given in the following definition and just says that $H_R = \left(\begin{smallmatrix} h_R & 0 \\ 0 & h_R^{-1} \end{smallmatrix}\right)$ is harmonic with parameter R for $\left(E = L \oplus L^{-1}, \Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)\right)$.

Definition 3.1.1. Let L be a holomorphic line bundle and consider the sections $\alpha \in \mathrm{H}^0(L^2K)$ and $\beta \in \mathrm{H}^0(L^{-2}K)$, and $R \in \mathbb{R}^+$. A metric h_R in L *solves the R-scaled vortex equation for* (α, β) if locally in a holomorphic frame

$$\partial_{\overline{z}}\partial_z \log h_R = R^2 \left(|\alpha|^2 h_R^2 - |\beta|^2 h_R^{-2}\right).$$

Remark 3.1.2. For $R = 0$ this is an equation for a metric of zero curvature. If such a Hermitian metric exists on L then $\deg(L) = 0$.

Example 3.1.3. If the Higgs bundle lies in a Hitchin components then

$$\left(E = K^{1/2} \oplus K^{-1/2}, \Phi = \left(\begin{smallmatrix} 0 & q \\ 1 & 0 \end{smallmatrix}\right)\right),$$

for a choice $K^{1/2}$ of square root of the canonical bundle K, from the 2^{2g} available, and $q \in \mathrm{H}^0(K^2)$ a quadratic differential. The R-scaled vortex equation is then just the R-scaled version of the abelian vortex equation as in Hitchin (Hitchin, 1987)

$$\partial_{\overline{z}}\partial_z \log h = R^2 \left(|q|^2 h^2 - h^{-2}\right).$$

In this case $g = h^{-2}$ is a metric in $(K^{1/2})^{-2} \cong K^{-1} \cong TX$, which satisfies

$$\partial_{\overline{z}}\partial_z \log g = 2R^2 g \left(1 - \frac{q\overline{q}}{g^2}\right). \tag{3.1.1}$$

For $R = 1$ and $q = 0$, this is the equation for a Riemannian metric g_0 of constant negative curvature -4.

In the case of $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles, the family of connections (2.2.1) that comes up in the conformal limit is thus

$$\begin{aligned}
\nabla_{\hbar,R} &= A_{H_R} + \hbar^{-1}\Phi + \hbar R^2 \Phi^{*_{H_R}} \\
&= d + \begin{pmatrix} \partial \log h_R & 0 \\ 0 & -\partial \log h_R \end{pmatrix} + \hbar^{-1} \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} + \hbar R^2 \begin{pmatrix} 0 & \overline{\beta} h_R^{-2} \\ \overline{\alpha} h_R^2 & 0 \end{pmatrix} \\
&= d + \begin{pmatrix} \partial \log h_R & \hbar^{-1}\alpha + \hbar R^2 \overline{\beta} h_R^{-2} \\ \hbar^{-1}\beta + \hbar R^2 \overline{\alpha} h_R^2 & -\partial \log h_R \end{pmatrix}.
\end{aligned} \tag{3.1.2}$$

To explicitly calculate this limit in the configuration space we make use of a symmetry of the equations. The argument relates the solutions of the scaled equation with the solutions of the unscaled one. We will give a more general argument valid for $\mathrm{SL}(2,\mathbb{C})$2-Higgs bundles in Proposition 3.6.1

Proposition 3.1.4. *Let L be a holomorphic line bundle, $\alpha \in \mathrm{H}^0(L^2K)$, $\beta \in \mathrm{H}^0(L^{-2}K)$, and $R \in \mathbb{R}^+$. A metric h_R is a solution of the R-scaled vortex equation for (α, β) if and only if $h := \frac{h_R}{R}$ is a solution of the unscaled vortex equation for $(R^2\alpha, \beta)$.*

Proof. Let $h := R^{-1}h_R$ be a solution of the unscaled vortex equation for $(R^2\alpha, \beta)$. This happens if and only if

$$\partial_{\overline{z}}\partial_z \log h = |R^2\alpha|^2 h^2 - |\beta|^2 h^{-2}.$$

Given the fact that R does not depend on z, i.e., $\partial_z \log h = \partial_z \log h_R - \partial_z \log R = \partial_z \log h_R$, the expression is equivalent to

$$\partial_{\overline{z}}\partial_z \log h_R = |R^2\alpha|^2 R^{-2} h_R^2 - |\beta|^2 R^2 h_R^{-2} = R^2 \left(|\alpha|^2 h_R^2 - |\beta|^2 h_R^{-2}\right),$$

and so h_R solves the R-scaled vortex equation for (α, β). □

Corollary 3.1.5. *Let $\deg(L) > 0$. Then the pointwise limit of $h := \frac{h_R}{R}$ as $R \to 0$ exists as a metric and it is a solution of the unscaled vortex equation for $(0, \beta)$.*

Proof. By Proposition 3.1.4 h is a solution of the unscaled vortex equations for $(R^2\alpha, \beta)$. This describes a continuous path $(E = L \oplus L^{-1}, \Phi(R))$ of polystable Higgs bundles, with

$\Phi(R) = \left(\begin{smallmatrix} 0 & R^2\alpha \\ \beta & 0 \end{smallmatrix}\right)$. Note that it is a well-defined path since, $(E, \Phi(R))$is stable for all R. This happens even for $R = 0$, since then the Higgs bundle is stable because $\deg(L) > 0$. (The subbundle L is maximally destabilizing.) Under the homeomorphism to the space of harmonic bundles, this is mapped to a continuous path of metrics, and $\lim_{R\to 0} h$ is the metric associated to $\Phi(0)$. ☐

Remark 3.1.6. If h_R is a solution of the R-scaled vortex equation for a fixed (α, β), it was already noted in (Dumitrescu et al., 2021, Theorem 3.2. and after) (for the Hitchin component) that $\lim_{R\to 0} h_R$ does not exist in general. In our case, this can be seen directly from the fact that a solution for the 0-scaled equation is just a metric with curvature equal to zero (cf. Remark 3.1.2), and thus it can only exist in the case $\deg(L) = 0$, but not for general L. More precisely, given the Corollary, it follows that $h = h_R/R$ tends to the solution of the unscaled vortex equations with $(0, \beta)$, and so $h_R = h\,R$ tends to zero as fast as $R \to 0$ for $\deg(L) > 0$. We note also that for $\deg(L) = 0$ the limit $\lim_{R\to 0} h$ does not exist, since in that case $(0, \beta)$ defines a non-polystable Higgs bundle (cf. Section 3.4 for further details).

Using Corollary 3.1.5 we can now calculate the conformal limit.

Theorem 3.1.7. *Let X be a closed Riemann surface of genus $g \geq 2$. Consider the vector bundle $E = L \oplus L^{-1}$, with $1 \leq \deg(L) \leq g-1$, and induced holomorphic structure $\overline{\partial}_E$ and Higgs field $\Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$, where $\alpha \in \mathrm{H}^0(L^2K)$ and $0 \neq \beta \in \mathrm{H}^0(L^2K)$.*

Then, the $\hbar$-conformal limit $\nabla_{\hbar,0}$ of the $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle $(\overline{\partial}_E, \Phi)$ exists. Using the holomorphic frame of E induced by a holomorphic frame of L, the coordinate representation of $\nabla_{\hbar,0}$ is

$$\nabla_{\hbar,0} = d + \begin{pmatrix} \partial \log h_0 & \hbar^{-1}\alpha + \hbar\overline{\beta} h_0^{-2} \\ \hbar^{-1}\beta & -\partial \log h_0 \end{pmatrix}, \tag{3.1.3}$$

where h_0 is the solution of the unscaled vortex equations for $(0, \beta)$.

Proof. By Proposition 3.1.4, for each $R \in \mathbb{R}^+$, we can write h_R as $h_R = h\,R$, where h is the solution of the unscaled vortex equation for $(R^2\alpha, \beta)$. By Corollary 3.1.5, the limit of h as $R \to 0$ exists and we have $\lim_{R\to 0} h = h_0$. We also note that $\partial_z \log R = 0$.

We are now able to compute the following limits:

$$\begin{aligned}
&\lim_{R\to 0} \partial_z \log h_R = \lim_{R\to 0} \partial_z \log(h\,R) = \lim_{R\to 0} (\partial_z \log h + \partial_z \log R) = \lim_{R\to 0} \partial_z \log h = \partial_z \log h_0, \\
&\lim_{R\to 0} R^2 h_R^2 = \lim_{R\to 0} R^4 h = 0, \\
&\lim_{R\to 0} R^2 h_R^{-2} = \lim_{R\to 0} h^{-2} = h_0^{-2}.
\end{aligned}$$

Taking $R \to 0$ in the family $\nabla_{\hbar,R}$ of (3.1.2) we then get the existence and explicit form of the conformal limit stated. ☐

Remark 3.1.8. Note that for $R \neq 0$, the family $\nabla_{\hbar,R}$ in 3.1.2, as a function of the parameters $\hbar$ and R into the configuration space of flat connections, is continuous. So, in fact, we have shown here that this continuity extends to $R = 0$.

Remark 3.1.9. Recall the discussion about the components of the moduli space of $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles (after Definition 2.1.8). There are 2^{2g} Hitchin components corresponding to the maximal Toledo invariant $|\deg(L)| = g-1$. Further, there are $g-1$ non-maximal components, corresponding to $0 \leq |\deg(L)| < g$. The existence of the conformal limit is thus established for all the components except for the minimal one, i.e., the one for which $\deg(L) = 0$. This will be done in Section 3.4.

3.2 The conformal limit is a partial oper

We now observe that the conformal limit calculated in Corollary 3.1.7 defines a partial oper. This was already proved in (Collier and Wentworth, 2018) in greater generality but it sets the stage for the construction of branched projective structures in Section 3.3.

Theorem 3.2.1. *The $\hbar$-conformal limit $\nabla_{\hbar,0}$ of the $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle $E = L \oplus L^{-1}$, with $1 \leq \deg(L) \leq g-1$, and Higgs field $\Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$, where $\alpha \in \mathrm{H}^0(L^2K)$ and $0 \neq \beta \in \mathrm{H}^0(L^2K)$ is a partial $\mathrm{SL}(2,\mathbb{C})$-oper*

$$0 \subset L \subset (E, \nabla_{\hbar,0})$$

with second fundamental form $\hbar^{-1}\beta$.

Proof. The expression (3.1.3) shows that $\nabla^{(0,1)}_{\hbar,0}$ preserves L which is therefore a holomorphic subbundle as required. Moreover, the second fundamental form is the lower left-hand corner of the matrix in (3.1.3) which is indeed the non-zero holomorphic section $\hbar^{-1}\beta$. □

Remark 3.2.2. This is the same as saying that the conformal limit yields a branched projective structure compatible with X. Its branching divisor is precisely the divisor of β in the Higgs field.

Remark 3.2.3. Observe also that in the case where $\alpha = 0$, the Higgs field Φ lies in the nilpotent cone. In this situation, the entire family $\nabla_{\hbar,R}$ in (3.1.2) can be seen directly to be a partial oper, since L still defines a $\nabla^{(0,1)}_{\hbar,0}$-holomorphic subbundle. (The matrix form of $\nabla^{(0,1)}_{\hbar,0}$ is upper triangular.) So the proof of the theorem actually works for non-zero R when $\alpha = 0$. We will see that there are more general conditions under which $\nabla_{\hbar,R}$ is the holonomy of a branched projective structure.

Remark 3.2.4. The extension class of the limit partial oper is represented by the upper right-hand corner of $\nabla^{0,1}_{\hbar,0}$ which reads $[\hbar\overline{\beta}h_0^{-2}] \in \mathrm{H}^{0,1}(L^2) \cong \mathrm{H}^1(L^2)$.

Remark 3.2.5. Since the branching divisor of the projective structure given by the conformal limit is the divisor of zeros of β, we see that the $\hbar$-conformal limit $\nabla_{\hbar,0}$ of the $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle $(\overline{\partial}_E, \Phi)$ is a full oper if and only if β is nowhere vanishing, i.e., if and only if $(\overline{\partial}_E, \Phi)$ lies in a Hitchin component.

It is important to note that partial opers correspond only to branched projective structures which are compatible with a fixed Riemann surface structure X. To study what happens along the conformal limit, i.e., as $R \to 0$ in $\nabla_{\hbar,R}$, and to check that these connections actually correspond to branched projective structures we need to vary the structure X. To this purpose, we will use the results of Section 2.4.

3.3 Branched projective structures coming from the conformal limit

We are now ready to carry out the main construction of branched projective structures coming from the conformal limit. We will define an appropriate Riemann surface structure X_μ for which $\nabla_{\hbar,R}$ is a partial oper, i.e., it defines a branched $\mathbb{CP}^1$-structure compatible with X_μ. This construction will carry through provided the Higgs field $\Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right)$ lies in the special locus where the divisors of zeros satisfy $\operatorname{div}(\alpha) \geq \operatorname{div}(\beta)$ and given also that $|\hbar^2 R^2| \leq 1$. In this section, we shall assume that $\deg(L) > 0$. The case $\deg(L) = 0$ will be treated in Section 3.4.

Theorem 3.3.1. *Fix $R \in \mathbb{R}_0^+$ and $\hbar \in \mathbb{C}^*$ such that $|\hbar^2 R^2| \leq 1$. Let L be a line bundle of degree $1 \leq \deg(L) \leq g-1$ and h_R be a solution of the R-scaled vortex equation for (α, β), where $\alpha \in \mathrm{H}^0(L^2 K)$ and $0 \neq \beta \in \mathrm{H}^0(L^2 K)$. Assume further that $\operatorname{div}(\alpha) \geq \operatorname{div}(\beta)$ and let $\mu = \hbar^2 R^2 \frac{\overline{\alpha}}{\beta} h_R^2$. Then μ is a Beltrami differential. Further $\nabla_{\hbar,R}$ is a partial oper for X_μ, with branching divisor $\operatorname{div}(\beta)$. In particular, it determines a branched projective structure compatible with X_μ.*

Remark 3.3.2. The case $\operatorname{div}(\alpha) = \operatorname{div}(\beta)$ implies that $\deg(L^2 K) = \deg(L^{-2} K)$ and thus $\deg(L) = 0$, which is excluded by the hypothesis on L. Moreover, if $\alpha = 0$, we consider that the condition $\operatorname{div}(\alpha) \geq \operatorname{div}(\beta)$ holds and then $\mu = 0$. In this case, everything is compatible with the base Riemann surface structure X, and $\nabla_{\hbar,R}$ is a partial oper (cf. Remark 3.2.3).

Proof. For $\alpha = 0$ there is nothing to prove in view of the preceding remark. So we treat the case of non-zero α. Note that $\mu = \hbar^2 R^2 \frac{\overline{\alpha}}{\beta} h_R^2$ is a smooth section of

$$\overline{L^2 K} \otimes (L^{-2} K)^{-1} \otimes (L^{-2} \overline{L}^{-2}) \cong \overline{K} \otimes K^{-1}.$$

Thus only $||\mu||_\infty < 1$ needs to be checked to prove that μ is a Beltrami differential. This will be done in Lemma 3.3.3 below. Now we can define a complex structure X_μ by Theorem 2.4.3 whose coordinates are given by solutions v of the Beltrami equation for μ. Using Equation (2.4.3) one knows that $dv = \nu(dz + \mu(z) d\overline{z})$, with $\nu = \partial_z v \neq 0$. Thus, writing $\alpha = \alpha(z) dz$ and $\beta = \beta(z) dz$, the flat connection $\nabla_{\hbar,R}$ can be written as in (3.1.2),

$$\nabla_{\hbar,R} - d = \begin{pmatrix} * & * \\ \hbar^{-1} \beta(z) \left(dz + \hbar^2 R^2 \frac{\overline{\alpha}(z)}{\beta(z)} h_R^2(z) d\overline{z} \right) & * \end{pmatrix} = \begin{pmatrix} * & * \\ \hbar^{-1} \frac{\beta(z)}{\nu} dv & * \end{pmatrix}. \tag{3.3.1}$$

This shows in particular that the holomorphic structure $\nabla_{\hbar,R}^{(0,1)_\mu}$ on E (as a bundle on X_μ) preserves L, since

$$\nabla_{\hbar,R}^{(0,1)_\mu} - \overline{\partial}^\mu = \begin{pmatrix} * & * \\ \hbar^{-1}\frac{\beta(z)}{\nu}dv & * \end{pmatrix}^{(0,1)_\mu} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

where $\overline{\partial}^\mu$ is the $\overline{\partial}$ operator of X_μ. This means that $L_\mu \subset E_\mu$ is in fact a holomorphic subbundle, where the subscript μ indicates that we are considering holomorphic bundles on X_μ with the holomorphic structure induced by the flat connection $\nabla_{\hbar,R}$.

It remains to show that the X_μ-holomorphic second fundamental form β_L^μ is non-zero. To this effect, we note that β_L^μ is the $\mathcal{O}_{X_\mu}$-localization of $q \circ \nabla_{\hbar,R} : \mathcal{O}_{X_\mu}(L_\mu) \to \mathcal{O}_{X_\mu}(E_\mu/L_\mu \otimes K_\mu)$, where K_μ is the canonical bundle of X_μ. But, observing the form of $\nabla_{\hbar,R}$ in (3.3.1), this is simply locally given by multiplication by $\hbar^{-1}\frac{\beta(z)}{\nu}$, which is non-zero. Further, the order of vanishing at each point is precisely the one of β, thus implying that the branching divisor is $\operatorname{div}(\beta)$. □

Lemma 3.3.3. *Let $|\hbar^2 R^2| \leq 1$ then $|\mu(z)|^2 = \left|\hbar^2 R^2 \frac{\overline{\alpha}(z)}{\beta(z)} h_R^2(z)\right|^2 < 1$ everywhere on X.*

Proof. We observe that, as $\operatorname{div}(\alpha) \geq \operatorname{div}(\beta)$, μ is smooth. We consider the function

$$u(z) = \log \frac{|\mu|^2}{|\hbar^2 R^2|^2} = \log \left| \frac{\alpha(z)}{\beta(z)} h_R^2(z) \right|^2.$$

This is simply the logarithm of the norm squared of the section $\frac{\alpha}{\beta} \in \mathrm{H}^0(L^4)$, where L is given the metric h_R which is a solution of the vortex equations (3.1.1). We will show that $u < 0$ everywhere on M, thus implying $|\mu(z)|^2 < |\hbar^2 R^2|^2$. As, by hypothesis $|\hbar^2 R^2| \leq 1$, the conclusion that $|\mu(z)|^2 < 1$ everywhere follows. To achieve the inequality $u < 0$ we use the maximum principle for elliptic operators (as in (Hitchin, 1992, proof of Theorem (11.2)), following (Li, 2019b, Claim 6.1)), which we set up as follows.

Consider the set $\{z_1, z_2, \cdots, z_k\}$ of zeros of $\frac{\alpha}{\beta}$, which is non-empty, because $\operatorname{div}(\alpha)$ is not everywhere equal to $\operatorname{div}(\beta)$. In this set, the function u has negative singularities, i.e., points where $\lim_{z \to z_j} u(z) = -\infty$. This means we can consider small closed disks around these points where $u(z)$ is as negative as we want. In particular, we can get disks where $u < 0$ (strict inequality). When we remove these disks from the surface M, we get a manifold with boundary U. The function u is smooth on the interior of U and continuous up to the boundary ∂U, where $u < 0$. These are part of the conditions to apply the maximum principle. Note that we only need to show that $u < 0$ on the interior of U, since in $M \backslash U$ the inequality already holds (possibly with $u = -\infty$).

