Structure, algorithmics and dynamics of endomorphisms for certain classes of groups

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Abstract

The study of endomorphisms of groups has been an important and very active area of research in the fields of Combinatorial and Geometric Group Theory for the past forty years. In this thesis, we study endomorphisms of groups from three main viewpoints: structural, algorithmic and dynamical. While playing a less relevant role in this work, we also devote ourselves to the study of subsets of groups, a topic lying within the intersection of Algebra with Computer Science.

We now summarize our results.

Structure of endomorphisms

We prove that every uniformly continuous endomorphism of a hyperbolic group with respect to a visual metric (or equivalently, every endomorphism of a hyperbolic group that admits a continuous extension to the Gromov boundary) satisfies a Hölder condition, answering a question of Araújo and Silva [2]. This result, combined with the work done by Paulin [87], implies that every such endomorphism has a finitely generated fixed subgroup. This was achieved by obtaining a geometric version of the Bounded Reduction Property (BRP), also known as the Bounded Cancellation Lemma, introduced by Cooper [32] in the context of free groups and used by many with great success. Another geometric connection following from this work concerns coarse-median spaces, which were introduced by Bowditch in [14]. Fioravanti [44] introduced the concept of coarse-median preserving automorphisms and was able to establish interesting properties, including finiteness results on the subgroup fixed by such an automorphism. We are able to prove that coarse-median preserving endomorphisms of hyperbolic groups are precisely the ones for which the BRP holds.

We also initiate the study of endomorphisms of automatic groups, by extending the geometric definition of the BRP previously introduced for hyperbolic groups to this class of groups. In doing so, there came the need to define two versions of it: synchronous and asynchronous. After that, we prove that every endomorphism with a finite kernel and a quasiconvex image satisfies (in some sense) the synchronous version of the BRP. We then show that if two endomorphisms φ, ψ satisfy the synchronous version of the BRP for automatic structures under certain (strict) conditions, then their equalizer is quasiconvex (and so automatic). Since this is always possible to do for inner automorphisms, it follows as a corollary an alternative proof to the fact that

centralizers of finite subsets of a biautomatic group are themselves biautomatic, which had previously been proved by Gersten and Short in [46].

Finally, we classify the endomorphisms of the direct product of two free groups of finite rank in seven different types and use this classification to prove algorithmic and dynamical results for endomorphisms of this class of groups.

Algorithmic problems

Using the description of endomorphisms of a direct product of two free groups, we prove decidability of several algorithmic problems for this class, such as the Whitehead problems and deciding whether the fixed (resp. periodic) subgroup of an endomorphism is finitely generated or not. In case it is, we also describe a procedure to compute a finite a set of generators for it. This was achieved by describing the fixed subgroup for each of the seven types of endomorphisms, showing that, when the fixed subgroup is finitely generated, it is the direct product of two free groups of finite rank.

Later, we consider the subgroup of eventually fixed (resp. eventually periodic) points of an endomorphism φ of a finitely generated virtually free group, which corresponds to the subgroup of points whose orbit through φ contains a fixed (resp. periodic) point. We prove that the length of finite orbits of an endomorphism of a virtually free group is bounded by a computable constant, which allowed us to show that it is decidable whether the subgroup of eventually fixed (resp. periodic) points is finitely generated or not and that, in case it is, we can effectively compute a finite set of generators. We also extend the result from [79] regarding computability of fixed points and of the stable image of an endomorphism of a free group to the class of virtually free groups.

Brinkmann's (resp. Brinkmann's conjugacy) problem consists on deciding, taking as input an automorphism φ of a group G and two elements $x, y \in G$, whether there is some $n \in \mathbb{N}$ such that $x\varphi^n = y$ (resp. $x\varphi^n \sim y$). We consider a *generalized* version of Brinkmann's conjugacy problem: given a subset $K \subseteq G$, an automorphism φ and an element $x \in G$, can we decide whether there is some $n \in \mathbb{N}$ such that $x\varphi^n$ has a conjugate in K? Following the ideas in [9], we are able to relate this problem with the generalized conjugacy problem: given a subset $K \subseteq G$ and an element $x \in G$, can we decide whether there is some conjugate of x in K? Using this connection and the fact that virtually polycyclic groups are conjugate separable, we show that the generalized Brinkmann's conjugacy problem is decidable for virtually polycyclic groups. Decidability of the simple versions of Brinkmann's problem and of the twisted conjugacy problem for virtually free groups is also proved.

Finally, we introduce the concepts of the φ-order of an element with respect to a subset of the group and of the φ-spectrum of a subset and prove that the φ-spectrum of a finite subset of a finitely generated virtually free group is computable.

Dynamics of endomorphisms

When considering a group G endowed with a metric such that the completion of the metric space is also compact, endomorphisms which admit a continuous extension to the completion are precisely the uniformly continuous ones. Some well-known topological spaces can be defined in this way. For example, the Gromov boundary can be obtained by considering the completion of a hyperbolic group when endowed with a visual metric. We will consider direct products of free groups of the form $F_n \times F_m$ and $\mathbb{Z}^m \times F_n$ when endowed with the product metric obtained by taking the prefix metric in each direct factor. We remark that the completion we obtain this way is the Roller completion ([15, 43, 44, 91]).

The family of free-abelian times free (FATF) groups, $\mathbb{Z}^m \times F_m$, despite looking simple, can present some nontrivial behavior. Making use of a classification of endomorphisms of FATF groups in two different types done by Delgado and Ventura in [36], we characterize uniformly continuous endomorphisms of a FATF group and prove that they coincide with the coarse-median preserving ones when the natural coarse median structure is taken. Then, we describe the dynamical behavior around infinite fixed points of automorphisms, i.e., fixed points of the continuous extension to the completion, classifying them as attractors, repellers or none of the two. We also introduce the concepts of recurrent and wandering points in the context of endomorphisms of groups. These are classical concepts in the theory of dynamical systems and so it makes sense to try and study them in the case where our dynamical system is given by (a continuous extension of) an endomorphism. We show that, for automorphisms and type II endomorphisms, every point in the completion is either periodic or wandering, which implies that, in these cases, the dynamics is asymptotically periodic, which had been proved to hold for automorphisms of the free group by Levitt and Lustig in [67].

We are also able to make use of the more intricate classification of endomorphisms of the direct product of two free groups of finite rank previously achieved and obtain a similar description of infinite fixed points as attractors/repellers for this class of groups.

Subsets of groups

We describe the structure of algebraic and context-free subsets of a group G relatively to the analogous structure for a finite index subgroup H. Using these results, we prove that a kind of Fatou property, previously studied by Berstel and Sakarovitch in the context of rational subsets in [7] and by Herbst in the context of algebraic subsets [53, 54], holds for context-free subsets if and only if the group is virtually free. We also exhibit a counterexample to a question Herbst posed in [53] concerning this property for algebraic subsets.

Keywords: Endomorphisms, Fixed points, Infinite fixed points, Algorithmic problems, Dynamics at the infinity, Bounded reduction, Subsets of groups

Resumo

O estudo de endomorfismos de grupos tem sido uma área muito ativa e importante na Teoria Combinatória e Geométrica de Grupos nos últimos quarenta anos. Nesta tese, estudamos endomorfismos de grupos sob três pontos de vista: estrutural, algorítmico e dinâmico. Embora desempenhando um papel menos relevante neste trabalho, também nos dedicamos ao estudo de subconjuntos de grupos, um tópico que se situa na interseção da Álgebra com a Ciência de Computadores.

Passamos agora a descrever de forma sucinta os nossos resultados.

Estrutura de endomorfismos

Provamos que todos os endomorfismos uniformemente contínuos de um grupo hiperbólico relativamente a uma métrica visual (ou, de forma equivalente, todos os endomorfismos que admitem uma extensão contínua ao bordo de Gromov) satisfazem uma condição de Hölder, respondendo assim a uma questão colocada por Araújo e Silva em [2]. Este resultado, combinado com os resultados obtidos por Paulin [87], implica que todos estes endomorfismos têm um subgrupo de pontos fixos finitamente gerado. Isto foi alcançado através da obtenção de uma versão geométrica da propriedade de redução limitada (PRL), também conhecida como Lema do Cancelamento Limitado, introduzida por Cooper [32] no contexto dos grupos livres e usada por muitos outros com sucesso. Uma outra conexão com a geometria que surge com este trabalho está relacionada com espaços de mediana grosseira, introduzidos por Bowditch em [14]. Fioravanti [44] introduziu o conceito de automorfismo que preserva a mediana grosseira, conseguindo obter propriedades interessantes, incluindo resultados de finitude acerca do subgrupo fixo por um destes automorfismos. Provamos que os endomorfismos de grupos hiperbólicos que preservam a mediana grosseira são precisamente os que satisfazem a PRL.

Iniciamos também o estudo de endomorfismos de grupos automáticos, estendendo a definição geométrica da PRL introduzida previamente para grupos hiperbólicos. Para o fazermos, surgiu a necessidade de definir duas versões da PRL: síncrona e assíncrona. Depois de o fazermos, provamos que todos os endomorfismos com núcleo finito e imagem quasiconvexa satisfazem (de alguma forma) a versão síncrona da PRL. Mostramos em seguida que, se dois endomorfismos φ, ψ satisfizerem a versão síncrona da PRL para estruturas automáticas que verifiquem certas condições (restritivas), então o igualizador é quasiconvexo (e, portanto, automático). Como as condições são sempre satisfeitas para automorfismos internos, segue como corolário uma

prova alternativa do facto de que os centralizadores de um subconjunto finito de um grupo biautomático são eles próprios biautomáticos, resultado esse que tinha sido provado previamente em [46].

Por fim, classificamos os endomorfismos do produto direto de dois grupos livres de dimensão finita em sete tipos diferentes e usamos esta classificação para provar resultados algorítmicos e dinâmicos para endomorfismos de grupos nesta classe.

Problemas algorítmicos

Usando a descrição dos endomorfismos do produto direto de dois grupos livres, provamos a decidibilidade de vários problemas algorítmicos para esta classe, tais como os problemas de Whitehead e o problema de decidir se o subgrupo fixo (resp. periódico) de um endomorfismo é finitamente gerado ou não. Caso seja, também descrevemos um procedimento para calcular um conjunto finito de geradores. Isto foi alcançado através da descrição do subgrupo fixo para cada um dos sete tipos de endomorfismos, mostrando que, quando o subgrupo fixo é finitamente gerado, é o produto direto de dois grupos livres finitamente gerados.

Depois consideramos o subgrupo de pontos eventualmente fixos (resp. eventualmente periódicos) de um endomorfismo φ, que corresponde ao subgrupo de pontos cuja órbita por φ contém um ponto fixo (resp. periódico). Provamos que o comprimento das órbitas finitas de um endomorfismo de um grupo virtualmente livre é majorado por uma constante computável, o que nos permitiu mostrar que é decidível se o subgrupo de pontos eventualmente fixos (resp. periódicos) é finitamente gerado ou não e, caso seja, calcular efetivamente um conjunto finito de geradores. Também estendemos os resultados de [79] que concernem a computabilidade dos pontos fixos e da imagem estável de um endomorfismo de um grupo livre à classe dos grupos virtualmente livres.

O problema de Brinkmann (resp. o problema da conjugação de Brinkmann) consiste em decidir, recebendo como entrada um automorfismo φ de um grupo G e dois elementos $x, y \in G$, se existe algum $n \in \mathbb{N}$ tal que $x\varphi^n = y$ (resp. $x\varphi^n \sim y$). Consideramos uma versão *generalizada* do problema da conjugação de Brinkmann: dado um subconjunto $K \subseteq G$, um automorfismo φ e um elemento $x \in G$, conseguimos decidir se existe algum $n \in \mathbb{N}$ tal que $x\varphi^n$ tenha um conjugado em K? Seguindo as ideias de [9], conseguimos relacionar este problema com o problema da conjugação generalizado: dados um subconjunto $K \subseteq G$ e um elemento $x \in G$, conseguimos decidir se há algum conjugado de x em K? Usando esta relação e o facto de que os grupos virtualmente policíclicos são separáveis por conjugação, mostramos que o problema da conjugação de Brinkmann generalizado é decidível para grupos virtualmente policíclicos. A decidibilidade das versões simples do problema de Brinkmann e do problema da conjugação torcida para grupos virtualmente livres é também demonstrada.

Por fim, introduzimos os conceitos de φ-ordem de um elemento relativamente a um subconjunto de um grupo e de φ-espetro de um subconjunto e provamos que o φ-espetro de um subconjunto finito de um grupo virtualmente livre finitamente gerado é computável.

Dinâmica de endomorfismos

Quando consideramos um grupo G munido de uma métrica tal que o completamento do espaço métrico é também compacto, os endomorfismos que admitem uma extensão contínua ao completamento são precisamente os uniformemente contínuos. Alguns espaços topológicos conhecidos podem ser definidos desta forma. Por exemplo, o bordo de Gromov pode ser obtido considerando o completamento de um grupo hiperbólico munido de uma métrica visual. Vamos considerar produtos diretos de grupos livres das formas $F_n \times F_m$ e $\mathbb{Z}^m \times F_n$ quando munidos da métrica produto obtida considerando a métrica dos prefixos em cada fator direto. Destacamos que o completamento obtido desta forma é o completamento de Roller [15, 43, 44, 91]).

A família de grupos abelianos-livres vezes livres (ALVL), $\mathbb{Z}^m \times F_n$, apesar de parecer ser simples, pode apresentar comportamento não trivial. Usando a classificação dos endomorfismos dos grupos ALVL obtida por Delgado e Ventura [36], caraterizamos os endomorfismos uniformemente contínuos de um grupo ALVL e provamos que coincidem com os endomorfismos que preservam a mediana grosseira natural. Descrevemos também o comportamento dinâmico em torno de pontos fixos infinitos de automorfismos, i.e., pontos fixos da extensão contínua ao completamento, classificando-os como atratores, repulsores, ou nenhum dos dois. Introduzimos também os conceitos de pontos recorrentes e errantes no contexto de endomorfismos de grupos. Estes são conceitos clássicos na teoria de sistemas dinâmicos e portanto faz sentido estudá-los no caso do nosso sistema dinâmico ser dado por (uma extensão contínua de) um endomorfismo. Mostramos que, para automorfismos e endomorfismos de tipo II, todos os pontos no completamento são periódicos ou errantes, o que implica que, nestes casos, a dinâmica seja assintoticamente periódica, resultado que já tinha sido provado para automorfismos do grupo livre em [67].

Conseguimos também usar a classificação mais intrincada dos endomorfismos do produto direto de dois grupos livres de dimensão finita previamente obtida para conseguir uma descrição semelhante dos pontos fixos infinitos como atratores ou repulsores para grupos nesta classe.

Subconjuntos de grupos

Descrevemos a estrutura de subconjuntos algébricos e livres de contexto de um grupo G relativamente à estrutura análoga para subgrupo H de índice finito. Usando estes resultados, provamos que uma espécie de propriedade de Fatou, previamente estudada por Berstel e Sakarovitch no contexto dos subconjuntos racionais em [7] e por Herbst no contexto dos subconjuntos algébricos [53, 54], se verifica para subconjuntos livres de contexto se e só se o grupo é virtualmente livre. Exibimos também um contraexemplo a uma questão de Herbst colocada em [53] acerca desta mesma propriedade para subconjuntos algébricos.

Palavras-chave: Endomorfismos, Pontos Fixos, Pontos fixos infinitos, Problemas algorítmicos, Dinâmica no infinito, Redução limitada, Subconjuntos de grupos

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Chapter 1

Introduction

1.1 A very brief history of endomorphisms

The study of the fixed points of endomorphisms of groups started with the (independent) work of Gersten [47] and Cooper [32], using respectively graph-theoretic and topological approaches. They proved that the subgroup of fixed points $\mathrm{Fix}(\varphi)$ of some fixed automorphism φ of F_n is always finitely generated. Moreover, Cooper succeeded on classifying from the dynamical viewpoint the fixed points of the continuous extension of φ to the boundary of F_n. Bestvina and Handel subsequently developed the theory of train tracks to prove that $\mathrm{Fix}(\varphi)$ has rank at most n in [8] and this result was generalized to (not necessarily injective) endomorphisms by Imrich and Turner in [60] using the stable image of an endomorphism. The problem of computing a basis for $\mathrm{Fix}(\varphi)$ had a tribulated history and was finally settled for automorphisms by Bogopolski and Maslakova in 2016 in [11] and by Mutanguha [79] for general endomorphisms of a free group. To prove this, Mutanguha presented an algorithm to compute the stable image of an endomorphism and with that in hand, computability of $\mathrm{Fix}(\varphi)$ can be reduced to computing the fixed subgroup of an automorphism, which was already known from the work of Bogopolski and Maslakova.

This line of research was quickly extended to wider classes of groups. For instance, Paulin proved in 1989 that the subgroup of fixed points of an automorphism of a hyperbolic group is finitely generated [87]. Many interesting results were also obtained for the class of right-angled Artin groups (RAAGs): the authors in [90] described which RAAGs are such that all endomorphisms have finitely generated fixed subgroups, and Fioravanti introduced in [44] the concept of coarse median preservation and proved that coarse-median preserving automorphisms of RAAGs have finitely generated and undistorted fixed subgroups. Fixed points of lamplighter groups are also investigated in [74], where it is shown that not even all inner automorphisms of $\mathcal{L}_2$ have finitely generated fixed subgroups.

Regarding the extension of an endomorphism to the completion, infinite fixed points of automorphisms of free groups were also discussed by Bestvina and Handel in [8] and by Gaboriau, Jaeger, Levitt and Lustig in [33]. The dynamics of free group automorphisms is proved to be asymptotically periodic in [67]. In [27], Cassaigne and Silva study the dynamics of infinite fixed points for monoids defined by special confluent rewriting systems (which contain free groups as a particular case). This was also achieved by Silva in [98] for virtually injective endomorphisms of virtually free groups. One of the essential tools used in proving some of these results is the bounded reduction property (also known as the bounded cancellation lemma) introduced in [32] and followed by many others.

From the algorithmic point of view, the main problems concerning endomorphisms that will matter to us are the twisted conjugacy problem, Brinkmann's (equality) problem and Brinkmann's conjugacy problem. The twisted conjugacy problem was proved to be decidable for automorphisms of a free group in [9] and for general endomorphisms in [68, 102]. Other results include, for example, decidability of the twisted conjugacy problem for endomorphisms of polycyclic groups [92] and automorphisms of braid groups [50]. Brinkmann's problems were introduced and proved to be decidable by Brinkmann for automorphisms of the free group in [17] and some variations were solved in [68] for endomorphisms. For other classes of groups, Brinkmann's problems were tackled for example by Kannan and Lipton for free-abelian groups [63], by González-Meneses and Ventura for braid groups [50] and for free-abelian-by-free groups in [26]. In [9], the authors prove that [f.g. free]-by-cyclic groups have solvable conjugacy problem by reducing this question to the twisted conjugacy problem and Brinkmann's conjugacy problem on free groups. This was later generalized to more general extensions of groups in [10], using orbit decidability, and very recently to ascending HNN-extensions of free groups in [68] using variations of the problems for injective endomorphisms.

1.2 Our main results

The research involved in this thesis lies in the areas of Combinatorial and Geometric Group Theory and is mostly focused on the study of endomorphisms of certain classes of groups from three points of view: structural, algorithmic and dynamical. Some results on subsets of groups involving Theoretical Computer Science, more specifically the area of Formal Languages and Automata, are also obtained. The work presented in this thesis led to the publication of five papers [20–24], the writing of two already available preprints [18, 19] and the writing of a preprint still in preparation [25].

We now describe our results.

1.2.1 Subsets of groups

Rational and recognizable subsets of groups are natural generalizations of finitely generated and finite index subgroups, respectively. Over the years, they have been studied from different points of view. From the structural viewpoint, presumably the most important result is Benois' Theorem, which provides us with a description of rational subsets of free groups in terms of reduced words. A description of rational subsets of free-abelian groups as the semilinear sets of $\mathbb{Z}^m$ is also known. Deep connections with the algebraic structure of the group have also been proved in many contexts. For example, rational (resp. recognizable) subgroups have been proved to be exactly the finitely generated (resp. finite index) ones, and some classes of groups can be described through the classification of their subsets. Another application of the study of subsets of groups is the generalization of some decision problems concerning finitely generated subgroups to a wider class of sets, such as the membership problem, the intersection problem or the generalized conjugacy problem as done, for example, in [62, 66, 69].

In [7], the authors prove that given a group G and a subgroup $H \leq G$, then

$$Rat(H) = \{K \subseteq H \mid K \in Rat(G)\}, \tag{1.1}$$

where $Rat(G)$ denotes the class of rational subsets of G. The authors call it *a kind of Fatou property for groups.*

While more complex, the context-free counterparts of rational and recognizable subsets, respectively algebraic and context-free subsets, also yield interesting results. We denote the classes of algebraic and context-free sets of G by $Alg(G)$ and $CF(G)$, respectively. In [53], Herbst studied these subsets and was able to characterize groups for which context-free subsets coincide with rational subsets as virtually cyclic groups. In the same paper, Herbst also proved the Fatou property for algebraic subsets in case H is a finite index normal subgroup of G and posed the question of whether this would hold in general. Later, Herbst proved that that was the case if G is a virtually free group. We exhibit a counterexample to this question, proving that the property does not hold for a free-abelian-by-cyclic group. We also consider the same question for recognizable and context-free subsets and prove that, in the first case, the property holds if and only if H is a finite index subgroup of G and in the latter it holds for all $H \leq_{f.g} G$ if and only if G is virtually free.

Theorem 3.2.3. Let G be a finitely generated group. Then G is virtually free if and only if

$$CF(H) = \{K \subseteq H \mid K \in CF(G)\},$$

for all $H \leq_{f.g.} G$.

To achieve this, we prove some results relating the structure of algebraic and context free-subsets of a group G with the structure of the corresponding subsets of a finite index subgroup H, obtaining the following structural results similar to the ones obtained for rational and recognizable subsets in [51, 96]:

Proposition 3.1.6. Let G be a finitely generated group and $H \leq_{f.i.} G$. If G is the disjoint union $G = \cup_{i=1}^{n} Hb_i$, then $CF(G)$ consists of all subsets of the form

$$\bigcup_{i=1}^{n} L_i b_i \qquad (L_i \in CF(H))$$

Proposition 3.1.13. Let G be a finitely generated group and $H \leq_{f.i.} G$. If G is the disjoint union $G = \cup_{i=1}^{n} Hb_i$, then $Alg(G)$ consists of all subsets of the form

$$\bigcup_{i=1}^{n} L_i b_i \qquad (L_i \in Alg(H))$$

1.2.2 Algorithmic results on endomorphisms

Virtually free groups

With respect to the class of virtually free groups, we extend the algorithmic results of [79] proving that the fixed subgroup $\mathrm{Fix}(\varphi)$ and the stable image $\varphi^{\infty}(G)$ are finitely generated and computable if φ is an endomorphism of a finitely generated virtually free group.

Theorem 4.1.1. There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G and output a finite set of generators for $\mathrm{Fix}(\varphi)$.

Theorem 4.1.4. There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G and output a finite set of generators for $\varphi^{\infty}(G)$.

From Theorem 4.1.1, decidability of the twisted conjugacy problem follows naturally.

Corollary 4.1.3. Let G be a finitely generated virtually free group. Then $TCP_{End}(G)$ is decidable.

In [80], Myasnikov and Shpilrain study finite orbits of elements of a free group under the action of an automorphism proving that, in a free group F_n, there is an orbit of cardinality k if and only if there is an element of order k in $\mathrm{Aut}(F_n)$. Moreover, the authors prove that this result does not hold for general endomorphisms, by providing an example of an endomorphism of F_3 for which there is a point whose orbit has 5 elements.

In this thesis, we introduce the study of finite orbits of elements under the action of an endomorphism of a finitely generated virtually free group. The orbit of an element $x \in G$ through an endomorphism φ is finite if and only if it intersects the subgroup of periodic points, $\mathrm{Per}(\varphi)$, of φ. The set of such points forms a subgroup of G and so do the points whose orbit intersects the fixed subgroup $\mathrm{Fix}(\varphi)$. We call these subgroups $\mathrm{EvPer}(\varphi)$ and $\mathrm{EvFix}(\varphi)$, respectively. It is easy to see that $\mathrm{EvFix}(\varphi)$ coincides with $\mathrm{Fix}(\varphi)$ if (and only if) φ is injective. For this reason, we mainly focus on noninjective endomorphisms. Despite the fact that the result in [80] cannot be generalized to endomorphisms, we prove that, replacing finite orbits by periodic orbits, the result holds for endomorphisms, i.e., there is an endomorphism of F_n with a periodic orbit of cardinality k if and only if there is an element of order k in $\mathrm{Aut}(F_n)$. Moreover, we prove that, for endomorphisms of finitely generated virtually free groups, finite orbits have bounded cardinality.

Corollary 4.4.6. There exists an algorithm with input a finitely generated virtually free group G and output a constant k such that

$$\max\{|\operatorname{Orb}_\varphi(x)| \mid \varphi \in \operatorname{End}(G)\,,\, x \in \operatorname{EvPer}(\varphi)\} \leq k.$$

This allows us to solve some algorithmic questions: we can decide if φ is a finite order element of $\operatorname{End}(G)$, if φ is aperiodic or not and, in case G is free, whether $\operatorname{EvFix}(\varphi)$ is a normal subgroup of G or not.

Unlike the case of $\operatorname{Fix}(\varphi)$ and $\operatorname{Per}(\varphi)$, the subgroups $\operatorname{EvFix}(\varphi)$ and $\operatorname{EvPer}(\varphi)$ are not necessarily finitely generated. However, we prove that we can always decide if that is the case and, if so, compute a set of generators.

Corollary 4.4.14. There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G that decides whether $\operatorname{EvFix}(\varphi)$ (resp. $\operatorname{EvPer}(\varphi)$) is finitely generated and, in case the answer is affirmative, computes a finite set of generators.

Brinkmann's problem consists on deciding, with input an endomorphism $\varphi \in \operatorname{End}(G)$ and two elements $x, y \in G$, whether there is some $n \in \mathbb{N}$ such that $x\varphi^n = y$ and we solve it for endomorphisms of a virtually free group.

Theorem 4.5.1. Let G be a finitely generated virtually free group. Then $BrP_{End}(\mathrm{G})$ is decidable.

Finally, we introduce the concepts of φ-order and φ-spectrum and prove that the φ-spectrum of a finite subset of a virtually free group is computable. For a subset $K \subseteq G$, we say that the *relative φ-order of g in K*, $\varphi\text{-ord}_K(g)$, is the smallest nonnegative integer n such that $g\varphi^n \in K$. If there is no such n, we say that $\varphi\text{-ord}_K(g) = \infty$. The *$\varphi$-spectrum* of a subset, $\varphi\text{-sp}(K)$, is the set of relative φ-orders of elements in K, i.e., $\varphi\text{-sp}(K) = \{\varphi\text{-ord}_K(g) \mid g \in G\}$. Our result is the following:

Theorem 4.5.10. There exists an algorithm with input a finitely generated virtually free group G, an endomorphism φ of G and a finite set $K = \{g_1, \ldots, g_k\} \subseteq G$ and output $\varphi\text{-sp}(K)$.

G-by-$\mathbb{Z}$ groups

Let $A = \{a_1, \ldots, a_n\}$, $G = \langle A \mid R\rangle$ be a group and $\varphi \in \operatorname{Aut}(G)$. A G-by-$\mathbb{Z}$ group has the form

$$G \rtimes_\varphi \mathbb{Z} = \langle A, t \mid R, t^{-1}a_i t = a_i\varphi\rangle. \tag{1.2}$$

Every element of $G \rtimes_\varphi \mathbb{Z}$ can be written in a unique way as an element of the form $t^a g$, where $a \in \mathbb{Z}$ and $g \in G$. Notice that this is a particular case of a HNN-extension with base group G and both associated subgroups equal to G.

Brinkmann's conjugacy problem consists on deciding, taking as input an automorphism φ of a group G and two elements $x, y \in G$, whether there is some $n \in \mathbb{N}$ such that $x\varphi^n \sim y$. In [9], Bogopolski, Martino, Maslakova and Ventura established an equivalence between decidability of Brinkmann's conjugacy problem, $BrCP$, and of the twisted conjugacy problem, TCP, in a group G and decidability of the conjugacy problem, CP, in $G \rtimes \mathbb{Z}$. Following this line, we consider a *generalized* version of these problems: the generalized Brinkmann's conjugacy problem, $GBrCP$, consists on, given a subset $K \subseteq G$, an automorphism φ and an element $x \in G$, deciding whether there is some $n \in \mathbb{N}$ such that $x\varphi^n$ has a conjugate in K; the generalized twisted conjugacy problem, $GTCP$, consists on, given a subset $K \subseteq G$, an automorphism φ and an element $x \in G$, deciding whether x has a φ-twisted conjugate in K; the generalized conjugacy problem, GCP, consists on, given a subset $K \subseteq G$ and an element $x \in G$, deciding whether there is a conjugate of x in K.

Whenever we want to consider an instance of an algorithmic problem with restrictions on the input, we will write the restrictions as indices. For example, if $H \leq G$, $\varphi \in \text{Aut}(G)$ and $x, y \in G$, we write $GTCP_{(yH,\varphi,x)}(G)$ to denote the problem of deciding whether x has a φ-twisted conjugate in yH or not. Notice that it might happen that not all components of the input (subset, automorphism and element) are restricted, but some are. For instance, we might write $GTCP_{(Rat(G),\varphi)}(G)$ meaning that we take as input a rational subset $K \subseteq G$ and an element $x \in G$ and we want to decide whether x has a φ-twisted conjugate element in K.

We remark that not much is known about these generalized problems. In [66], Ladra and Silva work in the context of rational subsets and solve $GCP_{Rat}(G)$ when G is a finitely generated virtually free group. They do so by solving $GTCP_{(Rat,\text{Via})}(F_n)$, where Via denotes the set of virtually inner automorphisms, i.e., automorphisms ϕ such that there is some $k \in \mathbb{N}$ for which $\phi^k \in \text{Inn}(F_n)$. In the survey [101], Ventura remarks that not much is known about $GBrP_{f.g.}(G)$ even when G is a free or a free-abelian group.

Given a subset $K \subseteq G \rtimes_\varphi \mathbb{Z}$ and $r \in \mathbb{Z}$, we define

$$K_r = \{x \in G \mid t^r x \in K\} = t^{-r}K \cap G.$$

Following the ideas in [9], we are able to obtain an analogous result for the generalized versions of the problems, connecting $GBrCP(G)$ and $GTCP(G)$ with $GCP(G \rtimes \mathbb{Z})$. The main result is the following:

Theorem 5.1.1. Let G be a group, $\varphi \in \text{Aut}(G)$, $K \subseteq G \rtimes_\varphi \mathbb{Z}$ and $t^r g \in G \rtimes_\varphi \mathbb{Z}$. Then:

1. if $r = 0$, then $GCP(G \rtimes_\varphi \mathbb{Z})$ outputs `YES` on input $(K, t^r g)$ if and only if $GBrCP(G)$ outputs `YES` on input (K_r, φ, g);

2. if $r \neq 0$, then $GCP(G \rtimes_\varphi \mathbb{Z})$ outputs `YES` on input $(K, t^r g)$ if and only $GTCP_(G)$ outputs `YES` on input $(K_r, \varphi^r, g\varphi^j)$, for some $0 \leq j \leq r - 1$.

Our main theorem is proved in a quite general form without imposing conditions on our target subsets and it provides us with an equivalence between an easier problem in $G \rtimes \mathbb{Z}$ and more complicated problems in G. However, as it will be made clear later, even when the target set K belongs to a well-behaved class of subsets, the subsets K_r can be wild, which makes it difficult to apply one of the directions in some cases. We obtain corollaries from both directions of this equivalence: proving $GBrCP(G)$ and $GTCP(G)$ to solve $GCP(G \rtimes \mathbb{Z})$ works better for recognizable and context-free subsets, while the converse works better for cosets of finitely generated subgroups, rational and algebraic subsets.

The main application concerns the class of virtually polycyclic groups. To do so, we prove the following theorem:

Theorem 5.2.1. Let G be a conjugate separable finitely presented group such that $MP_{f.g.}(G)$ is decidable. Then $GCP_{[f.g.\,coset]}(G)$ is decidable.

Using this connection and the fact that virtually polycyclic groups are conjugate separable, we obtain the following corollary:

Corollary 5.2.4. Let G be a virtually polycyclic group. Then $GBrCP_{[f.g.coset]}(G)$ and $GTCP_{[f.g.coset]}(G)$ are decidable.

This is the most general result on the $GBrCP$ to the knowledge of the author. Also, the solution of $GTCP$ seems to be new. However, the simple version of the twisted conjugacy problem is solvable even for general endomorphisms of polycyclic groups by [92].

Direct product of two free groups

Some algorithmic problems concerning endomorphisms of free-abelian times free groups $\mathbb{Z}^m \times F_n$ were solved in [36], namely computability of fixed subgroups and the Whitehead problems for endomorphisms, monomorphisms and automorphisms: $WhP_{\text{End}}(\mathbb{Z}^m \times F_n)$, $WhP_{\text{Mon}}(\mathbb{Z}^m \times F_n)$ and $WhP_{\text{Aut}}(\mathbb{Z}^m \times F_n)$.

The class of free times free groups, $F_n \times F_m$ can be algorithmically tricky due to some intricate structure of its subgroups. For instance, there is a finitely generated subgroup H of $F_2 \times F_2$ for which it is undecidable if a given element $g \in F_2 \times F_2$ belongs to H (see [76]). Surprisingly, endomorphisms of $F_n \times F_m$ can be described. Using that description, we can also solve the Whitehead problems and compute the fixed and periodic subgroups of endomorphisms.

Proposition 8.2.10. $WhP_{\text{Aut}}(F_n \times F_n)$, $WhP_{\text{Mon}}(F_n \times F_m)$ and $WhP_{\text{End}}(F_n \times F_m)$ are solvable.

Corollary 8.2.13. There is an algorithm which decides whether the fixed subgroup of a given endomorphism $\varphi \in \text{End}(F_n \times F_m)$ is finitely generated and computes a set of generators (recursively, in the infinite case).

1.2.3 Structural results on endomorphisms

Hyperbolic groups

For natural reasons, in order to study the dynamics of the continuous extension to a (compact) completion, in case the topology is defined by a given distance (as it is in the case of hyperbolic groups via visual metrics), it is of utmost importance to describe uniformly continuous endomorphisms for a certain class of groups, since those are precisely the ones for which a continuous extension exists.

Also, as said above, the bounded reduction property (BRP) introduced by Cooper [32] in the context of free group automorphisms has played an important role in that study. Another very interesting, although much more recent, tool is coarse-median preservation, which was introduced by Fioravanti in [44] and yielded important results in the context of RAAGs.

We started by defining and obtaining several geometric versions of the BRP for hyperbolic groups. Let H be a hyperbolic group and $\varphi : H \to H$ be a map. We say that *the BRP holds for* φ if, for every $p \geq 0$ there is some $q \geq 0$ such that: given two geodesics u and v such that

$$1 \xrightarrow{u} u \xrightarrow{v} uv$$

is a $(1, p)$-quasi-geodesic, we have that given any two geodesics α, β, from 1 to $u\varphi$ and from $u\varphi$ to $(uv)\varphi$, respectively, the path

$$1 \xrightarrow{\alpha} u\varphi \xrightarrow{\beta} (uv)\varphi$$

is a $(1, q)$-quasi-geodesic.

Theorem 6.2.7. Let $\varphi \in \mathrm{End}(H)$. The following conditions are equivalent:

1. The BRP holds for φ.
2. The BRP holds for φ when $p = 0$.
3. $\forall\, p > 0\, \exists\, q > 0\, \forall\, u, v \in H\; ((u|v) \leq p \implies (u\varphi|v\varphi) \leq q)$.
4. $\exists\, q > 0\, \forall\, u, v \in H\; ((u|v) = 0 \implies (u\varphi|v\varphi) \leq q)$.
5. there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x, y]$, we have that $\alpha\varphi$ is at Hausdorff distance at most N to every geodesic $[x\varphi, y\varphi]$.
6. there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x, y]$, we have that $\alpha\varphi \subseteq \mathcal{V}_N(\xi)$ for every geodesic $\xi = [x\varphi, y\varphi]$.
7. φ is coarse-median preserving.

A mapping $\varphi : (X, d) \to (X', d')$ between metric spaces satisfies a Hölder condition of exponent $r > 0$ if there exists a constant $K > 0$ such that

$$d'(x\varphi, y\varphi) \leq K(d(x, y))^r$$

for all $x, y \in X$. It clearly implies uniform continuity. Motivated by the possibility of defining new pseudometrics in the group of automorphisms of a hyperbolic group, the authors in [2] studied endomorphisms of hyperbolic groups satisfying a Hölder condition and conjectured that they were precisely the uniformly continuous ones. They proved the following:

Theorem 6.3.1 ([2], Araújo-Silva)**.** Let φ be a nontrivial endomorphism of a hyperbolic group G and let $d \in V^A(p, \gamma, T)$ be a visual metric on G. Then the following conditions are equivalent:

1. φ satisfies a Hölder condition with respect to d;
2. φ admits an extension to $\widehat{G}$ satisfying a Hölder condition with respect to $\widehat{d}$;
3. there exist constants $P > 0$ and $Q \in \mathbb{R}$ such that
$$P(g\varphi|h\varphi)_p^A + Q \geq (g|h)_p^A$$
for all $g, h \in G$;
4. φ is a quasi-isometric embedding of (G, d_A) into itself;
5. φ is virtually injective and $G\varphi$ is a quasiconvex subgroup of G.

Using the geometric versions of the BRP obtained, we can answer affirmatively to the conjecture by Araújo and Silva.

Theorem 6.3.5. Let $d \in V^A(p, \gamma, T)$ be a visual metric on H and let φ be an endomorphism of H. Then φ is uniformly continuous with respect to d if and only if the conditions from Theorem 6.3.1 hold.

We remark that this result, combined with previous work from Paulin suffices to show that every uniformly continuous endomorphism of a hyperbolic group has finitely generated fixed point subgroup.

Theorem 6.3.6. Let $\varphi \in \text{End}(H)$ be an endomorphism admitting a continuous extension $\hat{\varphi} : \widehat{H} \to \widehat{H}$ to the completion of H. Then, $\text{Fix}(\varphi)$ is finitely generated.

Automatic groups

Automatic groups were introduced in [42]. This class of groups can be hard to deal with, since it is very large. It contains the class of hyperbolic groups, but it is much larger. In particular, it is closed under taking direct products, which is very *anti-hyperbolic*. Indeed, a direct product of two finitely generated groups is hyperbolic if and only if one of them is finite and the other one is hyperbolic. Also, any group having $\mathbb{Z}^2$ as a subgroup cannot be hyperbolic. A natural subclass of automatic groups is the class of biautomatic groups. There are still some very natural questions about automatic groups yet to answer. For example, it is not known if every automatic group is biautomatic and if, whenever a direct product is automatic, then the direct factors are also automatic.

Despite being an interesting class with many developments and the fact that study of endomorphisms of groups plays an important role in the theory of finitely generated groups, not much has been done regarding the study of endomorphisms or automorphisms of automatic groups in its generality.

We adapt the geometric definition of the bounded reduction property proposed for the class of hyperbolic groups to the class of automatic groups. We also propose a *synchronous* version of the BRP and prove some technical results that help us prove the BRP for some endomorphisms. After all the technical work done, we focus on endomorphisms with an L-quasiconvex image and generalize a well-known result on hyperbolic groups to the class of automatic groups:

Theorem 7.3.8. Let G be an automatic group with an automatic stucture L for $\pi_1 : A^* \to G$ and φ be a virtually injective endomorphism with L-quasiconvex image. Then there is some automatic structure K such that the synchronous BRP holds for (φ, K, L).

After that, we apply the techniques developed in the previous section to obtain some finiteness results on equalizers (and so, on fixed subgroups) of endomorphisms of automatic groups:

Corollary 7.4.7. Let G_1 and G_2 be automatic groups with automatic structures L_1 and L_2 for $\pi_1 : A^* \to G_1$ and $\pi_2 : B^* \to G_2$, respectively. Let $\varphi, \psi : G_1 \to G_2$ be homomorphisms such that the synchronous BRP holds for (φ, L_1, L_2) and (ψ, L_1, L_2). Then $\mathrm{Eq}(\varphi, \psi) = \{x \in G_1 \mid x\varphi = x\psi\}$ is isomorphic to a $(L_2 \diamond L_2)$-quasiconvex subgroup of $G_2 \times G_2$. In particular, $\mathrm{Eq}(\varphi, \psi)$ is automatic.

Even though the hypotheses are quite strong, as we remark in Section 7.4 by proving exactly for which endomorphisms they hold in the case where the group is free and the structure considered is the structure of all geodesics, this result provides an alternative proof of a result from [46] which concerns the quasiconvexity of the centralizer of a finite subset:

Corollary 7.4.10. The centralizer of a finite subset of a biautomatic group is biautomatic.

We remark that although the results are obtained under strict hypotheses, very little was known about endomorphisms of automatic groups and this could be a first step in order to establishing a theory of endomorphisms of automatic groups.

Direct product of two free groups

Although the class of free groups and their endomorphisms is well known, some problems in the product $F_n \times F_m$ are not easily reduced to problems in each factor. In particular, when endomorphisms (and automorphisms) are considered, we have that many endomorphisms are not obtained by applying an endomorphism of F_n to the first component and one of F_m to the second, so some problems arise when the structure and dynamics of an endomorphism in the product are considered.

Similarly to what was done for free-abelian times free groups in [36], in which case there are two different types of endomorphisms, we describe endomorphisms of $F_n \times F_m$ and divide them in seven different types. With that classification done, we can describe the automorphism group of the direct product depending on the automorphism group of each factor.

Corollary 8.2.4. Let $m, n \neq 1$, $\theta_n \in \mathrm{Aut}(F_n \times F_n)$ be the involution defined by $(x, y) \mapsto (y, x)$. If $n \neq m$, then

$$\mathrm{Aut}(F_n \times F_m) \cong \mathrm{Aut}(F_n) \times \mathrm{Aut}(F_m).$$

If $n = m$, then $Aut(F_n \times F_n)$ is the semidirect product of $Aut(F_n) \times Aut(F_n)$ and $\langle \theta_n \rangle$.

After some work was done with respect to the structure of a fixed subgroup of an endomorphism for each of the seven types, we obtain a description of all possibilities:

Corollary 8.2.15. Let $\varphi \in \mathrm{End}(F_n \times F_m)$. Then:

1. if $\mathrm{Fix}(\varphi)$ is finitely generated, then $\mathrm{Fix}(\varphi)$ is a direct product of free groups of finite rank;

2. if $\mathrm{Fix}(\varphi)$ is not finitely generated, then there is some $K \in CF(F_m)$ such that $\mathrm{Fix}(\varphi) = \langle u \rangle \times K$, where u is the word given by the (type III) endomorphism φ.

1.2.4 Dynamical results on endomorphisms

We will study dynamics at the infinity in the following sense: we endow the group with a metric and complete the space (the completion, in our cases, will be a compact space). Then, we consider uniformly continuous endomorphisms with respect to that metric, which are precisely the ones admitting a continuous extension to the completion. Now, we view this extension as a dynamical system.

This setting includes the study of dynamics at the infinity of endomorphisms extended to the Gromov boundary of a hyperbolic group via metrization using a visual metric on the hyperbolic group. Many results have been obtained in this topic for hyperbolic and several other classes of groups (see, for example [8, 27, 33, 67, 98]).

Free-abelian times free groups

The class of free-abelian times free groups, despite looking relatively tame, has been the subject of some research, specially in the past few years, and presents some interesting behaviors [36–38, 93, 94].

In terms of endomorphisms, Delgado and Ventura proved in [36] that for $G = \mathbb{Z}^m \times F_n$, with $n \neq 1$, all endomorphisms of G are of one of the following forms:

(I) $\Psi_{\Phi,Q,P} : (a, u) \mapsto (aQ + \mathbf{u}P, u\Phi)$, where $\Phi \in End(F_n)$, $Q \in \mathcal{M}_m(\mathbb{Z})$, and $P \in \mathcal{M}_{n\times m}(\mathbb{Z})$.

(II) $\Psi_{z,\ell,h,Q,P} : (a, u) \mapsto (aQ + \mathbf{u}P, z^{a\ell^T + \mathbf{u}h^T})$, where $1 \neq z \in F_n$ is not a proper power, $Q \in \mathcal{M}_m(\mathbb{Z})$, $P \in \mathcal{M}_{n\times m}(\mathbb{Z})$, $\mathbf{0} \neq \ell \in \mathbb{Z}^m$, and $h \in \mathbb{Z}^n$,

where $\mathbf{u} \in \mathbb{Z}^n$ denotes the abelianization of the word $u \in F_n$.

The prefix metric on a free group is an ultrametric and its completion $(\hat{F}_n, \hat{d})$ is a compact space which can be described as the set of all finite and infinite reduced words on the alphabet $X \cup X^{-1}$. We will denote by ∂F_n the set consisting of only the infinite words and call it the *boundary* of F_n. Since the prefix metric is an example of a visual metric of the free group, $(\partial F_n, \hat{d})$ is homeomorphic to the Gromov boundary of F_n (and so to a Cantor set).

A free-abelian times free group is of the form $\mathbb{Z}^m \times F_n$, which we consider endowed with the product metric given by taking the prefix metric in each (free) component, i.e.,

$$d((a,u),(b,v)) = \max\{d(a_1,b_1),\dots,d(a_m,b_m),d(u,v)\},$$

where a_i and b_i denote the i-th component of a and b, respectively. This metric is also an ultrametric and $\widehat{\mathbb{Z}^m \times F_n}$ is homeomorphic to $\hat{\mathbb{Z}}^m \times \hat{F}_n$ by uniqueness of the completion (Theorem 24.4 in [104]). When seen as a CAT(0) cube complex, or alternatively, as a median algebra, this coincides with the Roller compactification (see [15, 29, 43, 44, 91]).

We focus our work on the uniformly continuous endomorphisms and their extensions to the completion. It is well known that for a free group F_n endowed with the prefix metric, an endomorphism $\varphi \in \mathrm{End}(F_n)$ is uniformly continuous if and only if it is either constant or injective. Characterizing and studying some properties of uniformly continuous endomorphisms has been done before for other classes of groups (see for example [28], [97], [98], [2]). An interesting property of this metric (and so, of this boundary) is that the uniformly continuous endomorphisms of $\mathbb{Z}^m \times F_n$ for this metric d are precisely the coarse-median preserving endomorphisms for the product coarse median obtained by taking the median operator induced by the metric ℓ_1 in $\mathbb{Z}^m$ and the one given by hyperbolicity of F_n:

Theorem 8.1.10. An endomorphism $\varphi \in \mathrm{End}(\mathbb{Z}^m \times F_n)$ is uniformly continuous with respect to the product metric d obtained by taking the prefix metric in each direct factor if and only if it is coarse-median preserving for the product coarse median μ obtained by taking the median operator μ_1 induced by the metric ℓ_1 in $\mathbb{Z}^m$ and the coarse median operator μ_2 given by hyperbolicity in F_n.

In [98], the author, considering automorphisms of virtually free groups, proves that infinite fixed points that belong to the topological closure $(\mathrm{Fix}(\varphi))^c$ are never attractors nor repellers, while infinite fixed points not belonging to $(\mathrm{Fix}(\varphi))^c$ must always be either attractors or repellers, generalizing a previously known result for free groups (see [32]).

Fixed points act naturally on infinite fixed points by left multiplication. It is known that, for free groups, the number of $(\mathrm{Fix}(\varphi))$-orbits of the set of attracting fixed points is finite (see [32]) and that is used to define the index of an automorphism in [33]. A similar result is obtained for virtually free groups in [98]. As highlighted in Section 8.1.3, a similar result cannot be obtained in general in this case. However, when φ is a type II uniformly continuous endomorphism, then infinite fixed (resp. periodic) points have $\sum\limits_{i=0}^{m} 2^i \binom{m}{i}$ $\mathrm{Fix}(\varphi)$-orbits (resp. $\mathrm{Per}(\varphi)$-orbits). This result can be seen as some sort of infinite version of Proposition 6.2 in [36].

We also classify infinite fixed points of a uniformly continuous automorphism of $\mathbb{Z}^m \times F_n$ in attractors or repellers, using the classification obtained for free groups:

Corollary 8.1.19. Let $\varphi \in Aut(\mathbb{Z}^m \times F_n)$ be a uniformly continuous automorphism such that $(a,u)\hat{\varphi} = (a\hat{\varphi}_1, b\hat{\phi})$, where φ_1 is given by a uniform matrix and $\phi \in Aut(F_n)$. Then a regular infinite fixed point $(a, u) \in \mathrm{Fix}(\hat{\varphi}) \setminus \mathrm{Fix}(\varphi)^c$ is an attractor (resp. repeller) if and only if $a \in Fix(\varphi_1)$ and u is an attractor (resp. repeller) for $\hat{\phi}$.

In the study of dynamical systems, the notion of *ω-limit set* plays a crucial role. Given a metric space X, a continuous function $f : X \to X$, and a point $x \in X$, the ω-limit set $\omega(x, f)$ of x consists of the accumulation points of the sequence of points in the orbit of x. Understanding the ω-limits gives us a grasp on the behavior of the system in the long term. If the space X is compact, then ω-limit sets are nonempty, compact and f-invariant.

In [67], the authors proved that in the case where f is the extension of a free group automorphism to the completion, then, for every point $x \in \hat{F}_n$, $\omega(x, f)$ is a periodic orbit. We prove that, for uniformly continuous automorphisms (i.e., the ones that extend to the completion) of $\mathbb{Z}^m \times F_n$ something stronger holds. Informally, a point is said to be *wandering* if it has some neighborhood such that, from some point on, its points leave the neighborhood and don't come back. Obviously, a wandering point cannot belong to an ω-limit set. We prove that, for a uniformly continuous automorphism, every point in the completion must be either periodic or wandering, showing that non-periodic points, when iterated long enough wander away from some neighborhood *carrying* the neighborhood with them. In particular, since ω-limit sets are nonempty, they must be periodic orbits.

Corollary 8.1.32. Let φ be a uniformly continuous automorphism of $\mathbb{Z}^m \times F_n$ defined by $(a, u) \mapsto (a\varphi_1, u\phi)$, where $\varphi_1 \in Aut(\mathbb{Z}^m)$ is given by a uniform matrix and $\phi \in Aut(F_n)$. Consider $\hat{\varphi}$, its continuous extension to the completion. Then every point $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ is either wandering or periodic.

In Section 8.1.4, we prove the same for type II endomorphisms:

Theorem 8.1.37. Let $\varphi \in End(\mathbb{Z}^m \times F_n)$ be a type II uniformly continuous endomorphism and consider $\hat{\varphi}$, its continuous extension to the completion. Then every point $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ is either wandering or periodic.

Direct product of two free groups

In the case of groups of the form $F_n \times F_m$, we have a classification of endomorphisms in seven different types. However, only types IV, VI and VII (see Subsection 8.2.1) can be uniformly continuous with respect to our metric.

For type IV endomorphisms, the nontrivial dynamical situation occurs for endomorphisms φ of the form $(u, v) \mapsto (v\phi, v\psi)$, where $\phi \in F_m \to F_n$ and $\psi \in End(F_m)$ are injective homomorphisms. In this case,

$$\mathrm{Fix}(\hat{\varphi}) = \{(v\hat{\phi}, v) \mid v \in \mathrm{Fix}(\hat{\psi})\}$$

and we can classify all the attractors in $\hat{\varphi}$ (repellers are not considered, because type IV endomorphisms are never invertible and so repellers are not well defined).

Proposition 8.2.22. Let $v \in \mathrm{Reg}(\hat{\psi})$. Then $(v\hat{\phi}, v)$ is an attractor if and only if v is an attractor for $\hat{\psi}$. Moreover, if $\alpha \in \mathrm{Sing}(\hat{\varphi})$, then α is not an attractor.

Automorphisms of type VI are of the form $(u, v) \mapsto (u\phi, u\psi)$, where $\phi \in \mathrm{Aut}(F_n)$ and $\psi \in \mathrm{Aut}(F_m)$. We also describe attractors, repellers and infinite fixed points that are not attractors nor repellers in this case.

Proposition 8.2.23. An infinite fixed point $\alpha = (u, v)$, where $u \in \mathrm{Fix}(\hat{\phi})$ and $v \in \mathrm{Fix}(\hat{\psi})$ is an attractor if and only if u and v are attractors for $\hat{\phi}$ and $\hat{\psi}$, respectively. If, additionally, φ is an automorphism, the same holds for repellers.

Automorphisms of type VII exist only when $m = n$, and so when the group has the form $F_n \times F_n$, and they are of the form $(u, v) \mapsto (v\psi, u\phi)$, where $\phi, \psi \in \mathrm{Aut}(F_n)$. In this case, we can prove that every regular fixed point is either an attractor or a repeller.

Corollary 8.2.27. Let φ be a type VII automorphism of $F_n \times F_n$. Then $\alpha \in \mathrm{Reg}(\hat{\varphi})$ is either an attractor or a repeller.

Chapter 2

Preliminaries

2.1 Formal languages and automata

An *alphabet* is just a set. To our purpose, alphabets will be finite, but they do not need to be in general. To the elements of an alphabet we call *letters* and *words* are tuples of letters, so a word of length $k > 0$ on the alphabet A is simply an element of A^k. We will also consider the *empty word* to be a word of length 0 and denote it by ε. The length of a word w will be denoted by $|w|$ and the number of occurrences of a letter a in a word w will be denoted by $n_a(w)$. The set of all words over an alphabet A is the *free monoid over A*

$$A^* = \{\varepsilon\} \cup \bigcup_{k>0} A^k,$$

when endowed with the operation given by concatenation: the product of two words u and v, uv, is the word we obtain by juxtaposing v after u. A *language* is simply a set of words, or equivalently, a subset of A^*.

Let A be a finite alphabet. An A-*automaton* is a tuple $\mathcal{A} = (Q, q_0, \delta, F)$, where

- Q is a set, called the set of *states*;
- $q_0 \in Q$ is the *initial state*;
- δ is called the *transition function*, and it is a function $\delta : Q \times A \to \mathcal{P}(Q)$;
- $F \subseteq Q$ is the set of *final states.*

If, for all $a \in A$ and $q \in Q$, $|\delta(q, a)| \leq 1$, the automaton is said to be *deterministic*. We say that an automaton is *finite* when Q is finite. Typically, an automaton is represented by a labelled directed graph, where Q is the vertex set and there is an edge labelled by $a \in A$ from p to q if and only if $q \in \delta(p, a)$. The vertex corresponding to the initial state is represented having an incoming arrow and the vertices corresponding to final states are represented with outgoing arrows. A *path* in $\mathcal{A}$ is a sequence

$$q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} q_n$$

where $q_{i+1} \in \delta(q_i, a_i)$, for all $i \in \{1, \cdots, n-1\}$. The *label* of the path is $a_1 \cdots a_{n-1} \in A^*$ and a path is said to be *successful* if the first vertex is q_0 and the last vertex belongs to F. We also consider paths of the form $q \xrightarrow{\varepsilon} q$, for all $q \in Q$, which we call *trivial*. The trivial path can only be successful for $q = q_0$ if $q_0 \in F$.

The *language* $L = L(\mathcal{A})$ *recognized by* $\mathcal{A}$ is the set of labels of successful paths in $\mathcal{A}$ and a language is said to be *rational* if it is recognized by some finite automaton. We can impose that the automata are deterministic, as that does not change the class of languages accepted.

Given a language L, we define L^* as the set of words over L, including the empty string, i.e.

$$L^* = \{\varepsilon\} \cup \bigcup_{k>0} L^k.$$

To the unary operator mapping L to L^* we call the *star operator*. Similarly, we define the *plus operator* analogously, but omitting the empty word, i.e.,

$$L^+ = \bigcup_{k>0} L^k.$$

The *reversal of a word* $w = w_1 \cdots w_n$, where w_i is a letter, for $i \in [n]$, is $w^r = w_n \cdots w_1$ and we define the *reversal of a language* L as

$$L^r = \{w^r \mid w \in L\}.$$

Given a class of languages $\mathcal{C}$, we say that $\mathcal{C}$ is *closed under morphism*, if given a language $L \subseteq A^*$ belonging to $\mathcal{C}$ and a monoid homomorphism $\varphi : A^* \to B^*$, we have that $L\varphi \in \mathcal{C}$. We say that $\mathcal{C}$ is *closed under inverse morphism*, if given a language $L \subseteq B^*$ in $\mathcal{C}$ and a monoid homomorphism $\varphi : A^* \to B^*$, we have that $L\varphi^{-1} = \{w \in A^* \mid w\varphi \in L\} \in \mathcal{C}$.

Kleene's theorem gives us another definition of rational languages.

Theorem 2.1.1. *[6] The family of rational languages over A is equal to the least family of languages over A containing the empty set and the singletons, and closed under union, product and the star operation.*

We also have the following closure properties for rational languages.

Proposition 2.1.2. *[6] Rational languages are closed under union, product, the star and the plus operation, intersection, complementation, reversal, morphism and inverse morphism.*

So far, we have two natural ways of proving that a certain language L is rational: we can either construct a finite automaton recognizing L or write L recursively starting with singletons and the empty set using only union, product and star finitely many times. We now present one of the main tools to prove that a certain language is not rational, the *pumping lemma for rational languages*.

Theorem 2.1.3 (Pumping lemma for rational languages)**.** *Let L be a rational language. Then there is an integer $p \geq 1$ such that every word $w \in L$ of length at least p admits a decomposition of the form $w = xyz$ such that $|y| \geq 1$, $|xy| \leq p$ and, for all $n \geq 0$, $xy^n z \in L$.*

Rational languages constitute the smallest class of languages in the Chomsky hierarchy. We will now define the class of languages in the second level of the hierarchy, *context-free* languages.

A *context-free grammar* is a tuple (V, A, P, S) where

- V is a finite set of *variables*;
- A is a set of *terminal symbols* disjoint from V;
- P is a finite subset of $V \times (V \cup A)^*$. An element of R is a *production*;
- $S \in V$ is the *starting symbol*.

A production (U, V) is often denoted by $U \to V$ and usually, when there are two or more productions with the same left-hand side, we denote them in the same line separated by a bar. So, for example, we write $U \to V \mid W$ to denote the productions $U \to V$ and $U \to W$.

Given $U, V \in (V \cup A)^*$, we write $U \Rightarrow V$ if there are $X, Y \in (V \cup A)^*$ and a production (Z, Z') such that $U = XZY$ and $V = XZ'Y$. We call $U \Rightarrow V$ a *derivation*. Also, we say that $U \Rightarrow^* V$ if there is some chain of derivations of the form

$$U = X_0 \Rightarrow X_1 \Rightarrow \cdots \Rightarrow X_m = V.$$

The *language of a context-free grammar G*, $L(G)$ is simply

$$L(G) = \{w \in A^* \mid S \Rightarrow^* w\}$$

and a language is *context-free* if it is the language of some context-free grammar.

Similarly to what happens with rational languages, automata are also a suitable model to define context-free languages.

Given an (infinite) set X, we define $\mathcal{P}_{fin}(X)$ to be the set of finite subsets of X.

An *A-nondeterministic pushdown automaton* $\mathcal{M}$ is a tuple $(Q, \Gamma, \delta, q_0, z, F)$, where

- Q is a set of *states*;
- Γ is the *stack alphabet*;
- $\delta : Q \times (A \cup \{\varepsilon\}) \times \Gamma \to \mathcal{P}_{fin}(Q \times \Gamma^*)$ is the *transition function*;
- $q_0 \in Q$ is the *initial state*;
- $z \in \Gamma$ is the *stack starting symbol*;
- $F \subseteq Q$ is the set of *final states*.

A triple $(p, w, x) \in Q \times A^* \times \Gamma^*$ is called an instantaneous description of the pushdown automaton. The transition function δ defines *moves* between two instantaneous descriptions, which will be denoted by $\vdash$. So $(q_1, aw, bx) \vdash (q_2, w, yx)$ if and only if $(q_2, y) \in \delta(q_1, a, b)$. Again, we denote moves involving finitely many steps by $\vdash^*$ and the language accepted by the automaton $\mathcal{M}$ is

$$L(M) = \{w \in A^* \mid (q_0, w, z) \vdash^* (p, \varepsilon, u),\, p \in F,\, u \in \Gamma^*\}.$$

Context-free languages are precisely the ones accepted by nondeterministic pushdown automata.

Proposition 2.1.4. *[6] Context-free languages are closed under union, product, star operation, reversal, morphism, inverse morphism and intersection with rational languages.*

Similarly to what happens in the rational case, there is also a version of the *pumping lemma for context-free languages*.

Theorem 2.1.5 (Pumping lemma for context-free languages)**.** *Let L be a context-free language. Then there is an integer $p \geq 1$ such that every word $w \in L$ of length at least p admits a decomposition of the form $w = uvzxy$ such that $|vx| \geq 1$, $|vzx| \leq p$ and, for all $n \geq 0$, $uv^n zx^n y \in L$.*

We denote by $\mathbb{N}$ the set of nonnegative integers. A subset of $\mathbb{N}^k$ is *linear* if it can be written as

$$u_0 + \mathbb{N}u_1 + \cdots + \mathbb{N}u_m,$$

for some $u_0, \cdots, u_m \in \mathbb{N}^k$ and a subset of $\mathbb{N}^k$ is *semilinear* if it is a union of finitely many linear sets.

In addition to the pumping lemma, another useful method to show that a certain language is not context-free is Parikh's theorem.

Write $A = \{a_1, \ldots, a_k\}$. We define the *Parikh function* $p : A^* \to \mathbb{N}^k$ by

$$w \mapsto (n_{a_1}(w), \ldots, n_{a_k}(w))$$

and the *Parikh vector* of a word w is $p(w)$.

Theorem 2.1.6 ([86], Parikh's theorem)**.** *Let L be a context-free language over $A = \{a_1, \ldots, a_k\}$. Then $P(L) = \{p(w) \in \mathbb{N}^k \mid w \in L\}$ is a semilinear subset of $\mathbb{N}^k$.*

We now present a classical example of a context-free language that is not rational.

Example 2.1.7. Let $A = \{a, b\}$ and $L = \{a^n b^n \in A^* \mid n \in \mathbb{N}\}$. It is easy to see that L is the language of the grammar $(\{S\}, A, P, S)$ with the following set of productions P:

$$S \to aSb \mid \varepsilon.$$

Hence, L is context-free. However, using the pumping lemma for rational languages we can see that L is not rational. Suppose it is and let p be the constant given by the pumping lemma. Then take the word $w = a^p b^p$ and write it as $w = xyz$ such that $|y| \geq 1$, $|xy| \leq p$ and, for all $n \geq 0$, $xy^n z \in L$. Since $|xy| \leq p$, the only letter occurring in xy is a, and since $|y| \geq 1$, it means that $y = a^k$ for some $k > 0$. But then $xy^2z = a^{p+k}b^p \notin L$, which is a contradiction.

2.2 Free groups and generalizations

2.2.1 Free groups

We will denote the *free group with basis* X by F_X and think of it as the set of reduced words over $X \cup X^{-1}$ with the operation of concatenation (followed by reduction). Given a word w on $X \cup X^{-1}$, we will denote its reduced form by $\overline{w}$. Occasionally, we will denote the free group with basis X by $\langle X | \rangle$, meaning that it is the group generated by X with no relations. Since two free groups are isomorphic if and only if they have the same rank, we will often refer to *the* free group of rank n and denote it by F_n. In general, the *rank* of a finitely generated group G is the cardinality of a minimal set of generators for G and will be denoted by $\text{rk}(G)$. Also, for a finitely generated group $G = \langle A \rangle$, we will denote its Cayley graph with respect to A by $\Gamma_A(G)$.

Given two words u and v in a free group, we write $u \wedge v$ to denote the longest common prefix of u and v. The prefix metric on a free group is defined by

$$d(u,v) = \begin{cases} 2^{-|u \wedge v|} \text{ if } u \neq v \\ 0 \text{ otherwise} \end{cases}.$$

The prefix metric on a free group is in fact an ultrametric and its completion $(\hat{F}_n, \hat{d})$ is a compact space which can be described as the set of all finite and infinite reduced words on the alphabet $A \cup A^{-1}$ (see [28]). We will denote by ∂F_n the set consisting of only the infinite words and call it the *boundary* of F_n. For free groups, it is known that an endomorphism $\varphi \in End(F_n)$ is uniformly continuous if and only if it is either trivial or injective [28, Corollary 8.5].

A reduced word $z = z_1 \cdots z_n$, with $z_i \in A \cup A^{-1}$, is said to be *cyclically reduced* if $z = 1$ or $z_1 \neq z_n^{-1}$. Every word admits a decomposition of the form $z = w\tilde{z}w^{-1}$, where $\tilde{z}$ is cyclically reduced. The word $\tilde{z}$ is called the *cyclically reduced core* of z.

When a subgroup $H \leq G$ has a special property, we denote it as a subscript. For instance, a finitely generated (resp. finite index) subgroup will be denoted by $H \leq_{f.g.} G$ (resp. $H \leq_{f.i.} G$).

A group is said to be *Howson* if the intersection of two finitely generated subgroups is again finitely generated. We remark that free groups are Howson [58] and so are finite extensions of Howson groups. Indeed, if we have a Howson group H such that $H \leq_{f.i.} G$, then taking $K_1, K_2 \leq_{f.g.} G$, we have that

$$K_1 \cap K_2 \cap H = (K_1 \cap H) \cap (K_2 \cap H).$$

Since, for $i = 1, 2$, $K_i \cap H \leq_{f.i.} K_i$ and K_i is finitely generated, then so is $K_i \cap H$. Since H is Howson, we have that $K_1 \cap K_2 \cap H$ is finitely generated and has finite index in $K_1 \cap K_2$. Hence $K_1 \cap K_2$ is finitely generated too.

A group is said to be *hopfian* if it is not isomorphic to any proper quotient of itself or, equivalently, if any surjective endomorphism of G is an automorphism. A group is said to be *cohopfian* if it is not isomorphic to any of its proper subgroups or, equivalently, if any injective endomorphism of G is an automorphism. It is well known that free and free-abelian groups are hopfian and not cohopfian. By [56, Corollary 2], finite extensions of hopfian groups are hopfian.

We now present two classical results concerning subgroups of free groups that will be important later.

Theorem 2.2.1 ([71], Nielsen-Schreier theorem)**.** *Every subgroup of a free group is free.*

Theorem 2.2.2 ([71], Marshall Hall's theorem)**.** *Let H be a finitely generated subgroup of a free group F. Then there is some finite index subgroup $H' \leq_{f.i.} F$ such that H is a free factor of H'.*

Endomorphisms of the free group have been studied by many authors through the years. A very important topic in this subject concerns the subgroup fixed by an endomorphism, $\text{Fix}(\varphi)$. In general, given an endomorphism $\varphi \in \text{End}(G)$, an element $x \in G$ is said to be a *fixed point* if $x\varphi = x$. The collection of all fixed points forms a subgroup, which we call the *fixed subgroup* of φ. A point $x \in G$ is said to be a *periodic point* if there is some $m > 0$ such that $x\varphi^m = x$. The set of all periodic points forms a subgroup, which we denote by $\text{Per}(\varphi)$. Obviously, we have that

$$\text{Per}(\varphi) = \bigcup_{k=1}^{\infty} \text{Fix}(\varphi^k).$$

Given $x \in G$, the *orbit of x through φ* is defined by

$$\text{Orb}_\varphi(x) = \{x\varphi^k \mid k \in \mathbb{N}\}.$$

The following theorem is a combination of the work of many ([8, 11, 32, 47, 60, 79]).

Theorem 2.2.3. *Let $\varphi \in \text{End}(F_n)$. Then* $\text{Fix}(\varphi)$ *is finitely generated,* $\text{rk}(\text{Fix}(\varphi)) \leq n$ *and a basis for* $\text{Fix}(\varphi)$ *is computable.*

In order to study the dynamics of the continuous extension of an endomorphism to a (compact) completion (as in this case, using the prefix metric), we must work with uniformly continuous endomorphisms, since those are precisely the ones for which a continuous extension exists. Indeed, it is well known by a general topology result [41, Section XIV.6] that every uniformly continuous mapping φ between metric spaces admits a unique continuous extension $\hat{\varphi}$ to the completion. The converse is obviously true by compactness: if a mapping between metric spaces admits a continuous extension to the completion, since the completion is compact, the extension must be uniformly continuous, and so does the restriction to the original mapping.

In terms of dynamics, we will focus on the dynamical behavior around *infinite* fixed and periodic points, which are fixed and periodic points belonging to the boundary of the group.

An infinite fixed point is said to be *singular* if it belongs to the topological closure $(\mathrm{Fix}(\varphi))^c$ of $\mathrm{Fix}(\varphi)$ and *regular* if it doesn't. We denote by $\mathrm{Sing}(\hat{\varphi})$ (resp. $\mathrm{Reg}(\hat{\varphi})$) the set of all singular (resp. regular) infinite fixed points of $\hat{\varphi}$.

Definition 2.2.4. An infinite fixed point $\alpha \in \mathrm{Fix}(\hat{\varphi})$ is

- an *attractor* if
$$\exists \varepsilon > 0\ \forall \beta \in \widehat{F_n}\ \left(d(\alpha, \beta) < \varepsilon \implies \lim_{n \to +\infty} \beta \hat{\varphi}^n = \alpha \right)$$

- a *repeller* if
$$\exists \varepsilon > 0\ \forall \beta \in \widehat{F_n}\ \left(d(\alpha, \beta) < \varepsilon \implies \lim_{n \to +\infty} \beta \hat{\varphi}^{-n} = \alpha \right)$$

The notion of a repeller will only be considered for automorphisms because it assumes the existence of an inverse. We will also consider the definition of an attractor in the cases where φ is not an automorphism but still admits an extension to the completion. In [33], the authors prove that an infinite fixed point of a free group automorphism is regular if and only if it is either an attractor or a repeller.

2.2.2 Virtually free groups

Given a property P, a group is said to be *virtually* P if it has a subgroup of finite index for which the property P holds. In particular, a group is said to be *virtually free* if it has a free subgroup of finite index, and so virtually free groups are a natural generalization of free groups that have been investigated by many over the years. There are many characterizations of virtually free groups with very different natures. We will enumerate some, but several others can be found in [1, 40].

Let G be a finitely generated group and A be a finite generating set. Then G is virtually free if and only if:

- $\Gamma_A(G)$ is quasi-isometric to a tree [48, Proposition 7.19];
- G has a context-free word problem [78];
- G is the fundamental group of a finite graph of finite groups [64];
- there exist some finite generating set B of G and some $k \geq 0$ such that every k-locally geodesic in $\Gamma_B(G)$ is a geodesic [49];
- G is polygon hyperbolic [3].

Since a finite index subgroup must contain a finite index normal subgroup and, by the Nielsen-Schreier theorem, subgroups of free groups are free, then a virtually free group must contain a free normal subgroup of finite index.

Endomorphisms of virtually free groups have also been a source of interest. For example, in [98], Silva studied endomorphisms of finitely generated virtually free groups using language theoretical techniques and proved that the subgroup of points fixed by an endomorphism of a finitely generated virtually free group is finitely generated itself (this is also a consequence of a previous result in [99]). Moreover, when endowing the virtually free group with a suitable metric (given in a concrete way), the uniformly continuous endomorphisms are precisely the trivial and the *virtually injective* ones. By a virtually injective endomorphism, we mean one with a finite kernel. Finally, Silva generalized the result in [33] to virtually free groups, proving that an infinite fixed point of a free group automorphism is regular if and only if it is either an (exponentially stable) attractor or an (exponentially stable) repeller.

2.3 Subsets of groups

The study of subgroups of groups is a classical topic in group theory. Subsets of groups, on the other hand, have only been considered more recently. Despite not having a natural algebraic description and having essentially no algebraic structure, sometimes, when they are defined by a suitable language theoretical condition, very interesting things can be said about them, and new (sometimes very useful) tools arise.

Let $G = \langle A \rangle$ be a finitely generated group, A be a finite generating set, A^{-1} be a set of formal inverses and $\tilde{A} = A \cup A^{-1}$. There is a canonical (surjective) homomorphism $\pi : \tilde{A}^* \to G$ mapping $a \in \tilde{A}$ (resp. $a^{-1} \in \tilde{A}$) to $a \in G$ (resp. $a^{-1} \in G$).

A subset $K \subseteq G$ is said to be *rational* if there is some rational language $L \subseteq \tilde{A}^*$ such that $L\pi = K$ and *recognizable* if $K\pi^{-1}$ is rational. We will denote by $Rat(G)$ and $Rec(G)$ the classes of rational and recognizable subsets of G, respectively.

As said above, these subsets do not need to have an algebraic structure. For example, every finite subset is rational, and one can take finite subsets that are not closed for the group operation, do not contain the identity, or inverses for some of the elements in the subset. So they can be far from being a subgroup. However, subgroups are also subsets and rational subsets generalize the notion of finitely generated subgroups.

Theorem 2.3.1. *[6, Theorem III.2.7] Let H be a subgroup of a group G. Then $H \in Rat(G)$ if and only if H is finitely generated.*

It is important to highlight that, in case we can decide whether an element belongs to the finitely generated subgroup H or not, the above theorem is constructive in the sense that, given a finite set of generators for a subgroup H, we can construct an automaton $\mathcal{A}$ such that $L(\mathcal{A})\pi = H$ and, conversely, given a finite automaton, we can construct a finite set of generators for $L(\mathcal{A})\pi$. In particular, for a finitely generated virtually free group, having an automaton recognizing a language L such that $L\pi = H$ for a finitely generated subgroup H is equivalent to having a finite set of generators for H, since the rational subset membership is decidable.

We remark that this a significant generalization, meaning that, in general, there are *many more* rational subsets than finitely generated subgroups.

For example, a nontrivial finite group of order n can be generated by a set S such that $|S| \leq \log_2(n)$. Indeed, let $s_1 \neq 1$ and put $G_1 = \langle s_1 \rangle$. Then, if G_i is defined and $G_i \neq G$, we choose $s_{i+1} \notin G_i$ and define $G_{i+1} = \langle G_i \cup \{s_{i+1}\} \rangle$. Since $s_{i+1} \notin G_i$, the cosets G_i and $s_{i+1}G_i$ are disjoint, and so $|G_{i+1}| \geq 2|G_i|$. Let $k = \lfloor \log_2(n) \rfloor$. If $G_i = G$ for some $i < k$, we are done since G_i is generated by i elements. If not, then $|G_k| \geq 2^k > \frac{n}{2}$, which, by Lagrange Theorem, implies that $G_k = G$. Hence, a (very rough) bound on the number of subgroups of a finitely generated group of order n is $1 + n^{\log_2(n)}$ (we added 1 for the trivial subgroup). Better bounds can be found in the literature (see [12, Corollary 1.6] for example). In a finite group, every subset is rational, so the number of rational subsets of a group of order n is 2^n. This way, we have that the probability that a rational subset of a group of order n is a subgroup is bounded above by $\frac{1+n^{\log_2(n)}}{2^n}$ and

$$\lim_{n\to\infty} \frac{1 + n^{\log_2(n)}}{2^n} = 0,$$

so for large enough finite groups, the number of finitely generated subgroups is very small when compared to the number of rational subsets.

Similarly, recognizable subsets generalize the notion of finitely index subgroups.

Proposition 2.3.2. *[6, Exercise III.1.3] Let H be a subgroup of a group G. Then $H \in Rec\ G$ if and only if H has finite index in G.*

In fact, it can be proved that if G is a group and K is a subset of G then K is recognizable if and only if K is a (finite) union of cosets of a subgroup of finite index. Similarly to the finitely generated case, if we can decide membership on finite index subgroups, this characterization of recognizable subsets is algorithmic. Also, using decidability of the membership problem for finite index subgroups, if given a recognizable subset as a union of cosets of a finite index subgroups $K = \cup_{i\in[k]} Hb_i$, we can compute a finite index normal subgroup N contained in H and write H as a disjoint union of cosets of N, so we can (algorithmically) write K as a union of cosets of a finite index normal subgroup N.

In case the group G is a free group with basis A, we define the set of reduced words of $L \subseteq \tilde{A}^*$ by

$$\overline{L} = \{\bar{w} \mid w \in L\}.$$

Benois' Theorem provides us with a useful characterization of rational subsets in terms of reduced words representing the elements in the subset.

Theorem 2.3.3 ([5], Benois' theorem)**.** *Let F be a free group with basis A. Then, a subset of $\overline{\tilde{A}^*}$ is a rational language of $\tilde{A}^*$ if and only if it is a rational subset of F.*

A natural generalization of these concepts concerns the class of context-free languages. A subset $K \subseteq G$ is said to be *algebraic* if there is some context-free language $L \subseteq \tilde{A}^*$ such that

$L\pi = K$ and *context-free* if $K\pi^{-1}$ is context-free. We will denote by $Alg(G)$ and $CF(G)$ the class of algebraic and context-free subsets of G, respectively. It follows from [53, Lemma 2.1] that $CF(G)$ and $Alg(G)$ do not depend on the alphabet $\tilde{A}^*$ or the surjective homomorphism π.

It is obvious from the definitions that $Rec(G)$, $Rat(G)$, $CF(G)$ and $Alg(G)$ are closed under union, since both rational and context-free languages are closed under union. The intersection case is distinct: from the fact that rational languages are closed under intersection, it follows that $Rec(G)$ must be closed under intersection too. However $Rat(G)$, $Alg(G)$ and $CF(G)$ might not be. Another important closure property is given by the following lemma from [53].

