Markov Chains

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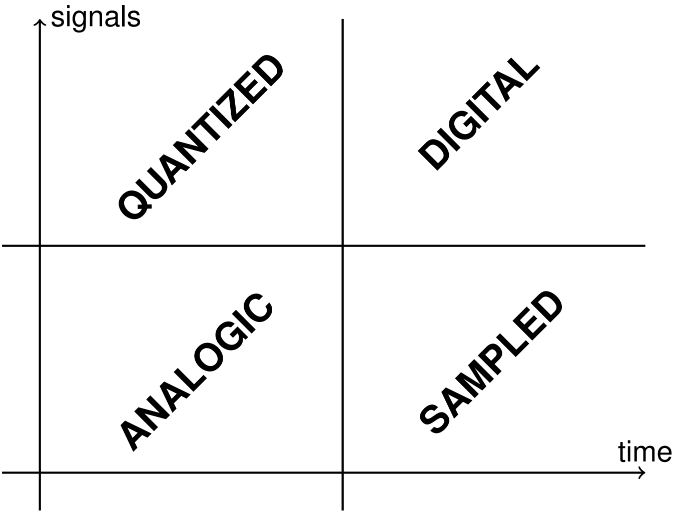
Estimation and Decision Systems, 2023

DEEC, FEUP

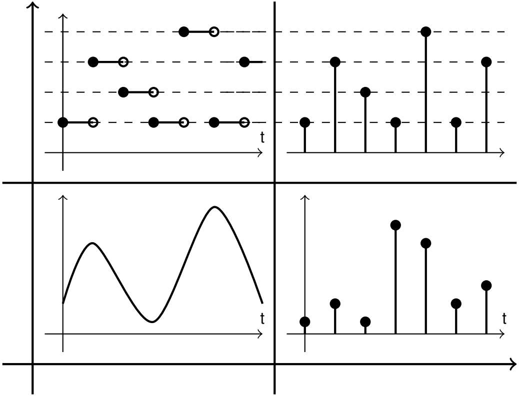
Outline

Signals



Signals

	Continuous time	Discrete time
Discrete Amp.	Quantized	**Digital**
Continuous Amp.	Analogic	Sampled



Systems

Deterministic: $y(t) = G(q)u(t)$

Stochastic: $y(t) = H(q)e(t),\ e(t)-$ white noise

Stochastic-deterministic: $y(t) = G(q)u(t) + H(q)e(t)$

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SISO: $u(t) \in \mathbb{R}\ , y(t) \in \mathbb{R}$

MIMO: $u(t) \in \mathbb{R}^{n_u},\ y(t) \in \mathbb{R}^{n_y}$

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Autonomous: $x_{k+1} = Ax_k$

Non-autonomous: $x_{k+1} = Ax_k + Bu_k$

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Autonomous: $x_{k+1} = Ax_k$

Linear: $x_{k+1} = Ax_k + Bu_k + Ke_k$

Non-linear: $x_{k+1} = f(x_k, u_k, e_k)$

Systems

Time invariant: $x_{k+1} = Ax_k + Bu_k.$

Time variant: $x_{k+1} = A(k)x_k + B(k)u_k$

Examples of Systems with quantized states:

- x_k is an integer (number of cars, number of persons in a queue, etc.)
- Robot mode $\{$ wait, search, recharge$\}$.

Discrete-time Markov chain: Definition

White noise sequence $\{x_k\}_k\,,\ k = 0, 1, \ldots$:

- Analysis is straightforward: There is **no memory** because $x_k,\ k = 0, 1, \ldots$ are independent variables.

Many real-life processes cannot be described by white noise processes.

Example:

- $x_k \equiv$stock price of a company: It is reasonable to assume that $x_k,\ k = 1, \ldots$ aren't statistically independent.

Discrete-time Markov chain: Definition

DISCRETE-TIME MARKOV PROCESS

A discrete-time Markov process is a stochastic process $\{x_k\}_k\,,\ k = 0, 1, \ldots,$ where

$$\text{Prob}\left(x_{k+1}|x_k, x_{k-1}, \ldots, x_0\right) = \text{Prob}\left(x_{k+1}|x_k\right)$$

with Prob$(a|b)$ denoting probability distribution of a given b.

In a Markov process the state is the **system memory**

DISCRETE-TIME MARKOV CHAIN

A Markov chain is a discrete-time Markov process whose possible values of the state (state-space) is a countable set.

Transition probability

Stationary discrete-time Markov chain:

$$\text{Prob}\left(x_{k+n+1} = j | x_{k+n} = i\right)_{\forall n \in \mathbb{N}} = \text{Prob}\left(x_{k+1} = j | x_k = i\right) = p_{ij}(k) = p_{ij}$$

- The distribution of x_k is the same for all k.

p_{ij} - **Transition probability**:

a) $p_{ij} \geq 0 \quad \forall i, j \in \mathbb{X}$

b) $\sum_{j \in \mathbb{X}} p_{ij} = 1 \quad \forall i \in \mathbb{X}.$

State transition matrix

If the state-space is finite, i.e., if there are only n states , then it can be defined the matrix,

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix},$$

denoted as **state transition matrix.**

It is characterized by the following:

- All its elements are positive.
- The sum of elements in any row equals 1.

State transition diagram

A Markov chain is usually shown by a **state transition diagram**.