Suppressing from the notation the dependence on z, we see that $e^u = h_R^4 |\frac{\alpha}{\beta}|^2$, and thus $e^{u/2} = h_R^2 |\frac{\alpha}{\beta}|$. So, writing $u = \log \frac{\alpha\overline{\alpha}}{\beta\overline{\beta}} h_R^4$, we can calculate that, away from the zeros of α (in particular in U),

$$\begin{aligned}
\partial_{\bar z}\partial_z u &= \partial_{\bar z}\partial_z \log\frac{\alpha}{\beta} + \partial_{\bar z}\partial_z \log\frac{\overline{\alpha}}{\overline{\beta}} + 4\partial_{\bar z}\partial_z \log h_R \\
&= 0 + 0 + 4\partial_{\bar z}\partial_z \log h_R \qquad\qquad (\text{because } \tfrac{\alpha}{\beta} \text{ is holomorphic}) \\
&= 4R^2\left(|\alpha|^2 h_R^2 - |\beta|^2 h_R^{-2}\right) \qquad\qquad (\text{by the vortex equation (3.1.1)}) \\
&= 4R^2|\alpha\beta|\left(\frac{|\alpha|}{|\beta|}h_R^2 - \frac{|\beta|}{|\alpha|}h_R^{-2}\right) \\
&= 4R^2|\alpha\beta|\left(e^{u/2} - e^{-u/2}\right) = 8R^2|\alpha\beta|\sinh(u/2).
\end{aligned}$$

Recalling that the Laplacian of X is $\Delta = \frac{4}{g_0}\partial_{\bar z}\partial_z$, where g_0 is the metric of constant negative curvature -4 (cf. Example 3.1.3), we have equivalently

$$\Delta u = 32R^2\frac{|\alpha\beta|}{g_0}\sinh(u/2).$$

This is a sinh-Gordon type equation, which can be written as

$$L[u] = \Delta u - 32R^2\frac{|\alpha\beta|}{g_0}\frac{\sinh(u/2)}{u}u = 0,$$

since $\lim_{u\to 0}\frac{\sinh(u/2)}{u} = \frac{1}{2}$ is finite. Here $L = \Delta - c$ is a linear differential operator, where $c = 32R^2\frac{|\alpha\beta|}{g_0}\frac{\sinh(u/2)}{u}$. In particular, $c \geq 0$, since $\sinh(u/2)$ and u have the same sign. This implies that u is a solution of a linear partial differential equation which is uniformly elliptic on U, and has $c \geq 0$. Since $u \leq 0$ on the boundary ∂U, we are thus in the conditions of the classical maximum principle (Gilbarg and Trudinger, 2001, Theorem 3.5) which then implies that either u is constant or it cannot attain a non-negative maximum in the interior of U. As u cannot be constant (since α has zeros but it is non-zero) we conclude that $u < 0$ in the interior of U, which finishes the proof. $\square$

We can also calculate the extension class of this partial $\mathrm{SL}(2,\mathbb{C})$-oper $\nabla_{\hbar,R}$.

Proposition 3.3.4. *The extension class of the partial* $\mathrm{SL}(2,\mathbb{C})$*-oper* $(E, \nabla_{\hbar,R})$ *over* X_μ*, with* $\mu = \hbar^2R^2\frac{\overline{\alpha}}{\beta}h_R^2$*, is the Dolbeault class in* $\mathrm{H}^1(X_\mu, L_\mu^2)$ *represented by*

$$\omega = \left(\frac{1 - |\mu|^2/|\hbar^2R^2|^2}{1-|\mu|^2}\right)\hbar R^2\overline{\beta}(z)h_R^{-2}\frac{d\overline{v}}{\overline{\nu}}.$$

Proof. We need to calculate the $(0,1)_\mu$-part of the upper right entry of $\nabla_{\hbar,R}$. Comparing with (3.1.2), this is $\omega = (\hbar^{-1}\alpha(z)dz + \hbar R^2\overline{\beta}(z)h_R^{-2}d\bar z)^{(0,1)_\mu}$. To calculate, we note that we can invert equations (2.4.3) and (2.4.4) to get

$$dz = \frac{1}{1-|\mu|^2}\left(\frac{dv}{\nu} - \mu\frac{d\overline{v}}{\overline{\nu}}\right)$$

$$d\overline{z} = \frac{1}{1-|\mu|^2}\left(-\overline{\mu}\frac{dv}{\nu} + \frac{d\overline{v}}{\overline{\nu}}\right).$$

This means ω has a term coming from α which is $\hbar^{-1}\alpha(z)\frac{-\mu}{1-|\mu|^2}\frac{d\overline{v}}{\overline{\nu}}$ and another one coming from β which is $\hbar R^2\overline{\beta}(z)h_R^{-2}\frac{1}{1-|\mu|^2}\frac{d\overline{v}}{\overline{\nu}}$. This implies

$$\begin{aligned}
\omega &= \left(\hbar R^2\overline{\beta}(z)h_R^{-2} - \mu\hbar^{-1}\alpha(z)\right)\frac{1}{1-|\mu|^2}\frac{d\overline{v}}{\overline{\nu}} \\
&= \hbar R^2\overline{\beta}(z)h_R^{-2}\left(1-\mu\frac{\hbar^{-1}\alpha(z)}{\hbar R^2\overline{\beta}(z)h_R^{-2}}\right)\frac{1}{1-|\mu|^2}\frac{d\overline{v}}{\overline{\nu}} \\
&= \hbar R^2\overline{\beta}(z)h_R^{-2}\left(1-\mu\frac{\overline{\hbar}^2R^2}{\overline{\hbar}^2R^2}\frac{\alpha(z)h_R^2}{\hbar^2R^2\overline{\beta}(z)}\right)\frac{1}{1-|\mu|^2}\frac{d\overline{v}}{\overline{\nu}} \\
&= \hbar R^2\overline{\beta}(z)h_R^{-2}\left(1-\mu\frac{\overline{\mu}}{|\hbar^2R^2|^2}\right)\frac{1}{1-|\mu|^2}\frac{d\overline{v}}{\overline{\nu}} \\
&= \left(\frac{1-|\mu|^2/|\hbar^2R^2|^2}{1-|\mu|^2}\right)\hbar R^2\overline{\beta}(z)h_R^{-2}\frac{d\overline{v}}{\overline{\nu}}.
\end{aligned}$$

□

3.4 The case of zero degree

In this section we consider the case $\deg(L) = 0$, where the construction has slightly different features. In particular, as already noted in Remark 3.1.6, a different argument is required for the existence of the conformal limit and we start with this.

3.4.1 The conformal limit

The conformal limit requires the polystability of the $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle. So we begin by studying the stability in this case.

Proposition 3.4.1. *Let L be a line bundle with $\deg(L) = 0$. Consider the $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle $(E = L \oplus L^{-1}, \Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0\end{smallmatrix}\right))$, with $\alpha \in \mathrm{H}^0(L^2K)$ and $\beta \in \mathrm{H}^0(L^{-2}K)$. Then*

i) if both $\alpha = 0$ and $\beta = 0$, $(E, \Phi = 0)$ is strictly polystable as an $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle;

ii) if both $\alpha \neq 0$ and $\beta \neq 0$, (E, Φ) is polystable. If further α and β are not proportional, (E, Φ) is stable;

iii) if one of α and β is zero but not the other one, (E, Φ) is unstable.

Proof. Case *i*) is immediate. For case *iii*) we note that if only one of α or β is non-zero, then the Higgs field Φ is nilpotent. This means Φ is not diagonalizable, and so (E, Φ) is not a direct sum of Higgs line bundles. This means it is not strictly polystable. It is also not stable, since either L or L^{-1} is Φ-invariant. We are left with case *ii*). Since E is polystable as a bundle, only degree zero subbundles can destabilize the Higgs bundle (E, Φ). Suppose there is a Φ-invariant holomorphic line subbundle S of degree zero. Write $s_1 : S \to L$ and $s_2 : S \to L^{-1}$ for the maps induced by the inclusion $s : S \hookrightarrow E = L \oplus L^{-1}$. Both of these maps are non-zero because neither L nor L^{-1} is Φ-invariant. Hence (since $\deg(S) = \deg(L) = 0$) we have $s_1 : S^{-1}L \cong \mathcal{O}$ and $s_2^{-1} : SL \cong \mathcal{O}$. Therefore $s_1/s_2 : L^2 \cong \mathcal{O}$. Now, the subbundle S being Φ-invariant means that $\Phi(s) = cs$ for a non-zero section c, i.e.,

$$\Phi(s) = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \alpha s_2 \\ \beta s_1 \end{pmatrix} = \begin{pmatrix} cs_1 \\ cs_2 \end{pmatrix} = cs.$$

Thus $c\alpha s_2^2 = c^2 s_1 s_2 = c\beta s_1^2$ and, in view of the isomorphism $s_1/s_2 : L^2 \cong \mathcal{O}$ we conclude that α and β are proportional sections of $L^2K \cong L^{-2}K \cong K$. Finally, we can include S^{-1} in $L \oplus L^{-1}$ using $\begin{pmatrix} s_2^{-1} \\ s_1^{-1} \end{pmatrix}$ and, since

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} s_2^{-1} \\ s_1^{-1} \end{pmatrix} = \begin{pmatrix} \alpha s_1^{-1} \\ \beta s_2^{-1} \end{pmatrix} = \begin{pmatrix} cs_2^{-1} \\ cs_1^{-1} \end{pmatrix}$$

by the above calculation, we conclude that S^{-1} is a Φ-invariant complement to S.

In conclusion, if there is a destabilizing Φ-invariant subbundle S, then the sections α and β are proportional and (E, Φ) decomposes as the direct sum of Higgs bundles $S \oplus S^{-1}$, with the induced Higgs fields. Thus (E, Φ) is strictly polystable. Otherwise, there are no such subbundles and (E, Φ) is stable. $\square$

Thus, in the case $\deg(L) = 0$, the conformal limit can be analyzed using the solution of the scaled vortex equations for either both $\alpha = 0 = \beta$ or both non-zero. In this special case, since the bundle $E = L \oplus L^{-1}$ itself is stable, the limit of the solution h_R of the scaled vortex equations as $R \to 0$ does exist, and it is simply a metric h_0 of zero curvature on L. The conformal limit is then directly calculated by taking the limit in the family (3.1.2) and it is $\nabla_{\hbar,0} = A_{h_0} + \hbar^{-1}\Phi$, where A_{h_0} is the diagonal Chern connection associated with the Hermitian metric h_0. It is a partial oper (since $E = \mathrm{Gr}(E)$ and it is semistable), and, when $\beta \neq 0$, it defines a branched complex projective structure compatible with X, just as in the previous section. In conclusion, we have the following.

Theorem 3.4.2. *Let* $(E = L \oplus L^{-1}, \Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right))$ *be a polystable* $\mathrm{SL}(2, \mathbb{R})$*-Higgs bundle with* $\deg(L) = 0$*. Then the* $\hbar$*-conformal limit of* (E, Φ) *exists and it has the structure of a partial* $\mathrm{SL}(2, \mathbb{C})$*-oper.*

Remark 3.4.3. Proposition 3.1.4 is no longer necessary to change the solution. This is consistent with the fact that in this case the vortex equations for $(\beta, 0)$ do not have a solution.

3.4.2 The projective structures coming from the conformal limit

In the zero degree case the condition $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$ implies that $\alpha = k\beta$, $k \in \mathbb{C}^*$, since both α and β must have the same number of zeros counted with multiplicity. We are thus in the situation *ii*) of Proposition 3.4.1, where $(E = L \oplus L^{-1}, \Phi = \left(\begin{smallmatrix} 0 & k\beta \\ \beta & 0 \end{smallmatrix}\right))$ is a polystable Higgs bundle. Note that the condition in particular implies that $L^4 \cong \mathcal{O}$, since $k = \frac{\alpha}{\beta} \in \mathrm{H}^0(L^4)$. The proof of Theorem 3.3.1 will carry through, except now the function u will read

$$u = \log \left| \frac{\alpha(z)}{\beta(z)} h_R^2(z) \right|^2 = \log \left| k h_R^2(z) \right|^2$$

and it will be constant by the maximum principle. Of course in this case one can directly check that $h_R = |k|^{-1/2}$ is a solution of the R scaled-vortex equations, for if $\alpha = k\beta$, the equation reads

$$\partial_{\overline{z}} \partial_z \log h_R = R^2 \left(|\alpha|^2 h_R^2 - |\beta|^2 h_R^{-2} \right) = R^2 \left(|k|^2 |\beta|^2 h_R^2 - |\beta|^2 h_R^{-2} \right).$$

The right hand side when $h_R = |k|^{-1/2}$ is $R^2 \left(|k|^2 |\beta|^2 |k|^{-1} - |\beta|^2 |k| \right) = 0$ which is precisely $\partial_{\overline{z}} \partial_z \log |k|^{-1/2} = 0$. The Beltrami differential will now be $\mu = \hbar^2 R^2 \overline{k} \frac{\overline{\beta}}{\beta} h_R^2 = \hbar^2 R^2 \frac{\overline{k}}{|k|} \frac{\overline{\beta}}{\beta}$. The subbundle L will determine a partial oper structure and, when $\beta \neq 0$, a complex projective structure compatible with X_μ. Thus we have the following result.

Theorem 3.4.4. *Fix $R \in \mathbb{R}^+$ and $\hbar, k \in \mathbb{C}^*$ such that $|\hbar^2 R^2| < 1$. Let L be a line bundle with $L^4 \cong \mathcal{O}$ (in particular $\deg(L) = 0$) and consider the polystable $\mathrm{SL}(2, \mathbb{R})$-Higgs bundle $(L \oplus L^{-1}, \Phi = \left(\begin{smallmatrix} 0 & k\beta \\ \beta & 0 \end{smallmatrix}\right))$, where $0 \neq \beta \in \mathrm{H}^0(L^2 K)$. Then the family (3.1.2) is*

$$\nabla_{\hbar, R} = d + \begin{pmatrix} 0 & \hbar^{-1} k\beta + \hbar R^2 \overline{\beta} |k| \\ \hbar^{-1} \beta + \hbar R^2 \frac{\overline{k}}{|k|} \overline{\beta} & 0 \end{pmatrix}.$$

Define $\mu = \hbar^2 R^2 \frac{\overline{k}}{|k|} \frac{\overline{\beta}}{\beta}$. Then μ is a Beltrami differential. Further $\nabla_{\hbar, R}$ determines a branched projective structure compatible with X_μ with branching divisor $\mathrm{div}(\beta)$. Its extension class $[\omega]$ is trivial.

3.5 Geometric interpretation of results

3.5.1 Curves in $\mathcal{B}(M)$

Let $(L \oplus L^{-1}, \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right))$ be a polystable $\mathrm{SL}(2, \mathbb{R})$-Higgs bundle with $\beta \neq 0$. Assume that $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$) and $|R^2 \hbar^2| \leq 1$ (or $|R^2 \hbar^2| < 1$ if $\deg(L) = 0$). Then, by Theorems 3.3.1 and 3.4.4, the partial oper structure on E given by the filtration $0 \subset L \subset E$ and the flat connection $\nabla_{\hbar, R}$ determines a branched projective structure. For each $\hbar$ and R, we will denote this structure by $\mathcal{P}_\beta^\alpha(\hbar, R) \in \mathcal{B}(M)$ or just by $\mathcal{P}(\hbar, R)$ whenever α and β are fixed. By construction these branched projective structures lift to $\mathrm{SL}(2, \mathbb{C})$, so by a slight abuse of notation we shall also write $\mathcal{P}(\hbar, R) = (\nabla_{\hbar, R}, L)$, i.e., as a pair consisting of a flat $\mathrm{SL}(2, \mathbb{C})$-connection and a transverse line subbundle (cf. Theorem1.12.4). Note that the dependence on $\hbar$ and R is continuous, also for $R = 0$, since the connection $\nabla_{\hbar, R}$ in the configuration space of flat connections depends continuously on the parameters (Remark 3.1.8). The structure $\mathcal{P}(\hbar, R)$ is compatible with the

Riemann surface X_μ, and thus the forgetful map $\mathcal{T} : \mathcal{B}(M) \to \mathcal{T}(M)$ takes $\mathcal{P}(\hbar, R) \mapsto [X_\mu]$. In all cases, regardless of the degree of L, the Beltrami differential μ is given by the expression

$$\mu = \hbar^2 R^2 \frac{\overline{\alpha}}{\beta} h_R^2 = \hbar^2 R^2 h_R^2 \frac{\overline{\alpha}\overline{\beta}}{|\beta|^2} \frac{|\alpha|}{|\alpha|} = \hbar^2 R^2 h_R^2 \frac{|\alpha|}{|\beta|} \frac{\overline{\alpha}\overline{\beta}}{|\alpha\beta|},$$

where h_R is the solution of the scaled vortex equations for (α, β), $\beta \neq 0$ and it is understood that $\mu = 0$ if $\alpha = 0$. Using Lemma 3.1.4 in the case $\deg(L) \neq 0$, or the fact that $\alpha = k\beta$ and $h_R = |k|^{-1/2}$, when $\deg(L) = 0$ (Proposition 3.4.4), for some constant $k \in \mathbb{C}^*$ we can write μ as:

$$\mu = \hbar^2 R^2 h_R^2 \frac{|\alpha|}{|\beta|} \frac{\overline{\alpha}\overline{\beta}}{|\alpha\beta|} = \begin{cases} \hbar^2 R^2 \frac{\overline{k}}{|k|} \frac{\overline{\beta}^2}{|\beta|^2} & \text{if } \deg(L) = 0 \\ \hbar^2 R^4 h^2 \frac{|\alpha|}{|\beta|} \frac{\overline{\alpha}\overline{\beta}}{|\alpha\beta|} & \text{if } \deg(L) \neq 0 \end{cases} \tag{3.5.1}$$

where h is the solution of the unscaled vortex equations for $(R^2\alpha, \beta)$. In conclusion, if we fix a pair (α, β) with $\operatorname{div}(\alpha) \geq \operatorname{div}(\beta)$ and $\beta \neq 0$ and denote the subset of valid parameters $\hbar$ and R by

$$\mathcal{D} = \{(\hbar, R) \in \mathbb{C}^* \times \mathbb{R}_0^+ \mid |\hbar^2 R^2| \leq 1\} \text{ or}$$
$$\mathcal{D} = \{(\hbar, R) \in \mathbb{C}^* \times \mathbb{R}_0^+ \mid |\hbar^2 R^2| < 1\} \text{ for } \deg(L) = 0,$$

we get a map $\mathcal{P}_\beta^\alpha : \mathcal{D} \to \mathcal{B}(M)$ given by $(\hbar, R) \mapsto \mathcal{P}_\beta^\alpha(\hbar, R)$. Thus we have continuous maps

$$\mathcal{D} \to \mathcal{B}(M) \to \mathcal{T}(M)$$
$$(\hbar, R) \mapsto \mathcal{P}_\beta^\alpha(\hbar, R) \mapsto X_{\mu(\hbar, R)},$$

where the associated complex structure $X_{\mu(\hbar,R)}$ is determined by $\mu = \mu(\hbar, R)$ in equation (3.5.1). We remark that the map is not injective. In particular, $\mu(\hbar, R) = \mu(\hbar', R')$ if $\hbar^2 R^2 = \hbar'^2 R'^2$ for degree zero, or $\hbar^2 R^4 = \hbar'^2 R'^4$, for non-zero degree. We can also fix $\hbar \in \mathbb{C}^*$, and in that case we obtain a curve $\mathcal{P}_\beta^\alpha(R) := \mathcal{P}_\beta^\alpha(\hbar, R)$ in $\mathcal{B}(M)$ projecting to a curve $\gamma(R) = X_\mu(\hbar, R)$ in Teichmüller space. In the next section, we study the geometry of this curve which, in the degree zero case, we show to be a Teichmüller geodesic.