Lemma 2.3.4. *[53] Let G be a finitely generated group, $R \in Rat(G)$ and $C \in \{Rat, Rec, Alg, CF\}$. If $K \in C(G)$, then $KR, RK \in C(G)$.*

This lemma will be used often in the particular case where R is a singleton.

The following lemma concerning the context-free subsets of a finitely generated subgroup was proved by Herbst and will be useful to us.

Lemma 2.3.5. *[53, Corollary 4.4 (b)] Let G be a finitely generated group and H be a finitely generated subgroup of G. Then*

$$K \in CF(G) \implies K \cap H \in CF(H).$$

A very similar property is also known for recognizable subsets.

Lemma 2.3.6. *[6, III, Exercise 1.1] Let G be a finitely generated group and H be a finitely generated subgroup of G. Then*

$$K \in Rec(G) \implies K \cap H \in Rec(H).$$

For a finitely generated group G, it is immediate from the definitions that

$$Rec(G) \subseteq CF(G) \subseteq Alg(G)$$

and that

$$Rec(G) \subseteq Rat(G) \subseteq Alg(G).$$

It is proved in [53] that

$$CF(G) = Alg(G) \iff CF(G) = Rat(G) \iff \text{G is virtually cyclic.} \tag{2.1}$$

However, there is no general inclusion between $Rat(G)$ and $CF(G)$. For example, if G is virtually abelian, then $CF(G) \subseteq Alg(G) = Rat(G)$ (and the inclusion is strict if the group is not virtually cyclic) and if the group is virtually free, then $Rat(G) \subseteq CF(G)$ (see [53, Lemma 4.2]).

In the free group case, Herbst proved an analogue of Benois' Theorem for context-free subsets, proving that for a subset $K \subseteq F_n$, then $K \in CF(F_n)$ if and only if the set of reduced words representing elements of K is context-free.

Lemma 2.3.7. *[53, Lemma 4.6] Let F be a finitely generated free group with basis A. Then, a subset of $\overline{\tilde{A}^*}$ is a context-free language of $\tilde{A}^*$ if and only if it is a context-free subset of F.*

It follows from the definition of a context-free subset and the fact that we can decide membership of a word in a context-free language that we can decide the membership problem in a context-free subset of a group G.

2.4 Algorithmic problems

Let G be a finitely presented group, A be a finite set of generators and $\pi : \tilde{A}^* \to G$ be the canonical surjective homomorphism. We now list some algorithmic problems that will be approached in this thesis. When we say that we take an endomorphism as an input, we mean that we are given images for the generators of the group.

- $WP(G)$ **- word problem:** taking as input a word w over $\tilde{A}$, decide whether $w\pi = 1$;
- $CP(G)$ **- conjugacy problem:** taking as input $x, y \in G$, decide whether $x \sim y$, i.e., if there is some $z \in G$ such that $y = z^{-1}xz$, in which case we say that x and y are *conjugate*;
- $IP(G)$ **- intersection problem:** taking as input two subsets $K_1, K_2 \subseteq G$, decide if they intersect or not;
- $TCP(G)$ **- twisted conjugacy problem:** taking as input an endomorphism $\varphi \in \text{End}(G)$ and two elements $x, y \in G$, decide whether x and y are φ-twisted conjugate, i.e., whether there is some $z \in G$ such that $y = (z^{-1}\varphi)xz$;
- $BrCP(G)$ **- Brinkmann's conjugacy problem:** taking as input an endomorphism $\varphi \in \text{End}(G)$ and two elements $x, y \in G$, decide whether there is some $n \in \mathbb{N}$ such that $x\varphi^n \sim y$;
- $BrP(G)$ **- Brinkmann's (equality) problem:** taking as input an endomorphism $\varphi \in \text{End}(G)$ and two elements $x, y \in G$, decide whether there is some $n \in \mathbb{N}$ such that $x\varphi^n = y$;
- $WhP(G)$ **- Whitehead problem:** taking as input two elements $x, y \in G$ decide whether there is some endomorphism $\varphi \in \text{End}(G)$ such that $x\varphi = y$.

Some *generalized* versions of the above problems will also be considered. Let $\mathcal{C}$ be a class of subsets of G. We define:

- $MP_{\mathcal{C}}(G)$ **- membership problem:** taking as input a subset $K \in \mathcal{C}$ and an element $x \in G$, decide whether $x \in K$;

- $GCP_{\mathcal{C}}(G)$ **- $\mathcal{C}$-generalized conjugacy problem:** taking as input a subset $K \in \mathcal{C}$ and an element $x \in G$, decide whether there is a conjugate of x in K;

- $GTCP_{\mathcal{C}}(G)$ **- $\mathcal{C}$-generalized twisted conjugacy problem:** taking as input a subset $K \in \mathcal{C}$, an endomorphism $\varphi \in \mathrm{End}(G)$ and an element $x \in G$, decide whether x has a φ-twisted conjugate in K, i.e., whether there is some $z \in G$ such that $(z^{-1}\varphi)xz \in K$;

- $GBrCP_{\mathcal{C}}(G)$ **- $\mathcal{C}$-generalized Brinkmann's conjugacy problem:** taking as input a subset $K \in \mathcal{C}$, an endomorphism $\varphi \in \mathrm{End}(G)$ and an element $x \in G$, decide whether there is some $k \in \mathbb{N}$ such that $x\varphi^k$ has a conjugate in K;

- $GBrP_{\mathcal{C}}(G)$ **- $\mathcal{C}$-generalized Brinkmann's problem:** taking as input a subset $K \in \mathcal{C}$, an endomorphism $\varphi \in \mathrm{End}(G)$ and an element $x \in G$, decide whether there is some $k \in \mathbb{N}$ such that $x\varphi^k$ belongs to K.

Some natural classes of subsets to consider are $f.g.$, *Rat*, *Rec*, *Alg* and CF, the classes of finitely generated subgroups, rational, recognizable, algebraic and context-free subsets, respectively. A seemingly less natural class to consider is the class of cosets of finitely generated groups, which we will denote by $[f.g.\, coset]$. As will become clear, in some contexts, this will be more adequate to us than the class of $f.g.$

2.5 Hyperbolic groups

A mapping $\varphi : (X, d) \to (X', d')$ between two metric spaces is called an *isometric embedding* if $d'(x\varphi, y\varphi) = d(x, y)$, for all $x, y \in X$. If, additionally, φ is surjective, then it is called an *isometry*. A (λ, K)*-quasi-isometric embedding* is a mapping $\varphi : (X, d) \to (X', d')$ such that there exist constants $\lambda \geq 1$ and $K \geq 0$ satisfying

$$\frac{1}{\lambda} d(x, y) - K \leq d'(x\varphi, y\varphi) \leq \lambda d(x, y) + K$$

for all $x, y \in X$ and a quasi-isometric embedding is a *quasi-isometry* if there is some $C \geq 0$ such that

$$\forall x' \in X' \, \exists x \in X : d'(x', x\varphi) \leq C.$$

A *geodesic* between two points $x, y \in X$ is an isometric embedding $\alpha : [0, s] \to X$ such that $0\alpha = x$ and $s\alpha = y$, where $[0, s] \subset \mathbb{R}$ is endowed with the usual metric of $\mathbb{R}$. Sometimes, we will also refer to $\mathrm{Im}(\alpha)$ as a geodesic. When the endpoint of a geodesic α coincides with the starting point of a geodesic β, we denote the concatenation of both geodesics by $\alpha + \beta$. We will write $[x, y]$ to represent an arbitrary geodesic between x and y.

A (λ, K)*-quasi-geodesic* of (X, d) is a (λ, K)-quasi-isometric embedding $\alpha : [0, s] \to X$ such that $0\alpha = x$ and $s\alpha = y$, where $[0, s] \subset \mathbb{R}$ is endowed with the usual metric of $\mathbb{R}$.

For $x, y, z \in X$, a *geodesic triangle* $[[x, y, z]]$ is a collection of three geodesics $[x, y]$, $[y, z]$ and $[z, x]$. Given $\delta \geq 0$, we say that X is *δ-hyperbolic* if every geodesic triangle is *δ-thin*, i.e., for all geodesic triangles $[[x, y, z]]$, we have that

$$\forall w \in [x, y]\, d(w, [y, z] \cup [z, x]) \leq \delta$$

holds.

Let (X, d) be a metric space and Y, Z nonempty subsets of X. We call the ε-neighborhood of Y in X and we denote by $\mathcal{V}_\varepsilon(Y)$ the set $\{x \in X \mid d(x, Y) \leq \varepsilon\}$. We call the *Hausdorff distance* between Y and Z and we denote by $\mathrm{Haus}(Y, Z)$, the number defined by

$$\inf\{\varepsilon > 0 \mid Y \subseteq \mathcal{V}_\varepsilon(Z) \text{ and } Z \subseteq \mathcal{V}_\varepsilon(Y)\}$$

if it exists. If it doesn't, we say that $\mathrm{Haus}(Y, Z) = \infty$.

A metric space (X, d) is *geodesic* if between any two points there is a geodesic.

Given a group $H = \langle A \rangle$, consider its Cayley graph $\Gamma_A(H)$ with respect to A endowed with the *geodesic metric* d_A, defined by letting $d_A(x, y)$ to be the length of the shortest path in $\Gamma_A(H)$ connecting x to y. This is not a geodesic metric space, since d_A only takes integral values. However, we can define the *geometric realization* $\bar{\Gamma}_A(H)$ of its Cayley graph $\Gamma_A(H)$ by embedding (H, d_A) isometrically into it. Then, edges of the Cayley graph become segments of length 1. With the metric induced by d_A, which we will also denote by d_A, $\bar{\Gamma}_A(H)$ becomes a geodesic metric space.

We say that a group H is *hyperbolic* if the metric space $(\bar{\Gamma}_A(H), d_A)$ is hyperbolic. We will simply write d instead of d_A when no confusion arises. Also, for $u \in H$ we will often denote $d_A(1, u)$ by $|u|$.

Recall that, given a finite alphabet A, we write $\widetilde{A} = A \cup A^{-1}$ and that $\widetilde{A}^*$ is the free monoid on $\widetilde{A}$. From now on, H will denote a finitely generated hyperbolic group generated by a finite set A and $\pi : \widetilde{A}^* \to H$ will be a matched epimorphism. A homomorphism $\pi : \widetilde{A}^* \to H$ is said to be *matched* if $a^{-1}\pi = (a\pi)^{-1}$.

An important property of the class of automatic groups, for which the class of hyperbolic groups is a subclass, is the *fellow traveler property*. Given a word $u \in \widetilde{A}^*$, we denote by $u^{[n]}$, the prefix of u with n letters. If $n > |u|$, then we consider $u^{[n]} = u$. As usual, $[n]$ will denote the set $\{1, 2, \ldots, n\}$. We say that the fellow traveler property holds for $L \subseteq \widetilde{A}^*$ if there is some $N \in \mathbb{N}$ such that, for every $u, v \in L$,

$$d_A(u\pi, v\pi) \leq 1 \implies d_A(u^{[n]}\pi, v^{[n]}\pi) \leq N,$$

for every $n \in \mathbb{N}$. By the triangle inequality, it follows that, for all $M \in \mathbb{N}$,

$$d_A(u\pi, v\pi) \leq M \implies d_A(u^{[n]}\pi, v^{[n]}\pi) \leq MN,$$

for every $n \in \mathbb{N}$.

Given $g, h, p \in H$, we define the *Gromov product of g and h* with basepoint p by

$$(g|h)_p^A = \frac{1}{2}(d_A(p,g) + d_A(p,h) - d_A(g,h)).$$

We will often write $(g|h)$ to denote $(g|h)_1^A$. Notice that, in the free group case, we have that $(g|h) = |g \wedge h|$.

Let G be a subgroup of a hyperbolic group $H = \langle A \rangle$ and $q \geq 0$. We say that G is *q-quasiconvex* (with respect to A) if

$$\forall x \in [g, g'] \quad d_A(x, G) \leq q$$

holds for every geodesic $[g, g']$ in $\overline{\Gamma}_A(H)$ with endpoints in G. Being quasiconvex does not depend on the generating set.

There are many ways to describe the *Gromov boundary ∂H* of H, such as being the equivalence classes of geodesic rays (infinite words on $\tilde{A}^*$ such that every finite factor is geodesic), when two rays are considered equivalent if the Hausdorff distance between them is finite. Another model for ∂H can be defined using *Gromov sequences*. We say that a sequence of points $(x_i)_{i\in\mathbb{N}}$ in H is a Gromov sequence if $(x_i|x_j) \to \infty$ as $i \to \infty$ and $j \to \infty$. Two such sequences $(x_i)_{i\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ are *equivalent* if

$$\lim_{i,j\to\infty} (x_i|y_j) = \infty$$

and the set of all the equivalence classes is a model for the boundary ∂H. Identifying an element h in H with the constant sequence $(h)_i$, we can extend the Gromov product to the boundary by putting, for all $\alpha, \beta \in \partial H$,

$$(\alpha|\beta)_p^A = \sup \left\{ \liminf_{i,j\to\infty} (x_i|y_j)_p^A \mid (x_i)_{i\in\mathbb{N}} \in \alpha, (y_j)_{j\in\mathbb{N}} \in \beta \right\}.$$

We define

$$\rho_{p,\gamma}^A(g,h) = \begin{cases} e^{-\gamma(g|h)_p^A} & \text{if } g \neq h \\ 0 & \text{otherwise} \end{cases}$$

for all $p, g, h \in H$.

Given $p \in H$, $\gamma > 0$ and $T \geq 1$, we denote by $V^A(p,\gamma,T)$ the set of all metrics d on H such that

$$\frac{1}{T}\rho_{p,\gamma}^A(g,h) \leq d(g,h) \leq T\rho_{p,\gamma}^A(g,h) \tag{2.2}$$

We refer to the metrics in some $V^A(p,\gamma,T)$ as the *visual metrics* on H. Let $d \in V^A(p,\gamma,T)$. The completion $(\widehat{H}, \hat{d})$ of the metric space (H, d) is a compact space and can be obtained by considering $\widehat{H} = H \cup \partial H$. It is a well-known fact that the topology induced by $\hat{d}$ on ∂H is the *Gromov topology* and that all visual metrics originate equivalent completions [16, Section III.H.3].

Considering the extension of $\rho^A_{p,\gamma}$ to the boundary, we define, for $\alpha, \beta \in \hat{H}$

$$\hat{\rho}^A_{p,\gamma}(\alpha,\beta) = \begin{cases} e^{-\gamma(\alpha|\beta)^A_p} & \text{if } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases}$$

By continuity, for all $\alpha, \beta \in \hat{H}$, the inequalities

$$\frac{1}{T}\hat{\rho}^A_{p,\gamma}(\alpha,\beta) \leq \hat{d}(\alpha,\beta) \leq T\hat{\rho}^A_{p,\gamma}(\alpha,\beta)$$

hold [16, Section III.H.3].

A metric space (X, d) is said to be a *median space* if, for all $x, y, z \in X$, there is some unique point $\mu(x, y, z) \in X$, known as the *median* of x, y, z, such that

- $d(x, y) = d(x, \mu(x, y, z)) + d(\mu(x, y, z), y)$;
- $d(y, z) = d(y, \mu(x, y, z)) + d(\mu(x, y, z), z)$;
- $d(z, x) = d(z, \mu(x, y, z)) + d(\mu(x, y, z), x)$.

We call $\mu : X^3 \to X$ the *median operator* of the median space X.

Coarse median spaces were introduced by Bowditch in [14]. Following the equivalent definition given in [85], we say that, given a metric space X, a *coarse median on* X is a ternary operation $\mu : X^3 \to X$ for which there exists a constant $C \geq 0$ such that, for all $a, b, c, x \in X$, we have that:

1. $\mu(a, a, b) = a$ and $\mu(a, b, c) = \mu(b, c, a) = \mu(b, a, c)$;
2. $d(\mu(\mu(a, x, b), x, c), \mu(a, x, \mu(b, x, c))) \leq C$;
3. $d(\mu(a, b, c), \mu(x, b, c)) \leq Cd(a, x) + C$.

Following the definitions in [44], two coarse medians $\mu_1, \mu_2 : X^3 \to X$ are said to be at *bounded distance* if there exists some constant C such that $d(\mu_1(x, y, z), \mu_2(x, y, z)) \leq C$ for all $x, y, z \in X$, and a *coarse median structure* on X is an equivalence class $[\mu]$ of coarse medians pairwise at bounded distance. When X is a metric space and $[\mu]$ is a coarse median structure on X, we say that $(X, [\mu])$ is a *coarse median space*. Following Fioravanti's definition, a *coarse median group* is a pair $(G, [\mu])$, where G is a finitely generated group with a geodesic metric d and $[\mu]$ is a G-invariant coarse median structure on G, meaning that for each $g \in G$, there is a constant $C(g)$ such that $d(g\mu(g_1, g_2, g_3), \mu(gg_1, gg_2, gg_3) \leq C(g)$, for all $g_1, g_2, g_3 \in G$. The author in [44] also remarks that this definition is stronger than the original definition from [14], that did not require G-invariance. Despite being better suited for this work, it is not quasi-isometry-invariant nor commensurability-invariant, unlike Bowditch's version.

An equivalent definition of hyperbolicity is given by the existence of a center of geodesic triangles (see, for example, [13]).

Lemma 2.5.1. *A group G is hyperbolic if and only if there is some constant $K \geq 0$ for which every geodesic triangle has a K-center, i.e., a point that, up to a bounded distance, depends only on the vertices, and is K-close to every edge of the triangle.*

Given three points, the operator that associates the three points to the K-center of a geodesic triangle they define is coarse median. In fact, by [85, Theorem 4.2] it is the only coarse-median structure that we can endow X with.

Finally, we present a lemma allowing us to define a coarse median operator on the direct product of two coarse median groups. This can naturally be extended to direct products with n factors.

Lemma 2.5.2. *Given two groups G_1 and G_2 endowed with coarse median operators μ_1 and μ_2, then $\mu : (G_1 \times G_2)^3 \to G_1 \times G_2$ defined by*

$$\mu((x_1, x_2), (y_1, y_2), (z_1, z_2)) = (\mu_1(x_1, y_1, z_1), \mu_2(x_2, y_2, z_2))$$

is a coarse median operator on $G_1 \times G_2$.

Proof. Indeed, let $(x_1, x_2), (y_1, y_2), (z_1, z_2), (w_1, w_2) \in G_1 \times G_2$, d_i be the metric for G_i and C_i be the constant for μ_i, for $i = 1, 2$. We will now check that μ satisfies the three conditions in the definition of a coarse median taking $G_1 \times G_2$ endowed with the product metric d.

1. We have that

$$\mu((x_1, x_2), (x_1, x_2), (y_1, y_2)) = (\mu_1(x_1, x_1, y_1), \mu_2(x_2, x_2, y_2)) = (x_1, x_2);$$

   and, for the second part, we only prove the first equality since the other is analogous:

$$\begin{aligned}\mu((x_1, x_2), (y_1, y_2), (z_1, z_2)) =&(\mu_1(x_1, y_1, z_1), \mu_2(x_2, y_2, z_2))\\ =&(\mu_1(y_1, z_1, x_1), \mu_2(y_2, z_2, x_2))\\ =&\mu((y_1, y_2), (z_1, z_2), (x_1, x_2)).\end{aligned}$$

2. Since

$$\begin{aligned}&\mu(\mu((x_1, x_2), (w_1, w_2), (y_1, y_2)), (w_1, w_2), (z_1, z_2))\\ =&\mu((\mu_1(x_1, w_1, y_1), \mu_2(x_2, w_2, y_2)), (w_1, w_2), (z_1, z_2))\\ =&(\mu_1(\mu_1(x_1, w_1, y_1), w_1, z_1), \mu_2(\mu_2(x_2, w_2, y_2), w_2, z_2)).\end{aligned}$$

   and μ_1 and μ_2 are coarse median operators, then

$$d_1(\mu_1(\mu_1(x_1, w_1, y_1), w_1, z_1), \mu_1(x_1, w_1, \mu_1(y_1, w_1, z_1))) \leq C_1$$

   and

$$d_2(\mu_2(\mu_2(x_2, w_2, y_2), w_2, z_2), \mu_2(x_2, w_2, \mu_2(y_2, w_2, z_2))) \leq C_2,$$

so we have that the distance d between

$$(\mu\,(\mu((x_1,x_2),(w_1,w_2),(y_1,y_2)),(w_1,w_2),(z_1,z_2)))$$

and

$$\mu\,((x_1,x_2),(w_1,w_2),\mu((y_1,y_2),(w_1,w_2),(z_1,z_2)))$$

is bounded above by $\max\{C_1,C_2\}$.

3. We have that

$$\begin{aligned}
&d(\mu((x_1,x_2),(y_1,y_2),(z_1,z_2)),\mu((w_1,w_2),(y_1,y_2),(z_1,z_2)))\\
&=\max\{d_1(\mu_1(x_1,y_1,z_1),\mu_1(w_1,y_1,z_1)),d_2(\mu_2(x_2,y_2,z_2),\mu_2(w_2,y_2,z_2))\}\\
&\leq \max\{C_1(d_1(x_1,w_1)+1),C_2(d_2(x_2,w_2)+1)\\
&\leq \max\{C_1,C_2\}(\max\{d_1(x_1,w_1),d_2(x_2,w_2)\}+1)\\
&=\max\{C_1,C_2\}d((x_1,x_2),(w_1,w_2))+\max\{C_1,C_2\}.
\end{aligned}$$

Hence, μ is a coarse median operator for the product metric d with constant $\max\{C_1,C_2\}$. □

We refer to μ defined as in the lemma as the *product coarse median operator*.

Chapter 3

Subsets of groups

In [7], the authors prove that, given a group G and a subgroup $H \leq G$, then

$$Rat(H) = \{K \subseteq H \mid K \in Rat(G)\}, \tag{3.1}$$

which they call *a kind of Fatou property for groups.* Notice that it is clear that $Rat(H) \subseteq \{K \subseteq H \mid K \in Rat(G)\}$. The difficult part is to prove the reverse inclusion. Herbst proved in [53] that the property (3.1) holds for algebraic subsets in case H is a finite index normal subgroup of G and posed the question of whether this would hold in general. Later, Herbst proved that that was the case if G is a virtually free group. We will discuss this property for recognizable, algebraic and context-free subsets. To do so, we will obtain some structural results relating the structure of algebraic and context free-subsets of a group G with the structure of the corresponding subsets of a finite index subgroup H, similar to the ones obtained for rational and recognizable subsets by Grunschlag and Silva (independently) in [51, 96].

Proposition 3.0.1. *[51, 96] Let G be a finitely generated group and $H \leq_{f.i.} G$. If G is the disjoint union $G = \cup_{i=1}^{n} Hb_i$, then $Rat(G)$ consists of all subsets of the form*

$$\bigcup_{i=1}^{n} L_i b_i \qquad (L_i \in Rat(H))$$

Proposition 3.0.2. *[51, 96] Let G be a finitely generated group and $H \leq_{f.i.} G$. If G is the disjoint union $G = \cup_{i=1}^{n} Hb_i$, then $Rec(G)$ consists of all subsets of the form*

$$\bigcup_{i=1}^{n} L_i b_i \qquad (L_i \in Rec(H))$$

It can be seen that, if we can construct a finite automaton recognizing the subgroup H (in particular, if $MP_{f.i.}(G)$ is decidable), the constructions above are algorithmic. We remark that $MP_{f.i.}(G)$ is known to be decidable for every finitely L-presented group by [52] and that, in [88], it is proved that for recursively presented groups, $MP_{f.i.}(G)$ is equivalent to having computable finite quotients (CFQ).

3.1 Finite index subgroups

In this section we will study how the structure of $CF(G)$ and $Alg(G)$ is related to the structure of $CF(H)$ and $Alg(H)$, where H is a finite index subgroup of G. The structural results obtained are similar to Propositions 3.0.1 and 3.0.2.

3.1.1 Context-free subsets

We will start by dealing with context-free subsets. We will state two technical lemmas that will be useful throughout the chapter. The first one follows immediately from Lemma 2.3.5.

Lemma 3.1.1. *Let G be a finitely generated group and $H \leq_{f.g.} G$. Then*

$$\{K \subseteq H \mid K \in CF(G)\} \subseteq CF(H).$$

Lemma 3.1.2. *Let G be a finitely generated group and $K_1 \in CF(G)$ and $K_2 \in Rec(G)$. Then $K_1 \cap K_2 \in CF(G)$.*

Proof. Let $G = \langle A \rangle$ and $\pi : A^* \to G$ be a surjective homomorphism. Then $(K_1 \cap K_2)\pi^{-1} = K_1\pi^{-1} \cap K_2\pi^{-1}$ is a context-free language, since context-free languages are closed under intersection with a rational language. $\square$

The following lemma is an immediate application of [53, Proposition 5.5(a)].

Lemma 3.1.3. *Let G be a finitely generated group and $H \trianglelefteq_{f.i} G$. Then $CF(H) \subseteq CF(G)$.*

We remark that we can remove the hypothesis of normality in this case.

Lemma 3.1.4. *Let G be a finitely generated group and $H \leq_{f.i} G$. Then $CF(H) \subseteq CF(G)$.*

Proof. Let $K \in CF(H)$. Since H has finite index, there exists a normal subgroup $F \leq H$ such that $F \trianglelefteq_{f.i.} G$ (and so $F \trianglelefteq_{f.i.} H$). Then H has a decomposition as a disjoint union

$$H = Fb_1 \cup \cdots \cup Fb_n,$$

for some $b_i \in H$ and K can be written as a disjoint union of the form

$$K = K \cap H = \bigcup_{i=1}^{n} Fb_i \cap K.$$

We will prove that for every $i \in [n]$, $Fb_i \cap K \in CF(G)$, which suffices since $CF(G)$ is closed under union.

So, let $i \in [n]$ and write $K_i = Fb_i \cap K$. Then $K_i b_i^{-1} \subseteq F \leq H$. Since F has finite index in H, then $F \in Rec(H)$, and so, by Lemma 2.3.4, $Fb_i \in Rec(H)$. Since $K \in CF(H)$, by Lemma 3.1.2, it follows that $K_i \in CF(H)$, which, again by Lemma 2.3.4, implies that $K_ib_i^{-1} \in CF(H)$. By Lemma 3.1.1, we have that $K_ib_i^{-1} \in CF(F)$. Using Lemma 3.1.3, we obtain that $K_ib_i^{-1} \in CF(G)$, which means that $K_i \in CF(G)$, by Lemma 2.3.4. □

Putting together Lemmas 3.1.1 and 3.1.4, we obtain the following corollary.

Corollary 3.1.5. *Let G be a finitely generated group and $H \leq_{f.i} G$. Then*

$$\{K \subseteq H \mid K \in CF(G)\} = CF(H).$$

We are now able to prove the previously announced structural result for context-free subsets. This gives an explicit description of $CF(G)$ based on $CF(H)$, where H is a finite index subgroup of G.

Proposition 3.1.6. *Let G be a finitely generated group and $H \leq_{f.i.} G$. If G is the disjoint union $G = \cup_{i=1}^n Hb_i$, then $CF(G)$ consists of all subsets of the form*

$$\bigcup_{i=1}^n L_ib_i \qquad (L_i \in CF(H))$$

Proof. Let $K \subseteq G$ be a subset of the form $\cup_{i=1}^n L_ib_i$ with $L_i \in CF(H)$. By Lemma 3.1.4, we have that $L_i \in CF(G)$ for all $i \in [n]$ and so $L_ib_i \in CF(G)$, by Lemma 2.3.4. Since $CF(G)$ is closed under union, then $K \in CF(G)$.

Conversely, let $K \in CF(G)$. Then K can be written as a disjoint union

$$K = K \cap G = \bigcup_{i=1}^n (Hb_i \cap K).$$

Put $K_i = Hb_i \cap K$ and let $i \in [n]$. Since $H \leq_{f.i.} G$, by Proposition 2.3.2, we have that $H \in Rec(G)$, and so, $Hb_i \in Rec(G)$, by Lemma 2.3.4. It follows from Lemma 3.1.2 that $K_i \in CF(G)$ and again by Lemma 2.3.4, $L_i = K_ib_i^{-1} \in CF(G)$ and $L_i \subseteq H$. By Lemma 3.1.1, we deduce that $L_i \in CF(H)$ and thus

$$K = K \cap G = \bigcup_{i=1}^n K_i = \bigcup_{i=1}^n L_ib_i.$$

□

Notice that if $H \leq_{f.i.} G$, then $H \in Rec(G)$ and if we are able to construct an automaton recognizing $H\pi^{-1}$ (in particular, if $MP_{f.i.}(G)$ is decidable), then the construction above is algorithmic, in the sense that, given a pushdown automaton recognizing $K\pi^{-1}$ for a subset $K \in CF(G)$, we can construct pushdown automata recognizing $L_i \in CF(H)$ such that $K = \cup_{i=1}^n L_ib_i$.

The proof of the first item of the following corollary is essentially the same as the proof of [96, Lemma 4.4], but we will present it for the sake of completeness.

Corollary 3.1.7. *Let G be a group and $H \leq_{f.i.} G$. Then we have that*

1. *$CF(G)$ is closed under intersection if and only if $CF(H)$ is closed under intersection;*

2. *$CF(G)$ is closed under complement if and only if $CF(H)$ is closed under complement.*

Proof. Write G as a disjoint union $G = \cup_{i=1}^{n} Hb_i$. We start by proving 1.

Suppose that $CF(G)$ is closed under intersection and let $K, K' \in CF(H)$. Then, by Lemma 3.1.4, $K, K' \in CF(G)$, and so $K \cap K' \in CF(G)$. Since $K \cap K' \subseteq H$, by Lemma 3.1.1, $K \cap K' \in CF(H)$.

Conversely, suppose that $CF(H)$ is closed under intersection and take $K, K' \in CF(G)$. Then

$$K = \bigcup_{i=1}^{n} L_i b_i \quad \text{and} \quad K' = \bigcup_{i=1}^{n} L_i' b_i,$$

for some $L_i, L_i' \in CF(H)$, $i \in [n]$. Since the cosets Hb_i are disjoint, it follows that

$$K \cap K' = \left(\bigcup_{i=1}^{n} L_i b_i\right) \cap \left(\bigcup_{i=1}^{n} L_i' b_i\right) = \bigcup_{i=1}^{n} (L_i \cap L_i') b_i.$$

Since $CF(H)$ is closed under intersection, then $L_i \cap L_i' \in CF(H)$, for all $i \in [n]$ and, by Proposition 3.1.6, $K \cap K' \in CF(G)$.

Now we prove 2. Suppose that $CF(G)$ is closed under complement. Since it is closed under union, it must be closed under intersection. Now, let $K \in CF(H)$. By Lemma 3.1.4, $K \in CF(G)$ and so $G \setminus K \in CF(G)$. Now, $H \leq_{f.i} G$ and so $H \in Rec(G)$. By Lemma 3.1.2, $H \setminus K = H \cap (G \setminus K) \in CF(G)$ and Lemma 3.1.1 yields that $H \setminus K \in CF(H)$.

Conversely, suppose that $CF(H)$ is closed under complement and let $K \in CF(G)$. Then $K = \bigcup_{i=1}^{n} L_i b_i$, for some $L_i \in CF(H)$, $i \in [n]$ and $G \setminus K = \bigcup_{i=1}^{n} (H \setminus L_i) b_i$. Since $CF(H)$ is closed under complement, then $H \setminus L_i \in CF(H)$, for all $i \in [n]$ and by Proposition 3.1.6, $G \setminus K \in CF(G)$. □

Since context-free languages are not closed under intersection and complement, it is not expected for the properties in the statement of Corollary 3.1.7 to hold often. Indeed, we conjecture that $CF(G)$ is only closed under intersection (resp. complement) if G is virtually cyclic. To support these conjectures, we present the following proposition.

Proposition 3.1.8. *Let G be a finitely generated group. If G is virtually abelian or virtually free, then $CF(G)$ is closed under intersection (resp. complement) if and only if G is virtually cyclic.*

Proof. If G is finite, then $CF(G)$ is obviously closed under intersection and complement. If G is virtually $\mathbb{Z}$, then, by (2.1), $CF(G) = Rat(G)$ and $Rat(G)$ is closed under intersection and complement since $Rat(\mathbb{Z})$ is closed under intersection and complement ([96, Propositions 3.6 and 3.9]).

Now we will prove that if $CF(\mathbb{Z}^m)$ and $CF(F_m)$ are closed under intersection (resp. complement), then $m = 1$ and that suffices by Corollary 3.1.7 and the fact that every finitely generated virtually abelian group has a free-abelian subgroup of finite index and every finitely generated virtually free group has a free subgroup of finite index.

We start with the free-abelian case. Suppose that $G = \mathbb{Z}^m$, for some $m > 1$. Let $A = \{e_1, \cdots, e_m\}$ be the canonical set of generators for $\mathbb{Z}^m$ and $\pi : \tilde{A}^* \to \mathbb{Z}^m$ be the standard surjective homomorphism. Let K_1 be the subset of $\mathbb{Z}^m$ given by the elements which have 0 in the first component and K_2 be the subset of elements having 0 in the second component. Then $K_1 \cap K_2$ is the set of elements that have 0 in both the first and second components. Then

$$K_1\pi^{-1} = \{w \in \tilde{A}^* \mid n_{e_1}(w) = n_{e_1^{-1}}(w)\},$$

$$K_2\pi^{-1} = \{w \in \tilde{A}^* \mid n_{e_2}(w) = n_{e_2^{-1}}(w)\}$$

and

$$(K_1 \cap K_2)\pi^{-1} = \{w \in \tilde{A}^* \mid n_{e_1}(w) = n_{e_1^{-1}}(w) \text{ and } n_{e_2}(w) = n_{e_2^{-1}}(w)\}.$$

It is well known that $K_1\pi^{-1}$ and $K_2\pi^{-1}$ are context-free languages of $\tilde{A}^*$. Suppose that $(K_1 \cap K_2)\pi^{-1}$ is context-free, let p be the constant given by the pumping lemma for context-free languages and let $z = e_1^p e_2^p e_1^{-p} e_2^{-p}$. Any factorization of z of the form $z = uvwxy$ with $|vwx| \leq p$ and $|vx| \geq 1$ is such that vx doesn't have occurrences neither of both e_1 and e_1^{-1} nor of both e_2 and e_2^{-1}, and so $uv^2wx^2y \notin (K_1 \cap K_2)\pi^{-1}$, which contradicts the assumption that $(K_1 \cap K_2)\pi^{-1}$ is context-free. Hence, $CF(\mathbb{Z}^m)$ is not closed under intersection if $m > 1$.

Now, suppose that $G = F_m$, for $m > 1$. Let $A = \{a_1, \cdots, a_m\}$ be a free basis for F_m and $\pi : \tilde{A}^* \to F_m$ be a surjective homomorphism. Let $K_1 = \{a_1^p a_2^p a_1^{-q} \in F_m \mid p, q \geq 0\}$ and $K_2 = \{a_1^p a_2^q a_1^{-q} : p, q \geq 0\}$. Then $\overline{K_1\pi^{-1}}$ and $\overline{K_2\pi^{-1}}$ are context-free languages of $\tilde{A}^*$ (K_1 is context-free since $L = \{a_1^p a_2^p \mid p \geq 0\}$ is context-free by Example 2.1.7 and $K_1 = L\{a_1^{-1}\}^*$; K_2 is analogous) but

$$\overline{(K_1 \cap K_2)\pi^{-1}} = \{a_1^n a_2^n a_1^{-n} \in \tilde{A}^* \mid n \geq 0\}$$

can easily be seen not to be context-free by the pumping lemma. Hence, by Lemma 2.3.7, $K_1, K_2 \in CF(F_m)$ but $K_1 \cap K_2 \notin CF(F_m)$.

Finally suppose that $CF(\mathbb{Z}^m)$ and $CF(F_m)$ are closed under complement. Then, since $CF(\mathbb{Z}^m)$ and $CF(F_m)$ are closed under union, they must also be closed under intersection and, by the above, it follows that $m = 1$.

Therefore, if G is a virtually abelian or virtually free group such that $CF(G)$ is closed under complement or intersection, then G must be virtually cyclic. $\square$

Combining Lemma 2.3.7 with Proposition 3.1.6, we get a description of context-free subsets of a finitely generated virtually free group.

Corollary 3.1.9. *Let G be a finitely generated virtually free group, $F = F_A$ be a finite index free subgroup of G with a free basis A, and $G = \cup_{i=1}^{n} Fb_i$ be a decomposition of G as a disjoint union of cosets of F. Then $CF(G)$ consists of the sets of the form*

$$\bigcup_{i=1}^{n} L_i b_i,$$

where $L_i \subseteq \tilde{A}^*$ is such that $\overline{L_i}$ is a context-free language of $\tilde{A}^*$.

3.1.2 Algebraic subsets

We will now replicate the approach we had for context-free subsets and obtain similar results for algebraic subsets.

Lemma 3.1.10. *Let G be a finitely generated group and $K_1 \in Alg(G)$ and $K_2 \in Rec(G)$. Then $K_1 \cap K_2 \in Alg(G)$.*

Proof. Let $G = \langle A \rangle$ and $\pi : A^* \to G$ be a surjective homomorphism. Let L be a context-free language such that $L\pi = K_1$. Then $L \cap K_2\pi^{-1}$ is a context-free language, since context-free languages are closed under intersection with a rational language and $(L \cap K_2\pi^{-1})\pi = K_1 \cap K_2$. Hence, $K_1 \cap K_2 \in Alg(G)$. $\square$

The following lemma follows directly from [53, Proposition 5.4].

Lemma 3.1.11. *Let G be a finitely generated group and $H \trianglelefteq_{f.i} G$. Then*

$$\{K \subseteq H \mid K \in Alg(G)\} \subseteq Alg(H).$$

We remark that we can remove the hypothesis of normality in this case.

Lemma 3.1.12. *Let G be a finitely generated group and $H \leq_{f.i} G$.*

$$\{K \subseteq H \mid K \in Alg(G)\} \subseteq Alg(H).$$

Proof. Let $K \in Alg(G)$ be such that $K \subseteq H$. There exists a normal subgroup $F \leq H$ such that $F \trianglelefteq_{f.i.} G$ (and so $F \trianglelefteq_{f.i.} H$). Then H has a decomposition as a disjoint union

$$H = Fb_1 \cup \cdots \cup Fb_n,$$

for some $b_i \in H$ and K can be written as a disjoint union of the form

$$K = K \cap H = \bigcup_{i=1}^{n} Fb_i \cap K.$$

We will prove that for every $i \in [n]$, $Fb_i \cap K \in Alg(H)$, which suffices since $Alg(H)$ is closed under union.

So, let $i \in [n]$ and write $K_i = Fb_i \cap K$. Then $K_i b_i^{-1} \subseteq F$. Since F has finite index in G, then $F \in Rec(G)$, and so, by Lemma 2.3.4, $Fb_i \in Rec(G)$. Since $K \in Alg(G)$, by Lemma 3.1.10, it follows that $K_i \in Alg(G)$, which implies that $K_i b_i^{-1} \in Alg(G)$. Since $K_i b_i^{-1} \subseteq F$ and F is a finite index normal subgroup of G, then Lemma 3.1.11 yields that $K_i b_i^{-1} \in Alg(F)$. Hence $K_i b_i^{-1} \in Alg(H)$ and $K_i \in Alg(H)$. □

Notice that, similarly to what happens in the rational case, the reverse inclusion holds for every finitely generated subgroup H (not necessarily of finite index). The analogous structural result to the one obtained for context-free subsets is now obtained in the algebraic case.

Proposition 3.1.13. *Let G be a finitely generated group and $H \leq_{f.i.} G$. If G is the disjoint union $G = \cup_{i=1}^n Hb_i$, then $Alg(G)$ consists of all subsets of the form*

$$\bigcup_{i=1}^{n} L_i b_i \qquad (L_i \in Alg(H))$$

Proof. Let $K \subseteq G$ be a subset of the form $\cup_{i=1}^n L_i b_i$ with $L_i \in Alg(H)$. Then, $L_i \in Alg(G)$ for all $i \in [n]$, and so $L_i b_i \in Alg(G)$. Since $Alg(G)$ is closed under union, then $K \in Alg(G)$.

Conversely, let $K \in Alg(G)$. Then K can be written as a disjoint union

$$K = K \cap G = \bigcup_{i=1}^{n} (Hb_i \cap K).$$

Put $K_i = Hb_i \cap K$ and let $i \in [n]$. Since $H \leq_{f.i.} G$, by Proposition 2.3.2, we have that $H \in Rec(G)$, and so, $Hb_i \in Rec(G)$, by Lemma 2.3.4. It follows from Lemma 3.1.10 that $K_i \in Alg(G)$ and so $L_i = K_i b_i^{-1} \in Alg(G)$ and $L_i \subseteq H$. By Lemma 3.1.12, we deduce that $L_i \in Alg(H)$ and thus

$$K = K \cap G = \bigcup_{i=1}^{n} K_i = \bigcup_{i=1}^{n} L_i b_i.$$

□

As in the case of context-free subsets, if we are able to construct an automaton recognizing $H\pi^{-1}$ (in particular, if $MP_{f.i.}(G)$ is decidable), then the construction above is algorithmic.

The proof of the following corollary is analogous to the one of Corollary 3.1.7.

Corollary 3.1.14. *Let G be a group and $H \leq_{f.i.} G$. Then we have that*

1. *$Alg(G)$ is closed under intersection if and only if $Alg(H)$ is closed under intersection;*
2. *$Alg(G)$ is closed under complement if and only if $Alg(H)$ is closed under complement.*

We will now use an example from [61], which also appears in [53], to prove that, unlike the case of context-free subsets [53, Corollary 4.7], algebraic subsets of F_n are not closed under intersection with rational subsets of F_n, for $n > 1$.

Example 3.1.15. Let F_n be the free group of rank $n > 1$ with basis $A = \{a_1, a_2, \dots, a_n\}$. Let $\mathcal{G} = (\{S, T\}, \tilde{A}, P, S)$ be a context-free grammar with the following set of productions P:

$$\begin{aligned}
&S \to a_1 S a_1^{-1}\\
&S \to T\\
&T \to a_1^{-1} T T a_1\\
&T \to a_2.
\end{aligned}$$

By definition, $L(\mathcal{G})$ is a context-free language and $K_1 = L(\mathcal{G})\pi \in Alg(F_n)$. The set $K_2 = \{a_2^n \in F_n \mid n \in \mathbb{N}\}$ is rational since $K_2 = (a_2^*)\pi$, but $K_1 \cap K_2 = \{a_2^{2^n} \in F_n \mid n \in \mathbb{N}\}$ (see [61, Example 1] or [53, Proposition 4.8]). We now prove that $K_1 \cap K_2 \notin Alg(F_n)$.

Let $p : \tilde{A}^* \to \mathbb{N}^{2n}$ be the function defined by

$$p(w) = (n_{a_1}(w), n_{a_1^{-1}}(w), n_{a_2}(w), n_{a_2^{-1}}(w), \dots, n_{a_n}(w), n_{a_n^{-1}}(w)).$$

Suppose that $K_1 \cap K_2 = L\pi$ for some context-free language L. By Theorem 2.1.6 (Parikh's Theorem), $p(L) = \{p(w) \mid w \in L\}$ is a semilinear set of $\mathbb{N}^{2n}$. Denoting by u_i the i-th coordinate of a vector $u \in \mathbb{N}^{2^n}$, we have that,

$$\{u_3 - u_4 \mid u \in p(L)\} = \{2^n \mid n \in \mathbb{N}\}, \tag{3.2}$$

because if $w \in \tilde{A}^*$ is such that $w\pi = a_2^{2^k}$, then $p(w)_3 - p(w)_4 = 2^k$.

If $p(L)$ is semilinear, then, by definition, $p(L) = \bigcup_{i=1}^k L_i$, for some linear sets L_i. Let $i \in [k]$ and write

$$L_i = u + \mathbb{N}v_1 + \dots + \mathbb{N}v_m,$$

for some $v_i \in \mathbb{N}^{2n}$, $i \in [m]$. Let $j \in [m]$. Then $u, u + v_j, u + 2v_j \in L_i \subseteq p(L)$. Hence, there are $k_0, k_1 \in \mathbb{N}$ such that

$$u_3 - u_4 = 2^{k_0} \quad \text{and} \quad (u + v_j)_3 - (u + v_j)_4 = u_3 + (v_j)_3 - u_4 - (v_j)_4 = 2^{k_1},$$

thus

$$(v_j)_3 - (v_j)_4 = 2^{k_1} - 2^{k_0}. \tag{3.3}$$

But now,

$$\begin{aligned}
(u + 2v_j)_3 - (u + 2v_j)_4 &= u_3 + 2(v_j)_3 - u_4 - 2(v_j)_4\\
&= u_3 - u_4 + 2((v_j)_3 - (v_j)_4)\\
&= 2^{k_0} + 2^{k_1+1} - 2^{k_0+1}\\
&= 2^{k_0}(2^{k_1+1-k_0} - 1).
\end{aligned}$$

Since $u + 2v_j \in p(L)$, then $2^{k_0}(2^{k_1+1-k_0} - 1)$ is a power of 2 with nonnegative exponent, which implies that $k_1 = k_0$. Indeed, if $k_1 + 1 - k_0 > 1$, then $2^{k_1+1-k_0} - 1$ is odd and if $k_1 + 1 - k_0 < 1$, then $2^{k_1+1-k_0} - 1$ is nonpositive. This, together with (3.3), implies that $(v_j)_3 - (v_j)_4 = 0$. Since j is arbitrary, we have that for all $j \in [m]$, $(v_j)_3 - (v_j)_4 = 0$. Hence, for all $x \in L_i$, $x_3 - x_4 = u_3 - u_4 = 2^{k_0}$.

Therefore, $\{u_3 - u_4 \mid u \in p(L)\}$ is finite, which contradicts (3.2).

Corollary 3.1.16. *Let G be a finitely generated virtually free group. Then $Alg(G)$ is closed under intersection (resp. complement) if and only if G is virtually cyclic.*

Proof. If G is virtually cyclic, then $Alg(G) = Rat(G)$, which is closed under intersection and complement.

If G is virtually free but neither finite nor virtually $\mathbb{Z}$, then G has a finite index free subgroup F of rank greater than 1. By Example 3.1.15, $Alg(F)$ is not closed under intersection and so by Corollary 3.1.14, $Alg(G)$ is not closed under intersection.

Since $Alg(G)$ is closed under union, closure under complement implies closure under intersection. □

Combining the results on the structure of $CF(G)$ and $Alg(G)$ we can also prove the equivalence of a natural decidability question on G and on H, similar to the one proved in the rational and recognizable case in [96, Theorem 4.8].

Proposition 3.1.17. *Let G be a group such that $MP_{f.i.}(G)$ is decidable and $H \leq_{f.i.} G$. Then the following are equivalent:*

1. *given $K \in Alg(G)$, it is decidable whether or not $K \in CF(G)$.*
2. *given $K \in Alg(H)$, it is decidable whether or not $K \in CF(H)$.*

Proof. Write G as a disjoint union of the form $\cup_{i=1}^{n} Hb_i$. Suppose that 1 holds and let $K \in Alg(H)$. Then $K \in Alg(G)$. Since $H \leq_{f.i.} G$ and $K \subseteq H$, then $K \in CF(G) \iff K \in CF(H)$, by Lemmas 3.1.1 and 3.1.4.

Conversely, suppose that 2 holds and let $K \in Alg(G)$. Then $K = \cup_{i=1}^{n} L_i b_i$, for some $L_i \in Alg(H)$. By Proposition 3.1.6, $K \in CF(G)$ if and only if each $L_i \in CF(H)$, for $i \in [n]$ and that we can decide from 2. □

3.2 Subsets of subgroups

The purpose of this section is to study the *kind of Fatou property* in (3.1) for recognizable, context-free and algebraic sets, completing the picture on this question for these four classes of subsets of a finitely generated group. The context-free case will lead to a new characterization of virtually free groups and the algebraic case will answer a question posed in [53], which was further developed in [54].

3.2.1 Recognizable subsets

In this section we deal with the easier case of recognizable subsets and are able to describe exactly when such a property holds.

Proposition 3.2.1. *Let G be a group and $H \leq G$ be a subgroup. Then*

$$\{K \subseteq H \mid K \in Rec(G)\} \subseteq Rec(H),$$

but the reverse inclusion holds if and only if $H \leq_{f.i.} G$.

Proof. Let $K \in Rec(G)$ be such that $K \subseteq H$. Then K is a (finite) union of cosets of some finite index subgroup $F \leq_{f.i.} G$:

$$K = \bigcup_{i=1}^{m} Fb_i,$$

for some $m \in \mathbb{N}$, $b_i \in K$ such that $Fb_i \neq Fb_j$ if $i \neq j$. Since $K \subseteq H$, then

$$K = K \cap H = \left(\bigcup_{i=1}^{m} Fb_i\right) \cap H = \bigcup_{i=1}^{m} (Fb_i \cap H).$$

Since $Fb_i \cap H$ is either empty or a coset of $F \cap H$, then

$$K = \bigcup_{i=1}^{m'} (F \cap H)b'_i,$$

for some $m' \leq m$ and $b'_i \in K$. Since F has finite index in G, then $F \cap H$ has finite index in H and K can be written as a union of cosets of a finite index subgroup of H. Thus, $K \in Rec(H)$.

If $[G : H] = \infty$, then $H \in Rec(H)$, but $H \notin Rec(G)$, and so $Rec(H) \not\subseteq \{K \subseteq H \mid K \in Rec(G)\}$. On the other hand, if $H \leq_{f.i.} G$, then $\{K \subseteq H \mid K \in Rec(G)\} = Rec(H)$, since every $K \in Rec(H)$ is the union of cosets of a finite index subgroup of H, and so it is the union of cosets of a finite index subgroup of G. $\square$

3.2.2 Context-free subsets

Now we will investigate the *kind of Fatou property* in (3.1) for context-free sets, which will lead to another characterization of virtually free groups. One of the inclusions is given by Lemma 3.1.1, so we only have to worry with the reverse inclusion. We will prove that it holds if and only if the group G is virtually free, making use of the Muller-Schupp theorem, which describes virtually free groups as being the ones with a context-free word problem (see [78]) and of the results of the previous section concerning finite index subgroups.

We will solve the free group case first.

Proposition 3.2.2. *Let F be a finitely generated free group. Then*

$$CF(H) = \{K \subseteq H \mid K \in CF(F)\},$$

for all $H \leq_{f.g.} F$.

Proof. Let $H \leq_{f.g} F$. By Marshall Hall's theorem, there is a subgroup $N \leq_{f.i.} F$ such that $N = H * H'$ for some $H' \leq F$. Let $K \in CF(H)$. We will prove that $K \in CF(N)$, which, by Lemma 3.1.4 is enough to show that $K \in CF(F)$. So let A be a free basis for H and A' be a free basis for H' disjoint from A (so $B = A \cup A'$ is a free basis for N) and let $\pi : \tilde{A}^* \to H$ and $\rho : \tilde{B}^* \to N$ be the standard surjective homomorphisms. So, $\rho|_{\tilde{A}^*} = \pi$. Since $K \in CF(H)$, by Lemma 2.3.7, $\overline{K\pi^{-1}}$ is a context-free language of $\tilde{A}^*$. But then $\overline{K\rho^{-1}} = \overline{K\pi^{-1}}$ is context free in $\tilde{B}^*$, which, again by Lemma 2.3.7, shows that $K \in CF(N)$. □

Now, we can prove the main result of this section which provides another language theoretical characterization of virtually free groups among finitely generated groups.

Theorem 3.2.3. *Let G be a finitely generated group. Then G is virtually free if and only if*

$$CF(H) = \{K \subseteq H \mid K \in CF(G)\},$$

for all $H \leq_{f.g.} G$.

Proof. Let $G = \langle A \rangle$ be a finitely generated group and $\pi : \tilde{A}^* \to G$ be the standard surjective homomorphism. If G is not virtually free, then by the Muller-Schupp theorem, we have that $\{1\}\pi^{-1}$ is not context-free, and so $\{1\} \notin CF(G)$. But taking $H = \{1\}$, we obviously have that $\{1\} \in CF(H)$, and so $CF(H) \neq \{K \subseteq H \mid K \in CF(G)\}$

Now, suppose that G is virtually free. Then G admits a decomposition as a disjoint union of the form

$$G = Fb_1 \cup \cdots \cup Fb_m,$$

for some free group $F \leq_{f.i.} G$ and $b_i \in G$, for $i \in [m]$. By Lemma 3.1.1, we only have to prove that $CF(H) \subseteq CF(G)$. Let $H \leq_{f.g} G$ and $K \in CF(H)$. We will show that $K \in CF(G)$.

We have that

$$H = H \cap G = H \cap \left(\bigcup_{i=1}^{m} Fb_i \right) = \bigcup_{i=1}^{m} (H \cap Fb_i) = \bigcup_{i=1}^{m'} (H \cap F)b_i',$$

for some $m' \leq m$ and $b_i' \in H$, since $H \cap Fb_i$ is either empty or a coset of $H \cap F$. Moreover, we can assume the cosets $(H \cap F)b_i'$ to be disjoint. Since $K \in CF(H)$, then

$$K = \bigcup_{i=1}^{m'} L_i b_i',$$

for some $L_i \in CF(H \cap F)$, $i \in [m']$. Since virtually free groups are Howson, then $H \cap F \leq_{f.g} F$ and, by Proposition 3.2.2, it follows that $L_i \in CF(F)$, for all $i \in [m']$. By Lemma 3.1.4, this implies that $L_i \in CF(G)$. Using Lemma 2.3.4, we get that $L_i b'_i \in CF(G)$, for all $i \in [m']$ and since $CF(G)$ is closed under union, then $K \in CF(G)$. $\square$

3.2.3 Algebraic subsets

Finally, we tackle the same problem for algebraic subsets.

Problem 3.2.4. *Let G be a group and $H \leq G$ be a subgroup. Is it true that $Alg(H) = \{K \subseteq H \mid K \in Alg(G)\}$?*

Notice that, similarly to what happens in the rational case, in the algebraic case the inclusion $Alg(H) \subseteq \{K \subseteq H \mid K \in Alg(G)\}$ is obvious, so we will only deal with the reverse inclusion.

This particular question was raised by Herbst in [53] and answered affirmatively in the particular case where G is a virtually free group in [54]. We also know that the answer is positive in the case where H is a finite index subgroup of G by Lemma 3.1.12. However, we will show that the answer is not always affirmative by constructing a specific counterexample.

Let $A = \{a_1, \ldots, a_n\}$, $G = \langle A \mid R\rangle$ be a group, $\varphi \in \mathrm{Aut}(G)$ and $g \in G$.

Consider the semidirect product

$$G \rtimes_\varphi \mathbb{Z} = \langle A, t \mid R, t^{-1} a_i t = a_i \varphi\rangle. \tag{3.4}$$

Using the relations, every element of $G \rtimes_\varphi \mathbb{Z}$ can be rewritten as an element of the form $t^a g$, where $a \in \mathbb{Z}$ and $g \in G$, in a unique way. These groups will be studied in more detail in Chapter 5.

Remark 3.2.5. *In a group of the form $G \rtimes_\varphi \mathbb{Z}$, $G\pi^{-1} = \{w \in \widetilde{A \cup \{t\}}^* \mid n_t(w) = n_{t^{-1}}(w)\}$ is context-free, and so $G \in CF(G \rtimes_\varphi \mathbb{Z})$. In particular, since G has infinite index in $G \rtimes_\varphi \mathbb{Z}$, this is an example of a group where recognizable and context-free subsets do not coincide.*

Theorem 3.2.6. *Let G be a finitely generated group and $\varphi \in \mathrm{Aut}(G)$. If $Alg(G) = \{K \subseteq G \mid K \in Alg(G \rtimes_\varphi \mathbb{Z})\}$, then the orbit through φ of every element is an algebraic subset of G.*

Proof. Let G be a group generated by a finite set $A = \{a_1, \ldots, a_n\}$ and $\pi : \tilde{A}^* \to G$ be the standard surjective homomorphism. Then $G \rtimes_\varphi \mathbb{Z}$ admits a presentation of the form (3.4) and there is a natural surjective homomorphism $\rho : \widetilde{A \cup \{t\}}^* \to G \rtimes_\varphi \mathbb{Z}$ such that $\rho|_A = \pi$ (identifying G with the subset $\{t^0 g \mid g \in G\}$). For every $w \in \tilde{A}^*$, the language $L = \{t^{-n} w t^n \mid n \in \mathbb{N}\} \subseteq \widetilde{A \cup \{t\}}^*$ is context-free and so $L\rho$ is an algebraic subset of $G \rtimes_\varphi \mathbb{Z}$. But $(t^{-n} w t^n)\rho = t^0 (w\pi)\varphi^n$, for all $n \in \mathbb{N}$, thus $L\rho = \{t^0 (w\pi)\varphi^n \mid n \in \mathbb{N}\} \subseteq G$. Since $Alg(G) = \{K \subseteq G \mid K \in Alg(G \rtimes_\varphi \mathbb{Z})\}$, then $\mathrm{Orb}_\varphi(w\pi) \in Alg(G)$, for every $w \in \tilde{A}^*$. $\square$

This allows us to construct a counterexample to Problem 3.2.4.

Example 3.2.7. Consider the group $\mathbb{Z}^2$ and $Q = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbb{Z})$

Since $\mathbb{Z}^2$ is abelian, we have that $Alg(\mathbb{Z}^2) = Rat(\mathbb{Z}^2)$. We will see that the orbit of $(1,0) \in \mathbb{Z}^2$ is not rational, which by Theorem 3.2.6, implies that $Alg(\mathbb{Z}^2) \neq \{K \subseteq G \mid K \in Alg(\mathbb{Z}^2 \rtimes_Q \mathbb{Z})\}$. Let $(f_n)_n \in \mathbb{N}$ be the Fibonacci sequence.

We have that, for $k \in \mathbb{N}$

$$Q^k = \begin{bmatrix} f_{2k+1} & f_{2k} \\ f_{2k} & f_{2k-1} \end{bmatrix}$$

This can be seen by induction on k. It holds for $k = 1$. Suppose that it holds for all integers up to some $r \in \mathbb{N}$. Then

$$Q^{r+1} = Q^r Q = \begin{bmatrix} f_{2r+1} & f_{2r} \\ f_{2r} & f_{2r-1} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2f_{2r+1} + f_{2r} & 2f_{2r} + f_{2r-1} \\ f_{2r} + f_{2r+1} & f_{2r} + f_{2r-1} \end{bmatrix} = \begin{bmatrix} f_{2r+3} & f_{2r+2} \\ f_{2r+2} & f_{2r+1} \end{bmatrix}.$$

The orbit of $(1,0)$ is the set of the first rows of powers of Q:

$$\mathrm{Orb}_Q((1,0)) = \{(1,0)\} \cup \{(f_{2k+1}, f_{2k}) \mid k \in \mathbb{N}\}.$$

Suppose that $\mathrm{Orb}_Q((1,0))$ is rational. Then so is its projection to the first component $\{f_{2k+1} \mid k \in \mathbb{N}\}$. Let $A = \{a, a^{-1}\}$. By Benois' Theorem, the language $L = \{a^{f_{2k+1}} \mid k \in \mathbb{N}\} \subseteq A^*$ is rational, which is absurd by the pumping lemma for rational languages.

Remark 3.2.8. *We know from [54] that if G is virtually free, then $Alg(H) = \{K \subseteq H \mid K \in Alg(G)\}$, for every $H \leq_{f.g.} G$. However, unlike the context-free case, we cannot expect this property to characterize virtually free groups, since, for example in abelian groups, the set of algebraic subsets and rational subsets coincide and property (3.1) always holds.*

Chapter 4

Virtually free groups

When, in an algorithm, we take a finitely generated virtually free group as input, we assume that we are given a decomposition as a disjoint union

$$G = Fb_1 \cup Fb_2 \cup \cdots \cup Fb_m, \tag{4.1}$$

where $F = F_A \trianglelefteq G$ is a finitely generated free group and a presentation of the form $\langle A, b_1, \ldots, b_m \mid R \rangle$, where the relations in R are of the form $b_i a = u_{ia} b_i$ and $b_i b_j = v_{ij} b_{r_{ij}}$, with $u_{ia}, v_{ij} \in F_A$ and $r_{ij} \in [m]$, $i, j = 1 \ldots, m$, $a \in A$.

A subgroup H of a group G is *fully invariant* if $\varphi(H) \subseteq H$ for every endomorphism φ of G. We start by presenting a technical lemma, which is simply an adaptation of [59, Lemma 2.2] with the additional condition of the subgroups being normal. The proof follows in the exact same way as theirs, but we will present it anyway, for completeness. This lemma will be a crucial tool throughout this chapter.

Lemma 4.0.1. *Let G be a group, n be a natural number and N be the intersection of all normal subgroups of G of index $\leq n$. Then N is fully invariant, and if G is finitely generated, then N has finite index in G.*

Proof. Let $\varphi \in \mathrm{End}(G)$ and

$$N = \bigcap_{\substack{H \trianglelefteq G \\ [G:H] \leq n}} H.$$

Now,

$$\varphi^{-1}(N) = \varphi^{-1}\left(\bigcap_{\substack{H \trianglelefteq G \\ [G:H] \leq n}} H \right) = \bigcap_{\substack{H \trianglelefteq G \\ [G:H] \leq n}} \varphi^{-1}(H)$$

and since the preimage of a normal subgroup of index at most n is itself a normal subgroup of index at most n, then $\varphi^{-1}(N)$ is the intersection of some (maybe not all) of these subgroups, which implies that $N \subseteq \varphi^{-1}(N)$, and so $\varphi(N) \subseteq N$. Moreover, if G is finitely generated, for each n, there are finitely many subgroups of index at most n, because there are finitely many groups of size $k \leq n$ and for each finite group G', there are finitely many homomorphisms $G \to G'$. Therefore, in this case, N is the intersection of finitely many finite index subgroups and so N has finite index itself. $\square$

Proposition 4.0.2. *Let G be a virtually free group. Then, we can compute generators for a finite index fully invariant free normal subgroup F' of G and write G as a disjoint union of the form*

$$G = F'b_1 \cup F'b_2 \cup \cdots \cup F'b_r$$

Proof. Take a decomposition as in (4.1). By Lemma 4.0.1, the intersection F' of all normal subgroups of G of index at most m is a fully invariant finite index subgroup, i.e. $[G : F'] < \infty$ and $F'\varphi \subseteq F'$ for all endomorphisms $\varphi \in \text{End}(G)$. Also, since $[G : F] = m$ and $F \trianglelefteq G$, then $F' \leq F$, and so F' is free. We will now prove that F' is computable. We start by enumerating all finite groups of cardinality at most m. For each such group $K = \{k_1, \ldots, k_s\}$ we enumerate all homomorphisms from G to K by defining images of the generators and checking all the relations. For each homomorphism $\theta : G \to K$, we have that $[G : \text{Ker}(\theta)] = |\,\text{Im}(\theta)| \leq |K| \leq m$. In fact, all normal subgroups of G of index at most m are of this form. We compute generators for the kernel of each θ, which is possible by Schreier's Lemma, since we can test membership in $\text{Ker}(\theta)$, which is a finite index subgroup. We can also find $a_2, \ldots, a_s \in G$ such that

$$G = \text{Ker}(\theta) \cup \text{Ker}(\theta)a_2 \cup \cdots \cup \text{Ker}(\theta)a_s,$$

taking a_i such that $a_i\theta = k_i$.

Hence, we can compute F', since it is a finite intersection of computable subgroups, and a decomposition of G as a disjoint union

$$G = F'b'_1 \cup Fb'_2 \cup \cdots \cup F'b'_r.$$

since the membership problem is decidable for finitely generated virtually free groups. $\square$

Notice that the subgroup F' in Proposition 4.0.2 is a subgroup of F, so we can use the theory of Stallings automata to find a basis for F'. This shows that when, in an algorithm, we take a finitely generated virtually free group as input, it is not a restriction to assume that we are given a decomposition as a disjoint union

$$G = Fb_1 \cup Fb_2 \cup \cdots \cup Fb_m, \tag{4.2}$$

where $F = F_A \trianglelefteq G$ is a fully invariant free subgroup of G and a presentation of the form $\langle A, b_1, \ldots, b_m \mid R \rangle$, where the relations in R are of the form $b_i a = u_{ia} b_i$ and $b_i b_j = v_{ij} b_{r_{ij}}$, with $u_{ia}, v_{ij} \in F_A$ and $r_{ij} \in [m]$, $i, j = 1 \ldots, m$, $a \in A$.

4.1 Fixed points and the stable image

Let G be a group and $\varphi \in \text{End}(G)$. We recall that a point $x \in G$ is said to be a fixed point if $x\varphi = x$. The set of all fixed points forms a subgroup which we denote by $\text{Fix}(\varphi)$. A point $x \in G$ is said to be a periodic point if there is some $m > 0$ such that $x\varphi^m = x$. The set of all periodic points forms a subgroup which we denote by $\text{Per}(\varphi)$. Obviously, we have that

$$\text{Per}(\varphi) = \bigcup_{k=1}^{\infty} \text{Fix}(\varphi^k).$$

The stable image of φ is

$$\varphi^\infty(G) = \bigcap_{i=1}^{\infty} \varphi^i(G).$$

This notion was introduced in [60], where it was proved that, if G is a finitely generated free group, then $\varphi^\infty(G)$ is finitely generated and $\varphi|_{\varphi^\infty(G)}$ is an automorphism.

Mutanguha proved in [79] that a basis for the stable image of an endomorphism of a finitely generated free group can be computed, which, combined with previous work by Bogopolski and Maslakova, implies that $\text{Fix}(\varphi)$ can be computed for general endomorphisms of a finitely generated free group.

The purpose of this section is to show how we can extend Mutanguha's results to finitely generated virtually free groups, providing an algorithm to compute the fixed subgroup of an endomorphism of a finitely generated virtually free group and another one to compute its stable image.

Theorem 4.1.1. *There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G and output a finite set of generators for* $\text{Fix}(\varphi)$*.*

Proof. Take a decomposition as in (4.2), writing G as a disjoint union

$$G = Fb_1 \cup Fb_2 \cup \cdots \cup Fb_m.$$

where F is a fully invariant free normal subgroup of G. Now, take $\psi = \varphi|_F$. Since F is fully invariant, then $\psi \in \text{End}(F)$. Now, for $u \in F$, put $X_u = \{x \in F \mid x\psi = xu\}$. We claim that X_u is computable. Since $[G : F] < \infty$, then F is finitely generated and so it has a finite basis X. Consider a new letter c not belonging to X, let $F' = F * \langle c|\rangle$ and $\psi' \in \text{End}(F')$ defined by mapping the letters $x \in X$ to $x\psi$ and c to $u^{-1}c$. By [79], we can compute a basis for $\text{Fix}(\psi')$. It is easy to see that $X_u c = \text{Fix}(\psi') \cap Fc$. Indeed, if $x \in X_u$, then

$$(xc)\psi' = (x\psi')(c\psi') = (x\psi)u^{-1}c = xuu^{-1}c = xc$$

and if $x \in \text{Fix}(\psi') \cap Fc$, then there is $y \in F$ such that $x = yc$ and

$$yc = (yc)\psi' = (y\psi)u^{-1}c,$$

which means that $y\psi = yu$ and so $y \in X_u$. Therefore, for all $u \in F$, X_uc (and so X_u) is computable. We claim that, for $i \in [m]$,

$$\mathrm{Fix}(\varphi) \cap Fb_i = \begin{cases} \emptyset & \text{if } b_i((b_i)^{-1}\varphi) \notin F \\ X_{b_i((b_i)^{-1}\varphi)}b_i & \text{if } b_i((b_i)^{-1}\varphi) \in F \end{cases}.$$

Suppose that $b_i((b_i)^{-1}\varphi) \in F$. Let $x \in X_{b_i((b_i)^{-1}\varphi)}$. Then, $x\psi = xb_i((b_i)^{-1}\varphi)$, and so $(xb_i)\varphi = xb_i$. Thus, $xb_i \in \mathrm{Fix}(\varphi) \cap Fb_i$. Now, let $x \in \mathrm{Fix}(\varphi) \cap Fb_i$. Then, $x(b_i)^{-1} \in F$ and

$$(x(b_i)^{-1})\psi = x\varphi(b_i)^{-1}\varphi = x((b_i)^{-1}\varphi) = x(b_i)^{-1}b_i((b_i)^{-1}\varphi)$$

and so $x(b_i)^{-1} \in X_{b_i((b_i)^{-1}\varphi)}$.

If $\mathrm{Fix}(\varphi) \cap Fb_i \neq \emptyset$, then there is some $x \in F$ such that $(x\varphi)(b_i\varphi) = (xb_i)\varphi = xb_i$ and so $x\varphi = xb_i((b_i)^{-1}\varphi)$. Since F is fully invariant, then $xb_i((b_i)^{-1}\varphi) \in F$, which yields that $b_i((b_i)^{-1}\varphi) \in F$.

Clearly,

$$\mathrm{Fix}(\varphi) = \bigcup_{i=1}^{m} \mathrm{Fix}(\varphi) \cap Fb_i,$$

and so $\mathrm{Fix}(\varphi)$ is computable, since it is a finite union of computable rational subsets. □

Bestvina and Handel proved in [8] the so-called Scott conjecture which stated that for an automorphism $\varphi \in \mathrm{Aut}(F_n)$ of a free group, the rank of the subgroup fixed by φ was bounded above by the rank of the free group, i.e., $\mathrm{rk}(\mathrm{Fix}(\varphi)) \leq n$. This was later shown to hold for general endomorphisms of the free group in [60]. We remark that the above proof does not give us a uniform bound on the rank of the virtually free group G, but it implies that the rank of the fixed subgroup by an endomorphism of a finitely generated virtually free group is bounded above by the sum of the rank the (fully invariant free normal) subgroup F and its index.

Remark 4.1.2. *Since* $\mathrm{Fix}(\psi) = \mathrm{Fix}(\varphi) \cap F$, *the proof above shows that we can compute* $m' \leq m$ *and* $c_i \in G$ *such that*

$$\mathrm{Fix}(\varphi) = \bigcup_{i=1}^{m'} \mathrm{Fix}(\psi)c_i.$$

Also, since $F \trianglelefteq G$, *then* $\mathrm{Fix}(\psi) \trianglelefteq \mathrm{Fix}(\varphi)$, *and so* $\left|{}^{\mathrm{Fix}(\varphi)}\!/_{\mathrm{Fix}(\psi)}\right| \leq m$, *which means that* ${}^{\mathrm{Fix}(\varphi)}\!/_{\mathrm{Fix}(\psi)}$ *has a generating set with at most* $\log_2(m)$ *elements, and so* $\mathrm{Fix}(\varphi)$ *has a generating set with at most* $\mathrm{rk}(\mathrm{Fix}(\psi)) + \log_2([G:F])$ *elements. By [60], it follows that* $\mathrm{rk}(\mathrm{Fix}(\varphi)) \leq \mathrm{rk}(F) + \log_2([G:F])$.

Bogopolski, Martino and Ventura proved in [10, Theorem 4.8] that the twisted conjugacy problem for automorphisms of a finitely generated virtually free group G, $TCP_{Aut}(G)$, is decidable. In fact, knowing that the computation of fixed subgroups for endomorphisms of virtually free groups is possible, we can easily extend this result to general endormorphisms of finitely generated virtually free groups. Notice that, in view of Proposition 3.0.1, computability of the intersection of two rational subsets of a finitely generated virtually free group can be reduced to the intersection of rational subsets of a free group, which is computable.

Corollary 4.1.3. *Let G be a finitely generated virtually free group. Then $TCP_{End}(G)$ is decidable.*

Proof. Let $\varphi \in \mathrm{End}(G)$ and $u, v \in G$ be our input. Let x, y be new letters not belonging to the generators of G and put $G_1 = G * \langle x, y | \rangle$. Then G_1 is virtually free, since it is a free product of virtually free groups. Let $\psi : G_1 \to G_1$ be defined by $g\psi = g\varphi$ for all $g \in G$; $x\psi = xv$; and $y\psi = u^{-1}y$. Then, there exists a twisted conjugator $w \in G$ such that $v = w^{-1}u(w\varphi)$ if and only if there exists a fixed point of ψ of the form xzy with $z \in G$. Indeed, if $v = w^{-1}u(w\varphi)$, then

$$(xw^{-1}y)\psi = xv(w^{-1}\varphi)u^{-1}y = xw^{-1}y$$

and, conversely, if there is a fixed point xzy of ψ with $z \in G$, then

$$xzy = (xzy)\psi = xv(z\varphi)u^{-1}y,$$

hence $z = v(z\varphi)u^{-1}$.

So, u and v are φ-twisted conjugates if and only if $x^{-1}\,\mathrm{Fix}(\psi)y^{-1} \cap G \neq \emptyset$, which can be decided, since G and $\mathrm{Fix}(\psi)$ are computable subgroups of G_1. □

Now we prove computability of stable images.

Theorem 4.1.4. *There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G and output a finite set of generators for $\varphi^\infty(G)$.*

Proof. Write G as a disjoint union of the form

$$G = Fb_1 \cup \cdots \cup Fb_m,$$

where F is a fully invariant free normal subgroup of G. Then

$$\begin{aligned}\varphi^\infty(G) &= \varphi^\infty(G) \cap G \\ &= (\varphi^\infty(G) \cap Fb_1) \cup \cdots \cup (\varphi^\infty(G) \cap Fb_m).\end{aligned}$$

We will prove that each of the subsets $\varphi^\infty(G) \cap Fb_i$ is rational and computable, which implies that $\varphi^\infty(G)$ is finitely generated and computable, since it is a finite union of (computable) rational subsets (see Theorem 2.3.1).

Fix $i \in [m]$. Since, for all $k \in \mathbb{N}$, $G\varphi^k = \bigcup_{j=1}^m F\varphi^k(b_j\varphi^k)$, we have that

$$\begin{aligned}
\varphi^\infty(G) \cap Fb_i &= \left(\bigcap_{k\in\mathbb{N}} G\varphi^k\right) \cap Fb_i \\
&= \bigcap_{k\in\mathbb{N}} (G\varphi^k \cap Fb_i) \\
&= \bigcap_{k\in\mathbb{N}} \bigcup_{j=1}^{m} (F\varphi^k(b_j\varphi^k) \cap Fb_i).
\end{aligned}$$

Since F is fully invariant, we have that, for all $k \in \mathbb{N}$, $(Fb_i)\varphi^k \subseteq F(b_i\varphi^k)$. Hence, the mapping $\theta : G/F \to G/F$ defined by $Fb_i \mapsto F(b_i\varphi)$ is a well-defined endomorphism. Since $|G/F| = m$ is finite, we can compute the orbit $\mathrm{Orb}_\theta(Fb_i)$ of Fb_i through θ. In particular, we can check if Fb_i is periodic.

If it is not, then $Fb_i \notin \mathrm{Im}(\theta^k)$, for $k > m$. This means that for $k > m$, and $j \in [m]$,

$$F\varphi^k(b_j\varphi^k) \cap Fb_i \subseteq F(b_j\varphi^k) \cap Fb_i = (Fb_j)\theta^k \cap Fb_i = \emptyset,$$

since $Fb_i \notin \mathrm{Im}(\theta^k)$ and so $(Fb_j)\theta^k$ and Fb_i are distinct, thus disjoint, cosets. Hence, $\varphi^\infty(G) \cap Fb_i = \emptyset$.

If Fb_i is periodic, say of period P, then we claim that

$$\varphi^\infty(G) \cap Fb_i = \bigcap_{k\in\mathbb{N}} (G\varphi^k \cap Fb_i) = \bigcap_{k>m} (G\varphi^k \cap Fb_i) = \bigcap_{k>m} F\varphi^{Pk} b_i \varphi^{Pk}. \tag{4.3}$$

The first two equalities are obvious and it is clear that

$$\bigcap_{k>m} F\varphi^{Pk} b_i \varphi^{Pk} \subseteq \bigcap_{k>m} (G\varphi^k \cap Fb_i)$$

since Fb_i is a periodic point of θ with period P. Now we prove the reverse inclusion. Let $x \in \bigcap_{k>m}(G\varphi^k \cap Fb_i)$ and fix $k > m$. Then $x \in G\varphi^{2Pk} \cap Fb_i$, which means that there are some $f \in F$ and $j \in [m]$ such that

$$(fb_j)\varphi^{2Pk} = (fb_j)\varphi^{Pk}\varphi^{Pk} = x \in Fb_i.$$

Since $k > m$, it follows that $(Fb_j)\theta^{Pk}$ must belong to the periodic orbit of Fb_i. Since it gets mapped to Fb_i after Pk applications of θ, we must have that $(Fb_j)\theta^{Pk} = Fb_i$, and so $(fb_j)\varphi^{Pk} \in Fb_i$ and

$$x = (fb_j)\varphi^{Pk}\varphi^{Pk} \in (Fb_i)\varphi^{Pk} = F\varphi^{Pk} b_i \varphi^{Pk}.$$

Since k is arbitrary, we have proved (4.3).

