ON EULER'S ROTATION THEOREM

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Abstract. It is well known that a rigid motion of the Euclidean plane can be written as the composition of at most three reflections. It is perhaps not so widely known that a rigid motion of n-dimensional Euclidean space can be written as the composition of at most $n + 1$ reflections.

The purpose of the present article is, firstly, to present a natural proof of this result in dimension 3 by explicitly constructing a suitable sequence of reflections, and, secondly, to show how a careful analysis of this construction provides a quick and pleasant geometric path to Euler's rotation theorem, and to the complete classification of rigid motions of space, whether orientation preserving or not. We believe that our presentation will highlight the elementary nature of the results and hope that readers, perhaps especially those more familiar with the usual linear algebra approach, will appreciate the simplicity and geometric flavour of the arguments.

1. On Euler's rotation theorem

1.1. Decomposition of 3-isometries. It is well known that a rigid motion of the Euclidean plane can be written as the composition of at most three reflections. It is perhaps not so widely known that a rigid motion of n-dimensional Euclidean space can be written as the composition of at most $n + 1$ reflections.

The purpose of the present article is, firstly, to present a natural proof of this result in dimension 3 by explicitly constructing a suitable sequence of reflections [1] and, secondly, to show how a careful analysis of this construction provides a quick and pleasant geometric path to Euler's rotation theorem, and to the complete classification of rigid motions of space, whether orientation preserving or not.

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In view of the topic of the article any list of references is bound to be inadequate, so we provide just two: our article [1] which deals with the case of the plane, and the article [2] which gives a thorough discussion

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[1] We opted to state and prove this theorem in three dimensions, but note that this proof can be easily adapted to fit any number of dimensions.

of Euler's theorem and, among several proofs, includes Euler's original one and a modern one using linear algebra.

Let $\pi = \textbf{plane}\,ABC$ be the plane through three noncollinear points, A, B and C. We write σ^{ABC} or σ^{π} for the reflection in π. If $A \neq B$ we write $\text{bis}\,\overline{AB}$ for the plane through the midpoint of $\overline{AB}$ perpendicular to this line, which is formed by the points at equal distance to A and B. In particular, if $A \neq B$, then $\sigma^{\text{bis}\overline{AB}}(A) = B$ and vice-versa.

Theorem 1. *Given points A, A', B, B', C and C' in space such that A, B and C are noncollinear and $|AB| = |A'B'|$, $|AC| = |A'C'|$ and $|BC| = |B'C'|$, there exist exactly two rigid motions sending A to A', B to B' and C to C'. The first of these can be written as the composition of three reflections. The second one is obtained composing the first one with the reflection in the plane $A'B'C'$.*

Proof. We start by noting that the location of any point P is determined by its distances to any four given non-coplanar points: indeed if $Q \neq P$ had the same distances to the four given points, then these would belong to $\text{bis}\,\overline{PQ}$ and thus be coplanar. It follows that a motion is determined by its action on any four non-coplanar points and, therefore, there exists no third motion with the stated properties.

Our proof now proceeds by first constructing a rigid motion i, written as the composition of three reflections and satisfying $i(A) = A'$, $i(B) = B'$ and $i(C) = C'$. The second motion will then be $j := \sigma^{A'B'C'} \circ i \neq i$.

In the generic situation, when $A \neq A'$, we define $\alpha = \text{bis}\,\overline{AA'}$, so that $A' = \sigma^{\alpha}(A)$.

Next, supposing that $B^* := \sigma^{\alpha}(B)$ is distinct from B', we define $\beta = \text{bis}\,\overline{B^*B'}$, so that $B' = \sigma^{\beta} \circ \sigma^{\alpha}(B)$. The key point is now to note that $\sigma^{\beta}(A') = A'$, because

$$|A'B^*| = |\sigma^{\alpha}(A)\;\sigma^{\alpha}(B)| = |AB| = |A'B'|\,.$$

Hence $\sigma^{\beta} \circ \sigma^{\alpha}$ sends A to A' and B to B'.

The final step is completely analogous. We define $C^* = \sigma^{\beta} \circ \sigma^{\alpha}(C)$, $\gamma = \text{bis}\,\overline{C^*C'}$ (here supposing $C^* \neq C'$). Again, $\sigma^{\gamma}(A') = A'$ and $\sigma^{\gamma}(B') = B'$, because

$$|A'C^*| = |\sigma^{\beta}\circ\sigma^{\alpha}(A)\;\sigma^{\beta}\circ\sigma^{\alpha}(C)| = |AC| = |A'C'|$$

and similarly

$$|B'C^*| = |\sigma^{\beta}\circ\sigma^{\alpha}(B)\;\sigma^{\beta}\circ\sigma^{\alpha}(C)| = |BC| = |B'C'|\,.$$

Hence, setting $i = \sigma^{\gamma} \circ \sigma^{\beta} \circ \sigma^{\alpha}$, we have $i(A) = A'$, $i(B) = B'$ and $i(C) = C'$.

Now, if $A = A'$ then we may instead define $\alpha = \textbf{plane}\,A'BC$, if $B^* = B'$ then we define $\beta = \sigma^{\alpha}(\textbf{plane}\,ABC) = \textbf{plane}\,A'B'\sigma^{\alpha}(C)$, and if $C^* = C'$ then we define $\gamma = \sigma^{\beta} \circ \sigma^{\alpha}(\textbf{plane}\,ABC) = \textbf{plane}\,A'B'C'$.