3.5.2 Teichmüller geodesics and disks

Recall the results of subsection 2.4.1, where we described geodesics for the Teichmüller metric in $\mathcal{T}(S)$. Observing the constructed μ in equation (3.5.1) for the case of $\deg(L) = 0$ we immediately conclude that the curve $\gamma(t)$ is a (reparameterization) of a Teichmüller geodesic ray.

Theorem 3.5.1. *Let $\mathcal{P}_\beta^{k\beta}(R)$ be the R-family of branched projective structures associated with the conformal limit of $E = L \oplus L^{-1}$, with $L^4 \cong \mathcal{O}$, and Higgs field $\varphi = \left(\begin{smallmatrix} 0 & k\beta \\ \beta & 0 \end{smallmatrix}\right)$, where $0 \neq \beta \in \mathrm{H}^0(L^2K)$ and $k \in \mathbb{C}^*$, and $\gamma(R) = X_{\mu(R)}$, with $\mu(R) = \hbar^2 R^2 \frac{\overline{k}}{|k|} \frac{\overline{\beta}^2}{|\beta|^2}$, be the curve of associated Riemann surface structures in $\mathcal{T}(S)$. Assume $|\hbar^2 R^2| < 1$. Then $\gamma(R)$ is (a reparameterization of) a geodesic ray.*

Remark 3.5.2. Note that the ray is associated to $-\arg(\hbar^2)k\beta^2 \in \mathrm{H}^0(L^4K^2) \cong \mathrm{H}^0(K^2)$. By allowing $\hbar$ to vary, we analogously get (a reparameterization of) the Teichmüller disk associated with $k\beta^2$.

3.5.3 Reality properties for nonzero degree

Let us now study the family $\mathcal{P}^\alpha_\beta(\hbar, R)$ for some specific parameters in the case where $\deg(L) \neq 0$. We will show that there exist values of $\hbar$ and R for which the structure $\mathcal{P}^\alpha_\beta(\hbar, R)$ is a branched hyperbolic structure, cf. Definition 1.5.2. We have seen before branched hyperbolic structures have real holonomy $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{PSL}(2,\mathbb{C})$. The condition of having real holonomy is, however, not enough to guarantee that the structure is hyperbolic, as there exist (even unbranched) projective structures with real holonomy which are not hyperbolic (cf. for example Hejhal (1975)).

The condition for the holonomy to be real can be written in gauge theoretic terms related to the flat bundle $(E, \nabla_{\hbar,R})$. Recall that the holonomy of this connection lifts to $\mathrm{SL}(2,\mathbb{C})$ the holonomy of the projective structure $\mathcal{P}^\alpha_\beta(\hbar, R)$. It is real if, up to gauge equivalence, it lies in $\mathrm{SL}(2,\mathbb{R})$, and this happens if $\nabla_{\hbar,R}$ preserves a real structure τ. A real structure τ is a $\mathbb{C}$-antilinear isomorphism of the bundle E such that $\tau^2 = \mathrm{Id}_E$. The condition that $\nabla_{\hbar,R}$ preserves τ means that $\nabla_{\hbar,R} \circ \tau = \tau_{T^*M} \circ \nabla_{\hbar,R}$, where τ_{T^*M} just acts as τ on the section part, and as complex conjugation, mapping $K \to \overline{K}$, on the form part. Equivalently τ_{T^*M} is the tensor product of τ and the real structure on the complex cotangent bundle T^*M. To write in gauge theoretic language the stronger condition that the image of d must be in $\mathbb{H}^2$, we note that this is the same as asking for the image of d to avoid the the real locus $\mathbb{RP}^1 \subset \mathbb{CP}^1$ which is the fixed point set of the involution $z \mapsto \overline{z}$ of $\mathbb{CP}^1$. This translates to the condition that the line bundle $L \subset E$ induced by d should avoid the fixed point locus of τ in E, i.e., the intersection should only be the zero section of L (cf. Alessandrini (2019)). Using this description we have the following.

Proposition 3.5.3. *Let* $\deg(L) \neq 0$ *and* $|\hbar|^2R^2 = 1$. *Then* $\mathcal{P}^\alpha_\beta(\hbar, R)$ *is a branched hyperbolic structure.*

Proof. The real structure τ defining the $\mathrm{SL}(2,\mathbb{R})$-structure in $E = L \oplus L^{-1}$ is given in a holomorphic frame by $\tau(v) = C\overline{v}$ with $C = \begin{pmatrix} 0 & h_R^{-1} \\ h_R & 0 \end{pmatrix}$. It is preserved by $\nabla_{\hbar,R} = d + B$ if and only if $\nabla_{\hbar,R} \circ \tau = \tau_{T^*M} \circ \nabla_{\hbar,R}$ which in this frame reads $dC + BC = C\overline{B}$. Using expression (3.1.2), one concludes that this happens if and only if

$$\begin{aligned} \hbar^{-1}\alpha h_R + \hbar R^2 \overline{\beta} h_R &= \overline{\hbar^{-1}\beta h_R^{-1} + \hbar R^2 \overline{\alpha} h_R} \\ \iff (\hbar^{-1} - \overline{\hbar}R^2)\alpha h_R &= (\hbar^{-1} - \overline{\hbar}R^2)\overline{\beta} h_R. \end{aligned} \tag{3.5.2}$$

In particular, if $|\hbar|^2R^2 = 1$ the equality is valid. Thus the holonomy of $\nabla_{\hbar,R}$ is real. To check that $\mathcal{P}^\alpha_\beta(\hbar, R)$ is branched hyperbolic we simply note that L avoids the fixed locus of τ. This happens since vectors fixed by the real structure, i.e., such that $\tau(v)=v$, are of the form $\binom{v_1}{\hbar v_1}$ and the vectors in L are multiples of $\binom{1}{0}$. □

Remark 3.5.4. In the particular case $\hbar=1$ and $R=1$, this result recovers the branched hyperbolic structures constructed (Biswas et al., 2021).

3.5.4 Deformations of geometric structures

Using the previous results one can interpret our construction in terms of geometric structures on X as follows. Start by fixing $\hbar$ with $|\hbar|=1$. Then the branched projective structure $\mathcal{P}^{\alpha}_{\beta}(1,R)$ interpolates between a branched hyperbolic structure, at $R=1$, and a partial oper, i.e., a complex projective compatible with X, at $R=0$. In the specific case of $\hbar=1$, the branched hyperbolic structure is exactly the one coming from the non-abelian Hodge correspondence.

In particular, take any Higgs bundle in a Hitchin component, i.e., one of the form

$$\left(E=K^{1/2}\oplus K^{-1/2}, \Phi=\begin{pmatrix}0 & q\\ 1 & 0\end{pmatrix}\right),$$

where $q\in \mathrm{H}^0(K^2)$ is a quadratic differential. In the previous notation $\alpha=q$ and $\beta=1$, and the condition $\operatorname{div}(\alpha)\geq\operatorname{div}(\beta)$ is satisfied, so our construction goes through. The non-abelian Hodge correspondence produces a connection which is the holonomy of the (unbranched) Fuchsian hyperbolic structure $\mathcal{P}^{q}_{1}(1,R=1)$. Then, for $\hbar=1$, and by varying R, we obtain the curve $\mathcal{P}^{q}_{1}(1,R)$ of complex projective structures. This curve interpolates between this Fuchsian structure, at $R=1$, and the structure of oper determined by q, at $R=0$, i.e., the complex projective structure compatible with X and which is obtained using the Schwarz parameterization with quadratic differential precisely equal to q.

3.6 A more general proof of the existence of the conformal limit

In this section, we give a more general proof of Lemma 3.1.4 which allows one to prove the existence and compute the conformal limit of polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles.

So we let $(\overline{\partial}_E,\Phi)$ be an $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle which is polystable. We denote by E de smooth underlying bundle which is fixed throughout. We also write $(E,\overline{\partial}_E)$ to denote the underlying holomorphic bundle. By the non-abelian Hodge correspondence (Theorem 2.1.6) there is a metric H_R harmonic with parameter R, i.e. H_R is a solution of the Hitchin's equation with parameter R for $(\overline{\partial}_E,\Phi)$. If the $(E,\overline{\partial}_E)$ is polystable as a vector bundle then $(\overline{\partial}_E,0)$ is a polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle. This means, as in Theorem 3.4.2, that there exists a metric H_0 satisfying Hitchin's equation for this Higgs bundle with zero Higgs field and we can calculate the limit directly as in the discussion just before that theorem. It is obtained by taking the limit in the family (2.2.1) and it is

$$\nabla_{\hbar,0} = A_{H_0} + \hbar^{-1}\Phi, \tag{3.6.1}$$

where A_{H_0} is the Chern connection associated to the Hermitian metric H_0. It is a partial oper (possibly in Simpson's terms, allowing the trivial filtration). We remark that in the strictly Higgs-polystable case, the metric H_0 is not unique but in fact unique up to the scaling of each factor. This is not a problem, as the limit connection is independent of the choice of such a scaling.

So we suppose now $(E, \overline{\partial}_E)$ is not polystable as a vector bundle, and fix $R \in \mathbb{R}^+$. Then E admits a maximally destabilizing holomorphic line subbundle $L \subset E$. Consider its orthogonal complement $L^\perp \cong E/L$ with respect to H_R. Then E admits a smooth decomposition of the form $E \cong L \oplus L^\perp$. The trivialization of $\det(E)$ induces an isomorphism $L^\perp \cong E/L \cong L^{-1}$, which gives a decomposition $E = L \oplus L^{-1}$. With respect to this decomposition, the harmonic metric is diagonal (because it is diagonal in $L \oplus L^\perp$). Since it is compatible with the trivialization of $\det(E)$ it can be written in $E = L \oplus L^{-1}$ as $H_R = \begin{pmatrix} h_R & 0 \\ 0 & h_R^{-1} \end{pmatrix}$, where h_R is a hermitian metric in L. Of course, in general, the decomposition is not a holomorphic one, i.e. $\overline{\partial}_E$ is written as $\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_L & \omega \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix}$, with $\omega \in \Omega^{(0,1)}(L^2)$ the Dolbeault representative of the extension class. With it, the Higgs field is written as $\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, where $a \in \Omega^{(1,0)}(K)$, $b \in \Omega^{(1,0)}(L^2K)$ and $c \in \Omega^{(1,0)}(L^2K)$, satisfying $\overline{\partial}_E(\Phi) = 0$. We will use this decomposition to prove a result similar to Proposition 3.1.4, but valid in this more general case. Of course, any polystable Higgs bundle (even when the underlying bundle is polystable as a bundle) admits a decomposition like this, for instead of the maximally destabilizing subbundle we just use any holomorphic line subbundle L. The point is that here the decomposition is not necessary to calculate the conformal limit, which we already know how to do in that case. We remark that this decomposition is reminiscent of the work in (Li, 2019a, §3.2).

Proposition 3.6.1. *Let $(E, \overline{\partial}_E, \Phi)$ be a polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle and $R \in \mathbb{R}^+$. Take the smooth decomposition $E = L \oplus L^{-1}$, where $L \subset E$ is a holomorphic line subbundle, and $(\overline{\partial}_E, \Phi)$ reads*

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_L & \omega \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},$$

with $\omega \in \Omega^{(0,1)}(L^2)$, $a \in \Omega^{(1,0)}(X)$, $b \in \Omega^{(1,0)}(L^2)$ and $c \in \Omega^{(1,0)}(L^{-2})$, satisfying $\overline{\partial}_E(\Phi) = 0$. Let $H_R = \begin{pmatrix} h_R & 0 \\ 0 & h_R^{-1} \end{pmatrix}$ be a hermitian metric in E, with h_R a hermitian metric in L.

Then H_R is a solution of Hitchin's equation with parameter R for $(\overline{\partial}_E, \Phi)$ if and only if $H = \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}$, with $h = \frac{h_R}{R}$, is a solution of the (unscaled) Hitchin's equation for

$$\overline{\partial}_{E'} = \begin{pmatrix} \overline{\partial}_L & R\omega \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Psi = \begin{pmatrix} Ra & R^2 b \\ c & -Ra \end{pmatrix}.$$

Proof. Throughout the proof we take a holomorphic local frame for L and one for L^{-1}, these induce a (non-holomorphic) frame on E. With respect to these frames, h_R and h are represented by functions, and $\overline{\partial}_L = \overline{\partial} = \overline{\partial}_{L^{-1}}$, where $\overline{\partial}$ is the usual $\overline{\partial}$-operator on forms. The sections a, b, c are represented by 1-forms. Inside matrices we will suppress the wedge product.

Let us start by showing that $\overline{\partial}_E(\Phi) = 0$ if and only if $\overline{\partial}_{E'}(\Phi) = 0$. This guarantees that the pair $(\overline{\partial}_{E'}, \Psi)$ is a Higgs bundle. The condition $\overline{\partial}_E(\Phi) = 0$ can be written as

$$\overline{\partial}_E(\Phi) = \overline{\partial}(\Phi) + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} \wedge \Phi - \Phi \wedge \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \overline{\partial} a + \omega c & \overline{\partial} b \\ \overline{\partial} c & -\overline{\partial} a - \omega c \end{pmatrix} = 0.$$

Note that Ψ and $\overline{\partial}_{E'}$ are obtained from Φ and $\overline{\partial}_E$ by the substitutions $a \rightsquigarrow Ra$, $b \rightsquigarrow R^2 b$, $c \rightsquigarrow c$ and $\omega \rightsquigarrow R\omega$. Upon performing these substitutions the above equation is kept unchanged. This is because the matrix on the left side of the last equality has its diagonal multiplied by R and the right upper corner multiplied by R^2, which are both non-zero terms that we can cancel out. We conclude that $\overline{\partial}_E(\Phi) = 0$ holds if and only if $\overline{\partial}_{E'}(\Phi) = 0$ does. For the rest of the proposition let us calculate

$$\Phi^{*_R} = H_R^{-1} \overline{\Phi}^T H_R = \begin{pmatrix} \overline{a} & h_R^{-2} \overline{c} \\ h_R^2 \overline{b} & -\overline{a} \end{pmatrix} =: \begin{pmatrix} a^{*_R} & c^{*_R} \\ b^{*_R} & -a^{*_R} \end{pmatrix}, \tag{3.6.2}$$

where the $*_R$ on the right are defined entry-wise and are just the adjoint with respect to the metric h_R on L. The commutator is

$$[\Phi, \Phi^{*_R}] = \Phi \wedge \Phi^{*_R} + \Phi^{*_R} \wedge \Phi = \begin{pmatrix} bb^{*_R} + c^{*_R} c & 2\left(a^{*_R} b - c^{*_R} a\right) \\ 2\left(b^{*_R} a - a^{*_R} c\right) & cc^{*_R} + b^{*_R} b \end{pmatrix}. \tag{3.6.3}$$

The Chern connection is A_{H_R}, with connection matrix which we still denote by A_{H_R} given by (cf. Lemma A.1.1)

$$A_{H_R} = H_R^{-1} \partial H_R + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} - H_R^{-1} \begin{pmatrix} 0 & 0 \\ \overline{\omega} & 0 \end{pmatrix} H_R = \begin{pmatrix} \partial \log h_R & \omega \\ -\omega^{*_R} & -\partial \log h_R \end{pmatrix}, \tag{3.6.4}$$

with $\omega^{*_R} = h_R^2 \overline{\omega}$. This connection has curvature $F_{A_{H_R}} = dA_{H_R} + A_{H_R} \wedge A_{H_R}$, i.e.,

$$F_{A_{H_R}} = \begin{pmatrix} \overline{\partial}\partial \log h_R & \partial \omega \\ -\overline{\partial} \omega^{*_R} & -\overline{\partial}\partial \log h_R \end{pmatrix} + \begin{pmatrix} \omega\omega^{*_R} & 2(\partial \log h_R)\omega \\ -2\omega^{*_R}(\partial \log h_R) & \omega^{*_R}\omega \end{pmatrix}.$$

The R-scaled Hitchin equation is $F_{A_{H_R}} + R^2[\Phi, \Phi^{*_R}] = 0$ and so it reads

$$\begin{pmatrix} \overline{\partial}\partial \log h_R & \partial \omega \\ -\overline{\partial} \omega^{*_R} & -\overline{\partial}\partial \log h_R \end{pmatrix} + \begin{pmatrix} \omega\omega^{*_R} & 2(\partial \log h_R)\omega \\ -2\omega^{*_R}(\partial \log h_R) & \omega^{*_R}\omega \end{pmatrix} + \\ + R^2 \begin{pmatrix} bb^{*_R} + c^{*_R} c & 2\left(a^{*_R} b - c^{*_R} a\right) \\ 2\left(b^{*_R} a - a^{*_R} c\right) & cc^{*_R} + b^{*_R} b \end{pmatrix} = 0. \tag{3.6.5}$$

Noting that $h_R = Rh$, we see that $\partial \log h_R = \partial \log(Rh) = \partial \log h + \partial \log R = \partial \log h$, since R is constant. Denoting by $*$ the adjoint with respect to h we see further that

$$a^{*_R} = \overline{a} = a^*$$

$$\begin{aligned} b^{*_R} &= h_R^2 \overline{b} = R^2 h^2 \overline{b} = R^2 b^* \\ c^{*_R} &= h_R^{-2} \overline{c} = R^{-2} h \overline{c} = R^{-2} c^* \\ \omega^{*_R} &= h_R^2 \overline{\omega} = R^2 h^2 \overline{\omega} = R^2 \omega^*. \end{aligned}$$

By using these identities, equation 3.6.5 is equivalent to

$$\begin{pmatrix} \overline{\partial}\partial \log h & \partial\omega \\ -R^2\overline{\partial}\omega^* & -\overline{\partial}\partial \log h \end{pmatrix} + \begin{pmatrix} R^2\omega\omega^* & 2(\partial \log h)\omega \\ -2R^2\omega^*(\partial \log h) & R^2\omega^*\omega \end{pmatrix} + \\ + R^2 \begin{pmatrix} R^2 bb^* + R^{-2}c^*c & 2\left(a^*b - R^{-2}c^*a\right) \\ 2\left(R^2b^*a - a^*c\right) & R^{-2}cc^* + R^2b^*b \end{pmatrix} = 0. \tag{3.6.6}$$

Distributing the R^2 in the last term, and rearranging we get

$$\begin{pmatrix} \overline{\partial}\partial \log h & R^{-1}\partial(R\omega) \\ -R\overline{\partial}(R\omega^*) & -\overline{\partial}\partial \log h \end{pmatrix} + \begin{pmatrix} R\omega\, R\omega^* & 2R^{-1}(\partial \log h)(R\omega) \\ -2R(R\omega^*)(\partial \log h) & R\omega^*\, R\omega \end{pmatrix} + \\ + \begin{pmatrix} R^2b\, R^2b^* + c^*c & 2R^{-1}\left(Ra^*\, R^2b - c^*\, Ra\right) \\ 2R\left(R^2b^*\, Ra - Ra^*\, c\right) & cc^* + R^2b^*\, R^2b \end{pmatrix} = 0. \tag{3.6.7}$$

If we multiply the upper right corner by the non-zero R and the lower left corner by R^{-1} we see that the equation is equivalent to

$$\begin{pmatrix} \overline{\partial}\partial \log h & \partial(R\omega) \\ -\overline{\partial}(R\omega^*) & -\overline{\partial}\partial \log h \end{pmatrix} + \begin{pmatrix} (R\omega)\,(R\omega^*) & 2(\partial \log h)(R\omega) \\ -2(R\omega^*)(\partial \log h) & (R\omega^*)\,(R\omega) \end{pmatrix} + \\ + \begin{pmatrix} (R^2b)\,(R^2b)^* + c^*c & 2\left((Ra)^*\,(R^2b) - c^*\,(Ra)\right) \\ 2\left((R^2b)^*\,(Ra) - (Ra)^*\,c\right) & cc^* + (R^2b)^*\,(R^2b) \end{pmatrix} = 0. \tag{3.6.8}$$

This is just the condition that H satisfies the unscaled Hitchin equation for $(\overline{\partial}_{E'}, \Psi)$. (To see this note this condition is obtained from 3.6.5 making the substitutions that take $(\overline{\partial}_E, \Phi)$ to $(\overline{\partial}_{E'}, \Psi)$, H_R to H, and by removing the R^2 factor before the last term). The conclusion is that H_R satisfies the scaled equation for $(\overline{\partial}_E, \Phi)$ if and only if H satisfies the unscaled equation for $(\overline{\partial}_{E'}, \Psi)$. □

We now proceed as in Section 3.1.