Now, since Fb_i is a periodic point of θ, this means that there is some $y \in F$ such that $b_i\varphi^P = yb_i$. Let c be a letter not belonging to F, $F' = F * \langle c| \rangle$ and $\psi : F' \to F'$ be the endomorphism that applies φ^P to the elements in F and maps c to yc. We will prove that

$$\varphi^\infty(G) \cap Fb_i = \bigcap_{k>m} F\varphi^{Pk} b_i \varphi^{Pk} = \left(\bigcap_{k>m} F\psi^k c\psi^k \right) c^{-1} b_i = (\psi^\infty(F') \cap Fc)c^{-1}b_i.$$

The first equality is simply (4.3). The second equality follows from the fact that

$$b_i \varphi^{Pk} = \left(\prod_{j=0}^{k-1} y\varphi^{(k-j-1)P} \right) b_i = (c\psi^k)c^{-1}b_i.$$

Finally, it is clear that $(\bigcap_{k>m} F\psi^k c\psi^k) = \bigcap_{k>m}(Fc)\psi^k \subseteq (\psi^\infty(F') \cap Fc)$, and so

$$(\bigcap_{k>m} F\psi^k c\psi^k)c^{-1}b_i \subseteq (\psi^\infty(F') \cap Fc)c^{-1}b_i.$$

We only have to prove that $(\psi^\infty(F') \cap Fc) \subseteq \bigcap_{k>m} F\psi^k c\psi^k$. We start by proving that

$$\forall k \in \mathbb{N},\ \forall x \in F'\ (x\psi^k \in Fc \implies x\psi^k \in (Fc)\psi^k)$$

Let $k \in \mathbb{N}$, $x \in F'$ and X be a basis for F. We proceed by induction on $|x|$. If $|x| = 1$, then $x\psi^k \in Fc$ implies that $x = c$, since $F\psi^k \subseteq F$ and so $x\psi^k = c\psi^k \in (Fc)\psi^k$. So assume the claim holds for all x such that $|x| \leq n$ and take x such that $|x| = n+1$. Write $x = x_1 \cdots x_{n+1}$ and $x\psi^k = w_1 \cdots w_{n+1}$, where $x_i \in \widetilde{X \cup \{c\}}$ and $w_i = x_i\psi^k$. Let $w = w_1 \cdots w_{n+1} \in (\widetilde{X \cup \{c\}})^*$. Then $n_c(x) = n_c(w)$ and $n_{c^{-1}}(w) = n_{c^{-1}}(x)$. Choose any cancellation order on $w_1 \cdots w_{n+1}$. We have that the result of cancellation, $\overline{w_1 \cdots w_{n+1}}$, belongs to Fc. The c that survives the cancellation must belong to some w_i and it appears as the last letter of w_i, by definition of ψ. If $i \leq n$, then we have that $x\psi^k = \overline{w_1 \cdots w_i} = (x_1 \cdots x_i)\psi^k$, and so, by the induction hypothesis, it follows that $x\psi^k = (x_1 \cdots x_i)\psi^k \in (Fc)\psi^k$. If, on the other hand the surviving c belongs to w_{n+1}, then either $n_c(w) = 1$ and $n_{c^{-1}}(w) = 0$, in which case the claim is obvious since $x \in Fc$, or $n_c(w) > 1$ and $n_{c^{-1}}(w) = n_c(w) - 1 > 0$ and there is cancellation between c's and c^{-1}'s. Consider the first cancellation occurring between c's and c^{-1}'s and let i, j be such that the cancelled c and the cancelled c^{-1} belong to w_i and w_j, respectively, and so $x_i = c$ and $x_j = c^{-1}$ and c (resp. c^{-1}) is the last (resp. first) letter of w_i (resp. w_j). Suppose that $i < j$. Then $j > i+1$, since otherwise x is not reduced. Thus, we have that $\overline{w_{i+1} \cdots w_{j-1}} = 1$, i.e., $x_{i+1} \cdots x_{j-1} \in \mathrm{Ker}(\psi^k)$ and $x\psi^k = x_1 \cdots x_i x_j \cdots x_{n+1}$. By the induction hypothesis, it follows that $x\psi^k = (x_1 \cdots x_i x_j \cdots x_{n+1})\psi^k \in (Fc)\psi^k$. If $i > j$, we proceed analogously. Again we have that $i > j+1$, since otherwise x is not reduced. Since the first letter of w_j cancels with the last letter of w_i, we have that $\overline{w_j \cdots w_i} = 1$, i.e., $x_j \cdots x_i \in \mathrm{Ker}(\psi^k)$ and $x\psi^k = x_1 \cdots x_{j-1}x_{i+1} \cdots x_{n+1}$. By the induction hypothesis, it follows that $x\psi^k = (x_1 \cdots x_{j-1}x_{i+1} \cdots x_{n+1})\psi^k \in (Fc)\psi^k$.

Hence, $\mathrm{Im}(\psi^k) \cap Fc \subseteq (Fc)\psi^k$ and

$$(\psi^\infty(F') \cap Fc) \subseteq \bigcap_{k \geq m} (Fc)\psi^k = \bigcap_{k \geq m} F\psi^k c\psi^k.$$

By [79], $\psi^\infty(F')$ is finitely generated (thus rational) and computable, so $\varphi^\infty(G) \cap Fb_i = (\psi^\infty(F') \cap Fc)c^{-1}b_i$ is also rational and can be effectively computed. □

4.2 Eventually fixed and eventually periodic points

Recall that, given $x \in G$, the orbit of x through φ is defined by

$$\mathrm{Orb}_\varphi(x) = \{x\varphi^k \mid k \in \mathbb{N}\}.$$

A point x is said to be *eventually periodic* if its orbit is finite, i.e., there is some $m \in \mathbb{N}$ such that $x\varphi^m \in \mathrm{Per}(\varphi)$ and similarly, x is said to be *eventually fixed* if there is some $m \in \mathbb{N}$ such that $x\varphi^m \in \mathrm{Fix}(\varphi)$ or, equivalently, such that $x\varphi^m = x\varphi^{m+1}$. In this case, for every $k \geq m$, we have that $x\varphi^k = x\varphi^m$. We denote by $\mathrm{EvPer}(\varphi)$ (resp. $\mathrm{EvFix}(\varphi)$) the set of all eventually periodic (resp. fixed) points of φ. It is clear from the definitions that

$$\mathrm{EvPer}(\varphi) = \bigcup_{k=1}^{\infty} (\mathrm{Per}(\varphi))\varphi^{-k} \quad \text{and} \quad \mathrm{EvFix}(\varphi) = \bigcup_{k=1}^{\infty} (\mathrm{Fix}(\varphi))\varphi^{-k}.$$

Proposition 4.2.1. *Let $\varphi \in \mathrm{End}(G)$. Then $\mathrm{EvPer}(\varphi)$ and $\mathrm{EvFix}(\varphi)$ are subgroups of G.*

Proof. Let $x_1, x_2 \in \mathrm{EvPer}(\varphi)$. Then, there are $m_1, m_2 \in \mathbb{N}$ such that $x_1\varphi^{k_1}$ and $x_1\varphi^{k_2}$ are periodic points for all $k_1 > m_1$ and $k_2 > m_2$. So, taking $M = \max\{m_1, m_2\}$, we have that $(x_1x_2)\varphi^M$ is periodic. Also, if there is some $m \in \mathbb{N}$ such that $x_1\varphi^m \in \mathrm{Per}(\varphi)$, then $x_1^{-1}\varphi^m \in \mathrm{Per}(\varphi)$.

Similarly, let $x_1, x_2 \in \mathrm{EvFix}(\varphi)$. Then, there are $m_i \in \mathbb{N}$ such that $x_i\varphi^{m_i} = x_i\varphi^{m_i+1}$, for $i = 1, 2$. Then, putting $M = \max\{m_1, m_2\}$, we have that $(xy)\varphi^{M+1} = x\varphi^{M+1}y\varphi^{M+1} = x\varphi^{m_1}y\varphi^{m_2} = x\varphi^M y\varphi^M$. Also, we have that $(x_1^{-1})\varphi^{m_1+1} = (x_1\varphi^{m_1+1})^{-1} = (x_1\varphi^m)^{-1} = x_1^{-1}\varphi^m$. □

We are interested in the study of these subgroups. We start by presenting some basic observations about $\mathrm{EvFix}(\varphi)$.

Lemma 4.2.2. *Let $\varphi \in \mathrm{End}(G)$. Then*

1. $\bigcup_{k=1}^{\infty} \mathrm{Ker}(\varphi^k) \trianglelefteq \mathrm{EvFix}(\varphi)$
2. $\mathrm{Fix}(\varphi) \leq \mathrm{EvFix}(\varphi)$
3. $\mathrm{EvFix}(\varphi) \cap \mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi)$

4. $\mathrm{EvFix}(\varphi) = \mathrm{Fix}(\varphi) \iff \varphi$ *is injective*

Proof. 1, 2 and 3 are obvious by definition. If φ is injective, then $(\mathrm{Fix}(\varphi))\varphi^{-k} = \mathrm{Fix}(\varphi)$ for every $k \in \mathbb{N}$, and so $\mathrm{EvFix}(\varphi) = \mathrm{Fix}(\varphi)$. If there exists $1 \neq x \in \mathrm{Ker}(\varphi)$, then $x \in \mathrm{EvFix}(\varphi) \setminus \mathrm{Fix}(\varphi)$. □

A natural question to ask is: is $\mathrm{EvFix}(\varphi)$ necessarily finitely generated? In case G is a finitely generated virtually free group, we know that $\mathrm{Fix}(\varphi)$ is finitely generated and computable, and so one might wonder if the same occurs for eventually fixed points. For instance, if φ is injective, then that must happen, by the previous lemma. However, it is not hard to find examples showing that both things can happen, even in the simpler case where G is a free group of finite rank.

Example 4.2.3. Let $\varphi : F_2 \to F_2$ be defined by $a \mapsto aba$ and $b \mapsto 1$. Then for $x \in F_2$, we have that $x\varphi = (aba)^{\lambda_a(x)}$, where $\lambda_a : F_2 \to \mathbb{Z}$ homomorphism defined by $a \mapsto 1$ and $b \mapsto 0$. So, $\mathrm{Fix}(\varphi)$ is trivial and $\mathrm{Ker}(\varphi) = \{w \mid \lambda_a(w) = 0\}$. Also $(\mathrm{Ker}(\varphi))\varphi^{-1} = \mathrm{Ker}(\varphi)$. So, $\mathrm{EvFix}(\varphi) = \mathrm{Ker}(\varphi)$, which is not finitely generated.

Example 4.2.4. Let $\varphi : F_2 \to F_2$ be defined by $a \mapsto bab^{-1}$ and $b \mapsto 1$. Then for $x \in F_2$, we have that $x\varphi = ba^{\lambda_a(x)}b^{-1}$. So, $\mathrm{Fix}(\varphi) = \{ba^k b^{-1} \mid k \in \mathbb{Z}\}$ and $\mathrm{EvFix}(\varphi) = (\mathrm{Fix}(\varphi))\varphi^{-1} = F_2$, which is obviously finitely generated.

Despite having both possibilities, we will show that we can decide whether $\mathrm{EvFix}(\varphi)$ and $\mathrm{EvPer}(\varphi)$ are finitely generated or not for endomorphisms of finitely generated virtually free groups.

Another natural question to ask is: can we decide whether $\mathrm{EvFix}(\varphi)$ is normal or not? In the next section, we will study this question in the case of free groups.

4.3 Normality of EvFix(φ) in free groups

The purpose of this section is to describe the cases where, for an endomorphism $\varphi \in \mathrm{End}(F_n)$, we have that $\mathrm{EvFix}(\varphi) \trianglelefteq F_n$. We start with a technical lemma.

Lemma 4.3.1. *Let $u \in F_n \setminus \{1\}$ be a nontrivial non proper power. If there exists $w \in F_n$ and $p, q \in \mathbb{Z}$ such that $w^{-1}u^p w = u^q$, then $p = q$ and $w \in \langle u \rangle$.*

Proof. Let $u \in F_n \setminus \{1\}$. If there exist $w \in F_n$ and $p, q \in \mathbb{Z}$ such that $w^{-1}u^p w = u^q$, then the cyclic reduced cores of u^p and u^q are equivalent under a cyclic permutation of their letters. This implies in particular that $p = q$ and w commutes with u^p. □

We are now able to describe the cases where $\mathrm{EvFix}(\varphi)$ is a normal subgroup of F_n.

Proposition 4.3.2. *Let $\varphi \in \mathrm{End}(F_n)$. Then one of the following holds:*

1. $\mathrm{EvFix}(\varphi) = F_n$

2. $\mathrm{EvFix}(\varphi) = \bigcup_{k=1}^{\infty} \mathrm{Ker}(\varphi^k)$

3. $\mathrm{EvFix}(\varphi)$ *is not a normal subgroup of* F_n*.*

Proof. Suppose that $\mathrm{EvFix}(\varphi) \neq F_n$, $\mathrm{EvFix}(\varphi) \neq \bigcup_{k=1}^{\infty} \mathrm{Ker}(\varphi^k)$ and that $\mathrm{EvFix}(\varphi) \trianglelefteq F_n$ and take $g \in F_n \setminus \mathrm{EvFix}(\varphi)$. Since $\mathrm{EvFix}(\varphi)$ is normal, then for every $x \in \mathrm{EvFix}(\varphi) \setminus \bigcup_{k=1}^{\infty} \mathrm{Ker}(\varphi^k)$ we have that $gxg^{-1} \in \mathrm{EvFix}(\varphi)$, and $g^{-1}xg \in \mathrm{EvFix}(\varphi)$, which means that there are some $n, m \in \mathbb{N}$ for which $(gxg^{-1})\varphi^{n+1} = (gxg^{-1})\varphi^n$ and $(g^{-1}xg)\varphi^{m+1} = (g^{-1}xg)\varphi^m$. Also, there is some $p \in \mathbb{N}$ such that $x\varphi^p = x\varphi^{p+1}$. So, letting $M = \max\{n, m, p\}$, it follows that

$$(gxg^{-1})\varphi^{M+1} = (gxg^{-1})\varphi^M, \tag{4.4}$$

$$(g^{-1}xg)\varphi^{M+1} = (g^{-1}xg)\varphi^M \tag{4.5}$$

and

$$x\varphi^M = x\varphi^{M+1}. \tag{4.6}$$

We can rewrite (4.4) as

$$(x\varphi^{M+1})(g^{-1}\varphi^{M+1})(g\varphi^M) = (g^{-1}\varphi^{M+1})(g\varphi^M)(x\varphi^M),$$

and so $(g^{-1}\varphi^{M+1})(g\varphi^M)$ and $x\varphi^M$ commute. Similarly, we can rewrite (4.5) as

$$(g\varphi^M)(g^{-1}\varphi^{M+1})(x\varphi^{M+1}) = (x\varphi^M)(g\varphi^M)(g^{-1}\varphi^{M+1}),$$

and so $(g\varphi^M)(g^{-1}\varphi^{M+1})$ and $x\varphi^M$ commute.

Since $g \notin \mathrm{EvFix}(\varphi)$, then $(g\varphi^M)(g^{-1}\varphi^{M+1}), (g^{-1}\varphi^{M+1})(g\varphi^M) \neq 1$, and since $x \notin \bigcup_{k=1}^{\infty} \mathrm{Ker}(\varphi^k)$, then $x\varphi^M \neq 1$. We then have that $(g\varphi^M)(g^{-1}\varphi^{M+1}), (g^{-1}\varphi^{M+1})(g\varphi^M)$ and $x\varphi^M$ are powers of the same primitive word $u \in F_n$. So, put

$$(g^{-1}\varphi^{M+1})(g\varphi^M) = u^q \tag{4.7}$$

$$(g\varphi^M)(g^{-1}\varphi^{M+1}) = u^k \tag{4.8}$$

and

$$x\varphi^M = u^r. \tag{4.9}$$

From (4.7) we get that

$$(g^{-1}\varphi^{M+1})(g\varphi^{M})(g^{-1}\varphi^{M+1})(g\varphi^{M}) = u^{2q}.$$

Applying (4.8), we obtain that $(g^{-1}\varphi^{M+1})u^k(g\varphi^M) = u^{2q}$, and so that

$$g\varphi^{M+1} = u^k(g\varphi^M)u^{-2q}.$$

But from (4.7), we know that

$$g\varphi^{M+1} = (g\varphi^M)u^{-q}.$$

Hence $(g\varphi^M)u^{-q} = u^k(g\varphi^M)u^{-2q}$, and $(g\varphi^M)u^q(g^{-1}\varphi^M) = u^k$. From Lemma 4.3.1, we know that $k = q$ and $g\varphi^M$ is a power of u.

Since $x\varphi^M \neq 1$, then $r \neq 0$. From (4.6) and (4.9), we know that $(u\varphi)^r = u^r\varphi = u^r$ and so $u \in \mathrm{Fix}(\varphi)$ and $g\varphi^M$ is fixed, which contradicts the assumption that $g \notin \mathrm{EvFix}(\varphi)$.

□

We now present some remarks on the previous Proposition.

Remark 4.3.3. *Condition 2 in Proposition 4.3.2 is equivalent to* $\mathrm{Fix}(\varphi)$ *being trivial. Indeed, if there is some nontrivial element* $x \in \mathrm{Fix}(\varphi)$*, then* $x\varphi^k = x \neq 1$*, for every* $k \in \mathbb{N}$*. If* $\mathrm{Fix}(\varphi) = 1$*, then 2 holds by definition.*

Remark 4.3.4. *If* $\mathrm{EvFix}(\varphi)$ *is finitely generated, then there must be a bound on the size of the orbits of eventually fixed points, but the converse is false by Example 4.2.3. Indeed, suppose that* $\mathrm{EvFix}(\varphi) = \langle w_1, \dots, w_k \rangle$*. Then*

$$M = \max\{|\operatorname{Orb}_\varphi(w_i)| \mid i \in [k]\}$$

is a bound on the size of orbits of eventually fixed points. We will see later that such a bound always exists if φ *is an endomorphism of a finitely generated virtually free group.*

Remark 4.3.5. *Conditions 1 and 2 are not mutually exclusive. However, if* $F_n = \bigcup_{k=1}^{\infty} \mathrm{Ker}(\varphi^k)$*, then every point is eventually sent to* 1 *and, by the observation above, orbit sizes must be bounded. So, this happens if and only if* φ *is a* vanishing endomorphism*, i.e, if there is some* r *such that* φ^r *maps every element to* 1*.*

So, we showed that $\mathrm{EvFix}(\varphi)$ is normal if and only if it is equal to the whole group F_n or to $\bigcup_{k=1}^{\infty} \mathrm{Ker}(\varphi^k)$, and both these conditions can hold simultaneously.

4.4 Finite Orbits

Given a finite orbit $\operatorname{Orb}_\varphi(x)$, we say that $\operatorname{Orb}_\varphi(x) \cap \mathrm{Per}(\varphi)$ is the *periodic part of the orbit* and $\operatorname{Orb}_\varphi(x) \setminus \mathrm{Per}(\varphi)$ is the *straight part of the orbit.*

$$\begin{tikzcd}
 & & & & & \cdots \arrow[dll, bend right] & \\
x \arrow[r] & x\varphi \arrow[r] & x\varphi^2 \arrow[r] & \cdots \arrow[r] & x\varphi^r \arrow[dr, bend right] & & x\varphi^{r+2} \arrow[ul, bend right] \\
 & & & & & x\varphi^{r+1} \arrow[ur, bend right] &
\end{tikzcd}$$

Figure 4.1 A finite orbit

In Figure 1, the straight part of the orbit corresponds to $\{x, x\varphi, \ldots, x\varphi^{r-1}\}$ and the periodic part of the orbit to $\{x\varphi^k \mid k \geq r\} = \{x\varphi^r, \ldots, x\varphi^{r+p-1}\}$, where p is the period of $x\varphi^r$.

In [80], the authors show that, for an automorphism of a free group F_n, there is an orbit of cardinality k if and only if there is an element of order k in $\mathrm{Aut}(F_n)$. Moreover, the authors prove that this result does not hold for general endomorphisms, by providing an example of an endomorphism of F_3 for which there is a point whose orbit has 5 elements. However, using a standard argument, we present a similar result for periodic parts of orbits of general endomorphisms of F_n and use this result to obtain a uniform bound on their size.

Lemma 4.4.1. *There is a periodic point of period k for some $\varphi \in \mathrm{End}(F_n)$ if and only if there is an element of order k in $\mathrm{Aut}(F_n)$.*

Proof. Let $x \in \mathrm{Per}(\varphi)$ be a periodic point of period k. Consider the stable image of φ,

$$S = \bigcap_{s \geq 1} F_n \varphi^s.$$

As highlighted in Section 4.1, we know that $\varphi|_S$ is an automorphism. Also, it is obvious that $\mathrm{Per}(\varphi) \subseteq S$, and so x is a point of S with a finite orbit of cardinality k. Therefore, by [80, Theorem 1.1], there is an element of order k in $\mathrm{Aut}(S)$. Since $\mathrm{rk}(S) = r \leq n$, then there is an automorphism of F_n of order k, which can be defined by applying the automorphism induced by $\varphi|_S$ to the first r letters and the identity in the remaining letters.

Conversely, if there is an element of order k in $\mathrm{Aut}(F_n)$, there is an orbit of cardinality k for some automorphism of F_n. Since finite orbits of automorphisms are periodic, the result follows. □

Corollary 4.4.2. *There is a computable constant k that bounds the size of the periodic parts of every orbit $\mathrm{Orb}_\varphi(x)$, when φ runs through $\mathrm{End}(F_n)$ and x runs through F_n.*

Proof. By [75] and [65], $\mathrm{Aut}(F_n)$ has an element of order $m = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \in \mathbb{N}$, where p_i's are different primes, if and only if $\sum_{i=1}^{s}(p_i^{\alpha_i} - p_i^{\alpha_i - 1}) \leq n$. We have that $\sum_{i=1}^{s}(p_i^{\alpha_i} - p_i^{\alpha_i-1}) = \sum_{i=1}^{s}(p_i - 1)p_i^{\alpha_i - 1}$ and so, if a natural number $m \in \mathbb{N}$ is the order of some automorphism of F_n, then it only admits in its factorization primes p such that $p - 1 \leq n$ and each of them can have exponent at most $\log_p(n) + 1$. There are finitely many integers in those conditions, and so, m

must be bounded above by some constant k that depends only on n. □

We now prove that, given an endomorphism of a finitely generated virtually free group, we can bound the size of periodic parts of finite orbits by a computable constant. We remark that this constant depends on the endomorphism unlike the one obtained in Corollary 4.4.2 for endomorphisms of free groups.

Proposition 4.4.3. *There exists an algorithm with input a finitely generated virtually free group G and output a constant k such that, for all $\varphi \in \mathrm{End}(G)$, the infinite ascending chain*

$$\mathrm{Fix}(\varphi) \subseteq \mathrm{Fix}(\varphi^{2!}) \subseteq \mathrm{Fix}(\varphi^{3!}) \subseteq \cdots$$

stabilizes after k steps. Equivalently, if $x \in \mathrm{EvPer}(\varphi)$, for some endomorphism φ of G, then the periodic part of the orbit of x has cardinality at most k.

Proof. Consider a decomposition

$$F = Fb_1 \cup \cdots \cup Fb_m,$$

where F is a fully invariant free normal subgroup of G. We want to compute k such that the ascending chain $\mathcal{C}$ defined by

$$\mathrm{Fix}(\varphi) \subseteq \mathrm{Fix}(\varphi^{2!}) \subseteq \mathrm{Fix}(\varphi^{3!}) \subseteq \cdots$$

stabilizes after at most k steps. For $i \in [m]$, consider the chains $\mathcal{C}_i$ given by

$$\mathrm{Fix}(\varphi) \cap Fb_i \subseteq \mathrm{Fix}(\varphi^{2!}) \cap Fb_i \subseteq \mathrm{Fix}(\varphi^{3!}) \cap Fb_i \subseteq \cdots$$

Since, for all $j \in \mathbb{N}$, we have that

$$\mathrm{Fix}(\varphi^{j!}) = \bigcup_{i\in[m]} (\mathrm{Fix}(\varphi^{j!}) \cap Fb_i),$$

it follows that $\mathcal{C}$ stabilizes after n steps if and only if all chains $\mathcal{C}_i$ stabilize after at most n steps.

We will prove that, for all $i \in [m]$, we can compute a constant k_i such that the chain $\mathcal{C}_i$ stabilizes after k_i steps and so, taking $k = \max\{k_i \mid i \in [m]\}$ suffices.

Let $i \in [m]$. Since F is fully invariant, we have that, for all $k \in \mathbb{N}$, $(Fb_i)\varphi^k \subseteq F(b_i\varphi^k)$. Hence, the mapping $\theta : G/F \to G/F$ defined by $Fb_i \mapsto F(b_i\varphi)$ is a well-defined endomorphism. Since G/F is finite, we can compute the orbit $\mathrm{Orb}_\theta(Fb_i)$ of Fb_i through θ. In particular, we can check if Fb_i is periodic. If it is not, then, $\mathrm{Fix}(\varphi^k) \cap Fb_i = \emptyset$, for all $k > 0$. Indeed, if there were some $k > 0$, $x \in F$ such that $(xb_i)\varphi^k = xb_i$, then, since $x\varphi^k \in F$, we have that

$$(Fb_i)\theta^k = F(b_i\varphi^k) = F((xb_i)\varphi^k) = F(xb_i) = Fb_i.$$

If Fb_i is periodic, then let p be its period and take $z \in F$ such that $b_i\varphi^p = zb_i$. Notice that $p \leq |G/F| = [G : F]$. Clearly, if $j > 0$ is such that $\mathrm{Fix}(\varphi^j) \cap Fb_i \neq \emptyset$, then p divides j. Also, let C be the bound given by Corollary 4.4.2 for $n = \mathrm{rk}(F) + 1$. Let c be a letter not belonging to the alphabet of F and $\psi : F * \langle c|\rangle \to F * \langle c|\rangle$ be defined by mapping the letters of the alphabet of F through φ^p and c to zc. Notice that, for all $j \in \mathbb{N}$,

$$c\psi^j = \left(\prod_{s=0}^{j-1} z\varphi^{(j-1-s)p}\right) c \quad \text{and} \quad b_i\varphi^{jp} = \left(\prod_{s=0}^{j-1} z\varphi^{(j-1-s)p}\right) b_i.$$

We claim that, for $x \in F$ and $q \in \mathbb{N}$,

$$(xc)\psi^q = xc \iff (xb_i)\varphi^{qp} = xb_i.$$

Indeed, let $x \in F$ and $q \in \mathbb{N}$ be such that

$$xc = (xc)\psi^q = x\varphi^{qp}c\psi^q = x\varphi^{qp}\left(\prod_{s=0}^{q-1} z\varphi^{(q-1-s)p}\right) c.$$

Then,

$$(xb_i)\varphi^{qp} = x\varphi^{qp}\left(\prod_{s=0}^{q-1} z\varphi^{(q-1-s)p}\right) b_i = xb_i.$$

The converse is analogous.

Since, by Corollary 4.4.2, the periods by the action of ψ are bounded above by C, then the periods of points in Fb_i by the action of φ are bounded above by Cp, which is computable since both C and p are. Hence, the chain $\mathcal{C}_i$ stabilizes after at most Cp steps. Notice that C depends only on the rank of F and $p \leq [G : F]$, and so our bound does not depend on the endomorphism φ. $\square$

An immediate consequence of the previous proposition is that studying the subgroup of eventually periodic points of φ is not more than studying the subgroup of eventually fixed points for a suitable power of φ. For this reason, we will focus on the study of $\mathrm{EvFix}(\varphi)$.

Corollary 4.4.4. *There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G and output a constant $k \in \mathbb{N}$ such that* $\mathrm{EvPer}(\varphi) = \mathrm{EvFix}(\varphi^{k!})$.

Proof. Let k be the constant given by Proposition 4.4.3. It is obvious that $\mathrm{EvFix}(\varphi^{k!}) \subseteq \mathrm{EvPer}(\varphi)$. Now, let $x \in \mathrm{EvPer}(\varphi)$. We have that there is some $s \in \mathbb{N}$ such that $x\varphi^s \in \mathrm{Per}(\varphi)$. By Proposition 4.4.3, the period of $x\varphi^s$ is bounded above by k, and so it divides $k!$. Take $n \in \mathbb{N}$ such that $nk! > s$. This way, we have that $x\varphi^{nk!}$ belongs to the periodic part of the orbit of x. Thus, $x\varphi^{nk!}\varphi^{k!} = x\varphi^{nk!}$ and so, $x \in \mathrm{EvFix}(\varphi^{k!})$. $\square$

Now we show that, for a fixed endomorphism, we can also bound the size of the straight part of the orbits by a computable constant, which, in combination with Proposition 4.4.3, gives us a way of computing an upper bound on the cardinality of finite orbits.

Proposition 4.4.5. *There exists an algorithm with input a finitely generated virtually free group G and output a constant k that bounds the size of the straight part of every finite orbit for all endomorphisms $\varphi \in \mathrm{End}(G)$.*

Proof. Let $\varphi \in \mathrm{End}(G)$. Consider a decomposition

$$G = Fb_1 \cup \cdots \cup Fb_m,$$

where F is a fully invariant free normal subgroup of G and write $\psi = \varphi|_F$. We can assume that $b_1 = 1$. For all $j \in \mathbb{N}$, consider the surjective mappings $\varphi_j : \mathrm{Im}(\varphi^j) \to \mathrm{Im}(\varphi^{j+1})$ and $\psi_j : \mathrm{Im}(\psi^j) \to \mathrm{Im}(\psi^{j+1})$ given by restricting φ. It suffices to prove that for some computable k, we have that φ_k is injective and this implies that the straight part of a finite orbit must contain at most k elements. Indeed, suppose that there is some $x \in \mathrm{EvPer}(\varphi)$ such that the straight part of $\mathrm{Orb}_\varphi(x)$ has $r > k$ elements. Put $y = x\varphi^r$ and let π be the period of y. Clearly, $y \in \mathrm{Im}(\varphi^r) \subseteq \mathrm{Im}(\varphi^k)$. Then $y = x\varphi^{r-1}\varphi_k$ and $y = y\varphi^{\pi-1}\varphi_k$. But $y\varphi^{\pi-1} \neq x\varphi^{r-1}$ since $x\varphi^{r-1}$ belongs to the straight part of the orbit and $y\varphi^{\pi-1}$ belongs to the periodic part. This contradicts the injectivity of φ_k.

So, it remains to prove that $\mathrm{Im}(\varphi^k) \simeq \mathrm{Im}(\varphi^{k+1})$ for some computable k, which, by hopfianity of $\mathrm{Im}(\varphi^k)$ implies that φ_k is injective. We have that, for all $i \in \mathbb{N}$,

$$G\varphi^i = F\varphi^i(b_1\varphi^i) \cup \cdots \cup F\varphi^i(b_m\varphi^i),$$

and so $F\varphi^i$ is a finite index subgroup of $G\varphi^i$ and $[G\varphi^i : F\varphi^i] \leq [G : F] = m$. Also, $0 \leq \mathrm{rk}(\mathrm{Im}(\psi^{i+1})) \leq \mathrm{rk}(\mathrm{Im}(\psi^i))$, for every $i \in \mathbb{N}$.

Now, we describe the algorithm to compute k. Start by computing the smallest positive integer $j_1 \in \mathbb{N}$ such that $\mathrm{rk}(\mathrm{Im}(\psi^{j_1+1})) = \mathrm{rk}(\mathrm{Im}(\psi^{j_1}))$. Clearly, j_1 is computable: we have generators for $\mathrm{Im}(\psi^i)$ for every $i \in \mathbb{N}$ and so we can compute its rank by computing the graph rank of its Stallings automaton. Also, $j_1 \leq \mathrm{rk}(F)$. If $\mathrm{rk}(\mathrm{Im}(\psi^{j_1})) = 0$, then ψ is a vanishing endomorphism and so $\mathrm{Im}(\varphi^{j_1})$ is finite. In that case, the orbits of elements in $\mathrm{Im}(\varphi^{j_1})$ must be finite, since $\mathrm{Im}(\varphi^k) \subseteq \mathrm{Im}(\varphi^{j_1})$, for $k > j_1$, and so after at most $|\,\mathrm{Im}(\varphi^{j_1})| \leq m$ iterations, we must reach a periodic point. We can compute the entire orbit of all the elements in $\mathrm{Im}(\varphi^{j_1})$, put M to be the cardinality of the largest orbit, $k = M + j_1$ and we are done. So, suppose that $\mathrm{Im}(\psi^{j_1})$ is nontrivial.

Since free groups are hopfian, a free group is not isomorphic to any of its proper quotients. Thus, ψ_{j_1} must be injective. If $\mathrm{Im}(\varphi^{j_1}) \simeq \mathrm{Im}(\varphi^{j_1+1})$, then, we are done. If not, then φ_{j_1} is not injective, and so there are some $f \in F$ and $i \in [m]$ such that $(fb_i)\varphi^{j_1} \neq 1$ and $((fb_i)\varphi^{j_1}\varphi) = 1$. Since ψ_{j_1} is injective, then $i \neq 1$. So, there is some $i \in \{2, \ldots, m\}$ such that $(fb_i)\varphi^{j_1+1} = 1$ and so $b_i\varphi^{j_1+1} \in F\varphi^{j_1+1}$, thus

$$[G\varphi^{j_1+1} : F\varphi^{j_1+1}] \leq m - 1$$

and for all $i \geq j_1$, $[G\varphi^i : F\varphi^i] \leq [G\varphi^{j_1+1} : F\varphi^{j_1+1}] \leq m-1$.

Now, we compute j_2, the second least positive integer with the property that $\mathrm{rk}(\mathrm{Im}(\psi^{j_2+1})) = \mathrm{rk}(\mathrm{Im}(\psi^{j_2}))$ and proceed as above. Notice that $j_2 \leq 2\,\mathrm{rk}(F)$. After m steps, we have either found k or we have that $G\varphi^{j_m} = F\varphi^{j_m} \simeq F\varphi^{j_m+1} = G\varphi^{j_m+1}$, and we are done. Therefore, the straight part of orbits through φ is bounded above by $\mathrm{rk}(F)m = \mathrm{rk}(F)[G : F]$. $\square$

Combining Proposition 4.4.3 and Proposition 4.4.5, we obtain the following corollaries.

Corollary 4.4.6. *There exists an algorithm with input a finitely generated virtually free group G and output a constant k such that*

$$\max\{|\operatorname{Orb}_\varphi(x)| \mid \varphi \in \operatorname{End}(G)\,,\, x \in \operatorname{EvPer}(\varphi)\} \leq k.$$

Given a finitely generated virtually free group G, we denote by C_G the computable constant that bounds the size of all finite orbits through endomorphisms of G.

Corollary 4.4.7. *There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G that decides whether* $\mathrm{EvFix}(\varphi)$ *is a normal subgroup of F_n or not.*

Proof. To decide if condition 1 in Proposition 4.3.2 holds, we check if the generators of the free group F_n are eventually fixed by computing the first C_{F_n} elements of their orbits. By Remark 4.3.3, condition 2 is equivalent to $\mathrm{Fix}(\varphi)$ being trivial which is known to be decidable (in fact, by [79], we can find a basis for $\mathrm{Fix}(\varphi)$). $\square$

An element x in a monoid S is said to be *aperiodic* if there is some $n > 0$ such that $x^n = x^{n+1}$. An element $x \in S$ has *finite order* if it generates a finite submonoid. Equivalently, x has finite order if there are distinct $p, q \in \mathbb{N}$ such that $x^p = x^q$.

Corollary 4.4.8. *There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G that decides whether φ is a finite order element of* $\mathrm{End}(G)$ *or not. We can also decide if φ is aperiodic or not.*

Proof. The endomorphism φ has finite order if there are distinct $p, q \in \mathbb{N}$ such that for every $x \in G$, $x\varphi^p = x\varphi^q$. Since the size of finite orbits is bounded above by a computable constant, we can check if the generators have finite orbits. If there is some generator a with infinite orbit, then $a\varphi^p \neq a\varphi^q$, for $p, q \in \mathbb{N}$ with $p \neq q$. If every generator is eventually periodic, then let p be the maximum length of the straight parts of the orbits of the generators, so that $a\varphi^p$ is a periodic point for every generator a and let m be the least common multipleof the lengths of the periodic parts. Then $a\varphi^p = a\varphi^{p+m}$ for every generator and so $\varphi^p = \varphi^{p+m}$.

Aperiodicity is similar. We have that there is an $m \in \mathbb{N}$ such that $\varphi^m = \varphi^{m+1}$ if and only if for each generator a there is a p such that $a\varphi^p = a\varphi^{p+1}$ and that is decidable simply by computing the orbits of the generators. $\square$

Corollary 4.4.9. *Let G be a finitely generated virtually free group. Then, for all $\varphi \in \mathrm{End}(G)$, the infinite ascending chain $\mathrm{Ker}(\varphi) \subseteq \mathrm{Ker}(\varphi^2) \subseteq \dots$ stabilizes and $\bigcup_{k=1}^{\infty} \mathrm{Ker}(\varphi^k) = \mathrm{Ker}(\varphi^{C_G})$.*

Proof. For $k > C_G$, if $x\varphi^k = 1$, then $x\varphi^{C_G} = 1$, since $|\,\mathrm{Orb}_\varphi(x)| \leq C_G$. □

The *join* of two subgroups $H_1, H_2 \leq G$ is the smallest subgroup of G containing H_1 and H_2. We will denote it by $H_1 \vee H_2 = \langle H_1 \cup H_2 \rangle$.

Corollary 4.4.10. *Let G be a finitely generated virtually free group. Then, for all $\varphi \in \mathrm{End}(G)$, $\mathrm{EvFix}(\varphi) = (\mathrm{Fix}(\varphi))\varphi^{-C_G} = \mathrm{Ker}(\varphi^{C_G}) \vee \mathrm{Fix}(\varphi)$ and $\mathrm{Fix}(\varphi) \simeq {}^{\mathrm{EvFix}(\varphi)}\!\big/\!_{\mathrm{Ker}(\varphi^{C_G})}$.*

Proof. We have that $\mathrm{EvFix}(\varphi)$ is the subgroup of points that get mapped to $\mathrm{Fix}(\varphi)$ by φ^{C_G}. Also, $\mathrm{Ker}(\varphi^{C_G}) \subseteq \mathrm{EvFix}(\varphi)$ and for every element $x \in \mathrm{EvFix}(\varphi)$, there is some $y \in \mathrm{Fix}(\varphi)$, such that $x\varphi^{C_G} = y = y\varphi^{C_G}$. So there must be some $z \in \mathrm{Ker}(\varphi^{C_G})$ such that $x = yz$. Thus, $\mathrm{EvFix}(\varphi) = \mathrm{Ker}(\varphi^{C_G}) \vee \mathrm{Fix}(\varphi)$.

Letting ψ denote the restriction of φ^{C_G} to $\mathrm{EvFix}(\varphi)$, we have that

$$\mathrm{Fix}(\varphi) = \mathrm{Im}(\psi) \simeq {}^{\mathrm{EvFix}(\varphi)}\!\big/\!_{\mathrm{Ker}(\psi)} = {}^{\mathrm{EvFix}(\varphi)}\!\big/\!_{\mathrm{Ker}(\varphi^{C_G})}.$$

□

We now present a proposition which, despite being easy and in the author's opinion, of independent interest, doesn't seem to appear in the literature. But first we present a well-known lemma.

Lemma 4.4.11. *Let G be a group and $H, K \leq G$. Then for all $x, y \in G$, $Hx \cap Ky$ is either empty or a coset of $H \cap K$.*

Proposition 4.4.12. *Let G be a finitely generated virtually free group having a free subgroup F of finite index, $H \leq_{f.g.} G$ and $\varphi \in \mathrm{End}(G)$ be an endomorphism. If $\mathrm{Ker}(\varphi)$ is finite, then $H\varphi^{-1}$ is finitely generated. If not, the following are equivalent:*

1. *$H\varphi^{-1}$ is finitely generated*
2. *$H\varphi^{-1} \cap F$ is a finite index subgroup of F*
3. *$H\varphi^{-1} \cap F$ is a finite index subgroup of G*
4. *$H\varphi^{-1}$ is a finite index subgroup of G*
5. *$H \cap G\varphi$ is a finite index subgroup of $G\varphi$*
6. *$H \cap F\varphi$ is a finite index subgroup of $F\varphi$*

Proof. It is easy to see that if $\mathrm{Ker}(\varphi)$ is finite, then $H\varphi^{-1}$ is finitely generated, since it is generated by the preimages of the generators of H together with the kernel.

So, assume that $\mathrm{Ker}(\varphi)$ is infinite. It is obvious that $2 \implies 3$ and that $3 \implies 4$. It is also well known that $4 \implies 1$. We will prove that $1 \implies 2$, $4 \iff 5$ and $5 \iff 6$ and that suffices.

We start by proving that $1 \implies 2$. Suppose that $H\varphi^{-1}$ is finitely generated. Since virtually free groups are Howson (free groups are Howson [58] and it is easy to see that the Howson property is preserved by taking finite extensions), then $H\varphi^{-1} \cap F$ is also finitely generated. Then, by Marshall Hall's Theorem, there is some finite index subgroup H' of F such that $H\varphi^{-1} \cap F$ is a free factor of H'. Take H'' such that

$$H' = (H\varphi^{-1} \cap F) * H''.$$

Since H' is a finite index subgroup of F, it is finitely generated. Let $A = \{a_1, \ldots, a_n\}$ be a basis of H' such that $H\varphi^{-1} \cap F = \langle a_1, \ldots, a_k \rangle$ and $H'' = \langle a_{k+1}, \ldots, a_n \rangle$. If H'' is trivial, then $H\varphi^{-1} \cap F = H'$ is a finite index subgroup of F and we are done. Suppose then that H'' is nontrivial. Obviously, since φ is noninjective, $\{1\} \neq \mathrm{Ker}(\varphi) \trianglelefteq H\varphi^{-1}$. Thus,

$$(\mathrm{Ker}(\varphi) \cap F) \trianglelefteq (H\varphi^{-1} \cap F).$$

Moreover, $\mathrm{Ker}(\varphi) \cap F$ is not trivial. Indeed, the fact that the kernel is infinite implies that $\varphi|_F$ is noninjective. Let $1 \neq x \in H''$ and $1 \neq y \in \mathrm{Ker}(\varphi) \cap F$. Then

$$xyx^{-1} \in (\mathrm{Ker}(\varphi) \cap F) \trianglelefteq (H\varphi^{-1} \cap F),$$

which is absurd, since the letters in x don't belong to the basis set of $(H\varphi^{-1} \cap F)$.

Now, we prove that $4 \implies 5$. Suppose that $H\varphi^{-1}$ is a finite index subgroup of G. Then there are $b_i \in G$, $i \in \{0, \ldots, k\}$, such that

$$G = b_0(H\varphi^{-1}) \cup \cdots \cup b_k(H\varphi^{-1}).$$

Clearly,

$$G\varphi = (b_0\varphi)(H \cap G\varphi) \cup \cdots \cup (b_k\varphi)(H \cap G\varphi)$$

and so $H \cap G\varphi$ is a finite index subgroup of $G\varphi$.

Similarly, if $H \cap G\varphi$ is a finite index subgroup of $G\varphi$, then there are $b_i \in G$, $i \in \{0, \ldots, k\}$, such that

$$G\varphi = (b_0\varphi)(H \cap G\varphi) \cup \cdots (b_k\varphi)(H \cap G\varphi).$$

Hence,

$$G = b_0(H\varphi^{-1}) \cup \cdots b_k(H\varphi^{-1}),$$

because, given $x \in G$, we have that there are some $y \in G$ and $i \in \{0, \ldots, k\}$ such that $y\varphi \in H$ and $x\varphi = (b_i\varphi)(y\varphi)$ and so $x = b_iyk$ for some $k \in \mathrm{Ker}(\varphi) \trianglelefteq H\varphi^{-1}$. So, $yk \in H\varphi^{-1}$. Hence $5 \implies 4$.

Finally, we prove that 5 $\iff$ 6. Assume that $H \cap G\varphi$ is a finite index subgroup of $G\varphi$. Then, there are $m > 0$ and elements $b_i \in G\varphi$ such that

$$G\varphi = (H \cap G\varphi)b_1 \cup \cdots \cup (H \cap G\varphi)b_m.$$

Thus,

$$F\varphi = G\varphi \cap F\varphi = ((H \cap G\varphi)b_1 \cap F\varphi) \cup \cdots \cup ((H \cap G\varphi)b_m \cap F\varphi)\,.$$

By Lemma 4.4.11, $H \cap F\varphi = H \cap G\varphi \cap F\varphi$ is a finite index subgroup of $F\varphi$ of index at most m. Conversely, since $F\varphi$ is a finite index subgroup of $G\varphi$, then, if $H \cap F\varphi$ is a finite index subgroup of $F\varphi$, it must also be a finite index subgroup of $G\varphi$, by transitivity. Since $H \cap G\varphi \geq H \cap F\varphi$, then $H \cap G\varphi$ is also a finite index subgroup of $G\varphi$. □

Notice that noninjective endomorphisms of free groups satisfy the hypothesis of Proposition 4.4.12. Also, the following corollary follows directly from the proof of Proposition 4.4.12.

Corollary 4.4.13. *The index of the subgroups in conditions 4 and 5 of Proposition 4.4.12 must coincide.*

Recall that, given a finitely generated virtually free group G, C_G is a bound to the size of all finite orbits of endomorphisms of G.

Corollary 4.4.14. *There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G that decides whether* $\mathrm{EvFix}(\varphi)$ *(resp.* $\mathrm{EvPer}(\varphi)$*) is finitely generated and, in case the answer is affirmative, computes a finite set of generators.*

Proof. Consider a decomposition

$$G = Fb_1 \cup \cdots \cup Fb_m,$$

where F is a fully invariant free normal subgroup of G and write $\psi = \varphi|_F$. We can assume that $b_1 = 1$.

Since F is fully invariant, we have that

$$G\varphi^{C_G} = F\varphi^{C_G}(b_1\varphi^{C_G}) \cup \cdots \cup F\varphi^{C_G}(b_m\varphi^{C_G}). \tag{4.10}$$

Now, it is easy to see that φ^{C_G} has finite kernel if and only if ψ^{C_G} is injective, which is decidable, since, by hopfianity, ψ^{C_G} is injective if and only if $\mathrm{rk}(F) = \mathrm{rk}(F\psi^{C_G})$. If φ^{C_G} has finite kernel, then it must be computable. Indeed, if, for $x \in F$, $(xb_i)\varphi^{C_G} = 1$, then $x\varphi^{C_G} = x\psi^{C_G} = b_i^{-1}\varphi^{C_G}$, and so for all $i \in [m]$, we check if $b_i^{-1}\varphi^{C_G} \in \mathrm{Im}(\psi^{C_G})$, and if it is, we compute $x \in F$ such that $x\psi^{C_G} = b_i^{-1}\varphi^{C_G}$. We then have that $\mathrm{EvFix}(\varphi) = (\mathrm{Fix}(\varphi))\varphi^{-C_G}$ is finitely generated and, since $\mathrm{Fix}(\varphi)$ is computable by Theorem 4.1.1 and $\mathrm{Ker}(\varphi^{C_G})$ is computable, a set of generators for $\mathrm{EvFix}(\varphi)$ can be computed. So, suppose that the kernel of φ^{C_G} is infinite.

We know that $\mathrm{EvFix}(\varphi) = (\mathrm{Fix}(\varphi))\varphi^{-C_G}$ and so, by Proposition 4.4.12, it is finitely generated if and only if $\mathrm{Fix}(\varphi) \cap F\varphi^{C_G}$ is a finite index subgroup of $F\varphi^{C_G}$. Using Theorem 4.1.1, we can compute a basis for $\mathrm{Fix}(\varphi)$, and so compute a set of generators for $\mathrm{Fix}(\varphi) \cap F\varphi^{C_G}$. Notice that, since F is fully invariant then $F\varphi^{C_G}$ (and so $\mathrm{Fix}(\varphi) \cap F\varphi^{C_G}$) is a subgroup of F. Then, we can decide if $\mathrm{Fix}(\varphi) \cap F\varphi^{C_G}$ has finite index on $F\varphi^{C_G}$ and compute right coset representatives $b'_i \in F\varphi^{C_G}$

$$F\varphi^{C_G} = \left(\mathrm{Fix}(\varphi) \cap F\varphi^{C_G}\right) b'_1 \cup \cdots \cup \left(\mathrm{Fix}(\varphi) \cap F\varphi^{C_G}\right) b'_k. \tag{4.11}$$

Combining (4.10) and (4.11), we have that

$$G\varphi^{C_G} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{k} (\mathrm{Fix}(\varphi) \cap F\varphi^{C_G}) b'_j (b_i \varphi^{C_G}),$$

and so

$$G\varphi^{C_G} = \bigcup_{i=1}^{m} \bigcup_{j=1}^{k} \mathrm{Fix}(\varphi) b'_j (b_i \varphi^{C_G}).$$

By testing membership in $\mathrm{Fix}(\varphi) \cap G\varphi^{C_G}$, we can check whether any two cosets coincide and refine the decomposition to obtain a proper subdecomposition where all cosets are distinct of the form (eventually relabeling the coset representatives)

$$G\varphi^{C_G} = \bigcup_{i=1}^{m'} \bigcup_{j=1}^{k'} (\mathrm{Fix}(\varphi) \cap G\varphi^{C_G}) b'_j (b_i \varphi^{C_G}) = \bigcup_{i=1}^{m'} \bigcup_{j=1}^{k'} (\mathrm{Fix}(\varphi)) b'_j (b_i \varphi^{C_G}),$$

where $k' \leq k$ and $m' \leq m$.

For all $(i,j) \in [m'] \times [k']$, we can compute $b_{i,j} \in G$ such that $b_{i,j}\varphi^{C_G} = b'_j(b_i\varphi^{C_G})$, and so, by the proof of Proposition 4.4.12,

$$G = \bigcup_{i=1}^{m'} \bigcup_{j=1}^{k'} (\mathrm{Fix}(\varphi)\varphi^{-C_G}) b_{i,j}.$$

Having the coset representative elements $b_{i,j}$ and being able to check membership on $\mathrm{Fix}(\varphi)\varphi^{-C_G}$, we can compute a set of generators for $\mathrm{Fix}(\varphi)\varphi^{-C_G} = \mathrm{EvFix}(\varphi)$. By Corollary 4.4.4, it is clear that the result also holds for $\mathrm{EvPer}(\varphi)$. □

Finally, we will prove that in the cases where $\mathrm{EvFix}(\varphi)$ is finitely generated, we can bound its rank. The rank of a finitely generated virtually free group G is defined as the minimal cardinality of a set of generators of G.

Proposition 4.4.15. *Let G be a finitely generated virtually free group, $\varphi \in \mathrm{End}(G)$ and F be a fully invariant free normal subgroup of G. If $\mathrm{EvFix}(\varphi)$ is finitely generated, then* $\mathrm{rk}(\mathrm{EvFix}(\varphi)) \leq \log_2([G : F]) + \max\{\mathrm{rk}(F), \mathrm{rk}(F)^2 - 3\,\mathrm{rk}(F) + 3\}$.

Proof. If φ is injective, then $\mathrm{rk}(\mathrm{EvFix}(\varphi)) = \mathrm{rk}(\mathrm{Fix}(\varphi)) \leq \mathrm{rk}(F) + \log_2([G:F])$, by Remark 4.1.2. So, assume that φ is not injective and put $\psi = \varphi|_F$. We have that $\mathrm{EvFix}(\psi) = \mathrm{EvFix}(\varphi) \cap F$ and $C_F \leq C_G$. Thus, by [4, Proposition 2.1],

$$[\mathrm{EvFix}(\varphi) : \mathrm{EvFix}(\psi)] \leq \log_2([G:F]).$$

It follows that $\mathrm{EvFix}(\psi)$ is finitely generated and $\mathrm{rk}(\mathrm{EvFix}(\varphi)) \leq \mathrm{rk}(\mathrm{EvFix}(\psi)) + \log_2([G:F])$.

If ψ is injective, then $\mathrm{rk}(\mathrm{EvFix}(\psi)) = \mathrm{rk}(\mathrm{Fix}(\psi)) \leq \mathrm{rk}(F)$. So, assume that ψ is noninjective.

If $\mathrm{rk}(F\varphi^{C_F}) = 0$, then ψ is a vanishing endomorphism, and so $\mathrm{EvFix}(\psi) = F$ and, in this case, we have that $\mathrm{rk}(\mathrm{EvFix}(\varphi)) \leq \mathrm{rk}(F) + \log_2([G:F])$.

If $\mathrm{rk}(F\varphi^{C_F}) = 1$, then $F\varphi^{C_F}$ is abelian, thus $\mathrm{Fix}(\psi) \trianglelefteq F\varphi^{C_F}$, and so

$$\mathrm{EvFix}(\psi) = \mathrm{Fix}(\psi)\psi^{-C_F} \trianglelefteq F.$$

By Proposition 4.3.2, we must have that either $\mathrm{EvFix}(\psi) = F$, in which case we are done, or $\mathrm{EvFix}(\psi) = \bigcup_{k=1}^{\infty} \mathrm{Ker}(\psi^k) = \mathrm{Ker}(\psi^{C_F})$. But we know that $\mathrm{EvFix}(\psi)$ is a finite index subgroup of F, because $\mathrm{EvFix}(\psi) = (\mathrm{Fix}(\psi))\psi^{-C_F}$ is finitely generated (see condition 4 of Proposition 4.4.12 with $G = F$). This means that $\mathrm{Im}(\psi^{C_F})$ is finite, which implies that ψ^{C_F} is trivial. Hence, in this case ψ is a vanishing endomorphism, and so $\mathrm{EvFix}(\psi) = F$. In this case, we have that $\mathrm{rk}(\mathrm{EvFix}(\varphi)) \leq \mathrm{rk}(F) + \log_2([G:F])$.

Now, suppose that $\mathrm{rk}(F\varphi^{C_F}) > 1$. Since $\mathrm{EvFix}(\psi)$ is finitely generated, then, by condition 5 of Proposition 4.4.12 (with $G = F$), $\mathrm{Fix}(\psi)$ must be a finite index subgroup of $F\psi^{C_F}$. By [71, Proposition 3.9] and [100, Corollary 2] , we have that

$$[F\varphi^{C_F} : \mathrm{Fix}(\psi)] = \frac{\mathrm{rk}(\mathrm{Fix}(\psi)) - 1}{\mathrm{rk}(F\varphi^{C_F}) - 1} \leq \mathrm{rk}(F) - 2$$

By Corollary 4.4.13, we have that $[F : \mathrm{EvFix}(\psi)] = [F\varphi^{C_F} : \mathrm{Fix}(\psi)] \leq \mathrm{rk}(F) - 2$. But now, using [71, Proposition 3.9] again, we get that

$$\frac{\mathrm{rk}(\mathrm{EvFix}(\psi)) - 1}{\mathrm{rk}(F) - 1} = [F : \mathrm{EvFix}(\psi)] \leq \mathrm{rk}(F) - 2,$$

and so $\mathrm{rk}(\mathrm{EvFix}(\psi)) \leq \mathrm{rk}(F)^2 - 3\,\mathrm{rk}(F) + 3$ and

$$\mathrm{rk}(\mathrm{EvFix}(\varphi)) \leq \mathrm{rk}(F)^2 - 3\,\mathrm{rk}(F) + 3 + \log_2([G:F]).$$

□

We ignore if this bound can be improved.

4.5 Brinkmann's problem and the φ-spectrum of a finite subset

In [17], Brinkmann proved that $BrP_{Aut}(F_n)$ is decidable, i.e., that given an automorphism $\varphi \in \mathrm{Aut}(F_n)$ and elements $x, y \in G$, we can decide whether there exists some $n \in \mathbb{N}$ such that $x\varphi^n = y$. Logan generalized this result for general endomorphisms in [68]. Using this result, we can prove decidability of $BrP_{End}(G)$ when G is a finitely generated virtually free group.

Theorem 4.5.1. *Let G be a finitely generated virtually free group. Then $BrP_{End}(G)$ is decidable.*

Proof. Consider a decomposition

$$F = Fb_1 \cup \cdots \cup Fb_m,$$

where F is a fully invariant free normal subgroup of G. Let $\varphi \in \mathrm{End}(G)$ and $g, h \in G$ be our input. We can assume that g, h are given as ub_i and vb_j where $u, v \in F$ and $i, j \in [m]$. So, we want to decide if there is some $k \in \mathbb{N}$ such that $(ub_i)\varphi^k = vb_j$. Since F is fully invariant, the mapping $\theta : G/F \to G/F$ defined by $Fb_i \mapsto F(b_i\varphi)$ is a well-defined endomorphism. Since G/F is finite, we can compute the entire θ-orbit of Fb_i, which must be finite. Hence, we can verify if $Fb_j \in \mathrm{Orb}_\theta(Fb_i)$, and the set $\{r \in \mathbb{N} \mid Fb_i\theta^r = Fb_j\}$ is either empty or of the form $s + p\mathbb{N}$ with computable $s, p \in \mathbb{N}$.

If, for some $k \in \mathbb{N}$, $(ub_i)\varphi^k = vb_j$, then $vb_j = (u\varphi^k)(b_i\varphi^k) \in F(b_i\varphi^k) = (Fb_i)\theta^k$. So, our candidate values for k are precisely $s + p\mathbb{N}$. Indeed, we denote by s the first time that Fb_j occurs in the θ-orbit of Fb_i and by p, the θ-period of Fb_j (which might be 0, in which case there is only one candidate). Compute $y \in F$ such that $(ub_i)\varphi^s = yb_j$ and $z \in F$ such that $b_j\varphi^p = zb_j$. It is easy to check by induction that, for all $d \in \mathbb{N}$, we have

$$b_j\varphi^{dp} = \left(\prod_{i=0}^{d-1} z\varphi^{(d-i-1)p}\right) b_j.$$

If $y = v$ then we answer yes and output s. Otherwise, let c be a new letter, not belonging to A and let $\psi : F * \langle c|\rangle \to F * \langle c|\rangle$ be defined by mapping x to $x\varphi^p$ for every $x \in F$ and c to zc. Also by induction, we can check that

$$c\psi^d = \left(\prod_{i=0}^{d-1} z\varphi^{(d-i-1)p}\right) c,$$

for all $d \in \mathbb{N}$.

We claim that there is some $k \in \mathbb{N}$ such that $(ub_i)\varphi^k = vb_j$ if and only if there is some $k \in \mathbb{N}$ such that $(yc)\psi^k = vc$, which can be decided using $BrP(F_n)$. We have that there is $k \in \mathbb{N}$ such that $(ub_i)\varphi^k = vb_j$ if and only if there is some $d \in \mathbb{N}$ such that $(ub_i)\varphi^{s+pd} = vb_j$, i.e., if there is some $d \in \mathbb{N}$ such that $(yb_j)\varphi^{pd} = vb_j$. We then have the following series of implications, which concludes the proof:

$$\begin{aligned}
& vb_j = (yb_j)\varphi^{pd} \\
\iff \quad & vb_j = (y\varphi^{pd})(b_j\varphi^{pd}) \\
\iff \quad & vb_j = (y\varphi^{pd})\left(\prod_{i=0}^{d-1} z\varphi^{(d-i-1)p}\right) b_j \\
\iff \quad & v = (y\varphi^{pd})\left(\prod_{i=0}^{d-1} z\varphi^{(d-i-1)p}\right) \\
\iff \quad & v = (y\psi^{d})\left(\prod_{i=0}^{d-1} z\varphi^{(d-i-1)p}\right) \\
\iff \quad & vc = (y\psi^{d})(c\psi^{d}) \\
\iff \quad & vc = (yc)\psi^{d}.
\end{aligned}$$

□

Clearly, for any group G and any subclass of endomorphisms $\mathcal{E}$, decidability of $BrP_{\mathcal{E}}(G)$ implies decidability of the Generalized Brinkmann's Problem $GBrP_{(\mathcal{E},Fin(G))}(G)$, where $Fin(G)$ denotes the class of finite subsets of G. Indeed, if we want to decide whether, given $\varphi \in \mathcal{E}$, $x \in G$ and $K = \{y_1, \dots, y_m\} \subseteq G$, there is some $n \in \mathbb{N}$ such that $x\varphi^n \in K$, we simply have to decide if there is some $n \in \mathbb{N}$ such that $x\varphi^n = y_i$, for $i \in [m]$.

Let G be a finitely generated virtually free group, $K \subseteq G$ be a subset of G, $g \in G$ be an element and $\varphi \in \text{End}(G)$ be an endomorphism. Inspired by the terminology in [38], we say that the *relative φ-order of g in K*, $\varphi\text{-ord}_K(g)$, is the smallest nonnegative integer k such that $g\varphi^k \in K$. If there is no such k, we say that $\varphi\text{-ord}_K(g) = \infty$. The *$\varphi$-spectrum* of a subset $\varphi\text{-sp}(K)$ is the set of relative φ-orders of elements in K, i.e., $\varphi\text{-sp}(K) = \{\varphi\text{-ord}_K(g) \mid g \in G\}$. A *$\varphi$-preorder* is the set of elements of a given relative φ-order in K. For a nonnegative integer $n \in \mathbb{N}$, we denote the φ-preorder of n in K by $\varphi\text{-pord}_K(n) = \{g \in G \mid \varphi\text{-ord}_K(g) = n\}$. We will now prove that, for a finite set K, $\varphi\text{-sp}(K)$ is computable.

It is clear from the definitions that

$$\varphi\text{-pord}_K(n) = K\varphi^{-n} \setminus \bigcup_{i=0}^{n-1} K\varphi^{-i}$$

and

$$\varphi\text{-sp}(K) = \{n \in \mathbb{N} \mid \varphi\text{-pord}_K(n) \neq \emptyset\}. \tag{4.12}$$

The following lemma is obvious by the definition.

Lemma 4.5.2. *Let G be a group, $K \subseteq G$, $\varphi \in \mathrm{End}(G)$ and $g \in G \setminus K$. Then,*

$$\varphi\text{-}ord_K(g\varphi) = \varphi\text{-}ord_K(g) - 1.$$

For $n \in \mathbb{N}$, we denote the set $\{0, 1, \cdots, n\}$ by $[n]_0$.

Corollary 4.5.3. *Let G be a group, $K \subseteq G$ and $\varphi \in \mathrm{End}(G)$. If, for some $n \in \mathbb{N}$, φ-pord$_K(n) = \emptyset$, then φ-pord$_K(m) = \emptyset$ for all $m \geq n$. Moreover,*

$$\varphi\text{-}sp(K) = \begin{cases} [n]_0 & \text{if } n = \min\{m \in \mathbb{N} \mid \varphi\text{-}pord_K(m+1) = \emptyset\} \\ \mathbb{N} & \text{if } \forall m \in \mathbb{N},\ \varphi\text{-}pord_K(m) \neq \emptyset \end{cases}$$

Proof. Suppose that φ-pord$_K(n) = \emptyset$. If φ-pord$_K(n+1) \neq \emptyset$, let $g \in \varphi$-pord$_K(n+1)$. Then φ-ord$_K(g\varphi) = n$ and so $g \in \varphi$-pord$_K(n)$, which is absurd. So, φ-pord$_K(n) = \emptyset$ implies that φ-pord$_K(n+1) = \emptyset$ and the result follows by induction.

The observation about φ-sp(K) can be proved in the same way. $\square$

We will start by the simpler case where K is a singleton. As done in the previous sections, we will denote the restriction of φ to $\mathrm{Im}(\varphi^k)$ by φ_k. Sometimes, we will also restrict the codomain to $\mathrm{Im}(\varphi^{k+1})$, or even $\mathrm{Im}(\varphi^k)$, but it should be clear from the situation what the codomain is.

We now present two technical lemmas related to kernels of powers of φ that will be useful later.

Lemma 4.5.4. *Let G be a group, $\varphi \in \mathrm{End}(G)$, $i, j \in \mathbb{N}$ be such that $i \geq j$ and $a_1, \ldots, a_n \in \mathrm{Ker}(\varphi^i)$. The following are equivalent:*

1. $\mathrm{Ker}(\varphi^i) = a_1 \mathrm{Ker}(\varphi^j) \cup \cdots \cup a_n \mathrm{Ker}(\varphi^j)$;
2. $\mathrm{Ker}(\varphi_j^{i-j}) = \{a_1\varphi^j, \ldots, a_n\varphi^j\}$.

Proof. Assume 1. Clearly, for all $r \in \{1, \ldots, n\}$, $a_r\varphi^j\varphi_j^{i-j} = a_r\varphi^i = 1$. Now, let $x \in \mathrm{Ker}(\varphi_j^{i-j})$. Then, there is some $y \in G$ such that $x = y\varphi^j$ and $y\varphi^i = y\varphi^{j+i-j} = 1$. Hence $y \in a_r \mathrm{Ker}(\varphi^j)$ for some $r \in \{1, \ldots, n\}$, and so $x = y\varphi^j = a_r\varphi^j$.

Now we prove that $2 \implies 1$. It is clear that $a_1 \mathrm{Ker}(\varphi^j) \cup \cdots \cup a_n \mathrm{Ker}(\varphi^j) \subseteq \mathrm{Ker}(\varphi^i)$. Now, let $x \in \mathrm{Ker}(\varphi^i)$. Then, $1 = x\varphi^i = x\varphi^j\varphi_j^{i-j}$ and so $x\varphi^j = a_r\varphi^j$ for some $r \in \{1, \ldots, n\}$ and so $x \in a_r \mathrm{Ker}(\varphi^j)$. $\square$

As a particular case, we obtain the following.

Lemma 4.5.5. *Let G be a group, $k \in \mathbb{N}$ and $\varphi \in \mathrm{End}(G)$. The following are equivalent:*

1. *φ_k is injective;*
2. $\mathrm{Ker}(\varphi^k) = \mathrm{Ker}(\varphi^i)$ *for all $i > k$;*

3. $\mathrm{Ker}(\varphi^k) = \mathrm{Ker}(\varphi^i)$ *for some* $i > k$.

Proof. If φ_k is injective, then $\mathrm{Ker}(\varphi_k^{i-k}) = \{1\varphi^k\}$, for all $i > k$. By Lemma 4.5.4, this implies that, for all $i > k$, $\mathrm{Ker}(\varphi^i) = \mathrm{Ker}(\varphi^k)$. So 1 implies 2. Since 2 trivially implies 3, we only have to verify that 3 $\implies$ 1. Suppose that there is some $i > k$ such that $\mathrm{Ker}(\varphi^k) = \mathrm{Ker}(\varphi^i)$. Then, by Lemma 4.5.4, it follows that φ_k^{i-k} is injective, and so must be φ_k, since $\mathrm{Ker}(\varphi_k) \leq \mathrm{Ker}(\varphi_k^{i-k})$. $\square$

Theorem 4.5.6. *There exists an algorithm with input a finitely generated virtually free group G and an endomorphism φ of G and output a constant C such that, for all $K = \{h\} \subseteq \mathrm{Per}(\varphi)$, then φ-sp$(K) = [n]_0$, for some $n \leq C$. Moreover, for such a K, φ-sp(K) is computable.*

Proof. Let G be a finitely generated virtually free group. From the proof of Proposition 4.4.5, it follows that there is a computable constant k such that the restriction of φ, $\varphi_k : \mathrm{Im}(\varphi^k) \to \mathrm{Im}(\varphi^{k+1})$, is injective. Also, from Proposition 4.4.3, there is a computable constant p such that every periodic orbit has length bounded by p. Let $C = p + k - 1$, $K = \{h\} \subseteq \mathrm{Per}(\varphi)$ and π_h denote the period of h. Suppose that there is an element $x \in G$ such that φ-$\mathrm{ord}_K(x) = C + 1$. Notice that $C + 1 - \pi_h = p + k - \pi_h \geq k$. Then,

$$x\varphi^{C+1-\pi_h}\varphi^{\pi_h} = x\varphi^{C+1-\pi_h}\varphi_k^{\pi_h} = h = h\varphi_k^{\pi_h}$$

and so, since $\varphi_k^{\pi_h}$ is injective, then $x\varphi^{C+1-\pi_h} = h$, which contradicts the assumption that φ-$\mathrm{ord}_K(x) = C + 1$. Hence, we have that φ-sp$(K) = [n]_0$, for some $n \leq C$.

Now, we have that $n \geq \pi_h - 1$, since φ-$\mathrm{ord}_K(h\varphi) = \pi_h - 1$. We will now prove that, given $\pi_h \leq i \leq C$, we can decide if $i \in \varphi$-sp(K) or not, and that suffices. Fix such i. By definition, $x \in G$ is a point of order i if and only if $x\varphi^i = h$ but $x\varphi^j \neq h$ for all $j < i$. This happens exactly when $x\varphi^i = h$ and $x\varphi^{i-\pi_h} \neq h$ because, if $x\varphi^j = h$, for some $j < i$, then, for $r > j$, $x\varphi^r = h$ if and only if $r = j + k\pi_h$. Write $i = k\pi_h + r$, with $0 \leq r < \pi_h$. The set of elements that get mapped to h through φ^i (resp. $\varphi^{i-\pi_h}$) is $h\varphi^{\pi_h - r}\,\mathrm{Ker}(\varphi^i)$ (resp. $h\varphi^{\pi_h-r}\,\mathrm{Ker}(\varphi^{i-\pi_h})$). It follows that there is an element of order i if and only if $\mathrm{Ker}(\varphi^i) \setminus \mathrm{Ker}(\varphi^{i-\pi_h}) \neq \emptyset$. Since $\mathrm{Ker}(\varphi^{i-\pi_h}) \subseteq \mathrm{Ker}(\varphi^i)$, then $\mathrm{Ker}(\varphi^i) \setminus \mathrm{Ker}(\varphi^{i-\pi_h}) \neq \emptyset$ if and only if $\mathrm{Ker}(\varphi^i) \neq \mathrm{Ker}(\varphi^{i-\pi_h})$. By Lemma 4.5.5, $\mathrm{Ker}(\varphi^i) = \mathrm{Ker}(\varphi^{i-\pi_h})$ if and only if $\varphi_{i-\pi_h}$ is injective, which we can decide, since it is equivalent to deciding whether $\mathrm{Im}(\varphi^{i-\pi_h}) \simeq \mathrm{Im}(\varphi^{i-\pi_h+1})$ or not: if $\varphi_{i-\pi_h}$ is injective, then obviously, $\mathrm{Im}(\varphi^{i-\pi_h}) \simeq \mathrm{Im}(\varphi^{i-\pi_h+1})$; we have that $\mathrm{Im}(\varphi^{i-\pi_h+1}) \simeq \mathrm{Im}(\varphi^{i-\pi_h}) / \mathrm{Ker}(\varphi_{i-\pi_h})$ and if $\mathrm{Im}(\varphi^{i-\pi_h}) \simeq \mathrm{Im}(\varphi^{i-\pi_h+1})$, we have that $\mathrm{Im}(\varphi^{i-\pi_h}) \simeq \mathrm{Im}(\varphi^{i-\pi_h}) / \mathrm{Ker}(\varphi_{i-\pi_h})$, which implies that $\mathrm{Ker}(\varphi_{i-\pi_h})$ is trivial, because virtually free groups are hopfian. $\square$

Corollary 4.5.7. *Let G be a finitely generated virtually free group $K = \{h\} \subseteq G$, and $\varphi \in \mathrm{End}(G)$. Then the following are equivalent:*

1. φ-*sp*$(K) = \mathbb{N}$;

2. $h \in \varphi^\infty(G) \setminus \mathrm{Per}(\varphi)$.

Proof. If φ-sp$(K) = \mathbb{N}$, then, obviously, $h \in \operatorname{Im}(\varphi^k)$, for all $k \in \mathbb{N}$. Also, $h \notin \operatorname{Per}(\varphi)$ by Proposition 4.5.6.

Conversely, if $h \in \varphi^\infty(G) \setminus \operatorname{Per}(\varphi)$, then, let $n \in \mathbb{N}$. Since $h \in \operatorname{Im}(\varphi^n)$, there is some x such that $x\varphi^n = h$. Since h is not periodic, we know that $x\varphi^k \neq h$, for every $k < n$, since otherwise we would have $h = x\varphi^n = x\varphi^k\varphi^{n-k} = h\varphi^{n-k}$. Thus φ-ord$_K(x) = n$ and so $n \in \varphi$-sp(K). □

Now, we can apply the computability results on the fixed subgroup and on the stable image of an endomorphism to compute the spectrum of a singleton

Corollary 4.5.8. *There exists an algorithm with input a finitely generated virtually free group G, an endomorphism φ of G and a singleton $K = \{h\} \subseteq G$ and output φ-sp(K).*

Proof. Let G be a finitely generated virtually free group. Then, we can compute $\varphi^\infty(G)$ by Theorem 4.1.4 and test if $h \in \varphi^\infty(G)$. If not, then by successively testing membership of h in $\operatorname{Im}(\varphi^k)$ we compute $m = \max\{k \in \mathbb{N} \mid h \in \operatorname{Im}(\varphi^k)\}$ (might be 0 if $h \notin \operatorname{Im}(\varphi)$), and we have that φ-sp$(K) = [m]_0$. So, assume that $h \in \varphi^\infty(G)$. Since the length of periodic orbits is bounded by a computable constant p and the fixed subgroup of φ is computable, then $\operatorname{Per}(\varphi) = \operatorname{Fix}(\varphi^{p!})$ is also computable by Theorem 4.1.1. If $h \notin \operatorname{Per}(\varphi)$, then, by Corollary 4.5.7, we have that φ-sp$(K) = \mathbb{N}$. If, on the other hand, $h \in \operatorname{Per}(\varphi)$, φ-sp(K) is computable by Theorem 4.5.6. □

Finally, after solving the case where K is a singleton, we can tackle the problem of computing the spectrum of an arbitrary finite set. We start with an easy lemma.

Lemma 4.5.9. *Let G be a group, $K \subseteq G$ and $\varphi \in \operatorname{End}(G)$. Then*

$$\varphi\text{-}sp(K) \subseteq \bigcup_{y \in K} \varphi\text{-}sp(\{y\}).$$

Proof. Let $n \in \varphi$-sp(K). There is some $x \in G$ such that $x\varphi^n \in K$ and $x\varphi^i \notin K$ for $i < n$. In particular, $x\varphi^i \neq x\varphi^n$ for $i < n$, so $n \in \varphi$-sp$(\{x\varphi^n\}) \subseteq \bigcup_{y \in K} \varphi$-sp$(\{y\})$. □

Theorem 4.5.10. *There exists an algorithm with input a finitely generated virtually free group G, an endomorphism φ of G and a finite set $K = \{g_1, \ldots, g_k\} \subseteq G$ and output φ-sp(K).*

Proof. We start by proving that φ-sp$(K) = \mathbb{N}$ if and only if φ-sp$(\{g_i\}) = \mathbb{N}$ for some $i \in \{1, \ldots, k\}$. Suppose that φ-sp$(K) = \mathbb{N}$. By Lemma 4.5.9, φ-sp$(K) \subseteq \bigcup_{i=1}^k \varphi$-sp$(\{g_i\})$, and so, there must be some $i \in \{1, \ldots, k\}$ for which φ-sp$(\{g_i\}) = \mathbb{N}$. Now suppose that φ-sp$(K) = [m]_0$, for some $m \in \mathbb{N}$ and that there is some $i \in \{1, \ldots, k\}$ such that φ-sp$(\{g_i\}) = \mathbb{N}$. Notice that it follows from Theorem 4.5.6 that g_i cannot be a periodic point. Put

$$I = \{j \in \{1, \ldots, k\} \mid 0 < \varphi\text{-ord}_{\{g_i\}}(g_j) < \infty\}.$$

If $I = \emptyset$, then, we can observe that $\mathbb{N} = \varphi$-sp$(\{g_i\}) \subseteq \varphi$-sp$(K)$, since, if $n \in \varphi$-sp$(\{g_i\})$, then there is some $x \in G$ such that φ-ord$_{\{g_i\}}(x) = n$. Clearly, φ-ord$_K(x) = n$ since, if there was some $j < n$ for which $x\varphi^j \in K$, then $x\varphi^j = g_r$ for some $r \in \{1, \ldots, k\}$ (and $r \neq i$ since $\varphi\text{-ord}_{\{g_i\}}(x) = n$) and

$$g_r\varphi^{n-j} = x\varphi^j\varphi^{n-j} = x\varphi^n = g_i$$

which, means that $0 < \varphi\text{-ord}_{\{g_i\}}(g_r) \leq n - j < \infty$ and that contradicts the assumption that $I = \emptyset$.

So, suppose that $I \neq \emptyset$ and let $C = \max\limits_{j \in I} \varphi\text{-ord}_{\{g_i\}}(g_j)$ and $x \in G$ be such that $M = \varphi\text{-ord}_{\{g_i\}}(x) > m + C$. Since $\varphi\text{-ord}_K(x) \leq m$, there are some $r \in [m]$ and $s \in \{1, \ldots, k\}$ such that $x\varphi^r = g_s$. Since

$$g_s\varphi^{M-r} = x\varphi^r\varphi^{M-r} = x\varphi^M = g_i,$$

then, since g_i is not periodic, $s \in I$ and, by definition of C, there must be some $p \leq C$ such that $g_s\varphi^p = g_i$. But, since $M - r > C$, then $M - r - p > 0$ and

$$g_i\varphi^{M-r-p} = (g_s\varphi^p)\varphi^{M-r-p} = g_s\varphi^{M-r} = g_i,$$

which is absurd since g_i is not periodic, by Theorem 4.5.6.

Therefore, we have proved that $\varphi\text{-sp}(K) = \mathbb{N}$ if and only if $\varphi\text{-sp}(\{g_i\}) = \mathbb{N}$ for some $i \in \{1, \ldots, k\}$, which is a decidable condition in view of Corollary 4.5.8.