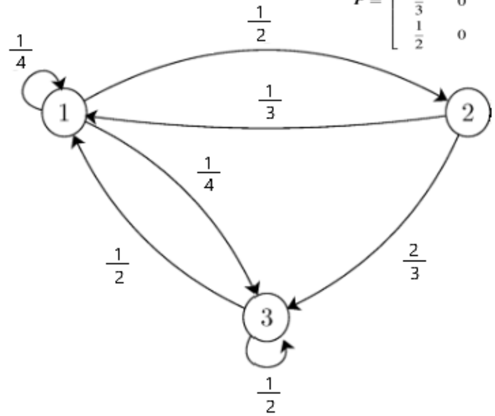
Example: Consider a Markov chain with the state transition matrix:

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

State transition diagram

Then it can be represented by the transition diagram:

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$



Exercise

Consider the Markov chain represented by the previous state transition diagram:

1. Find $P(x_4 = 3|x_3 = 2)$.
2. Find $P(x_3 = 1|x_2 = 1)$.
3. If we know $P(x_0 = 1) = \frac{1}{3}$, find $P(x_0 = 1, x_1 = 2)$.
4. If we know $P(x_0 = 1) = \frac{1}{3}$, find $P(x_0 = 1, x_1 = 1, x_2 = 3)$.

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Exercise

Consider the Markov chain represented by the previous state transition diagram:

1. Find $P(x_4 = 3|x_3 = 2) = p_{23} = \frac{2}{3}$.
2. Find $P(x_3 = 1|x_2 = 1)$.
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State probability distributions

- Markov chain: $\{x_k\}_{k=0,\ldots,\infty}$
- $x_k \in \mathbb{X} = \{1, 2, \ldots, n\}$.
- Define the probability distribution of x_0 as the row vector:

$$\pi^{(0)} = \begin{bmatrix} P(x_0 = 1) & P(x_0 = 2) & \cdots & P(x_0 = n) \end{bmatrix}$$

- What are the distributions of x_k, $k = 1, \ldots, \infty$?
  - $P(x_1 = j) = \sum_{i=1}^{n} P(x_1 = j | x_0 = i) P(x_0 = i) = \sum_{i=1}^{n} p_{ij} P(x_0 = i).$

$$= \begin{bmatrix} P(x_0 = 1) & P(x_0 = 2) & \cdots & P(x_0 = n) \end{bmatrix} \begin{bmatrix} p_{1,j} \\ p_{2,j} \\ \vdots \\ p_{n,j} \end{bmatrix}$$

State probability distributions

Hence:

$$\pi^{(1)} = \left[P(x_0 = 1) \quad P(x_0 = 2) \quad \cdots \quad P(x_0 = n)\right] \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

$$= \pi^{(0)} \boldsymbol{P}$$

- Similarly:

$$\begin{aligned} \pi^{(2)} &= \pi^{(1)} \boldsymbol{P} = \pi^{(0)} \boldsymbol{P}^2 \\ \pi^{(3)} &= \pi^{(2)} \boldsymbol{P} = \pi^{(0)} \boldsymbol{P}^3 \\ &\vdots \\ \pi^{(k+1)} &= \pi^{(k)} \boldsymbol{P} = \pi^{(0)} \boldsymbol{P}^{k+1} \end{aligned}$$

Exercise

Consider a system that can be in one of two possible states, $\mathbb{X} = \{0, 1\}$. Suppose that the transition matrix is given by

$$\boldsymbol{P} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Suppose that the system is in state 0 at time $k = 0$, i.e., $x_0 = 0$

1. Draw the transition diagram
2. Find the probability that the system is in state 1 at time $k = 3$

k-Step transition probabilities

- Markov chain: $\{x_k\}_{k=0,\ldots,\infty}$
- $x_k \in \mathbb{X}$
- $x_0 = i \Rightarrow P(x_1 = j) = p_{ij}$, probability of going from state i to state j in $\boldsymbol{k} = 1$ **step.**
- What is the probability of going from state i to state j in $\boldsymbol{k} = 2$ **steps**, i.e,

$$p_{ij}^{(2)} = P(x_2 = j | x_0 = i)?$$

- **Solution**:

$$p_{ij}^{(2)} = P(x_2 = j | x_0 = i) = \sum_{\ell \in \mathbb{X}} P(x_2 = j | x_1 = \ell, x_0 = i) =$$

$$= \sum_{\ell \in \mathbb{X}} P(x_2 = j | x_1 = \ell) P(x_1 = \ell | x_0 = i) = \sum_{\ell \in \mathbb{X}} p_{\ell j} p_{i\ell} = \sum_{\ell \in \mathbb{X}} p_{i\ell} p_{\ell j}.$$

k-Step transition probabilities

Two-step transition matrix:

$$\boldsymbol{P}^{(2)} = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} & \cdots & p_{1n}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} & \cdots & p_{2n}^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1}^{(2)} & p_{n2}^{(2)} & \cdots & p_{nn}^{(2)} \end{bmatrix}$$