Corollary 3.6.2. *Let* $(\overline{\partial}_E, \Phi)$ *be a polystable* $\mathrm{SL}(2,\mathbb{C})$*-Higgs bundle with underlying unstable bundle* (E, ∂_E)*, decomposed as* $E = L \oplus L^{-1}$ *as above with* L *the maximally destabilizing subbundle, and denote by* $H_R = \begin{pmatrix} h_R & 0 \\ 0 & h_R^{-1} \end{pmatrix}$ *the harmonic metric* H_R *with parameter* R *for*

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_L & \omega \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

Then the pointwise limit of $H := \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}$*, with* $h = \frac{h_R}{R}$*, as* $R \to 0$ *exists as a metric and it is a solution of the unscaled Hitchin equation for the polystable Higgs bundle*

$$\overline{\partial}_{E_0'} = \begin{pmatrix} \overline{\partial}_L & 0 \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Psi_0 = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

Proof. By Proposition 3.6.1, H is a solution of the unscaled Hitchin equation for

$$\overline{\partial}_{E'} = \begin{pmatrix} \overline{\partial}_L & R\omega \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Psi = \begin{pmatrix} Ra & R^2 b \\ c & -Ra \end{pmatrix}.$$

By varying $R \in \mathbb{R}_0^+$, this describes a continuous path $(E = L \oplus L^{-1}, \overline{\partial}_{E'}(R), \Psi(R))$ of polystable Higgs bundles. Note that it is a well defined path since, $(\overline{\partial}_{E'}(R), \Psi(R))$ is stable for all $R \in \mathbb{R}_0^+$. This happens even for $R = 0$. Firstly, because the pair $(\overline{\partial}_{E_0'}, \Psi_0)$ is a Higgs bundle, since $\overline{\partial}_{E_0'} \cdot \Psi_0 = 0$, given that a similar equality holds for $(\overline{\partial}_{E'}, \cdot\Psi)$ (cf. the proof of Proposition 3.6.1 for the calculation). And secondly because the pair $(\overline{\partial}_{E_0'}, \Psi_0)$ is polystable (in fact stable), because $\deg(L) > 0$. (The subbundle L is maximally destabilizing.) Under the homeomorphism to the space of harmonic bundles, this is mapped to a continuous path of metrics, and $\lim_{R\to 0} H$ is the metric associated to $(\overline{\partial}_{E_0'}, \Psi_0)$. □

Remark 3.6.3. It is clear the proof is not valid for polystable Higgs bundles with an underlying bundle that is polystable in the vector bundle sense. In that case, even though the bundle has a decomposition of the required form, the tentative limit $(\overline{\partial}_{E_0'}, \Psi_0)$ is not a polystable Higgs bundle, and so there is no harmonic metric. This could of course be seen directly from the definition of H because in this exceptional case, the limit H_R as $R \to 0$ does exist. This means the H defined in the statement does not have a limit as $R \to 0$. This is entirely analogous to the situation in Section 3.4.2.

We are now able to prove the existence of the conformal limit.

Theorem 3.6.4. *Let $(\overline{\partial}_E, \Phi)$ be a polystable* $\mathrm{SL}(2,\mathbb{C})$*-Higgs bundle with underlying unstable bundle (E, ∂_E), decomposed as $E = L \oplus L^{-1}$ as above, with L the maximally destabilizing subbundle, and denote by $H_R = \left(\begin{smallmatrix} h_R & 0 \\ 0 & h_R^{-1} \end{smallmatrix}\right)$ the harmonic metric H_R with parameter R for*

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_L & \omega \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

Then, the $\hbar$-conformal limit $\nabla_{\hbar,0}$ of $(\overline{\partial}_E, \Phi)$ exists. Using the holomorphic frame of E induced by a holomorphic frame of L, the coordinate representation of $\nabla_{\hbar,0}$ is

$$\nabla_{\hbar,0} = A_{H_0} + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} + \hbar^{-1}\Phi + \hbar\,\Psi_0^{*H_0} = d + \begin{pmatrix} \partial \log h_0 + \hbar^{-1}\,a & \omega + \hbar^{-1} b + \hbar\,\bar{c} h_0^{-2} \\ \hbar^{-1} c & -\partial \log h_0 - \hbar^{-1}\,a \end{pmatrix}, \quad (3.6.9)$$

where $H_0 := \left(\begin{smallmatrix} h_0 & 0 \\ 0 & h_0^{-1} \end{smallmatrix}\right)$ is a solution of the (unscaled) Hitchin equation for

$$\overline{\partial}_{E'_0} = \begin{pmatrix} \overline{\partial}_L & 0 \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Psi_0 = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

Proof. Recall that the conformal limit Equation 2.2.1 is

$$\nabla_{\hbar,R} = A_{H_R} + \hbar^{-1}\Phi + \hbar R^2 \Phi^{*H_R}.$$

By Proposition 3.6.1, for each $R \in \mathbb{R}^+$, we can write H_R as $H = \left(\begin{smallmatrix} h & 0 \\ 0 & h^{-1} \end{smallmatrix}\right)$, with $h = \frac{h_R}{R}$, is a solution of the (unscaled) Hitchin's equation for

$$\overline{\partial}_{E'} = \begin{pmatrix} \overline{\partial}_L & R\omega \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Psi = \begin{pmatrix} Ra & R^2 b \\ c & -Ra \end{pmatrix}.$$

By Corollary 3.6.2, the limit of H as $R \to 0$ exists and we have $\lim_{R\to 0} H = H_0$ and $\lim_{R\to 0} h = h_0$. We also note that $\partial_z \log R = 0$, and thus $\partial_z \log h_R = \partial_z \log h$. We are now able to compute the limits of each term appearing in the conformal limit. Writing the Chern connection (whose calculation is in Equation 3.6.4) in terms of H and $h = \frac{h_R}{R}$ we have:

$$A_{H_R} = \begin{pmatrix} \partial \log h_R & \omega \\ h_R^2 \overline{\omega} & -\partial \log h_R \end{pmatrix} = \begin{pmatrix} \partial \log h & \omega \\ R^2 h^2 \overline{\omega} & -\partial \log h \end{pmatrix}$$
$$\xrightarrow[R\to 0]{} \begin{pmatrix} \partial \log h_0 & \omega \\ 0 & -\partial \log h_0 \end{pmatrix} = A_{H_0} + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix}.$$

Now for the last term $\hbar R^2 \Phi^{*H_R}$ (whose calculation is in Equation 3.6.2) we have in terms of h:

$$R^2\Phi^{*H_R} = R^2 \begin{pmatrix} \overline{a} & h_R^{-2}\overline{c} \\ h_R^2\overline{b} & -\overline{a} \end{pmatrix} = R^2 \begin{pmatrix} \overline{a} & R^{-2}h^{-2}\overline{c} \\ R^2h^2\overline{b} & -\overline{a} \end{pmatrix} = \begin{pmatrix} R^2\overline{a} & h^{-2}\overline{c} \\ R^4h^2\overline{b} & -R^2\overline{a} \end{pmatrix}$$
$$\underset{R\to 0}{\longrightarrow} \begin{pmatrix} 0 & h_0^{-2}\overline{c} \\ 0 & 0 \end{pmatrix} = \Psi_0^{*H_0}.$$

Combining all these, the conclusion is that

$$\nabla_{\hbar,R} = A_{H_R} + \hbar^{-1}\Phi + \hbar R^2\Phi^{*H_R} \underset{R\to 0}{\longrightarrow} A_{H_0} + \begin{pmatrix} 0 & \omega \\ 0 & 0 \end{pmatrix} + \hbar^{-1}\Phi + \hbar\Psi_0^{*H_0}.$$

□

Remark 3.6.5. We recover the result in (Collier and Wentworth, 2018, Proposition 5.1).

To combine both cases, where the underlying bundle can be polystable as a bundle, we make a definition of a kind of associated graded Higgs bundle similar to the one in Definition 2.3.1. Note that there, the graded is associated with a filtration. Here, we will associate it to a Higgs bundle (basically by choosing a suitable filtration, but nonetheless the two definitions differ.) So, if the underlying bundle of $(\overline{\partial}_E, \Phi)$ is unstable we take the decomposition $E = L \oplus L^{-1}$ where L is the maximally destabilizing subbundle. Then we define the *graded associated to the Higgs bundle*

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_L & \omega \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

as the Higgs bundle $\operatorname{Gr} E = L \oplus L^{-1}$ with

$$\partial_{\operatorname{Gr} E} = \begin{pmatrix} \overline{\partial}_L & 0 \\ 0 & \overline{\partial}_{L^{-1}} \end{pmatrix} \qquad \Phi_{\operatorname{Gr} E} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}. \tag{3.6.10}$$

If the underlying E is polystable as a vector bundle, we define the graded associated to $(\overline{\partial}_E, \Phi)$ as $(\overline{\partial}_E, 0)$. In both cases, the associated graded is polystable. Then the conformal limit of a $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle is written in the unified form of the following corollary.

Corollary 3.6.6. *Let $(\overline{\partial}_E, \Phi)$ be a polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle. Write $H_{\operatorname{Gr} E}$ for the solution of the (unscaled) Hitchin's equation for the associated graded $(\overline{\partial}_{\operatorname{Gr}(E)}, \Phi_{\operatorname{Gr}(E)})$. Set $B_{\operatorname{Gr}(E)} = \overline{\partial}_E - \overline{\partial}_{\operatorname{Gr}(E)}$. Then the $\hbar$-conformal limit of $(\overline{\partial}_E, \Phi)$ exists in the configuration space and it is*

$$A_{H_{\operatorname{Gr} E}} + B_{\operatorname{Gr}(E)} + \hbar^{-1}\Phi + \hbar\Phi_{\operatorname{Gr} E}^{*H_{\operatorname{Gr} E}}. \tag{3.6.11}$$

Of course, when E is polystable as a bundle we have $B_{\operatorname{Gr}(E)} = 0$ and $\Phi_{\operatorname{Gr} E} = 0$, and we recover equation 3.6.1. When E is unstable we have $B_{\operatorname{Gr}(E)} = \left(\begin{smallmatrix} 0 & \omega \\ 0 & 0 \end{smallmatrix}\right)$ and $\Phi_{\operatorname{Gr} E} = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$, from where we recover Theorem 3.6.4.

Remark 3.6.7. Note that the associated graded is the limit as $t \to 0$ of the Higgs bundle $t \cdot (\overline{\partial}_E, \Phi)$, i.e. under the well-known $\mathbb{C}^*$ action on the moduli space. This means that Corollary 3.6.6 relates the conformal limit with the limit of this action.

This form of the result opens the door for a possible generalization to the higher rank case, which would be interesting to further pursue. We also note that there should be a construction similar to the one performed in chapter 3 in this more general case. If we write the lower left corner of the matrix representation of $\nabla_{\hbar,R} = A_{H_R} + \hbar^{-1}\Phi + \hbar R^2 \Phi^{*_{H_R}}$ in this case we find it to be equal to

$$-\overline{\omega} h_R^2 + \hbar^{-1} c + \hbar R^2 h_R^2 \overline{b} = \left(\hbar^{-1} c(z) - \overline{\omega}(z) h_R^2\right) \left(dz + \frac{\hbar^2 R^2 h_R^2 \overline{b}(z)}{c(z) - \hbar h_R^2 \overline{\omega}(z)} dz\right). \tag{3.6.12}$$

The quantity $\mu = \frac{\hbar^2 R^2 h_R^2 \overline{b}}{c - \hbar h_R^2 \overline{\omega}}$ is expected to play the same role as before, being a Beltrami differential under some restrictions on ω and b. In that case, essentially the same proof goes through to show that these define complex projective structures branched over c.

Chapter 4

Geometry of the harmonic map and elementary representations

This chapter collects several results about the geometry of the harmonic map, and it was born out of an attempt to better understand (non-)elementary representations, and also the origin of the Beltrami differential that does come up in the construction of projective structures in Chapter 3. Most of these results were already known, but we collect and present them in a slightly different format. As a general reference for this chapter, we point to the excellent lecture notes (Li, 2019a, Parts I and II).

4.1 Equivariant maps

In this section, we present some expressions related to the geometry of the harmonic map associated with a Higgs bundle. Most of these are known, but the presentation is non-standard. We start by recalling the non-abelian Hodge correspondence as in Section 2.1. So we let X be a closed Riemann surface of genus $g \geq 2$, and $(\overline{\partial}_E, \varphi)$ be an $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle which is polystable. By Theorem 2.1.6, there exists a Harmonic metric H that satisfies (the unscaled) Hitchin equation (i.e. with parameter $R = 1$)

$$F_{A_H} + [\Phi, \Phi^{*_H}] = 0,$$

where A_H is the Chern connection for $\overline{\partial}_E$ and H. This is the condition for the connection $\nabla = A_H + \Phi + \Phi^*$ to be flat, i.e. (E, ∇) is a flat vector bundle. Its holonomy is the homomorphism $\rho : \pi_1(X) \to \mathrm{SL}(n, \mathbb{C})$ obtained by ∇-parallel transport of a fiber $E_{x_0} \cong \mathbb{C}^n$ of E over the point $x_0 \in X$. In fact, parallel transport establishes an isomorphism of E with the flat $\mathrm{SL}(n, \mathbb{C})$-vector bundle $E_\rho = \widetilde{X} \times_\rho \mathbb{C}^n$. This isomorphism is not unique, since it depends on a choice of the point x_0 and of the identification $E_{x_0} \cong \mathbb{C}^n$, but it is canonical after these choices. This is an isomorphism of flat bundles, which means that the flat connection ∇ gets identified with the natural flat connection on E_ρ. Further, the metric H gets mapped to a metric in E_ρ. Note that locally, since the isomorphism is given by parallel transport, it amounts to choosing flat coordinates for E. We will choose, once and for all, such an isomorphism, and we will choose it with the extra property that the metric H_{x_0} at the point $x_0 \in X$ is mapped to the identity metric in $\mathbb{C}^n$. We can do this by changing the identification of the fiber $E_{x_0} \cong \mathbb{C}^n$ if needed. We will assume this isomorphism has been performed already. This means we work directly in E_ρ with the natural flat connection and still denote the metric H under the isomorphism by H. Note that the metric and the flat connection are enough to get back the holomorphic structure and the Higgs field, and it is only a matter of seeing how these objects are expressed under this isomorphism, which we will do now. We recall the well-known interpretation of the real Higgs field $\Psi = \Phi + \Phi^{*_H}$ as the derivative of the harmonic metric.

Lemma 4.1.1. *Let $(\overline{\partial}_E, \Phi)$ be a polystable Higgs bundle with associated flat connection $\nabla = A_H + \Psi$. Then for any sections $s, t \in \Omega^0(E)$ we have $\nabla(H)(s,t) = -2H(s, \Psi t)$.*

Remark 4.1.2. Chose a smooth frame for E and write $\nabla = d + B$, where B is the connection matrix, and use still H for the matrix representation of the metric, and Ψ for the one of the real Higgs field. Then this identity is written as $(d + B) \cdot H = -2H\Psi$ or equivalently $dH - HB - \overline{B}^T H = -2H\Psi$. (Recall that the action of ∇ is extended to H which is a 2-tensor.) When flat coordinates are chosen this is written as $dH = -2H\Psi$, since then $B = 0$. This is the usual way this result is written (cf. for example (Guichard, 2017, Lemma 10.12)).

Proof. Recall that A is compatible with H if $d(H(s,t)) = H(As,t) + H(s, At)$. The expression $\nabla(H)$ denotes the extension of H to 2-tensors. By definition it is

$$\begin{aligned}
(\nabla H)(s,t) &= d(H(s,t)) - H(\nabla(H)s,t) - H(s, \nabla(H)) = \\
&= d(H(s,t)) - H(As,t) - H(s,At) - H(\Psi s,t) - H(s,\Psi t) \\
&= -H(\Psi s,t) - H(s,\Psi t) = -H(\Psi s,t) - H(\Psi^{*_H} s,t) = -2H(\Psi s,t).
\end{aligned}$$

□

Remark 4.1.3. Note that this lemma is valid in general. Indeed if E is a complex vector bundle with a connection ∇ and Hermitian metric H, and we write the unique decomposition of $\nabla = A + \Psi$, where A is a connection compatible with H and Ψ is self adjoint, i.e. $\Psi^{*_H} = \Psi$, then, we will still have $\nabla(H)(s,t) = -2H(s, \Psi t)$, for any local sections $s, t \in \Omega^0(E)$. This is because the same proof holds in this situation.