So, assume that, for all $i \in \{1, \ldots, k\}$, $\varphi\text{-sp}(\{g_i\}) = [n_i]_0$, for some $n_i \in \mathbb{N}$. Then, by Lemma 4.5.9, $\varphi\text{-sp}(K) = [m]_0$ for some $m \leq \max_{i \in \{1,\ldots,k\}} n_i$. We will now prove that, given $n \leq \max_{i \in \{1,\ldots,k\}} n_i$ we can decide whether $n \in \varphi\text{-sp}(K)$ or not and that suffices. So, fix such an n. Put

$$\ell = \{i \in \{1, \ldots, k\} \mid \varphi\text{-sp}(\{g_i\}) \subsetneq [n]_0\} \quad \text{and} \quad L = \{1, \ldots, k\} \setminus \ell.$$

These sets are computable by Corollary 4.5.8. If there is some $x \in G$ such that $\varphi\text{-ord}_K(x) = n$, then $x\varphi^n = g_r$ for some $r \in L$. We will decide, for each $r \in L$, whether there is some $x \in G$ such that

$$x\varphi^n = g_r \;\wedge\; \forall\, 0 \leq i < n,\; x\varphi^i \notin K \tag{4.13}$$

or not. Clearly, $n \in \varphi\text{-sp}(K)$ if and only if there is some $r \in L$ and $x \in G$ for which condition (4.13) holds. Let $r \in L$ and put

$$I = \{j \in \{1, \ldots, k\} \mid 0 < \varphi\text{-ord}_{\{g_r\}}(g_j) \leq n\}.$$

Notice that I is computable since we only have to check if $g_j\varphi^i = g_r$ for $i \leq n_r$.

If $I = \emptyset$, then, it is easy to see that, for any point $x \in G$, $\varphi\text{-ord}_{\{g_r\}}(x) = n$ if and only if x satisfies condition (4.13). This means that there is such an x if and only if $n \in \varphi\text{-sp}(\{g_r\})$, which is decidable.

So, assume that $I \neq \emptyset$. For each $j \in I$, put $o_j = \varphi\text{-ord}_{\{g_r\}}(g_j) \leq n$. We now verify if g_r is periodic or not. If g_r is periodic with period π_r, we add it to I and put $o_r = \pi_r$. If g_r is not periodic, we proceed.

We claim that, for $x \in G$, condition (4.13) holds if and only if

$$x\varphi^n = g_r \ \wedge\ \forall j \in I,\ x\varphi^{n-o_j} \neq g_j. \tag{4.14}$$

Clearly, condition (4.13) implies condition (4.14). Conversely, if $x\varphi^n = g_r$ but $\varphi\text{-ord}_K(x) < n$, we will prove that there must be some $j \in I$ such that $x\varphi^{n-o_j} = g_j$. Clearly, there must be some $0 \leq s < n$ and some $p \in \{1, \dots, k\}$ such that $x\varphi^s = g_p$. Take s to be maximal. Then,

$$g_r = x\varphi^n = x\varphi^s\varphi^{n-s} = g_p\varphi^{n-s}$$

and so $p \in I$ (notice that this might mean $p = r$ if g_r is periodic). We must have that $n - s \geq o_p$, and so $s \leq n - o_p$. If $s = n - o_p$, we are done, since $x\varphi^{n-o_p} = x\varphi^s = g_p$. If $s < n - o_p$, then $x\varphi^{s+o_p} = g_p\varphi^{o_p} = g_r$, which contradicts the maximality of s.

Now, we will prove that we can decide if condition (4.14) holds for some $r \in L$ and $x \in G$. Compute $a \in g_r\varphi^{-n}$ and, for all $j \in I$, compute some $a_j \in g_j\varphi^{-(n-o_j)}$. We want to decide whether

$$a \operatorname{Ker}(\varphi^n) \subseteq \bigcup_{j \in I} a_j \operatorname{Ker}(\varphi^{n-o_j}). \tag{4.15}$$

Indeed, condition (4.14) holds for some $x \in G$ if and only if condition (4.15) does not hold. Obviously, (4.15) is equivalent to $\operatorname{Ker}(\varphi^n) \subseteq \bigcup_{j \in I} a^{-1}a_j \operatorname{Ker}(\varphi^{n-o_j})$. Now we refine the union, in order to include only the terms for which $a^{-1}a_j \in \operatorname{Ker}(\varphi^n)$ and such that the cosets in the union are all disjoint and we call I_2 the new set of indices. This can be done, since we can test membership in $\operatorname{Ker}(\varphi^n)$ and, since for $i \geq j$, we have that $\operatorname{Ker}(\varphi^j) \subseteq \operatorname{Ker}(\varphi^i)$, it follows that given two cosets $a \operatorname{Ker}(\varphi^i)$ and $b \operatorname{Ker}(\varphi^j)$, either $b \operatorname{Ker}(\varphi^j) \subseteq b \operatorname{Ker}(\varphi^i) = a \operatorname{Ker}(\varphi^i)$, in which case we remove the smaller one from the union, or $b \operatorname{Ker}(\varphi^j) \cap a \operatorname{Ker}(\varphi^i) = \emptyset$. Moreover, this can be easily checked. Also, if $a^{-1}a_j \notin \operatorname{Ker}(\varphi^n)$, then $a^{-1}a_j \operatorname{Ker}(\varphi^{n-o_j}) \subseteq a^{-1}a_j \operatorname{Ker}(\varphi^n)$ is disjoint from $\operatorname{Ker}(\varphi^n)$, and so it can be removed.

So, we will prove that we can decide if

$$\operatorname{Ker}(\varphi^n) = \bigcup_{j \in I_2} a^{-1}a_j \operatorname{Ker}(\varphi^{n-o_j}). \tag{4.16}$$

and that concludes the proof.

Assume then that (4.16) holds. Let $I_2 = \{j_1, \dots j_d\}$ where $o_{j_1} \geq \cdots \geq o_{j_d}$. Suppose that there is some $x \in \operatorname{Ker}(\varphi^{n-o_{j_2}}) \setminus \operatorname{Ker}(\varphi^{n-o_{j_1}})$. Then $a^{-1}a_{j_1}x \in \operatorname{Ker}(\varphi^n)$ and so, since $a^{-1}a_{j_1}x \notin a^{-1}a_{j_1} \operatorname{Ker}(\varphi^{n-o_{j_1}})$, there must be some $1 < r \leq d$ such that $a^{-1}a_{j_1}x \in a^{-1}a_{j_r} \operatorname{Ker}(\varphi^{n-o_{j_r}})$. Since $r \geq 2$, then $\operatorname{Ker}(\varphi^{n-o_{j_2}}) \subseteq \operatorname{Ker}(\varphi^{n-o_{j_r}})$, thus $x \in \operatorname{Ker}(\varphi^{n-o_{j_r}})$ and so $a^{-1}a_{j_1} \in a^{-1}a_{j_r} \operatorname{Ker}(\varphi^{n-o_{j_r}})$, which contradicts the fact that $a^{-1}a_{j_1} \operatorname{Ker}(\varphi^{n-o_{j_1}})$ and

$a^{-1}a_{j_r}\,\mathrm{Ker}(\varphi^{n-o_{j_r}})$ are disjoint. This implies that $\mathrm{Ker}(\varphi^{n-o_{j_2}}) = \mathrm{Ker}(\varphi^{n-o_{j_1}})$ and so, by Lemma 4.5.5,

$$\mathrm{Ker}(\varphi^n) = \mathrm{Ker}(\varphi^{n-o_{j_1}}) = \cdots = \mathrm{Ker}(\varphi^{n-o_{j_d}}),$$

which, since we are assuming that the union is disjoint implies that $|I_2| = 1$.

So, in order to decide (4.16) we only have to verify if $|I_2| = 1$ and if $\mathrm{Ker}(\varphi^{n-o_j}) = \mathrm{Ker}(\varphi^n)$, for $j \in I_2$,which can be done since it corresponds to checking injectivity of φ_{n-o_j}, by Lemma 4.5.5. □

Chapter 5

G-by-$\mathbb{Z}$ groups

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Let $A = \{a_1, \ldots, a_n\}$, $G = \langle A \mid R \rangle$ be a group and $\varphi \in \text{Aut}(G)$. Recall that a G-by-$\mathbb{Z}$ group has the form

$$G \rtimes_\varphi \mathbb{Z} = \langle A, t \mid R, t^{-1}a_it = a_i\varphi \rangle \tag{5.1}$$

and that every element of $G \rtimes_\varphi \mathbb{Z}$ can be written in a unique way as an element of the form $t^a g$, where $a \in \mathbb{Z}$ and $g \in G$. Notice that this is a particular case of a HNN-extension with base group G and both associated subgroups equal to G.

Given a subset $K \subseteq G \rtimes_\varphi \mathbb{Z}$ and $r \in \mathbb{Z}$, we define

$$K_r = \{x \in G \mid t^r x \in K\} = t^{-r}K \cap G.$$

In [9], the authors prove that [f.g. free]-by-cyclic groups have solvable conjugacy problem by reducing this question to the twisted conjugacy problem and Brinkmann's conjugacy problem on free groups. This was later generalized to other extensions of groups in [10], using orbit decidability, and very recently to ascending HNN-extensions of free groups in [68] using variants of the TCP and $BrCP$ for (nonsurjective) endomorphisms. Similar ideas have also been explored in [26] in the context of free-abelian times free groups, where it is proved that ascending HNN-extensions of free-abelian times free groups have solvable conjugacy problem.

In the same vein, we will relate the $GBrCP(G)$ and $GTCP(G)$ with $GCP(G \rtimes \mathbb{Z})$. We remark that these generalized problems are still somewhat unknown even in the cases where G is free or free-abelian (see [101]).

Lemma 5.0.1. *Let G be a finitely generated group, $\varphi \in \text{Aut}(G)$, $H \leq_{f.g.} G \rtimes_\varphi \mathbb{Z}$, $t^s g \in G \rtimes_\varphi \mathbb{Z}$ and $K = (t^s g)H$. Then, for all $r \in \mathbb{Z}$, K_r is either empty or a coset of $H \cap G$. Moreover, if we are given generators for H, we can decide whether K_r is empty or not and, in case it is nonempty, compute a coset representative for K_r.*

Proof. It is well known that the intersection of two cosets of a group is either empty or a coset of the intersection. Now, suppose that we are given generators for H, say $\{t^{k_1}g_1, \dots, t^{k_n}g_n\}$. We have that

$$K_r = \emptyset \iff (t^{s-r}g)H \cap G = \emptyset \iff H \cap t^{r-s}G = \emptyset \iff r - s \notin \langle k_1, \dots, k_n\rangle \leq \mathbb{Z},$$

which is clearly decidable. Moreover, in case K_r is nonempty, we can compute $\lambda_i \in \mathbb{Z}$ such that $r - s = \sum_{i=1}^{n} \lambda_i k_i$. Then, we compute $h \in G$ such that

$$t^r h = (t^s g) \prod_{i=1}^{n} (t^{k_i} g_i)^{\lambda_i} \in K \cap t^r G,$$

and so $h \in t^{-r}K \cap G = K_r$. Since K_r is a coset, then $K_r = (K \cap G)h$. $\square$

Notice that, if $K \leq_{f.g} G \rtimes_\varphi \mathbb{Z}$, then, in general, $K_r = t^{-r}K \cap G$ is not a subgroup of G. In fact, it is a subgroup if and only if $t^r \in K$: if $t^r \notin K$, then $1 \notin t^{-r}K$ and so $1 \notin t^{-r}K \cap G$. If $t^r \in K$, then $t^{-r}K \cap G = K \cap G$, which is a subgroup of G. In particular, $K_0 = K \cap G$ is always a subgroup, but not necessarily finitely generated. Obviously, if $G \rtimes_\varphi \mathbb{Z}$ is Howson, then K_0 is finitely generated.

We will also write $G \rtimes \mathbb{Z}$ to denote the whole class of G-by-$\mathbb{Z}$ groups and so $GCP(G \rtimes \mathbb{Z})$ will represent the uniform generalized conjugacy problem, i.e., taking the automorphism that defines the semidirect product as an input, while $GCP(G \rtimes_\varphi \mathbb{Z})$ will simply denote the generalized conjugacy problem for the group $G \rtimes_\varphi \mathbb{Z}$.

5.1 Generalized conjugacy problem on G-by-$\mathbb{Z}$ groups

The purpose of this section is to prove a generalized version of the result in [9], establishing a connection between $GCP(G \rtimes Z)$ and $GBrCP(G)$ and $GTCP(G)$ and discuss some possible applications for different classes of subsets.

5.1.1 The main result

In [9], the authors prove that [f.g. free]-by-cyclic groups have solvable conjugacy problem by reducing this to the twisted conjugacy problem and Brinkmann's conjugacy problem on free groups. We now prove a result analogous to theirs for the generalized version of the problems.

Theorem 5.1.1. *Let G be a group, $\varphi \in \mathrm{Aut}(G)$, $K \subseteq G \rtimes_\varphi \mathbb{Z}$ and $t^r g \in G \rtimes_\varphi \mathbb{Z}$. Then:*

1. *if $r = 0$, then $GCP(G \rtimes_\varphi \mathbb{Z})$ outputs* ***YES*** *on input $(K, t^r g)$ if and only if $GBrCP(G)$ outputs* ***YES*** *on input (K_r, φ, g);*

2. *if $r \neq 0$, then $GCP(G \rtimes_\varphi \mathbb{Z})$ outputs* ***YES*** *on input $(K, t^r g)$ if and only $GTCP(G)$ outputs* ***YES*** *on input $(K_r, \varphi^r, g\varphi^j)$, for some $0 \leq j \leq r - 1$.*

Proof. We start by proving 1, so suppose that $r = 0$. For $t^s v \in G \rtimes_\varphi \mathbb{Z}$, we have that

$$\begin{aligned} & v^{-1}t^{-s}t^0 g t^s v \in K \\ \iff\quad & v^{-1}(g\varphi^s)v \in K \\ \iff\quad & v^{-1}(g\varphi^s)v \in K \cap G, \end{aligned}$$

and so we have 1.

To prove 2, let $r \neq 0$. For $t^s v \in G \rtimes_\varphi \mathbb{Z}$, we have that

$$\begin{aligned} & v^{-1}t^{-s}t^r g t^s v \in K \\ \iff\quad & t^r(v^{-1}\varphi^r)(g\varphi^s)v \in K \\ \iff\quad & (v^{-1}\varphi^r)(g\varphi^s)v \in K_r \end{aligned}$$

Since

$$(v^{-1}\varphi^r)(g\varphi^s)v = ((v^{-1}(g\varphi^{s-r}))\varphi^r)(g\varphi^{s-r})(g^{-1}\varphi^{s-r})v,$$

then $g\varphi^s$ has a φ^r-twisted conjugate belonging to K_r if and only if $g\varphi^{s-r}$ has a φ^r-twisted conjugate belonging to K_r. Hence, it suffices to check the existence of φ^r-twisted conjugates for $0 \leq s \leq r-1$, i.e. if there are $s \in \mathbb{Z}$ and $v \in G$ such that $(v^{-1}\varphi^r)(g\varphi^s)v \in K_r$ if and only if there are $0 \leq s' \leq r-1$ and $v' \in G$ such that $(v'^{-1}\varphi^r)(g\varphi^{s'})v' \in K_r$ □

Our main theorem is proved in a quite general form without imposing conditions on our target subsets and it provides us with an equivalence between an easier problem in $G \rtimes \mathbb{Z}$ and more complicated problems in G. However, as it will be made clear, even when the target set K belongs to a resonably well-behaved class of subsets, the subsets K_r can be wild, which makes it difficult to apply one of the directions in some cases.

We obtain corollaries from both directions of this equivalence: proving $GBrCP(G)$ and $GTCP(G)$ to solve $GCP(G \rtimes \mathbb{Z})$ works better for recognizable and context-free subsets, while the converse works better for cosets of finitely generated groups, rational and algebraic subsets. We highlight that since, for $K \leq_{f.g} G \rtimes \mathbb{Z}$, K_r is not necessarily a subgroup, the coset setting will be more adequate to us than the finitely generated subgroup setting, in view of Lemma 5.0.1. In fact, this case will be the one for which we obtain the more relevant applications.

5.1.2 The case of cosets

The following corollary is an immediate application of Theorem 5.1.1 together with Lemma 5.0.1.

Corollary 5.1.2. *Let G be a finitely generated group such that $G \rtimes \mathbb{Z}$ is Howson and $\varphi \in \mathrm{Aut}(G)$. If for all $H \leq_{f.g} G \rtimes \mathbb{Z}$, we can compute a finite set of generators for $H \cap G$, then:*

1. *if* $GBrCP_{(f.g.,\varphi)}(G)$ *and* $GTCP_{[f.g.coset]}(G)$ *are decidable, then* $GCP_{f.g}(G \rtimes_\varphi \mathbb{Z})$ *is decidable;*

2. *if* $GBrCP_{f.g.}(G)$ *and* $GTCP_{[f.g.coset]}(G)$ *are decidable, then* $GCP_{f.g}(G \rtimes \mathbb{Z})$ *is decidable;*

The following corollaries show us how we can do the converse.

Corollary 5.1.3. *Let* G *be a group,* $\varphi \in \mathrm{Aut}(G)$*,* $K \subseteq G$ *and* $g \in G$*. Then:*

1. $GCP(G \rtimes_\varphi \mathbb{Z})$ *outputs* `YES` *on input* (K, g) *if and only if* $GBrCP(G)$ *outputs* `YES` *on input* (K, φ, g)*;*

2. $GCP(G \rtimes_\varphi \mathbb{Z})$ *outputs* `YES` *on input* (tK, tg) *if and only if* $GTCP(G)$ *outputs* `YES` *on input* (K, φ, g)*;*

Proof. 1 is an immediate application of condition 1 in Theorem 5.1.1. To see 2, notice that if $K \subseteq G$ and $K' = tK \subseteq G \rtimes_\varphi \mathbb{Z}$, then

$$K_1' = t^{-1}K' \cap G = t^{-1}tK \cap G = K \cap G = K.$$

Now the result follows immediately from condition 2 in Theorem 5.1.1. $\square$

Corollary 5.1.4. *Let* G *be a finitely generated group and* $\varphi \in \mathrm{Aut}(G)$*. Then the following hold:*

1. *if* $GCP_{f.g}(G \rtimes_\varphi \mathbb{Z})$ *is decidable, then* $GBrCP_{(f.g.,\varphi)}(G)$ *is decidable;*

2. *if* $GCP_{f.g}(G \rtimes \mathbb{Z})$ *is decidable, then* $GBrCP_{f.g.}(G)$ *is decidable;*

3. *if* $GCP_{[f.g.coset]}(G \rtimes_\varphi \mathbb{Z})$ *is decidable, then* $GBrCP_{([f.g.coset],\varphi)}(G)$ *and* $GTCP_{([f.g.coset],\varphi)}(G)$ *are decidable;*

4. *if* $GCP_{[f.g.coset]}(G \rtimes \mathbb{Z})$ *is decidable, then* $GBrCP_{[f.g.coset]}(G)$ *and* $GTCP_{[f.g.coset]}(G)$ *are decidable.*

Similar results hold for the *simple* versions of these problems. For example, in 1969, it was proved in [89] that polycyclic groups have solvable conjugacy problem. In particular $\mathbb{Z}^n \rtimes \mathbb{Z}$ has solvable conjugacy problem, and so $BrCP(\mathrm{GL}_m(\mathbb{Z}))$ is decidable, which only became known in 1986 after Kannan and Lipton ([63]) proved directly a (more general) version of Brinkmann's conjugacy problem for arbitrary matrices with rational entries.

5.1.3 Other natural cases

We saw how these problems relate when the class of subsets we consider is the class of cosets of finitely generated groups. In this case, one of the directions works *better* than the other, in the sense that Corollary 5.1.2 needs the (strong) additional hypothesis that $H \cap G$ is finitely generated and computable for all $H \leq_{f.g.} G$.

We will now see what we obtain when considering other natural classes of subsets of groups. In the rational and algebraic cases something similar will happen while in the recognizable and context-free cases we get the opposite as the analogous of Corollary 5.1.4 will be harder to use.

Recognizable subsets

We start with a technical lemma.

Lemma 5.1.5. *Let G be a finitely generated group, $\varphi \in \mathrm{Aut}(G)$ and $K \in Rec(G \rtimes_\varphi \mathbb{Z})$. Then $K_r \in Rec(G)$, for all $r \in \mathbb{Z}$,*

Proof. Let $r \in \mathbb{Z}$. If $K \in Rec(G \rtimes_\varphi \mathbb{Z})$, then $t^{-r}K \in Rec(G \rtimes_\varphi \mathbb{Z})$ and so $t^{-r}K \cap G \in Rec(G)$ by Lemma 2.3.6. □

The previous lemma, combined with Theorem 5.1.1, allows us to deduce that solving $GBrCP_{Rec}(G)$ and $GTCP_{Rec}(G)$ is enough to solve $GCP_{Rec}(G \rtimes \mathbb{Z})$.

Corollary 5.1.6. *Let G be a finitely generated group and $\varphi \in \mathrm{Aut}(G)$. Then:*

1. *if $GBrCP_{(Rec,\varphi)}(G)$ and $GTCP_{Rec}(G)$ are decidable, then $GCP_{Rec}(G \rtimes_\varphi \mathbb{Z})$ is decidable;*
2. *if $GBrCP_{Rec}(G)$ and $GTCP_{Rec}(G)$ are decidable, then $GCP_{Rec}(G \rtimes \mathbb{Z})$ is decidable.*

The converse implication is not easy to use, since a recognizable subset of G is not, in general, a recognizable subset of $G \rtimes \mathbb{Z}$ (for example $G \notin Rec(G \rtimes \mathbb{Z})$, since $[G \rtimes \mathbb{Z} : G] = \infty$).

We will now see another way of solving $GCP_{Rec}(G)$. Recall that a subset K of a group G is recognizable if and only if it is a (finite) union of cosets of some finite index subgroup of G.

Proposition 5.1.7. *If $MP_{f.i.}(G)$ is decidable, then $GCP_{Rec}(G)$ is decidable.*

Proof. Suppose that we can decide $MP_{f.i.}(G)$. Let $K \in Rec(G)$ and $g \in G$. We want to decide if there is some $x \in G$ such that $x^{-1}gx \in K$. We know that K is a (finite) disjoint union of cosets of some finite index normal subgroup $H \trianglelefteq_{f.i.} G$, i.e.,

$$K = \bigcup_{i=1}^{m} Hb_i,$$

for some $m \in \mathbb{N}$, $b_i \in K$.

So, we only have to decide, for each $i \in [m]$ whether there is some $x \in G$ such that $x^{-1}gx \in Hb_i$. But, since H is normal,

$$x^{-1}gx \in Hb_i \iff g \in Hxb_ix^{-1} \iff Hg = (Hx)(Hb_i)(Hx^{-1}),$$

so we only have to decide if Hg is conjugate to Hb_i in ${}^{G}\!/_{H}$. Since we can decide $MP_{f.i.}(G)$, we can compute ${}^{G}\!/_{H}$ (which is a finite group) and so we can decide $CP\left({}^{G}\!/_{H}\right)$. $\square$

As mentioned before, in [52], it is proved that every finitely L-presented group has solvable $MP_{f.i.}(G)$ and in [88], it is proved that for recursively presented groups, $MP_{f.i.}(G)$ is equivalent to having computable finite quotients (CFQ). Hence, if G is finitely L-presented or G is a recursively presented group with CFQ, then $GCP_{Rec}(G)$ is decidable.

Context-free subsets

We now prove the context-free analogous to Lemma 5.1.5.

Lemma 5.1.8. *Let G be a finitely generated group, $\varphi \in \operatorname{Aut}(G)$ and $K \in CF(G \rtimes_\varphi \mathbb{Z})$. Then $K_r \in CF(G)$, for all $r \in \mathbb{Z}$,*

Proof. If $K \in CF(G)$, by Lemma 2.3.4, we have that $t^{-r}K \in CF(G \rtimes_\varphi \mathbb{Z})$. By Lemma 2.3.5, $K_r = t^{-r}K \cap G \in CF(G)$. $\square$

So we have the following corollary.

Corollary 5.1.9. *Let G be a finitely generated group and $\varphi \in \operatorname{Aut}(G)$.*

1. *if $GBrCP_{(CF,\varphi)}(G)$ and $GTCP_{CF}(G)$ are decidable, then $GCP_{CF}(G \rtimes_\varphi \mathbb{Z})$ is decidable;*
2. *if $GBrCP_{CF}(G)$ and $GTCP_{CF}(G)$ are decidable, then $GCP_{CF}(G \rtimes \mathbb{Z})$ is decidable.*

Again, the converse implication is obstructed by the fact that a context-free subset of G is not necessarily a context-free subset of $G \rtimes_\varphi \mathbb{Z}$: it is enough to take $G = F_2$ and φ to be the identity. So $G \rtimes_\varphi \mathbb{Z} = F_2 \times \mathbb{Z}$ which is not virtually free. So, $\{1\} \in CF(F_2)$ and $\{1\} \notin CF(F_2 \times \mathbb{Z})$ by the Muller-Schupp theorem.

Rational and algebraic subsets

It is well known that $F_2 \rtimes_\varphi \mathbb{Z}$ is not Howson, and so the intersection of rational subsets might not be rational. The next example shows something stronger: $F_2 \rtimes_\varphi \mathbb{Z}$ has a rational subset K such that K_r is not a rational subset of F_2 for any $r \in \mathbb{Z}$.

Example 5.1.10. Take $F_2 = \langle a, b | \rangle$ and $\varphi = \lambda_b$. Then $K = (t \cup t^{-1})^* a (t \cup t^{-1})^*$ is a rational language of $\{\widetilde{a, b, t}\}^*$ and the subset of $F_2 \rtimes_\varphi \mathbb{Z}$ defined by K is

$$\{t^{m+n}(a\varphi^n) \mid m, n \in \mathbb{Z}\} = \{t^{m+n}b^{-n}ab^n \mid m, n \in \mathbb{Z}\}.$$

We have that, for every $r \in \mathbb{Z}$, $K_r = \{b^{-n}ab^n \in F_2 \mid n \in \mathbb{Z}\}$ which is not a rational subset of F_2, by Benois' Theorem.

Hence, similarly to what happens in the coset case, in the rational case, we cannot expect K_r to be a rational subset of G when K is a rational subset of $G \rtimes \mathbb{Z}$.

We remark that if $K \in Rat(G \rtimes_\varphi \mathbb{Z})$, then, $K_r \in Alg(G \rtimes_\varphi \mathbb{Z})$, for all $r \in \mathbb{Z}$. Indeed, let A be a set of generators for G, $\pi : \widetilde{A \cup \{t\}} \to G \rtimes_\varphi \mathbb{Z}$ be a surjective homomorphism and $K \in Rat(G \rtimes_\varphi \mathbb{Z})$. Then $t^{-r}K \in Rat(G \rtimes_\varphi \mathbb{Z})$. Hence, there is a rational language $L \subseteq \widetilde{A \cup \{t\}}^*$ such that $L\pi = t^{-r}K$. The language

$$L' = \{w \in L \mid n_t(w) = n_{t^{-1}}(w)\} = L \cap \{w \in \widetilde{A \cup \{t\}}^* \mid n_t(w) = n_{t^{-1}}(w)\}$$

is context-free since it is the intersection of a rational language with a context-free language and $L'\pi = t^{-r}K \cap G$. Hence, $K_r \in Alg(G \rtimes_\varphi \mathbb{Z})$. However, we might have that $K_r \notin Alg(G)$. In Example 3.2.7, it is proved that if $G = \mathbb{Z}^2$, $\varphi = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and K is the orbit of $(1,0)$ through φ, then K is rational in $\mathbb{Z}^2 \rtimes_\varphi \mathbb{Z}$, but K_0 is not algebraic in $\mathbb{Z}^2$. So, even for rational K, K_r might have a wild behavior. For this reason, it is hard to obtain an analogous of Corollary 5.1.2 in this case.

However, in the rational and algebraic case, unlike the recognizable and the context-free cases, the converse implication works without additional conditions. Indeed, $K \in Rat(G)$ (resp. $K \in Alg(G)$) implies that $K \in Rat(G \rtimes_\varphi \mathbb{Z})$ (resp. $K \in Alg(G \rtimes_\varphi \mathbb{Z})$), for every $\varphi \in \mathrm{Aut}(G)$ and so $xK \in Rat(G \rtimes_\varphi \mathbb{Z})$ (resp. $xK \in Alg(G \rtimes_\varphi \mathbb{Z})$), for all $\varphi \in \mathrm{Aut}(G)$ and $x \in G \rtimes_\varphi \mathbb{Z}$.

So we have the following corollary by directly applying Corollary 5.1.3.

Corollary 5.1.11. *Let G be a finitely generated group and $\varphi \in \mathrm{Aut}(G)$.*

1. *if $GCP_{Rat}(G \rtimes_\varphi \mathbb{Z})$ is decidable, then $GBrCP_{(Rat,\varphi)}(G)$ and $GTCP_{(Rat,\varphi)}(G)$ are decidable;*
2. *if $GCP_{Rat}(G \rtimes \mathbb{Z})$ is decidable, then $GBrCP_{Rat}(G)$ and $GTCP_{Rat}(G)$ are decidable;*
3. *if $GCP_{Alg}(G \rtimes_\varphi \mathbb{Z})$ is decidable, then $GBrCP_{(Alg,\varphi)}(G)$ and $GTCP_{(Alg,\varphi)}(G)$ are decidable;*
4. *if $GCP_{Alg}(G \rtimes \mathbb{Z})$ is decidable, then $GBrCP_{Alg}(G)$ and $GTCP_{Alg}(G)$ are decidable.*

An interesting consequence of Corollary 5.1.11 is that, in the case of the rational and algebraic versions of the conjugacy problem, decidability in G-by-$\mathbb{Z}$ groups yields the decidability in G-by-finite groups. To prove so, we start with a proposition that follows the strategy used in [66] to prove the rational generalized conjugacy problem in virtually free groups.

Recall that we denote by $\mathrm{Via}(G)$ the set of virtually inner automorphisms of a group G.

Proposition 5.1.12. *Let $\mathcal{C} \in \{Rec, Rat, CF, Alg\}$, H be a finitely generated group and G be an H-by-finite group. If $GTCP_{(\mathrm{Via},\mathcal{C})}(H)$ is decidable, then $GCP_{\mathcal{C}}(G)$ is decidable.*

Proof. We have that H is a finite index normal subgroup of G and so G admits a decomposition as a disjoint union

$$G = Hb_1 \cup \ldots \cup Hb_m,$$

for some $b_i \in G$. For each $i \in [m]$, we define $\varphi_i : H \to H$ by $h \mapsto b_i h b_i^{-1}$. Then, φ_i is clearly an automorphism of H and since ${G}/{H}$ is finite, then, for some $k \in \mathbb{N}$ we have that $b_i^k \in H$ and so $\varphi_i^k \in \mathrm{Inn}(H)$. Thus, $\varphi_i \in \mathrm{Via}(H)$.

Let $K \in \mathcal{C}(G)$, $h \in H$, $i \in [m]$ and assume that $GTCP_{(Via,\mathcal{C})}(H)$ is decidable. Then hb_i has a conjugate in K if and only if there are $h' \in H$ and $j \in [m]$ such that $(b_j^{-1}h'^{-1})(gb_i)(hb_j) \in K$. Now,

$$\begin{aligned}
&(b_j^{-1}h'^{-1})(gb_i)(hb_j) \in K\\
\iff\ & b_j^{-1}h'^{-1}g(h\varphi_i)b_ib_j \in K\\
\iff\ & (h'^{-1}g(h\varphi_i))\varphi_j^{-1} \in Kb_j^{-1}b_i^{-1}b_j\\
\iff\ & (h'^{-1}g(h\varphi_i))\varphi_j^{-1} \in (H \cap Kb_j^{-1}b_i^{-1}b_j)\\
\iff\ & h'^{-1}g(h\varphi_i) \in (H \cap Kb_j^{-1}b_i^{-1}b_j)\varphi_j.
\end{aligned}$$

Since $K \in \mathcal{C}(G)$, then $Kb_j^{-1}b_i^{-1}b_j \in \mathcal{C}(G)$ and $H \cap Kb_j^{-1}b_i^{-1}b_j \in \mathcal{C}(H)$ (see Lemmas 3.1.2 and 3.1.10 for the cases of CF and Alg and [6] for the cases of Rat and Rec) and so $(H \cap Kb_j^{-1}b_i^{-1}b_j)\varphi_j \in \mathcal{C}(H)$.

Therefore, our problem is now reduced to m instances of $GTCP_{(\mathrm{Via},\mathcal{C})}(H)$, one for each $j \in [m]$, which we can solve. □

Combining the previous proposition with Corollary 5.1.11, we obtain the following.

Corollary 5.1.13. *Let $\mathcal{C} \in \{Rat, Alg\}$, G be a finitely generated group and F be a finite group. If $GCP_{\mathcal{C}}(G \rtimes \mathbb{Z})$ is decidable, then $GCP_{\mathcal{C}}(G \rtimes F)$ is decidable.*

5.2 Virtually polycyclic groups

A group G is said to be *conjugate separable* if, for all $g, h \in G$,

$$g \text{ and } h \text{ are conjugate} \iff g \text{ and } h \text{ are conjugate in every finite quotient of } G.$$

If a group G is finitely presented and conjugate separable, then it has a decidable conjugacy problem (see [77]). We now show that, adding the solution to the finitely generated membership problem, we can also solve the $[f.g.\,coset]$-generalized conjugacy problem.

Theorem 5.2.1. *Let G be a conjugate separable finitely presented group such that $MP_{f.g.}(G)$ is decidable. Then $GCP_{[f.g.\,coset]}(G)$ is decidable.*

Proof. Let $a, b \in G$ and $H = \langle h_1, \dots, h_k \rangle \leq_{f.g.} G$ be our input. We want to decide if a has a conjugate in bH. To do so, we will run two partial algorithms: one that stops, answering `YES` if a has a conjugate in bH (but does not stop otherwise) and one that stops answering `NO` if a does not have a conjugate in bH (but does not stop otherwise).

The first one is simple: we enumerate all conjugates of a and check if they belong to bH using $MP_{f.g.}(G)$. If we find a positive answer, we stop and answer `YES`; while we don't we keep running.

Now we describe the second one. We start by enumerating all homomorphisms from G onto finite groups. For each surjective homomorphism $\varphi : G \to F$, where $F = \{x_1, \dots, x_m\}$ is a finite group, we compute $(bH)\varphi = b\varphi\langle h_1\varphi, \dots, h_k\varphi\rangle$ and the set X of all conjugates of $a\varphi$, which can be done since F is finite. Compute $y_i \in G$ such that $y_i\varphi = x_i$. We have that

$$G = y_1 \operatorname{Ker}(\varphi) \cup \cdots \cup y_m \operatorname{Ker}(\varphi)$$

and

$$bH = (y_1 \operatorname{Ker}(\varphi) \cap bH) \cup \cdots \cup (y_m \operatorname{Ker}(\varphi) \cap bH).$$

Each of the intersections $y_i \operatorname{Ker}(\varphi) \cap bH$ is either empty or a coset of $\operatorname{Ker}(\varphi) \cap H$. In fact, $y_i \operatorname{Ker}(\varphi) \cap bH \neq \emptyset$ if and only if $x_i \in (bH)\varphi$, which we can check. Moreover, for each nonempty intersection $y_i \operatorname{Ker}(\varphi) \cap bH$, we can compute a representative $h_i \in bH$ such that $y_i \operatorname{Ker}(\varphi) \cap bH = h_i(\operatorname{Ker}(\varphi) \cap H)$, by enumerating elements of bH until we find one whose image is x_i. Thus, we can write bH as a disjoint union of the form

$$bH = h_1(\operatorname{Ker}(\varphi) \cap H) \cup \cdots \cup h_n(\operatorname{Ker}(\varphi) \cap H),$$

where $n \leq m$ and $h_i\varphi = x_i$. Also, we can compute generators for $\operatorname{Ker}(\varphi) \cap H$ by Schreier's Lemma.

Now we check for each h_i whether or not $h_i\varphi \in X$ and write

$$bH \cap X\varphi^{-1} = \bigcup_{h_i\varphi \in X} h_i(\operatorname{Ker}(\varphi) \cap H).$$

The set $bH \cap X\varphi^{-1}$ gives us a set of candidate conjugators of a in bH in the sense that it contains all elements that are mapped by φ into a conjugator of $a\varphi$ (in F). So, if running through all homomorphisms φ, the intersection of the sets of the form $bH \cap X\varphi^{-1}$ becomes empty, then, by conjugate separability, a does not have a conjugate in bH.

We have that,

$$H \cap b^{-1}(X\varphi^{-1}) = \bigcup_{h_i\varphi \in X} b^{-1}h_i(\operatorname{Ker}(\varphi) \cap H).$$

This is a union of (computable) recognizable subsets of H, since $\operatorname{Ker}(\varphi) \cap H \leq_{f.i.} H$ and so $H \cap b^{-1}(X\varphi^{-1})$ is a (computable) recognizable subset of H. Moreover, given a set S of homomorphisms from G to finite groups,

$$\bigcap_{\varphi \in S} H \cap b^{-1}(X\varphi^{-1}) = \emptyset \iff \bigcap_{\varphi \in S} bH \cap X\varphi^{-1} = \emptyset.$$

Since the intersections of recognizable subsets is recognizable and computable, we can decide if this intersection becomes empty at some point. $\square$

A group is *polycyclic* if it admits a subnormal series

$$G = G_0 \triangleright G_1 \triangleright \dots G_n = \{1\}$$

such that ${G_{i-1}}/{G_i}$ is cyclic for $i \in [n]$.

In [89] and [45], Remeslennikov and Formanek proved that virtually polycyclic groups are conjugate separable. Mal'cev in [73] proved that polycyclic groups are subgroup separable, and so, they have decidable membership problem, which implies that virtually polycyclic groups do too. So, we have the following corollary.

Corollary 5.2.2. *Let G be a virtually polycyclic group. Then $GCP_{[f.g.coset]}(G)$ is decidable.*

If G is polycyclic with subnormal series

$$G = G_0 \triangleright G_1 \triangleright \dots G_n = \{1\},$$

such that ${G_{i-1}}/{G_i}$ is cyclic for $i \in [n]$, and $\varphi \in \mathrm{Aut}(G)$, then

$$G \rtimes_\varphi \mathbb{Z} \triangleright G_0 \triangleright G_1 \triangleright \dots G_n = \{1\}$$

and ${G \rtimes_\varphi \mathbb{Z}}/{G_0} \simeq \mathbb{Z}$. Hence, $G \rtimes_\varphi \mathbb{Z}$ is polycyclic.

We will now prove the same result for virtually polycyclic groups.

Proposition 5.2.3. *Let G be a virtually polycyclic group and $\varphi \in \mathrm{Aut}(G)$. Then $G \rtimes_\varphi \mathbb{Z}$ is virtually polycyclic.*

Proof. By Lemma 4.0.1, G has a fully invariant finite index normal polycyclic subgroup H: indeed if P is a finite index polycyclic subgroup, then P contains a finite index normal polycyclic subgroup N and letting H be the intersection of all normal polycyclic subgroups of index at most $[G : N]$ we obtain a fully invariant finite index normal polycyclic subgroup of G.

Since H is fully invariant, the restriction $\psi = \varphi|_H$ is an endomorphism of H. Write G as a disjoint union

$$G = H \cup Hb_1 \cup \dots \cup Hb_n,$$

for some $b_i \in G$. Then

$$G = G\varphi = H\varphi \cup H\varphi(b_1\varphi) \cup \dots \cup H\varphi(b_n\varphi),$$

and so $[G : H\varphi] \leq n = [G : H]$. Since $[G : H\varphi] = [G : H][H : H\varphi]$, then $[G : H\varphi] \geq [G : H]$. Hence, $[G : H\varphi] = [G : H]$ and $[H : H\varphi] = 1$. Therefore, ψ is bijective (injectivity is inherited from injectivity of φ).

Since H is polycyclic, then $H \rtimes_\psi \mathbb{Z}$ is polycyclic. Clearly $H \rtimes_\psi \mathbb{Z} \leq G \rtimes_\varphi \mathbb{Z}$ and

$$G \rtimes_\varphi \mathbb{Z} = (H \rtimes_\psi \mathbb{Z}) \cup (H \rtimes_\psi \mathbb{Z})t^0 b_1 \cup \cdots \cup (H \rtimes_\psi \mathbb{Z})t^0 b_n,$$

and so $G \rtimes_\varphi \mathbb{Z}$ is virtually polycyclic. □

From Corollaries 5.1.4 and 5.2.2 we deduce the following corollary.

Corollary 5.2.4. *Let G be a virtually polycyclic group. Then $GBrCP_{[f.g.coset]}(G)$ and $GTCP_{[f.g.coset]}(G)$ are decidable.*

Notice that Kannan and Lipton solved a version of Brinkmann's problem for matrices with entries in $\mathbb{Q}$. The generalized version of this algorithm is not easy: in [30], it is proved that it is decidable whether the orbit of a vector in $\mathbb{Q}^m$ by some matrix $A \in \mathbb{Q}^{m\times m}$ intersects a vector space of dimension at most 3; for greater dimensions, the problem remains open. The case where matrices belong to $\mathrm{GL}_m(\mathbb{Z})$ and the target set is a coset of a finitely generated subgroup is a consequence of Corollary 5.2.4 and, to the author's knowledge, was unknown. Moreover, since in abelian groups $GBrCP$ and $GBrP$ coincide, we have the following corollary. We remark that, as highlighted in [101], not much was known about $GBrP_{f.g}$ even for free-abelian groups.

Corollary 5.2.5. *Let G be finitely generated abelian group. Then $GBrP_{[f.g.coset]}(G)$ is decidable.*

5.3 Brinkmann's equality problem

In this section, we will relate the rational version of $GBrP(G)$ with the intersection problem of rational subsets of $G \rtimes \mathbb{Z}$ and solve $GBrP_{(Rat,\mathrm{Via})}(F)$ and $GBrCP_{(Rat,\mathrm{Via})}(F)$ for a free group F.

5.3.1 Generalized version

Naturally, one can reduce Brinkmann's problems restricted to Via to some conjugacy problems on the group. Using a result from [66], we can prove the following proposition.

Proposition 5.3.1. *Let F be a finitely generated free group. Then $GBrP_{(Rat,\mathrm{Via})}(F)$ and $GBrCP_{(Rat,\mathrm{Via})}(F)$ are decidable.*

Proof. Let $\varphi \in \mathrm{Via}(F)$ be such that $\varphi^r = \lambda_x$, $u \in F$ and $K \in Rat(F)$ be our input. Let $k \in \mathbb{N}$ and write $k = pr + q$ with $0 \leq q \leq r-1$. Then $u\varphi^k = (x^{-p}ux^p)\varphi^q$. Thus, there is some $k \in \mathbb{N}$ such that $u\varphi^k \in K$ (resp. $u\varphi^k$ has a conjugate in K) if and only if there are $p \in \mathbb{N}$, $q \in \{0, \dots, r-1\}$ such that $(x^{-p}ux^p)\varphi^q \in K$ (resp. $(x^{-p}ux^p)\varphi^q$ has a conjugate in K).

For each $0 \leq q \leq r-1$, the set $L = \{x^s \in F \mid s \in \mathbb{N}\}$ is rational and so $L\varphi^q$ is rational too. The problem of deciding whether there is some $p \in \mathbb{N}$ such that $x^{-p}ux^p \in K$ is an instance of the rational generalized conjugacy problem with rational constraints, which is decidable for F by [66, Theorem 4.3].

Moreover, deciding whether there is some $p \in \mathbb{N}$ such that $(x^{-p}ux^p)\varphi^q$ has a conjugate in K can also be done, since this is equivalent to deciding whether $u\varphi^q$ has a conjugate in K, which is an instance of the rational generalized conjugacy problem $\square$

We now observe that, if we can solve $IP_{Rat}(G \rtimes \mathbb{Z})$, then we can solve $GBrP_{Rat}(G)$.

Proposition 5.3.2. *If $IP_{Rat}(G \rtimes \mathbb{Z})$ is decidable, then $GBrP_{Rat}(G)$ is decidable.*

Proof. Let G be a group generated by a finite set $A = \{a_1, \dots a_n\}$ and $\pi : \tilde{A}^* \to G$ be the standard surjective homomorphism. Then $G \rtimes_\varphi \mathbb{Z}$ admits a presentation of the form (5.1) and there is a natural surjective homomorphism $\rho : \widetilde{A \cup \{t\}}^* \to G \rtimes_\varphi \mathbb{Z}$ such that $\rho|_A = \pi$ (identifying G with the subset $\{t^0 g \mid g \in G\}$). Let $K \in Rat(G)$, $\varphi \in \mathrm{Aut}(G)$ and $g \in G$. We want to decide if there is some $n \in \mathbb{N}$ such that $g\varphi^n \in K$. Since K is a rational subset of G, it is a rational subset of $G \rtimes_\varphi \mathbb{Z}$, and so is t^*K. Let $w \in \tilde{A}^*$ be a word such that $w\pi = g$ and put $L = \{wt^n \mid n \in \mathbb{N}\} \subseteq \widetilde{A \cup \{t\}}^*$. Then L is a rational language and $L\pi = \{t^n g\varphi^n \mid n \in \mathbb{N}\}$ is a rational subset of $G \rtimes_\varphi Z$. Now, there exists some $n \in \mathbb{N}$ such that $g\varphi^n \in K$ if and only if $L\pi \cap t^*K \neq \emptyset$, which we can decide by hypothesis. $\square$

Chapter 6

Hyperbolic groups

We will now study endomorphisms of hyperbolic groups. For free groups, the *bounded cancellation lemma* is said to hold for an endomorphism $\varphi \in \mathrm{End}(F_n)$ if there is some constant $N \in \mathbb{N}$ such that

$$|u^{-1} \wedge v| = 0 \implies |u^{-1}\varphi \wedge v\varphi| < N$$

holds for all $u, v \in F_n$.

We will propose geometric versions of this property for endomorphisms of hyperbolic groups and use it to characterize uniformly continuous endomorphisms of hyperbolic groups with respect to a visual metric. This can be important when studying dynamics at the infinity since these are precisely the endomorphisms admitting a continuous extension to the Gromov completion.

We start with a generalization of a well-known property of hyperbolic groups, the fellow traveler property.

6.1 Fellow Traveler Property for $(1, k)$-quasi-geodesics

The goal of this section is to present a slightly more general formulation of the fellow traveler property. The result is most likely known but since it is not easy to find a proof of the result, we include one here. While the property is usually stated considering geodesics, we can see that we get essentially the same result when the paths considered are $(1, k)$-quasi-geodesics for some $k \in \mathbb{N}$. We will present the result in a different version, considering quasi-geodesics with endpoints at distance at most one from one another, but we remark that the result holds as long as the distance between the endpoints is bounded by some constant.

Proposition 6.1.1. *Let H be a hyperbolic group, u be a $(1, r)$-quasi-geodesic and v be a $(1, s)$-quasi-geodesic with the same starting point. Then, there is a constant N depending on r, s, δ such that, for all $n \in \mathbb{N}$, we have that*

$$d_A(u\pi, v\pi) \leq 1 \implies d_A(u^{[n]}\pi, v^{[n]}\pi) \leq N.$$

Proof. Since the Cayley graph of H with respect to A is vertex-transitive, we can assume that 1 is the starting point of both u and v. Let p and q be the endpoints of u and v, and consider a geodesic w from q to p.

$$\begin{tikzcd} & p \\ 1 \arrow[ur, rightsquigarrow, "u"] \arrow[r, rightsquigarrow, "v"'] & q \arrow[u, "w"'] \end{tikzcd}$$

Let $k = \max\{r, s\}$. Then u, v and w are all $(1, k)$-quasi-geodesics. By [16, Corollary 1.8, Chapter III.H], there is a constant δ', depending only on r, s and δ, such that the triangle (u, v, w) is δ'-thin. Let $n \in \mathbb{N}$. We will prove that

$$d_A(u^{[n]}\pi, v^{[n]}\pi) \leq 3k + 2\delta' + 4.$$

Consider the factorizations of u and v given by

$$1 \overset{u^{[n]}}{\rightsquigarrow} p_n \overset{u_n}{\rightsquigarrow} p, \qquad 1 \overset{v^{[n]}}{\rightsquigarrow} q_n \overset{v_n}{\rightsquigarrow} q, \ .$$

Suppose that $v_n = 1$. This means that $q_n = q$, $d_A(1, q) \leq n$ and $d_A(1, p) \leq n + 1$. Also, notice that, since u is a $(1, k)$-quasi-geodesic, we have that

$$|u| - k \leq d_A(1, p) \leq |u|.$$

We have that

$$d_A(u^{[n]}\pi, p) \leq |u| - n \leq d_A(1, p) - n + k \leq k + 1 \leq 3k + 2\delta' + 4.$$

The case $u_n = 1$ is analogous, so we assume that $u_n, v_n \neq 1$.

Since (u, v, w) is a δ'-thin triangle, there is some $m \leq |u|$, such that

$$d_A(v^{[n]}\pi, u^{[m]}\pi) \leq \delta' + 1.$$

Again, since u and v are $(1, k)$-quasi-geodesics, we have that

$$d_A(1, q) - n - k \leq |v| - n - k \leq d_A(q_n, q) \leq |v| - n \leq d_A(1, q) - n + k$$

and

$$d_A(1, p) - m - k \leq |u| - m - k \leq d_A(p_m, p) \leq |u| - m \leq d_A(1, p) - m + k.$$

So,

$$\begin{aligned} d_A(1,q) - n - k \le d(q_n, q) &\le d(q_n, p_m) + d(p_m, p) + d(p, q) \\ &\le \delta' + 1 + d_A(1,p) - m + k + 1, \end{aligned}$$

thus, $m - n \le \delta' + 1 + d_A(1,p) - d_A(1,q) + 2k + 1 \le \delta' + 2k + 3$. Similarly, we have that

$$\begin{aligned} d_A(1,p) - m - k \le d(p_m, p) &\le d(p_m, q_n) + d(q_n, q) + d(q, p) \\ &\le \delta' + 1 + d_A(1,q) - n + 2k + 1, \end{aligned}$$

thus, $n - m \le \delta' + 1 + d_A(1,q) - d_A(1,p) + 2k + 1 \le \delta' + 2k + 3$.

Hence, we have that

$$d(p_n, q_n) \le d(p_n, p_m) + d(p_m, q_n) \le |m-n| + k + \delta' + 1 \le \delta' + 2k + 3 + k + \delta' + 1 = 3k + 2\delta' + 4.$$

□

6.2 Bounded Reduction Property

In this section, we will present three (equivalent) geometric versions of the Bounded Reduction Property (BRP), also known as the Bounded Cancellation Lemma, for hyperbolic groups.

Since, for $u, v \in F_n$, the Gromov product $(u|v)$ coincides with $u \wedge v$, a natural generalization for the hyperbolic case is

$$\exists N \in \mathbb{N} \left((u \mid v) = 0 \implies (u\varphi \mid v\varphi) \le N\right).$$

In [48, Proposition 15, Chapter 5], it is shown that if φ is a (λ, K)-quasi-isometric embedding, then there exists a constant A depending on λ, K and δ, with the property that

$$\frac{1}{\lambda}(u|v) - A \le (u\varphi|v\varphi) \le \lambda(u|v) + A. \tag{6.1}$$

So, if H is a hyperbolic group and $\varphi : H \to H$ is a quasi-isometric embedding, then for every $p \ge 0$, there exists $q \ge 0$ so that

$$(u \mid v) \le p \implies (u\varphi \mid v\varphi) \le q.$$

We will now present a geometric formulation of the inequality $(u|v) \le p$.

Lemma 6.2.1. *Let $u, v \in H$ and $p \in \mathbb{N}$. Then the following are equivalent:*

1. $(u|v) \le p$

2. *for any geodesics α and β from 1 to u^{-1} and v, respectively, we have that the concatenation*

$$1 \xrightarrow{\alpha} u^{-1} \xrightarrow{\beta} u^{-1}v$$

is a $(1,2p)$-quasi-geodesic

3. *there are geodesics α and β from 1 to u^{-1} and v, respectively, such that the concatenation*

$$1 \xrightarrow{\alpha} u^{-1} \xrightarrow{\beta} u^{-1}v$$

is a $(1,2p)$-quasi-geodesic

Proof. It is clear that $2 \implies 3$. Now we prove that $1 \implies 2$. Let $u, v \in H$ and $p \in \mathbb{N}$. Suppose that $(u|v) \leq p$ and take geodesics α and β from 1 to u^{-1} and v, respectively. Take the concatenation $\zeta : [0, |u| + |v|] \to H$, where $0\zeta = 1$, $(|u|)\zeta = u^{-1}$ and $(|u| + |v|)\zeta = u^{-1}v$.

We will prove that ζ is a $(1,2p)$-quasi-geodesic, i.e., for $0 \leq i \leq j \leq |u| + |v|$, we have that

$$j - i - 2p \leq d(i\zeta, j\zeta) \leq j - i + 2p.$$

We can assume that $0 \leq i \leq |u| \leq j \leq |u| + |v|$. Clearly, we have that $d(i\zeta, j\zeta) \leq j - i \leq j - i + 2p$.

Suppose that $|j\zeta| < j - 2(u|v)$ and consider a geodesic $\gamma : [0, |j\zeta|]$ from 1 to $j\zeta$ and concatenate it with $\zeta|_{[j,|u|+|v|]}$, which gives us a path of length

$$\begin{aligned}
|u| + |v| - j + |j\zeta| &< |u| + |v| - j + j - 2(u|v) \\
&= |u| + |v| - 2(u|v) \\
&= |u| + |v| - |u| - |v| + |u^{-1}v| \\
&= |u^{-1}v|
\end{aligned}$$

from 1 to $u^{-1}v$ and that contradicts the definition of d. Thus, $|j\zeta| \geq j - 2(u|v)$.

So,

$$\begin{aligned}
& d(u^{-1}, j\zeta) + |u^{-1}v| - |v| - d(i\zeta, j\zeta) - d(1, i\zeta) \\
\leq\ & d(u^{-1}, j\zeta) + |u^{-1}v| - |v| - |j\zeta| \\
=\ & j - |u| + d(u, v) - |v| - |j\zeta| \\
=\ & j - 2(u|v) - |j\zeta| \\
\leq\ & 0,
\end{aligned}$$

thus,

$$\begin{aligned} d(i\zeta, j\zeta) &\geq d(u^{-1}, j\zeta) + |u^{-1}v| - |v| - d(1, i\zeta) \\ &= j - |u| + |u^{-1}v| - |v| - i \\ &= j - i - 2(u|v) \\ &\geq j - i - 2p \end{aligned}$$

To prove that $3 \implies 1$, take geodesics α and β from 1 to u^{-1} and v, respectively, such that the concatenation

$$1 \xrightarrow{\alpha} u^{-1} \xrightarrow{\beta} u^{-1}v$$

is a $(1, 2p)$-quasi-geodesic. Then

$$d(u, v) = d(1, u^{-1}v) = d(0(\alpha + \beta), (|u| + |v|)(\alpha + \beta)) \geq |u| + |v| - 2p,$$

so $2(u|v) = |u| + |v| - d(u, v) \leq 2p$ and we are done. □

Let $\varphi : H \to H$ be a map. We say that *the BRP holds for* φ if, for every $p \geq 0$ there is some $q \geq 0$ such that: given two geodesics u and v such that

$$1 \xrightarrow{u} u \xrightarrow{v} uv$$

is a $(1, p)$-quasi-geodesic, we have that given any two geodesics α, β, from 1 to $u\varphi$ and from $u\varphi$ to $(uv)\varphi$, respectively, the path

$$1 \xrightarrow{\alpha} u\varphi \xrightarrow{\beta} (uv)\varphi$$

is a $(1, q)$-quasi-geodesic.

Proposition 6.2.2. *Let H be a hyperbolic group and $\varphi : H \to H$ be a mapping. Then, the following are equivalent:*

1. *BRP holds for φ;*
2. *for every $p \geq 0$ there is some $q \geq 0$ such that, for all $u, v \in H$, we have that*

$$(u \mid v) \leq p \implies (u\varphi \mid v\varphi) \leq q. \tag{6.2}$$

Proof. Let H be a hyperbolic group, $\varphi : H \to H$ be a mapping and p be a nonnegative integer. Suppose that the BRP holds for φ and take $u, v \in H$ such that $(u|v) \leq p$. Then, by Lemma 6.2.1 and the BRP, there is some $q \geq 0$ such that, given geodesics α and β from 1 to $u^{-1}\varphi$ and from $u^{-1}\varphi$ to $(u^{-1}v)\varphi$, respectively, the concatenation $\zeta : [0, |u^{-1}\varphi| + |v\varphi|] \to H$ is a $(1, 2q)$-quasi-geodesic. Using Lemma 6.2.1 again, we have that $(u\varphi|v\varphi) \leq q$.

Now, suppose that for every $p \geq 0$, there is some $q \geq 0$ such that (6.2) holds and take geodesics α, β such that the concatenation

$$1 \xrightarrow{\alpha} u \xrightarrow{\beta} uv$$

is a $(1,p)$-quasi-geodesic. In particular, it is also a $(1,2p)$-quasi-geodesic. Then by Lemma 6.2.1, we have that $(u^{-1}|v) \leq p$, so, using (6.2), we have that $(u^{-1}\varphi|v\varphi) \leq q$, so by Lemma 6.2.1, we have that the path

$$1 \xrightarrow{\alpha} u\varphi \xrightarrow{\beta} (uv)\varphi$$

is a $(1,2q)$-quasi-geodesic for every geodesics α, β as above. □

The following proposition is an immediate consequence of (6.1) and Proposition 6.2.2.

Proposition 6.2.3. *If $\varphi : H \to H$ is a quasi-isometric embedding, then the BRP holds for φ.*

The next proposition shows that the bounded reduction property can be reduced to the case where $p = 0$.

Proposition 6.2.4. *Let H be a hyperbolic group and $\varphi \in End(H)$. If the BRP holds for φ for $p = 0$, then it holds for every $p \in \mathbb{N}$.*

Proof. Let H be a hyperbolic group and $\varphi \in End(H)$ and assume that the BRP holds when $p = 0$. Let $p \in \mathbb{N}$, $u, v \in H$, and take two geodesics α and β from 1 to u and from u to uv, respectively, so that the concatenation $\alpha + \beta$ is a $(1,p)$-quasi-geodesic. By Proposition 6.1.1, there is some $N \in \mathbb{N}$ such that for a $(1,p)$-quasi-geodesic ξ starting in 1 and ending in uv, we have that

$$d(\xi^{[n]}, (\alpha+\beta)^{[n]}) < N, \tag{6.3}$$

for every $n \in \mathbb{N}$. We will prove that, there is some $q \in \mathbb{N}$ such that, given any two geodesics ξ_1 and ξ_2 from 1 to $u\varphi$ and from $u\varphi$ to $(uv)\varphi$, respectively, their concatenation is a $(1,q)$-quasi-geodesic.

Take γ to be a geodesic from 1 to uv and consider the factorization

$$1 \xrightarrow{\gamma^{[|u|]}} x \xrightarrow{\gamma_2} uv.$$

Notice that both $\gamma^{[|u|]}$ and γ_2 are geodesics and their concatenation, γ is also a geodesic. In particular γ is a $(1,p)$-quasi-geodesic with the same starting and ending points as the concatenation of α and β. So, by, (6.3) we have that $d(x,u) < N$.

Set $B_\varphi = \max\{|a\varphi| \mid a \in \widetilde{A}\}$. Then, we have that $d(x\varphi, u\varphi) \leq B_\varphi d(x,u) < B_\varphi N$. Let ζ_1 and ζ_2 be geodesics from 1 to $x\varphi$ and from $x\varphi$ to $(uv)\varphi$, respectively. Since the BRP holds for φ when $p = 0$, we have that there is some constant p_0 such that the concatenation $\zeta_1 + \zeta_2$ is a

$(1, p_0)$-quasi-geodesic and that constant is independent from the choice of γ. So, we have that

$$d(1,(uv)\varphi) \geq d(1,x\varphi) + d(x\varphi,(uv)\varphi) - p_0. \tag{6.4}$$

Since $d(u\varphi,(uv)\varphi) \leq d(u\varphi,x\varphi) + d(x\varphi,(uv)\varphi) \leq NB_\varphi + d(x\varphi,(uv)\varphi)$, we have that

$$d(x\varphi,(uv)\varphi) \geq d(u\varphi,(uv)\varphi) - NB_\varphi. \tag{6.5}$$

Similarly, we have that $d(1,u\varphi) \leq d(1,x\varphi) + d(x\varphi,u\varphi) \leq d(1,x\varphi) + NB_\varphi$, and so

$$d(1,x\varphi) \geq d(1,u\varphi) - NB_\varphi. \tag{6.6}$$

Combining (6.4) with (6.5) and (6.6), we have that

$$\begin{aligned} d(1,(uv)\varphi) &\geq d(1,u\varphi) - NB_\varphi + d(u\varphi,(uv)\varphi) - NB_\varphi - p_0 \\ &= d(1,u\varphi) + d(1,v\varphi) - 2NB_\varphi - p_0. \end{aligned}$$

Hence

$$\begin{aligned} 2\left((u\varphi)^{-1}|v\varphi\right) &= d(1,u\varphi) + d(1,v\varphi) - d((u\varphi)^{-1},v\varphi) \\ &= d(1,u\varphi) + d(1,v\varphi) - d(1,(uv)\varphi) \\ &\leq 2NB_\varphi + p_0. \end{aligned}$$

By Lemma 6.2.1, we have that the concatenation $\xi_1 + \xi_2$ is a $(1, 2NB_\varphi + p_0)$-quasi-geodesic. $\square$

If (X,d) is δ-hyperbolic and $\lambda \geq 1$, $K \geq 0$, it follows from [16, Thm 1.7, Section III.H.3] that there exists a constant $R(\delta,\lambda,K)$, depending only on δ,λ,K, such that any geodesic and (λ,K)-quasi-geodesic in X having the same initial and terminal points lie at Hausdorff distance $\leq R(\delta,\lambda,K)$ from each other. This constant will be used in the proof of the next result.

We recall that for a geodesic $\alpha : [0,n] \to H$, we will often denote its image by α as well. We are now ready to present two more (equivalent) geometric formulations of the BRP.

Theorem 6.2.5. *Let $\varphi \in \mathrm{End}(H)$. The following conditions are equivalent:*

1. *the BRP holds for φ.*
2. *there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x,y]$, we have that $\alpha\varphi$ is at Hausdorff distance at most N to every geodesic $[x\varphi, y\varphi]$.*
3. *there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x,y]$, we have that $\alpha\varphi \subseteq \mathcal{V}_N(\xi)$ for every geodesic $\xi = [x\varphi, y\varphi]$.*

Proof. Clearly $2 \implies 3$.

$1 \implies 2$: Let $x, y \in H$ and $N \in \mathbb{N}$ given by the BRP when $p = 0$. Consider geodesics $\alpha = [x,y]$ and $\xi = [x\varphi, y\varphi]$. Let $u \in \xi$ and $k = d(x\varphi, u)$. So, clearly, $d(x\varphi, y\varphi) \geq k$. Put

$B_\varphi = \max\{|a\varphi| \mid a \in \widetilde{A}\}$. We may assume that $k > B_\varphi$ since, otherwise $d(u, \alpha\varphi) \leq B_\varphi$. Since $d(x\varphi, x\varphi) = 0 < k$ and $d(y\varphi, x\varphi) \geq k$, there is some $n_k > 0$ such that $d(x\varphi, (x(\alpha^{[n_k]}\pi))\varphi) < k$ and $d(x\varphi, (x(\alpha^{[n_k+1]}\pi))\varphi) \geq k$ (notice that $n_k > 0$ since $d(x\varphi, (x(\alpha^{[1]}\pi))\varphi) \leq B_\varphi < k$). Consider the following factorization of α

$$x \xrightarrow{\alpha^{[n_k]}} x_k \xrightarrow{\alpha_k} y\ .$$

Using the BRP, we have that, given geodesics β, γ from $x\varphi$ to $x_k\varphi$ and from $x_k\varphi$ to $y\varphi$, respectively, the concatenation

$$x\varphi \xrightarrow{\beta} x_k\varphi \xrightarrow{\gamma} y\varphi$$

is a $(1, N)$-quasi-geodesic. Set $x_{k+1} = x(\alpha^{[n_k+1]}\pi)$. We know that $d(x_k\varphi, x_{k+1}\varphi) \leq B_\varphi$ and so

$$k \leq d(x\varphi, x_{k+1}\varphi) \leq d(x\varphi, x_k\varphi) + B_\varphi.$$

Thus, we have that $d(x_k\varphi, x\varphi) \geq k - B_\varphi$.

Let $z = (\beta + \gamma)^{[k]}$. Notice that, since $d(x_k\varphi, x\varphi) < k$, then $z \in \gamma$.

$$\begin{tikzcd}
& u \arrow[dr, bend left, "\xi_k"] & \\
x\varphi \arrow[ur, bend left, "\xi^{[k]}"] \arrow[r, bend left, "\beta"'] & x_k\varphi \arrow[r, bend left, "\gamma"'] & y\varphi
\end{tikzcd}$$

Using the fellow traveler property for $(1, N)$-quasi-geodesics, there is some constant M depending only on N and δ such that $d(u, z) < M$. Since γ is a geodesic, then $d(x_k\varphi, z) = k - d(x\varphi, x_k\varphi) \leq k - (k - B_\varphi) = B_\varphi$. Thus,

$$d(u, x_k\varphi) \leq d(u, z) + d(z, x_k\varphi) < M + B_\varphi$$

and $u \in \mathcal{V}_{M+B_\varphi}(\alpha\varphi)$. Since u is an arbitrary element of ξ such that $d(u, \alpha\varphi) > B_\varphi$, we have that $\xi \subseteq \mathcal{V}_{M+B_\varphi}(\alpha\varphi)$

Now, let $v \in \alpha$. Since the BRP holds, taking any geodesics, β', γ' from $x\varphi$ to $v\varphi$ and from $v\varphi$ to $y\varphi$, the concatenation

$$x\varphi \xrightarrow{\beta'} v\varphi \xrightarrow{\gamma'} y\varphi$$

is a $(1, N)$-quasi-geodesic, so $\mathrm{Haus}((\beta' + \gamma'), \xi) < R(\delta, 1, N)$. In particular, $d(v\varphi, \xi) \leq R(\delta, 1, N)$. Since v is arbitrary, we have that $\alpha\varphi \subseteq \mathcal{V}_{R(\delta,1,N)}(\xi)$.

Hence $\mathrm{Haus}(\xi, \alpha\varphi) \leq \max\{R(\delta, 1, N), M + B_\varphi\}$.

$3 \implies 1$: Take N such that condition 3 holds and $u, v \in H$. Let $\alpha = [1, u]$, $\beta = [u, uv]$ be such that $\alpha + \beta$ is a geodesic and consider $\gamma = [1, u\varphi]$ and $\zeta = [u\varphi, (uv)\varphi]$. We want to prove that there is some $M \in \mathbb{N}$ such that $\gamma + \zeta$ is a $(1, M)$-quasi-geodesic and that suffices by Proposition 6.2.4. From condition 3, we have that $\alpha\varphi \subseteq \mathcal{V}_N(\gamma)$, $\beta\varphi \subseteq \mathcal{V}_N(\zeta)$ and $(\alpha + \beta)\varphi \subseteq \mathcal{V}_N(\gamma + \zeta)$. Since $\alpha + \beta$ is a geodesic, given a geodesic $\xi = [1, (uv)\varphi]$, we have that $((\alpha + \beta)\varphi) \subseteq \mathcal{V}_N(\xi)$ too.

$$\begin{tikzcd}
1 \arrow[r, "\gamma"'] \arrow[r, squiggly, bend left, "\alpha\varphi"] \arrow[rr, bend right, "\xi"'] & u\varphi \arrow[r, "\zeta"'] \arrow[r, squiggly, bend left, "\beta\varphi"] & (uv)\varphi
\end{tikzcd}$$

Now, we have that there is some $x_u \in \xi$ such that $d(u\varphi, x_u) < N$. Suppose that $d(1, x_u) \geq |\gamma|$ and denote $\xi^{[|\gamma|]}$ by y. So, γ and $\xi_u = \xi^{[|x_u|]}$ are two geodesics with the same starting point that end at bounded distance. By the fellow traveler property, there is some $K \in \mathbb{N}$ such that $d\,(u\varphi, y) < K$. So, we have that,

$$|\zeta| = d(u\varphi, (uv)\varphi) \leq d(u\varphi, y) + d(y, (uv)\varphi) \leq K + d(y, (uv)\varphi).$$

Now,

$$|\xi| = d(1, y) + d(y, (uv)\varphi) = |\gamma| + d(y, (uv)\varphi) \geq |\gamma| + |\zeta| - K.$$

If $d(1, x_u) < |\gamma|$, the same inequality can be obtained analogously, considering the geodesics ζ^{-1} and ξ^{-1}, since $|\gamma| + |\zeta| \geq |\xi|$.

So, we have that

$$\begin{aligned}
(u^{-1}\varphi | v\varphi) &= \frac{1}{2}(d(1, u\varphi) + d(1, v\varphi) - d(u^{-1}\varphi, v\varphi)) \\
&= \frac{1}{2}(|\gamma| + |\zeta| - d(1, (uv)\varphi)) = \frac{1}{2}(|\gamma| + |\zeta| - |\xi|) \\
&\leq \frac{K}{2}.
\end{aligned}$$

By Lemma 6.2.1, we have that $\gamma + \zeta$ is a $(1, K)$-quasi-geodesic. □

Recall that an equivalent definition of hyperbolicity is given by the existence of a center of geodesic triangles (see Lemma 2.5.1).

Given three points, the operator that associates the three points to the K-center of a geodesic triangle they define is coarse median and, by [85, Theorem 4.2.], it is the only coarse-median structure that we can endow X with.

We will now show that the BRP coincides with coarse-median preservation.

Theorem 6.2.6. *Let G be a hyperbolic group and $\varphi \in \mathrm{End}(G)$. Then the BRP holds for φ if and only if φ is coarse-median preserving.*

Proof. Let $H = \langle A \rangle$ be a hyperbolic group and $\varphi \in \mathrm{End}(H)$. Suppose that the BRP holds for φ and take N given by version 2 of the BRP given by Theorem 6.2.5, K be the constant given by Lemma 2.5.1, $B_\varphi = \max\{d_A(1, a\varphi) \mid a \in A\}$, $x, y, z \in G$ and geodesics $\alpha = [x, y]$, $\beta = [y, z]$,

$\gamma = [z, x]$. Then by definition of the coarse-median operator, $d(\mu(x, y, z), \alpha) \leq K$, and so there is some $x' \in \alpha$ such that $d(x', \mu(x, y, z)) \leq K$ and so

$$d(x'\varphi, \mu(x, y, z)\varphi) \leq KB_\varphi. \tag{6.7}$$

Now consider the triangle defined by $\alpha' = [x\varphi, y\varphi], \beta' = [y\varphi, z\varphi]$ and $\gamma' = [z\varphi, x\varphi]$. By the BRP, we have that $\alpha\varphi$ is at a Hausdorff distance at most N from α', so there is some $x'' \in \alpha'$ such that

$$d(x'', x'\varphi) \leq N. \tag{6.8}$$

Combining (6.7) and (6.8), we get that $d(\mu(x, y, z)\varphi, x'') \leq KB_\varphi + N$, and so $d(\mu(x, y, z)\varphi, \alpha') \leq KB_\varphi + N$. Similarly, we can prove that $d(\mu(x, y, z)\varphi, \beta') \leq KB_\varphi + N$ and that $d(\mu(x, y, z)\varphi, \gamma') \leq KB_\varphi + N$ and so $\mu(x, y, z)\varphi$ is a $(KB_\varphi + N)$-center of the triangle defined by α', β' and γ'. Since $\mu(x\varphi, y\varphi, z\varphi)$ is a K-center of the triangle, it is also a $(KB_\varphi + N)$-center and by [13, Lemma 3.1.5], we have that the distance between $\mu(x\varphi, y\varphi, z\varphi)$ and $\mu(x, y, z)\varphi$ is bounded and so φ is coarse-median preserving.

Now, suppose that φ is coarse-median preserving with constant $C \geq 0$. Take two points $x, y \in H$ and consider a geodesic $\alpha = [x, y]$. Let $x_i \in \alpha$. Then $\mu(x, x_i, y) = x_i$. Now consider a geodesic triangle given by $\alpha' = [x\varphi, x_i\varphi], \beta' = [x_i\varphi, y\varphi]$ and $\gamma' = [x\varphi, y\varphi]$. Since φ is coarse-median preserving, we have that

$$\begin{aligned} d(x_i\varphi, \gamma') &\leq d(x_i\varphi, \mu(x\varphi, x_i\varphi, y\varphi)) + d(\mu(x\varphi, x_i\varphi, y\varphi), \gamma') \\ &\leq d(\mu(x, x_i, y)\varphi, \mu(x\varphi, x_i\varphi, y\varphi)) + K \\ &\leq C + K. \end{aligned}$$

Since $x_i \in \alpha$ is arbitrary, this means that $\alpha\varphi \subseteq \mathcal{V}_{C+K}(\gamma')$, which means that the BRP holds for φ, by Theorem 6.2.5. $\square$

So, combining Propositions 6.2.2 and 6.2.4 with Theorems 6.2.5 and 6.2.6, we have proved the following result:

Theorem 6.2.7. *Let $\varphi \in \mathrm{End}(H)$. The following conditions are equivalent:*

1. *The BRP holds for φ.*
2. *The BRP holds for φ when $p = 0$.*
3. $\forall\, p > 0\, \exists\, q > 0\, \forall\, u, v \in H\, ((u|v) \leq p \implies (u\varphi|v\varphi) \leq q)$.
4. $\exists\, q > 0\, \forall\, u, v \in H\, ((u|v) = 0 \implies (u\varphi|v\varphi) \leq q)$.
5. *there is some $N \in \mathbb{N}$ such that, for all $x, y \in H$ and every geodesic $\alpha = [x, y]$, we have that $\alpha\varphi$ is at Hausdorff distance at most N to every geodesic $[x\varphi, y\varphi]$.*

6. *there is some* $N \in \mathbb{N}$ *such that, for all* $x, y \in H$ *and every geodesic* $\alpha = [x, y]$*, we have that* $\alpha\varphi \subseteq \mathcal{V}_N(\xi)$ *for every geodesic* $\xi = [x\varphi, y\varphi]$.

7. φ *is coarse-median preserving.*

6.3 Uniformly continuous endomorphisms

We will start by describing when an endomorphism of a hyperbolic group is uniformly continuous with respect to a visual metric. Recall that, since the completion is compact, those are precisely the endomorphisms admitting a continuous extension to the completion, as seen in Subsection 2.2.1.

A mapping $\varphi : (X, d) \to (X', d')$ between metric spaces satisfies a *Hölder condition* of exponent $r > 0$ if there exists a constant $K > 0$ such that

$$d'(x\varphi, y\varphi) \leq K(d(x, y))^r$$

for all $x, y \in X$. It clearly implies uniform continuity. We will show that in case of hyperbolic groups, the converse also holds.

In [2], the authors thoroughly study endomorphisms of hyperbolic groups satisfying a Hölder condition. In particular, they find several properties equivalent to satisfying a Hölder condition.

Theorem 6.3.1 ([2], Araújo-Silva)**.** *Let* φ *be a nontrivial endomorphism of a hyperbolic group* G *and let* $d \in V^A(p, \gamma, T)$ *be a visual metric on* G*. Then the following conditions are equivalent:*

1. φ *satisfies a Hölder condition with respect to* d*;*

2. φ *admits an extension to* $\widehat{G}$ *satisfying a Hölder condition with respect to* $\widehat{d}$*;*

3. *there exist constants* $P > 0$ *and* $Q \in \mathbb{R}$ *such that*

$$P(g\varphi|h\varphi)^A_p + Q \geq (g|h)^A_p$$

for all $g, h \in G$*;*

4. φ *is a quasi-isometric embedding of* (G, d_A) *into itself;*

5. φ *is virtually injective and* $G\varphi$ *is a quasiconvex subgroup of* G*.*

The authors in [2] conjecture that every uniformly continuous endomorphism satisfies a Hölder condition. We will give a positive answer to that problem later in this section.

We now present a natural result following from Theorem 6.2.5.

Corollary 6.3.2. *Let* $\varphi \in \mathrm{End}(H)$ *such that the BRP holds for* φ*. Then* $H\varphi$ *is quasiconvex.*

Proof. Let $x, y \in H$ and take $N \in \mathbb{N}$ given by condition 2 of Theorem 6.2.5. Consider geodesics $\alpha = [x, y]$ and $\xi = [x\varphi, y\varphi]$. We have that $\mathrm{Haus}(\xi, \alpha\varphi) \leq N$. Let $u \in \xi$. Then

$$d(u, H\varphi) \leq d(u, \alpha\varphi) \leq \mathrm{Haus}(\xi, \alpha\varphi) \leq N.$$

□

Lemma 4.1 in [2] states that a uniformly continuous endomorphism is virtually injective. So, next we will prove that uniform continuity implies the BRP. In that case, it follows that uniformly continuous endomorphisms are precisely the ones satisfying a Hölder condition.