Calculate

$$\boldsymbol{P}^2 = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{i1} & p_{i2} & \cdots & p_{in} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1n} \\ p_{21} & \cdots & p_{2j} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{\ell 1} & \cdots & p_{\ell j} & \cdots & p_{\ell n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n1} & \cdots & p_{nj} & \cdots & p_{nn} \end{bmatrix}$$

k-Step transition probabilities

$$\boldsymbol{P}^2 = \begin{bmatrix} \sum_{\ell=1}^{n} p_{1\ell}p_{\ell 1} & \cdots & \sum_{\ell=1}^{n} p_{1\ell}p_{\ell j} & \cdots & \sum_{\ell=1}^{n} p_{1\ell}p_{\ell n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{\ell=1}^{n} p_{i\ell}p_{\ell 1} & \cdots & \sum_{\ell=1}^{n} p_{i\ell}p_{\ell j} & \cdots & \sum_{\ell=1}^{n} p_{i\ell}p_{\ell n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{\ell=1}^{n} p_{n\ell}p_{\ell 1} & \cdots & \sum_{\ell=1}^{n} p_{n\ell}p_{\ell j} & \cdots & \sum_{\ell=1}^{n} p_{n\ell}p_{\ell n} \end{bmatrix}$$

k-Step transition probabilities

$$
\boldsymbol{P}^2 = \begin{bmatrix}
\sum_{\ell=1}^{n} p_{1\ell}p_{\ell 1} & \cdots & \sum_{\ell=1}^{n} p_{1\ell}p_{\ell j} & \cdots & \sum_{\ell=1}^{n} p_{1\ell}p_{\ell n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_{\ell=1}^{n} p_{i\ell}p_{\ell 1} & \cdots & \color{red}{\sum_{\ell=1}^{n} p_{i\ell}p_{\ell j}} & \cdots & \sum_{\ell=1}^{n} p_{i\ell}p_{\ell n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
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\end{bmatrix}
$$

k-Step transition probabilities

$$P^2 = \begin{bmatrix} \sum_{\ell=1}^{n} p_{1\ell}p_{\ell 1} & \cdots & \sum_{\ell=1}^{n} p_{1\ell}p_{\ell j} & \cdots & \sum_{\ell=1}^{n} p_{1\ell}p_{\ell n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{\ell=1}^{n} p_{i\ell}p_{\ell 1} & \cdots & p_{ij}^{(2)} & \cdots & \sum_{\ell=1}^{n} p_{i\ell}p_{\ell n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{\ell=1}^{n} p_{n\ell}p_{\ell 1} & \cdots & \sum_{\ell=1}^{n} p_{n\ell}p_{\ell j} & \cdots & \sum_{\ell=1}^{n} p_{n\ell}p_{\ell n} \end{bmatrix}$$

k-Step transition probabilities

$$\boldsymbol{P}^2 = \begin{bmatrix} p_{11}^{(2)} & \cdots & p_{1j}^{(2)} & \cdots & p_{1n}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{i1}^{(2)} & \cdots & p_{ij}^{(2)} & \cdots & p_{in}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n1}^{(2)} & \cdots & p_{nj}^{(2)} & \cdots & p_{nn}^{(2)} \end{bmatrix} = \boldsymbol{P}^{(2)}$$

k-Step transition probabilities

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- k-step transition probability:

$$p_{ij}^{(k)} = P(x_k = j | x_0 = i), \quad k = 0, 1, \ldots, \infty$$

k-Step transition probabilities

Let k and m be two positive integers and assume $x_0 = i$. In order to go to state j in $(m+k)$ steps, the chain will be at some intermediate state ℓ after m steps. To obtain $p_{ij}^{(m+k)}$, we sum over all possible intermediate states:

$$p_{ij}^{(m+k)} = P(x_{m+k} = j | x_0 = i) = \sum_{\ell = \mathbb{X}} p_{i\ell}^{(m)} p_{\ell j}^{(k)}$$

k-Step transition probabilities

Chapman-Kolmogorov equation:

$$p_{ij}^{(m+k)} = P(x_{m+k} = j | x_0 = i) = \sum_{\ell \in \mathbb{X}} p_{i\ell}^{(m)} p_{\ell j}^{(k)}$$

The k-step transition matrix is given by:

$$\boldsymbol{P}^{(k)} = \boldsymbol{P}^k.$$

Classification of states

- The state j is **accessible** from state i, denoted as $i \to j$, if $p_{ij}^{(k)} > 0$ for some k. Every state is accessible from itself because $p_{ii}^{(0)} = 1$.
- The states i and j **communicate**, denoted as $i \leftrightarrow j$, if they are **accessible** from each other, i.e.,

$$i \leftrightarrow j \Leftrightarrow \begin{cases} i \to j \\ j \to i \end{cases}$$

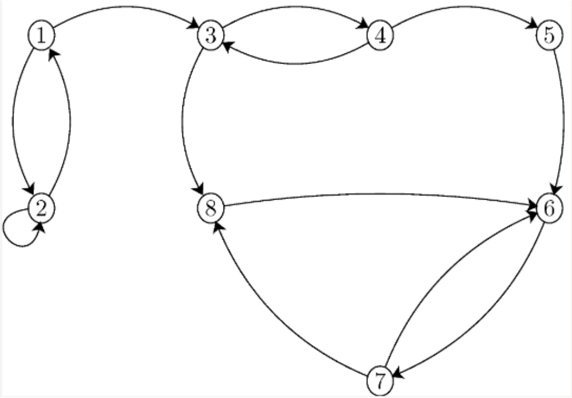
Communication is an **equivalence** relation:

1. $i \leftrightarrow i$
2. if $i \leftrightarrow j$ then $j \leftrightarrow i$
3. if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$

Equivalence

Exercise

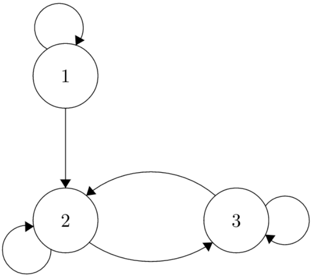
Find the equivalence classes of the Markov chain shown in the figure (It is assumed that $p_{ij} > 0$ when there is an arrow from state i to j).