We can now use this to write Φ in terms of ∇ and H. Indeed, taking the $(1,0)$-part, we have immediately $\nabla^{1,0}(H)(s,t) = H(s, \Phi t)$. This is enough to express all objects of interest in E_ρ. So denote by $[\widetilde{x}, v]$ the points of $E_\rho = \widetilde{X} \times_\rho \mathbb{C}^n$, with $\widetilde{x} \in \widetilde{X}$ a lift of $x \in X$ and $v \in \mathbb{C}^n$. The sections $s \in \Omega^0(E_\rho)$ are of the form $x \mapsto [\widetilde{x}, \widetilde{s}(\widetilde{x})]$, where $\widetilde{s} : \widetilde{X} \to \mathbb{C}^n$ is a ρ-equivariant map. The correspondence $s \mapsto \widetilde{s}$ is the well-known identification of sections of a flat bundle with the equivariant maps to the model fiber. The metric H is in the same fashion represented by $x \mapsto [\widetilde{x}, \widetilde{H}_{\widetilde{x}}]$, where $\widetilde{H} : \widetilde{X} \to (\overline{\mathbb{C}^n})^* \otimes (\mathbb{C}^n)^* \cong \mathbb{C}^{n \times n}$, with $\widetilde{x} \mapsto \widetilde{H}_{\widetilde{x}}$, is a ρ-equivariant map, where the action of $\mathrm{SL}(n,\mathbb{C})$ on $\mathbb{C}^n$ is extended to the dual spaces and tensor products as usual. In particular $G \in \mathrm{SL}(n,\mathbb{C})$ acts on the matrix representing the metric $\widetilde{H}_{\widetilde{x}}$ as $\overline{G}^{-T}\widetilde{H}_{\widetilde{x}}G^{-1}$. Note also that the trivialization of $\det(E_\rho) = \widetilde{X} \times_\rho \wedge^n\mathbb{C}^n \cong \widetilde{X} \times_\rho \mathbb{C} = X \times \mathbb{C}$ (since $\det(\rho) = 1$) is represented by a constant section $c \in \mathbb{C}$. Recalling that we have chosen the identification of the fiber in such a way that $H_{x_0} = Id$, and that H induces this trivialization on $\det(E_\rho)$, we conclude that there is a point where $c = 1$, which then holds globally. This means, in particular, that $\det(\widetilde{H}) = 1$, i.e. the matrix $\widetilde{H}_{\widetilde{x}}$ is positive-definite and with unit determinant. We will denote the set of such matrices by $\mathcal{D} \subset \mathbb{C}^{n\times n}$. The flat connection ∇ is the natural connection

$$\begin{aligned}\nabla : \Omega^0(E_\rho) &\to \Omega^1(E_\rho)\\ [\widetilde{x}, \widetilde{s}(\widetilde{x})] &\mapsto [\widetilde{x}, d\widetilde{s}(\widetilde{x})].\end{aligned}$$

Now the Higgs field satisfies the identity $\nabla^{1,0}(H)(s,t) = H(s, \Phi t)$, and mapping this to E_ρ, we get, from the expression above of ∇ as d, that $\nabla(\widetilde{H}) = d\widetilde{H}$, since the connection matrix of ∇ is trivial, and thus

$$[\widetilde{x}, \nabla^{1,0}(\widetilde{H})_{\widetilde{x}}] = [\widetilde{x}, \partial\widetilde{H}_{\widetilde{x}}] = -2[\widetilde{x}, \widetilde{H}_{\widetilde{x}}\widetilde{\Phi}_{\widetilde{x}}].$$

This identifies the trace-free Higgs Field $\Phi \in \Omega^0(\mathrm{End}_0(E_\rho))$ as the map

$$x \mapsto [\widetilde{x}, \widetilde{\Phi}_{\widetilde{x}}] = [\widetilde{x}, -\frac{1}{2}\widetilde{H}_{\widetilde{x}}^{-1}\partial\widetilde{H}_{\widetilde{x}}]$$

where $\widetilde{\Phi} = -\frac{1}{2}\widetilde{H}^{-1}\partial\widetilde{H} : \widetilde{X} \to \mathrm{End}_0(\mathbb{C}^n)$ is the corresponding ρ-equivariant map. Note that it follows from $\Phi^* = H^{-1}\overline{\Phi}^T\Phi H$ that Φ^* corresponds to the equivariant map $\widetilde{\Phi}^* = \widetilde{H}^{-1}\overline{\widetilde{\Phi}}^T\widetilde{H} = -\frac{1}{2}\widetilde{H}^{-1}\overline{\partial}\widetilde{H}$, since $\widetilde{H}$ is a hermitian matrix. Finally the holomorphic structure $\overline{\partial}_E = \nabla^{(0,1)} - \Phi^*$ corresponds to the operator $\overline{\partial} - \widetilde{\Phi}^* = \overline{\partial} + \frac{1}{2}\widetilde{H}^{-1}\overline{\partial}\widetilde{H}$, i.e.

$$\begin{aligned}\overline{\partial}_E : \Omega^0(E_\rho) &\to \Omega^{(0,1)}(E_\rho)\\ [\widetilde{x}, \widetilde{s}(\widetilde{x})] &\mapsto [\widetilde{x}, (\overline{\partial} - \widetilde{\Phi}^*)\widetilde{s}(\widetilde{x})] = [\widetilde{x}, (\overline{\partial} + \frac{1}{2}\widetilde{H}^{-1}\overline{\partial}\widetilde{H})\widetilde{s}(\widetilde{x})].\end{aligned}$$

Now we recall that A_H is the Chern connection of H and $\overline{\partial}_E$. This means $A_H^{(0,1)} = \overline{\partial}_E$ and $A_H(H) = 0$ (equivalent to the compatibility of A_H and H, $dH(s,t) = H(A_H s, t) + H(s, A_H t)$). Using these expressions, the Chern connection is identified with the operator $A_H = d + \widetilde{A}$ which has connection matrix (cf. Lemma A.1.1)

$$\begin{aligned}A = (-\widetilde{\Phi}^*) - (-\widetilde{\Phi}^*)^* + \widetilde{H}^{-1}\partial\widetilde{H} = \frac{1}{2}\widetilde{H}^{-1}\overline{\partial}\widetilde{H} - \frac{1}{2}\widetilde{H}^{-1}\partial\widetilde{H} + \widetilde{H}^{-1}\partial\widetilde{H} &= \frac{1}{2}\widetilde{H}^{-1}\overline{\partial}\widetilde{H} + \frac{1}{2}\widetilde{H}^{-1}\partial\widetilde{H}\\ &= \frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}, \text{ i.e.}\end{aligned}$$

$$\begin{aligned}A_H : \Omega^0(E_\rho) &\to \Omega(E_\rho)\\ [\widetilde{x}, \widetilde{s}(\widetilde{x})] &\mapsto [\widetilde{x}, (d + \widetilde{A})\widetilde{s}(\widetilde{x})] = [\widetilde{x}, (d + \frac{1}{2}\widetilde{H}^{-1}d\widetilde{H})\widetilde{s}(\widetilde{x})].\end{aligned}$$

Let us collect the information about the equivariant maps in Table 4.1.

Table 4.1 Table representing the relations between objects in E_ρ and their equivariant counterparts. The metric H on E_ρ corresponds to the map $\widetilde{H}$ in $\widetilde{X}$ which is valued in the space of positive-definite Hermitian matrices in $\mathbb{C}^{n\times n}$ with unit determinant. The Higgs fields Φ, Φ^* and Ψ are trace-free $\mathrm{End}_0(E_\rho)$-valued one forms corresponding to $\widetilde{\Phi}$, $\widetilde{\Phi}^*$ and $\widetilde{\Psi}$ which are equivariant $\mathrm{End}_0(\mathbb{C}^n) = \mathfrak{sl}_n^{\mathbb{C}}$ valued one-forms on $\widetilde{X}$. The operators ∇, A_H and $\overline{\partial}_E$ are defined on sections $s \in \Omega^0(E_\rho)$ or, equivalently, by the expressions on the corresponding spaces of equivariant mappings $\widetilde{s} : \widetilde{X} \to \mathbb{C}^n$ given in the table.

Object in E_ρ	Equivariant object
H	$\widetilde{H} : \widetilde{X} \to \mathcal{D}$
∇	d
Φ	$\widetilde{\Phi} = -\frac{1}{2}\widetilde{H}^{-1}\partial\widetilde{H}$
Φ^*	$\widetilde{\Phi}^* = -\frac{1}{2}\widetilde{H}^{-1}\overline{\partial}\widetilde{H}$
Ψ	$\widetilde{\Psi} = -\frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}$
$\overline{\partial}_E$	$\overline{\partial} + \frac{1}{2}\widetilde{H}^{-1}\overline{\partial}\widetilde{H}$
A_H	$d + \widetilde{A} = d + \frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}$

Finally, we are only left with the Hitchin equation, which corresponds to the flatness of ∇. Since the connection ∇ is simply the trivial connection d, the flatness is automatic (for we are using flat sections to build the isomorphism with E_ρ). Thus Hitchin equation boils down only to the holomorphy of the Higgs field, i.e $\overline{\partial}_E \cdot \Phi = 0$ which is written equivariantly as

$$-\frac{1}{2}(\overline{\partial} + \frac{1}{2}\widetilde{H}^{-1}\overline{\partial}\widetilde{H}) \cdot \widetilde{H}^{-1}\partial\widetilde{H} = 0. \tag{4.1.1}$$

Remark 4.1.4. Note that from the point of view of the original bundle E the map $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ is not canonically defined since it depends on the isomorphism $E \cong E_\rho$ we have fixed from the beginning. Nonetheless, changing this isomorphism (the point x_0 and the identification of the fiber) corresponds to changing $\widetilde{H}$ by the global action of G on $\mathcal{D}$ (and possibly the realization of $\pi_1(X)$ as group of deck transformations of $\widetilde{X}$). This means that the geometry of the situation is preserved, and we can safely refer to *the* map $\widetilde{H}$.

4.2 The geometry of the target space

Let us consider the equivariant map induced by the Harmonic metric $\widetilde{H}$. This is a map $\widetilde{H} : \widetilde{X} \to \mathcal{D} \subset \mathbb{C}^{n\times n}$, where $\mathcal{D}$ is the set of positive definite Hermitian matrices with determinant one,

$$\mathcal{D} = \left\{ H \in \mathbb{C}^{n\times n} | H = \overline{H}^T, H \succ 0 \text{ and } \det(H) = 1 \right\}. \tag{4.2.1}$$

This is the smooth manifold whose geometry determines the possible harmonic metrics, and whose study occupies this subsection.

Remark 4.2.1. Note that $\mathcal{D}$ is just a model for the homogeneous space $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$, which is more commonly used. These spaces are diffeomorphic by the map $\mathsf{F} : \mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n) \to \mathcal{D}$ sending $[G] \mapsto \overline{G}^{-T}G^{-1}$. This map is equivariant, changing the left multiplication in $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$ to the left action on metrics, where G act as $\overline{G}^{-T}HG^{-1}$. Indeed

$$\mathsf{F}(G_1[G_2]) = \overline{G_1G_2}^{-T}(G_1G_2)^{-1} = \overline{G_1G_2}^{-T}G_2^{-1}G_1^{-1} = \overline{G_1}^{-T}\mathsf{F}(G_2)G^{-1}.$$

Also note that $\mathcal{D}$ *is not* a complex manifold. For example, in the case of $n = 2$, one can show this is a model for hyperbolic space $\mathbb{H}^3$.

Since $H \succ 0$ is an open condition, the tangent bundle to $\mathcal{D}$ is obtained by linearizing the identities $\det(H) = 1$ and $H = \overline{H}^T$. Since the second is linear, only in the first one we calculate $d(\det(H)) = \mathrm{tr}(\mathrm{adj}(H)dH) = \det(H)\,\mathrm{tr}(H^{-1}dH) = \mathrm{tr}(H^{-1}dH)$, where we have used Jacobi's formula in the first equality together with the fact that $\det(H) = 1$. This means we have an identification as

$$T_H^{\mathbb{R}}\mathcal{D} = \left\{M \in \mathbb{C}^{n\times n} | M = \overline{M}^T \text{ and } \mathrm{tr}(H^{-1}M) = 0\right\}. \tag{4.2.2}$$

The complexification of the tangent bundle $T\mathcal{D}^{\mathbb{C}} = T\mathcal{D}$ is simply $T^{\mathbb{R}}\mathcal{D} \otimes_{\mathbb{R}} \mathbb{C}$ and it has a natural trivialization given by

$$\xi : T\mathcal{D}^{\mathbb{C}} \to \mathcal{D} \times \mathrm{End}_0(\mathbb{C}^n) \tag{4.2.3}$$
$$M_H \otimes \alpha \mapsto (H, -\frac{1}{2}\alpha H^{-1}M_H). \tag{4.2.4}$$

The choice of the constant $-\frac{1}{2}$ is used to simplify the expressions that will follow.

This maps the canonical real structure τ_{can} sending $M_H \otimes \alpha \mapsto M_H \otimes \overline{\alpha}$ to the one given by $\tau(H, \Psi) = (H, \Psi^{*_H}) = (H, H^{-1}\overline{\Psi}^T H)$, since $\xi \circ \tau_{can} = \tau \circ \xi$, as the following computation shows:

$$\begin{aligned}\xi \circ \tau_{can}(M_H \otimes \alpha) = \xi(M_H \otimes \overline{\alpha}) &= -\frac{1}{2}\overline{\alpha}H^{-1}M_H = -\frac{1}{2}\overline{\alpha}H^{-1}\overline{M_H}^T H^{-1}H = \\ &= -\frac{1}{2}\overline{\alpha}H^{-1}\overline{H^{-1}M_H}^T H = -\frac{1}{2}(\alpha H^{-1}M_H)^{*_H} = \tau(H, \frac{1}{2}\alpha H^{-1}M_H) = \tau \circ \xi(M_H \otimes \alpha).\end{aligned}$$

The real bundle $T^{\mathbb{R}}\mathcal{D} \subset T\mathcal{D}^{\mathbb{C}}$ is thus identified with the fixed locus of τ in $\mathcal{D} \times \mathrm{End}_0(\mathbb{C}^n)$. The tangent action on $T\mathcal{D}^{\mathbb{C}}$ gets changed to the $\mathrm{SL}(n,\mathbb{C})$ action on $\mathcal{D} \times \mathrm{End}_0(\mathbb{C}^n)$. Indeed, if we denote the tangent action by $G \cdot M_H = \overline{G}^{-T}M_H G^{-1} \in T_{\overline{G}^{-T}HG^{-1}}\mathcal{D}^{\mathbb{C}}$ we have

$$\xi(G \cdot M_H) = \xi(\overline{G}^{-T}M_H G) = -\frac{1}{2}GH^{-1}\overline{G}^T\overline{G}^{-T}M_H G = -\frac{1}{2}G^{-1}M_H G = G^{-1}\xi(M_H)G.$$

The Killing form on $\mathrm{SL}(n,\mathbb{C})$ is $B(Y_1,Y_2)=2n\ \mathrm{tr}(Y_1Y_2)$. It descends to $\mathfrak{sl}_n^{\mathbb{C}}/\mathfrak{su}_n$, the tangent space of $\mathrm{SL}(n,\mathbb{C})/\,\mathrm{SU}(n)$ to $[id]$. This will then induce a left-invariant metric on $T^{\mathbb{R}}\mathcal{D}$. Indeed to calculate its value at $T^{\mathbb{R}}_H\mathcal{D}$ recall the equivariant diffeomorphism $\mathsf{F}([G])=\overline{G}^{-T}G^{-1}$ of Remark 4.2.1. We map $T^{\mathbb{R}}_H\mathcal{D}$ to $T^{\mathbb{R}}_{Id}\mathcal{D}$ using the left action on $\mathcal{D}$ and then pull-back by the differential of F at $[G]=[id]$. So if $H=\overline{G}^{-T}G^{-1}$ then a vector M in $T^{\mathbb{R}}_H\mathcal{D}$ gets mapped to $\overline{G}^TMG$ in $T^{\mathbb{R}}_{Id}\mathcal{D}$ by using G^{-1}. The differential of F at $[G]=[id]$ is $d\mathsf{F}|_{id}=d(\overline{G}^{-T}G^{-1})|_{id}=-dG-\overline{dG}^T$, and thus it is the map $Y\mapsto -Y-\overline{Y}^T$, for $Y\in\mathfrak{sl}_n^{\mathbb{C}}/\mathfrak{su}_n$. Its inverse is the map $Z\mapsto -\frac{1}{2}Z+\mathfrak{su}_n\in\mathfrak{sl}_n^{\mathbb{C}}/\mathfrak{su}_n$. Composing this with the action we get $-\frac{1}{2}\overline{G}^TMG+\mathfrak{su}_n$. Thus one obtains the left-invariant Riemannian metric on $T^{\mathbb{R}}\mathcal{D}$ which, since $H=\overline{G}^{-T}G^{-1}$, is

$$\langle M_1,M_2\rangle_H=\frac{1}{4}B\left(\overline{G}^TM_1G\,,\,\overline{G}^TM_2G\right)=\frac{n}{2}\,\mathrm{tr}(G\overline{G}^TM_1G\overline{G}^TM_2)=\frac{n}{2}\,\mathrm{tr}(H^{-1}M_1H^{-1}M_2).$$

This metric can be extended as a hermitian metric on $T\mathcal{D}^{\mathbb{C}}$ as

$$\langle M_1\otimes\alpha_1,M_2\otimes\alpha_2\rangle_H=\overline{\alpha_1}\alpha_2\,\frac{n}{2}\,\mathrm{tr}(H^{-1}M_1H^{-1}M_2)=\frac{n}{2}\,\mathrm{tr}\left(H^{-1}\overline{M_1\otimes\alpha_1}^TH^{-1}M_2\otimes\alpha_2\right).$$

Using ξ we can induce a metric in $\mathcal{D}\times\mathrm{End}_0(\mathbb{C}^n)$ that makes this trivialization isometric, simply by setting $\langle\!\langle\Psi_1,\Psi_2\rangle\!\rangle_H:=\langle\xi^{-1}(\Psi_1),\xi^{-1}(\Psi_2)\rangle_H$. We can calculate ξ^{-1} to obtain $\xi^{-1}(H,\Psi)=-2H\Psi$ at H. Thus we get

$$\langle\!\langle\Psi_1,\Psi_2\rangle\!\rangle_H=4\langle H\Psi_1,H\Psi_2\rangle_H=2n\,\mathrm{tr}\left(H^{-1}\overline{\Psi_1}^THH^{-1}H\Psi_2\right)=2n\,\mathrm{tr}(\Psi_1^{*_H}\Psi_2).\qquad(4.2.5)$$

The equivariance of ξ and the fact that $\mathrm{SL}(n,\mathbb{C})$ acts by isometries in $\mathcal{D}$ guarantee that $\mathrm{SL}(n,\mathbb{C})$ also acts by isometries on both factors of $\mathcal{D}\times\mathrm{End}_0(\mathbb{C}^n)$.

4.3 Geometric features of the induced equivariant map

So we are ready to study the differential geometry of the equivariant map induced by the harmonic metric $\widetilde{H}:\widetilde{X}\to\mathcal{D}$, which we call harmonic map for short (and will be justified later in Section 4.6). Recall that the derivative of a map f between manifolds can be calculated in coordinates as the Jacobian matrix $\left(\frac{\partial f^j}{\partial x^k}\right)$. An equivalent way to write this matrix makes use of the entry-wise differential (df^j), where we take the exterior derivative of the coordinate representation(f^j) of f. It is then contracted to $(df^j)_{\lrcorner\frac{\partial}{\partial x^k}}$, which yields the same matrix. The image of a vector-field $Y=Y^k\frac{\partial}{\partial x^k}$ under the differential is simply $\sum_k\left(\frac{\partial f^j}{\partial x^k}Y^k\right)$ which using this second representation reads

$$\sum_k(df^j)_{\lrcorner\frac{\partial}{\partial x^k}}Y^k=(df^j)_{\lrcorner\sum_kY^k\frac{\partial}{\partial x^k}}=(df^j)_{\lrcorner Y}.$$

Of course inside linear spaces the coordinate representations of maps are global, and we write f as a global tuple of functions. When we are dealing with matrix spaces, f is represented by a matrix and the entry-wise differential is simply df. Thus the derivative applied to a vector field

Y is simply $df_{\lrcorner Y}$. We will use this notation from now onward. So the derivative of $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ is simply the matrix of one-forms in $\widetilde{X}$ acting by contraction,

$$d\widetilde{H} : T^{\mathbb{R}}\widetilde{X} \to T^{\mathbb{R}}\mathcal{D} \tag{4.3.1}$$

$$Y \mapsto d\widetilde{H}_{\lrcorner Y}. \tag{4.3.2}$$

This extends linearly to the complexification as $d\widetilde{H} : T^{\mathbb{C}}\widetilde{X} \to T^{\mathbb{C}}\mathcal{D}$ with the same expression $Y \mapsto d\widetilde{H}_{\lrcorner Y}$. We can compose it with the trivialization ξ to obtain a map to $\mathcal{D} \times \mathrm{End}_0(\mathbb{C}^n)$ which is simply

$$(Id, -\frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}) : T\widetilde{X} \to \mathcal{D} \times \mathrm{End}_0(\mathbb{C}^n) \tag{4.3.3}$$

$$Y \mapsto (\widetilde{H}, -\frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}_{\lrcorner Y}). \tag{4.3.4}$$

Since the trivialization swaps the tangent action with the adjoint action on $\mathrm{End}_0(\mathbb{C}^n)$ we conclude that for any Y the second component $-\frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}_{\lrcorner Y}$ is equivariant. This means it can be interpreted as a section of the bundle $\Omega^0(\mathrm{End}(E_\rho))$. The conclusion is that, even though the derivative per se is not well defined as a section of a bundle over the surface X, after the trivialization, it becomes a well-defined section of a bundle over X. Thus the ρ-equivariant harmonic map $\widetilde{H}$ shares similarities with maps out of compact surfaces, in the sense that, even though $\widetilde{H}$ itself is not defined on the compact surface X, its tangent data is determined by a bundle over X.