Let $p, q, x, y \in H$. We have that

$$\begin{aligned}(x|y)_p &= \frac{1}{2}(d_A(p,x) + d_A(p,y) - d_A(x,y)) \\ &\leq \frac{1}{2}(d_A(q,x) + d_A(q,y) - d_A(x,y) + 2d_A(p,q)) \\ &= (x|y)_q + d_A(p,q)\end{aligned}$$

Similarly, $(x|y)_q \leq (x|y)_p + d_A(p,q)$, so

$$(x|y)_q - d_A(p,q) \leq (x|y)_p \leq (x|y)_q + d_A(p,q) \tag{6.9}$$

Proposition 6.3.3. *Let $G = \langle A \rangle$ and $H = \langle B \rangle$ be hyperbolic groups and consider visual metrics $d_1 \in V^A(p, \gamma, T)$ and $d_2 \in V^B(p', \gamma', T')$ on G and H, respectively. Let $\varphi : (G, d_1) \to (H, d_2)$ be an injective uniformly continuous homomorphism. Then, for every $M \geq 0$, there is some $N \geq 0$ such that*

$$(u|v)^A \leq M \implies (u\varphi|v\varphi)^B \leq N$$

holds for every $u, v \in G$.

Proof. Since φ is uniformly continuous, by a general topology result it admits a continuous extension $\hat{\varphi} : (\hat{G}, \hat{d}_1) \to (\hat{H}, \hat{d}_2)$. Since $\hat{\varphi}$ is a continuous map between compact spaces, it is closed, and so it has a closed (thus compact) image.

Now, restricting the codomain of $\hat{\varphi}$ to the image, we have a continuous bijection between compact spaces, and so it is a homeomorphism. Its inverse, $\psi : \mathrm{Im}(\hat{\varphi}) \to \hat{G}$ is a continuous map between compact spaces, hence uniformly continuous. So, the restriction $\psi' : (\mathrm{Im}(\varphi), d_2) \to (G, d_1)$ is also uniformly continuous, i.e.,

$$\forall \varepsilon > 0\, \exists \delta > 0\, (d_2(x,y) < \delta \implies d_1(x\psi', y\psi') < \varepsilon),$$

which, by construction of ψ', means that

$$\forall \varepsilon > 0\, \exists \delta > 0\, (d_2(x\varphi, y\varphi) < \delta \implies d_1(x,y) < \varepsilon). \tag{6.10}$$

Using (2.2), we have that (6.10) is equivalent to

$$\forall M \in \mathbb{N}\, \exists N \in \mathbb{N}\, ((x\varphi|y\varphi)^B_{p'} > N \implies (x|y)^A_p > M). \tag{6.11}$$

Since p and p' are fixed, we can change the basepoint to 1 using (6.9). So, (6.11) becomes equivalent to

$$\forall M \in \mathbb{N}\, \exists N \in \mathbb{N}\, ((x|y)^A \leq M \implies (x\varphi|y\varphi)^B \leq N).$$

□

Proposition 6.3.4. *Let $d \in V^A(p, \gamma, T)$ be a visual metric on H and let φ be a uniformly continuous endomorphism of H with respect to d. Then, the BRP holds for φ.*

Proof. If φ is injective it follows from Proposition 6.3.3. Now, in case φ is not injective, by Lemma 4.1 in [2], it must have finite kernel K. Consider $\pi : H \to H/K$ to be the projection and the geodesic metric $d_{A\pi}$ on the quotient. Let $\varphi' : \left(H/K, d_{A\pi}\right) \to (H, d_A)$ be the injective homomorphism induced by φ.

$$\begin{tikzcd} H \arrow[r, "\varphi"] \arrow[d, "\pi"'] & H \\ H/K \arrow[ur, "\varphi'"] & \end{tikzcd}$$

Let

$$L = \max\{d_A(1, x) \mid x \in K\}.$$

Let $g, h \in H$. We claim that

$$d_A(g, h) - L \leq d_{A\pi}(g\pi, h\pi) \leq d_A(g, h). \tag{6.12}$$

Since $h = ga_1 \ldots a_n$ implies $h\pi = (ga_1 \ldots a_n)\pi$ for all $a_1, \ldots, a_n \in \widetilde{A}$, we have $d_{A\pi}(g\pi, h\pi) \leq d_A(g, h)$.

Write $h\pi = (gw)\pi$, where w is a word on $\widetilde{A}$ of minimal length. Then $h = gwx$ for some $x \in K$ and so

$$d_A(g, h) \leq d_A(g, gw) + d_A(gw, gwx) = d_A(1, w) + d_A(1, x) \leq |w| + L.$$

By minimality of w, we have actually $|w| = d_{A\pi}(g\pi, h\pi)$ and thus (6.12) holds.

This means that (H, d_A) and $\left(H/K, d_{A\pi}\right)$ are quasi-isometric. In particular, it yields that H/K is hyperbolic.

Now, take $d' \in V^{A\pi}(p'\pi, \gamma', T')$ to be a visual metric on ${}^{H}\!/\!_{K}$. For every $u, v, p \in H$, we have that

$$\begin{aligned}(u\pi|v\pi)^{A\pi}_{p\pi} &= \frac{1}{2}(d_{A\pi}(p\pi, u\pi) + d_{A\pi}(p\pi, v\pi) - d_{A\pi}(u\pi, v\pi)) \\ &\leq \frac{1}{2}(d_A(p, u) + d_A(p, v) - d_A(u, v) + L) \\ &= (u|v)^A_p + \frac{L}{2}\end{aligned}$$

From uniform continuity of φ with respect to d, we get that

$$\forall M \in \mathbb{N}\, \exists N \in \mathbb{N}\, ((x|y)^A_p > N \implies (x\varphi|y\varphi)^A_p > M).$$

It follows that

$$\forall M \in \mathbb{N}\, \exists N \in \mathbb{N}\, ((x\pi|y\pi)^{A\pi}_{p\pi} > N \implies (x\varphi|y\varphi)^A_p > M),$$

and so φ' is uniformly continuous with respect to d' and d. It follows from Proposition 6.3.3 that for every $p \geq 0$, there is some $q \geq 0$ such that

$$(u\pi|v\pi)^{A\pi} \leq p \implies (u\pi\varphi'|v\pi\varphi')^A \leq q \tag{6.13}$$

holds for all $u\pi, v\pi \in {}^{H}\!/\!_{K}$.

Take $u, v \in H$ such that $(u|v)^A = 0$. Then

$$(u\pi|v\pi)^{A\pi} \leq (u|v)^A + \frac{L}{2} = \frac{L}{2}.$$

So, by (6.13), there is some q which does not depend on u, v such that $(u\varphi|v\varphi)^A \leq q$. By Proposition 6.2.4, the BRP holds for φ.

□

We can now answer Problem 6.1 left by the authors in [2].

Theorem 6.3.5. *Let $d \in V^A(p, \gamma, T)$ be a visual metric on H and let φ be an endomorphism of H. Then φ is uniformly continuous with respect to d if and only if the conditions from Theorem 6.3.1 hold.*

Proof. It is straightforward to see that condition 1 from Theorem 6.3.1 implies uniform continuity. Now, if φ is uniformly continuous, by [2, Lemma 4.1] it must be virtually injective and combining Proposition 6.3.4 with Corollary 6.3.2, we have that $H\varphi$ is quasiconvex, so condition 5 of Theorem 6.3.1 holds. □

We now present a visual representation of these properties, where the shaded region represents the nontrivial uniformly continuous endomorphisms. Indeed, we have proved that every nontrivial uniformly continuous endomorphism satisfies the BRP (Proposition 6.3.4) and that every endomorphism satisfying the BRP must have quasiconvex image (Corollary 6.3.2). Theorem 6.2.6 establishes an equivalence between the BRP and coarse-median preservation (CMP). Also, every virtually injective endomorphism with quasiconvex image must be uniformly continuous by Theorem 6.3.1. In [2], the authors give an example of an injective endomorphism of a torsion-free hyperbolic group with non quasiconvex image. So, unlike the case of virtually free groups, the BRP does not hold in general for injective endomorphisms of hyperbolic groups, not even when restricted to torsion-free hyperbolic groups. In the virtually free groups case, that does not happen, as every virtually injective endomorphism is uniformly continuous [98].

Taking $\varphi : F_3 \to F_3$ defined by $a \mapsto a$, $b \mapsto b$ and $c \mapsto 1$, we have that $F_3\varphi = \langle a, b\rangle$. Since $F_3\varphi$ is finitely generated and the standard embedding $\langle a, b\rangle \hookrightarrow F_3$ is a quasi-isometric embedding, then φ has quasiconvex image. But the BRP does not hold for φ since $|cb^n \wedge b^n| = 0$ and $|(cb^n)\varphi \wedge b^n\varphi| = |b^n \wedge b^n| = n$, which can be arbitrarily large.

It is easy to find examples of endomorphisms for which the BRP holds that are not virtually injective, even for virtually free groups, by taking an endomorphism with finite image. For example, take $H = \mathbb{Z} \times \mathbb{Z}_2$ and φ defined by $(n, 0) \mapsto (0, 0)$ and $(n, 1) \mapsto (0, 1)$. Then, the BRP holds for φ and its kernel is infinite (in particular, it can't be uniformly continuous).

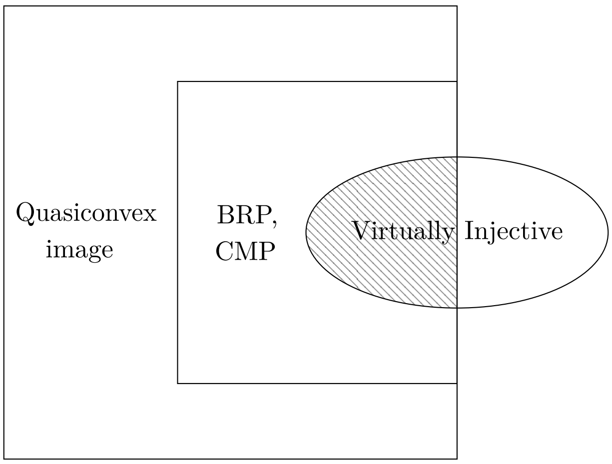

Figure 6.1 Nontrivial uniformly continuous endomorphisms of hyperbolic groups

So, for hyperbolic groups, we have the figure above in which every region is nonempty. Notice that, for virtually free groups, the only difference is that nontrivial uniformly continuous endomorphisms are precisely the virtually injective ones and the BRP holds for all of them.

In the case of free groups, it is even simpler as, for every nontrivial endomorphism, the properties of being injective, uniformly continuous and satisfying the BRP are equivalent. Indeed, it is well-known that injective endomorphisms coincide with uniformly continuous ones and that the BRP holds for this class. It is easy to see that the converse also holds. For

$n \geq 2$, let $X = \{x_1, \dots, x_n\}$ be a finite alphabet and $F_n = \langle X \rangle$ be a free group of rank n. If a nontrivial endomorphism $\varphi \in \mathrm{End}(F_n)$ is not injective, then there is some $w \in \mathrm{Ker}(\varphi)$ such that $w \neq 1$ and some letter a such that $a\varphi \neq 1$. Let $p = |w|$. We have that w is not a proper power of a since, in that case we would have that $w\varphi$ would be a proper power of $a\varphi$ and so, nontrivial. So, we have that, for arbitrarily large $m \in \mathbb{N}$, $|wa^m \wedge a^m| < |w|$, but $|(wa^m)\varphi \wedge a^m\varphi| = |a^m\varphi|$, which is arbitrarily large. So, nontrivial endomorphisms for which the BRP holds in a free group of finite rank are precisely the injective ones.

In [87], Paulin proved that $\mathrm{Fix}(\varphi)$ is finitely generated if $\varphi \in \mathrm{Aut}(H)$. We remark that his proof also yields the result for quasi-isometric embeddings. So, we have proved the following result.

Theorem 6.3.6. *Let $\varphi \in \mathrm{End}(H)$ be an endomorphism admitting a continuous extension $\hat{\varphi} : \widehat{H} \to \widehat{H}$ to the completion of H. Then,* $\mathrm{Fix}(\varphi)$ *is finitely generated.* □

Chapter 7

Automatic groups

7.1 Preliminaries

We now introduce some definitions and properties of automatic groups. For more details, the reader is referred to [42] and [57]. Let G be a group, A be a finite alphabet and $\pi : A^* \to G$ be a surjective homomorphism. A language $L \subseteq A^*$ is said to be a *section of* π if $L\pi = G$ and if $\pi|_L$ is bijective, we say that L is a *transversal of* π. Let \$ be a symbol that doesn't belong to A and consider

$$A_\$ = (A \times A) \cup (A \times \{\$\}) \cup (\{\$\} \times A).$$

Given two words $u, v \in A^*$, the convolution $u \diamond v$ is the only word in $A_\$^*$ such that the projection on the first (respectively second) component belongs to $u\$^*$ (respectively $v\$^*$).

The language L is an *automatic structure for* π if it is a rational section of π and, for every $x \in A \cup \{1\}$, there are finite state automata M_x over $A_\$$ such that $L(M_x) = \{u \diamond v \mid u, v \in L,\ v\pi = (ux)\pi\}$. If, additionally, L is a transversal, then we say that L is an *automatic structure with uniqueness for* π. Moreover, if for every $x \in A$ there is a finite state automaton ${}_xM$ over $A_\$$ such that $L({}_xM) = \{u \diamond v \mid u, v \in L,\ v\pi = (xu)\pi\}$, L is a *biautomatic structure for* π.

It is well known that, if L is an automatic structure for π, then the set of all shortlex minimal representatives of elements of G in L forms an automatic structure with uniqueness for π.

Given a group $G = \langle A \rangle$, consider its Cayley graph $\Gamma_A(G)$ with respect to A endowed with the geodesic metric d_A. We will slightly abuse notation when we write d_A. Indeed, when we are given an automatic structure L for $\pi : A^* \to G$, we have that $G = \langle A\pi \rangle$ and we will write d_A to denote $d_{A\pi}$, except when necessary: when the same alphabet is read in different ways (through different homorphisms).

For a language $L \subseteq A^*$ we say that the *fellow traveler property holds for* L if there is some $N \in \mathbb{N}$ such that, for all $u, v \in L$,

$$d_A(u\pi, v\pi) \leq 1 \implies d_A(u^{[n]}\pi, v^{[n]}\pi) \leq N,$$

for every $n \in \mathbb{N}$.

We say that $u, v \in A^*$ are *p-fellow travelers*, or that u and v *p-fellow travel*, in $\Gamma_A(G)$ if $d_A(u^{[n]}\pi, v^{[n]}\pi) \leq p$ for every $n \in \mathbb{N}$.

We will consider similar definitions to the ones above when the paths remain at bounded distance but asynchronously. We say that two words u and v in L *asynchronously p-fellow travel* if they p-fellow travel up to some reparametrisations, meaning that there are nondecreasing surjective functions $\phi, \psi : \mathbb{N} \to \mathbb{N}$ such that for all $t \in \mathbb{N}$ we have that $d_A\left(u^{[\phi(t)]}\pi, v^{[\psi(t)]}\pi\right) \leq p$. We say that a rational section L is an *asynchronous automatic structure* if the asynchronous version of the fellow traveler property holds for L.

It is also a well-known fact that a rational section L is an automatic structure for π if and only if the fellow traveler property holds for L. Similarly, biautomatic structures can be characterized by a variant of the fellow traveler property, where points might start at distance at most one apart. If $L_1 \subseteq A^*$ and $L_2 \subseteq B^*$ are synchronous or asynchronous automatic structures, then $L_1 \cup L_2$ is rational on $(A \cup B)^*$. Following [81], we say that L_1 and L_2 are *equivalent* if $L_1 \cup L_2$ is an asynchronous automatic structure. In hyperbolic groups, all structures are equivalent. We will also consider the notion of *synchronous equivalence* when $L_1 \cup L_2$ is a synchronous automatic structure. This notion is stricter than the notion of equivalence. For example in $\mathbb{Z}$, the structure given by $(aa^{-1}a)^* \cup (a^{-1}aa^{-1})^*$ is not synchronous equivalent to the usual structure given by $a^* \cup (a^{-1})^*$. In fact, taking $\mathbb{Z}$ with a single generator $a = 1$, there are infinitely many synchronous equivalence classes. Indeed, for $n \in \mathbb{N}$, the structures $L_n = (a(a^{-1}a)^n)^* \cup (a^{-1}(aa^{-1})^n)^*$ belong to different synchronous equivalence classes.

We say that an automatic structure has the *Hausdorff closeness property* if there is some $N \in \mathbb{N}$ such that, for all $u, v \in L$ such that $d_A(u\pi, v\pi) \leq 1$, the paths in the Cayley graph defined by u and v are at Hausdorff distance bounded by N.

We will also use a characterization of a *boundedly asynchronous automatic structure* from [42, Theorem 7.2.8]. We will only use the result to prove that a certain structure is asynchronous automatic and may take this characterization as a definition.

If G is an automatic group with automatic structure L for $\pi : A^* \to G$, a *departure function* for (G, L) is any function $D : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that, if $w \in L$, $r, s \geq 0$, $t \geq D(r)$ and $s + t \leq |w|$, then $d_A(w^{[s]}\pi, w^{[s+t]}\pi) > r$.

Definition 7.1.1. Let G be a group, A be an alphabet, $\pi : A^* \to G$ be a surjective homomorphism and L be a rational section of π. Then L is a *boundedly asynchronous structure on G* if and only if there exists a departure function D for (G, L) and L has the Hausdorff closeness property.

While an automatic structure is obviously asynchronous automatic and satisfies the Hausdorff closeness property, it is not true that it always admits a departure function. For example, take a finite group G and the alphabet $A = G$. Then take $L = A^*$ and $\pi : A^* \to G$. Let $p = \operatorname{diam}(G)$. It is obvious that any two words p-fellow travel, thus L is an automatic structure. However, it is easy to see that it can never admit a departure function. Clearly, the constant function equal to 0 is not a departure function, since, in that case, for $r > p$, we would have that, for any $w \in L$, $s,t \geq 0$ such that $s+t \leq |w|$, $d_A(w^{[s]}\pi, w^{[s+t]}\pi) > p$, which is absurd. If there was one nonzero departure function, say $D : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, then we could take $a \in A$, $x \in \mathbb{R}_0^+$ such that $D(x) \neq 0$ and the word $w = a^{\lceil D(x) \rceil |A|}$ and taking the prefixes of size a multiple of $\lceil D(x) \rceil$, $a^{k\lceil D(x) \rceil}$, for $k = 0, \ldots, A$, then we must have two distinct ones, say $a^{i\lceil D(x) \rceil}$ and $a^{j\lceil D(x) \rceil}$, with $0 \leq i < j \leq |A|$ representing the same element in G. This means that for $r = x$, $s = i\lceil D(x) \rceil$, $t = (j-i)\lceil D(x) \rceil > \lceil D(x) \rceil \geq D(x)$, we have that $s+t \leq |w|$ and $d_A(w^{[s]}\pi, w^{[s+t]}\pi) = 0$.

However, if L is an automatic structure, there is some $L' \subseteq L$ such that L' is boundedly asynchronous automatic. Also, every boundedly asynchronous automatic structure is in particular an asynchronous automatic structure.

When we work with an automatic structure, given an element $g \in G$, we will denote by $\overline{g}$ an arbitrary minimal length element of L such that $\overline{g}\pi = g$.

The following lemma, known as the bounded length difference lemma, will be crucial to our work.

Lemma 7.1.2. *[42, Lemma 2.3.9] Let G be an automatic group, A be an alphabet, $\pi : A^* \to G$ be a surjective homomorphism and L be an automatic structure π. Then, there is a constant $K > 0$ such that, if $w \in L$ and $g \in G$ is a vertex of the Cayley graph at distance at most one from $w\pi$, we have the following situation.*

1. *g has some representative of length at most $|w|+K$ in L; and*

2. *if some representative of g in L has length greater that $|w|+K$, there are infinitely many representatives of g in L.*

As a corollary, we have that if g is at bounded distance from $w\pi$, then there is some representative of bounded length. In particular, it follows that, for all $g, h \in G$,

$$\left| |\bar{g}| - |\bar{h}| \right| \leq K d_A(g,h).$$

If L is an automatic structure for $\pi : A^* \to G$ such that the empty word ε does not belong to L, we can consider $L' = L \cup \{\varepsilon\}$, which is obviously a rational section and satisfies the fellow traveler property. Thus, for all $g \in G$, we have that

$$|\bar{g}| \leq K d_A(1,g). \tag{7.1}$$

A subgroup $H \leq G$ is said to be *L-quasiconvex* if there exists some N such that, for all $h, h' \in H$ and $w \in L$ such that $h(w\pi) = h'$, we have that

$$d_A(h(w^{[n]}\pi), H) \leq N,$$

holds for all $n \in \mathbb{N}$.

7.1.1 A remark on asynchronous automatic structures and equivalence

Our definition of asynchronous automatic structure coincides with the one used in [55]. It is slightly different from the most common definition (see, for example [57],[83],[95]) because we take functions from $\mathbb{N}$ to $\mathbb{N}$ instead of taking the paths parametrized by arc length and nondecreasing surjective maps $\phi, \psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $t \in \mathbb{R}_{\geq 0}$ we have that

$$d_A\left(u(\phi(t)), v(\psi(t))\right) \leq p.$$

We now show that these two definitions coincide.

We start by showing that if the discrete version holds, then so does the continuous definition, using linear interpolation. Let $u, v \in L$ ending at distance at most one apart and take nondecreasing surjective $\phi, \psi : \mathbb{N} \to \mathbb{N}$ such that

$$d(u^{[\phi(t)]}, v^{[\psi(t)]}) \leq C,$$

for all $t \in \mathbb{N}$. Define $\phi', \psi' : [0, \infty) \to [0, \infty)$ by

$$\phi'(x) = (\phi(\lceil x \rceil) - \phi(\lfloor x \rfloor))(x - \lfloor x \rfloor) + \phi(\lfloor x \rfloor)$$

and

$$\psi'(x) = (\psi(\lceil x \rceil) - \psi(\lfloor x \rfloor))(x - \lfloor x \rfloor) + \psi(\lfloor x \rfloor).$$

We have that both ϕ' and ψ' are surjective, nondecreasing and they coincide with ϕ and ψ, respectively, in $\mathbb{N}$. Also, $\phi(\lfloor x \rfloor) \leq \phi'(x) \leq \phi(\lceil x \rceil)$, $\psi(\lfloor x \rfloor) \leq \psi'(x) \leq \psi(\lceil x \rceil)$ and

$$\begin{aligned} d(u(\phi'(t)), v(\psi'(t))) &\leq d(u(\phi'(t)), u(\phi(\lfloor t \rfloor))) + d(u(\phi(\lfloor t \rfloor)), v(\psi(\lfloor t \rfloor))) \\ &\quad + d(v(\psi(\lfloor t \rfloor)), v(\psi'(t))) \\ &\leq C + 2. \end{aligned}$$

Conversely, suppose that L is an asynchronous automatic structure for the continuous version of the definition. Let $u, v \in L$ ending at distance at most one apart and take nondecreasing surjective $\phi, \psi : [0, \infty) \to [0, \infty)$ such that

$$d(u(\phi(t)), v(\psi(t))) \leq C,$$

for all $t \in [0, \infty)$. Let t_i be such that $\phi(t_i) = i$. Notice that we can take $t_0 = \phi(t_0) = \psi(t_0) = 0$.

For $k \in \mathbb{N}$, let $z_k = \lfloor \psi(t_k) \rfloor + k$. Since the sequence $(z_k)_{k \in \mathbb{N}}$ is increasing and starts in 0, for each $n \in \mathbb{N}$, there is exactly one $i \in \mathbb{N}$ such that

$$\lfloor \psi(t_i) \rfloor + i \leq n < \lfloor \psi(t_{i+1}) \rfloor + i + 1.$$

We define $\phi', \psi' : \mathbb{N} \to \mathbb{N}$ as $\phi'(n) = i$ and $\psi'(n) = n - i$.

We have that ϕ' and ψ' are nondecreasing. Indeed in each interval of the form $\lfloor\psi(t_i)\rfloor + i \leq n < \lfloor\psi(t_{i+1})\rfloor + i + 1$, ϕ' is constant and increases by 1 when we move to the next interval and ψ' increases by 1 in each step inside the interval and remains constant when we change interval. Clearly, they are both surjective as well.

Now let $n \in \mathbb{N}$ and take $i \in \mathbb{N}$ such that $\lfloor\psi(t_i)\rfloor + i \leq n < \lfloor\psi(t_{i+1})\rfloor + i + 1$. Then, $\lfloor\psi(t_i)\rfloor \leq n - i < \lfloor\psi(t_{i+1})\rfloor + 1$, and since $n - i$ is an integer, then,

$$\lfloor\psi(t_i)\rfloor \leq n - i \leq \lfloor\psi(t_{i+1})\rfloor. \tag{7.2}$$

Now, if $\psi(t_i) \leq n - i$, there is some $t \in [t_i, t_{i+1}]$ such that $\psi(t) = n - i$. Fix such a t. Since ϕ is nondecreasing, then $i = \phi(t_i) \leq \phi(t) \leq \phi(t_{i+1}) = i + 1$. It follows that

$$\begin{aligned} d(u^{[\phi'(n)]}, v^{[\psi'(n)]}) &= d(u^{[i]}, v^{[n-i]}) \\ &= d(u^{[i]}, v(\psi(t))) \\ &\leq d(u^{[i]}, u(\phi(t))) + d(u(\phi(t)), v(\psi(t))) \\ &\leq d(u(\phi(t_i)), u(\phi(t))) + C \\ &\leq C + 1. \end{aligned}$$

If $n - i < \psi(t_i)$, then, by (7.2), $\psi(t_i) - (n - i) < 1$. In this case, let t be such that $\psi(t) = n - i$. We have that $\psi(t_i) - \psi(t) < 1$. Then

$$\begin{aligned} d(u^{[\phi'(n)]}, v^{[\psi'(n)]}) &= d(u^{[i]}, v^{[n-i]}) \\ &= d(u^{[i]}, v(\psi(t))) \\ &\leq d(u(\phi(t_i)), v(\psi(t_i))) + d(v(\psi(t_i)), v(\psi(t))) \\ &\leq C + 1. \end{aligned}$$

There is also another common way to define an asynchronous automatic structure (see, for example, [81], [84]). According to this definition, L is an asynchronous automatic structure if there is some $p > 0$ such that for all words u and v in L ending at distance at most 1, there is a nondecreasing surjective function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $t \in \mathbb{R}_{\geq 0}$, we have that

$$d_A\left(u(t), v\left(\phi(t)\right)\right) \leq p. \tag{7.3}$$

In [82], the author gives the following definition and lemma:

Definition 7.1.3. If X is a metric space and $\delta > 0$, two paths $\gamma_1, \gamma_2 : [0, \infty) \to X$ will be said to *δ-track* if there exists a homeomorphism $\phi : [0, \infty) \to [0, \infty)$ such that $d(\gamma_1(t), \gamma_2(\phi(t))) \leq \delta$ for all $t \in [0, \infty)$.

Lemma 7.1.4 ([82], Neumann)**.** *Suppose δ is a positive integer. If paths in $\Gamma_A(G)$ defined by words $w, v \in A^*$ δ-track, then there exist surjective nondecreasing functions $t \mapsto t'$ and $t \mapsto t''$*

mapping $\mathbb{N} \to \mathbb{N}$ *such that* $d(w^{[t']}, v^{[t'']}) \leq \delta$ *for all* $t \in \mathbb{N}$. *Conversely, if the latter condition is satisfied then the paths* $(\delta + 1)$*-track.*

Now we can see that this definition of an asynchronous automatic structure coincides with the previous ones. By Lemma 7.1.4, we have that if L is an asynchronous automatic structure, then for all words u, v ending at distance at most 1 there is a surjective nondecreasing function $\phi : [0, \infty) \to [0, \infty)]$ such that

$$d(u(t), v(\phi(t))) \leq C.$$

The converse also holds: Suppose that, for words u, v there is a surjective nondecreasing function $\phi : [0, \infty) \to [0, \infty)]$ such that

$$d(u(t), v(\phi(t))) \leq C.$$

It is possible to deform ϕ so that it becomes strictly increasing. In that case, an increasing surjective map must be a homeomorphism.

We remark that this notion of an asynchronous automatic structure, when considered in its discrete version does not coincide with the continuous version (nor with the previous definition). Similarly, using linear interpolation, we can see that if some L verifies the discrete condition, then it verifies the continuous one, but the converse is not true.

For example, if we take $\mathbb{Z}$ generated by $A = \{a, a^{-1}, a^2\}$, $\pi : A^* \to \mathbb{Z}$ defined in the natural way and put $L = a^* \cup (a^{-1})^* \cup (a^2)^*$, then L is rational and $\pi|_L$ is surjective. Let $u, v \in L$ be such that $d_A(u, v) \leq 1$.

If $\{u\pi, v\pi\} = \{-1, 1\}$, then $\{u, v\} = \{a, a^{-1}\}$ and $d(u(t), v(t)) \leq 1$, for all $t \in [0, \infty)$. If not, we must have that $(u\pi)(v\pi) \geq 0$.

If $u\pi, v\pi \leq 0$, then it is obvious that $d(u(t), v(t)) \leq 1$, for all $t \in [0, \infty)$.

If either $u\pi$ or $v\pi = 0$, then we also have that $d(u(t), v(t)) \leq 1$, for all $t \in [0, \infty)$.

So, suppose that $u\pi, v\pi > 0$. If $u = a^{k_1}$ and $v = a^{k_2}$ with $k_1, k_2 \in \mathbb{N}$, then $|k_1 - k_2| \leq 2$ and again $d(u(t), v(t)) \leq 1$, for all $t \in [0, \infty)$. Similarly, if $u = (a^2)^{k_1}$ and $v = (a^2)^{k_2}$, with $k_1, k_2 \in \mathbb{N}$, then $d(u(t), v(t)) \leq 1$, for all $t \in [0, \infty)$. If $u = (a^2)^{k_1}$ and $v = a^{k_2}$ with $k_1, k_2 \in \mathbb{N}$, then, $|2k_1 - k_2| \leq 2$ and letting $\phi(x) = 2x$, we have that $d(u(t), v(\phi(t))) \leq 1$. Similarly, if $u = a^{k_1}$ and $v = (a^2)^{k_2}$, we get the same with $\phi(x) = \frac{1}{2}x$. So, it is an asynchronous automatic structure according to (7.3). However, in a discrete version of (7.3), for every $C \in \mathbb{N}$, if we take $u = (a^2)^{C+1}$ and $v = a^{2C+2}$, for every surjective, nondecreasing $\phi : \mathbb{N} \to \mathbb{N}$, we have that $d(u(C+1), v(\phi(C+1))) \geq d(a^{2C+2}, a^{C+1}) = \frac{C+1}{2} > \frac{C}{2}$, because $\phi(n) \leq n$ for all $n \in \mathbb{N}$.

Shapiro, in the last remark of [95], states that it is easy to see that the notion of equivalence of automatic structures is indeed an equivalence relation. However, we couldn't find immediate to verify transitivity without using Lemma 7.1.4.

Let $\pi : A^* \to G$ and $\pi_2 : B^* \to G$ be surjective homomorphisms. We define

$$N_{A,B} = \max(\{d_B(1, a\pi_1) \mid a \in A\} \cup \{d_A(1, b\pi_2) \mid b \in B\}$$

and it follows that

$$\frac{1}{N_{A,B}} d_B(g,h) \leq d_A(g,h) \leq N_{A,B} d_B(g,h) \tag{7.4}$$

holds for all $g, h \in G$.

Let L be an asynchronous automatic structure (according to (7.3)) for $\pi : A^* \to G$ with asynchronous fellow travel constant C. We start by proving that for every $M > 0$ and for all words $u, v \in L$, there is a surjective nondecreasing map $\phi : [0, \infty) \to [0, \infty)$ for which the following holds:

$$d(u\pi, v\pi) \leq M \implies d(u(t), v(\phi(t))) \leq MC, \tag{7.5}$$

for all $t \in [0, \infty)$. We prove this by induction on M. If $M = 1$, this is clear by the fact that L is an asynchronous automatic structure. Now, suppose that this holds for $M \leq n$ and take words $u, v \in L$ such that $d(u\pi, v\pi) = n + 1$. Then, there is some word $w = w_1 \cdots w_{n+1} \in A^*$ such that $(uw)\pi = v\pi$. Take $w' \in L$ such that $w'\pi = (uw_1 \cdots w_n)\pi$. We have that $d(u\pi, w'\pi) \leq n$ and $d(w'\pi, v\pi) \leq 1$. So, there are nondecreasing and sujective maps ϕ_1 and ϕ_2 such that

$$d(u(t), w'(\phi_1(t))) \leq nC \quad \text{and} \quad d(w'(t), v(\phi_2(t))) \leq C, \tag{7.6}$$

for all $t \in [0, \infty)$. It follows that

$$\begin{aligned} d(u(t), v(\phi_2(\phi_1(t)))) &\leq d(u(t), w'(\phi_1(t))) + d(w'(\phi_1(t)), v(\phi_2(\phi_1(t)))) \\ &\leq (n+1)C. \end{aligned}$$

Now, we are ready to prove transitivity of the relation. Take asynchronous automatic structures L_1, L_2, L_3, for $\pi_1 : A^* \to G$, $\pi_2 : B^* \to G$ and $\pi_3 : C^* \to G$, respectively, such that $L_1 \sim L_2$ and $L_2 \sim L_3$. Also, let C_1, C_2 be the asynchronous fellow traveler constants (according to (7.3)) satisfied by $L_1 \cup L_2$ and $L_2 \cup L_3$, respectively.

We will prove that $L_1 \cup L_3$ is an asynchronous automatic structure for $\pi_4 : (A \cup C)^* \to G$ defined naturally. Let $u, v \in L_1 \cup L_3$ be such that $d_{A\cup C}(u\pi_4, v\pi_4) \leq 1$. We may suppose w.l.o.g. that $u \in L_1$ and $v \in L_3$, since, if u and v both belong to the same structure, then they asynchronously fellow travel in the respective Cayley graphs, and so they do in $\Gamma_{A\cup C}(G)$. Let $w \in L_2$ be such that $w\pi_2 = v\pi_3$. We have that $L_1 \cup L_2$ is an asynchronous automatic structure and

$$d_{A\cup B}(u\pi_1, w\pi_2) \leq N_{A\cup B, A\cup C} d_{A\cup C}(u\pi_1, v\pi_3) \leq N_{A\cup B, A\cup C},$$

so, by (7.5), there is some surjective nondecreasing $\phi_1 : [0, \infty) \to [0, \infty)$ such that

$$d_{A\cup B}(u(t), w(\phi_1(t))) \leq C_1 N_{A\cup B, A\cup C},$$

for all $t \in [0, \infty)$. Thus,

$$d_{A\cup C}(u(t), w(\phi_1(t))) \leq C_1 N^2_{A\cup B, A\cup C},$$

for all $t \in [0, \infty)$.

Also, since $L_2 \cup L_3$ is an asynchronous automatic structure, we have that there is some surjective nondecreasing $\phi_2 : [0, \infty) \to [0, \infty)$ such that

$$d_{A\cup C}(w(t), v(\phi_2(t))) \leq C_2,$$

for all $t \in [0, \infty)$. It follows that

$$\begin{aligned} d_{A\cup C}(u(t), v(\phi_2(\phi_1(t)))) &\leq d_{A\cup C}(u(t), w(\phi_1(t))) + d_{A\cup C}(w(\phi_1(t))v(\phi_2(\phi_1(t)))) \\ &\leq C_1 N^2_{A\cup B, A\cup C} + C_2 \end{aligned}$$

holds for all $t \in [0, \infty)$. Since $\phi_2 \circ \phi_1$ is surjective and nondecreasing, we are done.

7.2 Bounded Reduction Property

The purpose of this section is to explore the notion of bounded reduction for endomorphisms of automatic groups. We will adapt the definition proposed in for hyperbolic groups (see Theorem 6.2.7) and propose a new one, establishing some relations between them.

Let G be an automatic group, A be a finite alphabet and $\pi : A^* \to G$ be a surjective homomorphism. Let $L \subseteq A^*$ be an automatic structure for π and $\varphi \in \text{End}(G)$. Whenever we are able to, we will try to state the results as generally as possible, avoiding restrictions to the class of automatic structures that the group admits.

Given $u \in L$, $x \in G$, let $\theta_u^x : \{0, \dots, |u|\} \to G$ be the function that maps n to $x(u^{[n]}\pi)$.

Given a homomorphism $\varphi : G_1 \to G_2$ between two automatic groups G_1 and G_2 with automatic structures $L_1 \subseteq A^*$ and $L_2 \subseteq B^*$ for $\pi_1 : A^* \to G$ and $\pi_2 : B^* \to G$, respectively, we say that the *BRP holds for* (φ, L_1, L_2) if there is some $N > 0$ such that

$$\text{Haus}\left(\text{Im}(\theta_\alpha^x \varphi), \text{Im}(\theta_\beta^{x\varphi})\right) \leq N,$$

for all $x \in G$, $\alpha \in L_1$ and $\beta \in L_2$ such that $\beta\pi_2 = \alpha\pi_1\varphi$. Notice that this Hausdorff distance is not well defined since it is not clear what metric it refers to. However, that distinction is not relevant because of (7.4). We will also consider a synchronous version of the BRP. We say that the *synchronous BRP holds for* (φ, L_1, L_2) if there is some $N > 0$ such that

$$d\left((\theta_\alpha^x(n))\varphi, \theta_\beta^{x\varphi}(n)\right) \leq N,$$

for all $n \in \mathbb{N}$, $x \in G$, $\alpha \in L_1$ and $\beta \in L_2$ such that $\beta\pi_2 = \alpha\pi_1\varphi$.

Now, we prove that these versions of the bounded reduction property are preserved when we compose endomorphisms.

Lemma 7.2.1. *Let G_1, G_2 and G_3 be automatic groups with automatic structures $L_1 \subseteq A_1^*$, $L_2 \subseteq A_2^*$, and $L_3 \subseteq A_3^*$ for π_1, π_2 and π_3, respectively, and $\varphi_1 : G_1 \to G_2$ and $\varphi_2 : G_2 \to G_3$ be homomorphisms such that the (synchronous) BRP holds for (φ_1, L_1, L_2) and the (synchronous) BRP holds for (φ_2, L_2, L_3). Then the (synchronous) BRP holds for $(\varphi_1\varphi_2, L_1, L_3)$.*

Proof. First we deal with the BRP. Let $x \in G$, $\alpha \in L_1$ and $\gamma \in L_3$ such that $\gamma\pi_3 = \alpha\pi_1\varphi_1\varphi_2$. Let $\beta \in L_2$ such that $\beta\pi_2 = \alpha\pi_1\varphi_1$ and put $V_{\varphi_2} = \max\{d_{A_3}(1, a\varphi_2) \mid a \in A_2\}$. Since the BRP holds for (φ_1, L_1, L_2), there is some $N_1 > 0$ for which

$$\text{Haus}\left(\text{Im}(\theta_\alpha^x\varphi_1), \text{Im}(\theta_\beta^{x\varphi_1})\right) \leq N_1.$$

This yields that

$$\text{Haus}\left(\text{Im}(\theta_\alpha^x\varphi_1\varphi_2), \text{Im}(\theta_\beta^{x\varphi_1}\varphi_2)\right) \leq N_1 V_{\varphi_2}.$$

Since $\gamma\pi_3 = \alpha\pi_1\varphi_1\varphi_2 = \beta\pi_2\varphi_2$, by application of the BRP for (φ_2, L_2, L_3), there is some $N_2 > 0$ for which

$$\text{Haus}\left(\text{Im}(\theta_\beta^{x\varphi_1}\varphi_2), \text{Im}(\theta_\gamma^{x\varphi_1\varphi_2})\right) \leq N_2.$$

Thus,

$$\text{Haus}\left(\text{Im}(\theta_\alpha^x\varphi_1\varphi_2), \text{Im}(\theta_\gamma^{x\varphi_1\varphi_2})\right) \leq N_1 V_{\varphi_2} + N_2$$

and the BRP holds for $(\varphi_1\varphi_2, L_1, L_3)$.

Now, we deal with the synchronous BRP. Let $x \in G$, $\alpha \in L_1$ and $\gamma \in L_3$ be such that $\gamma\pi_3 = \alpha\pi_1\varphi_1\varphi_2$. Let $\beta \in L_2$ be such that $\beta\pi_2 = \alpha\pi_1\varphi_1$, $n \in \mathbb{N}$ and N_i be the constant given by the synchronous BRP for $(\varphi_i, L_i, L_{i+1})$ for $i = 1, 2$.

We want to prove that $d_{A_3}(\alpha^{[n]}\pi_1\varphi_1\varphi_2, \gamma^{[n]}\pi_3)$ is bounded. Since the synchronous BRP holds for (φ_1, L_1, L_2), we have that

$$d_{A_2}(\alpha^{[n]}\pi_1\varphi_1, \beta^{[n]}\pi_2) \leq N_1$$

and so

$$d_{A_3}(\alpha^{[n]}\pi_1\varphi_1\varphi_2, \beta^{[n]}\pi_2\varphi_2) \leq N_1 V_{\varphi_2}.$$

By application of the synchronous BRP for (φ_2, L_2, L_3), we obtain that

$$d_{A_3}(\beta^{[n]}\pi_2\varphi_2, \gamma^{[n]}\pi_3) \leq N_2.$$

Hence, we have that

$$d_{A_3}(\alpha^{[n]}\pi_1\varphi_1\varphi_2, \gamma^{[n]}\pi_3) \leq N_1 V_{\varphi_2} + N_2.$$

□

We now see that the definition of equivalent automatic structures can be relaxed, which will simplify some arguments.

Lemma 7.2.2. *Two automatic structures L_1 and L_2 are equivalent if and only if $L_1 \cup L_2$ has the Hausdorff closeness property.*

Proof. Let L_1, L_2 be automatic structures for $\pi_1 : A^* \to G$ and $\pi_2 : B^* \to G$. Clearly, if L_1 and L_2 are equivalent, then $L_1 \cup L_2$ has the Hausdorff closeness property. Now, suppose that $L_1 \cup L_2$ has the Hausdorff closeness property. We want to prove that L_1 and L_2 are equivalent, i.e., $L = L_1 \cup L_2 \subseteq (A \cup B)^*$ is an asynchronous automatic structure for $\pi_3 : (A \cup B)^* \to G$ defined naturally. Consider the boundedly asynchronous automatic structures $L_1' \subseteq L_1$ and $L_2' \subseteq L_2$ given by [42, Theorem 7.2.4]. We have that L_1' is equivalent to L_1 and L_2' is equivalent to L_2, so we will prove that L_1' is equivalent to L_2' and that suffices by transitivity. To do so, we will prove that $L_1' \cup L_2'$ satisfies both conditions of Definition 7.1.1. Hausdorff closeness is satisfied by hypothesis. We now prove the existence of a departure function for $L_1' \cup L_2'$. Let p_1, p_2 be the (synchronous) fellow traveler constants satisfied by L_1 and L_2 (and consequently by L_1' and L_2'), respectively, and D_1, D_2 be departure functions for the structures (L_1', π_1), (L_2', π_2), respectively. Take $D : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ defined by

$$D(x) = \max\{D_1(N_{A,A\cup B}x), D_1(N_{B,A\cup B}x), D_2(N_{A,A\cup B}x), D_2(N_{B,A\cup B}x)\},$$

for every $x \in \mathbb{R}_0^+$. Let $w \in L_1' \cup L_2'$, $r, s \geq 0$, $t \geq D(r)$ and $s + t \leq |w|$. Suppose w.l.o.g. that $w \in L_1'$. Then

$$d_{A\cup B}(w^{[s]}\pi_3, w^{[s+t]}\pi_3) \geq \frac{1}{N_{A,A\cup B}} d_A(w^{[s]}\pi_1, w^{[s+t]}\pi_1) \geq r.$$

□

7.3 Quasiconvex subgroups

In this section, we will focus on endomorphisms whose image is L-quasiconvex. The main result of the section proves that, in some sense, the synchronous BRP holds for endomorphisms with finite kernel and L-quasiconvex image.

Gersten and Short proved in [46] that quasiconvex subgroups of automatic groups are automatic and gave a structure for the subgroup. We will replicate their definitions and follow the lines in their proof. Let $L \subseteq A^*$ be an automatic structure for $\pi_1 : A^* \to G$ and H be an L-quasiconvex subgroup with constant k. Take a word $w = a_1 \dots a_n$ in $L'(H) = L \cap \pi_1^{-1}(H)$. Since H is quasiconvex, and $w\pi_1 \in H$, we have that for every $i \in [n]$ there is some word $g_i \in A^*$ such that $|g_i| \leq k$ and $w^{[i]}\pi_1 g_i \pi_1 \in H$. So, $w\pi_1 = \prod_{i=1}^{n} (g_{i-1}\pi_1)^{-1}(a_i\pi_1)(g_i\pi_1)$, where $g_0 = g_n = \varepsilon$. Hence, each element in H can be written as a product of elements of norm at most $2k+1$. We take B to be the set of those words together with their inverses and let $L''(H)$ be the set of words in $L'(H)$ rewritten as words in B^*. Notice that every $b \in B$ represents an element of H. When the quasiconvex subgroup H is clearly set, we usually write L' and L'' instead of

$L'(H)$ and $L''(H)$, respectively. We remark that this notation will be adopted throughout the chapter, so, whenever we write L' or L'', we will always be referring to this construction and will mostly be used when the subgroup is the image of an endomorphism.

We now present a technical lemma that will be very useful later on.

Lemma 7.3.1. *Let G be an automatic group with automatic structure L for $\pi : A^* \to G$ and H be an L-quasiconvex subgroup with constant k and automatic structure L'' for $\pi_2 : B^* \to H$. There is $K \in \mathbb{N}$ such that, for all $x, y \in H$,*

$$\frac{1}{2k+1} d_A(x,y) \leq d_B(x,y) \leq K d_A(x,y).$$

Proof. Take $K \in \mathbb{N}$ given by Lemma 7.1.2 applied to L and let $x, y \in H$. By construction of L'', elements of B are words of length at most $2k+1$ in A^*. So,

$$\frac{1}{2k+1} d_A(x,y) \leq d_B(x,y).$$

Take w to be an arbitrary representative of $x^{-1}y$ of minimal length in L. By (7.1), there is K such that $|w| \leq K d_A(x,y)$. By construction of the rewriting process, there is a word $w' \in L'' \subseteq B^*$ of length equal to $|w|$ such that $w'\pi_2 = x^{-1}y$. Hence,

$$d_B(x,y) \leq |w'| = |w| \leq K d_A(x,y).$$

□

Proposition 7.3.2. *Let G be an automatic group with automatic structure L for $\pi_1 : A^* \to G$ and H be an L-quasiconvex subgroup with constant k. Let B be the canonical set of generators of L'' with $\pi_2 : B^* \to H$ and $\pi_3 : (A \cup B)^* \to G$, defined naturally. Then, there is some $N \in \mathbb{N}$ satisfying the following property:*

> *for words $u, v \in L' \cup L''$ such that $d_A(u\pi_3, v\pi_3) \leq 1$, the inequality*
>
> $$d_A(u^{[n]}\pi_3, v^{[n]}\pi_3) \leq N$$
>
> *holds for all $n \in \mathbb{N}$.*

Proof. By [46, Theorem 3.1], L'' is an automatic structure for $\pi_2 : B^* \to H$ defined by $b\pi_2 = b\pi_1$. Let $\tilde{L} = L' \cup L''$. Let K be given by Lemma 7.3.1, M and M'' be the constants given by the fellow traveler property of L and L'', respectively, and take $u, v \in \tilde{L}$ such that $d_A(u\pi_3, v\pi_3) \leq 1$. If both u and v belong to $L' \subseteq L$ they M-fellow travel in $\Gamma_A(G)$ and if $u, v \in L''$, then $d_B(u\pi_2, v\pi_2) \leq K$ and so u and v (KM'')-fellow travel in $\Gamma_B(H)$. By Lemma 7.3.1, the result follows.

So suppose w.l.o.g. that $u \in L'$ and $v \in L''$. There is some word $w \in L''$ obtained by rewriting u as a word in B^*. Also, by construction, we have that

$$d_A(u^{[n]}\pi_1, w^{[n]}\pi_2) \leq k, \tag{7.7}$$

for all $n \in \{0, \dots |u|\}$, where w is seen as a word in B^*. In this sense, $w^{[1]}$ is a word of length at most $2k+1$ in A^*. But then, w is a word in B^* such that $d_A(w\pi_3, v\pi_3) \leq 1$, thus $d_B(w\pi_2, v\pi_2) \leq K$. We know that paths starting in the same point ending at bounded distance fellow travel in $\Gamma_B(H)$. Hence, we have that

$$d_B(w^{[n]}\pi_2, v^{[n]}\pi_2) \leq KM''$$

and so

$$d_A(w^{[n]}\pi_3, v^{[n]}\pi_3) \leq (2k+1)d_B(w^{[n]}\pi_2, v^{[n]}\pi_2) \leq (2k+1)KM'' \tag{7.8}$$

for all $n \in \mathbb{N}$. Combining (7.7) with (7.8), we get that

$$d_A(u^{[n]}\pi_3, v^{[n]}\pi_3) = d_A(u^{[n]}\pi_1, v^{[n]}\pi_2) \leq (2k+1)KM'' + k,$$

for all $n \in \mathbb{N}$. □

Let G_1, G_2 be automatic groups and L be an automatic structure for $\pi : A^* \to G_1$. Consider $\varphi : G_1 \to G_2$ to be a homomorphism. We say that L *induces an automatic structure through* φ if L is an automatic structure of $G\varphi$ for $\pi\varphi$. We call this the *automatic structure induced by L through* φ and denote it by $L^{(\varphi)}$. In particular, the existence of an induced structure implies that $G\varphi$ is automatic.

In the proof of the following lemma we will always specify how letters in A are read to avoid confusion, since we are dealing with the same language read through different surjective homomorphisms.

Lemma 7.3.3. *Let G be an automatic group, L be an automatic structure for $\pi_1 : A^* \to G$ and $\varphi \in \operatorname{End}(G)$ be a virtually injective endomorphism of G. Then ${}^{G}\!/_{\operatorname{Ker}(\varphi)}$ is automatic and L is an automatic structure for $\pi_1\pi$, where $\pi : G \to {}^{G}\!/_{\operatorname{Ker}(\varphi)}$ denotes the projection onto the quotient.*

Proof. Let K denote $\operatorname{Ker}(\varphi)$. We have that $G = \langle A\pi_1 \rangle$ and ${}^{G}\!/_{K} = \langle A\pi_1\pi \rangle$. Take $M = \max\{d_{A\pi_1}(1, x) \mid x \in K\}$. Since π_1 and π are sujective, so is $\pi_1\pi$. We will now see that the fellow traveler property holds. So, take two words $u, v \in L$ such that $d_{A\pi_1\pi}(u\pi_1\pi, v\pi_1\pi) \leq 1$. Then, there is some $a \in A \cup \{1\}$ such that $u\pi_1\pi = (v\pi_1\pi)(a\pi_1\pi)$, which means, by definition of π, that $u\pi_1\varphi = (va)\pi_1\varphi$ and so $(u\pi_1)^{-1}(va)\pi_1 \in K$. By definition of M, $d_{A\pi_1}(u\pi_1, (va)\pi_1) = d_{A\pi_1}(1, (u\pi_1)^{-1}(va)\pi_1) \leq M$. So, there is some N given by the fellow traveler property for π_1

for L-words ending at distance at most M such that

$$d_{A\pi_1}(u^{[n]}\pi_1, v^{[n]}\pi_1) \leq N$$

holds for all $n \in \mathbb{N}$ and so

$$d_{A\pi_1\pi}(u^{[n]}\pi_1\pi, v^{[n]}\pi_1\pi) \leq N$$

holds for all $n \in \mathbb{N}$. $\square$

Corollary 7.3.4. *Let G be an automatic group and $\varphi \in \mathrm{End}(G)$ be virtually injective. Then L induces an automatic structure through φ.*

Proof. Let $\varphi' : G \to G\varphi$ be the homomorphism obtained by restricting the codomain of φ to the image. Then $\varphi' = \pi\bar{\varphi}$, where $\bar{\varphi} : {}^{G}\!/_{\mathrm{Ker}(\varphi)} \to G\varphi$ is an isomorphism and so taking an automatic structure L for $\pi_1 : A^* \to G$, by Lemma 7.3.3, we have that L is an automatic structure for $\pi_1\pi\bar{\varphi}$. $\square$

We remark that the converse does not hold. To see that, it suffices to consider an endomorphism with finite image.

Theorem 7.3.5. *Let G_1 and G_2 be automatic groups with automatic structures L_1 and L_2 for $\pi_1 : A^* \to G_1$ and $\pi_2 : B^* \to G_2$, respectively. Let $\varphi : G_1 \to G_2$ be a homomorphism such that $G_1\varphi$ is L_2-quasiconvex and suppose that L_1 induces an automatic structure through φ. Then $L_1^{(\varphi)}$ and $L_2''(G_1\varphi)$ are (synchronous) equivalent if and only if the (synchronous) BRP holds for (φ, L_1, L_2).*

Proof. Let C be the standard set of generators of L_2''.

Asynchronous case. Suppose that $L_1^{(\varphi)}$ and L_2'' are equivalent and put $L = L_1^{(\varphi)} \cup L_2''$. Then L is an asynchronous automatic structure for $\pi_3 : (A \cup C)^* \to G_1\varphi$. Take p to be the asynchronous fellow travel constant satisfied by L and q to be the quasiconvexity constant.

Let $\alpha \in L_1$, $x \in G_1$, and $\beta \in L_2$ be such that $\beta\pi_2 = \alpha\pi_1\varphi$, $k \in \{0, \ldots, |\alpha|\}$ and $y = (\theta_\alpha^x(k))\varphi$. Take $N \in \mathbb{N}$ given by Proposition 7.3.2 and $\beta' \in L_2''$ such that $\beta'\pi_3 = \alpha\pi_1\varphi$. By Proposition 7.3.2, we have that

$$d_B(\beta^{[n]}\pi_2, \beta'^{[n]}\pi_3) \leq N$$

holds for all $n \in \mathbb{N}$. By hypothesis, we have that α, when read in $L_1^{(\varphi)}$, asynchronously p-fellow travels β' in $\Gamma_{A\cup C}(G_1\varphi)$. Let ϕ, ψ be the reparametrisation functions for α and β' and take $t \in \mathbb{N}$ such that $\phi(t) = k$. This way, we have that $y = (x\alpha^{[\phi(t)]\pi_1})\varphi$. By construction, elements of C are words of length at most $2q+1$ in B^*. Hence, putting $\lambda = \max(\{2q+1\} \cup \{d_B(1, a\pi_1\varphi) \mid a \in A\})$, we have that

$$\begin{aligned}
d_B\left(y, \mathrm{Im}(\theta_\beta^{x\varphi})\right) &\leq d_B\left(y, x\varphi\left(\beta^{[\psi(t)]}\pi_2\right)\right)\\
&\leq d_B\left(x\varphi\left(\alpha^{[\phi(t)]}\pi_1\varphi\right), x\varphi\left(\beta'^{[\psi(t)]}\pi_2\right)\right)\\
&\quad + d_B\left(x\varphi\left(\beta'^{[\psi(t)]}\pi_2\right), x\varphi\left(\beta^{[\psi(t)]}\pi_2\right)\right)\\
&\leq \lambda d_{A\cup C}\left(\alpha^{[\phi(t)]}\pi_1\varphi, \beta'^{[\psi(t)]}\pi_2\right) + N\\
&\leq \lambda p + N
\end{aligned}$$

Now, let $k' \in \{0, \dots |\beta|\}$, $z = \theta_\beta^{x\varphi}(k')$ and k'' such that $\psi(k'') = k'$. This way, $z = x\varphi(\beta^{[\psi(k'')]}\pi_2)$. We have that

$$\begin{aligned}
d_B\left(z, \mathrm{Im}(\theta_\alpha^x\varphi)\right) &\leq d_B\left(z, x\varphi\left(\alpha^{[\phi(k'')]}\pi_1\varphi\right)\right)\\
&\leq d_B\left(z, x\varphi\left(\beta'^{[\psi(k'')]}\pi_2\right)\right) + d_B\left(x\varphi\left(\beta'^{[\psi(k'')]}\pi_2\right), x\varphi\left(\alpha^{[\phi(k'')]}\pi_1\varphi\right)\right)\\
&\leq d_B\left(x\varphi\left(\beta^{[\psi(k'')]}\pi_2\right), x\varphi\left(\beta'^{[\psi(k'')]}\pi_2\right)\right)\\
&\quad + \lambda d_{A\cup C}\left(x\varphi\left(\beta'^{[\psi(k'')]}\pi_2\right), x\varphi\left(\alpha^{[\phi(k'')]}\pi_1\varphi\right)\right)\\
&\leq N + \lambda p
\end{aligned}$$

It follows that

$$\mathrm{Haus}\left(\mathrm{Im}(\theta_\alpha^x\varphi), \mathrm{Im}(\theta_\beta^{x\varphi})\right) \leq N + \lambda p.$$

Now, suppose that $L_1 \subseteq A^*$ and $L_2 \subseteq B^*$ are such that there is some $K \in \mathbb{N}$ satisfying the following condition: for all $\alpha \in L_1$, $x \in G$, we have that

$$\mathrm{Haus}\left(\mathrm{Im}(\theta_\alpha^x\varphi), \mathrm{Im}(\theta_\beta^{x\varphi})\right) \leq K,$$

for every $\beta \in L_2$ such that $\beta\pi_2 = \alpha\pi_1\varphi$. We want to check that $L_1^{(\varphi)}$ and L_2'' are equivalent. In view of Lemma 7.2.2, it suffices to see that $L_1^{(\varphi)} \cup L_2''$ has the Hausdorff closeness property.

Consider two words $w_1, w_2 \in L_1^{(\varphi)} \cup L_2''$, whose images under the map π_3 are at distance at most one apart in $\Gamma_{A\cup C}(G_1\varphi)$.

If $w_1, w_2 \in L_1^{(\varphi)}$ or $w_1, w_2 \in L_2''$, then their Hausdorff distance is bounded by the synchronous fellow traveler constant of $L_1^{(\varphi)}$ or L_2'', respectively. So, suppose w.l.o.g. that $w_1 \in L_1^{(\varphi)}$ and $w_2 \in L_2''$. Take $w_2'' \in L_2''$ such that $w_2''\pi_2 = w_1\pi_1\varphi$ and $w_2' \in L_2' \subseteq L_2$ such that w_2'' is obtained by rewriting w_2' in C^*. By hypothesis, we have that

$$\mathrm{Haus}\left(\mathrm{Im}(\theta_{w_1}^1\varphi), \mathrm{Im}(\theta_{w_2'}^1)\right) \leq K.$$

Also, by construction, $d_B(w_2''^{[n]}\pi_2, w_2'^{[n]}\pi_3) \leq q$, for all $n \in \mathbb{N}$, thus w_1 and w_2'' are $(K+q)$-Hausdorff close in $\Gamma_B(G_2)$. By Lemma 7.3.1, it follow that they are Hausdorff close in $\Gamma_C(G_1\varphi)$.

Since w_2'' and w_2 are two words in L_2'' ending at distance at most one apart, they fellow travel in $\Gamma_C(G_1\varphi)$. So, w_1 and w_2 are Hausdorff close in $\Gamma_C(G_1\varphi)$ and thus in $\Gamma_{A\cup C}(G_1\varphi)$.

Hence, $L_1^{(\varphi)}$ and L_2'' are equivalent.

Synchronous case. Suppose that $L_1^{(\varphi)}$ and L_2'' are synchronous equivalent and put $L = L_1^{(\varphi)} \cup L_2''$. Then L is an automatic structure for $\pi_3 : (A \cup C)^* \to G_1\varphi$. Take p to be the synchronous fellow travel constant satisfied by L and q to be the quasiconvexity constant.

Let $\alpha \in L_1$ and $\beta \in L_2$ be such that $\beta\pi_2 = \alpha\pi_1\varphi$ and $n \in \mathbb{N}$. We want to prove that $d_B(\alpha^{[n]}\pi_1\varphi, \beta^{[n]}\pi_2)$ is bounded.

Take $N \in \mathbb{N}$ given by Proposition 7.3.2 and $\beta' \in L_2''$ obtained by rewriting β in C^*. Then, by construction, we have that

$$d_B(\beta'^{[n]}\pi_3, \beta^{[n]}\pi_2) \leq q.$$

By hypothesis, we have that α, when read in $L_1^{(\varphi)}$, synchronously p-fellow travels β' in $\Gamma_{A\cup C}(G_1\varphi)$, so

$$d_{A\cup C}(\alpha^{[n]}\pi_1\varphi, \beta'^{[n]}\pi_3) \leq p.$$

Hence, putting $\lambda = \max(\{2q+1\} \cup \{d_B(1, a\pi_1\varphi) \mid a \in A\})$, we have that

$$d_B(\alpha^{[n]}\pi_1\varphi, \beta'^{[n]}\pi_3) \leq \lambda p.$$

Thus,

$$d_B(\alpha^{[n]}\pi_1\varphi, \beta^{[n]}\pi_2) \leq q + \lambda p.$$

To prove the converse, suppose that the synchronous BRP holds for (φ, L_1, L_2) with constant K. We want to prove that $L = L_1^{(\varphi)} \cup L_2''$ is an automatic structure for $G_1\varphi$ with the homomorphism $\pi_3 : (A \cup C)^* \to G_1\varphi$ defined in the natural way. Take words $\alpha, \beta \in L$ such that $d_{A\cup C}(\alpha\pi_3, \beta\pi_3) \leq 1$. If both α and β belong to either $L_1^{(\varphi)}$ or L_2'', then they fellow travel in $\Gamma_A(G_1\varphi)$ or $\Gamma_C(G_1\varphi)$, respectively and so they do in $\Gamma_{A\cup C}(G_1\varphi)$. So suppose w.l.o.g. that $\alpha \in L_1^{(\varphi)}$ and $\beta \in L_2''$. Take $\gamma'' \in L_2''$ such that $\gamma''\pi_3 = \alpha\pi_3$ and $\gamma' \in L_2' \subseteq L_2$ such that γ'' is obtained by rewriting γ' in C^*. For every $n \in \mathbb{N}$, we have that

$$d_B(\alpha^{[n]}\pi_3, \gamma'^{[n]}\pi_2) = d_B(\alpha^{[n]}\pi_1\varphi, \gamma'^{[n]}\pi_2) \leq K,$$

by hypothesis. Also, by construction, we have that

$$d_B(\gamma'^{[n]}\pi_2, \gamma''^{[n]}\pi_3) \leq q,$$

for all $n \in \mathbb{N}$. Applying (7.4), we have that

$$d_{A\cup C}(\alpha^{[n]}\pi_3, \gamma''^{[n]}\pi_3) \leq d_A(\alpha^{[n]}\pi_3, \gamma''^{[n]}\pi_3) \leq N_{A,B}(q+K)$$

and since γ'' and β are words in L''_2 ending at bounded distance, they synchronously fellow travel in $\Gamma_C(G_1\varphi)$ and so, they do in $\Gamma_{A\cup C}(G_1\varphi)$ and the result follows. $\square$

Corollary 7.3.6. *Let G be an automatic group with automatic structures L_1 and L_2 for $\pi_1 : A^* \to G$ and $\pi_2 : B^* \to G$ respectively, and consider the identity mapping $\iota : (G, L_1) \to (G, L_2)$. Then L_1 and L_2 are (synchronous) equivalent if and only if the (synchronous) BRP holds for (ι, L_1, L_2). In particular, if G is hyperbolic, then the BRP holds for (ι, L_1, L_2).*

Corollary 7.3.7. *Let G be an automatic group with automatic structure L for $\pi_1 : A^* \to G$ and $\varphi \in \mathrm{End}(G)$ be an endomorphism inducing an automatic structure $L^{(\varphi)}$ on $G\varphi$. Then the (synchronous) BRP holds for $(\varphi', L, L^{(\varphi)})$, where $\varphi' : G \to G\varphi$ is the homomorphism obtained by restricting the codomain of φ to the image.*

Combining Corollary 7.3.6 with Lemma 7.2.1, we get that, if G is hyperbolic, then the BRP is independent of the structures considered, since we can compose with the identity mapping on the left (resp. right) to change the structure considered in the domain (resp. codomain) and the BRP will be preserved. Also, if G is an automatic group with automatic structure L for $\pi_1 : A^* \to G$ and $\varphi \in \mathrm{End}(G)$ is an endomorphism with L-quasiconvex image such that L induces an automatic structure through φ, then the BRP holds for (φ, L, L) if and only if $L^{(\varphi)}$ equivalent to L''. In particular, if the image is hyperbolic then the BRP must hold for (φ, L, L). This yields the already known result that if G is a hyperbolic group and $\varphi \in \mathrm{End}(G)$ is virtually injective and has quasiconvex image, i.e., a quasi-isometric embedding, then the BRP holds for φ (Proposition 6.2.3). Finally, we can generalize this by proving that in some sense, the synchronous BRP always holds for virtually injective endomorphisms with quasiconvex image.

Theorem 7.3.8. *Let G be an automatic group with an automatic stucture L for $\pi_1 : A^* \to G$ and φ be a virtually injective endomorphism with L-quasiconvex image. Then there is some automatic structure $\tilde{L}$ such that the synchronous BRP holds for $(\varphi, \tilde{L}, L)$.*

Proof. Let q be the quasiconvexity constant, K be given by Lemma 7.3.1, put $V_\varphi = \max\{d_A(1, a\varphi) \mid a \in A\}$ and $M = \max\{d_A(1, g) \mid g \in \mathrm{Ker}(\varphi)\}$. Consider $L'' = L''(G\varphi)$, which is an automatic structure for $\pi_2 : B^* \to G\varphi$. Put

$$S = B\pi_2 = \{b\pi_2 \mid b \in B\}.$$

We fix a total ordering of A. For every $s \in S$, $s\varphi^{-1}$ is finite and we denote the shortlex minimal word in A^* that represents an element in $s\varphi^{-1}$ by $\tilde{s}_{\varphi^{-1}}$. Let

$$C = \{\tilde{s}_{\varphi^{-1}} \mid s \in S\},$$

which is finite since S is finite. Notice that it might be the case where $\varepsilon \in S$. Also, let $N_C = \max\{d_A(1, c\pi_1) \mid c \in C\}$ and

$$L_{\mathrm{Ker}} = \{\overline{g} \in A^* \mid g \in \mathrm{Ker}(\varphi)\},$$

which is obviously finite. Now, we will define a language $\tilde{L}$ on $(A \cup C)^*$ and prove that it defines an automatic structure for $\pi_3 : (A \cup C)^* \to G$ defined by $w\pi_3 = w\pi_1$, where letters in C are viewed as words in A^*, such that $\tilde{L}^{(\varphi)}$ is synchronous equivalent to L''. Take a word $w = w_1 w_2 \cdots w_n \in L''$ and put $x_i = w_i \pi_2$, for $i = 1, \ldots, n$. Consider all words of the form $\widetilde{x_1}_{\varphi^{-1}} \cdots \widetilde{x_n}_{\varphi^{-1}} z \in (A \cup C)^*$, where $z \in L_{\mathrm{Ker}}$. So, each word is rewritten in $|\mathrm{Ker}(\varphi)|$ different ways and let $\tilde{L}$ be the language of the words obtained by rewriting all words in L''. We will prove that:

1. $\tilde{L}$ is rational;
2. $\pi_3|_{\tilde{L}}$ is a surjective homomorphism;
3. $\tilde{L}$ satisfies the fellow traveler property;
4. $\tilde{L}^{(\varphi)}$ and L'' are synchronous equivalent.

To prove 1, let $\mathcal{A}'' = (Q, q_0, T, \delta)$ be a finite state automaton recognizing L''. Then we replace every transition labelled by $b \in B$ by a transition labelled $\widetilde{(b\pi_2)}_{\varphi^{-1}}$. Then we add a new terminal state q_T and for each terminal state $q \in T$, add paths from q to q_T labelled by all words in L_{Ker}. The language accepted by the new automaton is precisely $\tilde{L}$.

To prove 2, let $g \in G$ and take a word $w = w_1 \cdots w_n \in L''$ representing $g\varphi$. Put $x_i = w_i \pi_2$, for $i = 1, \ldots, n$ and consider the word $w' = \widetilde{x_1}_{\varphi^{-1}} \cdots \widetilde{x_n}_{\varphi^{-1}} \in C^*$. Then, by definition of $\widetilde{x_i}_{\varphi^{-1}}$ and since φ is a homomorphism, we have that w' represents (via π_3) an element $g' \in G$ such that $g'\varphi = g\varphi$, i.e., there is some $k \in \mathrm{Ker}(\varphi)$ such that $g = g'k$. By construction of $\tilde{L}$, $w'\overline{k} \in \tilde{L}$ and it represents g when read through π_3. Since g is arbitrary, we have that $\pi_3|_{\tilde{L}}$ is a surjective homomorphism.

To prove 3, take two words $u, v \in \tilde{L}$ such that $d_{A\cup C}(u\pi_3, v\pi_3) \leq 1$. Then, by construction $u = u_C u_A$ and $v = v_C v_A$ where $u_C, v_C \in C^*$ and $u_A, v_A \in L_{\mathrm{Ker}}$.

Let $u'', v'' \in L''$ be words from which u, v are obtained through rewriting.

$$\begin{aligned} d_B(u''\pi_2, v''\pi_2) &= d_B(u\pi_3\varphi, v\pi_3\varphi) \\ &\leq K d_A(u\pi_3\varphi, v\pi_3\varphi) \\ &\leq K V_\varphi d_A(u\pi_3, v\pi_3) \\ &\leq K V_\varphi N_C d_{A\cup C}(u\pi_3, v\pi_3) \\ &\leq K V_\varphi N_C. \end{aligned}$$

Since L'' is an automatic structure, there is some N depending only on K, V_φ, N_C such that $u'' = u_1 \cdots u_r$ and $v'' = v_1 \cdots v_s$ N-fellow travel in $\Gamma_B(G\varphi)$. Take

$$P = \max\{d_A(1, g) \mid g \in G, d_A(1, g\varphi) \leq (2q+1)N\},$$

which is well defined since the kernel is finite. We now claim that u_C and v_C $(P+M)$-fellow travel in $\Gamma_{A\cup C}(G)$. Indeed, let $n \in \mathbb{N}$. For $i > r$, put $u_i = \varepsilon$ and for $i > s$, put $v_i = \varepsilon$. Then, by

Lemma 7.3.1, we have that

$$\begin{aligned}
&d_A\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\varphi,\left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\varphi\right)\\
\leq&(2q+1)d_B\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\varphi,\left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\varphi\right)\\
=&(2q+1)d_B(u''^{[n]}\pi_2,v''^{[n]}\pi_2)\\
\leq&(2q+1)N.
\end{aligned}$$

Thus, there are letters $a_1,\ldots,a_k\in A$ for some $k\leq(2q+1)N$ such that

$$\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\varphi(a_1\cdots a_k)\pi_3=\left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\varphi$$

and $(a_1\cdots a_k)\pi_3\in G\varphi$. Since $k\leq(2q+1)N$, then $(a_1\cdots a_k)\pi_3=\alpha\pi_3\varphi$, for some $\alpha\in A^*$ such that

$$|\alpha|\leq P. \tag{7.9}$$

Hence,

$$\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\alpha\right)\pi_3\right)^{-1}\left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\in\mathrm{Ker}(\varphi),$$

so

$$d_A\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\alpha\right)\pi_3,\left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\right)\leq M \tag{7.10}$$

Using (7.9) and (7.10), we get that

$$\begin{aligned}
&d_A\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\right)\pi_3,\left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\right)\\
\leq\ &d_A\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\right)\pi_3,\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\alpha\right)\pi_3\right)\\
&+d_A\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\alpha\right)\pi_3,\left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\right)\\
\leq\ &d_A(1,\alpha\pi_3)+M\\
\leq\ &|\alpha|+M\\
\leq\ &P+M.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&d_{A\cup C}(u_C^{[n]}, v_C^{[n]})\\
&= d_{A\cup C}\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\right)\pi_3, \left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\right)\\
&\leq d_{A}\left(\left(\widetilde{(u_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(u_n\pi_2)}_{\varphi^{-1}}\right)\pi_3, \left(\widetilde{(v_1\pi_2)}_{\varphi^{-1}}\cdots\widetilde{(v_n\pi_2)}_{\varphi^{-1}}\right)\pi_3\right)\\
&\leq P+M.
\end{aligned}$$

It follows that u and v $(P+3M)$-fellow travel because $|u_A|, |v_A|\leq M$.

Now, 4 is essentially obvious by construction. Consider $\tilde{L}^{(\varphi)} \cup L''$ and define $\pi_4 : (A \cup C \cup B)^* \to G\varphi$ naturally. Take words $u, v \in \tilde{L}^{(\varphi)} \cup L''$ such that $d_{A\cup C\cup B}(u\pi_4, v\pi_4) \leq 1$. If they both belong to $\tilde{L}^{(\varphi)}$ or L'', then they fellow travel in $\Gamma_{A\cup C}(G\varphi)$ or $\Gamma_B(G\varphi)$, respectively and so, they do in $\Gamma_{A\cup C\cup B}(G\varphi)$. So, suppose w.l.o.g that $u \in \tilde{L}^{(\varphi)}$ and $v \in L''$ and consider the factorization $u = u_C u_A$ as done above. Then $u\pi_4 = u\pi_3\varphi = u_C\pi_4$ since $u_A\pi_4 \in \mathrm{Ker}(\varphi)$. Consider the word $u'' \in L''$ from which u is obtained through rewriting. Since u'' and v fellow travel in $\Gamma_B(G\varphi)$, so do u_C (when read through $\pi_3\varphi$) and v. Since $|u_A|\leq M$, then u and v fellow travel in $\Gamma_B(G\varphi)$, and so they do in $\Gamma_{A\cup C\cup B}(G\varphi)$.

These four points, combined with Theorem 7.3.5, yield the desired result. □

7.4 Some applications

The goal of this section is to apply the techniques developed in the previous sections in order to prove quasiconvexity of interesting subgroups defined by endomorphisms.

Let A, B be finite alphabets. Take the alphabet

$$C = (A \times B) \cup (A \times \{\$\}) \cup (\{\$\} \cup B)$$

and define the convolution of two words $u \in A^*, v \in B^*$ as the only word in C^* whose projection on the first (resp. second) component belongs to $u\$^*$ (resp. $v\$^*$). The convolution can be defined naturally for languages $K \subseteq A^*$ and $L \subseteq B^*$ by taking

$$K \diamond L = \{u \diamond v \mid u \in K,\, v \in L\}.$$

We start by defining a natural structure on the direct product of two automatic groups.

Proposition 7.4.1. *Let G_1 and G_2 be automatic groups and take automatic structures L_1 for $\pi_1 : A^* \to G_1$ and L_2 for $\pi_2 : B^* \to G_2$. Then $L_1 \diamond L_2$ is an automatic structure for $G_1 \times G_2$.*

Proof. We start by proving that $L_1 \diamond L_2$ is a rational language of C^*. Define the obvious projection-like homomorphisms $\rho_1 : C^* \to A^*$ and $\rho_2 : C^* \to B^*$. We have that

$$A^* \diamond B^* = (A \times B)^*((A \times \{\$\})^* \cup (\{\$\} \times B)^*)$$

and so it is rational. Since

$$L_1 \diamond L_2 = L_1\rho_1^{-1} \cap L_2\rho_2^{-1} \cap (A^* \diamond B^*),$$

it is also rational. Now, define $\pi_3 : C^* \to G_1 \times G_2$ by $(x, y) \mapsto (x\pi_1, y\pi_2)$ if $(x, y) \in A \times B$, $(x, \$) \mapsto (x\pi_1, 1)$ and $(\$, y) \mapsto (1, y\pi_2)$. It is clearly surjective and $\pi_3|_{L_1 \diamond L_2}$ is still surjective. We only have to check that the fellow traveler property holds. So, take words $x = u_1 \diamond v_1$ and $y = u_2 \diamond v_2$ in $L_1 \diamond L_2$ such that $d_C(x\pi_3, y\pi_3) \leq 1$. Notice that $d_C(x\pi_3, y\pi_3) = \max\{d_A(u_1\pi_1, u_2\pi_1), d_B(v_1\pi_2, v_2\pi_2)\}$ and so there is some $r > 0$ such that u_1, u_2 r-fellow travel in $\Gamma_A(G_1)$ and v_1, v_2 r-fellow travel in $\Gamma_B(G_2)$. Let $n \in \mathbb{N}$. We have that

$$d_C(x^{[n]}\pi_3, y^{[n]}\pi_3) = \max\{d_A(u_1^{[n]}\pi_1, u_2^{[n]}\pi_1), d_B(v_1^{[n]}\pi_2, v_2^{[n]}\pi_2)\} \leq r.$$

□

Proposition 7.4.2. *Let G be an automatic group with automatic structures L_1 and L_2 for $\pi_1 : A^* \to G$ and $\pi_2 : B^* \to G$, respectively. Consider an endomorphism $\varphi : G \to G$ such that L_1 induces an automatic structure on $G\varphi$ and the BRP holds for (φ, L_1, L_2). Then,* Ker(φ) *is isomorphic to a $(L_1 \diamond L_1^{(\varphi)})$-quasiconvex subgroup of $G \times G\varphi$. In particular,* Ker(φ) *is automatic.*

Proof. We start by showing that we can assume that $L_2 \cap 1\pi_2^{-1} = \{\varepsilon\}$. If the BRP holds for (φ, L_1, L_2), then it holds for (φ, L_1, L_2'), where $L_2' \subseteq L_2$ is an automatic structure with uniqueness. If we replace the unique representative of 1 in L_2' by ε, we obtain a new automatic structure L_3 equivalent to L_2, and so, by Corollary 7.3.6, the BRP holds for (φ, L_1, L_3).