Irreducible Markov chains

- A Markov chain is **irreducible** if all states communicate with each other.

Consider the Markov chain:



- State **1** is **transient**.
- States $\mathbf{2}$ and $\mathbf{3}$ are **recurrent**.

Recurrent and transient states

- define $f_{ii} = P(x_k = i, \text{for some } k \geq 1 | x_0 = i) = 1$.
  - State i is **recurrent** if $f_{ii} = 1$.
  - State i is **transient** if $f_{ii} < 1$.
- If two states are **in the same class**, either **both of them** are **recurrent**, or **both of them** are **transient.**

A class is recurrent if its states are recurrent.

A class is transient if its states are transient.

Recurrent and transient states

Let V be the total number of visits to the state i of a Markov chain. Then

- if i is a recurrent state then

$$P\left(V = \infty | x_0 = i\right) = 1.$$

- if i is a transient state with probability of returning equal f_{ii} then

$$V | x_0 = i \sim \text{Geometric}(1 - f_{ii}).$$

Exercise

Show that in a finite Markov chain, there is at least one recurrent class.

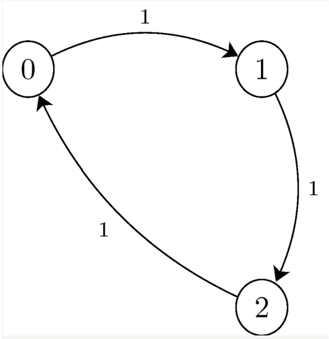
Recurrent and transient states

Solution

- Consider a finite Markov chain with r states, $S = \{1, 2, \dots, r\}$
- Suppose that all states are transient.
- Then, starting from time 0, the chain might visit state 1 several times, but at some point the chain will leave state 1 and will never return to it. That is, there exists an integer $M_1 > 0$ such that $x_k \neq 1$, for all $k \geq M_1$
- Similarly, there exists an integer $M_2 > 0$ such that $x_k \neq 2$, for all $k \geq M_2$, and so on. Now, if you choose $k \geq \max(M_1, M_2, \dots, M_r)$, then x_k cannot be equal to any of the states $1, 2, \dots, r$. This is a contradiction, so we conclude that there must be at least one recurrent state, which means that there must be at least one recurrent class.

Periodicity

- If we Start from state **0** we only return to it at times $k = 3, 6, \ldots$
- In other words, $p_{00}^{(k)} = 0$ if k is not divisible by 3
- Such a state is called a periodic state with period $d(0) = 3$.



Periodicity

Period of a state

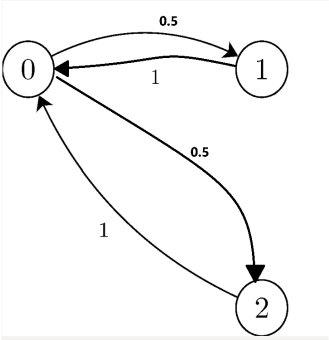
The period of a state i is the largest integer d satisfying the following property: $p_{ii}^{(k)} = 0$ whenever k is not divisible by d. The period of i is denoted by $d(i)$. If $p_{ii}^{(k)} = 0$, for all $k > 0$, then we $d(i) = \infty$.

- If $d(i) > 1$, state i is periodic.
- If $d(i) = 1$, state i is aperiodic.

- All states in **the same class** have the **same period.**
- A class is **periodic** if **its states are periodic.**
- A class is **aperiodic** if **its states are aperiodic.**

Periodicity

The states of the Markov chain in the Figure are periodic with period $d = 2$.



Periodicity

- If $p_{ii}^{(n)} > 0$ and $p_{ii}^{(m)} > 0$ then $p_{ii}(\ell) > 0$ where ℓ is the greatest common divider of m and n
- If $\ell > 1$, i.e., if m and n have common factors, then $\boldsymbol{i}$ is periodic with period $\boldsymbol{d(i) = \ell}$.
- If $\ell = 1$, i.e, if m and n are co-prime, then $\boldsymbol{i}$ is aperiodic.

Aperiodicity conditions in an irreducible Markov chains

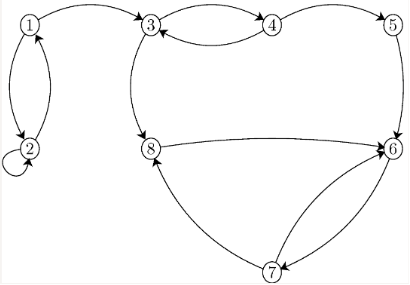
- Existence of at least a self transition ($p_{ii} > 0$ for some state $\boldsymbol{i}$).
- If $p_{ii}^{(n)} > 0$, $p_{ii}^{(m)} > 0$ and m and n are co-prime,
- If exists an integer n such that tha matrix $\boldsymbol{P^n}$ is strictly positive, i.e., if

$$P_{ij}^{(n)} > 0, \text{ for all } i \text{ and } j.$$

Periodicity

Exercise

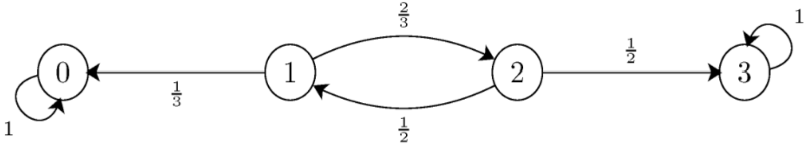
For the Markov chain of the Figure:



1. Is Class 1=$\{1, 2\}$ aperiodic?
2. Is Class 2=$\{3, 4\}$ aperiodic?
3. Is Class 4=$\{6, 7, 8\}$ aperiodic?

Absorbing states

Question: How many classes are in the Markov chain of the Figure?