If we let Y vary, this is actually the one form $-\frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}$ on $\widetilde{S}$ with values in $\mathrm{End}(E_\rho)$, acting by contraction. We can now compare with Table 4.1 and observe that this is in fact the equivariant map $\widetilde{\Psi}$ associated with the real Higgs field Ψ. This means that the derivative of the Harmonic map is identified with the real Higgs field, where the image of a vector field on the surface is obtained by contraction. From now on we will stop using the explicit distinction between the equivariant maps and the objects in E_ρ.

Lemma 4.3.1. *Under the previous identification, the derivative of the harmonic map $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ corresponds to the real Higgs field Ψ. The image of a vector field is obtained by contraction with Ψ.*

Note that we can just take the $(1,0)$ and the $(0,1)$ parts to obtain the identifications with Φ and Φ^*. The endomorphism part of Φ is simply $\Phi_{\lrcorner \frac{\partial}{\partial z}}$. By the lemma, this is simply the image of $\frac{\partial}{\partial z}$ under the derivative of the harmonic map $\widetilde{H}$, since $\Phi^*_{\lrcorner \frac{\partial}{\partial z}} = 0$. Analogously, the image of $\frac{\partial}{\partial \bar{z}}$ is $\Phi^*_{\lrcorner \frac{\partial}{\partial \bar{z}}}$, the endomorphism part of Φ^*. We are in conditions to study the rank of the harmonic map. To ease notation we write $\Phi_{\lrcorner \frac{\partial}{\partial z}} = \phi$ and $\Phi^*_{\lrcorner \frac{\partial}{\partial \bar{z}}} = \phi^*$, that is

$$\Phi = \phi\, dz \quad \text{and} \quad \Phi^* = \phi^*\, d\bar{z}.$$

Note also that, since the action of $\mathrm{SL}(n,\mathbb{C})$ in $\mathcal{D}$ is made by diffeomorphisms, it preserves the rank. This means that the set of points of $\widetilde{X}$ where the rank of the derivative of $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ is constant, i.e the level sets $\{rank = c\}_{c=0,1,2}$, is, in fact, invariant. Thus it covers a set in X, which we call $R^c \subset X$. By a slight abuse of language, we refer to the rank of the derivative of $\widetilde{H}$ at points of X (and not on the universal cover $\widetilde{X}$).

Theorem 4.3.2. *At the point $z \in X$, the rank of the harmonic map $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ is the $\mathbb{C}$-dimension of the vector space generated by ϕ and ϕ^* inside the fiber $(\mathrm{End}_0(E_\rho))_z$, where the Higgs field is $\Phi = \phi\, dz$. More precisely, at z the map has*

1. *rank 2 if and only if ϕ and ϕ^* are linearly independent;*

2. *rank 1 if and only if $\phi = k\,\phi^*$, for $k \in \mathbb{C}^*$ with $|k| = 1$, but $\phi \neq 0 \neq \phi^*$;*

3. *rank 0 if and only if $\phi = 0 = \phi^*$.*

Proof. This is simply a consequence of the fact that ϕ and ϕ^* are the images of the complex vectors ∂_z and $\partial_{\overline{z}}$ under the (complexification of) the derivative of the harmonic map. Since $\{\partial_z, \partial_{\overline{z}}\}$ form a $\mathbb{C}$-basis of $T_z\widetilde{X}$ and the real rank is the same as the complex rank of the complexification, the result follows. $\square$

In conclusion, the local features of the harmonic map are controlled by the endomorphism parts of the Higgs field and its adjoint. It is a particularly interesting question to study how the representation interacts with the harmonic map, and concretely for which representations and base Riemann surface structure X is $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ an immersion. Thus it might be particularly useful to have some more expressions relating the Higgs field and its adjoint with the *degeneracy locus* $R = R^0 \cup R^1$ of the harmonic map, i.e. the set of points in X where the map has non-maximal rank.

Lemma 4.3.3. *For all points $z \in X$ we have*

$$\mathrm{tr}(\phi\phi^*)^2 - \left|\mathrm{tr}(\phi^2)\right|^2 \geq 0,$$

with equality if and only if the point lines in the degeneracy locus i.e. $z \in R$.

Remark 4.3.4. Recall that ϕ is the endomorphism part of the Higgs field, and so the square ϕ^2 is an endomorphism, and thus not automatically zero, which contrasts with the two form $\Phi \wedge \Phi = 0$.

Proof. We will use the equivariant objects to prove this. Recall the hermitian metric on $T^{\mathbb{C}}\mathcal{D} \cong \mathcal{D} \times \mathrm{End}_0(E_\rho)$ is $\langle\!\langle \Psi_1, \Psi_2 \rangle\!\rangle_H = 2n\,\mathrm{tr}(\Psi_1^{*_H}\Psi_2)$. The Cauchy–Schwarz for this metric is

$$|\langle\!\langle \Psi_1, \Psi_2 \rangle\!\rangle_H|^2 \leq \langle\!\langle \Psi_1, \Psi_1 \rangle\!\rangle_H \langle\!\langle \Psi_2, \Psi_2 \rangle\!\rangle_H,$$

with equality if and only if Ψ_1 and Ψ_2 are linearly dependent. This inequality for the metric can be pulled-back under $\widetilde{H} : \widetilde{X} \to \mathcal{D}$, i.e. it can be calculated at $\Psi_1 = \widetilde{\phi}^*$ and $\Psi_2 = \widetilde{\phi}$ reading

$$|\operatorname{tr}\widetilde{\phi}^2|^2 = |\operatorname{tr}(\widetilde{\phi}\widetilde{\phi})|^2 \leq \operatorname{tr}(\widetilde{\phi}\widetilde{\phi}^*)\operatorname{tr}(\widetilde{\phi}^*\widetilde{\phi}) = \left(\operatorname{tr}(\widetilde{\phi}\widetilde{\phi}^*)\right)^2,$$

since $\operatorname{tr}(\widetilde{\phi}\widetilde{\phi}^*)$ is real. This is the inequality to be proved. Equality holds if and only if $\widetilde{\phi}$ and $\widetilde{\phi}^*$ are linearly dependent. By Theorem 4.3.2 this happens if and only if the derivative of the harmonic map has non-maximal rank, i.e. if and only if z is in the degeneracy locus R. □

Remark 4.3.5. Since the harmonic metric is actually real-analytic this realizes R as a real-analytic space, i.e a real analytic subvariety of the closed Riemann surface X given by the zero locus of the real-analytic function $\operatorname{tr}(\phi\phi^*)^2 - |\operatorname{tr}(\phi^2)|^2$.

4.4 The induced Riemannian metric

The harmonic map $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ can be used to pull back the ambient Riemannian metric on $\mathcal{D}$ to the surface $\widetilde{X}$, and, as $\mathrm{SL}(n, \mathbb{C})$ acts by isometries, this tensor is actually invariant and it descends to X. It is a Riemannian metric at the immersed points of $\widetilde{X}$. This happens precisely at the points where the derivative of the harmonic map has maximal rank. At the level of the compact surface X, this induces a Riemannian metric on $X - R$, i.e. outside the degeneracy locus. We call this tensor the *induced* metric (even though it degenerates at R). It is useful to have an explicit expression for this metric, which we calculate. So the pullback of the metric is just obtained by applying the derivative of the Harmonic metric in Equation 4.3.4 to the entries of the hermitian metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ given in Equation 4.2.5:

$$h(Y_1, Y_2) = \langle\!\langle \Psi_1, \Psi_2 \rangle\!\rangle = 2n \operatorname{tr}(\Psi_1^{*H} \Psi_2) \qquad \text{with } \Psi_j = -\frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_j}.$$

Recall that the Hermitian metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ was the Hermitian extension of the real Riemannian metric (i.e. antilinear in the first factor). This means that the bilinear extension of the Riemannian metric is $g(Y_1, Y_2) = h(\overline{Y}_1, Y_2)$. Using the real structure τ in $\mathcal{D} \times \operatorname{End}_0 \mathbb{C}^n$ (cf. paragraph after the definition of the trivialization ξ, Equation 4.2.4) this gets mapped to

$$g(Y_1, Y_2) = \langle\!\langle \Psi_1^*, \Psi_2 \rangle\!\rangle = 2n \operatorname{tr}(\Psi_1, \Psi_2) \qquad \text{with } \Psi_j = -\frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_j}. \tag{4.4.1}$$

This is the $\mathbb{C}$-bilinear tensor which is the complexification in $TX = T^{\mathbb{C}}X$ of the Riemannian metric in $T^{\mathbb{R}}X$. We conclude the following result.

Proposition 4.4.1. *(Li, 2019b, Equation (5.1)) If the Higgs field is $\Phi = \phi\, dz$ the metric induced by the harmonic map on X is*

$$g = 2n\left(\operatorname{tr}(\phi^2)\, dz^2 + \operatorname{tr}(\phi\phi^*)\,(dz \otimes d\overline{z} + d\overline{z} \otimes dz) + \overline{\operatorname{tr}(\phi^2)}\, d\overline{z}^2\right).$$

Proof. If g is the (bilinear complexification) of the Riemannian metric, then its local expression is

$$g\left(\frac{\partial}{\partial z},\frac{\partial}{\partial z}\right)dz\otimes dz+g\left(\frac{\partial}{\partial \overline{z}},\frac{\partial}{\partial z}\right)d\overline{z}\otimes dz+g\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \overline{z}}\right)dz\otimes d\overline{z}+g\left(\frac{\partial}{\partial \overline{z}},\frac{\partial}{\partial \overline{z}}\right)d\overline{z}\otimes d\overline{z}. \tag{4.4.2}$$

Recall that $\phi=\Phi_{\lrcorner\frac{\partial}{\partial z}}=-\frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}_{\lrcorner\frac{\partial}{\partial z}}$ and that $\phi^*=\Phi^*_{\lrcorner\frac{\partial}{\partial \overline{z}}}=-\frac{1}{2}\widetilde{H}^{-1}d\widetilde{H}_{\lrcorner\frac{\partial}{\partial \overline{z}}}$ (by equation 4.3.4, Lemma 4.3.1 and the discussion after it). So we can substitute this in Equation 4.4.1 defining g, to calculate the expression in the proposition. □

We can perform the check-up calculation to verify that indeed this metric degenerates only on R as defined by Lemma 4.3.3.

Remark 4.4.2. Note that the conformal part of the metric is $2n\,\mathrm{tr}(\phi\phi^*)\,(dz\otimes d\overline{z}+d\overline{z}\otimes dz)$. The function $e(\phi)=2n\,\mathrm{tr}(\phi\phi^*)$ is the *energy density* of the field. We can calculate it as the the two form $2n\,\mathrm{tr}(\Psi\wedge\Psi^*)=e(\phi)dz\wedge d\overline{z}$.

One of the most relevant features of this induced metric is that it is not a conformal metric. This happens if and only if the dz^2 term vanishes, or equivalently, $\mathrm{tr}(\phi^2)=0$. If this equality holds at a point, the harmonic map is infinitesimally conformal there (i.e. it preserves angles in that tangent space). If this equality holds everywhere the harmonic map $\widetilde{H}:\widetilde{X}\to\mathcal{D}$ is a conformal map. Of course, this is to be interpreted at points outside R, for in that set, the map lowers its rank, and the metric degenerates. One might perform the usual construction (see for example (Jost, 2002, eg. Lemma 3.11.1)) to obtain conformal coordinates for g.

Proposition 4.4.3. *Let $\Phi=\phi\,dz$ be the Higgs field and g the induced metric on X by the harmonic map,*

$$g=2n\left(adz^2+b(dz\otimes d\overline{z}+d\overline{z}\otimes dz)+\overline{a}d\overline{z}^2\right),\qquad a=\mathrm{tr}(\phi^2)\quad b=\mathrm{tr}(\phi\phi^{*_H}).$$

Then g can be written as

$$g=\sigma\left(\eta\otimes\overline{\eta}+\overline{\eta}\otimes\eta\right),$$

where $\eta=dz+\mu\,d\overline{z}$, $\mu\in\mathbb{C}$ with $\mu\in\mathbb{C}$ and $\sigma\in\mathbb{R}$. These quantities are given by the expressions

$$\sigma=2n\frac{b+\sqrt{b^2-|a|^2}}{2}\qquad\text{and}\qquad\mu=\frac{\overline{a}}{b+\sqrt{b^2-|a|^2}}.$$

The set of points where g degenerates is precisely the one where μ is such that $|\mu|=1$.

Proof. Upon simplifying $\eta=dz+\mu\,d\overline{z}$ on $g=\sigma\left(\eta\otimes\overline{\eta}+\overline{\eta}\otimes\eta\right)$ we get

$$g=2\sigma\overline{\mu}\,dz^2+\sigma(1+|\mu|^2)(dz\otimes d\overline{z}+d\overline{z}\otimes dz)+2\sigma\mu\,d\overline{z}^2.$$

We start by noting that $2\sigma\mu = 2n\overline{a}$, and $2\sigma\overline{\mu} = 2na$ (take notice that σ is real). So only the calculation of the middle term is left. Using this first equality again we notice

$$\begin{aligned}\sigma(1+|\mu|^2) = \sigma + \sigma\overline{\mu}\mu = 2n\frac{b+\sqrt{b^2-|a|^2}}{2} + na\frac{\overline{a}}{b+\sqrt{b^2-|a|^2}} &= \\ = \frac{n}{b+\sqrt{b^2-|a|^2}}\left(\left(b+\sqrt{b^2-|a|^2}\right))^2+|a|^2\right) &= \\ = \frac{n}{b+\sqrt{b^2-|a|^2}}\left(b^2+b^2+2b\sqrt{b^2-|a|^2}\right) &= 2nb,\end{aligned}$$

where we have used the inequality $b^2 - |a|^2 \geq 0$ of Lemma 4.3.3, on the first to last equality. Finally $|\mu| = 1$ if and only if

$$\begin{aligned}|a|^2 = \left(b+\sqrt{b^2-|a|^2}\right)^2 &\iff \\ \iff |a|^2 - b^2 - b^2 + |a|^2 - 2b\sqrt{b^2-|a|^2} &= 0 \\ \iff |a|^2 - b^2 - b\sqrt{b^2-|a|^2} &= 0.\end{aligned}$$

writing $r = \sqrt{b^2-|a|^2}$ this is equivalent to the quadratic equation $r^2 - br = 0$ which has solutions $b = 0$ or $r = 0$. The first case corresponds to $b = \operatorname{tr}(\phi\phi^*) = 0$ which means the norm of the Higgs field is zero. This happens if and only if the Higgs field is zero, i.e. in the case where the metric is everywhere degenerate. In particular, we also have $r = 0$ in this case. On the non-degenerate case, the $r = 0$ condition is precisely the one that describes the set where g degenerates. □

Remark 4.4.4. When $|\mu| < 1$, the coordinates v satisfying the Beltrami equation $\frac{\overline{\partial v}}{\partial \overline{z}} = \mu\frac{\partial v}{\partial z}$ are conformal for g (compare with Section 2.4).

It is also possible to calculate the Gaussian curvature (along with the connection matrix) of this metric, but the computations are rather cumbersome. We include for reference the result, whose calculation can be retrieved from the one in the proof of (Hitchin, 1987, Theorem (11.2)), and shall be explored further somewhere else.

Proposition 4.4.5. *The Lévi-Civita connection of the metric g induced by the harmonic map is*

$$\theta = d + \frac{1}{1-|\mu|^2}\left((C_1 + \overline{\mu}C_2)\, dz + (C_2 + \mu C_1)\, d\overline{z}\right), \tag{4.4.3}$$

with $C_1 = \partial_{\overline{z}}\mu - \partial^{\mu}\log(\sigma)$ and $C_2 = -\partial_z\mu$, where $\partial^{\mu} = \frac{1}{1-|\mu|^2}\left(\partial_z - \overline{\mu}\partial_{\overline{z}}\right)$ and the rest of the notation is as in Proposition 4.4.3.

Of course, the curvature form is then $d\theta + \theta \wedge \theta$.

4.5 Second fundamental form

After the calculation of the Riemannian metric (first fundamental form), one can calculate the second fundamental form of $\widetilde{H} : \widetilde{X} \to \mathcal{D}$. Recall that for general submanifolds this is the 2-tensor valued in the normal bundle $N\widetilde{H} \subset T^{\mathbb{R}}\mathcal{D}$

$$\mathrm{II}^{\mathbb{R}}(Y_1, Y_2) = \left(\nabla^{\mathcal{D}}_{\widetilde{H}_* Y_1}(\widetilde{H}_* Y_2)\right)^{\perp} = \nabla^{\mathcal{D}}_{\widetilde{H}_* Y_1}(\widetilde{H}_* Y_2) - \widetilde{H}_* \nabla^X_{Y_1} Y_2,$$

where $\nabla^{\mathcal{D}}$ is the Levi-Civita in $\mathcal{D}$ and ∇^X the original Levi-Civita connection on X or on $\widetilde{X}$ (we pick the metric g_0 compatible with the Riemann surface structure). This is just the classical Gauss Formula, cf. for example (Dajczer and Tojeiro, 2019, Section 1.1). Of course, we will prefer working with the complexifications of these objects. For the $\mathbb{C}$-linear extensions, the same equality holds inside $T\mathcal{D} = T^{\mathbb{C}}\mathcal{D}$, but we can now apply the trivialization ξ in Equation 4.2.4 and look at its second component. This will be a $2-$tensor on $\widetilde{X}$ valued on $\mathrm{SL}(n, \mathbb{C})$. As before, by equivariance, it will correspond to a tensor valued on $\mathrm{End}_0(E_\rho)$ which we will try to identify in the context of Higgs bundles, as we have been doing throughout this section. So our goal is to make more explicit the tensor

$$\mathrm{II}(Y_1, Y_2) = \xi_2\left(\nabla^{\mathcal{D}}_{\widetilde{H}_* Y_1}(\widetilde{H}_* Y_2) - \widetilde{H}_* \nabla^X_{Y_1} Y_2\right) \in \mathrm{End}_0(\mathbb{C}^n),$$

where ξ_2 is the second component of ξ. To begin with we need to calculate the Levi-Civita connection of $\mathcal{D}$. Recall that $T\mathcal{D}$ is seen as a matrix space (cf. the comments at the beginning of Subsection 4.3), this means that a vector field in $T\mathcal{D}$ is a matrix Z which is a function of $H \in \mathcal{D}$. The action of a vector field Z_1 on another one Z_2 is $Z_1 \cdot (Z_2) = dZ_{2\lrcorner Z_1} = \left(\frac{\partial Z_2^j}{\partial a^k} Z_1^k\right)_{j,k}$, where a^k are matrix coordinates on $\mathcal{D}$.