Let $N \in \mathbb{N}$ be the constant given by the BRP. Since L_1 induces an automatic structure on $G\varphi$, by Proposition 7.4.1, we have that $G \times G\varphi$ is an automatic group and $L_1 \diamond L_1^{(\varphi)}$ is an automatic structure of $G \times G\varphi$. Now, put $H = \{(x, x\varphi) \mid x \in G\} \leq G \times G\varphi$. We have that

$$\mathrm{Ker}(\varphi) \simeq \{(x, x\varphi) \mid x \in \mathrm{Ker}(\varphi)\} = (G \times \{1\}) \cap H.$$

Now we will prove that both H and $G \times \{1\}$ are $(L_1 \diamond L_1^{(\varphi)})$-quasiconvex subgroups of $G \times G\varphi$. It is obvious by construction that H is $(L_1 \diamond L_1^{(\varphi)})$-quasiconvex. Indeed, take a word $u \diamond v \in L_1 \diamond L_1^{(\varphi)}$ representing $(x, x\varphi)$ for some $x \in G$. We have that $u\pi_1\varphi = x\varphi = v\pi_1\varphi$ and so u and v fellow travel in $\Gamma_A(G\varphi)$, reading through $\pi_1\varphi$. Thus,

$$d\left((u^{[n]}\pi_1, v^{[n]}\pi_1\varphi), H\right) \leq d\left((u^{[n]}\pi_1, v^{[n]}\pi_1\varphi), (u^{[n]}\pi_1, u^{[n]}\pi_1\varphi)\right),$$

which is bounded by the fellow traveler constant satisfied by $L_1^{(\varphi)}$.

Also, to prove that $G \times \{1\}$ is $(L_1 \diamond L_1^{(\varphi)})$-quasiconvex with constant N, observe that, taking a word $u \diamond v \in L_1 \diamond L_1^{(\varphi)}$ representing an element in $G \times \{1\}$, we have that v must be a word in L_1 such that $v\pi_1 \in \mathrm{Ker}(\varphi)$. The path defined by v in $\Gamma_A(G\varphi)$ where letters are read through $\pi_1\varphi$ is at a Hausdorff distance smaller than N from $\{1\}$, i.e., $d_B(1, v^{[n]}\pi_1\varphi) \leq N$, because ε is the only representative of 1 in L_2 and the BRP holds for (φ, L_1, L_2). Thus, every point of the path defined by $u \diamond v \in L_1 \diamond L_1^{(\varphi)}$ is of the form (g, h), where $g \in G$ and $d_B(1, h) \leq N$. Letting $M = \max\{d_A(1, x) \mid x \in G\varphi,\, d_B(1, x) \leq N\}$, where letters of A are read through $\pi_1\varphi$, we have that $G \times \{1\}$ is $(L_1 \diamond L_1^{(\varphi)})$-quasiconvex with quasiconvexity M.

Since both subgroups are $(L_1 \diamond L_1^{(\varphi)})$-quasiconvex, they are $(L_1 \diamond L_1^{(\varphi)})$-rational and so their intersection is also $(L_1 \diamond L_1^{(\varphi)})$-rational, thus $(L_1 \diamond L_1^{(\varphi)})$-quasiconvex. □

Proposition 7.4.3. *Let G be an automatic group with automatic structures L_1 and L_2 for $\pi_1 : A^* \to G$ and $\pi_2 : B^* \to G$, respectively and consider an endomorphism $\varphi \in \mathrm{End}(G)$ such that the synchronous BRP holds for (φ, L_1, L_2). Then L_1 induces an automatic structure through φ.*

Proof. Let N be the constant given by the synchronous BRP for (φ, L_1, L_2), $V_\varphi = \max\{d_B(1, a\varphi) \mid a \in A\}$ and K be the fellow traveler constant satisfied by words in L_2 ending at distance at most V_φ. The only thing we need to check is that $L_1^{(\varphi)}$ satisfies the fellow traveler property. So let $u, v \in L_1$ be such that $d_A(u\pi_1\varphi, v\pi_1\varphi) \leq 1$. Let $u', v' \in L_2$ be such that $u'\pi_2 = u\pi_1\varphi$ and $v'\pi_2 = v\pi_1\varphi$. Since the synchronous BRP holds for (φ, L_1, L_2), we have that, for all $n \in \mathbb{N}$,

$$d_B(u^{[n]}\pi_1\varphi, u'^{[n]}\pi_2) \leq N \quad \text{ and } d_B(v^{[n]}\pi_1\varphi, v'^{[n]}\pi_2) \leq N.$$

Also, we have $d_B(u'\pi_2, v'\pi_2) \leq V_\varphi$, thus u' and v' K-fellow travel in $\Gamma_B(G)$. So, for every $n \in \mathbb{N}$, we have that

$$\begin{aligned} d_B(u^{[n]}\pi_1\varphi, v^{[n]}\pi_1\varphi) &\leq d_B(u^{[n]}\pi_1\varphi, u'^{[n]}\pi_2) + d_B(u'^{[n]}\pi_2, v'^{[n]}\pi_2) \\ &\quad + d_B(v'^{[n]}\pi_2, v^{[n]}\pi_1\varphi) \\ &\leq 2N + K \end{aligned}$$

and so, letting $M = \max\{d_A(1, x) \mid x \in G\varphi,\, d_B(1, x) \leq 2N + K\}$, where letters in A are read through $\pi_1\varphi$, we have that $L_1^{(\varphi)}$ satisfies the M-fellow traveler property. □

Remark 7.4.4. *The last two propositions combined show that having the synchronous BRP for any pair of languages L_1 and L_2 is enough to have an automatic, thus finitely presented, kernel. Indeed, if the synchronous BRP holds for (φ, L_1, L_2), then it holds for (φ, L_1, L_2') where L_2' is an automatic structure with uniqueness by Corollary 7.3.6. If we replace the unique representative of 1 in L_2' by ε, we obtain a new automatic structure L_3 such that the synchronous BRP holds for (φ, L_1, L_3). Now, φ, L_1 and L_3 satisfy the hypothesis of Proposition 7.4.2. This immediately* *shows that this is a very strong condition to impose on a general endomorphism. For example, in the case of virtually free groups, having finitely generated kernel implies that either the kernel or the image is finite.*

We will prove something similar for fixed points with the additional hypothesis that L_1 and L_2 belong to the same synchronous equivalence class, which seems a strong condition to impose.

Proposition 7.4.5. *Let G_1 and G_2 be automatic groups with automatic structures L_1 and L_2 for $\pi_1 : A^* \to G_1$ and $\pi_2 : B^* \to G_2$, respectively. Let $\varphi, \psi : G_1 \to G_2$ be homomorphisms such that the synchronous BRP holds for (φ, L_1, L_2) and (ψ, L_1, L_2). Let $\theta : G_1 \to G_2 \times G_2$ be the homomorphism defined by $x \mapsto (x\varphi, x\psi)$. Then the synchronous BRP holds for $(\theta, L_1, L_2 \diamond L_2)$.*

Proof. Let N_1 and N_2 be the constants given by the synchronous BRP holding for (φ, L_1, L_2) and (ψ, L_1, L_2), respectively, take $N = \max\{N_1, N_2\}$, and put, as usual, $C = (B \times B) \cup (B \times \{\$\}) \cup (\{\$\} \cup B)$ and $\pi_3 : C^* \to G_2 \times G_2$. Let $x \in G_1$, $u \in L_1$ representing x and $v \diamond w \in L_2 \diamond L_2$ representing $(x\varphi, x\psi)$. Since the synchronous BRP holds for (φ, L_1, L_2) and (ψ, L_1, L_2), we have that, for all $n \in \mathbb{N}$,

$$d_B(u^{[n]}\pi_1\varphi, v^{[n]}\pi_2) \leq N$$

and

$$d_B(u^{[n]}\pi_1\psi, w^{[n]}\pi_2) \leq N.$$

Thus,

$$d(u^{[n]}\pi_1\theta, (v \diamond w)^{[n]}\pi_3) = \max\{d_B(u^{[n]}\pi_1\varphi, v^{[n]}\pi_2), d_B(u^{[n]}\pi_1\psi, w^{[n]}\pi_2)\} \leq N$$

and the synchronous BRP holds for $(\theta, L_1, L_2 \diamond L_2)$. □

Example 7.4.6. The fact that the BRP is synchronous for both homomorphisms is crucial to the proof above. Indeed, consider $\mathbb{Z}$ with the structure L given by the geodesics and let $\theta : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $a \mapsto (a, a^2)$. Then, the image is not $(L \diamond L)$-quasiconvex (and so the BRP does not hold for $(\varphi, L, L \diamond L)$) despite being of the form $x \mapsto (x\varphi, x\psi)$, where the synchronous BRP holds for (φ, L, L) and the BRP holds for (ψ, L, L).

Corollary 7.4.7. *Let G_1 and G_2 be automatic groups with automatic structures L_1 and L_2 for $\pi_1 : A^* \to G_1$ and $\pi_2 : B^* \to G_2$, respectively. Let $\varphi, \psi : G_1 \to G_2$ be homomorphisms such that the synchronous BRP holds for (φ, L_1, L_2) and (ψ, L_1, L_2). Then $Eq(\varphi, \psi) = \{x \in G_1 \mid x\varphi = x\psi\}$ is isomorphic to a $(L_2 \diamond L_2)$-quasiconvex subgroup of $G_2 \times G_2$. In particular, $Eq(\varphi, \psi)$ is automatic.*

Proof. It is obvious that the diagonal subgroup $\Delta = \{(x, x) \in G_2 \times G_2 : x \in G_2\}$ is an $(L_2 \diamond L_2)$-quasiconvex subgroup of $G_2 \times G_2$. The subgroup $H = \{(x\varphi, x\psi) \mid x \in G_1\}$ is also $(L_2 \diamond L_2)$-quasiconvex since it is the image of the homomorphism in Proposition 7.4.5, for which the (synchronous) BRP holds. Since $\mathrm{Eq}(\varphi) \simeq \Delta \cap H$, the result follows. □

Corollary 7.4.8. *Let G be an automatic group with synchronous equivalent automatic structures L_1 and L_2 for $\pi_1 : A^* \to G$ and $\pi_2 : B^* \to G$, respectively. Let φ be an endomorphism such that the synchronous BRP holds for (φ, L_1, L_2). Then $\mathrm{Fix}(\varphi)$ is isomorphic to a $(L_2 \diamond L_2)$-quasiconvex subgroup of $G \times G$. In particular, Fix(φ) is automatic.* □

In view of Corollary 7.3.6, the hypothesis needed to apply Corollary 7.4.8 is equivalent to the existence of some automatic structure L such that the synchronous BRP holds for (φ, L, L). This seems to be quite strong since it requires the endomorphism to distort points only a bounded amount, in some sense. We will now prove that for biautomatic groups, inner automorphisms satisfy this property for any biautomatic structure and see what happens in case the group is free and L is the structure given by the geodesics.

Proposition 7.4.9. *Let G be a biautomatic group, L be a biautomatic structure for $\pi : A^* \to G$ and $\varphi \in \mathrm{Inn}(G)$ be an inner automorphism of G. Then, the synchronous BRP holds for (φ, L, L).*

Proof. For $a \in A$, let λ_a be the inner automorphism given by $x \mapsto axa^{-1}$. We will see that, for every $a \in A$, the synchronous BRP holds for (λ_a, L, L) and the result will follow by Lemma 7.2.1. Let $a \in A$, $w \in L$, N be given by the fellow traveler property for paths in $\Gamma_A(G)$ labelled by words in L starting and ending at distance at most one and take $w' \in L$ such that $w'\pi = a(w\pi)a^{-1}$. Consider the paths α starting in 1 labelled by w and β starting in a^{-1} labelled by w'. They start and end at distance at most 1 (it is exactly one if $a\pi \neq 1$), thus they N-fellow travel in $\Gamma_A(G)$. This means that for all $n \in \mathbb{N}$,

$$d_A\left(a^{-1}(w'^{[n]}\pi), w^{[n]}\pi\right) \leq N,$$

and so

$$d_A\left(w'^{[n]}\pi, a(w^{[n]}\pi)\right) \leq N,$$

then having

$$d_A\left(w'^{[n]}\pi, w^{[n]}\pi\varphi\right) = d_A\left(w'^{[n]}\pi, a(w^{[n]}\pi)a^{-1}\right) \leq N + 1.$$

□

Even though the hypothesis of Corollary 7.4.8 are strong, it yields an alternative proof to [46, Proposition 4.3].

Corollary 7.4.10. *The centralizer of a finite subset of a biautomatic group is biautomatic.*

Proof. The centralizer of an element is the fixed subgroup of the inner automorphism defined by that element. Since, by Proposition 7.4.9, inner automorphisms of biautomatic groups satisfy the synchronous BRP for (φ, L, L), where L is a biautomatic structure, then we can apply Corollary 7.4.8 to get that the centralizer is quasiconvex (and thus, biautomatic). Now, the centralizer of a finite subset is a finite intersection of quasiconvex subgroups, and so quasiconvex. □

We now prove that in the case of free groups, not many endomorphisms besides the inner automorphisms satisfy the hypothesis of Corollary 7.4.8.

Given a free group with basis X, we call *letter permutation automorphism* (relatively to the basis X) to an automorphism that permutes the elements of X.

Proposition 7.4.11. *For a free group $F = F_A$, the endomorphisms φ for which the synchronous BRP holds for $(\varphi, Geo_A(F), Geo_A(F))$ are precisely the automorphisms in the subgroup generated by the inner and the letter permutation automorphisms.*

Proof. Let A be a finite alphabet, $F = F_A$ be the free group over A and let H be the subgroup of $\text{Aut}(F)$ generated by the inner and the letter permutation automorphisms. For $u \in F$, we will denote by λ_u the inner automorphism defined by $x \mapsto uxu^{-1}$.

It is obvious that the synchronous BRP holds for automorphisms consisting of permutations of letters with constant 0 and Proposition 7.4.9 shows that it also holds for inner automorphisms, since $Geo_A(F)$ is a biautomatic structure of F. By Lemma 7.2.1, if φ belongs to H, then the synchronous BRP holds for $(\varphi, Geo_A(F), Geo_A(F))$.

Now, we will prove the converse. Let $\varphi \in \text{End}(F)$ be an endomorphism and let $a \in A$. Suppose that the cyclically reduced core of $a\varphi$ has length greater than 1. We have that $|a^n\varphi| - n > n$, and so, letting w be a reduced word representing $a^n\varphi$, it follows that $d_A(w^{[n]}, a^n\varphi)$ is unbounded, thus the synchronous BRP cannot hold for $(\varphi, Geo_A(F), Geo_A(F))$. So, if the synchronous BRP holds for $(\varphi, Geo_A(F), Geo_A(F))$, then the image of a letter x is of the form $w_x y_x w_x^{-1}$ for some $w_x \in F$, $y_x \in A$.

If, for every $x \in A$, w_x is trivial, then the endomorphism is induced by a permutation of the generators. Notice that, in this case, we must have that $y_x \neq y_z$ for all $x, z \in A$ with $x \neq z$, because otherwise we would have that $(x^n z^{-n})\varphi = 1$ and $d((x^n z^{-n})^{[n]}\varphi, 1) = d(x^n\varphi, 1) = d(y_x^n, 1) = n$, which is unbounded and so the synchronous BRP cannot hold for $(\varphi, Geo_A(F), Geo_A(F))$.

So suppose that there is some $x \in A$ such that $w_x \neq \varepsilon$. Let $z \in A$ be such that $|w_z| = \max\{|w_a| \mid a \in A\}$. If there is some $x \in A$ such that $w_x \neq \varepsilon$ and w_x is not a prefix of w_z, then the cyclically reduced core of $(zx)\varphi = w_z y_z w_z^{-1} w_x y_x w_x^{-1}$ has length greater than 3. Thus,

$$|(zx)^n\varphi| = |(w_z y_z w_z^{-1} w_x y_x w_x^{-1})^n| > 3n$$

and so, letting w be a geodesic representing $(zx)^n\varphi$, we have that $d_A((zx)^n\varphi, w^{[2n]})$ is unbounded. So, if the synchronous BRP holds for $(\varphi, Geo_A(F), Geo_A(F))$, then for every $x \in A$, w_x is a prefix of w_z (it might be the case where $w_x = \varepsilon$).

We now proceed by induction on $m = \max\{|w_a| \mid a \in A\}$. If $m = 0$, then we are done. Suppose now that for an endomorphism φ such that $m \leq n$, if the synchronous BRP holds for $(\varphi, Geo_A(F), Geo_A(F))$, then φ belongs to H. Take φ such that $\max\{|w_a| \mid a \in A\} = n+1$, fix $z \in A$ such that $|w_z| = n+1$ and let b be the first letter in w_z.

Suppose that there is some $x \in A$ such that $w_x = \varepsilon$. In this case, if $y_x \neq b^{-1}, b$, then $(zx)\varphi = w_z y_z w_z^{-1} y_x$ is cyclically irreducible and has length greater than 3, thus the same argument as above shows that the synchronous BRP cannot hold for φ. This means that every $x \in A$ having $w_x = \varepsilon$ must be such that $y_x = b^{-1}$ or $y_x = b$. Obviously, we have that $\varphi = (\varphi\lambda_b^{-1})\lambda_b$. The image of a letter x through $\varphi\lambda_b^{-1}$ is equal to $v_x y_x v_x^{-1}$, where v_x is just w_x without the first letter (might be empty), if $|w_x| \neq \varepsilon$, and it is equal to b or b^{-1} if $x\varphi = b$ or $x\varphi = b^{-1}$, respectively. In view of Lemma 7.2.1 and Proposition 7.4.9, the synchronous BRP holds for φ if and only if it holds for $\varphi\lambda_b^{-1}$. Applying the induction hypothesis to $\varphi\lambda_b^{-1}$, we get that if the synchronous BRP holds for $\varphi\lambda_b^{-1}$, then $\varphi\lambda_b^{-1} \in H$ and so does φ. $\square$

In [46], Gersten and Short give an example of an automorphism of a biautomatic group whose fixed subgroup is not finitely generated: letting $G = F_2 \times \mathbb{Z}$, the automorphism φ of G given by $(a, 0) \mapsto (a, 0)$, $b \mapsto (b, 1)$ and $(0, 1) \mapsto (0, 1)$ has fixed subgroup $N \times \mathbb{Z}$, where N is the normal closure of a in F_2 . Thus the fixed subgroup of φ is not finitely generated. In Theorem 7.3.8, it is proved that, given an automorphism φ, there are structures L_1 and L_2 such that the synchronous BRP holds for (φ, L_1, L_2). However, L_1 and L_2 might not be synchronous equivalent, as this example shows.

7.5 Classification of endomorphisms

In the previous chapter, we considered the properties of having quasiconvex image, satisfying the BRP, being uniformly continuous for a visual metric and being virtually injective and obtained implications between them and gave counterexamples for the ones that do not hold for nontrivial endomorphisms of hyperbolic groups. As seen above, in hyperbolic groups, quasiconvexity and the BRP are independent of the automatic structures. When the synchronous BRP is considered or the class of groups is expanded to the whole class of automatic groups, this is not the case. When we simply say that the (synchronous) BRP holds for an endomorphism, we mean that it holds for some pair of structures L_1 and L_2 and having quasiconvex image will mean that there is some automatic structure L such that the image of the endomorphism is L-quasiconvex. We will consider in detail the cases of automatic groups.

It is proved in Theorem 7.3.8 that the synchronous BRP holds for a virtually injective endomorphism with quasiconvex image. The facts that the synchronous BRP implies the BRP and that the BRP implies quasiconvex image are obvious. We know from the hyperbolic case that every region except the region between the BRP and the synchronous BRP is nonempty. We ignore if this new region is empty or not for automatic groups. So we have the following figure:

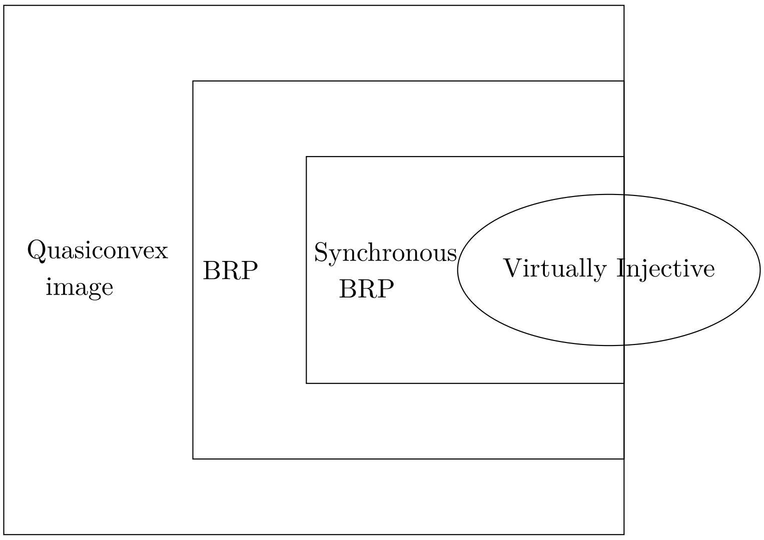

Figure 7.1 Nontrivial endomorphisms of automatic groups

Chapter 8

Direct products of free groups

Now, we will study the dynamics of endomorphisms of direct products of free groups of the form $\mathbb{Z}^m \times F_n$ and $F_n \times F_m$.

Free-abelian times free groups are of the form $\mathbb{Z}^m \times F_n$ and we consider them endowed with the product metric d given by taking the prefix metric d' in each (free) component, i.e., for $a, b \in \mathbb{Z}^m$ and $u, v \in F_n$,

$$d((a,u),(b,v)) = \max\{d'(a_1,b_1),\ldots,d'(a_m,b_m),d'(u,v)\},$$

where a_i and b_i denote the i-th component of a and b, respectively. This metric is an ultrametric and $\widehat{\mathbb{Z}^m \times F_n}$ is homeomorphic to $\widehat{\mathbb{Z}^m} \times \widehat{F_n}$ by uniqueness of the completion ([104, Theorem 24.4]).

When seen as a CAT(0) cube complex, or alternatively, as a median algebra, this coincides with the *Roller compactification* (see [15, 29, 43, 44, 91]). Indeed, the Roller boundary and the Gromov boundary coincide in the free group and the behavior of the Roller compactification when taking direct products is the same as the one of the completion of metric spaces, i.e., denoting by $\bar{X}$ the Roller compactification of X, we have that

$$\bar{X} = \bigcup_{i=1}^{m} \bar{X}_1 \times \ldots \times \bar{X}_m.$$

Obviously, when we consider *free times free* groups, $F_n \times F_m$, endowed with the product metric given by taking the prefix metric in each component and take the completion, we also obtain the Roller compactification of $F_n \times F_m$.

8.1 Free-abelian times free groups

In this section we will deal with free-abelian times free groups, so groups of the form $\mathbb{Z}^m \times F_n$. Each direct factor is a free group: there are m free groups of rank 1 and one of rank n. A very important tool for us is the classification of endomorphisms of $\mathbb{Z}^m \times F_n$ obtained in [36]: for $G = \mathbb{Z}^m \times F_n$, with $n \neq 1$, all endomorphisms of G are of one of the following forms:

(I) $\Psi_{\Phi,Q,P} = (a,u) \mapsto (aQ + \mathbf{u}P, u\Phi)$, where $\Phi \in \text{End}(F_n)$, $Q \in \mathcal{M}_m(\mathbb{Z})$, and $P \in \mathcal{M}_{n\times m}(\mathbb{Z})$.

(II) $\Psi_{z,\ell,h,Q,P} = (a,u) \mapsto (aQ + \mathbf{u}P, z^{a\ell^T + \mathbf{u}h^T})$, where $1 \neq z \in F_n$ is not a proper power, $Q \in \mathcal{M}_m(\mathbb{Z})$, $P \in \mathcal{M}_{n\times m}(\mathbb{Z})$, $\mathbf{0} \neq \ell \in \mathbb{Z}^m$, and $h \in \mathbb{Z}^n$,

where $\mathbf{u} \in \mathbb{Z}^n$ denotes the abelianization of the word $u \in F_n$.

We will refer to endomorphisms of the form (I) (resp. (II)) as *type I (resp. type II)* endomorphisms.

8.1.1 Uniform continuity of endomorphisms

We will focus on the dynamical study of the continuous extension of an endomorphism to the completion, when it exists, i.e., when the endomorphism is uniformly continuous with respect to the metric d. Hence, it makes sense to start by obtaining conditions for an endomorphism of a free-abelian group to be uniformly continuous. Whenever the metric is not mentioned, we assume that it is the product metric given by taking the prefix metric in each direct factor. We present a proof of the following trivial lemma for sake of completeness.

Lemma 8.1.1. *Consider groups G_i endowed with metrics d_i, for $i = 1,2,3,4$. Let $\varphi_1 : G_1 \to G_3$ and $\varphi_2 : G_2 \to G_4$ be homomorphisms of groups. The homomorphism $\varphi : G_1 \times G_2 \to G_3 \times G_4$ given by $(x,y)\varphi = (x\varphi_1, y\varphi_2)$ is uniformly continuous with respect to the product metrics d and d' if and only if φ_1 is uniformly continuous with respect to d_1 and d_3 and φ_2 is uniformly continuous with respect to d_2 and d_4.*

Proof. Consider the homomorphism $\varphi : G_1 \times G_2 \to G_3 \times G_4$ given by $(x,y)\varphi = (x\varphi_1, y\varphi_2)$ and suppose that it is uniformly continuous with respect to the product metrics d and d'. Let $\varepsilon > 0$ and take δ such that for every $(x_1,x_2),(y_1,y_2) \in G_1 \times G_2$ such that $d((x_1,x_2),(y_1,y_2)) < \delta$, we have that $d'((x_1\varphi_1, x_2\varphi_2),(y_1\varphi_1, y_2\varphi_2)) < \varepsilon$. We know that for every $x_1, y_1 \in G_1$ such that $d_1(x_1,y_1) < \delta$ we have $d_3(x_1\varphi_1, y_1\varphi_1) < \varepsilon$, since $d((x_1,1),(y_1,1)) = d_1(x_1,y_1) < \delta$ and so

$$d_3(x_1\varphi_1, y_1\varphi_1) = d'((x_1\varphi_1, 1\varphi_2),(y_1\varphi_1, 1\varphi_2)) < \varepsilon.$$

The case of φ_2 is analogous.

Conversely, if both φ_1, φ_2 are uniformly continuous, then taking $\varepsilon > 0$, there are δ_i such that for every $x, y \in G_i$ such that $d_i(x,y) < \delta_i$, we have $d_{i+2}(x\varphi_i, y\varphi_i) < \varepsilon$, for $i = 1,2$. Taking $\delta = \min \delta_i$, we know that for every $(x_1,y_1),(x_2,y_2) \in G_1 \times G_2$ such that $d((x_1,x_2),(y_1,y_2)) < \delta$, then $d_1(x_1,y_1) < \delta \leq \delta_1$ and $d_2(x_2,y_2) < \delta \leq \delta_2$, thus $d_{i+2}(x\varphi_i, y\varphi_i) < \varepsilon$ and

$$d'((x_1,x_2)\varphi,(y_1,y_2)\varphi) = d'((x_1\varphi_1, x_2\varphi_2),(y_1\varphi_1, y_2\varphi_2)) < \varepsilon.$$

□

Proposition 8.1.2. *Let $u \in \mathbb{Z}^m$ and $\varphi : \mathbb{Z}^m \to \mathbb{Z}$ be a homomorphism given by $v \mapsto vu^T$. Then φ is uniformly continuous if and only if u has at most one nonzero entry.*

Proof. If u=0, then φ is uniformly continuous. It is clear that if u has a single nonzero entry, then φ is uniformly continuous. Indeed, take $u \in \mathbb{Z}^m$ such that $u_k = \lambda \neq 0$ for some $k \in [m]$ and $u_j = 0$ for all $j \in [m] \setminus \{k\}$. Take $\varepsilon > 0$. Set $\delta = \varepsilon$ and take $a, b \in \mathbb{Z}^m$ such that $d(a, b) < \delta$. Notice that $au^T = \lambda a_k$ and $bu^T = \lambda b_k$. If $a_k = b_k$, then $d(au^T, bu^T) = 0 < \varepsilon$. If not, since $d(a, b) < \delta$ then $d(a_i, b_i) < \delta$ for all $i \in [m]$. In particular $d(a_k, b_k) < \delta$. This means that $|a_k \wedge b_k| > \log_2(\frac{1}{\delta})$, i.e. $a_k b_k > 0$ and $|a_k|, |b_k| > \log_2(\frac{1}{\delta})$. But then, $au^T bu^T = \lambda^2 a_k b_k > 0$ and $|\lambda a_k| = |\lambda||a_k| \geq |a_k| > \log_2(\frac{1}{\delta})$. Similarly, $|\lambda b_k| > \log_2(\frac{1}{\delta})$. This means that $d(au^T, bu^T) \leq d(a, b) < \delta = \varepsilon$

Suppose now that u has at least one positive and one negative entry. Let $u_{i_1}, \ldots, u_{i_r}$ be the nonnegative entries and $u_{j_1}, \ldots, u_{j_s}$ be the negative entries and suppose w.l.o.g. that $\sum\limits_{x=1}^{r} u_{i_x} \geq -\sum\limits_{x=1}^{s} u_{j_x}$. We will show that for every $\delta > 0$, there are $v, w \in \mathbb{Z}^m$ such that $d(v, w) < \delta$ and $d(vu^T, wu^T) = 1$ and so φ is not uniformly continuous. Take $\delta > 0$, v such that

$$v_i = 1 + \left\lceil \log_2 \left(\frac{1}{\delta} \right) \right\rceil, \text{ for every } i \in [m]$$

and w such that

$$w_{i_k} = 1 + \left\lceil \log_2 \left(\frac{1}{\delta} \right) \right\rceil, \text{ for all } k \in [r]$$

and

$$w_{j_k} = \left(\sum_{x=1}^{r} u_{i_x} \right) \left(1 + \left\lceil \log_2 \left(\frac{1}{\delta} \right) \right\rceil \right), \text{ for all } k \in [s].$$

Then $d(v, w) < \delta$ since, for every $i \in [m]$, $v_i w_i > 0$ and $|v_i|, |w_i| > \log_2(\frac{1}{\delta})$ (notice that $\sum\limits_{x=1}^{r} u_{i_x} \geq 1$). Also, $vu^T = \sum v_i u_i \geq 0$, since we are assuming that $\sum\limits_{x=1}^{r} u_{i_x} \geq -\sum\limits_{x=1}^{s} v_{j_x}$. We have that

$$wu^T = \left(\sum_{x=1}^{r} u_{i_x} \right) \left(1 + \left\lceil \log_2 \left(\frac{1}{\delta} \right) \right\rceil \right) \left(1 + \sum_{x=1}^{s} v_{j_x} \right) \leq 0.$$

Thus, $d(vu^T, wu^T) = 1$.

Now, suppose that $u \in (\mathbb{Z}_0^+)^m$ has at least two nonzero entries (the nonpositive case is analogous). Let u_k be a minimal nonzero entry. As above, we will show that for every $\delta > 0$, there are $v, w \in \mathbb{Z}^m$ such that $d(v, w) < \delta$ and $d(vu^T, wu^T) = 1$ and so φ is not uniformly continuous. Take $\delta > 0$, v such that

$$v_i = 1 + \left\lceil \log_2 \left(\frac{1}{\delta} \right) \right\rceil \text{ for every } i \in [m] \setminus \{k\} \quad \text{and} \quad v_k = -1 - \left\lceil \log_2 \left(\frac{1}{\delta} \right) \right\rceil$$

and w such that

$$w_i = 1 + \left\lceil \log_2 \left(\frac{1}{\delta} \right) \right\rceil \text{ for every } i \in [m] \setminus \{k\}$$

and

$$w_k = -\left(\sum_{i\in[m]} u_i\right)\left(1+\left\lceil \log_2\left(\frac{1}{\delta}\right)\right\rceil\right).$$

Now, $w_i v_i > 0$ and $|w_i|, |v_i| > \log_2(\frac{1}{\delta})$ for every $i \in [m]$, so $d(v,w) < \delta$. Also,

$$vu^T = \left(1+\left\lceil \log_2\left(\frac{1}{\delta}\right)\right\rceil\right)\left(\sum_{i\neq k} u_i - u_k\right) \geq 0$$

by minimality of u_k. We have that

$$wu^T = \left(1+\left\lceil \log_2\left(\frac{1}{\delta}\right)\right\rceil\right)\left(\sum_{i\neq k} u_i - u_k \sum_{i\in[m]} u_i\right) \leq 0.$$

Thus, $d(vu^T, wu^T) = 1$. $\square$

Corollary 8.1.3. *Let $Q \in M_m(\mathbb{Z})$ and $\varphi \in \mathrm{End}(\mathbb{Z}^m)$ to be given by $u \mapsto uQ$. Then φ is uniformly continuous if and only if every column Q_i of Q has at most one nonzero entry.*

Proof. Consider the homomorphisms $\varphi_i : \mathbb{Z}^m \to \mathbb{Z}$ defined by $u \mapsto uQ_i$. Then

$$\varphi(u) = (\varphi_1(u), \cdots, \varphi_m(u)).$$

$\square$

We are now capable of obtaining conditions of uniform continuity for endomorphisms of type I.

Proposition 8.1.4. *Let $G = \mathbb{Z}^m \times F_n$, with $n > 1$ and consider an endomorphism φ of type I, mapping (a,u) to $(aQ + \mathbf{u}P, u\Phi)$. Denote by ψ the endomorphism of $\mathbb{Z}^m$ defined by $a \mapsto aQ$. Then the following conditions are equivalent:*

1. *φ is uniformly continuous.*

2. *$P = 0$, ψ is uniformly continuous and Φ is either constant or injective.*

Proof. 1 $\implies$ 2. Consider the alphabet of F_n to be $\{x_1, \ldots, x_n\}$. Suppose $P \neq 0$ and pick entries $p_{rs} \neq 0$ and p_{ts} with $r \neq t$. We will prove that φ is not uniformly continuous, by showing that $\forall \delta > 0$ there exists $X, Y \in G$ such that $d(X,Y) < \delta$ and $d(X\varphi, Y\varphi) = 1$. We may assume $\delta \leq 1$, so pick such δ and, as usual, set $q = 1 + \lceil \log_2(\frac{1}{\delta}) \rceil$. Take $\beta \in \mathbb{Z}$ to be such that $sgn(p_{ts}q) \neq sgn(p_{ts}q + \beta p_{rs})$ (if $p_{ts} = 0$, put $\beta = 1$). Let $X = (0, x_t^q)$ and $Y = (0, x_t^q x_r^\beta)$. To simplify notation, write u and v for the free parts of X and Y, respectively, so $u = x_t^q$ and $v = x_t^q x_r^\beta$.

Since the free abelian parts coincide, $d(X,Y) = d(u,v) = 2^{-q} < \delta$.

We have that $d((\mathbf{u}P, u\Phi), (\mathbf{v}P, v\Phi)) = \max\{d(u\Phi, v\Phi), d(\mathbf{u}P, \mathbf{v}P)\} \geq d(\mathbf{u}P, \mathbf{v}P)$. But

$$\mathbf{u}P = \begin{bmatrix} 0 & \cdots & q & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & & \\ p_{n1} & \cdots & p_{nm} \end{bmatrix} = \begin{bmatrix} p_{ti}q \end{bmatrix}_{i\in[m]}$$

and

$$\mathbf{v}P = \begin{bmatrix} 0 & \cdots & q & \cdots & \beta & \cdots & 0 \end{bmatrix} \begin{bmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & & \\ p_{n1} & \cdots & p_{nm} \end{bmatrix} = \begin{bmatrix} p_{ti}q + \beta p_{ri} \end{bmatrix}_{i\in[m]}$$

thus, $d(\mathbf{u}P, \mathbf{v}P) = \max\{d((\mathbf{u}P)_i, (\mathbf{v}P)_i)\} \geq d((\mathbf{u}P)_s, (\mathbf{v}P)_s) = d(p_{ts}q, p_{ts}q + \beta p_{rs}) = 1$, by definition of β.

The remaining conditions follow by Lemma 8.1.1.

$2 \implies 1$. This implication is obvious by Lemma 8.1.1 since both $a \mapsto aQ$ and $u \mapsto u\Phi$ are uniformly continuous. $\square$

Now we deal with the type II endomorphisms. Recall that a reduced word $z = z_1 \cdots z_n$, with $z_i \in A \cup A^{-1}$, is said to be cyclically reduced if $z_1 \neq z_n^{-1}$ and that every word admits a decomposition of the form $z = w\tilde{z}w^{-1}$, where $\tilde{z}$ is cyclically reduced.

Proposition 8.1.5. *Let $G = \mathbb{Z}^m \times F_n$, with $n > 1$ and consider an endomorphism φ of type II, mapping (a, u) to $(aQ + \mathbf{u}P, z^{a\ell^T + \mathbf{u}h^T})$. Denote by ψ_1 the endomorphism of $\mathbb{Z}^m$ defined by $a \mapsto aQ$ and $\psi_2 : \mathbb{Z}^m \to \mathbb{Z}$ the homomorphism defined by $a \mapsto a\ell^T$. Then the following conditions are equivalent:*

1. *φ is uniformly continuous.*
2. *$P = 0$, $h = 0$ and both ψ_1 and ψ_2 are uniformly continuous.*

Proof. Write $z = w\tilde{z}w^{-1}$, where $\tilde{z}$ is the cyclically reduced core of z.

$1 \implies 2$: The proof that $P = 0$ is the same as in the previous proposition.

Now, we will prove that if $h \neq 0$, then for all $\delta > 0$, there are $X, Y \in G$ such that $d(X, Y) < \delta$ and $d(X\varphi, Y\varphi) \geq 2^{-|w|}$. Suppose then that $h \neq 0$ and pick entries $h_k \neq 0$ and h_t, with $t \neq k$ and some $\delta > 0$. Set $q = 1 + \lceil \log_2(\frac{1}{\delta}) \rceil$ and take $X = (0, x_t^q)$ and $Y = (0, x_t^q x_k^\alpha)$ for some $\alpha \in \mathbb{Z}$ such that $sgn(h_t q + \alpha h_k) \neq sgn(h_t q)$ (if $h_t = 0$ put $\alpha = 1$). Then $d(X, Y) < \delta$ and

$$1 \geq d(X\varphi, Y\varphi) \geq d(z^{h_t q}, z^{\alpha h_k + h_t q}) = 2^{-|w|}.$$

The proof that ψ_1 is uniformly continuous is analogous the the one in the previous proposition.

Now, suppose ψ_2 is not uniformly continuous. There exists $\varepsilon > 0$ such that for every $\delta > 0$, there are $a, b \in \mathbb{Z}^m$ such that $d(a,b) < \delta$ and

$$d\left(\sum_{i\in[m]} a_i\ell_i, \sum_{i\in[m]} b_i\ell_i\right) \geq \varepsilon. \tag{8.1}$$

We now show that for every $\delta > 0$, there are $X, Y \in G$ such that $d(X,Y) < \delta$ and $d(X\varphi, Y\varphi) \geq 2^{-|w|-|\tilde{z}|\lceil \log_2(\frac{1}{\varepsilon})\rceil}$. Notice that $\tilde{z}$ and w^{-1} don't share a prefix. Take $\delta > 0$ and take $a, b \in \mathbb{Z}^m$ such that $d(a,b) < \delta$ satisfying (8.1). Now, consider $X = (a,1)$ and $Y = (b,1)$. Clearly $d(X,Y) = d(a,b) < \delta$ and

$$d(X\varphi, Y\varphi) = d\left(\left(aQ, z^{a\ell^T}\right), \left(bQ, z^{b\ell^T}\right)\right) \geq d\left(z^{a\ell^T}, z^{b\ell^T}\right)$$

We know that (8.1) holds, so, either

$$a\ell^T b\ell^T = \sum_{i\in[m]} a_i\ell_i \sum_{i\in[m]} b_i\ell_i \leq 0$$

and in that case $d\left(z^{a\ell^T}, z^{b\ell^T}\right) \geq 2^{-|w|} \geq 2^{-|w|-|\tilde{z}|\lceil \log_2(\frac{1}{\varepsilon})\rceil}$, or

$$\sum_{i\in[m]} a_i\ell_i \sum_{i\in[m]} b_i\ell_i > 0 \text{ and } 2^{-\min\left\{\left|a\ell^T\right|,\left|b\ell^T\right|\right\}} \geq \varepsilon,$$

which means that

$$\min\left\{\left|a\ell^T\right|, \left|b\ell^T\right|\right\} \leq \log_2\left(\frac{1}{\varepsilon}\right).$$

In this case, we have that

$$\begin{aligned} d(z^{a\ell^T}, z^{b\ell^T}) &= 2^{-|z^{a\ell^T} \wedge z^{b\ell^T}|} = 2^{-|w\tilde{z}^{a\ell^T}w^{-1} \wedge w\tilde{z}^{b\ell^T}w^{-1}|} \\ &= 2^{-|w|-|\tilde{z}|\min\{|a\ell^T|,|b\ell^T|\}} \geq 2^{-|w|-|\tilde{z}|\lceil \log_2(\frac{1}{\varepsilon})\rceil}. \end{aligned}$$

$2 \implies 1$: Straightforward. □

So, type I uniformly continuous endomorphisms are of the form $(a,u) \mapsto (aQ, u\phi)$ where $Q \in \mathcal{M}_m(\mathbb{Z})$ has at most one nonzero entry in each column and $\phi \in \text{End}(F_n)$ is either constant or injective. Type II endomorphisms are uniformly continuous if they map (a,u) to $(aQ, z^{\lambda a_k})$ where $Q \in \mathcal{M}_m(\mathbb{Z})$ has at most one nonzero entry in each column, $k \in [m]$, $z \in F_n \setminus \{1\}$ is not a proper power and $\lambda \in \mathbb{Z} \setminus \{0\}$. Notice that $\lambda \neq 0$, since by definition of a type II endomorphism we have that $\ell \neq 0$.

Remark 8.1.6. *In [36], the authors prove that an endomorphism $\varphi \in \text{End}(\mathbb{Z}^m \times F_n)$ is an automorphism if it is of type I and of the form $(a,u) \mapsto (aQ + \mathbf{u}P, u\phi)$, with $\phi \in \text{Aut}(F_n)$ and $Q \in GL_m(\mathbb{Z})$. In the case where φ is uniformly continuous, every nonzero entry of Q is either*

1 or -1 and $Q = AD$, where D is diagonal and A is a permutation matrix. There are $2^m m!$ such matrices, which we call uniform. *So, a uniformly continuous automorphism of $\mathbb{Z}^m \times F_n$ is defined as $(a, u) \mapsto (aQ, u\phi)$, where Q is a uniform matrix and $\phi \in \mathrm{Aut}(F_n)$.*

8.1.2 Coarse-median preserving endomorphisms

In $\mathbb{Z}^m$, the ℓ_1 metric is defined by

$$d_{\ell_1}(a, b) = \sum_{i=1}^{m} |a_i - b_i|.$$

Consider $\mathbb{Z}^m$ endowed with the ℓ_1 metric. Given three points $a, b, c \in \mathbb{Z}^m$, consider $\mu_1(a, b, c)$ to be the point having in the i-th component the (numerical) median of $\{a_i, b_i, c_i\}$. Then μ is a median operator for $(\mathbb{Z}^m, d_{\ell_1})$. Recall that, in a hyperbolic group (and so, in particular, in a free group), given three points, the operator μ_2 that associates the three points to the K-center of a geodesic triangle they define is a coarse median. We will consider the product coarse median operator μ in $\mathbb{Z}^m \times F_n$ obtained by considering the factors $(\mathbb{Z}^m, \mu_1)$ and (F_n, μ_2).

Remark 8.1.7. *From [44, Example 2.26] we have that, in $\mathbb{Z}^m$, when taking μ to be the median operator associated with the ℓ_1 metric, the coarse median preserving automorphisms are the ones given by uniform matrices, which correspond to the uniformly continuous automorphisms of $\mathbb{Z}^m$ when the product metric is taken with the prefix metric in each component (recall Remark 8.1.6). In the case of hyperbolic groups, with the coarse median defined above, every automorphism is coarse-median preserving. In fact, Theorem 6.2.7 states that, given a hyperbolic group G and an endomorphism $\varphi \in \mathrm{End}(G)$, then the bounded reduction property (BRP) holds for φ if and only if φ is coarse-median preserving.*

We will now see that, in some sense, coarse-median preserving endomorphisms coincide with the uniformly continuous ones for $\mathbb{Z}^m \times F_n$. We start with a fairly obvious technical lemma. Observe that, given two groups $G_1 = \langle A \rangle$ and $G_2 = \langle B \rangle$ endowed with geodesic metrics d_1 and d_2, respectively, then the product metric d is a geodesic metric for the generators $(A \cup \{1\}) \times (B \cup \{1\})$.

Lemma 8.1.8. *Let G_i be groups endowed with geodesic metrics d_i and coarse medians μ_i, and consider endomorphisms $\phi_i \in \mathrm{End}(G_i)$, for $i = 1, \ldots, k$. The endomorphism $\varphi \in \mathrm{End}(G_1 \times \cdots \times G_k)$ defined by $(x_1, \ldots x_k) \mapsto (x_1\phi_1, \ldots, x_k\phi_k)$ is coarse-median preserving with respect to the product coarse median operator and the product metric d if and only if ϕ_i is coarse-median preserving for μ_i and d_i for every $i \in [k]$.*

Proof. Suppose that, for every $i \in [k]$, the endomorphism ϕ_i is coarse-median preserving and take $C = \max\{C_i \mid i \in [k]\}$, where C_i is the constant given by this property for ϕ_i. Then, we have that, for every $i \in [k]$,

$$d_i(\mu_i(x_i, y_i, z_i)\phi_i, \mu_i(x_i\phi_i, y_i\phi_i, z_i\phi_i)) \leq C_i \leq C, \tag{8.2}$$

for all $x_i, y_i, z_i \in G_i$. Thus, for all $(x_1, \dots, x_k), (y_1, \dots, y_k), (z_1, \dots, z_k) \in G_1 \times \cdots \times G_k$, we have that

$$\begin{aligned}&\mu((x_1, \dots, x_k)\varphi, (y_1, \dots, y_k)\varphi, (z_1, \dots, z_k)\varphi)\\=&(\mu_1(x_1\phi_1, y_1\phi_1, z_1\phi_1), \dots, \mu_k(x_k\phi_k, y_k\phi_k, z_k\phi_k))\end{aligned}$$

and

$$(\mu((x_1, \dots x_k), (y_1, \dots, y_k), (z_1, \dots, z_k)))\,\varphi = ((\mu_1(x_1, y_1, z_1))\phi_1, \dots, (\mu_k(x_k, y_k, z_k))\phi_k).$$

From (8.2), it follows that

$$\mu((x_1, \dots, x_k)\varphi, (y_1, \dots, y_k)\varphi, (z_1, \dots, z_k)\varphi)$$

is C-close to

$$\mu((x_1, \dots x_k), (y_1, \dots, y_k), (z_1, \dots, z_k)))\varphi$$

with respesct to the product metric d, and so φ is coarse-median preserving.

The converse is proved similarly. $\square$

Let $\varphi \in \mathrm{End}(\mathbb{Z}^m)$ defined by $a \mapsto aQ$ be a uniformly continuous endomorphism. From Corollary 8.1.3 we know that Q has at most one nonzero entry in each column. Given a column Q_j, if $Q_j \neq 0$, we call λ_j to its nonzero entry and denote by α_j the corresponding row, so that $q_{ij} = \lambda_j$ if $i = \alpha_j$ and $q_{ij} = 0$ otherwise. If $Q_j = 0$ we put $\lambda_j = 0$ and $\alpha_j = 1$. Then, we have that $[a_i]_{i\in[m]}$ is mapped to $[\lambda_i a_{\alpha_i}]_{i\in[m]}$ and $a \in \mathrm{Fix}(\varphi)$ if $a_i = \lambda_i a_{\alpha_i}$ for every $i \in [m]$. This notation will be kept throughout this section.

Lemma 8.1.9. *An endomorphism $\varphi \in \mathrm{End}(\mathbb{Z}^m)$ is uniformly continuous with respect to the product metric d obtained by taking the prefix metric in each direct factor if and only if it is coarse-median preserving for the median operator μ induced by the metric ℓ_1 in $\mathbb{Z}^m$.*

Proof. Let $\varphi \in \mathrm{End}(\mathbb{Z}^m)$ be given by $Q \in \mathcal{M}_m(\mathbb{Z})$ and suppose that it is coarse-median preserving. We proceed in a similar way to what we did in the proof of Proposition 8.1.2. If φ is not uniformly continuous with respect to d, then there is some column Q_j and nonzero entries q_{rj} and q_{sj} with $s > r$. If they are both positive, suppose w.l.o.g. that $q_{sj} > q_{rj}$. For every $n \in \mathbb{N}$, let $x^{(n)} \in \mathbb{Z}^m$ such that $x_r^{(n)} = 1$, $x_s^{(n)} = n$ and all the other entries are 0; $y^{(n)} \in \mathbb{Z}^m$ such that $y_r^{(n)} = 1$ and all the other entries are 0; and $z^{(n)} \in \mathbb{Z}^m$ such that $z_r^{(n)} = 1 + 2nq_{sj}$, $z_s^{(n)} = -1$ and all the other entries are 0. Then $\mu(x^{(n)}, y^{(n)}, z^{(n)})$ has 1 in the r-th entry and all the other entries are 0, so $[\mu(x^{(n)}, y^{(n)}, z^{(n)})\varphi]_j = q_{r_j}$. But

$$[\mu(x^{(n)}\varphi, y^{(n)}\varphi, z^{(n)}\varphi)]_j = \mu(q_{rj} + nq_{sj}, q_{rj}, q_{rj} + (2nq_{rj} - 1)q_{sj}) = q_{rj} + nq_{sj}$$

and so $d_{\ell^1}(\mu(x^{(n)}\varphi, y^{(n)}\varphi, z^{(n)}\varphi), \mu(x^{(n)}, y^{(n)}, z^{(n)})\varphi) \geq nq_{sj}$, and that contradicts the fact that φ is coarse-median preserving. If both q_{rj} and q_{sj} are negative, we can reach a contradiction in the same way. If, suppose $q_{rj} > 0$ and $q_{sj} < 0$, putting, for all $n \in \mathbb{N}$, $x^{(n)} \in \mathbb{Z}^m$ such that $x_r^{(n)} = -nq_{sj}$, $x_s^{(n)} = nq_{rj}$ and all the other entries are 0; $y^{(n)} \in \mathbb{Z}^m$ such that $y_r^{(n)} = -2nq_{sj}$, $y_s^{(n)} = -nq_{rj}$ and all the other entries are 0; and $z^{(n)} \in \mathbb{Z}^m$ such that $z_r^{(n)} = -3nq_{sj}$, $z_s^{(n)} = 4nq_{rj}$ and all the other entries are 0, we get that $\mu(x^{(n)}, y^{(n)}, z^{(n)})$ has $-2nq_{sj}$ in the r-th entry, nq_{rj} in the s-th entry and all the other entries are 0, so $[\mu(x^{(n)}, y^{(n)}, z^{(n)})\varphi]_j = -nq_{rj}q_{sj}$. But

$$[\mu(x^{(n)}\varphi, y^{(n)}\varphi, z^{(n)}\varphi)]_j = \mu(0, -3nq_{rj}q_{sj}, nq_{rj}q_{sj}) = 0$$

and so $d_{\ell^1}(\mu(x^{(n)}\varphi, y^{(n)}\varphi, z^{(n)}\varphi), \mu(x^{(n)}, y^{(n)}, z^{(n)})\varphi) \geq nq_{rj}q_{sj}$, and that contradicts the fact that φ is coarse-median preserving.

Hence, we have that φ is uniformly continuous with respect to d.

Conversely, suppose that φ is uniformly continuous with respect to d and recall the notation introduced above. Consider three points $a, b, c \in \mathbb{Z}^m$. Then

$$\mu(aQ, bQ, cQ) = [\mu(\lambda_i a_{\alpha_i}, \lambda_i b_{\alpha_i}, \lambda_i c_{\alpha_i})]_{i\in[m]} = [\lambda_i \mu(a_{\alpha_i}, b_{\alpha_i}, c_{\alpha_i})]_{i\in[m]} = \mu(a, b, c)Q.$$

Thus φ is coarse-median preserving. □

We are now able to prove the main result of this subsection.

Theorem 8.1.10. *An endomorphism $\varphi \in \mathrm{End}(\mathbb{Z}^m \times F_n)$ is uniformly continuous with respect to the product metric d obtained by taking the prefix metric in each direct factor if and only if it is coarse-median preserving for the product coarse median μ obtained by taking the median operator μ_1 induced by the metric ℓ_1 in $\mathbb{Z}^m$ and the coarse median operator μ_2 given by hyperbolicity in F_n.*

Proof. An endomorphism of $\mathbb{Z}^m \times F_n$ is of the form $(a, u) \mapsto (aQ + \mathbf{u}P, (a, u)\psi)$, where $Q \in \mathcal{M}_m(\mathbb{Z})$, $P \in \mathcal{M}_{n\times m}(\mathbb{Z})$ and ψ is a homomorphism from $\mathbb{Z}^m \times F_n$ to F_n. We start by proving that if φ is coarse-median preserving for μ, then, $P = 0$. Suppose then that $P \neq 0$ and pick entries $p_{rs} \neq 0$ and p_{ts} with $r \neq t$. Take $\{x_1, \ldots, x_n\}$ to be a basis of F_n and let d_2 be the geodesic metric defined by this set of generators. Now, we have that

$$(\mu((0, x_t), (0, x_t x_r^{C+1}), (0, x_r^{C+1} x_t)))\varphi = (0, x_t)\varphi = ([p_{ti}]_{i\in[m]}, (0, x_t)\psi)$$

and

$$\begin{aligned}
&\mu\left((\mathbf{x_t}P, (0, x_t)\psi), (\mathbf{x_t}\mathbf{x_r}^{C+1}P, (0, x_t x_r^{C+1})\psi), (\mathbf{x_r}^{C+1}\mathbf{x_t}P, (0, x_r^{C+1}x_t)\psi)\right)\\
&= \left(\mathbf{x_t}\mathbf{x_r}^{C+1}P, \mu_2((0, x_t)\psi, (0, x_t x_r^{C+1})\psi, (0, x_r^{C+1}x_t)\psi)\right)\\
&= \left([p_{ti} + p_{ri}(C+1)]_{i\in[m]}, \mu_2((0, x_t)\psi, (0, x_t x_r^{C+1})\psi, (0, x_r^{C+1}x_t)\psi)\right).
\end{aligned}$$

But taking the ℓ_1 metric in $\mathbb{Z}^m$, we have that

$$d_{\ell_1}([p_{ti}]_{i\in[m]}, [p_{ti} + p_{ri}(C+1)]_{i\in[m]}) \geq |(C+1)p_{rs}| > C,$$

which contradicts the fact that φ is coarse-median preserving with constant C. So, P must be zero.

If φ is a type I coarse-median preserving endomorphism for μ, then φ is of the form $(a, u) \mapsto (aQ, u\phi)$, where $Q \in \mathcal{M}_m(\mathbb{Z})$ and $\phi \in \mathrm{End}(F_n)$. By Lemma 8.1.8, we get that both the endomorphism θ of $\mathbb{Z}^m$ defined by Q and $\phi \in \mathrm{End}(F_n)$ must be coarse-median preserving. By Lemma 8.1.9, θ is uniformly continuous and by Remark 8.1.7, we have that the BRP holds for ϕ, and so ϕ is uniformly continuous. By Proposition 8.1.4, φ is uniformly continuous for the metric d.

Conversely, if φ is a type I uniformly continuous with respect to the metric d, then, by Proposition 8.1.4, φ is of the form $(a, u) \mapsto (aQ, u\phi)$, where $Q \in \mathcal{M}_m(\mathbb{Z})$ has at most one nonzero entry in each column and $\phi \in \mathrm{End}(F_n)$ is uniformly continuous for the prefix metric. Combining Lemmas 8.1.8 and 8.1.9 with Remark 8.1.7, we get that φ is coarse-median preserving for μ.

Now we deal with the type II endomorphisms. Suppose that φ is a coarse-median preserving type II endomorphism with constant $C \geq 0$. Then, it must be of the form $(a, u) \mapsto (aQ, z^{a\ell^T + \mathbf{u}h^T})$, for some $\mathbf{0} \neq \ell \in \mathbb{Z}^m$ and $h \in \mathbb{Z}^n$. We will prove, proceeding in the same way we did above, that h must be zero and that ℓ must have at most one nonzero entry. Suppose that h is not zero and pick $h_s \neq 0$ and h_t, with $t \neq s$. Then

$$(\mu((0, x_t), (0, x_t x_r^{C+1}), (0, x_r^{C+1} x_t)))\varphi = (0, x_t)\varphi = (0, z^{h_t})$$

and

$$\begin{aligned} &\mu\left((0, z^{h_t}), (0, z^{h_t+(C+1)h_s}), (0, z^{h_t+(C+1)h_s})\right) \\ =&(0, z^{h_t+(C+1)h_s}) \end{aligned}$$

But we have that

$$d_2(z^{h_t}, z^{h_t+(C+1)h_s}) > C,$$

which contradicts the fact that φ is coarse-median preserving with constant C. So, h must be zero. To prove that ℓ must have at most one nonzero entry, we proceed as in the proof of Lemma 8.1.9. For all $n \in \mathbb{N}$, we define $x^{(n)}$, $y^{(n)}$ and $z^{(n)}$ in the same way replacing q_{rj} and q_{sj} by l_r and l_s, respectively. We have that the distance of the free component between $\mu((x^{(n)}, 1), (y^{(n)}, 1), (z^{(n)}, 1))\varphi$ and $\mu((x^{(n)}, 1)\varphi, (y^{(n)}, 1)\varphi, (z^{(n)}, 1)\varphi)$ will be unbounded, contradicting coarse-median preservation of φ.

Conversely, if φ is a uniformly continuous type II endomorphism, then it is of the form $(a, u) \mapsto (aQ, z^{\lambda a_k})$, for some $Q \in \mathcal{M}_m(\mathbb{Z})$ with at most on nonzero entry in each column, $k \in [m]$, $1 \neq z \in F_n$, which is not a proper power and $0 \neq \lambda \in \mathbb{Z}$. Then, taking $(a, u), (b, v), (c, w) \in \mathbb{Z}^m \times F_n$, and letting $\mu_3 : \mathbb{Z}^3 \to \mathbb{Z}$ denote the usual median in $\mathbb{Z}$, we have that

$$\mu((a, u), (b, v), (c, w))\varphi = (\mu_1(a, b, c)Q, z^{\lambda\mu_3(a_k, b_k, c_k)})$$

and
$$\mu((a,u)\varphi,(b,v)\varphi,(c,w)\varphi) = (\mu_1(aQ,bQ,cQ),\mu_2(z^{\lambda a_k},z^{\lambda b_k},z^{\lambda c_k})).$$

Since Q defines a uniformly continuous endomorphism of $\mathbb{Z}^m$, it follows from Lemma 8.1.9 that it is coarse-median preserving, thus $\mu_1(aQ,bQ,cQ)$ and $\mu_1(a,b,c)Q$ are close. Letting $z = w\tilde{z}w^{-1}$, where $\tilde{z}$ is the cyclically reduced core of z, we have that $w\tilde{z}^{\mu_3(\lambda a_k,\lambda b_k,\lambda c_k)} = w\tilde{z}^{\lambda\mu_3(a_k,b_k,c_k)}$ belongs to every edge of the geodesic triangle defined by $z^{\lambda a_k}$, $z^{\lambda b_k}$ and $z^{\lambda c_k}$. Since

$$d_2(z^{\lambda\mu_3(a_k,b_k,c_k)}, w\tilde{z}^{\lambda\mu_3(a_k,b_k,c_k)}) = d_2(w\tilde{z}^{\lambda\mu_3(a_k,b_k,c_k)}w^{-1}, w\tilde{z}^{\lambda\mu_3(a_k,b_k,c_k)}) = |w|,$$

we have that the distance between $z^{\lambda\mu_3(a_k,b_k,c_k)}$ and $\mu_2(z^{\lambda a_k},z^{\lambda b_k},z^{\lambda c_k})$ is bounded. $\square$

8.1.3 Infinite fixed and periodic points

In this subsection, we will study infinite fixed and periodic points of $\hat{\varphi}$, for $\varphi \in \mathrm{End}(\mathbb{Z}^m \times F_n)$ uniformly continuous. Recall that an infinite fixed point is said to be singular if it belongs to the topological closure $(\mathrm{Fix}(\varphi))^c$ of $\mathrm{Fix}(\varphi)$ and regular if it doesn't. We denote by $\mathrm{Sing}(\hat{\varphi})$ (resp. $\mathrm{Reg}(\hat{\varphi})$) the set of all singular (resp. regular) infinite fixed points of $\hat{\varphi}$.

We start by obtaining finiteness conditions on infinite fixed and periodic points, counting the number of $\mathrm{Fix}(\varphi)$-orbits of $\mathrm{Fix}(\hat{\varphi})$ and the number of $\mathrm{Per}(\varphi)$-orbits of $\mathrm{Per}(\hat{\varphi})$, under natural actions of $\mathrm{Fix}(\varphi)$ and $\mathrm{Per}(\varphi)$ on $\mathrm{Fix}(\hat{\varphi})$ and $\mathrm{Per}(\hat{\varphi})$, respectively when φ is a uniformly continuous type II endomorphism. In particular, the number of orbits is finite, which can be seen as some sort of infinite version of [36, Proposition 6.2], which states that a type II endomorphism of a free-abelian times free group has a finitely generated fixed subgroup.

Finally, we classify regular infinite fixed points as attractors or repellers.

Finiteness Conditions on Infinite Fixed Points

Let φ be a type I uniformly continuous endomorphism of $\mathbb{Z}^m \times F_n$, with $n > 1$. Then $\varphi : \mathbb{Z}^m \times F_n \to \mathbb{Z}^m \times F_n$ is given by $(a,u) \mapsto (aQ, u\phi)$, for some $Q \in \mathcal{M}_m(\mathbb{Z})$ such that every column of Q contains at most one nonzero entry and some either constant or injective $\phi \in \mathrm{End}(F_n)$. Consider $\varphi_1 \in \mathrm{End}(\mathbb{Z}^m)$ to be defined as $a \mapsto aQ$. Clearly, $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_1) \times \mathrm{Fix}(\phi)$ and it is finitely generated.

Let $\hat{\varphi} : \widehat{\mathbb{Z}^m} \times \widehat{F_n} \to \widehat{\mathbb{Z}^m} \times \widehat{F_n}$ be its continuous extension to the completion. By uniqueness of the extension, we have that $\hat{\varphi}$ is given by $(a,u) \mapsto (a\widehat{\varphi_1}, u\widehat{\phi})$. Then, we have that $\mathrm{Fix}(\hat{\varphi}) = \mathrm{Fix}(\hat{\varphi}_1) \times \mathrm{Fix}(\hat{\phi})$; $\mathrm{Sing}(\hat{\varphi}) = \mathrm{Sing}(\hat{\varphi}_1) \times \mathrm{Sing}(\hat{\phi})$ and $\mathrm{Reg}(\hat{\varphi}) = \mathrm{Fix}(\hat{\varphi}_1) \times \mathrm{Reg}(\hat{\phi}) \cup \mathrm{Reg}(\widehat{\varphi_1}) \times \mathrm{Fix}(\hat{\phi})$. There is no hope of finding a finiteness condition in this case that holds in general since, if $n = 2$ and ϕ is the identity mapping, then $\mathrm{Sing}(\hat{\phi})$ is uncountable, thus so are both $\mathrm{Reg}(\hat{\varphi})$ and $\mathrm{Sing}(\hat{\varphi})$.

However, we will see that this is not the case when dealing with type II endomorphisms.

We start by studying the free-abelian case. Let $\varphi \in \text{End}(\mathbb{Z}^m)$ be uniformly continuous and let $\hat{\varphi} : \widehat{\mathbb{Z}^m} \to \widehat{\mathbb{Z}^m}$ be its continuous extension to the completion.

Given $a \in \widehat{\mathbb{Z}^m}$, we set $i_1, \ldots, i_r$ to be the indices such that $a_{i_1} = \ldots = a_{i_r} = +\infty$; $j_1, \ldots, j_s$ to be the indices such that $a_{j_1} = \ldots = a_{j_s} = -\infty$ and $k_1, \ldots, k_t$ to be the indices such that $a_{k_1}, \ldots a_{k_t} \notin \{-\infty, +\infty\}$. Define, for every $n \in \mathbb{N}$, $a_n \in \mathbb{Z}^m$ such that

$$a_{n_{i_l}} = n, \text{ for } l \text{ in } [r]; \quad a_{n_{j_l}} = -n, \text{ for } l \text{ in } [s] \quad \text{and} \quad a_{n_{k_l}} = a_{k_l} \text{ for } l \text{ in } [t].$$

We have that, given $\varepsilon > 0$, for every $n > \log_2(\frac{1}{\varepsilon})$, $d(a_n, a) < \varepsilon$, so $(a_n) \to a$. Thus, $a\hat{\varphi} = (\lim a_n)\hat{\varphi} = \lim(a_n\varphi)$. Since $(a_n\varphi)$ is such that $(a_n\varphi)_i = \lambda_i a_{\alpha_i}$ if $a_{\alpha_i} \notin \{+\infty, -\infty\}$, $(a_n\varphi)_i = n\lambda_i$ if $a_{\alpha_i} = +\infty$, and $(a_n\varphi)_i = -n\lambda_i$ if $a_{\alpha_i} = -\infty$, we have that $a\hat{\varphi} = [\lambda_i a_{\alpha_i}]$, assuming that $0 \times \infty = 0$. So, $a \in \text{Fix}(\hat{\varphi})$ if and only if $a_i = \lambda_i a_{\alpha_i}$ for every $i \in [m]$.

Defining the sum of an integer with infinite in the natural way, we have that the subgroup $\text{Fix}(\varphi) \leq \mathbb{Z}^m \times F_n$ acts on $\text{Fix}(\hat{\varphi})$ by left multiplication. Given $a \in \text{Fix}(\varphi)$ and $b \in \text{Fix}(\hat{\varphi})$, then $(a+b)\hat{\varphi} = a\varphi + b\hat{\varphi} = a + b \in \text{Fix}(\hat{\varphi})$. We now count the orbit of this action.

Proposition 8.1.11. *Let $\varphi \in \text{End}(\mathbb{Z}^m)$ be a uniformly continuous endomorphism. Then $\text{Fix}(\hat{\varphi})$ has at most $\sum\limits_{i=0}^{m} 2^i \binom{m}{i}$ $\text{Fix}(\varphi)$-orbits.*

Proof. Let $a \in \text{Fix}(\hat{\varphi})$ and define $r, s, t \in \{0, \ldots, m\}$, and i_l, j_l, k_l as above. We will prove that, for $b \in \text{Fix}(\hat{\varphi})$, we have that $b \in (\text{Fix}\,\varphi)a$ if and only if $b_{i_l} = a_{i_l}$ for every $l \in [r]$, $b_{j_l} = a_{j_l}$, for every $l \in [s]$ and $b_{k_l} \notin \{+\infty, -\infty\}$, for every $l \in [t]$, i.e., if their infinite entries coincide. If that is the case, then every orbit is defined by the position and the signal of their infinite entries. Obviously, for $i \in \{0, \ldots, m\}$, there are $\binom{m}{i}$ choices for i infinite entries and each of them can be $+\infty$ or $-\infty$, hence the 2^i factor.

Start by supposing that $b \in \text{Fix}(\hat{\varphi})$ is such that $b \in (\text{Fix}\,\varphi)a$. Then, there is some $c \in \text{Fix}\,\varphi$ such that $b = c + a$. This means that for every $l \in [r]$, we have that $b_{i_l} = c_{i_l} + a_{i_l} = c_{i_l} + (+\infty) = +\infty$, since $c \in \mathbb{Z}^m$. Similarly, we have that $b_{j_l} = a_{j_l}$, for every $j \in [s]$ and $b_{k_l} \notin \{+\infty, -\infty\}$. It is clear that $|b_{k_l}| < \infty$ for $l \in [t]$.

Now, suppose $b \in \text{Fix}(\hat{\varphi})$ is such that $b_{i_l} = +\infty$, for $l \in [r]$, $b_{j_l} = -\infty$, for $l \in [s]$ and $b_{k_l} \notin \{+\infty, -\infty\}$, for $l \in [t]$. Consider $c \in \mathbb{Z}^m$ defined by $c_{k_l} = b_{k_l} - a_{k_l}$ and all other entries are 0. Clearly $b = c + a$. We only have to check that $c \in \text{Fix}(\varphi)$, i.e., $\lambda_i c_{\alpha_i} = c_i$ for every $i \in [m]$. For i such that $a_i = \pm\infty$, we have that $c_i = 0$ and $a_{\alpha_i} = sgn(\lambda_i) a_i = \pm\infty$, which implies that $c_{\alpha_i} = 0$. If not, $c_i = b_i - a_i = \lambda_i(b_{\alpha_i} - a_{\alpha_i}) = \lambda_i c_{\alpha_i}$ and we are done. □

Now, let φ be a type II uniformly continuous endomorphism of $\mathbb{Z}^m \times F_n$, with $n > 1$. Then $\varphi : \mathbb{Z}^m \times F_n \to \mathbb{Z}^m \times F_n$ is given by $(a, u) \mapsto (aQ, z^{\lambda a_k})$, for some $Q \in \mathcal{M}_m(\mathbb{Z})$ such that every column of Q contains at most one nonzero entry, $0 \neq \lambda \in \mathbb{Z}$ and $k \in [m]$. Consider $\varphi_1 \in \text{End}(\mathbb{Z}^m)$ to be defined as $a \mapsto aQ$ and $\varphi_2 : \mathbb{Z}^m \to F_n$ that maps a to $z^{\lambda a_k}$, which are both uniformly continuous. Observe that $\text{Fix}(\varphi) = \{(a, a\varphi_2) \mid a \in \text{Fix}(\varphi_1)\}$ and it is finitely generated (see [36, Proposition 6.2]).

By uniqueness of extension, we have that $\hat{\varphi} : \widehat{\mathbb{Z}^m} \times \widehat{F_n} \to \widehat{\mathbb{Z}^m} \times \widehat{F_n}$ is defined by $(a, u) \mapsto (a\hat{\varphi}_1, a\hat{\varphi}_2)$, thus $\mathrm{Fix}(\hat{\varphi}) = \{(a, a\hat{\varphi}_2) \mid a \in \mathrm{Fix}(\hat{\varphi}_1)\}$.

Proposition 8.1.12. *Let φ be a type II uniformly continuous endomorphism of $\mathbb{Z}^m \times F_n$, with $n > 1$. Then, Sing($\hat{\varphi}$) $= \{(a, a\hat{\varphi}_2) \mid a \in$ Sing$(\hat{\varphi}_1)\}$. Consequentely, Reg($\hat{\varphi}$) $= \{(a, a\hat{\varphi}_2) \mid a \in$ Reg$(\hat{\varphi}_1)\}$.*

Proof. We start by showing that $\mathrm{Sing}(\hat{\varphi}) \subseteq \{(a, a\hat{\varphi}_2) \mid a \in \mathrm{Sing}(\hat{\varphi}_1)\}$. Take some $(a, a\hat{\varphi}_2) \in (\mathrm{Fix}(\varphi))^c$ with $a \in \mathrm{Fix}(\hat{\varphi}_1)$. Then, for every $\varepsilon > 0$, the open ball of radius ε centered in $(a, a\hat{\varphi}_2)$ contains an element $(b_\varepsilon, b_\varepsilon\varphi_2) \in \mathrm{Fix}(\varphi)$, with $b_\varepsilon \in \mathrm{Fix}(\varphi_1)$. Notice that $d(a, b_\varepsilon) \leq d((a, a\hat{\varphi}_2), (b_\varepsilon, b_\varepsilon\varphi_2)) < \varepsilon$, thus $a \in (\mathrm{Fix}(\varphi_1))^c$.

For the reverse inclusion, take some $a \in \mathrm{Sing}(\hat{\varphi}_1)$. As above, we know that for every $\varepsilon > 0$, there is some $b_\varepsilon \in B(a; \varepsilon) \cap \mathrm{Fix}(\varphi_1)$. Notice that, since $\hat{\varphi}_2$ is uniformly continuous, for every $\varepsilon > 0$, there is some δ_ε such that, for all $a, b \in \widehat{\mathbb{Z}^m}$ such that $d(a, b) < \delta_\varepsilon$, we have that $d(a\hat{\varphi}_2, b\hat{\varphi}_2) < \varepsilon$. We want to prove that $(a, a\hat{\varphi}_2) \in (\mathrm{Fix}(\varphi))^c$, by showing that, for every $\varepsilon > 0$, the ball centered in $(a, a\hat{\varphi}_2)$ contains a fixed point of φ. So, let $\varepsilon > 0$ and consider $\delta = \min\{\delta_\varepsilon, \varepsilon\}$. We have that $(b_\delta, b_\delta\hat{\varphi}_2) \in B((a, a\hat{\varphi}_2); \varepsilon)$ since, by definition of b_δ, we have that $d(a, b_\delta) < \delta \leq \varepsilon$ and also, $d(a, b_\delta) < \delta_\varepsilon$ means that $d(a\hat{\varphi}_2, b_\delta\hat{\varphi}_2) < \varepsilon$. $\square$

Corollary 8.1.13. *Let $\varphi \in \mathrm{End}(\mathbb{Z}^m \times F_n)$ be a uniformly continuous type II endomorphism. Then $\mathrm{Fix}(\hat{\varphi})$ has at most $\sum\limits_{i=0}^{m} 2^i \binom{m}{i}$ $\mathrm{Fix}(\varphi)$-orbits.*

Proof. For $(a, a\hat{\varphi}_2), (b, b\hat{\varphi}_2) \in \mathrm{Fix}(\hat{\varphi})$, we have that $(a, a\hat{\varphi}_2)$ belongs to $(\mathrm{Fix}\,\varphi)(b, b\hat{\varphi}_2)$ if and only if there is some $(c, c\varphi_2) \in \mathrm{Fix}\,\varphi$ such that $(a, a\hat{\varphi}_2) = (c, c\varphi_2)(b, b\hat{\varphi}_2)$, i.e., a and b belong to the same orbit of $\mathrm{Fix}(\varphi_1)$. $\square$

Finiteness Conditions on Infinite Periodic Points

We proceed in a similar way in the case of periodic points.

We know that $\mathrm{Per}(\varphi)$ acts on $\mathrm{Per}(\hat{\varphi})$ on the left, since $a\varphi^p = a$ and $b\hat{\varphi}^q = b$ implies that $(a + b)\hat{\varphi}^{pq} = a\hat{\varphi}^{pq} + b\hat{\varphi}^{pq} = a + b$ and we want to count the orbits of such action.

When we consider a type I endomorphism, we have the same issue we had in the fixed points case, in the sense that we have $\mathrm{Per}\,\hat{\varphi} = \mathrm{Per}\,\hat{\varphi}_1 \times \mathrm{Per}\,\hat{\phi}$, which may also be uncountable.

To obtain a result for type II endomorphisms, we start as above, by dealing with the free-abelian part first. Let $\varphi \in \mathrm{End}(\mathbb{Z}^m)$ defined by $a \mapsto aQ$ be a uniformly continuous endomorphism. As above, given a column Q_j, if $Q_j \neq 0$, we call λ_j to its nonzero entry of column and α_j to its row, and if $Q_j = 0$ we put $\lambda_j = 0$ and $\alpha_j = 1$. Also, we will define the mapping $\psi : [m] \to [m]$ mapping i to α_i. Take $\hat{\varphi} : \widehat{\mathbb{Z}^m} \to \widehat{\mathbb{Z}^m}$ to be the continuous extension of φ to the completion.

This way, we have that $aQ = [\lambda_i a_{\alpha_i}]_{i\in[m]}$ and

$$aQ^r = \left[\left(\prod_{j=1}^{r} \lambda_{i\psi^{j-1}}\right) a_{i\psi^r}\right]_{i\in[m]}.$$

To lighten notation, for $i \in [m]$ and $r \in \mathbb{N}$ we will write

$$\pi_i^{(r)} := \prod_{j=1}^{r} \lambda_{i\psi^{j-1}} \tag{8.3}$$

This notation will be used throughout the rest of this section.

Proposition 8.1.14. *Let $\varphi \in \operatorname{End}(\mathbb{Z}^m)$ be a uniformly continuous endomorphism. Then* $\operatorname{Per}(\hat{\varphi})$ *has at most* $\sum_{i=0}^{m} 2^i \binom{m}{i}$ $\operatorname{Per}(\varphi)$*-orbits.*

Proof. Let $a \in \operatorname{Per}(\hat{\varphi})$. As done in the fixed point case, we will prove that, for $b \in \operatorname{Per}(\hat{\varphi})$, we have that $b \in (\operatorname{Per}\varphi)a$ if and only if their infinite entries coincide, and that suffices.