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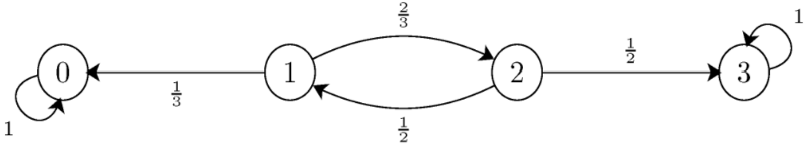
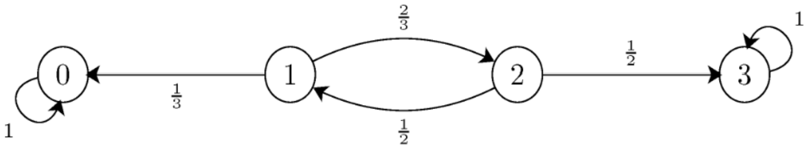


Answer: 3 classes

- **Class 1:** State $\mathbf{0}$: **Recurrent**;
- **Class 2:** States $\mathbf{1}$ and $\mathbf{2}$: **Transient**;
- **Class 3:** State $\mathbf{3}$: **Recurrent**;

Absorbing states

Question: How many classes are in the Markov chain of the Figure?



Answer: 3 classes

- **Class 1:** State $\mathbf{0}$: **Recurrent**;
- **Class 2:** States $\mathbf{1}$ and $\mathbf{2}$: **Transient**;
- **Class 3:** State $\mathbf{3}$: **Recurrent**;

States $\mathbf{0}$ and $\mathbf{3}$ are **absorbing**

Once you enter these states you never leave them.

Absorption probabilities

What are the aborption probabilities of states $\mathbf{2}$ and $\mathbf{3}$?

1. Define the conditional probabilities of absorption in $\mathbf{0}$:
   - $a_0 = P\left(\text{absorption in } \mathbf{0} | x_0 = \mathbf{0}\right) = 1,$
   - $a_1 = P\left(\text{absorption in } \mathbf{0} | x_0 = \mathbf{1}\right) = p_{10}a_0 + p_{12}a_2 = \frac{1}{3}a_0 + \frac{2}{3}a_2,$
   - $a_2 = P\left(\text{absorption in } \mathbf{0} | x_0 = \mathbf{2}\right) = p_{21}a_1 + p_{23}a_3 = \frac{1}{2}a_1 + \frac{1}{2}a_3,$
   - $a_3 = P\left(\text{absorption in } \mathbf{0} | x_0 = \mathbf{3}\right) = 0.$

   Solving the equations:

$$\begin{cases} a_1 = \left.\frac{1}{3}a_0\right|_{a_0=1} + \frac{2}{3}a_2 \\ a_2 = \frac{1}{2}a_1 + \left.\frac{1}{2}a_3\right|_{a_3=0} \end{cases} \Rightarrow \begin{cases} a_1 = \frac{1}{2} \\ a_2 = \frac{1}{4} \end{cases}$$

Absorption probabilities

1. Define the conditional probabilities of absorption in $\mathbf{3}$:
   - $b_0 = P\left(\text{absorption in } \mathbf{3} | x_0 = \mathbf{0}\right) = 0$,
   - $b_1 = P\left(\text{absorption in } \mathbf{3} | x_0 = \mathbf{1}\right) = p_{10}b_0 + p_{12}b_2 = \frac{1}{3}p_0 + \frac{2}{3}p_2$,
   - $b_2 = P\left(\text{absorption in } \mathbf{3} | x_0 = \mathbf{2}\right) = p_{21}b_1 + p_{23}b_3 = \frac{1}{2}b_1 + \frac{1}{2}b_3$,
   - $b_3 = P\left(\text{absorption in } \boldsymbol{b} | x_0 = \mathbf{3}\right) = 1$.

   Solving the equations:

$$\begin{cases} b_1 &= \left.\frac{1}{3}a_0\right|_{b_0=0} + \frac{2}{3}b_2 \\ b_2 &= \frac{1}{2}b_1 + \left.\frac{1}{2}b_3\right|_{b_3=1} \end{cases} \Rightarrow \begin{cases} b_1 = \frac{1}{2} \\ b_2 = \frac{3}{4} \end{cases}$$

Absorption probabilities

Aborption probabilities

Consider a finite Markov chain $\{x_k, k = 0, 1, 2, \dots\}$ with state-space $\boldsymbol{S} = \{\mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n}\}$. Suppose that all states are either absorbing or transient. Let $\boldsymbol{r} \in \boldsymbol{S}$ be an absorbing state. Define

$$a_i = P\,(\text{absorption in } \boldsymbol{r} | x_0 = \boldsymbol{i})\,, \text{for all } \boldsymbol{i} \in \boldsymbol{S}.$$

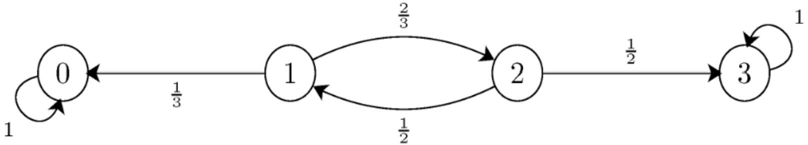
By this definition $a_r = 1$ and $a_j = 0$ if $\boldsymbol{j}$ is any other absorbing state. The unknown values of a_i can be found by solving the equations:

$$a_i = \sum_{\ell} a_\ell p_{i\ell}, \quad \text{for all} \boldsymbol{i} \in \boldsymbol{S}.$$