Lemma 4.5.1. *The (complexification of the) Levi-Civita connection of $\mathcal{D}$ is*

$$\nabla^{\mathcal{D}} : \mathfrak{X}(\mathcal{D}) \times \mathfrak{X}(\mathcal{D}) \to \mathfrak{X}(\mathcal{D}) \tag{4.5.1}$$

$$(Z_1, Z_2)_H \mapsto \nabla^{\mathcal{D}}_{Z_1} Z_2 = dZ_{2\lrcorner Z_1} - \frac{1}{2}\left(dH_{\lrcorner Z_1} H^{-1} Z_2 + Z_2 H^{-1} dH_{\lrcorner Z_1}\right). \tag{4.5.2}$$

Proof. We need to verify that this is torsion-free and compatible with the metric. Observe that the operator d is taken in the manifold $\mathcal{D}$. Further, the point $H \in \mathcal{D}$ appears as its representation in coordinates on the right-hand side. That is, the entries of H, a^{ij} are coordinates in $\mathcal{D}$. This means $H = (a^{jk})$ is just the identity map in $\mathcal{D}$. Thus dH is the matrix of differentials (da^{jk}). It follows that $(dH_{\lrcorner Z_1})^{jk} = (da^{jk})_{\lrcorner Z_1} = (Z_1)^{jk}$, i.e. $dH_{\lrcorner Z_1} = Z_1$. So, for the torsion we calculate

$$\begin{aligned}\nabla^{\mathcal{D}}_{Z_1} Z_2 - \nabla^{\mathcal{D}}_{Z_2} Z_1 &= dZ_{2\lrcorner Z_1} - dZ_{1\lrcorner Z_2} - \frac{1}{2}\left(Z_1 H^{-1} Z_2 + Z_2 H^{-1} Z_1 - Z_2 H^{-1} Z_1 - Z_1 H^{-1} Z_2\right) \\ &= [Z_1, Z_2] + 0.\end{aligned}$$

This means the connection is torsion-free. Now for the compatibility with the metric. Write

$$\nabla^{\mathcal{D}} Z = dZ - \frac{1}{2}\left(dHH^{-1}Z + ZH^{-1}dH\right).$$

We will prove that for real vector fields Z_1 and Z_2 we have

$$d\langle Z_1, Z_2\rangle = \langle \nabla^{\mathcal{D}} Z_1, Z_2\rangle + \langle Z_1, \nabla^{\mathcal{D}} Z_2\rangle.$$

Recall that $\langle Z_1, Z_2\rangle = \frac{n}{2}\operatorname{tr}(H^{-1}Z_1H^{-1}Z_2)$ is the Riemannian metric in $T^{\mathbb{R}}\mathcal{D}$. So the left-hand side reads

$$\begin{aligned} d\langle Z_1, Z_2\rangle = \frac{n}{2} d\operatorname{tr}(H^{-1}Z_1H^{-1}Z_2) = \\ = -\frac{n}{2}\operatorname{tr}(H^{-1}dHH^{-1}Z_1H^{-1}Z_2) + \frac{n}{2}\operatorname{tr}(H^{-1}dZ_1H^{-1}Z_2) \\ -\frac{n}{2}\operatorname{tr}(H^{-1}Z_1H^{-1}dHH^{-1}Z_2) + \frac{n}{2}\operatorname{tr}(H^{-1}Z_1H^{-1}dZ_2). \end{aligned}$$

The right-hand side reads

$$\begin{aligned} &\langle \nabla^{\mathcal{D}} Z_1, Z_2\rangle + \langle Z_1, \nabla^{\mathcal{D}} Z_2\rangle = \\ &= \frac{n}{2}\operatorname{tr}\left(H^{-1}\left(dZ_1 - \frac{1}{2}\left(dHH^{-1}Z_1 + Z_1H^{-1}dH\right)\right)H^{-1}Z_2\right) \\ &\quad + \frac{n}{2}\operatorname{tr}\left(H^{-1}Z_1H^{-1}\left(dZ_2 - \frac{1}{2}\left(dHH^{-1}Z_2 + Z_2H^{-1}dH\right)\right)\right) = \\ &= \frac{n}{2}\operatorname{tr}\left(H^{-1}dZ_1H^{-1}Z_2\right) + \frac{n}{2}\operatorname{tr}\left(H^{-1}Z_1H^{-1}dZ_2\right) \\ &\quad - \frac{n}{2}\operatorname{tr}\left(H^{-1}Z_1H^{-1}dHH^{-1}Z_2\right) - \frac{n}{2}\operatorname{tr}\left(H^{-1}dHH^{-1}Z_1H^{-1}Z_2\right). \end{aligned}$$

Since the two sides are equal, this proves the real connection is compatible with the metric. Since the Lévi-Civita connection is the unique torsion-free connection compatible with the metric, this concludes the proof. □

This clarifies a computation already appearing in (Donaldson, 1987, Lemma a)).

We can now map this under ξ to obtain the connection

$$\xi_2\left(\nabla^{\mathcal{D}}_{Z_1} Z_2\right) = -\frac{1}{2}\left(H^{-1}dZ_{2\lrcorner Z_1} - \frac{1}{2}\left(H^{-1}dH_{\lrcorner Z_1}H^{-1}Z_2 + H^{-1}Z_2H^{-1}dH_{\lrcorner Z_1}\right)\right). \tag{4.5.3}$$

We will now pull-back the Levi-Civita connection $\nabla^{\mathcal{D}}$ under $\widetilde{H}$ to obtain the term $\nabla^{\mathcal{D}}_{\widetilde{H}_* Y_1}(\widetilde{H}_* Y_2)$. Recalling that the derivative of the harmonic map in $T\mathcal{D}$ is $d\widetilde{H}$ acting on vectors $Y \in T\widetilde{X}$ as $d\widetilde{H}_{\lrcorner Y}$ (cf. Equation 4.3.1) we just need to substitute $Z_j = d\widetilde{H}_{\lrcorner Y_j}$ in the expression. We have to be careful about the notation here, since in the expression of the connection $\xi_2 \circ \nabla^{\mathcal{D}}$ the d represents the differential in $\mathcal{D}$, i.e. for the coordinates a of $\mathcal{D}$, and in the derivative of the harmonic metric $d\widetilde{H}_{\lrcorner Y}$ it represents the the operator d in $\widetilde{X}$, i.e. with respect to z coordinates.

If we write d^a for the first case, the chain rule reads $d^a f_{d\widetilde{H}_{\lrcorner Y}} = d(f \circ \widetilde{H})_{\lrcorner Y}$, which can be seen in a coordinate computation. Thus we obtain

$$\xi_2\left(\nabla^{\mathcal{D}}_{\widetilde{H}_* Y_1}(\widetilde{H}_* Y_2)\right) = -\frac{1}{2}\widetilde{H}^{-1} d\left(d\widetilde{H}_{\lrcorner Y_2}\right)_{\lrcorner Y_1} + \frac{1}{4}\left(\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_1}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_2} + \widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_2}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_1}\right).$$

To simplify the first term we note the identity

$$d\left(H^{-1} d\widetilde{H}_{\lrcorner Y_2}\right)_{\lrcorner Y_1} = -\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_1} \widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_2} + \widetilde{H}^{-1} d\left(d\widetilde{H}_{\lrcorner Y_2}\right)_{\lrcorner Y_1}$$

which implies

$$-\frac{1}{2}\widetilde{H}^{-1} d\left(d\widetilde{H}_{\lrcorner Y_2}\right)_{\lrcorner Y_1} = -\frac{1}{2} d\left(H^{-1} d\widetilde{H}_{\lrcorner Y_2}\right)_{\lrcorner Y_1} - \frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_1}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_2}.$$

Noting that by Lemma 4.3.1, the derivative $-\frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}$ corresponds to the real Higgs field $\widetilde{\Psi} = \psi dz$ we write $-\frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_j} = \psi_j$. So the identity reads

$$-\frac{1}{2}\widetilde{H}^{-1} d\left(d\widetilde{H}_{\lrcorner Y_2}\right)_{\lrcorner Y_1} = d\psi_{2\lrcorner Y_1} - 2\psi_1\psi_2.$$

Substituting back in the equation for connection we get

$$\xi_2\left(\nabla^{\mathcal{D}}_{\widetilde{H}_* Y_1}(\widetilde{H}_* Y_2)\right) = d\psi_{2\lrcorner Y_1} - 2\psi_1\psi_2 + \psi_1\psi_2 + \psi_2\psi_1 = d\psi_{2\lrcorner Y_1} - \psi_1\psi_2 + \psi_2\psi_1.$$

Going back to Table 4.1 we note that the Chern connection A for the harmonic metric gets identified with $d - \frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}$ which means $d_{\lrcorner Y_1} - \psi_1 = d_{\lrcorner Y_1} + \frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner Y_1} = A_{\lrcorner Y_1}$. Thus the connection is just identified with the action of A on the endomorphism ψ_2, i.e. $A_{\lrcorner Y_1} \cdot (\psi_2)$. We conclude that the pull-back of the Levi-Civita gets identified with the connection A acting on endomorphisms.

Lemma 4.5.2. *The pull-back of the Levi-Civita $\nabla^{\mathcal{D}}$ with respect to the harmonic map $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ is identified with the Chern connection A of $\overline{\partial}_E$ and H, acting on* $\mathrm{End}(E)$.

For the second term $\xi_2(\widetilde{H}_* \nabla^X_{Y_1} Y_2)$ in the second fundamental form $\mathrm{II}(Y_1, Y_2)$ we cannot simplify it much without explicitly writing the Levi-Civita ∇^X. So, using the fact that $-\frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}$ identifies with the real Higgs field (Lemma 4.3.1), we have $\xi_2(\widetilde{H}_* \nabla^X_{Y_1} Y_2) = -\frac{1}{2}\widetilde{H}^{-1} d\widetilde{H}_{\lrcorner \nabla^X_{Y_1} Y_2} = \Psi_{\lrcorner \nabla^X_{Y_1} Y_2}$. The conclusion is the invariant description of the second fundamental form.

Proposition 4.5.3. *The complex second fundamental form of the harmonic map $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ is identified with the* $\mathrm{End}_0(E)$*-valued* 2*-tensor*

$$\mathrm{II}(Y_1, Y_2) = (A_{\lrcorner Y_2} \cdot (\Psi_{\lrcorner Y_1}))^{\perp} = A_{\lrcorner Y_2} \cdot (\Psi_{\lrcorner Y_1}) - \Psi_{\lrcorner \nabla^X_{Y_1} Y_2}, \tag{4.5.4}$$

where A is the Chern connection of the Higgs bundle with real Higgs field Ψ and ∇^X is the Levi-Civita connection of the conformal metric g_0 on X.

4.6 The harmonic equation and the second fundamental form

We can now relate the geometry we have described with the classical theory of harmonic maps. In general, a map $f : X \to M$ from a Riemann surface X to a Riemannian manifold M is called *harmonic* if its tension field is zero. The *tension field* is defined as

$$\tau(f) = 4\nabla_{\frac{\partial}{\partial \bar{z}}} \left(\frac{\partial f}{\partial z}\right), \tag{4.6.1}$$

where ∇ is the pull-back of the Levi-Civita connection of M and $\frac{\partial f}{\partial z}$ is the section $\frac{\partial f}{\partial z} = f_* \left(\frac{\partial}{\partial z}\right)$ of the pullback bundle f^*TM. It is well known that this definition is equivalent to the usual definition of a Harmonic map as a critical point of an energy functional, or as one for which the trace of the second fundamental form vanishes. For the expression presented here see for example (Jost, 2017, Equation 10.1.13.). Under the corresponding representations we have set up, for $f = \widetilde{H}$ the pull back connection ∇ is identified with the connection A on $\mathrm{End}_0(E)$ and the section $f_* \left(\frac{\partial}{\partial z}\right)$ with the image of $\frac{\partial}{\partial z}$ under the derivative of the harmonic map, i.e with $\Psi_{\lrcorner\frac{\partial}{\partial z}} = \Phi_{\lrcorner\frac{\partial}{\partial z}}$. Thus the tension field is represented by

$$\tau(\widetilde{H}) = A_{\lrcorner\frac{\partial}{\partial \bar{z}}} \cdot \left(\Psi_{\lrcorner\frac{\partial}{\partial z}}\right) = \overline{\partial}_{E_{\lrcorner\frac{\partial}{\partial \bar{z}}}} \left(\Phi_{\lrcorner\frac{\partial}{\partial z}}\right), \tag{4.6.2}$$

and it vanishes if and only if the Higgs field is holomorphic. Thus we conclude that, in this setup, the holomorphicity of the Higgs field is equivalent to the harmonicity of $\widetilde{H}$ in the classical sense. Note that this is the only restriction needed, for Hitchin's equation is automatically satisfied, since it is equivalent to the flatness of the connection $\nabla = A + \Phi + \Phi^*$ which was already used to build the harmonic map $\widetilde{H}$ (under the guise of the isomorphism of E with E_ρ). This is also the trace of the second fundamental form

$$\operatorname{tr}\mathrm{II} = \mathrm{II}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) + \mathrm{II}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) = 4\,\mathrm{II}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = \overline{\partial}_{E_{\lrcorner\frac{\partial}{\partial \bar{z}}}} \left(\Phi_{\lrcorner\frac{\partial}{\partial z}}\right),$$

since the mixed Christoffel symbols $\nabla^X_{\frac{\partial}{\partial \bar{z}}} \frac{\partial}{\partial z}$ of the complexification of the Levi-Civita connection on X are zero (Ballmann, 2006, Equation 4.36). Note that this means the only non-zero term in the second fundamental form is $\mathrm{II}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \overline{\mathrm{II}\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial \bar{z}}\right)}$. This is the *traceless second fundamental form*. It also means that for the harmonic map, II is zero if and only if the traceless second fundamental form is. A map for which the second fundamental form vanishes is called *totally geodesic*. This is because it sends (parameterized) geodesics in the domain to (parameterized) geodesics in the target ambient space (Xin, 1996, after Definition 1.2.1). Noting that the Levi-Civita connection is $\nabla^X_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \partial_z \log g_0 \frac{\partial}{\partial z}$, the conclusion is that

Proposition 4.6.1. *The harmonic map $\widetilde{H} : \widetilde{X} \to \mathcal{D}$ for the Higgs field $\Phi = \phi\, dz$ is totally geodesic if and only if its traceless second fundamental form vanishes, i.e.,*

$$\mathrm{II}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = A_{\lrcorner\frac{\partial}{\partial z}} \cdot \left(\Phi_{\lrcorner\frac{\partial}{\partial z}}\right) - \Psi_{\lrcorner\partial_z \log g_0 \frac{\partial}{\partial z}} = A^{(1,0)}_{\lrcorner\frac{\partial}{\partial z}} \cdot (\phi) - \partial_z \log g_0 \phi = 0. \tag{4.6.3}$$

Remark 4.6.2. Recall that the $(1,0)$-part of the connection $A^{(1,0)}$ acts on ϕ with the commutator as usual.

4.7 Examples with $\mathrm{SL}(2,\mathbb{R})$ representations

4.7.1 Hitchin component and uniformization

For a quick application let us consider the Hitchin component in $\mathrm{SL}(2,\mathbb{C})$ as in Example 3.1.3. So the Higgs bundle is of the form

$$\left(E = K^{1/2} \oplus K^{-1/2}, \Phi = \left(\begin{smallmatrix} 0 & q \\ 1 & 0 \end{smallmatrix}\right)\right),$$

for a choice $K^{1/2}$ of square root of the canonical bundle K and $q \in \mathrm{H}^0(K^2)$ a quadratic differential.

Proposition 4.7.1. *The harmonic map for Higgs fields in the* $\mathrm{SL}(2,\mathbb{C})$*-Hitchin component is an immersion.*

Proof. This Higgs field is nowhere zero (because of the 1 in the lower left corner), and so by Proposition 4.3.2, the Harmonic map has non-zero rank. The harmonic metric is known to diagonalize and so the adjoint Higgs field is $\Phi^* = \left(\begin{smallmatrix} 0 & h^{-2} \\ \overline{q}h^2 & 0 \end{smallmatrix}\right)$. Note that the endomorphism part of the fields cannot be proportional. Indeed, the constant of proportionality $k(z)$ would have to satisfy

$$\begin{aligned} q(z) &= k(z)h^{-2}(z) \\ k(z)\overline{q}h^2 &= 1. \end{aligned}$$

By multiplying both equalities we would have $q\overline{q}h^2 = h^{-2}$, which is impossible, since $||q||^2 = q\overline{q}h^4 < 1$, by the maximum principle, as shown by Hitchin in (Hitchin, 1987, Theorem 11.2). (Or verify that $||q||^2$ is the norm of the Beltrami differential in Theorem 3.3.1). This means that, by Proposition4.3.2, the harmonic map cannot have rank one. $\square$

We can further calculate that the connection A is as before. Expressing the connection in terms of the metric $g = h^{-2}$ in TX as in Equation 3.1.1 we have $d + \left(\begin{smallmatrix} -\frac{1}{2}\partial \log g & 0 \\ 0 & \frac{1}{2}\partial \log g \end{smallmatrix}\right)$. Writing $\Phi = \phi\, dz$ this means that the traceless second fundamental form is

$$\begin{aligned} \mathrm{II}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) &= \partial_z\phi + \begin{pmatrix} -\frac{1}{2}\partial_z \log g & 0 \\ 0 & \frac{1}{2}\partial_z \log g \end{pmatrix}\phi - \phi\begin{pmatrix} -\frac{1}{2}\partial_z \log g & 0 \\ 0 & \frac{1}{2}\partial_z \log g \end{pmatrix} - \partial_z \log(g_0)\phi \\ &= \begin{pmatrix} 0 & \partial_z q(z) - q(z)\partial_z \log g(z) - q(z)\partial_z \log g_0(z) \\ \partial_z \log g(z) - \partial_z \log g_0(z) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \partial_z q(z) - q(z)\partial_z \log\left(g(z)g_0(z)\right) \\ \partial_z \log \frac{g(z)}{g_0(z)} & 0 \end{pmatrix}. \end{aligned}$$

The conclusion is that the harmonic map is totally geodesic for $q = 0$, since the solution metric g is simply equal to g_0, as the Hitchin equations reduce to the negative curvature equations. If we see $\mathcal{D}$ as the hyperbolic space $\mathbb{H}^3$, the harmonic map is a totally geodesic embedding onto $\mathbb{H}^2 \subset \mathbb{H}^3$. Of course, this is expected since we are dealing with the uniformizing Higgs bundle. The pull-back metric also coincides with g_0, (up to factor) since $\mathrm{tr}(\phi^2) = 2q = 0$. An interpretation of the proof of the uniformization theorem from the point of view of the harmonic map could then be stated as: «A Riemann surface X is uniformizable by a Fuchsian representation ρ because the unique ρ-equivariant harmonic map to the homogeneous space $\mathcal{D} \cong \mathbb{H}^3$ is a conformal and totally-geodesic embedding.»