Clearly, if $b \in \operatorname{Per}(\hat{\varphi})$ is such that $b = c + a$ for some $c \in \operatorname{Per}(\varphi)$, then the infinite entries of a and b coincide.

Now, suppose a and b are two infinite periodic points whose infinite entries coincide. Then, there are $p, q > 0$ such that $a\hat{\varphi}^p = a$ and $b\hat{\varphi}^q = b$, so $a\hat{\varphi}^{pq} = a$ and $b\hat{\varphi}^{pq} = b$. Consider $c \in \mathbb{Z}^m$ defined by $c_i = 0$ if $a_i, b_i \in \{+\infty, -\infty\}$ and $c_i = b_i - a_i$ otherwise. Clearly, $b = c + a$. We only have to check that $c \in \operatorname{Per}(\varphi)$ and for that, we will show that $c\varphi^{pq} = c$. We have that, for $r > 0$, the mapping φ^r is defined by

$$[c_i]_{i\in[m]} \mapsto \left[\pi_1^{(r)} c_{i\psi^r}\right]_{i\in[m]}.$$

Now, we only have to see that, for every $i \in [m]$, we have that $c_i = \pi_i^{(pq)} c_{i\psi^{pq}}$. Let $i \in [m]$ such that $a_i, b_i \in \{+\infty, -\infty\}$ and so $c_i = 0$. Since $|a_i| = \infty$, $a_i = (a\hat{\varphi}^{pq})_i = \pi_i^{(pq)} a_{i\psi^{pq}}$ and all λ_k's are finite, we have that $a_{i\psi^{pq}} \in \{+\infty, -\infty\}$ (and the same holds for $b_{i\psi^{pq}}$), thus, $c_{i\psi^{pq}} = 0$. Then, we have that $\pi_i^{(pq)} c_{i\psi^{pq}} = 0 = c_i$. Now, take $i \in [m]$ such that $a_i, b_i \notin \{+\infty, -\infty\}$. Then $c_i = b_i - a_i = (b\hat{\varphi}^{pq})_i - (a\hat{\varphi}^{pq})_i = \pi_i^{(pq)} b_{i\psi^{pq}} - \pi_i^{(pq)} a_{i\psi^{pq}} = \pi_i^{(pq)}(b_{i\psi^{pq}} - a_{i\psi^{pq}}) = \pi_i^{(pq)} c_{i\psi^{pq}}$, since $a_{i\psi^{pq}}$ and $b_{i\psi^{pq}}$ are both finite. $\square$

Now, let φ be a type II uniformly continuous endomorphism of $\mathbb{Z}^m \times F_n$, with $n > 1$. Then $\varphi : \mathbb{Z}^m \times F_n \to \mathbb{Z}^m \times F_n$ is given by $(a, u) \mapsto (aQ, z^{\lambda a_k})$, for some $Q \in \mathcal{M}_m(\mathbb{Z})$ such that every column of Q contains at most one nonzero entry and $0 \neq \lambda \in \mathbb{Z}$, $k \in [m]$. Consider $\varphi_1 \in \operatorname{End}(\mathbb{Z}^m)$ to be defined as $a \mapsto aQ$ and $\varphi_2 : \mathbb{Z}^m \to F_n$ that maps a to $z^{\lambda a_k}$, which are both uniformly continuous.

By uniqueness of extension, we have that $\hat{\varphi} : \widehat{\mathbb{Z}^m} \times \widehat{F_n} \to \widehat{\mathbb{Z}^m} \times \widehat{F_n}$ is defined by $(a, u) \mapsto (a\hat{\varphi}_1, a\hat{\varphi}_2)$. Hence, if $(a, u) \in \operatorname{Per}(\hat{\varphi})$, then $a \in \operatorname{Per}(\hat{\varphi}_1)$.

Proposition 8.1.15. *Let $\varphi \in \mathrm{End}(\mathbb{Z}^m \times F_n)$ be a uniformly continuous type II endomorphism. Then* $\mathrm{Per}(\hat{\varphi})$ *has* $\sum\limits_{i=0}^{m} 2^i \binom{m}{i}$ $\mathrm{Per}(\varphi)$*-orbits.*

Proof. It is easy to see, by induction on r that, for every $r > 0$, we have that $(a,u)\hat{\varphi}^r = (a\hat{\varphi}_1^r, a\hat{\varphi}_1^{r-1}\hat{\varphi}_2)$. Indeed, it is true for $r = 1$ and if we have that $(a,u)\hat{\varphi}^r = (a\hat{\varphi}_1^r, a\hat{\varphi}_1^{r-1}\hat{\varphi}_2)$, then $(a,u)\hat{\varphi}^{r+1} = (a\hat{\varphi}_1^r, a\hat{\varphi}_1^{r-1}\hat{\varphi}_2)\hat{\varphi} = (a\hat{\varphi}_1^{r+1}, a\hat{\varphi}_1^r\hat{\varphi}_2)$. So we have that

$$(a,u)\hat{\varphi}^r = \left([\pi_i^{(r)} a_{i\psi^r}]_{i\in[m]}, z^{\lambda\pi_k^{r-1} a_{k\psi^{r-1}}}\right). \tag{8.4}$$

Let $(a,u),(b,v) \in \mathrm{Per}(\hat{\varphi})$. We have that $(b,v) \in (\mathrm{Per}\,\varphi)(a,u)$ if and only if $b \in (\mathrm{Per}(\varphi_1))a$. Indeed, if $b \in \mathrm{Per}(\varphi_1))a$, let $c \in \mathrm{Per}(\varphi_1)$ defined as in the proof of Proposition 8.1.14, such that $b = c + a$ and denote by p, q the periods of a and b, respectively.

Then $(c, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2)\varphi^{pq} = (c, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2)$, so $(c, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) \in \mathrm{Per}(\varphi)$ and

$$\begin{aligned}
(b,v) &= (b\hat{\varphi}_1^{pq}, b\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) = ((c+a)\hat{\varphi}_1^{pq}, (c+a)\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) \\
&= (c\hat{\varphi}_1^{pq} + a\hat{\varphi}_1^{pq}, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2 a\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) \\
&= (c\hat{\varphi}_1^{pq}, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2)(a\hat{\varphi}_1^{pq}, a\hat{\varphi}_1^{pq-1}\hat{\varphi}_2) \\
&= (c, c\hat{\varphi}_1^{pq-1}\hat{\varphi}_2)(a,u).
\end{aligned}$$

Hence, we have that $\mathrm{Per}(\hat{\varphi})$ has at most $\sum\limits_{i=0}^{m} 2^i \binom{m}{i}$ $\mathrm{Per}(\varphi)$-orbits. □

Classification of the Infinite Fixed Points for Automorphisms

Recall that a uniformly continuous automorphism of $\mathbb{Z}^m \times F_n$ is defined as $(a,u) \mapsto (aQ, u\phi)$, where Q is a uniform matrix and $\phi \in \mathrm{Aut}(F_n)$. As above, we define $\varphi_1 : \mathbb{Z}^m \to \mathbb{Z}^m$ that maps a to aQ and we have that $\mathrm{Fix}(\hat{\varphi}) = \mathrm{Fix}(\hat{\varphi}_1) \times \mathrm{Fix}(\hat{\phi})$

We are interested in classifying infinite fixed points as attractors or repellers. We will only consider these concepts regarding automorphisms because this definition of a repeller assumes the existence of an inverse. It is well known that, for an automorphism of a (virtually) free group, singular fixed points cannot be attractors nor repellers (see [33, Proposition 1.1]) and that every regular fixed point must be either an attractor or a repeller (see [98]). We will start by investigating what happens in the free-abelian part.

Proposition 8.1.16. *Let $\varphi \in \mathrm{End}(\mathbb{Z}^m)$ be an endomorphism defined by $a \mapsto aQ$, where Q is a uniform matrix. Then* $\mathrm{Sing}(\hat{\varphi}) = \mathrm{Fix}(\hat{\varphi})$.

Proof. By definition, $\mathrm{Sing}(\hat{\varphi}) \subseteq \mathrm{Fix}(\hat{\varphi})$.

Let $\pi \in S_m$ be a permutation such that φ maps $[a_i]_{i\in[m]}$ to $[\lambda_i a_{\pi(i)}]_{i\in[m]}$, and $\lambda_i = \pm 1$. Then $\mathrm{Fix}(\varphi) = \{a \in \mathbb{Z}^m \mid \forall i \in [m],\, a_i = \lambda_i a_{\pi(i)}\}$ and $\mathrm{Fix}(\hat{\varphi}) = \{a \in \widehat{\mathbb{Z}^m} \mid \forall i \in [m],\, a_i = \lambda_i a_{\pi(i)}\}$.

Given $a \in \mathrm{Fix}(\hat{\varphi})$ and $\varepsilon > 0$, choosing some

$$n > \max_{a_i \in \mathbb{Z}} \left\{ |a_i|, \left\lceil \log_2 \left(\frac{1}{\varepsilon} \right) \right\rceil \right\},$$

consider $b \in \mathbb{Z}^m$ such that $b_i = n$ if $a_i = +\infty$; $b_i = -n$ if $a_i = -\infty$ and $b_i = a_i$, otherwise. Then $b_i \in \mathrm{Fix}(\varphi)$ and $d(a, b) < \varepsilon$. Thus, b is a point of closure of $\mathrm{Fix}(\varphi)$ and we are done. $\square$

Proposition 8.1.17. *Let $\varphi \in \mathrm{End}(\mathbb{Z}^m)$ be defined by $a \mapsto aQ$, where Q is a uniform matrix. Then an infinite fixed point $a \in \mathrm{Fix}(\hat{\varphi}) \setminus \mathrm{Fix}(\varphi)$ is neither an attractor nor a repeller.*

Proof. Let $a \in \mathrm{Fix}\,\hat{\varphi} \setminus \mathrm{Fix}\,\varphi$. Let $\varepsilon > 0$ and define $q = \lceil \log_2(\frac{1}{\varepsilon}) \rceil$. Take $p = \max\limits_{a_i \in \mathbb{Z}} \{q, |a_i|\}$. Take $b \in \mathbb{Z}^m$ such that $b_i = p$ for every i such that $a_i = +\infty$; $b_i = -p$ for every i such that $a_i = -\infty$ and $b_i = a_i$ otherwise. Then $d(a, b) < \varepsilon$ but $b\hat{\varphi}^n \not\to a$ since $\max\limits_{i \in [m]} |b_i| = p$ and applying $\hat{\varphi}$ simply changes order and signal of the entries, so for every $n \in \mathbb{N}$, we have that $\max\limits_{i \in [m]} \{|(b\hat{\varphi}^n)_i|\} = p$. Hence $d(a, b\hat{\varphi}^n) \geq 2^{-p}$ since there is some k such that $a_k \in \{+\infty, -\infty\}$ and $d(a_k, b_k) \geq d(a_k, sgn(a_k)|p|) = 2^{-p}$. The repeller case is analogous, since the inverse of a uniform matrix is uniform. $\square$

So, when an endomorphism is given by a uniform matrix, no infinite fixed point is an attractor or a repeller. The next result shows how that impacts the case of a general uniformly continuous automorphism, providing a full classification of infinite fixed points.

Theorem 8.1.18. *An infinite fixed point $\alpha = (a, u)$, where $a \in \mathrm{Fix}(\hat{\varphi}_1)$ and $u \in \mathrm{Fix}(\hat{\phi})$ is an attractor (resp. repeller) if and only if a and u are attractors (resp. repellers) for $\hat{\varphi}_1$ and $\hat{\phi}$, respectively.*

Proof. Let $\alpha = (a, u)$ be an infinite fixed point, where $a \in \mathrm{Fix}(\hat{\varphi}_1)$ and $u \in \mathrm{Fix}(\hat{\phi})$. Clearly if $a \in \mathrm{Fix}(\hat{\varphi}_1)$ and $u \in \mathrm{Fix}(\hat{\phi})$ are attractors, then, $(a, u) \in \mathrm{Fix}(\hat{\varphi})$ is an attractor. Indeed, in that case, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\forall b \in \widehat{\mathbb{Z}^m}, \left(d(a, b) < \varepsilon_1 \implies \lim_{n \to +\infty} b\hat{\varphi}_1^n = a \right)$$

and

$$\forall v \in \widehat{F_n}, \left(d(u, v) < \varepsilon_2 \implies \lim_{n \to +\infty} v\hat{\phi}^n = u \right).$$

Thus, taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have that

$$\begin{aligned} \forall (b, v) \in \widehat{\mathbb{Z}^m \times F_n}\ (d((a, u), (b, v)) < \varepsilon &\implies d(a, b) < \varepsilon \wedge d(u, v) < \varepsilon \\ &\implies \lim_{n \to +\infty} b\hat{\varphi}_1^n = a \wedge \lim_{n \to +\infty} v\hat{\phi}^n = u \\ &\implies \lim_{n \to +\infty} (b, v)\hat{\varphi}^n = (a, u). \end{aligned}$$

Conversely, suppose w.l.o.g that a is not an attractor for $\hat{\varphi}_1$. Then, for every $\varepsilon > 0$, there is some $b_\varepsilon \in \widehat{\mathbb{Z}^m}$ such that $d(a,b) < \varepsilon$ but $b_\varepsilon \hat{\varphi}_1 \not\to a$. In this case, we have that, for every $\varepsilon > 0$, $d((a,u),(b_\varepsilon,u)) < \varepsilon$ and $(b_\varepsilon,u)\hat{\varphi}^n = (b_\varepsilon \hat{\varphi}_1^n, u) \not\to (a,u)$. □

Corollary 8.1.19. *Let $\varphi \in \mathrm{Aut}(\mathbb{Z}^m \times F_n)$ be a uniformly continuous automorphism such that $(a,u)\hat{\varphi} = (a\hat{\varphi}_1, b\hat{\phi})$, where φ_1 is given by a uniform matrix and $\phi \in \mathrm{Aut}(F_n)$. Then a regular infinite fixed point $(a,u) \in \mathrm{Fix}(\hat{\varphi}) \setminus \mathrm{Fix}(\varphi)^c$ is an attractor (resp. repeller) if and only if $a \in Fix(\varphi_1)$ and u is an attractor (resp. repeller) for $\hat{\phi}$.*

Notice that, given an infinite attractor (resp. repeller) $u \in \widehat{F_n}$ and denoting by S_u the set of points attracted (resp. repelled) to it, then the set of points attracted (resp. repelled) by (a,u) is given by $T_{(a,u)} = \{(a,y) \mid y \in S_u\}$.

8.1.4 Dynamics of infinite points

This subsection is devoted to the study of the dynamics of the extension of a uniformly continuous endomorphism to the completion. We will prove that if φ is a uniformly continuous automorphism or a uniformly continuous type II endomorphism, then every point in the completion is either periodic or wandering, which implies that, in these cases, the dynamics is asymptotically periodic.

The automorphism case

We have that a uniformly continuous automorphism of $\mathbb{Z}^m \times F_n$ is defined as $(a,u) \mapsto (aQ, u\phi)$, where Q is a uniform matrix and $\phi \in \mathrm{Aut}(F_n)$.

We start by observing that, for the abelian part, the dynamics is simple in the sense that every point is periodic.

Proposition 8.1.20. *Let $\varphi \in \mathrm{Aut}(\mathbb{Z}^m)$ be defined by $a \mapsto aQ$, where Q is a uniform matrix and consider $\hat{\varphi}$ to be its continuous extension to the completion. Then, there is some constant $p \leq 2^m m!$ such that $\hat{\varphi}^p = Id$. Hence, $\mathrm{Per}(\hat{\varphi}) = \widehat{\mathbb{Z}^m}$ and the period of every element divides p.*

Proof. There are only $2^m m!$ distinct uniform $m \times m$ matrices, so there are $0 < p < q \leq 2^m m! + 1$ such that $Q^p = Q^q$, thus $I_m = Q^{q-p}$. □

We now present some standard dynamical definitions, that will be useful to the classification of infinite points.

Definition 8.1.21. Let G be a group and $\varphi \in \mathrm{End}(G)$. A point $x \in G$ is said to be a φ-*wandering point* if there is a neighborhood U of x and a positive integer N such that for all $n > N$, we have that $U\varphi^n \cap U = \emptyset$. When it is clear, we simply say that x is a wandering point.

Definition 8.1.22. Let G be a group and $\varphi \in \text{End}(G)$. A point $x \in G$ is said to be a *φ-recurrent point* if, for every neighborhood U of x, there exists $n > 0$ such that $x\varphi^n \in U$. When it is clear, we simply say that x is a recurrent point.

Definition 8.1.23. Let f be a homeomorphism of a compact space K. Given $y \in K$, the *ω-limit set* $\omega(y, f)$, or simply $\omega(y)$, is the set of limit points of the sequence $f^n(y)$ as $n \to +\infty$.

Definition 8.1.24. Let G be a group. A uniformly continuous endomorphism $\varphi \in \text{End}(G)$ has *asymptotically periodic dynamics* on $\widehat{G}$ if there exists $q \geq 1$ such that, for every $x \in \widehat{G}$, the sequence $x\hat{\varphi}^{qn}$ converges to a fixed point of $\hat{\varphi}^q$.

To the knowledge of the author, this is the first application of the concepts of recurrent and wandering points in the context of group theory. The next proposition shows how being $\hat{\varphi}$-recurrent (resp. wandering) relates with being $\hat{\phi}$-recurrent (resp. wandering).

Proposition 8.1.25. *Let $\varphi \in \text{End}(\mathbb{Z}^m \times F_n)$ be a uniformly continuous endomorphism defined by $(a, u) \mapsto (a\varphi_1, u\phi)$, where $\varphi_1 \in \text{End}(\mathbb{Z}^m)$ and $\phi \in \text{End}(F_n)$ and $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$. We have the following:*

1. *(a, u) is $\hat{\varphi}$-periodic $\implies$ u is $\hat{\phi}$-periodic.*
2. *(a, u) is $\hat{\varphi}$-wandering $\impliedby$ u is $\hat{\phi}$-wandering.*
3. *(a, u) is $\hat{\varphi}$-recurrent $\implies$ u is $\hat{\phi}$-recurrent.*

Proof.

1. This is obvious. Observe that if Q is uniform, which is the case when we deal with automorphisms, then the reverse implication holds as well, by Propostion 8.1.20.
2. Suppose u is $\hat{\phi}$-wandering. Then, take $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all $n > N$, we have that $B(u; \varepsilon)\hat{\phi}^n \cap B(u; \varepsilon) = \emptyset$. Let $n > N$ and take $V = B((a, u); \varepsilon)$ and $(b, v) \in V$. Then $v \in B(u; \varepsilon)$, thus $v\hat{\phi}^n \notin B(u; \varepsilon)$ and $(b, v)\hat{\varphi}^n = (b\hat{\varphi}_1^n, v\hat{\phi}^n) \notin V$. Since (b, v) is an arbitrary point of V, we have that (a, u) is $\hat{\phi}$-wandering.
3. Suppose (a, u) is $\hat{\varphi}$-recurrent. Take $\varepsilon > 0$. There is some $n > 0$ such that $(a, u)\hat{\varphi}^n \in B((a, u); \varepsilon)$ and so $u\hat{\phi}^n \in B(u; \varepsilon)$. □

Notice that, in case $u \in F_n$ and $\phi \in \text{End}(F_n)$, if u is nonwandering, then it must be periodic, since we can take $U = \{u\}$, we have that there is some n such that $U\hat{\phi}^n \cap U \neq \emptyset$ and so, it is periodic. So, in case $\phi \in \text{Aut}(\mathbb{Z}^m \times F_n)$, we have that a point $(a, u) \in \widehat{\mathbb{Z}^m} \times F_n$ must be periodic or wandering, by Proposition 8.1.25.

We now present two results from [67] regarding automorphisms of free groups, that will be very useful in this case.

Lemma 8.1.26 ([67], Levitt-Lustig)**.** *Let f be a homeomorphism of a compact space K. Given $y \in K$ and $q \geq 1$, the following conditions are equivalent:*

1. *$\omega(y)$ is finite and has q elements.*
2. *$\omega(y)$ is a periodic orbit of order q.*
3. *The sequence $f^{qn}(y)$ converges as $n \to +\infty$, and q is minimal for this property.*

Given $p \geq 2$, the set $\omega(y, f^p)$ is finite if and only if $\omega(y, f)$ is finite.

If these equivalent conditions hold, we say that the point y is *asymptotically periodic*. If every point is asymptotically periodic, then the endomorphism has asymptotically periodic dynamics (this definition is equivalent to Definition 8.1.24).

Theorem 8.1.27 ([67], Levitt-Lustig)**.** *Every automorphism $\alpha \in \mathrm{Aut}(F_n)$ has asymptotically periodic dynamics.*

In Theorem 6.3.5, it is proved that every uniformly continuous endomorphism φ of a hyperbolic group G with respect to a visual metric satisfies a Hölder condition. From [2, Proposition 4.2], $\hat{\varphi}$ also satisfies a Hölder condition, i.e., there are $K, r > 0$ such that

$$\hat{d}(x\hat{\varphi}, y\hat{\varphi}) \leq K\hat{d}(x,y)^r,$$

for all $x, y \in \widehat{G}$. We will prove that, in this setting, every nonwandering point is recurrent.

Lemma 8.1.28. *Let (X, d) be a metric space and $\phi : (X, d) \to (X, d)$ be a Hölder mapping with constants (K, r). Then*

$$d(x\phi^n, y\phi^n) \leq K^{\frac{1-r^n}{1-r}} d(x,y)^{r^n}$$

for all $x, y \in X$, $n > 0$.

Proof. We prove this by induction on n. By hypothesis, we have that $d(x\phi, y\phi) \leq Kd(x,y)^r$, for all $x, y \in X$. Now suppose that the result holds for all $n \leq k$, for some $k > 0$. Let $x, y \in X$. We have that

$$\begin{aligned} d(x\phi^{k+1}, y\phi^{k+1}) &\leq Kd(x\phi^k, y\phi^k)^r \\ &\leq K\left(K^{\frac{1-r^k}{1-r}} d(x,y)^{r^k}\right)^r \\ &\leq K^{1+r\frac{1-r^k}{1-r}} d(x,y)^{r^{k+1}} \\ &\leq K^{\frac{1-r}{1-r}+\frac{r-r^{k+1}}{1-r}} d(x,y)^{r^{k+1}} \\ &\leq K^{\frac{1-r^{k+1}}{1-r}} d(x,y)^{r^{k+1}}. \end{aligned}$$

□

Proposition 8.1.29. *Let (X,d) be a metric space and $\phi : (X,d) \to (X,d)$ be a Hölder mapping. Then every nonwandering point is recurrent.*

Proof. Let (K,r) be the constants given by the Hölder condition and $x \in X$ be a nonwandering point. Then

$$\forall \varepsilon > 0 \;\; \forall N \in \mathbb{N} \;\; \exists n > \mathbb{N} : B(x,\varepsilon)\phi^n \cap B(x,\varepsilon) \neq \emptyset.$$

Let $\varepsilon \leq K^{\frac{1}{1-r}}$. We then have that

$$\forall N \in \mathbb{N} \;\; \exists n > N \;\; \exists y_n \in X : (d(y_n,x) \leq \varepsilon \;\; \wedge \;\; d(y_n\phi^n,x) \leq \varepsilon)\,.$$

Let $N \in \mathbb{N}$ and take n given by the condition. By Lemma 8.1.28, we have that

$$d(x\phi^n, y_n\phi^n) \leq K^{\frac{1-r^n}{1-r}} d(x,y_n)^{r^n} \leq K^{\frac{1-r^n}{1-r}} \varepsilon^{r^n}.$$

Thus,

$$\begin{aligned}
d(x\phi^n, x) &\leq d(x\phi^n, y_n\phi^n) + d(y_n\phi^n, x)\\
&\leq K^{\frac{1-r^n}{1-r}} \varepsilon^{r^n} + \varepsilon\\
&\leq \varepsilon(1 + K^{\frac{1-r^n}{1-r}} \varepsilon^{r^n-1})\\
&\leq \varepsilon\left(1 + K^{\frac{1-r^n}{1-r}} \left(K^{\frac{1}{1-r}}\right)^{r^n-1}\right)\\
&= 2\varepsilon.
\end{aligned}$$

So, fixing $\varepsilon \leq K^{\frac{1}{1-r}}$, we have arbitrarily large integers n such that $d(x\phi^n, x) \leq 2\varepsilon$. Thus, let $\varepsilon > 0$ and take $\varepsilon' = \min\{\frac{\varepsilon}{2}, \frac{K^{\frac{1}{1-r}}}{2}\}$. Then

$$\forall N \in \mathbb{N} \;\; \exists n > N : x\phi^n \in B(x, 2\varepsilon') \subseteq B(x,\varepsilon),$$

and so x is recurrent. $\square$

Corollary 8.1.30. *Let φ be a uniformly continuous endomorphism of a hyperbolic group G with respect to a visual metric d. Then every nonwandering point of $\widehat{G}$ is recurrent.*

In particular, if φ is an automorphism of a free group, we can apply Corollary 8.1.30. Thus, the following results follow:

Corollary 8.1.31. *Let φ be an automorphism of F_n and $\hat{\varphi}$, its continuous extension to the completion. Then every point $u \in \widehat{F_n}$ is either wandering or periodic.*

Proof. By Corollary 8.1.30, nonwandering points are recurrent. Since recurrent points belong to its own ω-limit, by Theorem 8.1.27, they must be periodic. $\square$

Corollary 8.1.32. *Let φ be a uniformly continuous automorphism of $\mathbb{Z}^m \times F_n$ defined by $(a, u) \mapsto (a\varphi_1, u\phi)$, where $\varphi_1 \in \operatorname{Aut}(\mathbb{Z}^m)$ is given by a uniform matrix and $\phi \in \operatorname{Aut}(F_n)$. Consider $\hat{\varphi}$, its continuous extension to the completion. Then every point $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ is either wandering or periodic.*

Proof. Let $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ be a nonwandering point. Then, by Proposition 8.1.25, u is a nonwandering point for ϕ, which means that u is periodic, by Corollary 8.1.31. Since a is also periodic by Proposition 8.1.20, then so is (a, u). □

Corollary 8.1.33. *Every uniformly continuous automorphism $\varphi \in \operatorname{Aut}(\mathbb{Z}^m \times F_n)$ has asymptotically periodic dynamics on $\widehat{\mathbb{Z}^m \times F_n}$.*

Proof. Since $\widehat{\mathbb{Z}^m \times F_n}$ is compact then ω-limits are nonempty. Moreover, if a point (a, u) belongs to an ω-limit set, then it must be nonwandering, which, by Corollary 8.1.32, means that (a, u) is periodic. □

Type II Endomorphisms

The purpose of this subsubsection is to establish the dichotomy periodic vs wandering for type II endomorphisms. Recall the notation introduced before Lemma 8.1.9. Also, recall (8.4) and the decomposition $z = w\tilde{z}w^{-1}$where $\tilde{z}$ is the cyclically reduced core of z.

Remark 8.1.34. *Assume $n > 1$. The set of periodic points of the extension of a uniformly continuous type II endomorphism to the completion is not dense in the entire space, even when we restrict ourselves to the boundary. Indeed, if we take a point (a, u) such that u does not share a prefix with z and z^{-1}, then $B\left((a, u); \frac{1}{2}\right)$ does not contain a periodic point. Also, the system does not admit the existence of a dense orbit: Indeed, given $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$, choosing a point $(b, v) \in \widehat{\mathbb{Z}^m \times F_n}$ such that $b \neq a$ and v doesn't share a prefix of size with neither z nor z^{-1}, we have that $B((b, v); \frac{1}{2})$ does not contain any point in the orbit.*

Also, in the automorphism case, we have that the first component is always periodic, so there is not a dense orbit, even when restricted to the boundary.

We will now prove two technical lemmas that will be very useful for proving the main result.

Lemma 8.1.35. *Consider a uniformly continuous endomorphism φ of a free-abelian group $\mathbb{Z}^m$ and take $i \in [m]$ and some positive integer $r > m$. Then, the following conditions are equivalent:*

1. $\exists N \in \mathbb{N} \, \forall p > N \; |\lambda_{i\psi^p}| = 1$
2. $\exists N \leq m \, \forall p > N \; |\lambda_{i\psi^p}| = 1$
3. $|\pi^{(r)}_{i\psi^{tr}}| = 1,$ *for every positive integer t*

4. $|\pi^{(r)}_{i\psi^{tr}}| = 1$*, for some positive integer* t

Proof. It is obvious that $2 \implies 1$, $2 \implies 3$ and $3 \implies 4$. We will prove that $1 \implies 2$ and that $4 \implies 1$.

$1 \implies 2$: Suppose that there is some $N > m$ such that for every $p > N$ we have that $|\lambda_{i\psi^p}| = 1$. We have that ψ maps $[m]$ to a subset of $[m]$, and so, for every $i \in [m]$, there is some $k_i \leq m$ such that $i\psi^{m+1} = i\psi^{k_i}$. This way, we have a periodic orbit (can be fixed) of ψ given by $\{i\psi^{k_i}, \dots i\psi^m\}$. So, for every $p \geq N$, we define $j_p \in \{k_i, \dots, m\}$ to be such that $i\psi^{j_p} = i\psi^p$. Also, if for some $j > j_N$, we had $|\lambda_{i\psi^j}| > 1$, we could obtain p arbitrarily large such that $|\lambda_{i\psi^p}| > 1$, which is absurd. So, we have that,

$$\forall p > j_N \; |\lambda_{i\psi^p}| = 1$$

and $j_N \leq m$.

$4 \implies 1$: If, for some $i \in [m]$, we have that $|\pi^{(r)}_{i\psi^{tr}}| = 1$, for some $t > 0$, then for every $j \in \{tr, \dots, (t+1)r - 1\}$, we have that $|\lambda_{i\psi^j}| = 1$. In this case, for some $s \geq (t+1)r$, we have that $i\psi^s = i\psi^j$ for some $j \in \{tr, \dots, (t+1)r-1\}$ and so $|\lambda_{i\psi^s}| = 1$. Thus, 1 holds for $N = r$. □

Lemma 8.1.36. *Consider a nonwandering point* $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ *such that* a *has finite entries. Let* $\delta = \max\limits_{a_i \in \mathbb{Z}} \{|a_i|\}$ *and* $U = B\left((a, z^{+\infty}); \frac{1}{2^\delta}\right)$*. Consider a point* $(b, v) \in U$ *and a positive integer* $r > m$ *such that* $(b, v)\hat{\varphi}^r \in U$*. Then the conditions from Lemma 8.1.35 hold for every index* $i \in [m]$ *such that* $a_i \neq \pi^{(r)}_i a_{i\psi^r}$*.*

Moreover, if $u = z^{+\infty}$ *and* $|a_{k\psi^{r-1}}| < \infty$*, then the conditions from Lemma 8.1.35 hold when* $i = k$*.*

Proof. Consider a nonwandering point $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ such that a has finite entries, let $\delta = \max\limits_{a_i \in \mathbb{Z}} \{|a_i|\}$ and $U = B((a, u); \frac{1}{2^\delta})$. Consider a point $(b, v) \in U$ and a positive integer $r > m$ such that $(b, v)\hat{\varphi}^r \in U$. So, for $i \in [m]$, we have that:

$$\text{if } |a_i| < \infty, \text{ then } a_i = b_i = \pi^{(r)}_i b_{i\psi^r} \tag{8.5}$$

and

$$\text{if } |a_i| = \infty, \text{ then } sgn(a_i) = sgn(b_i) = sgn(\pi^{(r)}_i b_{i\psi^r}) \text{ and } |b_i|, |\pi^{(r)}_i b_{i\psi^r}| > \delta. \tag{8.6}$$

If $a\hat{\varphi}^r_1 = a$, then $a_i = \pi^{(r)}_i a_{i\psi^r}$ for every $i \in [m]$ and the first part of the lemma trivially holds.

If not, take $q \in [m]$ such that $a_q \neq \pi^{(r)}_q a_{q\psi^r}$. We start by observing that a_q must be infinite since if that is not the case, then, by (8.5), we have that $a_q = b_q = \pi^{(r)}_q b_{q\psi^r}$. If $\pi^{(r)}_q = 0$, then $a_q = \pi^{(r)}_q b_{q\psi^r} = 0 = \pi^{(r)}_q a_{q\psi^r}$. If not, then $|b_{q\psi^r}| \leq |b_q|$, which means that $|b_{q\psi^r}| \leq \delta$ and by (8.6), we have that $a_{q\psi^r}$ is finite and by (8.5), it follows that $a_{q\psi^r} = b_{q\psi^r}$, so $a_q = \pi_q^{(r)} b_{q\psi^r} = \pi_q^{(r)} a_{q\psi^r}$. Also, $a_{q\psi^r}$ must be finite, since, if it is infinite, then by (8.6) we have that $sgn(a_{q\psi^r}) = sgn(b_{q\psi^r})$ and $sgn(a_q) = sgn(\pi_q^{(r)} b_{q\psi^r})$, thus $sgn(a_q) = sgn(\pi_q^{(r)} a_{q\psi^r})$ and that implies that $a_q = \pi_q^{(r)} a_{q\psi^r}$, since $\pi_q^{(r)} \neq 0$.

So, using (8.5) with $i = q\psi^r$, we have that

$$a_{q\psi^r} = b_{q\psi^r} = \pi_{q\psi^r}^{(r)} b_{q\psi^r\psi^r} = \pi_{q\psi^r}^{(r)} b_{q\psi^{2r}}. \tag{8.7}$$

We have that a_q is infinite, so, by (8.6), we know that $|\pi_q^{(r)} b_{q\psi^r}| > \delta$. This means in particular that $a_{q\psi^r} \neq 0$, because otherwise we would have $b_{q\psi^r} = 0$, by (8.5).

Suppose now that for every positive integer t, we have that $|\pi_{q\psi^{tr}}^{(r)}| \neq 1$. If $\pi_{q\psi^r}^{(r)} = 0$, then $a_{q\psi^r} = 0$, which is absurd. Then, we have that $|\pi_{q\psi^r}^{(r)}| > 1$, and $|b_{q\psi^{2r}}| < |b_{q\psi^r}| < \infty$ and again, using (8.5) with $i = q\psi^{2r}$, we get that $a_{q\psi^{2r}} = b_{q\psi^{2r}} = \pi_{q\psi^{2r}}^{(r)} b_{q\psi^{3r}}$. Again, if $\pi_{q\psi^{2r}}^{(r)} = 0$, then $a_{q\psi^{2r}} = 0$ and by (8.5), it follows that $b_{q\psi^{2r}} = 0$. From (8.7), we reach a contradiction. So, we must have $|\pi_{q\psi^r}^{(r)}| > 1$ and $|b_{q\psi^{3r}}| < |b_{q\psi^{2r}}| < \infty$. Proceeding like this, since the value of $|b_{q\psi^{pr}}|$, for $p \in \mathbb{N}$ cannot decrease indefinitely, then we must have that $b_{q\psi^r} = 0$ and so $a_{q\psi^r} = 0$, which is absurd.

So, the conditions from Lemma 8.1.35 hold when $i = q$.

If we have that $u = z^{+\infty}$ and $|a_{k\psi^{r-1}}| < \infty$, then by (8.5), we have that $a_{k\psi^{r-1}} = b_{k\psi^{r-1}} = \pi_{k\psi^{r-1}}^{(r)} b_{k\psi^{2r-1}}$. If the conditions from Lemma 8.1.35 do not hold when $i = k$, then using the same argument as above, we obtain $a_{k\psi^{r-1}} = b_{k\psi^{r-1}} = 0$, which is absurd since $\left| z^{+\infty} \wedge z^{\lambda \pi_k^{(r-1)} b_{k\psi^{r-1}}} \right| > \delta$. □

Notice that for every i for which the conditions from Lemma 8.1.35 hold, there is some constant $B_i \geq 1$ such that any product of the form $\prod_{j=s}^{t} \lambda_{i\psi^j}$, with $t \geq s$ is bounded above by B_i. Also, we remark that if follows from the proof that for q such that $a_q \neq \pi_q^{(r)} a_{q\psi^r}$ we must have that $\pi_q^{(r)} \neq 0$, a_q is infinite and $a_{q\psi^r}$ is finite.

Theorem 8.1.37. *Let $\varphi \in \text{End}(\mathbb{Z}^m \times F_n)$ be a type II uniformly continuous endomorphism defined by $(a, u) \mapsto (aQ, z^{\lambda a_k})$, for some $k \in [m]$, $1 \neq z \in F_n$, which is not a proper power and Q such that $a \mapsto aQ$ is uniformly continuous. Consider $\hat{\varphi}$, its continuous extension to the completion. Then every point $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$ is either wandering or periodic.*

Proof. Let $(a, u) \in \widehat{\mathbb{Z}^m \times F_n}$. Clearly, if (a, u) is wandering, it is not periodic. To prove the reverse inclusion, we will consider several cases.

Case 1: $u \in F_n$. Start by supposing that every entry in a is infinite. In this case, (a, u) is never periodic, so we will prove it is wandering. Take $U = B\left((a, u); \frac{1}{2^{|u|}}\right)$, $r > 0$ and $(b, v) \in U$. We have that, if $(b, v) \in U$, then $d(u, v) < \frac{1}{2^{|u|}}$, which means $u = v$ and for every $i \in [m]$, we have that $a_i = b_i$, or $a_i b_i > 0$ and $|a_i|, |b_i| > |u|$.

Then, we have that $(b,v)\hat{\varphi}^r = \left(\left[\pi_i^{(r)} b_{i\psi^r}\right]_{i\in[m]}, z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}}\right)$. This means that $\pi_k^{(r)} \geq 1$, because, if $\pi_k^{(r)} = 0$, then $\pi_k^{(r)} b_{k\psi^r} = 0 < |u|$, which is absurd. Since $|b_{k\psi^{r-1}}| > |u|$, then

$$\left|z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}}\right| = 2|w| + \left|\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}\right| |\tilde{z}| > |u|$$

and $(b,v)\hat{\varphi}^r \notin U$. So, in this case, (a,u) is wandering.

Now, we deal with the case where a has finite entries. Suppose (a,u) is not wandering. Then, for every neighborhood U of (a,u), we have that $U\varphi^r \cap U \neq \emptyset$ for arbitrarily large r. Set $\delta = \max\limits_{a_i\in\mathbb{Z}}\{|a_i|, |u|\}$ and consider $U = B\left((a,u); \frac{1}{2^\delta}\right)$. We have that, if $(b,v) \in U$, then $u = v$ and for $i \in [m]$, if a_i is finite we have (8.5) and if a_i is infinite, then we have (8.6). Take $r > m$, $(b,v) \in U$ such that $(b,v)\varphi^r \in U$. If $a\hat{\varphi}_1^r = a$, then if $\pi_k^{(r-1)} = 0$, we have that

$$u = z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} = 1 = z^{\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}}} = a\hat{\varphi}_2^r.$$

If $\pi_k^{(r-1)} \geq 1$, then, since $u = z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}}$, we have that $|b_{k\psi^{r-1}}| < |u| \leq \delta$ and so $a_{k\psi^{r-1}}$ must be finite by (8.6). Thus, by (8.5), we have that $a_{k\psi^{r-1}} = b_{k\psi^{r-1}}$ and

$$u = z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} = z^{\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}}} = a\hat{\varphi}_2^r.$$

So, we have that if $a\hat{\varphi}_1^r = a$, then (a,u) is periodic. If not, then by Lemma 8.1.36, we have that the conditions from Lemma 8.1.35 hold for every i such that $a_i \neq \pi_i^{(r)} a_{i\psi^r}$. Thus, the set

$$X = \{j \in [m] \mid \text{the conditions from Lemma 8.1.35 hold for } i = j\}$$

is nonempty.

Now, take $\tau = \max\{B_q \mid q \in X\}$ and $U' = B\left((a,u); \frac{1}{2^{\tau\delta}}\right)$. Notice that $U' \subseteq U$ and so Lemma 8.1.36 can be applied. Since (a,u) is nonwandering, there is some $r' > m$ and $(b',v') \in U'$ such that $(b',v')\hat{\varphi}^{r'} \in U'$. We will prove that $a\hat{\varphi}_1^{r'} = a$. Suppose not and take $q \in [m]$ such that $a_q \neq \pi_q^{(r')} a_{q\psi^{r'}}$. So $q \in X$ and from the proof of the Lemma 8.1.36 it follows that $\pi_q^{(r')} \neq 0$, a_q is infinite and $a_{q\psi^r}$ is finite. But then, since $(b',v')\hat{\varphi}^{r'} \in U'$, we must have $|\pi_q^{(r')} b'_{q\psi^{r'}}| > \delta\tau$, which is absurd since $\pi_q^{(r')} \leq \tau$ and $b'_{q\psi^{r'}} = a_{q\psi^{r'}} \leq \delta$.

As done above, we can check that $a\hat{\varphi}_2^{r'} = u$ and so (a,u) is periodic.

Case 2: $u \in \partial F_n \setminus \{z^{+\infty}, z^{-\infty}\}$. In this case (a,u) is never periodic, so we will prove it is wandering. Take $\delta = |w| + \max\{|z^{-\infty} \wedge u|, |z^{+\infty} \wedge u|\}$ and consider $U = B\left((a,u); \frac{1}{2^\delta}\right)$. Let $(b,v) \in U$. We have that $|v \wedge u| > \delta$ and for every $i \in [m]$, $a_i = b_i$ or $a_i b_i > 0$ and $|a_i|, |b_i| > \delta$.

So, for every $r > 0$, we have that $(b, v)\hat{\varphi}^r \notin U$, since

$$\left| z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} \wedge u \right| \leq \begin{cases} |z^{+\infty} \wedge u| + |w| & \text{if } \lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} > 0 \\ 0 & \text{if } \lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} = 0 \\ |z^{-\infty} \wedge u| + |w| & \text{if } \lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} < 0. \end{cases}$$

Case 3: $u \in \{z^{+\infty}, z^{-\infty}\}$. Suppose (a, u) is not wandering and assume w.l.o.g. that $u = z^{+\infty}$. Suppose first that every entry of a is infinite and consider $U = B\left((a, u), \frac{1}{2^{|w|}}\right)$. Take $r > m$, $(b, v) \in U$ such that $(b, v)\varphi^r \in U$. Denote the first letter of $\tilde{z}$ by $\tilde{z}_1$. We have that $w\tilde{z}_1$ is a prefix of v and for every $i \in [m]$, either $a_i = b_i$ or $a_i b_i > 0$ and $|a_i|, |b_i| > |w|$.

From $(b, v)\hat{\varphi}^r \in U$ we deduce that $w\tilde{z}_1$ is a prefix of $z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}}$, so $\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} > 0$ and $\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}} = +\infty$, since it has the same sign and every entry of a is infinite. Also, $a\hat{\varphi}_1^r = a$ because $a_i b_i > 0$ and $a_i(b\hat{\varphi}_1^r)_i > 0$, so $\hat{\varphi}_1^r$ doesn't change the signs of the entries in b, thus it also does not change the ones in a. In that case, $(a, u)\hat{\varphi}^r = (a, u)$ and a is periodic.

To complete the proof, we take $(a, z^{+\infty})$ such that a has finite entries and suppose it is not wandering nor periodic. Take $\delta = \max\limits_{a_i \in \mathbb{Z}}\{|a_i|\}$ and $U = B\left((a, z^{+\infty}); \frac{1}{2^\delta}\right)$. Take $r > m$ and $(b, v) \in U$ such that $(b, v)\hat{\varphi}^r \in U$. We now consider two subcases:

Subcase 3.1: $|a_{k\psi^{r-1}}| = \infty$. We have that

$$\left| z^{+\infty} \wedge z^{\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}}} \right| > \delta$$

and so $\lambda\pi_k^{(r-1)} b_{k\psi^{r-1}} > 0$. Thus, using (8.6) with $i = k\psi^{r-1}$, we have that $\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}} > 0$ and, since $|a_{k\psi^{r-1}}| = \infty$, we have that $z^{\lambda\pi_k^{(r-1)} a_{k\psi^{r-1}}} = z^{+\infty}$. Since (a, u) is not periodic, we have that $a\hat{\varphi}_1^r = [\pi_i^{(r)} a_{i\psi^r}]_{i\in[m]} \neq a$. Take q such that $a_q \neq \pi_q^{(r)} a_{q\psi^r}$. By Lemma 8.1.36, we have that the conditions from Lemma 8.1.35 hold when $i = q$. Thus, the set $X = \{j \in [m] \mid$ the conditions from Lemma 8.1.35 hold for $i = j\}$ is nonempty.

Consider $\delta = \max\limits_{a_i \in \mathbb{Z}}\{|a_i|\}$, $\tau = \max\{B_q \mid q \in X\}$ and $\delta' = 2|w| + \lambda\tau\sigma|\tilde{z}|$ and let $U' = B((a, u); \frac{1}{2^{\delta'}})$. Since (a, u) is nonwandering, there is some $r' > m$ and $(b', v') \in U'$ such that $(b', v')\hat{\varphi}^{r'} \in U'$. Notice that $U' \subseteq U$ and so Lemma 8.1.36 can be applied. We will prove that $(a, u)\hat{\varphi}^{r'} = (a, u)$, which is absurd.

We start by showing that $z^{\lambda\pi_k^{(r'-1)} a_{k\psi^{r'-1}}} = z^{+\infty}$. Indeed, if $|a_{k\psi^{r'-1}}| < \infty$, then by the last statement of Lemma 8.1.36, we have that $k \in X$. Also, we know by (8.5) that $b'_{k\psi^{r'-1}} = a_{k\psi^{r'-1}}$, so

$$\left| z^{\lambda\pi_k^{(r'-1)} b'_{k\psi^{r'-1}}} \right| = 2|w| + |\lambda\pi_k^{(r'-1)} b'_{k\psi^{r'-1}}||\tilde{z}| \leq 2|w| + \lambda\tau\delta|\tilde{z}| = \delta',$$

which is absurd because $(b,v)\hat{\varphi}^r \in U'$ implies that

$$\left| z^{+\infty} \wedge z^{\lambda \pi_k^{(r'-1)} b'_{k\psi^{r'-1}}} \right| > \delta'. \tag{8.8}$$

So this can never happen and so we must have $|a_{k\psi^{r'-1}}| = \infty$. Since we have (8.8), it follows that $\lambda\pi_k^{(r'-1)} b'_{k\psi^{r'-1}} > 0$. Thus, by (8.6), we have that $\lambda\pi_k^{(r'-1)} a_{k\psi^{r'-1}} > 0$ and, since $|a_{k\psi^{r'-1}}| = \infty$, we have that $z^{\lambda\pi_k^{(r'-1)} a_{k\psi^{r'-1}}} = z^{+\infty}$.

We only have to see that $a\varphi_1^{r'} = a$. If that is not the case, then there is some $q \in [m]$ such that $a_q \neq \pi_q^{(r')} a_{q\psi^{r'}}$. By Lemma 8.1.36, we have that $q \in X$, which implies that a_q is infinite and that $a_{q\psi^{r'}}$ is finite. It follows from (8.6) that $\pi_q^{(r')} b'_{q\psi^{r'}} > \delta'$, which is absurd since $\pi_q^{(r')} \leq \tau$ and, by (8.5), we have that $b'_{q\psi^{r'}} = a_{q\psi^{r'}} \leq \delta$.

Subcase 3.2: $|a_{k\psi^{r-1}}| < \infty$ By Lemma 8.1.36, we know that $k \in X$. Consider $\delta'' = 2|w| + \lambda\delta B_k|\tilde{z}|$ and $U'' = B\left((a,u); \frac{1}{2^{\delta''}}\right)$. As usual, take $r'' > m$ and $(b'', v'') \in U''$ such that $(b'', v'')\hat{\varphi}^{r''} \in U''$. We have that

$$\left| z^{+\infty} \wedge z^{\lambda \pi_k^{(r''-1)} b''_{k\psi^{r''-1}}} \right| > \delta''.$$

But that cannot happen since

$$\left| z^{\lambda \pi_k^{(r''-1)} b''_{k\psi^{r''-1}}} \right| = 2|w| + |\lambda\pi_k^{(r''-1)} b''_{k\psi^{r''-1}}||\tilde{z}| \leq \delta''.$$

□

Since $\widehat{\mathbb{Z}^m \times F_n}$ is compact, every ω-limit set is nonempty. Since such a set cannot contain wandering points, then, for every point, its limit set is a periodic orbit, which means that every point is asymptotically periodic.

We also remark that most of these results should be easily extended to the product of a free group with a finitely generated abelian group. So, consider $P = \mathbb{Z}_{p_1} \times \cdots \mathbb{Z}_{p_r}$ for some $r > 0$ and powers of (not necessarily distinct) prime numbers p_i, for $i \in [r]$, endowed with the product metric given by taking the discrete metric in each component. This is a complete space. Take $G = F_n \times \mathbb{Z}^m \times P$ and $\varphi \in \mathrm{End}(G)$. We have that a point of the form $(1, 0, x_1, \ldots, x_r)$ must be mapped to a point of the same form. Indeed, setting $(1, 0, x_1, \ldots, x_r)\varphi = (w, a, y_1, \ldots y_r)$, we have that, for $p = |P|$, $(1, 0, x_1, \ldots, x_r)\varphi = (1, 0, x_1, \ldots, x_r)^{p+1}\varphi$, thus $w = w^{p+1}$ and $a = (p+1)a$, so $w = 1$ and $a = 0$. It follows that $(u, a, p)\varphi = ((u, a)\psi_1, (u, a, p)\psi_2)$, for some $\psi_1 \in \mathrm{End}(F_n \times \mathbb{Z}^m)$ and $\psi_2 : F_n \times \mathbb{Z}^m \times P \to P$, i.e., the P-component has no influence in the $F_n \times \mathbb{Z}^m$-component of the image.

If we want φ to be uniformly continuous, we can see that φ must be given by $(u,a,p)\varphi = ((u,a)\psi_1, p\psi_2)$, for some $\psi_1 \in \mathrm{End}(F_n \times \mathbb{Z}^m)$ and $\psi_2 \in \mathrm{End}(P)$, i.e., the $F_n \times \mathbb{Z}^m$-component has no influence in the P-component of the image. Indeed, for every element x in the basis of F_n, setting $(x,0,0,\dots,0)\varphi = (w,a,n_1,\dots n_r)$, we have that every n_i must be equal to 0 because if that was not the case taking $\varepsilon < 1$, for every δ, choosing q such that $qp_1 \cdots p_r > \log_2(\frac{1}{\delta})$ we would have that

$$d((x,0,\dots 0)^{qp_1\cdots p_r}, (x,0,\dots 0)^{qp_1\cdots p_r+1}) < \delta$$

but

$$d((x,0,\dots 0)^{qp_1\cdots p_r}\varphi, (x,0,\dots 0)^{qp_1\cdots p_r+1}\varphi) = 1$$

and the same happens when we consider an element in the basis of $\mathbb{Z}^m$.

Now observe that every point in ψ_2 must be periodic, since P is finite. So, if the dichotomy periodic vs wandering holds for ψ_1 (in particular, if ψ_1 is type II), taking a nonwandering point $(w,a,p) \in G$, we have that (w,a) must be a nonwandering point of ψ_1, thus periodic, and p is periodic and so (w,a,p) is φ-periodic.

8.2 Free times free groups

We will now describe the endomorphisms of a direct product of two free groups of finite rank. We will assume that the ranks of both free groups are at least two since the cases $\mathbb{Z}^2$ and $\mathbb{Z} \times F_n$ are free-abelian and free-abelian times free, respectively, and so already known.

8.2.1 Endomorphisms and Automorphisms

In this subsection, we describe the endomorphisms and automorphisms of $F_n \times F_m$ and solve the Whitehead problems for $F_n \times F_m$.

Endomorphisms

Consider an endomorphism $\varphi : F_n \times F_m \to F_n \times F_m$ defined by $(a_i, 1) \mapsto (x_i, y_i)$ and $(1, b_j) \mapsto (z_j, w_j)$, for $i \in [n]$ and $j \in [m]$. We define $X = \{x_i \mid i \in [n]\}$, $Y = \{y_i \mid i \in [n]\}$, $Z = \{z_j \mid j \in [m]\}$ and $W = \{w_j \mid j \in [m]\}$. We say that these sets are *trivial* if they are singletons containing only the empty word and *nontrivial* otherwise.

For φ to be well defined, we must have that $x_i z_j = z_j x_i$ and $y_i w_j = w_j y_i$, for every $i \in [n]$ and $j \in [m]$. By [70, Proposition 1.3.2], we have that two words in a free group commute if and only if they are powers of the same word, thus, for every $(i,j) \in [n] \times [m]$, we have

$$x_i = 1 \,\vee\, z_j = 1 \,\vee\, (\exists\, u \in F_n \setminus \{1\}\, \exists\, p_i, r_j \in \mathbb{Z} \setminus \{0\} : x_i = u^{p_i} \wedge z_j = u^{r_j}) \tag{8.9}$$

and a similar condition holds for y_i and w_j:

$$y_i = 1 \,\vee\, w_j = 1 \,\vee\, (\exists\, v \in F_m \setminus \{1\}\, \exists\, q_i, s_j \in \mathbb{Z} \setminus \{0\} : y_i = v^{q_i} \wedge w_j = v^{s_j}) \tag{8.10}$$

Lemma 8.2.1. *Suppose X and Z are nontrivial. Then, there is some $1 \neq u \in F_n$ such that $X \cup Z \subseteq \{u^k \mid k \in \mathbb{Z}\}$. Similarly, if Y and W are nontrivial, then, there is some $1 \neq v \in F_n$ such that $Y \cup W \subseteq \{v^k \mid k \in \mathbb{Z}\}$.*

Proof. Just consider $\tilde{X} = X \setminus \{1\}$ and $\tilde{Z} = Z \setminus \{1\}$ which are both finite sets and apply (8.9) or (8.10) . ☐

So, we will consider several different cases:

(I) All sets X, Y, Z and W are nontrivial

(II) X is the only trivial set

(III) Y is the only trivial set

(IV) X and Y are the only trivial sets

(V) X and Z are the only trivial sets

(VI) Y and Z are trivial sets

(VII) X and W are trivial sets.

These cases are sufficient since every other case is analogous to one of the above, by swapping order of the factors (notice we are not assuming any relation between m and n). Indeed, if W is the only trivial set, we reduce to the second case; if Z is the only trivial set, we reduce to the third case; if Z and W are the only trivial sets, we reduce to the fourth case; if Y and W are the only trivial sets, we reduce to the fifth case. If three or more sets are trivial, then we must fall into one of the two last cases.

We define $P = \{p_i \in \mathbb{Z} \mid i \in [n]\}$, $Q = \{q_i \in \mathbb{Z} \mid i \in [n]\}$, $R = \{r_j \in \mathbb{Z} \mid j \in [m]\}$ and $S = \{s_j \in \mathbb{Z} \mid j \in [m]\}$, where p_i, q_i, r_j, s_j are the numbers from (8.9) and (8.10). So, matching the numeration above, φ must have one of the following forms:

(I) $(a_i, 1) \mapsto (u^{p_i}, v^{q_i})$ and $(1, b_j) \mapsto (u^{r_j}, v^{s_j})$, for some $1 \neq u \in F_n$, $1 \neq v \in F_m$ and integers $p_i, q_i, r_j, s_j \in \mathbb{Z}$ for $(i, j) \in [n] \times [m]$, such that $P, Q, R, S \neq \{0\}$.

(II) $(a_i, 1) \mapsto (1, v^{q_i})$ and $(1, b_j) \mapsto (z_j, v^{s_j})$, for some $1 \neq v \in F_m$, $z_j \in F_n$ and integers $q_i, s_j \in \mathbb{Z}$ for $(i, j) \in [n] \times [m]$, such that $Q, S \neq \{0\}$ and $Z \neq \{1\}$. We will denote by ϕ the homomorphism from F_m to F_n mapping b_j to z_j, $j \in [m]$.

(III) $(a_i, 1) \mapsto (u^{p_i}, 1)$ and $(1, b_j) \mapsto (u^{r_j}, w_j)$, for some $1 \neq u \in F_n$, $w_j \in F_n$ and integers $p_i, r_j \in \mathbb{Z}$ for $(i, j) \in [n] \times [m]$, such that $P, R \neq \{0\}$ and $W \neq \{1\}$. We will denote by ϕ the endomorphism of F_m mapping b_j to w_j, $j \in [m]$.

(IV) $(a_i, 1) \mapsto (1, 1)$ and $(1, b_j) \mapsto (z_j, w_j)$, for some $(z_j, w_j) \in F_n \times F_m$ such that $Z, W \neq \{1\}$. We will denote the component mappings $\phi : F_m \to F_n$ and $\psi \in \mathrm{End}(F_m)$, defined by $b_j \mapsto z_j$ and $b_j \mapsto w_j$, $j \in [m]$, respectively.

(V) $(a_i,1)\mapsto(1,v^{q_i})$ and $(1,b_j)\mapsto(1,v^{s_j})$, for some $1\neq v\in F_m$, and integers $q_i,s_j\in\mathbb{Z}$ for $(i,j)\in[n]\times[m]$, such that $Q,S\neq\{0\}$.

(VI) $(a_i,1)\mapsto(x_i,1)$ and $(1,b_j)\mapsto(1,w_j)$, for some $x_i\in F_n$ and $w_j\in F_n$ for $(i,j)\in[n]\times[m]$. We will denote the component mappings $\phi\in\mathrm{End}(F_n)$ and $\psi\in\mathrm{End}(F_m)$, defined by $a_i\mapsto x_i$, $i\in[n]$ and $b_j\mapsto w_j$, $j\in[m]$, respectively.

(VII) $(a_i,1)\mapsto(1,y_i)$ and $(1,b_j)\mapsto(z_j,1)$, for some $y_i\in F_m$ and $z_j\in F_n$ for $(i,j)\in[n]\times[m]$. We will denote the component mappings $\phi:F_n\to F_m$ and $\psi:F_m\to F_n$, defined by $a_i\mapsto y_i$, $i\in[n]$ and $b_j\mapsto z_j$, $j\in[m]$, respectively.

For $(i,j)\in[n]\times[m]$, we define $\lambda_i:F_n\to\mathbb{Z}$ as the endomorphism given by $a_k\mapsto\delta_{ik}$ and $\tau_j:F_m\to\mathbb{Z}$ given by $b_k\mapsto\delta_{jk}$, where δ_{ij} is the Kronecker symbol.

Automorphisms

We are now able to describe the automorphisms of $F_n\times F_m$.

Proposition 8.2.2. *An endomorphism $\varphi:F_n\times F_m$ is surjective if and only if it is of type VI or type VII such that the component mappings ϕ and ψ are surjective. An endomorphism $\varphi:F_n\times F_m$ is injective if and only if it is of type VI or type VII such that the component mappings ϕ and ψ are injective.*

Proof. It is easy to check that type I, II and III endomorphisms are neither injective nor surjective.

A type IV endomorphism φ is never surjective: we have that F_n is in the kernel of φ and there is no surjective homomorphism from F_m to $F_n\times F_m$: $F_n\times F_m$ surjects to $\mathbb{Z}^{m+n}$ (through abelianization) which has rank $n+m$. Also, it is never injective since for every $(x,y)\in F_n\times F_m$, we have that $(1,y)\varphi=(x,y)\varphi$.

Type V endomorphisms are clearly not surjective, since for every $1\neq u\in F_n$, we have that $(u,1)\notin\mathrm{Im}(\varphi)$. Also, $(a_1a_2,1)\varphi=(1,v^{q_1+q_2})=(a_2a_1,1)\varphi$, so it is not injective.

It is obvious that a type VI endomorphism is surjective (resp. injective) if and only if both ϕ and ψ are surjective (resp. injective) and the same holds for type VII endomorphisms. □

Corollary 8.2.3. *$\varphi\in\mathrm{End}(F_n\times F_m)$ is an automorphism if and only if one of the following holds:*

1. *φ is a type VI endomorphism such that both ϕ and ψ are automorphisms.*
2. *$n=m$ and φ is a type VII endomorphism such that both ϕ and ψ are automorphisms.*

Corollary 8.2.4. *Let $m,n\neq1$, $\theta_n\in\mathrm{Aut}(F_n\times F_n)$ be the involution defined by $(x,y)\mapsto(y,x)$. If $n\neq m$, then*

$$\mathrm{Aut}(F_n\times F_m)\cong\mathrm{Aut}(F_n)\times\mathrm{Aut}(F_m).$$

If $n=m$, then $Aut(F_n\times F_n)$ is the semidirect product of $Aut(F_n)\times Aut(F_n)$ and $\langle\theta_n\rangle$.

Proof. If $n \neq m$, the claim follows from Corollary 8.2.3.

Suppose now that $n = m$ and denote by $\mathrm{Aut}_6(F_n \times F_n)$ the subgroup of $\mathrm{Aut}(F_n \times F_n)$ whose elements are type VI automorphisms and by $\mathrm{Aut}_7(F_n \times F_n)$ the subset of type VII automorphisms. It is easy to check that $\mathrm{Aut}_6(F_n \times F_n) \cong \mathrm{Aut}(F_n) \times \mathrm{Aut}(F_n)$ is a normal subgroup of $\mathrm{Aut}(F_n \times F_n)$ of index 2, so, since θ_n has order 2, we have that $\mathrm{Aut}(F_n \times F_n)$ is the semidirect product of $\mathrm{Aut}(F_n) \times \mathrm{Aut}(F_n)$ and $\langle \theta_n \rangle$. $\square$

Corollary 8.2.5. $\mathrm{Aut}(F_n \times F_m)$ *is finitely presented.* $\square$

Corollary 8.2.6. $F_n \times F_m$ *is hopfian but not cohopfian.*

Proof. If $\varphi \in \mathrm{End}(F_n \times F_m)$ is type VII and surjective, then $m = n$ and the component mappings are surjective endomorphisms of F_n. Since F_n is hopfian, we have that the components are injective, so φ is injective. If φ is type VI, then the component mappings ϕ and ψ are surjective endomorphisms of F_n and F_m, respectively, thus injective and we have that φ is injective.

It is not cohopfian since neither F_n nor F_m are cohopfian. Indeed take ϕ and ψ injective and nonsurjective endomorphisms of F_n and F_m respectively and consider the type VI endomorphism $\varphi \in \mathrm{End}(F_n \times F_m)$ defined by taking ϕ and ψ as the component mappings. We have that φ is injective but not surjective. $\square$

Whitehead Problems

We denote by $\mathrm{Mon}(G)$ the monoid of monomorphisms (i.e., injective endomorphisms) of a group G.

In [34] and [35], the author generalizes the techniques from Whitehead and applies them to partially commutative groups, being able to solve $WhP_{\mathrm{Aut}}(G)$, when G is partially commutative, which obviously solves $WhP_{\mathrm{Aut}}(F_n \times F_m)$. Also, since it is possible to solve equations in such groups (see [39]), $WhP_{\mathrm{End}}(F_n \times F_m)$ can also be seen to be decidable. The author is not aware of any previous result implying the solution of $WhP_{\mathrm{Mon}}(F_n \times F_m)$. We will present an alternative way to solving the Whitehead problems using the classification of endomorphisms already achieved and the already known solution of the problems for free groups.

Theorem 8.2.7. *For $n \geq 2$:*

1. *([103]) $WhP_{\mathrm{Aut}}(F_n)$ is solvable*

2. *([31]) $WhP_{\mathrm{Mon}}(F_n)$ is solvable*

3. *([72]) $WhP_{\mathrm{End}}(F_n)$ is solvable*

Corollary 8.2.8. *Given $m, n \in \mathbb{N}$, free groups F_n, F_m and elements $u \in F_n$, $v \in F_m$, the problem of deciding whether there exists a homomorphism $\phi : F_n \to F_m$ such that $u\phi = v$ and in case it does, finding it, is decidable.*

Proof. If $n = m$, we have the Whitehead problem $WhP_{\text{End}}(F_n)$ for free groups, which is solvable.

If $n < m$, then consider $\psi : F_n \to F_m$ to be the natural embedding. Then, we can decide if we have an endomorphism $\varphi \in \text{End}(F_m)$ such that $u\psi\varphi = v$. If it exists, then we take $\phi = \psi\varphi$. If there is no such φ, then there is no $\phi : F_n \to F_m$ such that $u\phi = v$. Indeed, if there was some $\phi : F_n \to F_m$ such that $u\phi = v$, defining $\varphi \in \text{End}(F_m)$ to be the endomorphism that maps the first n letters in the basis of F_m through the endomorphism induced by ϕ and the $m - n$ extra elements in the basis of F_m to 1, we would have that $\phi = \psi\varphi$.

In case $n > m$, we proceed in a similar way, extending the codomain instead of the domain. So, take $\psi : F_m \to F_n$ to be the natural embedding. If there exists a homomorphism $\phi : F_n \to F_m$ such that $u\phi = v$, then $\phi\psi$ is an endomorphism of F_n such that $u\phi\psi = v\psi$. Now, suppose that there is an endomorphism $\varphi \in \text{End}(F_n)$ such that $u\varphi = v\psi$. Consider $\theta : F_n \to F_m$ such that $\psi\theta = id$. It follows that, putting $\phi = \varphi\theta$, we have that $u\phi = v$. □

Following the proof for free groups monomorphisms in [31] step by step, the same result follows when the free groups have different ranks.

Corollary 8.2.9. *Given $m, n \in \mathbb{N}$, free groups F_n, F_m and elements $u \in F_n$, $v \in F_m$, the problem of deciding whether there exists an injective homomorphism $\phi : F_n \to F_m$ mapping u to v is decidable.*

We remark that, given $u \in F_n$, we can compute $U = \{k \mid \exists \alpha \in F_n : u = \alpha^k\}$.

Proposition 8.2.10. *$WhP_{\text{Aut}}(F_n \times F_n)$, $WhP_{\text{Mon}}(F_n \times F_m)$ and $WhP_{\text{End}}(F_n \times F_m)$ are solvable.*

Proof. Let $(x, y), (z, w) \in F_n \times F_m$. We want to check if there is an endomorphism (resp. monomorphism, automorphism) φ such that $(x, y)\varphi = (z, w)$.

$WhP_{\text{Aut}}(F_n \times F_m)$ follows directly from Corollary 8.2.3 and condition 1 in Theorem 8.2.7.

For $WhP_{\text{Mon}}(F_n \times F_m)$, we decide if there is a monomorphism of type VI and if not, we check if there is a monomorphism of type VII. For type VI endomorphisms, it follows from Proposition 8.2.2 and condition 2 in Theorem 8.2.7. For type VII monomorphisms, we use Proposition 8.2.2 and Corollary 8.2.9.

For $WhP_{\text{End}}(F_n \times F_m)$, we will check if there is a type I endomorphism such that $(x, y)\varphi = (z, w)$. If there is, we stop. If not, we check the existence of a type II endomorphism. If there is one, we stop. If not, we check the existence of endomorphisms of a type III endomorphism, and so on.

So, we want to check if, given $(x, y), (z, w) \in F_n \times F_m$, there is a type I endomorphism φ such that $(x, y)\varphi = (z, w)$, i.e., if there are $u \in F_n$, $v \in F_m$, $p_i \in \mathbb{Z}$, $q_i \in \mathbb{Z}$, $r_j \in \mathbb{Z}$ and $s_j \in \mathbb{Z}$, for $(i, j) \in [n] \times [m]$ such that

$$\begin{cases} z = u^{\sum\limits_{i\in[n]} \lambda_i(x)p_i + \sum\limits_{j\in[m]} \tau_j(y)r_j} \\ w = v^{\sum\limits_{i\in[n]} \lambda_i(x)q_i + \sum\limits_{j\in[m]} \tau_j(y)s_j} \end{cases}$$

We compute the values of $\lambda_i(x)$, $\tau_j(y)$ for $(i,j) \in [n] \times [m]$, $Z = \{k \mid \exists \alpha \in F_n : z = \alpha^k\}$ and $W = \{k \mid \exists \alpha \in F_m : w = \alpha^k\}$. Then for every $(k,\ell) \in Z \times W$ we see if the linear Diophantine system

$$\begin{cases} k = \sum\limits_{i\in[n]} \lambda_i(x)p_i + \sum\limits_{j\in[m]} \tau_j(y)r_j \\ \ell = \sum\limits_{i\in[n]} \lambda_i(x)q_i + \sum\limits_{j\in[m]} \tau_j(y)s_j \end{cases}$$

has a solution on p_i, q_i, r_j, s_j. If it does, then there is an endomorphism given by the solution of the system together with u, v such that $u^k = z$ and $v^\ell = w$. If not, there is no endomorphism and we check the existence of a type II endomorphism.

To check the existence of such a type II endomorphism, we use Corollary 8.2.8 to check if there is an endomorphism $\phi : F_m \to F_n$ such that $z = y\phi$. If there is not, then we stop and see if there is a type III endomorphism. If there is, we compute the values of $\lambda_i(x)$, $\tau_j(y)$ for $(i,j) \in [n] \times [m]$ and $W = \{k \mid \exists \alpha \in F_m : w = \alpha^k\}$. Then for every $k \in W$, we see if the linear Diophantine equation

$$k = \sum_{i\in[n]} \lambda_i(x)q_i + \sum_{j\in[m]} \tau_j(y)s_j$$

has a solution on q_i, s_j. If it does, then there is an endomorphism defined by

$$(\alpha, \beta) \mapsto \left(\beta\phi, v^{\sum\limits_{i\in[n]} \lambda_i(\alpha)q_i + \sum\limits_{j\in[m]} \tau_j(\beta)s_j}\right),$$

for $v \in F_m$ such that $w = v^k$. If not, there is no endomorphism.

The type III case is entirely analogous to the type II.

Type IV reduces to Corollary 8.2.8 and condition 3 in Theorem 8.2.7.

Type V follows from decidability of the word problem in F_n and the same argument as above for the second component.

Type VI follows from condition 3 in Theorem 8.2.7 for F_n and F_m.

Type VII follows from Corollary 8.2.8. □

Remark 8.2.11. *In the multiple Whitehead problem, we are given two k-tuples of elements of $F_n \times F_m$, $(g_1, \ldots, g_k)$ and $(h_1, \ldots, h_k)$, and we want to decide whether there exists an endomorphism (or monomorphism or automorphism) that maps the g_i to h_i, for all $i \in [k]$. We remark that, proceeding as above and using the corresponding known results for free groups, the multiple Whitehead problems for endomorphisms, monomorphisms, and automorphisms are also decidable in $F_n \times F_m$.*

8.2.2 Fixed subgroup of an endomorphism

In this subsection, we aim at giving conditions for the fixed subgroup of an endomorphism to be finitely generated. We will deal with each type of endomorphisms one by one. We now state the main result from this subsection.

Given a Diophantine equation $\sum_{i=1}^{k} a_i x_i = 0$, we denote its solution set (as a subset of $\mathbb{Z}^k$) by $\mathrm{Sol}(\sum_{i=1}^{k} a_i x_i)$. Also, we denote the abelianization map by $\rho : F_m \to \mathbb{Z}^m$.

Theorem 8.2.12. *Let $\varphi \in \mathrm{End}(F_n \times F_m)$. Then, $\mathrm{Fix}(\varphi)$ is finitely generated if and only if one of the following holds:*

1. *φ is a type I, II, IV, V, VI or VII endomorphism*
2. *φ is a type III endomorphism such that $\sum\limits_{i\in[n]} \lambda_i(u)p_i \neq 1$*
3. *φ is a type III endomorphism such that $\sum\limits_{i\in[n]} \lambda_i(u)p_i = 1$ and*
$$\mathrm{rk}\left((\mathrm{Fix}(\phi))\rho \cap \mathrm{Sol}\left(\sum_{j\in[m]} r_j x_j = 0\right)\right) = \mathrm{rk}((\mathrm{Fix}(\phi))\rho),$$
4. *φ is a type III endomorphism such that $\sum\limits_{i\in[n]} \lambda_i(u)p_i = 1$ and $\mathrm{Fix}(\phi) = 1$*
5. *φ is a type III endomorphism such that $\sum\limits_{i\in[n]} \lambda_i(u)p_i = 1$, $\mathrm{Fix}(\phi)$ is cyclic, $(\mathrm{Fix}(\varphi))\rho \neq \mathbf{0}$ and $(\mathrm{Fix}(\phi))\rho \cap \mathrm{Sol}\left(\sum\limits_{j\in[m]} r_j x_j = 0\right) = \mathbf{0}$*

Type I endomorphisms

We will study the fixed point subgroup of such an endomorphism. To start, we obviously have that a fixed point $(u, v) \in F_n \times F_m$ must be of the form $(u^a, v^b) \in F_n \times F_m$ for some $a, b \in \mathbb{Z}$. For $i \in [n]$, we have that φ is defined by

$$(x, y) \mapsto \left(u^{\sum\limits_{i\in[n]} \lambda_i(x)p_i + \sum\limits_{j\in[m]} \tau_j(y)r_j}, v^{\sum\limits_{i\in[n]} \lambda_i(x)q_i + \sum\limits_{j\in[m]} \tau_j(y)s_j} \right).$$

For $x \in F_n, y \in F_m$, we denote $\sum\limits_{i\in[n]} \lambda_i(x)p_i$ by x^P; $\sum\limits_{j\in[m]} \tau_j(y)r_j$ by y^R; $\sum\limits_{i\in[n]} \lambda_i(x)q_i$ by x^Q and $\sum\limits_{j\in[m]} \tau_j(y)s_j$ by y^S. We will keep this notation throughout this section.

So

$$(u^a, v^b) \in \mathrm{Fix}(\varphi) \iff \begin{cases} a = \sum\limits_{i\in[n]} \lambda_i(u^a)p_i + \sum\limits_{j\in[m]} \tau_j(v^b)r_j \\ b = \sum\limits_{i\in[n]} \lambda_i(u^a)q_i + \sum\limits_{j\in[m]} \tau_j(v^b)s_j \end{cases}$$

$$\iff \begin{cases} a = au^P + bv^R \\ b = au^Q + bv^S \end{cases}$$

$$\iff \begin{cases} a\left(-1 + u^P\right) + bv^R = 0 \\ au^Q + b\left(-1 + v^S\right) = 0 \end{cases}$$

So, consider the matrix given by

$$M_\varphi = \begin{bmatrix} -1 + u^P & v^R \\ u^Q & -1 + v^S \end{bmatrix}$$

and we have that $\mathrm{Fix}(\varphi) = \{(u^a, v^b) \in F_n \times F_m \mid (a, b) \in \mathrm{Ker}(M_\varphi)\}$. It is in fact isomorphic to $\mathrm{Ker}(M_\varphi)$ as a subgroup of the free-abelian group $\mathbb{Z}^2$, which is finitely generated and has a computable basis. In particular, if $\det(M_\varphi) \neq 0$, we have that the fixed subgroup of φ is trivial.

Type II endomorphisms

Consider a homomorphism $\phi : F_m \to F_n$ given by $b_j \mapsto z_j$, $j \in [m]$. Then, we have that φ is defined by

$$(x, y) \mapsto \left(y\phi, v^{\sum\limits_{i\in[n]} \lambda_i(x)q_i + \sum\limits_{j\in[m]} \tau_j(y)s_j} \right).$$

So a fixed point must be of the form $(v^b\phi, v^b)$, for some $b \in \mathbb{Z}$. We have that

$$(v^b\phi, v^b) \in \mathrm{Fix}(\varphi) \iff b = \sum_{i\in[n]} \lambda_i(v^b\phi)q_i + \sum_{j\in[m]} \tau_j(v^b)s_j$$
$$\iff b\left(-1 + (v\phi)^Q + v^S\right) = 0$$

So, if $(v\phi)^Q + v^S \neq 1$, we have that $\mathrm{Fix}(\varphi)$ is trivial and otherwise, $\mathrm{Fix}(\varphi)$ is given by $\{(v^b\phi, v^b) \in F_n \times F_m \mid b \in \mathbb{Z}\} \cong \mathbb{Z}$.

Type III endomorphisms

Consider $\phi \in \mathrm{End}(F_m)$ given by $b_j \mapsto w_j$, $j \in [m]$. Then, we have that φ is defined by

$$(x, y) \mapsto \left(u^{\sum\limits_{i\in[n]} \lambda_i(x)p_i + \sum\limits_{j\in[m]} \tau_j(y)r_j}, y\phi \right).$$

So a fixed point must be of the form $(u^a, y) \in F_n \times F_m$, for some $a \in \mathbb{Z}$. We have that

$$\begin{aligned}(u^a, y) \in \mathrm{Fix}(\varphi) &\iff a = \sum_{i\in[n]} \lambda_i(u^a)p_i + \sum_{j\in[m]} \tau_j(y)r_j \quad \wedge \quad y \in \mathrm{Fix}(\phi)\\ &\iff a\left(-1+u^P\right) + y^R = 0 \quad \wedge \quad y \in \mathrm{Fix}(\phi)\end{aligned}$$

So, if $\sum_{i\in[n]} \lambda_i(u)p_i \neq 1$, then putting

$$G = \left\{ y \in \mathrm{Fix}(\phi) \;\middle|\; \left(1-u^P\right) \text{ divides } y^R \right\} \leq \mathrm{Fix}(\phi)$$

we get that

$$\mathrm{Fix}(\varphi) = \left\{ \left(u^{\frac{y^R}{1-u^P}}, y \right) \mid y \in G \right\} \cong G.$$

We now prove that G is always finitely generated. Set

$$H = \left\{ y \in F_m \;\middle|\; \left(1-u^P\right) \text{ divides } y^R \right\} \leq F_m.$$

We have that $G = \mathrm{Fix}(\phi) \cap H$ and $\mathrm{Fix}(\phi)$ is finitely generated.