Absorption probabilities

- A finite Markov chain can exhibit multiple transient and recurrent classes.
- As time increases, the chain becomes absorbed in one of its recurrent classes, where it will remain indefinitely.
- Using the outlined approach, we can determine the probability of absorption into each recurrent class.
- We can substitute each recurrent class with an absorbing state.
- the transformed chain consists exclusively of transient and absorbing states.
- Employing the aforementioned method we can calculate the absorption probabilities

Mean hitting times



t_i -Number of steps needed until the chain hits state $\mathbf{0}$ or $\mathbf{3}$ (i.e., an obsorbing state) given that $x_0 = i$:

- $t_0 = 0$ (the chain is alreay in an aborbing state).
- $t_1 = 1 + \frac{1}{3}t_0 + \frac{2}{3}t_2 = 1 + \frac{2}{3}t_2$ (because if $x_0 = 1$, after one step $x_1 = 1$ or $x_1 = 2$ with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively and from $i = 0$ takes t_0 steps and from $i = 2$ takes t_2 steps).
- $t_2 = 1 + \frac{1}{2}t_1 + \frac{1}{2}t_3 = 1 + \frac{1}{2}t_1.$

Mean hitting times

Solving the equation:

$$\begin{cases} t_1 = 1 + \dfrac{2}{3} t_2 \\ \\ t_2 = 1 + \dfrac{1}{2} t_1 \end{cases} \Rightarrow \begin{cases} t_1 = \dfrac{5}{2} \\ \\ t_2 = \dfrac{9}{4} \end{cases}$$

Mean hitting times

Consider a finite Markov chain $\{x_k, k = 0, 1, 2, \dots\}$ with state space $\boldsymbol{S} = \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, k\}$. Let $\boldsymbol{A} \subset \boldsymbol{S}$ and $\boldsymbol{T}$ be the first time the chain visits $\boldsymbol{A}$. Define $t_i = \boldsymbol{E}\{\boldsymbol{T} | x_0 = \boldsymbol{i}\}$. By definition $t_j = 0$ if $\boldsymbol{j} \in \boldsymbol{A}$. If $\boldsymbol{i} \notin \boldsymbol{A}$ then

$$t_i = 1 + \sum_{\ell} t_\ell p_{i\ell}.$$

Mean return time

Assume the chain is in state ℓ.
The **mean return time** is the expected number of steps required for the chain return to state ℓ.

Consider the subsequent series of states in a Markov chain:

k	0	1	2	3	4	5	6	7	$\cdots$
x_k	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{4}$	$\mathbf{3}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{2}$	$\mathbf{3}$	$\dots$

Define:
$R_2 =$ First return to state $\mathbf{2} = 4$ because it happens at $k = 4$.
Then

$$r_2 = \boldsymbol{E}\left\{R_2 | x_0 = \mathbf{2}\right\}$$

is the mean return time to state $\mathbf{2}$.

Mean return time

Mean return time

Let

$$R_i = \min\left\{k \geq 1 : x_k = \boldsymbol{i}\right\}.$$

Then

$$r_i = \boldsymbol{E}\left\{R_i | x_0 = \boldsymbol{i}\right\}$$

is the mean return time to $\boldsymbol{i}$.

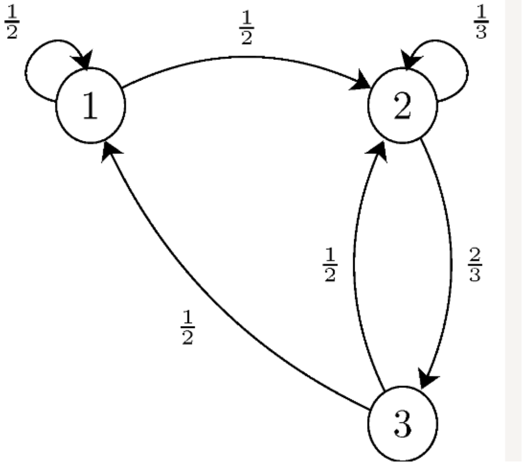
- By definition $R_i \geq 1 \Rightarrow r_i \geq 1$.
- $r_i = 1$ if and only if $\boldsymbol{i}$ is an aborbing state.
- Define t_{ji} as the expected time until the chain reaches the state $\boldsymbol{i}$ for the first time, given that $x_0 = \boldsymbol{j}$. Then

$$r_i = 1 + \sum_j p_{ij} t_{ji} \quad \text{with } t_{ji} = 1 + \sum_\ell p_{j\ell} t_{\ell j}.$$

Mean return time

Exercise

Find t_{11}, t_{21}, t_{31} and r_1.