4.7.2 General case and the Beltrami differential in the construction

It is instructive to repeat the previous calculations for more general $\mathrm{SL}(2,\mathbb{R})$-Higgs bundles. In this case, we let $(E = L \oplus L^{-1}, \Phi = \left(\begin{smallmatrix} 0 & \alpha \\ \beta & 0 \end{smallmatrix}\right))$ be a polystable $\mathrm{SL}(2,\mathbb{R})$-Higgs bundle with $0 \leq \deg(L) \leq g-1$. Note that the polystable case with $\beta = 0$ can only happen when $\deg(L) = 0$, and Proposition 3.4.1 implies that $\alpha = 0$ in that case. Thus the Higgs field is zero everywhere. By Proposition 4.3.2, the harmonic map has rank zero, i.e., it is constant, mapping $\widetilde{X}$ to a single point. So we assume $\beta \neq 0$. Now to study the injectivity in general observe that the rank is zero at the simultaneous zeros of α and β. In particular when the condition $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$ is imposed, as in Chapter 3, it implies that the derivative of the harmonic map is zero precisely at the branching divisor $\mathrm{div}(\beta)$. Now for the locus of points of rank 1. These points are described in Lemma 4.3.3 as the ones for which

$$\mathrm{tr}(\phi\phi^*)^2 - \left|\mathrm{tr}(\phi^2)\right|^2 = 0,$$

but α and β are not simultaneously zero. So we calculate

$$\phi = \begin{pmatrix} 0 & \alpha(z) \\ \beta(z) & 0 \end{pmatrix} \qquad \phi^2 = \begin{pmatrix} \alpha(z)\beta(z) & 0 \\ 0 & \alpha(z)\beta(z) \end{pmatrix} \qquad a = \mathrm{tr}(\phi^2) = 2\alpha(z)\beta(z).$$

And also

$$\phi^* = \begin{pmatrix} 0 & \overline{\beta}(z)h^{-2}(z) \\ \overline{\alpha}(z)h^2(z) & 0 \end{pmatrix} \qquad \phi\phi^* = \begin{pmatrix} |\beta|^2(z)h^{-2}(z) & 0 \\ 0 & |\alpha|^2(z)h^2(z) \end{pmatrix},$$

from where $b = \mathrm{tr}(\phi\phi^*) = |\beta|^2(z)h^{-2}(z) + |\alpha|^2(z)h^2(z)$. Thus the degeneracy locus is the set of points for which

$$b^2 - |a|^2 = \left(|\beta|^2h^{-2} + |\alpha|^2h^2\right)^2 - 4\,|\alpha\beta|^2 = 0 \Longleftrightarrow \tag{4.7.1}$$

$$\Longleftrightarrow \left(|\beta|^2h^{-2}\right)^2 + \left(|\alpha|^2h^2\right)^2 + 2\,|\alpha\beta|^2 - 4\,|\alpha\beta|^2 = 0 \tag{4.7.2}$$

$$\Longleftrightarrow \left(|\beta|^2h^{-2} - |\alpha|^2h^2\right)^2 = 0 \tag{4.7.3}$$

$$\Longleftrightarrow |\beta|^2h^{-2}(z) - |\alpha|^2(z)h^2(z) = 0. \tag{4.7.4}$$

Under the condition $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$ and $\deg(L) > 0$ it follows from the maximum principle in Lemma 3.3.3 that $|\alpha|^2(z)h^2(z) < |\beta|^2(z)h^{-2}(z)$ (just take $\hbar = 1 = R$ there). Thus, with this condition, the equality can only be satisfied at the zeros of β, where both α and β vanish. The conclusion is that under these hypotheses the harmonic map has rank 2 except at the zeros of β where it has rank 0. For the case of $\deg(L) = 0$, the condition $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$ translates to $\alpha = k\beta$. We have already seen that in this case the metric $|k|^{-1/2}$ is a solution of Hitchin's equation, and thus the equality holds everywhere (cf. the discussion before Theorem 3.4.4), i.e. the map has rank 1 everywhere except at the zeros of β where it has zero derivative. Let us now study the induced metric. Because of the degeneracy, let us assume we are in the case where $\deg(L) > 0$, the condition $\mathrm{div}(\alpha) \geq \mathrm{div}(\beta)$, and so the harmonic map is a branched immersion. The induced metric is

$$g = 4\left(2\alpha\beta\, dz^2 + |\beta|^2 h^{-2} + |\alpha|^2 h^2\left(dz \otimes d\bar{z} + d\bar{z} \otimes dz\right) + 2\overline{\alpha\beta}\, d\bar{z}^2\right). \tag{4.7.5}$$

If we apply Proposition 4.4.3 we discover that the Beltrami differential that gives conformal coordinates in this case is precisely

$$\mu = \frac{\overline{a}}{b + \sqrt{b^2 - |a|^2}} = \frac{2\overline{\alpha\beta}}{|\beta|^2 h^{-2} + |\alpha|^2 h^2 + \sqrt{b^2 - |a|^2}}.$$

Noting that we have already calculated $(b^2 - |a|^2)$ which is the condition that determines the branching locus, Equation 4.7.4. We have $\sqrt{b^2 - |a|^2} = |\beta|^2(z)h^{-2}(z) - |\alpha|^2(z)h^2(z)$ and thus

$$\mu = \frac{2\overline{\alpha\beta}}{|\beta|^2 h^{-2} + |\alpha|^2 h^2 + |\beta|^2 h^{-2} - |\alpha|^2 h^2} = \frac{\overline{\alpha}}{\beta} h^2. \tag{4.7.6}$$

This is exactly the same Beltrami differential that appears in the main construction in Chapter 3, for $\hbar = 1 = R$. This provides a more geometric explanation for the introduction of such a Beltrami differential: It is the Beltrami differential that renders conformal the induced metric from the harmonic immersion. We can use Proposition 4.4.5 to calculate the curvature of the metric. When we carry out this computation we obtain a metric with constant negative curvature, and this provides another way to check that these representations correspond to branched hyperbolic structures as in (Biswas et al., 2021) and in Section 3.5.3.

4.8 Elementary Higgs bundles

We include here a short note on elementary Higgs bundles. Let X be a Riemann surface of genus $g \geq 2$. We describe the $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles on X associated with elementary representations $\rho : \pi_1(X) \to \mathrm{SL}(2,\mathbb{C})$ which are reductive. We make use of Higgs bundles for non-connected groups, for which we don't provide any details and refer to (Barajas et al., 2023). So we take (∂_E, Φ) a polystable Higgs bundle.

Type I

If (∂_E, Φ) is a Higgs bundle associated with an elementary representation of type I, i.e. a unitary representation, then it is of the form $(E, 0)$. This is a consequence of the Narasimhan–Seshadri theorem.

Type II

If (E, Φ) is a Higgs bundle associated to a reductive elementary representation ρ of type II, its image $\rho(\pi_1(X))$ lies in the subgroup $\mathbb{C}^* \hookrightarrow \mathrm{SL}(2, \mathbb{C})$ $\lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}$. This is because type II representations are Euclidean and the reductivity condition excludes translations. As so, the bundle E must split as a direct sum $E = L \oplus L^{-1}$, where L is a holomorphic line bundle, and the Higgs field must be diagonal $\Phi = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, with $\alpha \in \Omega^{1,0}(X)$. The fact ρ is reductive guarantees that (E, Φ) is polystable. As (L, α) is a Higgs bundle, we see that (∂_E, Φ) decomposes as a direct sum of Higgs bundles. Since E has degree zero, it follows from polystability that (L, α) should have $\deg(L) = 0$.

Type III

If (E, Φ) is a Higgs bundle associated to an elementary representation ρ of type III, its image $\rho(\pi_1(X))$ lies in the subgroup G of $\mathrm{SL}(2, \mathbb{C})$ generated by Λ and R with

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix}, \lambda \in \mathbb{C}^* \quad \text{and} \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.8.1}$$

This G is a non-connected Lie group whose identity component is $G_0 \cong \mathbb{C}^* \hookrightarrow \mathrm{SL}(2, \mathbb{C})$ embedded via $\lambda \mapsto \Lambda$. Since it has 2 connected components (G_0 and the one where R lies) it follows that G is the extension

$$1 \to G_0 \cong \mathbb{C}^* \to G \to \pi_0(G) \cong \mathbb{Z}_2 \to 1$$

which does not split. Consider the (non-homomorphic) section $s : \pi_0(G) \cong \mathbb{Z}_2 \to G$ given by $[1] \mapsto 1$ and $[R] \mapsto R$. Define the homomorphism

$$\begin{aligned} \theta : \pi_0(G) &\to \mathrm{Aut}(G_0) \\ \gamma &\mapsto \theta_\gamma, \qquad \text{with } \theta_\gamma(g) = s(\gamma) g s(\gamma)^{-1}, \end{aligned}$$

and the cocycle

$$c : \pi_0(G) \times \pi_0(G) \to G_0$$

by the condition

$$s(\gamma_1)s(\gamma_2) = c(\gamma_1, \gamma_2)s(\gamma_1\gamma_2).$$

Then one can verify that G is isomorphic to $G_0 \times_{(\theta,c)} \pi_0(G)$, the product group with twisted multiplication:

$$(g_1, \gamma_1) \cdot (g_2, \gamma_2) = (g_1\theta_{\gamma_1}(g_2)c(\gamma_1, \gamma_2), \gamma_1\gamma_2).$$

The isomorphism is given by $\mathbb{C}^* \times_{(\theta,c)} \mathbb{Z}_2 \cong G_0 \times_{(\theta,c)} \pi_0(G) \to G$, with $(g, \gamma) \mapsto gs(\gamma)$. Let now E be any rank 2 vector bundle over X with structure group G, and denote by P its associated principal bundle. The quotient $Y := P/G$ is then a $\pi_0(G)$ principal bundle. Since $\pi_0(G) \cong \mathbb{Z}_2$ is discrete, Y is a covering of X, whose covering group is precisely $\mathbb{Z}_2$. Denote by $\pi : Y \to X$ the 2-to-1 quotient map, which is invariant under $\mathbb{Z}_2$. One can show that the bundle π^*E has structure group G_0. Using the decomposition $\mathbb{C}^* \times_{(\theta,c)} \mathbb{Z}_2 \cong G_0 \times_{(\theta,c)} \pi_0(G) \to G$, one can further show that π^* establishes a one-to-one correspondence between sections of E over X and $\mathbb{Z}_2$-equivariant sections of π^*E over Y. This correspondence extends to Higgs bundles, and we refer to (Barajas et al., 2023) for further details.

Note that Y is not connected if and only if P admits a reduction of structure group to G_0. In this situation we are back at the previous case of a reductive elementary representation of type II. Otherwise, Y is connected.

Applying this correspondence we conclude that Higgs bundles of elementary representations of type III are of the form (E, Φ) such that there exists a 2-fold cover $\pi : Y \to X$ with $(\pi^*E, \pi^*\Phi) = (L \oplus L^{-1}, \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix})$, where L is a line bundle and $\alpha \in \Omega^{1,0}(Y)$ is $\mathbb{Z}_2$-equivariant. Note that polystability is preserved by the correspondence since solutions of Hitchin's equations for (E, Φ) give rise to solutions for $(\pi^*E, \pi^*\Phi)$. And so L has degree zero.

4.8.1 Stability

Proposition 4.8.1. *If (E, Φ) is a polystable Higgs bundle corresponding to a reductive representation of elementary type III which is not of type II. Then the underlying bundle E is stable.*

Proof. Suppose (E, Φ) is elementary of type III. Then there exists a double cover $\pi : Y \to X$ such that

$$(\pi^*E, \pi^*\Phi) = (L \oplus L^{-1}, \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}), \deg(L) = 0 \text{ and } \alpha \in \Omega^{1,0}(Y).$$

Suppose that U is a line subbundle of E. We need to show that $\deg(U) < 0$. Assume by contradiction that $\deg(U) > 0$ then $\deg(\pi^*U) = 2\deg(U) > 0$. This means $\pi^*U \subset \pi^*E$ is a destabilizing subbundle which cannot exist since $\pi^*E = L \oplus L^{-1}$ is in particular semistable as a bundle. Assume now that $\deg(U) = 0$ then $\deg(\pi^*U) = 2\deg(U) = 0$. By the uniqueness of the decomposition of $\pi^*E = L \oplus L^{-1}$ we conclude that $U \cong L$ or $U \cong L^{-1}$. In either case

$\pi^*E \cong \pi^*U \oplus \pi^*U^{-1}$, from where it follows that $E = U \oplus U^{-1}$. This contradicts the hypothesis of E being not of type II. $\square$

4.9 The harmonic map for elementary representations

We include the proof of how a restriction on the rank of the harmonic map implies the elementarity of the representation. Namely, we show that if the ρ-equivariant harmonic map has rank less than one everywhere, the representation is elementary. To prove this result we need an equivalent characterization of elementary subgroups of Möbius transformations. Recall that the action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{CP}^1$, seen as $\mathbb{C} \cup \{\infty\} = \partial\mathbb{H}^3$, can be extended to $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. Here $\mathbb{CP}^1$ is identified with the xy-plane in $\mathbb{R}^3 \cup \{\infty\}$ and $\mathbb{H}^3$ with the upper half-space. This extension is called the *Poincaré extension* (Beardon, 1983, cf. Equation 4.1.4). When we are using the ball model of hyperbolic space, $\mathbb{CP}^1$ is identified with its boundary sphere, and the action is an extension to the closed ball. See, for example, (Beardon, 1983, §5.1) or (Ratcliffe, 2019, §5.5) for the proof of the following proposition.

Proposition 4.9.1. *A group G of Möbius transformations is elementary if and only if it has a finite orbit in $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. Equivalently, G is of type*

(I) if it fixes a point inside $\mathbb{H}^3$;

(II) if it fixes a point in $\mathbb{CP}^1 = \partial\mathbb{H}^3$;

(III) if it preserves a hyperbolic geodesic in $\mathbb{H}^3$.

After this characterization, we also need further properties of Harmonic maps. In particular, we need the following classical result.

Theorem 4.9.2. *(Sampson, 1978, Theorem 3) Let $f : M \to N$ be a harmonic map between connected Riemannian manifolds. If f has rank 1 on an open set then it maps M to a geodesic arc in N and it has rank 1 on an open dense set.*

Combining these can conclude the following.

Proposition 4.9.3. *Let $(\overline{\partial}_E, \Phi)$ be a polystable $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle and consider the associated harmonic map $\widetilde{H} : \widetilde{X} \to \mathcal{D} \cong \mathbb{H}^3$. If $\widetilde{H}$ has non-maximal rank everywhere then the representation associated with $(\overline{\partial}_E, \Phi)$ is elementary (and reductive).*

Proof. The set of points where the harmonic map has rank zero R^0 matches the zero set of the Higgs field by Proposition 4.3.2. This means it is either a discrete set or all of $\widetilde{X}$ since the Higgs field is holomorphic. In this second case, the map is a constant point in $\mathcal{D} \cong \mathbb{H}^3$. And this is the only possibility for rank 0 everywhere. Suppose now the rank is everywhere less than one, but not everywhere zero. This means that it is one on an open set. By Theorem 4.9.2 then $\widetilde{H}$ maps $\widetilde{X}$ to a geodesic. Since the image of the harmonic map is preserved by the representation, this means that the representation fixes either a point in $\mathbb{H}^3$ or a geodesic, and by Theorem 4.9.1 it is an elementary representation. $\square$

4.9.1 Representations associated to the construction with $\deg(L) = 0$

Using the results of Section 3.4.2 and the proposition we can give further information about the kind of representations that come in the construction of branched projective structures for the $\deg(L) = 0$ case. The condition $\operatorname{div}\alpha \geq \operatorname{div}\beta$ in this case implies $\alpha = k\beta$ and the Higgs bundle is thus $E = L \oplus L^{-1}$ with Higgs field

$$\Phi = \begin{pmatrix} 0 & k\beta \\ \beta & 0 \end{pmatrix}.$$

We have seen already, just before Theorem 3.4.4, that $|k|^{-1/2}$ is a solution of the Hitchin equation. The degeneracy locus is given by Equation 4.7.4 which is identically satisfied. Thus the rank of the harmonic map $\widetilde{H}$ is one everywhere except at the zeros of β where it is zero. By Proposition 4.9.3, these correspond to elementary representations. In fact, elementary of type (III), except when $\beta = 0$, for they preserve the image of the harmonic map which is a geodesic.

Appendix A

Some technical results and calculations

A.1 Chern connection

We include the expression of the Chern connection in a general (non-holomorphic) frame, as it seems rather delicate to find it in the literature.

Lemma A.1.1. *Let (E, ∂_E) be a holomorphic vector bundle over a Riemann surface X. Let H be a Hermitian metric on E. Take a local frame for E (not necessarily holomorphic). Denote still by H the matrix representation of the metric and the one of the $\overline{\partial}$-operator as $\overline{\partial}_E = \overline{\partial} + C$, where C is a matrix of $(0,1)$-forms. Then the Chern connection A for $\overline{\partial}_E$ and H has matrix representation*

$$A_H = d + H^{-1}\partial H + C - C^{*_H} = H^{-1}\partial H + C - H^{-1}\overline{C}^T H.$$

Proof. We need to show that A_H is compatible with both H and $\overline{\partial}_E$. For the compatibility with $\overline{\partial}_E$, just take the $(0,1)$-part of A_H and observe it is $\overline{\partial} + C$ which is exactly the representation of $\overline{\partial}_E$. For the compatibility with the metric, note that A_H is compatible with H if and only if $d(H(s,t)) = H(A_H s, t) + H(s, A_H t)$, for local sections s, t. In matrix form, i.e, taking $s, t = e_j$, for e_j in the frame, the equality reads

$$dH = \overline{A}^T H + HA,$$

where $A_H = d + A$. Substituting back the expression for A we get

$$dH = \overline{\partial} H + \overline{C}^T H - HC + \partial H + HC - \overline{C}^T H,$$

because $H = \overline{H}^T$ is Hermitian. This identity is always satisfied, and so A_H is the Chern connection. □

A.2 Parameters for reproduction of figures

The figures in Chapter 1 of domains of discontinuity were reproduced with the algorithms in (Mumford et al., 2002). We list the parameters used here.

A.2.1 Schottky groups

For the Schottky groups, let us use the following notation for circles in the complex plane $\mathbb{C}$: we denote a circle C as a pair $C = (P, r)$ where $P \in \mathbb{C}$ is its center and $r \in \mathbb{R}^+$ is its radius. We start by noting that a Möbius transformation that maps the interior of the circle $C = (P, r)$ to the exterior of circle $C' = (Q, s)$ is given by (Mumford et al., 2002, pag. 90)

$$z \mapsto s\frac{\overline{u}(z-P) + r\overline{u}}{u(z-P) + rv} + Q$$

where u and v are parameters such that $|u|^2 - |v|^2 = 1$. These parameters correspond to the freedom inside $\mathrm{PSL}(2, \mathbb{C})$ of choosing a Möbius transformation performing such a pairing of C and C'. In all constructions, we have used the same fixed parameters u, v for all pairs of circles appearing in the group. So a Schottky group in our images is determined by a pair of circles for each generator, and two global parameters u and v. This is the information we reproduce here. Each pair of circles, together with the parameters, is to be replaced in the expression to obtain the Möbius transformation corresponding to the generator. The values used for the production of Figure 1.2.11 counting from the left are as follows.

First picture

$$C_1 = (-0.80 + 0.80i, 1.99) \qquad C_1' = (0.30 - 1.3i, 0.37)$$
$$C_2 = (0.95 - 0.60i, 0.25) \qquad C_2' = (1.20 - 1.35i, 0.528)$$
$$u = -1 - 1i \qquad v = 1$$

Second picture

$$C_1 = (1.00 + i, 0.99) \qquad C_1' = (1.00 - i, 0.99)$$
$$C_2 = (-0.20, 0.19) \qquad C_2' = (-1 + i, 0.99)$$
$$C_3 = (-1.00 - i, 0.99) \qquad C_3' = (0.20, 0.19)$$
$$u = 2 \qquad v = \sqrt{3}$$

Third picture

$$C_1 = (1.00 + i, 0.99) \qquad C_1' = (1.00 - i, 0.99)$$
$$C_2 = (-1.00 - i, 0.99) \qquad C_2' = (-1.00 + i, 0.99)$$
$$u = 2 \qquad v = \sqrt{3}$$

A.2.2 Quasi-Fuchsian group

For the quasi-Fuchsian group we used the recipe "Grandma's special parabolic commutator groups" on Box 21 of (Mumford et al., 2002, pag. 229). This is a family of two generators depending on parameters t_a and t_b which, for the picture in Figure 1.2, were

$$t_a = 1.91 + 0.05i \qquad t_b = 1.91 + 0.05i.$$

We search the limit set up until the threshold $\varepsilon = 0.005$.

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