The subgroup H has index $|1-u^P|$, and so it is finitely generated. Indeed, it follows from the fact that τ_j is a homomorphism for every $j \in [m]$, that

$$(xy)^R = \sum_{j=1}^m \tau_j(xy)r_j = \sum_{j=1}^m \tau_j(x)r_j + \sum_{j=1}^m \tau_j(y)r_j = x^R + y^R$$

holds for all $x, y \in F_m$ and so $xH = \{z \in F_m \mid x \equiv_{1-u^P} z\}$. Since free groups are Howson, we have that G is finitely generated.

If, on the other hand, $\sum_{i\in[n]} \lambda_i(u)p_i = 1$, putting

$$H = \left\{ y \in \mathrm{Fix}(\phi) \mid \sum_{j\in[m]} \tau_j(y)r_j = 0 \right\} \leq \mathrm{Fix}(\phi),$$

we have that

$$\mathrm{Fix}(\varphi) = \{(u^a, y) \mid y \in H, a \in \mathbb{Z}\} \cong \mathbb{Z} \times H$$

and it is finitely generated if and only if H is finitely generated. Notice this might not be the case. Indeed, if $m = 2$, $r_1 = 1$, $r_2 = -1$ and $\phi = Id$, we have that

$$H = \left\{ y \in \mathrm{Fix}(\phi) \mid \sum_{j\in[m]} \tau_j(y)r_j = 0 \right\} = \{y \in F_2 \mid \tau_1(y) = \tau_2(y)\}.$$

The language it defines is not rational, so, by Theorem 2.3.1, H is not finitely generated.

Observe that, since $\tau_j(uvu^{-1}) = \tau_j(v)$ for all $u, v \in F_m$, $j \in [m]$, then H is a normal subgroup of $\mathrm{Fix}(\phi)$ (not of F_n) and so it is finitely generated if and only if it is trivial or has finite index in $\mathrm{Fix}(\phi)$. Consider the restriction $\rho' : \mathrm{Fix}(\phi) \to (\mathrm{Fix}(\phi))\rho$ of the abelianization map ρ. Since $x\rho = (\tau_1(x), \dots, \tau_m(x))$, we have that

$$H\rho' = (\mathrm{Fix}(\phi))\rho' \cap \mathrm{Sol}\left(\sum_{j\in[m]} r_j x_j = 0\right).$$

Also, $H = H\rho'\rho'^{-1}$: the fact that $H \subseteq H\rho'\rho'^{-1}$ is obvious and we have that $H\rho'\rho'^{-1} \subseteq H$ because $\mathrm{Ker}(\rho') \subseteq H$. By [36, Lemma 3.2 (ii)], we have that $H = H\rho'\rho'^{-1}$ has finite index in $\mathrm{Fix}(\phi)$ if and only if $H\rho'$ has finite index in $(\mathrm{Fix}(\phi))\rho'$, i.e. if and only if condition 3 holds:

$$\mathrm{rk}\left((\mathrm{Fix}(\phi))\rho \cap \mathrm{Sol}\left(\sum_{j\in[m]} r_j x_j = 0\right)\right) = \mathrm{rk}((\mathrm{Fix}(\phi))\rho).$$

Now, we will describe the cases where H is trivial. Since $\mathrm{Ker}(\rho') \subseteq H$, if H is trivial then ρ' is injective. If $\mathrm{rk}(\mathrm{Fix}(\phi)) \geq 2$, then ρ' cannot be injective since the commutator of two free generators is mapped to 1. Thus, ρ' is injective if and only if $\mathrm{Fix}(\phi)$ is trivial or $\mathrm{Fix}(\phi)$ is cyclic not abelianizing to zero.

Also, if $\mathrm{Fix}(\phi)$ is trivial (condition 4), then obviously H is trivial and if $\mathrm{Fix}(\phi)$ is cyclic not abelianizing to zero, then H is trivial if and only if $H\rho'$ is trivial (condition 5).

Although the set of reduced words in H might not be a rational language, we can prove that it is always a context-free language constructing a pushdown automaton recognizing $H' = \left\{y \in F_m \mid \sum_{j\in[m]} \tau_j(y) r_j = 0\right\}$. Since the words in $\mathrm{Fix}(\phi)$ form a rational language, and $H = H' \cap \mathrm{Fix}(\phi)$, it follows that H is context-free, since context-free languages are closed under intersection with rational languages. To do so, consider $\Gamma = \{z, X, X^{-1}\}$ to be the stack alphabet and z to be the starting symbol. When b_i is read, if r_i is positive, then if the top symbol of the stack has X or z, we add X^{r_i} to the stack. If the stack has at least r_i X^{-1}'s on the top, we remove them and if there are $k < r_i$ symbols X^{-1} at the top, we remove them and add X^{r_i-k}. If r_i is negative, we do as above switching X and X^{-1}. Whenever b_i^{-1} is read, we do the same switching X and X^{-1}. If we read the starting symbol and have nothing to add, we reach a final state. This way, only a finite amount of memory is required, so H' is context-free. By Lemma 2.3.7, we have that $H \in CF(F_m)$.

Type IV endomorphisms

Consider $\phi : F_m \to F_n$ defined by $b_j \mapsto z_j$, $j \in [m]$ and $\psi \in \mathrm{End}(F_m)$ given by $b_j \mapsto w_j$, $j \in [m]$. Then $(x, y)\varphi = (y\phi, y\psi)$. In particular, $\mathrm{Fix}(\varphi) = \{(y\phi, y) \in F_n \times F_m \mid y \in \mathrm{Fix}(\psi)\}$ which is finitely generated. In fact, it is isomorphic to $\mathrm{Fix}(\psi)$ which is finitely generated and from Theorem 2.2.3, it has a computable basis.

Type V endomorphisms

In this case, we have that φ is defined by

$$(x,y) \mapsto \left(1, v^{\sum\limits_{i\in[n]} \lambda_i(x)q_i + \sum\limits_{j\in[m]} \tau_j(y)s_j}\right).$$

In particular, we get that a fixed point must be of the form $(1, v^b)$, for some $b \in \mathbb{Z}$. We have that

$$(1, v^b) \in \mathrm{Fix}(\varphi) \iff b = b \sum_{j\in[m]} \tau_j(v)s_j.$$

It follows that, if $v^S = 1$, then $\mathrm{Fix}(\varphi) \cong \mathbb{Z}$, since it is given by $\{(1, v^b) \mid b \in \mathbb{Z}\}$. If $v^S \neq 1$, then $\mathrm{Fix}(\varphi)$ is trivial.

Type VI endomorphisms

This case reduces to the free group case, since we have that φ is given by $(x,y) \mapsto (x\phi, y\psi)$, for some $\phi \in \mathrm{End}(F_n)$ and $\psi \in \mathrm{End}(F_m)$. It follows that $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\phi) \times \mathrm{Fix}(\psi)$, and so it is finitely generated and, from Theorem 2.2.3, it has a computable basis.

Type VII endomorphisms

This case is also similar to the previous, since we have that φ is given by $(x,y) \mapsto (y\psi, x\phi)$, for $\phi : F_n \to F_m$ defined by $a_i \mapsto y_i$, $i \in [n]$ and $\psi : F_m \to F_n$ defined by $b_j \mapsto z_j$, $j \in [m]$. So a fixed point must be (x,y) such that $x = y\psi$ and $y = x\phi$ and so $x = x\phi\psi$ and $y = y\psi\phi$. So, $\mathrm{Fix}(\varphi) \subseteq \mathrm{Fix}(\phi\psi) \times \mathrm{Fix}(\psi\phi)$. Also, for $x \in \mathrm{Fix}(\phi\psi)$, we have that $(x, x\phi)\varphi = (x\phi\psi, x\phi) = (x, x\phi)$. Similarly, we have that $(y\psi, y)\varphi = (y\psi, y)$ for $y \in \mathrm{Fix}(\psi\phi)$. So,

$$\mathrm{Fix}(\varphi) = \{(x, x\phi) \mid x \in \mathrm{Fix}(\phi\psi)\} = \{(y\psi, y) \mid y \in \mathrm{Fix}(\psi\phi)\} \cong \mathrm{Fix}(\phi\psi) \cong \mathrm{Fix}(\psi\phi)$$

and thus it is finitely generated and, from Theorem 2.2.3, it has a computable basis.

We remark that this also yields a somewhat trivial proof that, for homomorphisms $\phi : F_n \to F_m$ and $\psi : F_m \to F_n$, the restrictions of ϕ and ψ are explicit (mutually inverse) isomorphisms between $\mathrm{Fix}(\phi\psi)$ and $\mathrm{Fix}(\psi\phi)$.

Since all the calculations above are explicit, we obtain the following algorithmic corollary.

Corollary 8.2.13. *There is an algorithm which decides whether the fixed subgroup of a given endomorphism $\varphi \in \mathrm{End}(F_n \times F_m)$ is finitely generated, and computes a set of generators (recursively, in the infinite case).*

In the case of free groups, all endomorphisms have a finitely generated fixed subgroup, where in the case of free-abelian times free groups not even all automorphisms have a finitely generated fixed subgroup (see [36, Example 6.4]). In this class of groups, we have a situation in between these two situations.

Corollary 8.2.14. *Let $\varphi \in \mathrm{Aut}(F_n \times F_m)$. Then,* $\mathrm{Fix}(\varphi)$ *is finitely generated.*

Also, since the proof above is constructive, we can describe finitely generated fixed subgroups of endomorphisms of $F_n \times F_m$.

Corollary 8.2.15. *Let $\varphi \in \mathrm{End}(F_n \times F_m)$. Then:*

1. *if* $\mathrm{Fix}(\varphi)$ *is finitely generated, then* $\mathrm{Fix}(\varphi)$ *is a direct product of free groups of finite rank;*
2. *if* $\mathrm{Fix}(\varphi)$ *is not finitely generated, then there is some $K \in CF(F_m)$ such that* $\mathrm{Fix}(\varphi) = \langle u \rangle \times K$*, where u is the word given by the (type III) endomorphism φ.*

Proof. If φ is a type I endomorphism, then the fixed subgroup must be a subgroup of $\mathbb{Z}^2$ and so it must be trivial, infinite cyclic, or isomorphic to $\mathbb{Z}^2$. For type II endomorphisms, $\mathrm{Fix}(\varphi)$ must be trivial or isomorphic to $\mathbb{Z}$. For type III endomorphisms such that $\sum\limits_{i\in[n]} \lambda_i(u)p_i \neq 1$, $\mathrm{Fix}(\varphi)$ is a finitely generated (in fact, finite index) subgroup of F_m and so it is a finitely generated free group. If $\sum\limits_{i\in[n]} \lambda_i(u)p_i = 1$, then it is isomorphic to $\mathbb{Z} \times H$, where H is either trivial, a subgroup of a free group (hence free) or context-free. For type IV endomorphisms, $\mathrm{Fix}(\varphi)$ is isomorphic to the fixed subgroup of an endomorphism of F_m. For type V endomorphisms, $\mathrm{Fix}(\varphi)$ must be either isomorphic to $\mathbb{Z}$ or trivial. For type VI endomorphisms, it must be a direct product of fixed subgroups of endomorphisms of free groups and finally, for type VII, it is isomorphic to a fixed subgroup of an endomorphism of F_m. □

8.2.3 Periodic points

The purpose of this subsection is to describe the cases where $\mathrm{Per}(\varphi)$ is finitely generated. We will proceed as in the previous subsection, dealing with each type of endomorphisms one by one. We start by recalling that, in the case of free groups, periodic points have their period bounded by a computable constant depending only on the rank of the free group (see Corollary 4.4.2). We will denote by P_n the constant that bounds the periods for endomorphisms of F_n.

We now state the main result of this subsection.

Theorem 8.2.16. *Let $\varphi \in \mathrm{End}(F_n \times F_m)$. Then,* $\mathrm{Per}(\varphi)$ *is finitely generated if and only if one of the following holds:*

1. *φ is a type I, II, IV, V, VI or VII endomorphism*
2. *φ is a type III endomorphism such that* $\left|\sum\limits_{i\in[n]} \lambda_i(u)p_i\right| \neq 1$
3. *φ is a type III endomorphism such that* $\left|\sum\limits_{i\in[n]} \lambda_i(u)p_i\right| = 1$ *and* $\mathrm{Fix}(\varphi^{(2P_m)!})$ *is finitely generated.*

Type I endomorphisms

Consider a type I endomorphism given by

$$(x,y)\mapsto\left(u^{\sum\limits_{i\in[n]}\lambda_i(x)p_i+\sum\limits_{j\in[m]}\tau_j(y)r_j},v^{\sum\limits_{i\in[n]}\lambda_i(x)q_i+\sum\limits_{j\in[m]}\tau_j(y)s_j}\right).$$

We start to determine the orbit of a point $(x,y)\in F_n\times F_m$. Define sequences in $\mathbb{Z}$ by $a_1(x,y)=x^P+y^R$, $b_1(x,y)=x^Q+y^S$ and $a_{n+1}(x,y)=a_n(x,y)u^P+b_n(x,y)v^R$ and $b_{n+1}(x,y)=a_n(x,y)u^Q+b_n(x,y)v^S$. We have that, for every $k>0$, $(x,y)\varphi^k=(u^{a_k(x,y)},v^{b_k(x,y)})$. Indeed, for $k=1$, we have that $(x,y)\varphi=(u^{a_1(x,y)},v^{b_1(x,y)})$. Assume that $(x,y)\varphi^r=(u^{a_r(x,y)},v^{b_r(x,y)})$, for every $r\leq k$. We have that

$$\begin{aligned}(x,y)\varphi^{k+1}&=(x,y)\varphi^k\varphi=(u^{a_k(x,y)},v^{b_k(x,y)})\varphi\\&=(u^{a_k(x,y)u^P+b_k(x,y)v^R},v^{a_k(x,y)u^Q+b_k(x,y)v^S})\\&=(u^{a_{k+1}(x,y)},v^{b_{k+1}(x,y)}).\end{aligned}$$

Clearly, a periodic point must be of the form $(x,y)=(u^a,v^b)$. In this case, $x^P=au^P$, $y^R=bv^R$, $x^Q=au^Q$ and $y^S=bv^S$, so putting $a_0=a$, $b_0=b$, $a_{n+1}=a_nu^P+b_nv^R$ and $b_{n+1}=a_nu^Q+b_nv^S$, we have that $(u^a,v^b)\varphi^k=(u^{a_k},v^{b_k})$. Defining the matrix

$$M_\varphi=\begin{bmatrix}u^P & v^R\\ u^Q & v^S\end{bmatrix}$$

and denoting also by M_φ the endomorphism of $\mathbb{Z}^2$ defined by the matrix, we have that

$$\mathrm{Per}(\varphi)=\{(u^a,v^b)\in F_n\times F_m\mid(a,b)\in\mathrm{Per}(M_\varphi)\}\cong\mathrm{Per}(M_\varphi),$$

which is finitely generated, as every subgroup of $\mathbb{Z}^2$.

Type II endomorphisms

Consider a type II endomorphism defined by

$$(x,y)\mapsto\left(y\phi,v^{\sum\limits_{i\in[n]}\lambda_i(x)q_i+\sum\limits_{j\in[m]}\tau_j(y)s_j}\right).$$

We want to study the orbit of a point $(x,y)\in F_n\times F_m$. We define a sequence of integers $(a_n)_n$ by $a_1(x,y)=x^Q+y^S$, $a_2(x,y)=a_1(x,y)v^S+(y\phi)^Q$ and for $n>2$, $a_n(x,y)=a_{n-1}(x,y)v^S+a_{n-2}(x,y)(v\phi)^Q$. We will now prove that, for $k\geq 2$, we have that $(x,y)\varphi^k=(v^{a_{k-1}(x,y)}\phi,v^{a_k(x,y)})$. For $k=2$, we have that

$$(x,y)\varphi^2=(y\phi,v^{x^Q+y^S})\varphi=(v^{a_1(x,y)}\phi,v^{(y\phi)^Q+a_1(x,y)v^S})=(v^{a_1(x,y)}\phi,v^{a_2(x,y)}).$$

Now, assume the statement holds for every $r \leq k$. We have that

$$\begin{aligned}(x,y)\varphi^{k+1} &= (v^{a_{k-1}(x,y)}\phi, v^{a_k(x,y)})\varphi = (v^{a_k(x,y)}\phi, v^{a_{k-1}(x,y)(v\phi)^Q + a_k(x,y)v^S})\\ &= (v^{a_k(x,y)}\phi, v^{a_{k+1}(x,y)}).\end{aligned}$$

So, a periodic point has the form $(v^b\phi, v^a)$, for $a, b \in \mathbb{Z}$.

Suppose that $v\phi \neq 1$ and consider

$$H = \{(b,a) \in \mathbb{Z}^2 \mid (v^b\phi, v^a) \in \operatorname{Per}(\varphi)\}.$$

Clearly, H is a subgroup of $\mathbb{Z}^2$, thus finitely generated and $\operatorname{Per}(\varphi) \cong H$.

Now, if $v\phi = 1$, then a periodic point must be of the form $(x,y) = (1, v^a)$ which is isomorphic to a subgroup of $\mathbb{Z}$, hence finitely generated.

Type III endomorphisms

As in the fixed point case, the subgroup of periodic points of a type III endomorphism is more complex than the subgroup of periodic points of endomorphisms of other types. In particular, it might not be finitely generated, as seen in the fixed point case.

Clearly, a periodic point must be of the form $(x,y) = (u^a, w) \in F_n \times F_m$. We will prove by induction on k that

$$(u^a, w)\varphi^k = \left(u^{a(u^P)^k + \sum\limits_{t=0}^{k-1} (w\phi^t)^R (u^P)^{k-t-1}}, w\phi^k\right). \tag{8.11}$$

For $k = 1$, we have that $(u^a, w)\varphi = (u^{au^P + w^R}, w\phi)$. Now, assume the statement holds for every $r \leq k$. We have that

$$\begin{aligned}(u^a, w)\varphi^{k+1} =& (u^a, w)\varphi^k\varphi = \left(u^{a(u^P)^k + \sum\limits_{t=0}^{k-1} (w\phi^t)^R (u^P)^{k-t-1}}, w\phi^k\right)\varphi\\ =& \left(u^{\left(a(u^P)^k + \sum\limits_{t=0}^{k-1} (w\phi^t)^R (u^P)^{k-t-1}\right)u^P + (w\phi^k)^R}, w\phi^{k+1}\right)\\ =& \left(u^{a(u^P)^{k+1} + \sum\limits_{t=0}^{k} (w\phi^t)^R (u^P)^{k-t}}, w\phi^{k+1}\right).\end{aligned}$$

If $u^P = 1$, then put

$$H = \left\{ y \in \operatorname{Per}(\phi) \mid \exists s > 0 : \sum_{t=0}^{s\pi_y - 1} (y\phi^t)^R = 0 \right\},$$

where π_y denotes the period of y. Observe that for $s>0$, $y\in \mathrm{Per}(\phi)$, we have that $\sum\limits_{t=0}^{s\pi_y-1}(y\phi^t)^R = s\sum\limits_{t=0}^{\pi_y-1}(y\phi^t)^R$, so

$$H=\left\{y\in \mathrm{Per}(\phi)\mid \sum_{t=0}^{\pi_y-1}(y\phi^t)^R=0\right\}.$$

We have that $\mathrm{Per}(\varphi)=\{(u^a,w)\mid w\in H, a\in\mathbb{Z}\}\cong \mathbb{Z}\times H$, and we know that H might not be finitely generated, as seen in the fixed points case.

Moreover, it follows that if (u^a,y) is periodic, then its period is equal to the period of y, which is bounded by P_m, and so, in this case, $\mathrm{Per}(\varphi)=\mathrm{Fix}(\varphi^{P_m!})=\mathrm{Fix}(\varphi^{(2P_m)!})$.

If $u^P=-1$, we will consider two cases. Let $y\in\mathrm{Per}(\phi)$ with even period π_y and take $a\in\mathbb{Z}$ and $s>0$. We have that

$$(u^a,y)\varphi^{s\pi_y}=\left(u^{a+\sum\limits_{t=0}^{s\pi_y-1}(y\phi^t)^R(-1)^{t+1}},y\right).$$

So, $(u^a,y)\in\mathrm{Per}(\varphi)$ if and only if there is $s>0$, such that

$$\sum_{t=0}^{s\pi_y-1}(y\phi^t)^R(-1)^{t+1}=0.$$

Since,

$$\sum_{t=0}^{s\pi_y-1}(y\phi^t)^R(-1)^{t+1}=s\sum_{t=0}^{\pi_y-1}(y\phi^t)^R(-1)^{t+1},$$

then putting

$$H=\left\{y\in\mathrm{Per}(\phi)\mid \pi_y\equiv_2 0\wedge \sum_{t=0}^{\pi_y-1}(-1)^{t+1}(y\phi^t)^R=0\right\},$$

we have that

$$(u^a,y)\in\mathrm{Per}(\varphi)\iff y\in H.$$

It follows that, in this case, the period of (u^a,y) is the same as the period of y, which is bounded by P_m, and so, in this case, $\mathrm{Per}(\varphi)=\mathrm{Fix}(\varphi^{P_m!})=\mathrm{Fix}(\varphi^{(2P_m)!})$.

Now, let $y\in\mathrm{Per}(\phi)$ with odd period π_y and take $a\in\mathbb{Z}$ and $s>0$. Then

$$(u^a,y)\varphi^{2\pi_y}=\left(u^{a+\sum\limits_{t=0}^{2\pi_y-1}(-1)^{t+1}(y\phi^t)^R},y\right)=(u^a,y),$$

and the same equality is not true with any other positive exponent smaller than $2\pi_y$.

So, $\text{Per}(\varphi) = \{(u^a, y) \in F_n \times \text{Per}(\phi) \mid y \in H \vee \pi_y \equiv_2 1\}$. Notice that, if every periodic point of ϕ has odd period, then $\text{Per}(\varphi)$ is finitely generated, since in that case, $\text{Per}(\varphi) \cong \mathbb{Z} \times \text{Per}(\phi)$. However, it might be the case where $\text{Per}(\varphi)$ is not finitely generated. For example, let $m = 2$ and $\phi \in \text{End}(F_2)$ defined by $a \mapsto b$ and $b \mapsto a$. Then $\text{Per}(\phi) = F_m$ and every point has even period. If $r_1 = 1$ and $r_2 = -1$, then

$$\begin{aligned} H &= \{y \in F_2 \mid -y^R + (y\phi)^R = 0\} \\ &= \{y \in F_2 \mid -\lambda_1(y) + \lambda_2(y) + \lambda_2(y) - \lambda_1(y) = 0\} \\ &= \{y \in F_2 \mid \lambda_1(y) - \lambda_2(y) = 0\} \end{aligned}$$

which is not finitely generated.

In this case, if (u^a, y) is periodic, then its period is at most $2\pi_y$, which is bounded by $2P_m$, and so, in this case, $\text{Per}(\varphi) = \text{Fix}(\varphi^{(2P_m)!})$.

In case $|u^P| \neq 1$, we will show that $\text{Per}(\varphi)$ is always finitely generated. To do so, we start by proving that if φ is a type III endomorphism such that $|u^P| \neq 1$, then for every $k > 0$, we have that φ^k has finitely generated fixed subgroup. Indeed,

$$(a_i, 1)\varphi^k = \left(u^{p_i (u^P)^{k-1}}, 1\right) \quad \text{and} \quad (1, b_j)\varphi^k = \left(u^{\sum\limits_{t=0}^{k-1} (b_j\phi^t)^R (u^P)^{k-t-1}}, b_j\phi^k \right),$$

so if φ^k is a type III endomorphism, then

$$\left| \sum_{i \in [n]} \lambda_i(u) p_i (u^P)^{k-1} \right| = \left| (u^P)^k \right| \neq 1.$$

So, we know that $\text{Fix}(\varphi^k)$ is finitely generated for every $k > 0$. We will now show that there is a bound for the periods of periodic points of φ. To do that, we will show that the period of (u^a, w) is the same as the period of w through ϕ, π_w, which is bounded by P_m. This suffices to prove that $\text{Per}(\varphi)$ is finitely generated since, in this case

$$\text{Per}(\varphi) = \bigcup_{k=1}^{P_m!} \text{Fix}(\varphi^k),$$

which is a finite union of finitely generated groups.

We have that

$$(u^a, w)\varphi^k = (u^a, w) \implies k = s\pi_w, \text{ for some } s > 0.$$

By (8.11) putting

$$S = \left\{ w \in \mathrm{Per}(\phi) \mid \exists s_w \in N : (1-(u^P)^{s_w\pi_w}) \text{ divides } \sum_{t=0}^{s_w\pi_w-1} (w\phi^t)^R (u^P)^{s_w\pi_w-t-1} \right\},$$

we have that

$$\mathrm{Per}(\varphi) = \left\{ \left(u^{\frac{\sum_{t=0}^{s_w\pi_w-1}(w\phi^t)^R(u^P)^{s_w\pi_w-t-1}}{1-(u^P)^{s_w\pi_w}}}, w \right) \mid w \in S \right\}.$$

Now, for $w \in S$, reordering the summands, we have that

$$\sum_{t=0}^{s_w\pi_w-1} (w\phi^t)^R (u^P)^{s_w\pi_w-t-1}$$

is equal to

$$\begin{array}{llll}
w^R(u^P)^{s_w\pi_w-1} & +\ (w\phi)^R(u^P)^{s_w\pi_w-2} & +\cdots+ & (w\phi^{\pi_w-1})^R(u^P)^{(s_w-1)\pi_w} \\
+w^R(u^P)^{(s_w-1)\pi_w-1} & +\ (w\phi)^R(u^P)^{(s_w-1)\pi_w-2} & +\cdots+ & (w\phi^{\pi_w-1})^R(u^P)^{(s_w-2)\pi_w} \\
 & \quad\vdots & \quad\vdots & \\
+w^R(u^P)^{\pi_w-1} & +\ (w\phi)^R(u^P)^{\pi_w-2} & +\cdots+ & (w\phi^{\pi_w-1})^R \\
=w^R\sum_{t=1}^{s_w}(u^P)^{t\pi_w-1} & +\ (w\phi)^R\sum_{t=1}^{s_w}(u^P)^{t\pi_w-2} & +\cdots+ & (w\phi^{\pi_w-1})^R\sum_{t=1}^{s_w}(u^P)^{(t-1)\pi_w} \\
=w^R(u^P)^{\pi_w-1}\frac{1-(u^P)^{s_w\pi_w}}{1-(u^P)^{\pi_w}} & +\ (w\phi)^R(u^P)^{\pi_w-2}\frac{1-(u^P)^{s_w\pi_w}}{1-(u^P)^{\pi_w}} & +\cdots+ & (w\phi^{\pi_w-1})^R\frac{1-(u^P)^{s_w\pi_w}}{1-(u^P)^{\pi_w}}.
\end{array}$$

So, we have that

$$\sum_{t=0}^{s_w\pi_w-1} (w\phi^t)^R (u^P)^{s_w\pi_w-t-1} = \frac{1-(u^P)^{s_w\pi_w}}{1-(u^P)^{\pi_w}} \left(\sum_{t=0}^{\pi_w-1} (w\phi^t)^R (u^P)^{\pi_w-t-1} \right),$$

and

$$1-(u^P)^{s_w\pi_w} \text{ divides } \sum_{t=0}^{s_w\pi_w-1} (w\phi^t)^R (u^P)^{s_w\pi_w-t-1}$$

if and only if

$$1-(u^P)^{\pi_w} \text{ divides } \sum_{t=0}^{\pi_w-1} (w\phi^t)^R (u^P)^{\pi_w-t-1},$$

which is a property of w that does not depend on s_w.

So, we proved that given a periodic point (u^a, w), we have that the existence of $s > 0$ such that $(u^a, w)\varphi^{s\pi_w} = (u^a, w)$ depends only on w and if there is one, then $(u^a, w)\varphi^{s\pi_w} = (u^a, w)$

for every $s \in \mathbb{N}$. Thus, the period of (u^a, w) is equal to the period of w through ϕ, which is bounded.

Type IV endomorphisms

It is easy to see by induction that $(u,v)\varphi^r = (v\psi^{r-1}\phi, v\psi^r)$, for every $r > 0$. Indeed, it is true for $r = 1$ and if we have that $(u,v)\varphi^r = (v\psi^{r-1}\phi, v\psi^r)$, then

$$(u,v)\varphi^{r+1} = (v\psi^{r-1}\phi, v\psi^r)\varphi = (v\psi^r\phi, v\psi^{r+1}).$$

So $\mathrm{Per}(\varphi) = \{(y\psi^{\pi_y - 1}\phi, y) \mid y \in \mathrm{Per}(\psi)\} \cong \mathrm{Per}(\psi)$, where π_y denotes the period of y through ψ.

Type V endomorphisms

Consider a type V endomorphism φ defined by

$$(x,y) \mapsto \left(1, v^{\sum\limits_{i\in[n]} \lambda_i(x)q_i + \sum\limits_{j\in[m]} \tau_j(y)s_j}\right).$$

We prove by induction that $(x,y)\varphi^k = \left(1, v^{(x^Q+y^S)(v^S)^{k-1}}\right)$. For $k = 1$, it is clear it holds. Now, assume the statement holds for every $r \leq k$. We have that

$$(x,y)\varphi^{k+1} = (x,y)\varphi^k\varphi = \left(1, v^{(x^Q+y^S)(v^S)^{k-1}}\right)\varphi = \left(1, v^{(x^Q+y^S)(v^S)^{k}}\right).$$

So, a periodic point must have the form $(1, v^b)$ for some $b \in \mathbb{Z}$. In this case,

$$(1, v^b)\varphi^k = (1, v^{bv^S(v^S)^{k-1}}) = \left(1, v^{b(v^S)^k}\right).$$

So, if $v^S \notin \{-1, 1\}$, then, $\mathrm{Per}(\varphi)$ is trivial. If $v^S \in \{-1, 1\}$, then $\mathrm{Per}(\varphi) \cong \mathbb{Z}$.

Type VI endomorphisms

We have that $(x,y)\varphi^k = (x\phi^k, y\psi^k)$, so

$$\mathrm{Per}(\varphi) = \mathrm{Per}(\phi) \times \mathrm{Per}(\psi).$$

Type VII endomorphisms

Consider an endomorphism φ defined by $(x,y)\varphi = (y\psi, x\phi)$, for $\phi : F_n \to F_m$ and $\psi : F_m \to F_n$. We will prove by induction that for $k \in \mathbb{N}$, $(x,y)\varphi^{2k} = (x(\phi\psi)^k, y(\psi\phi)^k)$ and $(x,y)\varphi^{2k+1} = (y(\psi\phi)^k\psi, x(\phi\psi)^k\phi)$. We have that $(x,y)\varphi = (y\psi, x\varphi)$. Now suppose the claim holds for $(x,y)\varphi^r$

for every r up to some k. We have that

$$(x,y)\varphi^{k+1} = (x,y)\varphi^k\varphi = \begin{cases} (x(\phi\psi)^{\frac{k}{2}}, y(\psi\phi)^{\frac{k}{2}})\varphi, & \text{if } k \text{ is even} \\ (y(\psi\phi)^{\frac{k-1}{2}}\psi, x(\phi\psi)^{\frac{k-1}{2}}\phi)\varphi, & \text{if } k \text{ is odd} \end{cases}$$
$$= \begin{cases} (y(\psi\phi)^{\frac{k}{2}}\psi, x(\phi\psi)^{\frac{k}{2}}\phi), & \text{if } k+1 \text{ is odd} \\ (x(\phi\psi)^{\frac{k+1}{2}}, y(\psi\phi)^{\frac{k+1}{2}}), & \text{if } k+1 \text{ is even} \end{cases}$$

So, the periodic points are given by $\mathrm{Per}(\phi\psi) \times \mathrm{Per}(\psi\phi)$, which is finitely generated.

Corollary 8.2.17. *There is an algorithm which decides whether the periodic subgroup of a given endomorphism of $F_n \times F_m$ is finitely generated.*

Notice that if $\mathrm{Per}(\varphi)$ is finitely generated, then the periods must be bounded above by the least common multiple of the periods of the generators. The following corollary is immediate from the proof of Theorem 8.2.16.

Corollary 8.2.18. *Let $\varphi \in \mathrm{End}(F_n \times F_m)$. There is a constant $C_\varphi > 0$ such that* $\mathrm{Per}(\varphi) = \mathrm{Fix}(\varphi^{C_\varphi})$.

8.2.4 Dynamics of infinite points

For $i = 1, 2$, denote by π_i the projections in the i-th component and by φ_i the homomorphism $\varphi\pi_i$. We have that φ is uniformly continuous if and only if both φ_1 and φ_2 are. Also, recall the definition of the sets X, Y, Z, W introduced in Subsection 8.2.1. We will consider $F_n \times F_m$ endowed with the product metric given by taking the prefix metric in each component, as done in the previous section.

We start by obtaining conditions for an endomorphism of $F_n \times F_m$ to be uniformly continuous and see how some questions typically studied can be reduced to the free group case.

Proposition 8.2.19. *Let $\varphi \in \mathrm{End}(F_n \times F_m)$. The following conditions are equivalent:*

1. *φ is uniformly continuous.*
2. *φ is of type IV, VI or VII with uniformly continuous components functions ϕ and ψ.*

Proof. It is obvious that $2 \implies 1$.

$1 \implies 2$: Suppose that φ is uniformly continuous. We start by proving that $1 \in X \implies X = 1$. Indeed, if $x_i = 1$ for some $i \in [n]$ and $x_j \neq 1$ for some $j \in [n]$, then for every $\delta > 0$, we have

$$d\left(\left(a_i^{\lceil \log_2(\frac{1}{\delta})\rceil}, 1\right), \left(a_i^{\lceil \log_2(\frac{1}{\delta})\rceil}a_j, 1\right)\right) < \delta$$

and

$$d\left(\left(a_i^{\lceil \log_2(\frac{1}{\delta})\rceil}, 1\right)\varphi_1, \left(a_i^{\lceil \log_2(\frac{1}{\delta})\rceil}a_j, 1\right)\varphi_1\right) = d(1, x_j) = 1.$$

The same holds for Y, Z and W.

We now prove that we can't have $Y \neq \{1\}$ and $W \neq \{1\}$ (types I, II and V). Suppose we do. Then $1 \notin Y \cup W$. We know that there is some word $1 \neq v \in F_m$ and nonzero exponents q_i and s_j such that $y_i = v^{q_i}$ and $w_j = v^{s_j}$. Take $\delta > 0$ and set $k = \lceil \log_2(\frac{1}{\delta}) \rceil$. We have that

$$d\left((a_i^{ks_j}, b_j^{-kq_i}), (a_i^{ks_j+1}, b_j^{-kq_i})\right) < \delta$$

and

$$d\left((a_i^{ks_j}, b_j^{-kq_i})\varphi_2, (a_i^{ks_j+1}, b_j^{-kq_i})\varphi_2\right) = d(1, v^{q_i}) = 1.$$

The same argument shows that we can't have $X \neq \{1\}$ and $Z \neq \{1\}$, so type III is also done.

It is obvious that an endomorphism of type IV, VI or VII is uniformly continuous if and only if both ϕ and ψ are. $\square$

Following the proof in [28] step by step, where the following lemma is proved for endomorphisms of a free group, we can actually prove the same result for homomorphisms of free groups with different ranks.

Lemma 8.2.20. *Let $\phi : F_n \to F_m$ be a homomorphism. Then ϕ is uniformly continuous if and only if it is either trivial or injective.*

Type IV uniformly continuous endomorphisms

Let $\varphi : F_n \times F_m \to F_n \times F_m$ be a type IV uniformly continuous endomorphism. If both component mappings ϕ and ψ are trivial, then φ is trivial. If ϕ is trivial and ψ is injective, then $(u, v)\varphi^r = (1, v\psi^r)$ and φ has essentially the same dynamical behavior as ψ. If ψ is trivial and ϕ is injective, then $(u, v)\varphi = (v\phi, 1)$ and $(u, v)\varphi^r = (1, 1)$, for $r > 1$.

So, take injective $\phi \in F_m \to F_n$ and $\psi \in \text{End}(F_m)$ such that $(u, v) \mapsto (v\phi, v\psi)$. We have that $(u, v)\varphi^r = (v\psi^{r-1}\phi, v\psi^r)$, for every $r > 0$. By uniqueness of extension, we have that $\hat{\varphi}$ is given by $(u, v) \mapsto (v\hat{\phi}, v\hat{\psi})$. As above, we have that $(u, v)\hat{\varphi}^r = (v\hat{\psi}^{r-1}\hat{\phi}, v\hat{\psi}^r)$.

So

$$\text{Fix}(\hat{\varphi}) = \{(v\hat{\phi}, v) \mid v \in \text{Fix}(\hat{\psi})\}$$

and

$$\text{Per}(\hat{\varphi}) = \{(v\hat{\psi}^{\pi_v - 1}\hat{\phi}, v) \mid v \in \text{Per}(\hat{\psi})\},$$

where π_v denotes the period of v.

Proposition 8.2.21. *Let φ be a type IV uniformly continuous endomorphism of $F_n \times F_m$, with $m, n > 1$. Then, Sing$(\hat{\varphi}) = \{(v\hat{\phi}, v) \mid v \in$ Sing$(\hat{\psi})\}$. Consequentely, Reg$(\hat{\varphi}) = \{(v\hat{\phi}, v) \mid v \in$ Reg$(\hat{\psi})\}$.*

Proof. We start by showing that $\text{Sing}(\hat{\varphi}) \subseteq \{(v\hat{\phi}, v) \mid v \in \text{Sing}(\hat{\psi})\}$. Take some $(v\hat{\phi}, v) \in (\text{Fix}(\varphi))^c$ with $v \in \text{Fix}(\hat{\psi})$. Then, for every $\varepsilon > 0$, the open ball of radius ε centered in $(v\hat{\phi}, v)$ contains an element $(u_\varepsilon\phi, u_\varepsilon) \in \text{Fix}(\varphi)$, with $u_\varepsilon \in \text{Fix}(\psi)$. Notice that $d(v, u_\varepsilon) \leq d((v\hat{\phi}, v), (u_\varepsilon\phi, u_\varepsilon)) < \varepsilon$, thus $v \in (\text{Fix}(\psi))^c$.

For the reverse inclusion, take some $v \in \mathrm{Sing}(\hat{\psi})$. As above, we know that for every $\varepsilon > 0$, there is some $u_\varepsilon \in B(v;\varepsilon)\cap \mathrm{Fix}(\psi)$. Notice that, since $\hat{\phi}$ is uniformly continuous, for every $\varepsilon > 0$, there is some δ_ε such that, for all $x, y \in \widehat{F_m}$ such that $d(x,y) < \delta_\varepsilon$, we have that $d(x\hat{\phi}, y\hat{\phi}) < \varepsilon$. We want to prove that $(v\hat{\phi}, v) \in (\mathrm{Fix}(\varphi))^c$, by showing that, for every $\varepsilon > 0$, the ball centered in $(v\hat{\phi}, v)$ with radius ε contains a fixed point of φ. So, let $\varepsilon > 0$ and consider $\delta = \min\{\delta_\varepsilon, \varepsilon\}$. We have that $(u_\delta\hat{\phi}, u_\delta) \in B((v\hat{\phi}, v);\varepsilon)$ since, by definition of u_δ, we have that $d(v, u_\delta) < \delta \leq \varepsilon$ and also, $d(v, u_\delta) < \delta_\varepsilon$ means that $d(v\hat{\phi}, u_\delta\hat{\phi}) < \varepsilon$. $\square$

Two infinite fixed points $(u\hat{\phi}, u)$ and $(v\hat{\phi}, v)$ are in the same $\mathrm{Fix}(\varphi)$-orbit if and only if there is some (finite) fixed point $(w\phi, w)$ such that $(v\hat{\phi}, v) = (w\phi, w)(u\hat{\phi}, u)$, i.e., if and only if u and v are in the same $\mathrm{Fix}(\psi)$-orbit. It is known that $\mathrm{Reg}(\hat{\psi})$ has finitely many $\mathrm{Fix}(\psi)$-orbits (see [98]), so $\mathrm{Reg}(\hat{\varphi})$ has finitely many $\mathrm{Fix}(\varphi)$-orbits.

Moreover, if ψ is an automorphism, then every regular infinite fixed point v of $\hat{\psi}$ must be either an attractor or a repeller and a singular infinite fixed point can never be an attractor nor a repeller.

Proposition 8.2.22. *Let $v \in \mathrm{Reg}(\hat{\psi})$. Then $(v\hat{\phi}, v)$ is an attractor if and only if v is an attractor for $\hat{\psi}$. Moreover, if $\alpha \in \mathrm{Sing}(\hat{\varphi})$, then α is not an attractor.*

Proof. Let $(v\hat{\phi}, v) \in \mathrm{Reg}(\hat{\varphi})$ such that v is an attractor for ψ. We have that $\hat{\phi}$ is uniformly continuous, so, for every $\varepsilon > 0$, there is some $\delta_\varepsilon > 0$ such that, for every $x, y \in \widehat{F_m}$ such that $d(x,y) < \delta_\varepsilon$ we have that $d(x\hat{\phi}, y\hat{\phi}) < \varepsilon$. Also, let $\tau > 0$ be such that

$$\forall \beta \in \widehat{F_m}(d(v,\beta) < \tau \implies \lim_{n\to+\infty} \beta\hat{\psi}^n = v).$$

We will prove that, for every $x \in \widehat{F_n}$, $y \in B(v;\tau)$, we have $(x,y)\hat{\varphi}^n \to (v\hat{\phi}, v)$. Let $\varepsilon > 0$ and take $\delta = \min\{\varepsilon, \delta_\varepsilon\}$. Take $N \in \mathbb{N}$ such that for every $n > N$ we have that $d(v, y\hat{\psi}^n) < \delta$, which also implies that $d(v\hat{\phi}, y\hat{\psi}^n\hat{\phi}) < \varepsilon$. Thus, for every $n > N+1$, we have that

$$(x,y)\hat{\varphi}^n = (y\hat{\psi}^{n-1}\hat{\phi}, v\hat{\psi}^n) \in B((v\hat{\phi}, v);\varepsilon).$$

Thus, $(x,y)\hat{\varphi}^n \to (v\hat{\phi}, v)$.

Hence, $(v\hat{\phi}, v)$ is an attractor and the set of points it attracts is given by $\widehat{F_n} \times \mathcal{A}_{v,\hat{\psi}}$, where $\mathcal{A}_{v,\hat{\psi}}$ denotes the set of points v attracts through ψ.

Now, we prove that, if $(w\hat{\phi}, w) \in \mathrm{Fix}(\hat{\varphi})$ is an attractor, then w is an attractor for ψ and that completes the proof. Indeed, take $\varepsilon > 0$ such that

$$\forall (x,y) \in \widehat{F_n \times F_m}(d((w\hat{\phi}, w), (x,y)) < \varepsilon \implies \lim_{n\to+\infty}(x,y)\hat{\varphi}^n = (w\hat{\phi}, w)).$$

For every $y \in B(w;\varepsilon)$, we have that $d((w\hat{\phi}, w), (w\hat{\phi}, y)) < \varepsilon$, so

$$\lim_{n\to+\infty}(y\psi^{n-1}\phi, y\psi^n) = \lim_{n\to+\infty}(w\hat{\phi}, y)\hat{\varphi}^n = (w\hat{\phi}, w),$$

thus $y\psi^n \to w$.

Moreover, if $\alpha \in \mathrm{Sing}(\hat{\varphi})$, then

$$\forall \varepsilon > 0\, \exists x_\varepsilon \in \mathrm{Fix}(\varphi) \cap B(\alpha; \varepsilon).$$

So, $x_\varepsilon \varphi^n = x_\varepsilon$, for all $n \in \mathbb{N}$, thus $x_\varepsilon \varphi^n \not\to \alpha$. $\square$

Type VI uniformly continuous endomorphisms

Let $\varphi : F_n \times F_m \to F_n \times F_m$ be a type VI uniformly continuous endomorphism and take $\phi \in \mathrm{End}(F_n)$, $\psi \in \mathrm{End}(F_m)$ such that $(u, v) \mapsto (u\phi, v\psi)$.

By uniqueness of extension, we have that $\hat{\varphi}$ is given by $(u, v) \mapsto (u\hat{\phi}, v\hat{\psi})$. We then have that $\mathrm{Fix}(\hat{\varphi}) = \mathrm{Fix}(\hat{\phi}) \times \mathrm{Fix}(\hat{\psi})$, $\mathrm{Sing}(\hat{\varphi}) = \mathrm{Sing}(\hat{\phi}) \times \mathrm{Sing}(\hat{\psi})$ and $\mathrm{Reg}(\hat{\varphi}) = \mathrm{Fix}(\hat{\phi}) \times \mathrm{Reg}(\hat{\psi}) \cup \mathrm{Reg}(\hat{\phi}) \times \mathrm{Fix}(\hat{\psi})$. There is no hope of finding a finiteness condition that holds in general, since if $n = 2$ and ϕ is the identity mapping, then $\mathrm{Sing}(\hat{\phi})$ is uncountable. In this case, both $\mathrm{Reg}(\hat{\varphi})$ and $\mathrm{Sing}(\hat{\varphi})$ are uncountable.

Proposition 8.2.23. *An infinite fixed point $\alpha = (u, v)$, where $u \in \mathrm{Fix}(\hat{\phi})$ and $v \in \mathrm{Fix}(\hat{\psi})$ is an attractor if and only if u and v are attractors for $\hat{\phi}$ and $\hat{\psi}$, respectively. If, additionally, φ is an automorphism, the same holds for repellers.*

Proof. Let $\alpha = (u, v)$ be an infinite fixed point, where $u \in \mathrm{Fix}(\hat{\phi})$ and $v \in \mathrm{Fix}(\hat{\psi})$. Clearly if $u \in \mathrm{Fix}(\hat{\phi})$ and $v \in \mathrm{Fix}(\hat{\psi})$ are attractors, then, $(u, v) \in \mathrm{Fix}(\hat{\varphi})$ is an attractor. Indeed, in that case, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\forall x \in \widehat{F_n},\ \left(d(u, x) < \varepsilon_1 \implies \lim_{n \to +\infty} x\hat{\phi}^n = u \right)$$

and

$$\forall y \in \widehat{F_m},\ \left(d(v, y) < \varepsilon_2 \implies \lim_{n \to +\infty} y\hat{\psi}^n = v \right).$$

Thus, taking $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we have that

$$\begin{aligned} \forall (x, y) \in \widehat{F_n \times F_m}(d((u, v), (x, y)) < \varepsilon &\implies d(u, x) < \varepsilon \wedge d(v, y) < \varepsilon \\ &\implies \lim_{n \to +\infty} x\hat{\phi}^n = u \wedge \lim_{n \to +\infty} y\hat{\psi}^n = v \\ &\implies \lim_{n \to +\infty} (x, y)\hat{\varphi}^n = (u, v). \end{aligned}$$

Conversely, suppose w.l.o.g that u is not an attractor for $\hat{\phi}$. Then, for every $\varepsilon > 0$, there is some $x_\varepsilon \in \widehat{F_n}$ such that $d(u, x_\varepsilon) < \varepsilon$ but $x_\varepsilon \hat{\phi}^n \not\to u$. In this case, we have that, for every $\varepsilon > 0$, $d((u, v), (x_\varepsilon, v)) < \varepsilon$ and $(x_\varepsilon, v)\hat{\varphi}^n = (x_\varepsilon \hat{\phi}^n, v) \not\to (u, v)$.

In case, φ is an automorphism, the repellers case is analogous. $\square$

Type VII uniformly continuous automorphisms

Let $\varphi : F_n \times F_m \to F_n \times F_m$ be a type VII uniformly continuous endomorphism and take $\phi : F_n \to F_m$, $\psi : F_m \to F_n$ such that $(u, v) \mapsto (v\psi, u\phi)$.

By uniqueness of extension, we have that $(x, y)\hat{\varphi} = (y\hat{\psi}, x\hat{\phi})$. As done in the finite case, we have that an infinite fixed point must be $(x, y) \in \widehat{F_n \times F_m}$ such that $x = y\hat{\psi}$ and $y = x\hat{\phi}$ and so $x = x\hat{\phi}\hat{\psi}$ and $y = y\hat{\psi}\hat{\phi}$. So, $\mathrm{Fix}(\hat{\varphi}) \subseteq \mathrm{Fix}(\hat{\phi}\hat{\psi}) \times \mathrm{Fix}(\hat{\psi}\hat{\phi})$. Also, for $x \in \mathrm{Fix}(\hat{\phi}\hat{\psi})$, we have that $(x, x\hat{\phi})\hat{\varphi} = (x\hat{\phi}\hat{\psi}, x\hat{\phi}) = (x, x\hat{\phi})$. Similarly, we have that $(y\hat{\psi}, y)\hat{\varphi} = (y\hat{\psi}, y)$ for $y \in \mathrm{Fix}(\hat{\psi}\hat{\phi})$. So, $\mathrm{Fix}(\hat{\varphi}) = \{(x, x\hat{\phi}) \mid x \in \mathrm{Fix}(\hat{\phi}\hat{\psi})\} = \{(y\hat{\psi}, y) \mid y \in \mathrm{Fix}(\hat{\psi}\hat{\phi})\}$.

Notice that, by uniqueness of extension, we have that $\hat{\phi}\hat{\psi} = \widehat{\phi\psi}$ and $\hat{\psi}\hat{\phi} = \widehat{\psi\phi}$.

Proposition 8.2.24. *Let φ be a type VII uniformly continuous endomorphism of $F_n \times F_m$, with $m, n > 1$. Then, Sing$(\hat{\varphi}) = \{(x, x\hat{\phi}) \mid x \in \mathrm{Sing}(\hat{\phi}\hat{\psi})\}$. Consequentely, Reg$(\hat{\varphi}) = \{(x, x\hat{\phi}) \mid x \in \mathrm{Reg}(\hat{\phi}\hat{\psi})\}$.*

Proof. We start by showing that $\mathrm{Sing}(\hat{\varphi}) \subseteq \{(x, x\hat{\phi}) \mid x \in \mathrm{Sing}(\hat{\phi}\hat{\psi})\}$. Take some $(x, x\hat{\phi}) \in (\mathrm{Fix}(\varphi))^c$ with $x \in \mathrm{Fix}(\hat{\phi}\hat{\psi})$. Then, for every $\varepsilon > 0$, the open ball of radius ε centered in $(x, x\hat{\phi})$ contains an element $(y_\varepsilon, y_\varepsilon\phi) \in \mathrm{Fix}(\varphi)$, with $y_\varepsilon \in \mathrm{Fix}(\phi\psi)$. Notice that $d(x, y_\varepsilon) \leq d((x, x\hat{\phi}), (y_\varepsilon, y_\varepsilon\phi)) < \varepsilon$, thus $x \in (\mathrm{Fix}(\phi\psi))^c$.

For the reverse inclusion, take some $x \in \mathrm{Sing}(\hat{\phi}\hat{\psi})$. As above, we know that for every $\varepsilon > 0$, there is some $y_\varepsilon \in B(x; \varepsilon) \cap \mathrm{Fix}(\phi\psi)$. Notice that, since $\hat{\phi}$ is uniformly continuous, for every $\varepsilon > 0$, there is some δ_ε such that, for all $x, y \in \widehat{F_n}$ such that $d(x, y) < \delta_\varepsilon$, we have that $d(x\hat{\phi}, y\hat{\phi}) < \varepsilon$. We want to prove that $(x, x\hat{\phi}) \in (\mathrm{Fix}(\varphi))^c$, by showing that, for every $\varepsilon > 0$, the ball centered in $(x, x\hat{\phi})$ with radius ε contains a fixed point of φ. So, let $\varepsilon > 0$ and consider $\delta = \min\{\delta_\varepsilon, \varepsilon\}$. We have that $(y_\delta, y_\delta\phi) \in B((x, x\hat{\phi}); \varepsilon)$ since, by definition of y_δ, we have that $d(x, y_\delta) < \delta \leq \varepsilon$ and also, $d(x, y_\delta) < \delta_\varepsilon$ means that $d(x\hat{\phi}, y_\delta\hat{\phi}) < \varepsilon$. ☐

As done in the type IV case, observing that two infinite fixed points $(x, x\hat{\phi})$ and $(y, y\hat{\phi})$ are in the same $\mathrm{Fix}(\varphi)$-orbit if and only if x and y are in the same $\mathrm{Fix}(\phi\psi)$-orbit, we have that $\mathrm{Reg}(\hat{\varphi})$ has finitely many $\mathrm{Fix}(\varphi)$-orbits.

Proposition 8.2.25. *Suppose that $m = n$ and both ϕ and ψ are automorphisms. Let $x \in \mathrm{Fix}(\hat{\phi}\hat{\psi})$. Then $x \in \mathrm{Sing}(\hat{\phi}\hat{\psi})$ if and only if $x\hat{\phi} \in \mathrm{Sing}(\hat{\psi}\hat{\phi})$. Consequentely, $x \in \mathrm{Reg}(\hat{\phi}\hat{\psi})$ if and only if $x\hat{\phi} \in \mathrm{Reg}(\hat{\psi}\hat{\phi})$.*

Proof. Let $x \in \mathrm{Sing}(\hat{\phi}\hat{\psi})$. Then $x\hat{\phi} \in \mathrm{Fix}(\hat{\psi}\hat{\phi})$.

Since $\hat{\phi}$ and $\hat{\phi}^{-1}$ are uniformly continuous, we have that for every $\varepsilon > 0$, there are some δ_ε, δ'_ε such that for every $z, w \in \widehat{F_n}$, we have

$$d(z, w) < \delta_\varepsilon \implies d(z\hat{\phi}, w\hat{\phi}) < \varepsilon$$

and

$$d(z, w) < \delta'_\varepsilon \implies d(z\hat{\phi}^{-1}, w\hat{\phi}^{-1}) < \varepsilon.$$

Now, let $\varepsilon > 0$. We want to prove that there is some $y \in \text{Fix}(\psi\phi) \cap B(x\hat{\phi};\varepsilon)$. Since $x \in \text{Sing}(\hat{\phi}\hat{\psi})$, then there is some y_ε such that $y_\varepsilon \in \text{Fix}(\phi\psi) \cap B(x;\delta_\varepsilon)$, so

$$y_\varepsilon\phi \in \text{Fix}(\psi\phi) \cap B(x\hat{\phi};\varepsilon).$$

Now, suppose $x \in \text{Reg}(\hat{\phi}\hat{\psi})$. There is some $\varepsilon > 0$ such that $B(x;\varepsilon) \cap \text{Fix}(\phi\psi) = \emptyset$. We will now prove that $B(x\hat{\phi};\delta'_\varepsilon) \cap \text{Fix}(\psi\phi) = \emptyset$. Suppose there is some $y \in B(x\hat{\phi};\delta'_\varepsilon) \cap \text{Fix}(\psi\phi)$. Then, $y\phi^{-1} \in B(x;\varepsilon)$. But

$$y\phi^{-1}(\phi\psi) = y\psi = y\psi\phi\phi^{-1} = y\phi^{-1},$$

so $y\phi^{-1} \in B(x;\varepsilon) \cap \text{Fix}(\phi\psi)$, a contradiction. □

Proposition 8.2.26. *Suppose that $m = n$ and both ϕ and ψ are automorphisms. Let $x \in \text{Reg}(\hat{\phi}\hat{\psi})$. Then $(x, x\hat{\phi})$ is an attractor (resp. repeller) if and only if x is an attractor (resp. repeller) for $\hat{\phi}\hat{\psi}$. Moreover, if $\alpha \in \text{Sing}(\hat{\varphi})$, then α is not an attractor nor a repeller.*

Proof. Let $(x, x\hat{\phi}) \in \text{Reg}(\hat{\phi}\hat{\psi})$ be such that x is an attractor for $\hat{\phi}\hat{\psi}$. Then there is some $\tau > 0$ such that

$$\forall \beta \in \widehat{F_n}(d(x,\beta) < \tau \implies \lim_{n\to+\infty} \beta(\hat{\phi}\hat{\psi})^n = x). \tag{8.12}$$

Also, since $\hat{\phi}$, $\hat{\phi}^{-1}$ and $\hat{\psi}$ are uniformly continuous, we have that for every $\varepsilon > 0$, there is some δ_ε, such that for every $z, w \in \widehat{F_n}$, we have

$$d(z,w) < \delta_\varepsilon \implies \quad d(z\hat{\phi}, w\hat{\phi}) < \varepsilon \wedge d(z\hat{\phi}^{-1}, w\hat{\phi}^{-1}) < \varepsilon \wedge d(z\hat{\psi}, w\hat{\psi}) < \varepsilon.$$

We start by proving that $x\hat{\phi}$ is an attractor for $\hat{\psi}\hat{\phi}$ by proving that

$$\forall \beta \in \widehat{F_n}(d(x\hat{\phi},\beta) < \delta_\tau \implies \lim_{n\to+\infty} \beta(\hat{\psi}\hat{\phi})^n = x\hat{\phi}). \tag{8.13}$$

Indeed, if $d(x\hat{\phi},\beta) < \delta_\tau$ then $d(x, \beta\hat{\phi}^{-1}) = d(x\hat{\phi}\hat{\phi}^{-1}, \beta\hat{\phi}^{-1}) < \tau$, so

$$\lim_{n\to+\infty} \beta(\hat{\psi}\hat{\phi})^{n-1}\hat{\psi} = \lim_{n\to+\infty} \beta\hat{\phi}^{-1}(\hat{\phi}\hat{\psi})^n = x.$$

By continuity of $\hat{\phi}$, we have that $\lim\limits_{n\to+\infty} \beta(\hat{\psi}\hat{\phi})^{n-1}\hat{\psi}\hat{\phi} = \lim\limits_{n\to+\infty} \beta(\hat{\psi}\hat{\phi})^n = x\hat{\phi}$.

We want to prove that $(x, x\hat{\phi})$ is an attractor for $\hat{\varphi}$. Let $\tau' = \min\{\tau, \delta_\tau\}$. We will prove that, for every $(z,w) \in B((x, x\hat{\phi}); \tau')$, we have $(z,w)\hat{\varphi}^n \to (x, x\hat{\phi})$. Let $\varepsilon > 0$ and take $\delta = \min\{\varepsilon, \delta_\varepsilon\}$. By (8.12), we can take $N \in \mathbb{N}$ such that for every $n > N$ we have that

$$d(x, z(\hat{\phi}\hat{\psi})^n) < \delta, \quad \text{and so} \quad d(x\hat{\phi}, z(\hat{\phi}\hat{\psi})^n\hat{\phi}) < \varepsilon \tag{8.14}$$

and by (8.13), we can take $N' \in \mathbb{N}$ such that for every $n > N'$ we have that

$$d(x\hat{\phi}, w(\hat{\psi}\hat{\phi})^n) < \delta, \qquad \text{and so} \qquad d(x\hat{\phi}\hat{\psi}, w(\hat{\psi}\hat{\phi})^n\hat{\psi}) = d(x, w(\hat{\psi}\hat{\phi})^n\hat{\psi}) < \varepsilon. \tag{8.15}$$

Thus, for every $n > 2\max\{N, N'\} + 1$, we have that, if n is even,

$$(z, w)\hat{\varphi}^n = (z(\hat{\phi}\hat{\psi})^{\frac{n}{2}}, w(\hat{\psi}\hat{\phi})^{\frac{n}{2}}) \in B(x, x\hat{\phi})$$

by the first parts of both (8.14) and (8.15); if n is odd, then

$$(z, w)\hat{\varphi}^n = (w(\hat{\psi}\hat{\phi})^{\frac{n-1}{2}}\hat{\psi}, z(\hat{\phi}\hat{\psi})^{\frac{n-1}{2}}\hat{\phi}) \in B(x, x\hat{\phi})$$

by the second parts of both (8.14) and (8.15).

Finally, we only have to prove that, if $(x, x\hat{\phi})$ is an attractor, for $\hat{\varphi}$, then x is an attractor for $\hat{\phi}\hat{\psi}$. Suppose then that there is some $\tau > 0$ such that

$$\forall (z, w) \in \widehat{F_n \times F_n} \left(d((z, w), (x, x\hat{\phi})) < \tau \implies \lim_{n \to +\infty} (z, w)\hat{\varphi}^n = (x, x\hat{\phi}) \right).$$

We will prove that

$$\forall y \in \widehat{F_n} \left(d(x, y) < \tau \implies \lim_{n \to +\infty} y(\hat{\phi}\hat{\psi})^n = x \right)$$

which is equivalent to

$$\forall y \in \widehat{F_n} \left(d(x, y) < \tau \implies \forall \varepsilon > 0\ \exists N \in \mathbb{N} : \forall n > N \quad d(y(\hat{\phi}\hat{\psi})^n, x) < \varepsilon \right)$$

Let $y \in \widehat{F_n}$ be such that $d(x, y) < \tau$ and $\varepsilon > 0$. We know that $d((y, x\hat{\phi}), (x, x\hat{\phi})) < \tau$, there is $N \in N$ such that for every $n > N$, $d((y, x\hat{\phi})\hat{\varphi}^n, (x, x\hat{\phi})) < \varepsilon$. Let $n > N$. Then

$$d\left((y, x\hat{\phi})\hat{\varphi}^{2n}, (x, x\hat{\phi})\right) = d\left((y(\hat{\phi}\hat{\psi})^n, x\hat{\phi}(\hat{\psi}\hat{\phi})^n), (x, x\hat{\phi})\right) < \varepsilon,$$

so $d(y(\hat{\phi}\hat{\psi})^n, x) < \varepsilon$.

The repellers case is analogous noticing that $(x, y)\hat{\varphi}^{-1} = (y\hat{\psi}^{-1}, x\hat{\phi}^{-1})$ and so

$$\begin{aligned}
\operatorname{Reg}(\hat{\varphi}^{-1}) &= \left\{(x, x\hat{\psi}^{-1}) \mid x \in \operatorname{Reg}(\hat{\psi}^{-1}\hat{\phi}^{-1})\right\} \\
&= \left\{(x, x\hat{\psi}^{-1}) \mid x \in \operatorname{Reg}\left((\hat{\phi}\hat{\psi})^{-1}\right)\right\}.
\end{aligned}$$

□

Corollary 8.2.27. *Let φ be a type VII automorphism of $F_n \times F_n$. Then $\alpha \in \operatorname{Reg}(\hat{\varphi})$ is either an attractor or a repeller.*

Proof. Let φ be a type VII automorphism of $F_n \times F_n$ and take $\alpha \in \text{Reg}(\hat{\varphi})$. From Proposition 8.2.24, we have that $\alpha = (x, x\hat{\phi})$, for some $x \in \text{Reg}(\hat{\phi}\hat{\psi})$. We know, that x is either an attractor or a repeller. The result follows directly from Proposition 8.2.26. $\square$

Chapter 9

Further work

The following are the main results of this thesis:

In Chapter 3, we proved that the Fatou property holds for context-free subsets if and only if the group is virtually free and that it is not true that it holds in general for algebraic subsets.

In Chapter 4, we generalized several important results, previously known for free groups, to the class of virtually free groups: we proved that the fixed subgroup and the stable image of an endomorphism are finitely generated and computable and solved the TCP and the BrP for endomorphisms. We also introduced the concepts of eventually fixed and eventually periodic points and of φ-spectrum and φ-order, proving several positive algorithmic results related to these concepts.

In Chapter 5, we relate $GCP(G \rtimes \mathbb{Z})$ with $GBrCP(G)$ and $GTCP(G)$ and prove that $GBrCP(G)$ is decidable if G is a virtually polycyclic group, which implies in particular that $GBrP(G)$ is decidable if G is a finitely generated abelian group.

In Chapter 6, we present a geometric version of the BRP for hyperbolic groups and use it to prove that uniformly continuous endomorphisms with respect to a visual metric satisfy a Hölder condition, which, combined with previous work of Paulin, implies that they must have a finitely generated fixed subgroup.

In Chapter 7, we propose two geometric versions of the BRP inspired by the hyperbolic case: a synhronous and an asynchronous one. We then prove that endomorphisms with a finite kernel and L-quasiconvex image satisfy the synchronous version of the BRP. Using these ideas, we are able to prove that the centralizer of a finite subset of a biautomatic group is biautomatic.

In Chapter 8, we considered direct products of free groups of two different kinds: free-abelian times free groups $\mathbb{Z}^m \times F_n$ and free times free groups $F_n \times F_m$. We consider these groups endowed with the product metric obtained by taking the prefix metric in each direct factor (which is a free group) and study the dynamics of the continuous extension of an endormorphism to the completion, when it exists. We prove that every point of $\widehat{\mathbb{Z}^m \times F_n}$ is $\hat{\varphi}$-periodic or $\hat{\varphi}$-wandering when φ is an automorphism or a type II uniformly continuous endomorphism. We also study dynamics around infinite fixed points in both cases. Moreover, in the case $F_n \times F_m$, we also prove computability of the fixed and periodic subgroups of an endomorphism.

9.1 Questions left to answer

Several questions arise from this work. We now list some of them, indicating the chapter they refer to.

9.1.1 Chapter 3

Regarding the work on subsets of groups (Chapter 3), we have two conjectures:

Question 9.1.1. *Let G be a finitely generated group such that $CF(G)$ is closed under intersection. Must G be virtually cyclic?*

Question 9.1.2. *Let G be a finitely generated group such that $CF(G)$ is closed under complement. Must G be virtually cyclic?*

Since the class of context-free languages is neither closed under intersection nor complement, it should be expected that the condition translates into a very restrictive algebraic property. Recall that we present some evidence supporting this conjecture in Proposition 3.1.8 by proving that the answer to both questions is affirmative for virtually free and virtually abelian groups.

9.1.2 Chapter 4

The main question left open by the work in Chapter 4 concerns the generalization of these results for other classes of groups.

Question 9.1.3. *Let G be a (torsion-free) hyperbolic group and $\varphi \in \mathrm{End}(G)$. Is it decidable whether $\mathrm{EvFix}(\varphi)$ (resp. $\mathrm{EvPer}(\varphi)$) is finitely generated and, in case the answer is affirmative, can a set of generators be effectively computed?*

Another potentially interesting problem could be checking the existence of a better bound to the rank of $\mathrm{EvFix}(\varphi)$ than the one provided by Proposition 4.4.15.

Question 9.1.4. *Is the bound given by Proposition 4.4.15 sharp?*

In Corollary 4.1.3 and Theorem 4.5.1, we prove that $TCP(G)$ and $BrP(G)$ are decidable for a virtually free group G. A very natural question is the following:

Question 9.1.5. *Let G be a finitely generated virtually free group. Is $BrCP(G)$ decidable?*

Notice that answering affirmatively to the question above even for automorphisms would yield an affirmative answer to the conjugacy problem in [f.g. virtually free]-by-cyclic groups due to the work in [9].

Finally, it would be very interesting to go beyond the finite case in computing the φ-spectrum. Even though it is much more complex (even computing orders is very difficult, since solving $GBrP(G)$ is not known for the majority of the groups G), there is a much better looking condition in case our endomorphism is bijective.

Given an element $g \in G$ and an endomorphism $\varphi \in \text{End}(G)$, we call *$\varphi$-tail of g* to a sequence $(x_{-i})_{i\in\mathbb{N}}$ such that $x_0 = g$ and, for all $i > 0$, $x_{-i+1} = x_{-i}\varphi$. We will also refer to finite arrays of this kind as φ-tail of g. So, $(x, x\varphi, \dots, x\varphi^n)$, where $x\varphi^n = g$ is a φ-tail of g.

Let $t = (x_{-i})_i$ be a φ-tail of g and $K \subseteq G$. We define

$$\lambda_K(t) := \begin{cases} |t| & \text{if for all } i > 0,\ x_{-i} \notin K \\ \min\{i > 0 \mid x_{-i} \in K\} & \text{otherwise} \end{cases}$$

Notice that $\lambda_K(t) \leq |t|$ and that it is possible that $\lambda_K(t) = \infty$.

In case φ is an automorphism, every φ-tail of h is contained in the infinite φ-tail $t = (\dots, h\varphi^{-2}, h\varphi^{-1}, h)$.

Lemma 9.1.6. *Let G be a group, $K \subseteq G$ and $\varphi \in \text{End}(G)$. Then the following are equivalent:*

1. *φ-sp$(K) = [n]_0$*

2. $\max\{\lambda_K(t) \mid g \in K, t$ *is a φ-tail of g*$\} = n + 1$

Proof. Suppose that φ-sp$(K) = [n]_0$ and put $S = \{\lambda_K(t) \mid g \in K, t \text{ is a } \varphi\text{-tail of } g\}$. Then,

$$K\varphi^{-n-1} \setminus \bigcup_{i=0}^{n} K\varphi^{-i} = \varphi\text{-pord}_K(n+1) = \emptyset,$$

which means that $K\varphi^{-n-1} \subseteq \bigcup_{i=0}^{n} K\varphi^{-i}$. So let $g \in K$ and $t = (t_{-i})_i$ be a φ-tail of g. If $|t| \leq n+1$, then $\lambda_K(t) \leq n+1$. If $|t| > n+1$, by definition of φ-tail, we have that $t_{-n-1} \in K\varphi^{-n-1} \subseteq \bigcup_{i=0}^{n} K\varphi^{-i}$ and so $t_{-n-1}\varphi^k \in K$, for some $k \in [n]_0$, which means that $t_{-n-1+k} \in K$ and so $\lambda_K(t) \leq n+1-k \leq n+1$. Hence, S is bounded above by $n+1$. Clearly, since φ-sp$(K) = [n]_0$, there is an element $x \in G$ with φ-order equal to n, i.e., such that $x\varphi^n \in K$ and $x\varphi^k \notin K$ for any $0 \leq k < n$. Then $t = (x, x\varphi, \dots, x\varphi^n)$ is a φ-tail of $x\varphi^n \in K$ and $\lambda_K(t) = |t| = n+1$. So, $\max\{\lambda_K(t) \mid h \in K, t \text{ is a } \varphi\text{-tail of } h\} = n+1$.

Conversely, suppose that $\max S = n+1$. If there was an element of order $n+1$, arguing as above, we could construct a φ-tail t of an element $x \in K$ such that $\lambda_K(t) = n+2$, which proves that φ-sp$(K) = [k]_0$ for some $k \leq n$. Since $\max S = n+1$, there is a tail t ending in some $g \in K$ such that $\lambda_K(t) = n+1$, and so t_{-n} has order n. Therefore φ-sp$(K) = [n]_0$. □

Naturally, φ-sp$(K) = \mathbb{N}$ if and only if $\{\lambda_K(t) \mid h \in K, t \text{ is a } \varphi\text{-tail of } h\}$ is unbounded.

Since, in the case of automorphisms, every tail ending in an certain element is contained in the unique infinite tail defined by that element, a better version of Lemma 9.1.6 is possible.

Proposition 9.1.7. *Let G be a finitely generated group, $\varphi \in \text{Aut}(G)$ and $K \subseteq G$. Then, the following are equivalent:*

1. *φ-sp$(K) = [n]_0$;*

2. n is the smallest natural number such that

$$K \subseteq \bigcup_{k=1}^{n+1} K\varphi^k. \tag{9.1}$$

Proof. Suppose that φ-sp$(K) = [n]_0$ and let $g \in K$. Then, consider the φ-tail of g of length $n+2$: $t = (g\varphi^{-n-1}, g\varphi^{-n}, \cdots, g\varphi^{-1}, g)$. By Lemma 9.1.6, $\lambda_K(t) \leq n+1$, which means that there is some $1 \leq k \leq n+1$ such that $g\varphi^{-k} \in K$ and so $g \in K\varphi^k \subseteq \bigcup_{k=1}^{n+1} K\varphi^k$. Now, suppose that $K \subseteq \bigcup_{k=1}^{i} K\varphi^k$ for some $i \leq n$ and take $x \in G$ such that φ-ord$_K(x) = n$. Then, $x\varphi^n$ belongs to $K\varphi^k$ for some $1 \leq k \leq i \leq n$ which means that $x\varphi^{n-k} \in K$, which contradicts the fact that φ-ord$_K(x) = n$.

Conversely, suppose that n is the smallest natural number such that $K \subseteq \bigcup_{k=1}^{n+1} K\varphi^k$. Let $x \in K \setminus \bigcup_{k=1}^{n} K\varphi^k$. Then, $x \in K\varphi^{n+1} \setminus \bigcup_{k=1}^{n} K\varphi^k$. Hence, the φ-tail of x with length $n+2$ starts and ends in K and has no other element in K, which means that φ-ord$_K(x\varphi^{-n}) = n$. Also,

$$\varphi\text{-pord}_K(n+1) = K\varphi^{-n-1} \setminus \bigcup_{i=0}^{n} K\varphi^{-i} = \emptyset,$$

because $K \subseteq \bigcup_{k=1}^{n+1} K\varphi^k$ and so, by 4.12, we have that $n+1 \notin \varphi$-sp(K). □

Condition (9.1) looks more tractable for automorphisms of well-behaved classes of groups and might lead to answering affirmatively the following question with respect to some class $\mathcal{C}$.

Question 9.1.8. *Let G be a finitely generated group in $\mathcal{C}$, $\varphi \in \mathrm{Aut}(G)$ and $K \leq_{f.g} G$. Can we compute φ-sp(K)?*

9.1.3 Chapter 7

Following Figure 7.1, the following questions arise naturally:

Question 9.1.9. *Is there an endomorphism of an automatic group for which the BRP holds and the synchronous BRP does not?*

Question 9.1.10. *Is there an endomorphism of an automatic group for which the BRP holds but the kernel is not finitely generated? By Remark 7.4.4, an affirmative answer to this question would also yield an affirmative answer to Question 9.1.9.*

In virtually free groups, we know that finitely generated normal subgroups are finite or have finite index, so having a finitely generated kernel is the same as having a finite kernel or a finite image. We wonder if something similar to the virtually free groups case might hold in the case of hyperbolic groups.

Question 9.1.11. *Is there an endomorphism of a hyperbolic group with infinite image and infinite kernel for which the (synchronous) BRP holds?*

In the case of automatic groups, we can answer affirmatively to Question 9.1.11: consider $\mathbb{Z} \times \mathbb{Z}$, put $a = (1,0)$ and $b = (0,1)$ and take the structure $L = a^*b^*$. Let $\varphi \in \mathrm{End}(\mathbb{Z} \times \mathbb{Z})$ defined by $(n,m) \mapsto (n,0)$. Then it is easy to see that the synchronous BRP holds for (φ, L, L), but $\mathrm{Ker}(\varphi) \simeq \mathbb{Z}$ and $\mathrm{Im}(\varphi) \simeq \mathbb{Z}$.

The remaining questions arising from this work concern the satisfiability of the hypothesis in Corollary 7.4.8.

Question 9.1.12. *In [46], the authors ask if a finite extension of a biautomatic group is also biautomatic. They also remark that it is a necessary condition that the fixed subgroup of an automorphism of finite order is biautomatic. Can we use Corollary 7.4.8 to prove that, i.e., does such an automorphism admit a language L such that the synchronous BRP holds for (φ, L, L)?*

Question 9.1.13. *Getting knowledge on the synchronous equivalence classes for some classes of automatic groups might be useful to prove fixed point results using Corollary 7.4.8. There are infinitely many, but can we describe them in some sense?*

9.1.4 Chapter 8

Free-abelian times free groups

Although we were able to establish the dichotomy wandering/periodic for automorphisms and type II endomorphisms, we weren't able to go very far in answering this question for type I endomorphisms (not bijective).

Question 9.1.14. *Let $\varphi \in \mathrm{End}(\mathbb{Z}^m \times F_n)$ be a uniformly continuous endomorphism and $\alpha \in \widehat{\mathbb{Z}^m \times F_n}$. Must α be either periodic or wandering?*

It is likely that a necessary step to answer to the above question is answsering to the following question.

Question 9.1.15. *Let $\varphi \in \mathrm{Mon}(F_n)$ be an injective endomorphism and $\alpha \in \hat{F}_n$. Must α be either periodic or wandering?*

We remark that the obvious approach using stable images to reduce the question above to a question about automorphisms (which we know how to answer) does not seem to work easily. Indeed, all nonwandering points must be recurrent by Corollary 8.1.30, so the only thing to prove is that recurrent points are periodic. So, consider $\varphi \in \mathrm{Mon}(F_n)$ and let $\alpha \in \hat{F}_n$ be a recurrent point. We know that $\psi = \varphi|_{\varphi^\infty(F_n)}$ is an automorphism and so every $\hat{\psi}$-recurrent point is periodic. So it would be enough to show that α is $\hat{\psi}$-recurrent. The problem is that we can have $\hat{\varphi}$-recurrent points not belonging to $\widehat{\varphi^\infty(F_n)}$. So, consider, for example $\varphi : F_2 \to F_2$ given by $a \mapsto a$ and $b \mapsto b^2$. It can be seen that $\varphi^\infty(F_2) = \langle a \rangle$ and so $\widehat{\varphi^\infty(F_2)} = \{a^{+\infty}, a^{-\infty}\} \cup \langle a \rangle$. The point ab^∞ is $\hat{\varphi}$-recurrent (in fact it is fixed), and it does not belong to the completion of the stable image.

Another interesting question could be obtaining conditions on the properties $\phi \in End(F_n)$ must satisfy so that the converse implications of the ones in Proposition 8.1.25 also hold.

Free times free groups

Finally, regarding free times free groups, it should be possible, given all the tools provided by the classification of endomorphisms to study the dychotomy wandering/periodic for uniformly continuous endomorphisms of $F_n \times F_m$.

Question 9.1.16. *Let $\varphi \in \mathrm{End}(F_n \times F_m)$ be a uniformly continuous endomorphism and $\alpha \in \widehat{F_n \times F_m}$. Must α be either periodic or wandering?*

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