Mean return time

Solution:

$$\begin{cases} t_{11} &= 0 \text{ by definition} \\ t_{21} &= 1 + \frac{1}{3}t_{21} + \frac{2}{3}t_{31} \\ t_{31} &= 1 + \frac{1}{2}t_{11} + \frac{1}{2}t_{21} \end{cases} \Leftrightarrow \begin{cases} t_{11} = 0 \\ t_{21} = 5 \\ t_{31} = \frac{7}{2} \end{cases}$$

and

$$r_1 = 1 + p_{12}t_{21} = 1 + \frac{5}{2} = \frac{7}{2}.$$

Stationary and Limiting Distributions

Problem: Find the fraction of time that a Markov chain occupies each state as time goes to infinity.
More specifically, study the distribution:

$$\pi^{(k)} = \left[P\left(x_k = \mathbf{0}\right) \quad P\left(x_k = \mathbf{1}\right) \quad \cdots \right]$$

Example: Condider a Markov chain with state-space $\boldsymbol{S} = \{\mathbf{0}, \mathbf{1}\}$ and transition matrix

$$\boldsymbol{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}, \quad a, b \in [0,1] \Rightarrow 0 < a+b < 2.$$

If $P(x_0 = 0) = \alpha$ then

$$\pi^{(0)} \quad = \quad \left[p(x_0) = 0 \quad p(x_0) = 1 \right] = \left[\alpha \quad 1-\alpha \right]$$

Stationary and Limiting Distributions

On the other hand

$$\boldsymbol{P}^k = \mathcal{Z}^{-1}\left\{(z\boldsymbol{I}-\boldsymbol{P})^{-1}\right\}.$$

Given that

$$\begin{aligned}(z\boldsymbol{I}-\boldsymbol{P})^{-1} &= \begin{bmatrix} z-(1-a) & -a \\ -b & z-1-b \end{bmatrix}^{-1} = \\ &= \frac{1}{z^2-(2-a-b)z+1-a-b}\begin{bmatrix} z-1-b & a \\ b & z-1-a \end{bmatrix} = \\ &= \begin{bmatrix} \frac{z-(1-b)}{(z-1)(z-(1-a-b))} & \frac{a}{(z-1)(z-(1-a-b))} \\ \frac{b}{(z-1)(z-(1-a-b))} & \frac{z-(1-a)}{(z-1)(z-(1-a-b))} \end{bmatrix}\end{aligned}$$

Stationary and Limiting Distributions

If $|1-a-b|<1$, then, from the z-transform final value theorem

$$\lim_{k\infty} \boldsymbol{P}^k = \lim_{z\to 1}(z-1)\left(z\boldsymbol{I}-\boldsymbol{P}\right)^{-1} = \begin{bmatrix} \dfrac{b}{a+b} & \dfrac{a}{a+b} \\ \dfrac{b}{a+b} & \dfrac{a}{a+b} \end{bmatrix}$$

and

$$\lim_{k\to\infty} \pi^{(k)} = \lim_{k\to\infty} \left[\pi^{(0)}\boldsymbol{P}^k\right] = \pi^{(0)} \lim_{k\to\infty} \boldsymbol{P}^k = \begin{bmatrix} \dfrac{b}{a+b} & \dfrac{a}{a+b} \end{bmatrix}$$

Stationary and Limiting Distributions

In this example:

$$\lim_{k\to\infty} P\left(x_k = \mathbf{0} | x_0 = \boldsymbol{i}\right) = = \lim_{k\to\infty} p_{i0}^{(k)} = \frac{b}{a+b}, \quad \boldsymbol{i} = 0, 1$$

$$\lim_{k\to\infty} P\left(x_k = \mathbf{1} | x_0 = \boldsymbol{i}\right) = \lim_{k\to\infty} p_{i1}^{(k)} = \frac{a}{a+b}, \quad \boldsymbol{i} = 0, 1$$

Limiting Distributions

The probability distribution $\pi = \begin{bmatrix} \pi_0 & \pi_1 & \pi_2 & \cdots \end{bmatrix}$ is called **limiting distribution** of the Markov chain $\{x_k\}_{k=0,\ldots}$ with state-space $\boldsymbol{S}$ if

$$\pi_j = \lim_{k\to\infty} P\left(x_k = \boldsymbol{j} | x_0 = \boldsymbol{i}\right)$$

for all $\boldsymbol{i},\boldsymbol{j} \in \boldsymbol{S}$, and

$$\sum_{j\in\boldsymbol{S}} \pi_j = 1.$$

Stationary and Limiting Distributions

Exercise

Find the mean return times of a Markov chain with state-space $S = \{0, 1\}$ and transition matrix

$$\boldsymbol{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Stationary and Limiting Distributions

Exercise

Find the mean return times of a Markov chain with state-space $S = \{0, 1\}$ and transition matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Solution:

$$\begin{aligned} r_0 &= 1 + t_{10}p_{01} \\ r_1 &= 1 + t_{01}p_{10} \end{aligned}$$

where t_{ij} is the expected time until the chain reach the state j, given that $x_0 = i$:

$$\begin{cases} t_{10} = 1 + p_{10}t_{00} + p_{11}t_{10} = 1 + (1-b)t_{10} \\ t_{01} = 1 + p_{00}t_{01} + p_{01}t_{11} = 1 + (1-a)t_{01} \end{cases} \Rightarrow \begin{cases} t_{10} = 1/b \\ t_{01} = 1/a. \end{cases}$$

Stationary and Limiting Distributions

$$r_0 = 1 + \frac{a}{b} = \frac{a+b}{b} = \frac{1}{\pi_0}$$

$$r_1 = 1 + \frac{b}{a} = \frac{a+b}{a} = \frac{1}{\pi_1}$$

If in the previous example $a = b = 1$ then

$$\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow x_k + 2 = \boldsymbol{P}^2 x_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k.$$

The Markov chain is periodic. In particular

$$x_k = \begin{cases} x_0 & \text{if } k \text{ is even} \\ x_1 & \text{if } k \text{ is odd} \end{cases}$$

if $a = b = 0$ the chain will consist of two disconnected nodes. In this case,

$$x_k = x_0 \quad \text{for all } k.$$

In these cases the chain does not have a limiting distribution.

Finite Markov Chains

- If a finite Markov chain has more than one recurrent class, then the chain will get absorbed in one of the recurrent classes.
- Consider a irreducible Markov chain with only one recurrent class. In this case the chain has a limiting distribution:

$$\pi = \lim_{k\to\infty} \pi^{(k)} = \lim_{k\to\infty} \left[\pi^{(0)} \boldsymbol{P}^k\right].$$

- Similarly

$$\pi = \lim_{k\to\infty} \pi^{(k+1)} = \lim_{k\to\infty} \left[\pi^{(0)} \boldsymbol{P}^{k+1}\right] \lim_{k\to\infty} \left[\pi^{(0)} \boldsymbol{P}^k \boldsymbol{P}\right] = \left[\pi^{(0)} \boldsymbol{P}^k\right] \boldsymbol{P} = \pi \boldsymbol{P}.$$

- Hence

$$\pi_j = \sum_i \pi_i p_{ij}$$

Finite Markov Chains

Exercise

Find the limiting distribution of the Markov chain with $\boldsymbol{S} = \{\mathbf{0}, \mathbf{1}\}$ and transition matrix

$$\boldsymbol{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

using the relation $\pi = \pi\boldsymbol{P}$.

Finite Markov Chains

Exercise

Find the limiting distribution of the Markov chain with $\boldsymbol{S} = \{\mathbf{0}, \mathbf{1}\}$ and transition matrix

$$\boldsymbol{P} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

using the relation $\pi = \pi \boldsymbol{P}$.

Solution:

$$\pi = \pi \boldsymbol{P} = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} = \begin{bmatrix} (1-a)\pi_0 + b\pi_1 & a\pi_0 + (1-b)\pi_1 \end{bmatrix}.$$

Therefore

$$\left\{ \begin{array}{rcl} \pi_0 & = & (1-a)\pi_0 + b\pi_1 \\ \pi_1 & = & a\pi_0 + (1-b)\pi_1 \end{array} \right. \Rightarrow a\pi_0 = b\pi_1$$

Finite Markov Chains

Solve:

$$\begin{cases} \pi_0 + \pi_1 &= 1 \\ a\pi_0 - b\pi_1 &= 0 \end{cases} \begin{cases} \pi_0 &= \dfrac{b}{a+b} \\ \pi_1 &= \dfrac{a}{a+b}. \end{cases}$$

Finite Markov Chains

Consider a Markov chain $\{x_k\}_{k=0,1,\dots}$ with state-space $\boldsymbol{S} = \{\mathbf{0}, \mathbf{1}, \dots, \boldsymbol{n}\}$. Then,

1. It has a limiting distribution if and only if the set of equations:

$$\begin{cases} \pi = \pi \boldsymbol{P} \\ \sum_{j=1}^{n} \pi_j = 1 \end{cases}$$

   has a unique solution.

2. This unique solution is the limiting distribution of the Markov chain.
3. The mean return time to state $\boldsymbol{j}$ is

$$r_j = \frac{1}{\pi_j}, \quad \text{for all } \boldsymbol{j} \in \boldsymbol{S}.$$

Countably Infinite Markov Chains:

Positive recurrent and null recurrent

Let $\boldsymbol{i}$ be a recurrent state. Assuming $x_0 = \boldsymbol{i}$, let $\boldsymbol{R}_i$ be the number of transitions needed to return to state i , i.e.,

$$\boldsymbol{R}_i = \min\{k \geq 1 : x_k = \boldsymbol{i}\}$$

if $r_i = \boldsymbol{E}\{\boldsymbol{R}_i | x_0 = \boldsymbol{i}\} < \infty$ then $\boldsymbol{i}$ is **positive recurrent**, otherwise is **null recurrent.**

Countably Infinite Markov Chains:

Theorem

Consider a Markov chain $\{x_k\}_{k=0,1,\ldots}$ with state-space $\boldsymbol{S} = \{\mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{n}\}$. Assume that the chain is **irreducible** and **aperiodic**. Then, one of the following cases can occur:

1. All states are transient, and
$$\lim_{k\to\infty} P\left(x_k = \boldsymbol{j} | x_0 = \boldsymbol{i}\right) = 0, \quad \text{for all } \boldsymbol{i}, \boldsymbol{j}.$$
2. All states are null recurrent, and
$$\lim_{k\to\infty} P\left(x_k = \boldsymbol{j} | x_0 = \boldsymbol{i}\right) = 0, \quad \text{for all } \boldsymbol{i}, \boldsymbol{j}.$$
3. All states are positive recurrent. In this case exists a limiting distribution $\pi = \begin{bmatrix} \pi_0 & \pi_1 & \cdots \end{bmatrix}$ where
$$\pi_j = \lim_{k\to\infty} P\left(x_k = \boldsymbol{j} | x_0 = \boldsymbol{i}\right) > 0, \quad \text{for all } \boldsymbol{i}, \boldsymbol{j}.$$

Countably Infinite Markov Chains:

Theorem - continuation

This limiting distribution is the unique solution of

$$\begin{cases} \sum_{i=0}^{\infty} p_{ij}\pi_i & = & \pi_j \\ \sum_{j=0}^{\infty} \pi_j & = & 1. \end{cases}$$

Also

$$r_j = \frac{1}{\pi_j}, \quad \text{for all } j = 0, 1, \ldots,$$

where r_j is the return times to state j.