In all these cases [2], one still has $i(A) = A'$, $i(B) = B'$ and $i(C) = C'$. Note that if $A = A'$, $B = B'$ and $C = C'$ then $i = \sigma^{ABC}$. □

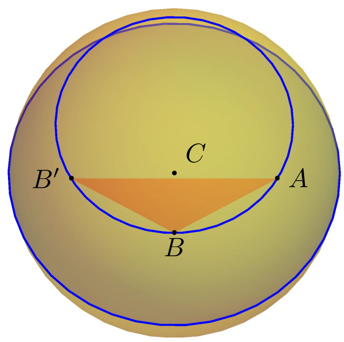
1.2. **Euler's rotation theorem.** Recall that a rotation about a line ℓ is a composition $\rho = \sigma^{\pi_2} \circ \sigma^{\pi_1}$ of reflections, where the planes π_1 and π_2 intersect along the line ℓ, forming an angle which is half that of the rotation.

Theorem 2. *Let $\mathfrak{m}$ be a rigid motion in space with a fixed point, C, and suppose $\mathfrak{m}$ is not the identity.*

- (Euler) *If $\mathfrak{m}$ is an orientation preserving isometry then $\mathfrak{m}$ is a rotation about a line through C.*
- *If $\mathfrak{m}$ does not preserve orientations then $\mathfrak{m}$ is either an inversion in C, a reflection in a plane through C, or a* rotary reflection [3]*, a reflection in a plane π through C followed (or preceded) by a rotation about a line through C perpendicular to π.*

Proof. Let A be such that $B = \mathfrak{m}(A) \neq A$. Since $|AC| = |BC|$, if for every point A the points A, B and C are collinear then for every point A, C is the midpoint of $\overline{AB}$. Hence, $\mathfrak{m}$ is the inversion in C. Thus we may assume that A, B and C are non-collinear.

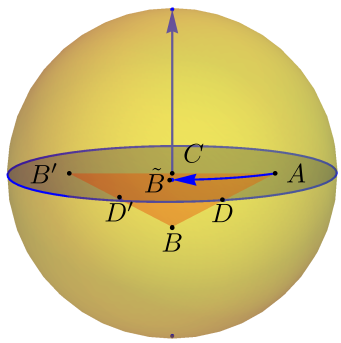
Let us construct i and j as defined in the proof of Theorem 1, using the points A, B and C, and their images $A' = B$, B' and $C' = C$ under $\mathfrak{m}$. Since C is fixed by $\mathfrak{m}$, we have $|AC| = |BC| = |B'C|$. Since $\alpha = \text{bis}\,\overline{AB}$ we have $\sigma^\alpha(C) = C$ and, moreover, $B^* = \sigma^\alpha(B) = A$. We can now see that $C \in \beta$: in the case $A \neq B'$ because $\beta = \text{bis}\,\overline{AB'}$ and $|AC| = |B'C|$, and in the case $B' = A$ because $\beta = \textbf{plane}\,A'B'C$. It follows that $C^* = \sigma^\beta \circ \sigma^\alpha(C) = C$.



Thus $\gamma = \textbf{plane}\,A'B'C'$, and we conclude that $j = \sigma^{A'B'C'} \circ \sigma^\gamma \circ \sigma^\beta \circ \sigma^\alpha = \sigma^\beta \circ \sigma^\alpha$, which is a rotation around the line $\alpha \cap \beta$.

[2]In each of the cases of coincidence, if we were to instead simply omit the respective reflection from our sequence, we would still obtain a motion with the desired properties. The reason why we do not do so, is that having the specific sequence of reflections will be essential in our proof of Theorem 2.

[3]Note that an inversion is the special case of a rotary reflection corresponding to the composition of reflections in 3 perpendicular planes.



It remains to identify i. If $B' = A$, then $\beta = \gamma$ and $i = \sigma^{\mathrm{bis}\,AB}$ is a reflection, so let us assume $B' \neq A$. Let D be the midpoint of $\overline{AB}$, let $D' = \mathfrak{m}(D)$ be the midpoint of $\overline{BB'}$ and let $\delta = \mathbf{plane}\,CDD'$. To see that i is a rotary reflection, we shall show that $\mathfrak{m}' := \sigma^\delta \circ i$ is a rotation around the line ℓ through C perpendicular to δ.

Let $\tilde{B} = \sigma^\delta(B) = \mathfrak{m}'(A)$. If $\tilde{B} = B$, we have $\delta = \gamma$ and we have the desired conclusion, noting that $i = \sigma^\gamma \circ j$.

Assume from now on that $\tilde{B} \neq B$. Arguing as in the first part of the proof for $\mathfrak{m}'$ (using that $\mathfrak{m}'(A) = \tilde{B}$, $\mathfrak{m}'(\tilde{B}) = B'$ and $\mathfrak{m}'(C) = C$) we see that there are two possibilities. The first one is that

$$\mathfrak{m}' = \sigma^{\mathrm{bis}\,\overline{AB'}} \circ \sigma^{\mathrm{bis}\,\overline{A\tilde{B}}}.$$

Now observe that the planes $\mathrm{bis}\,\overline{AB'}$ and $\mathrm{bis}\,\overline{A\tilde{B}}$ are perpendicular to $\mathbf{plane}\,A\tilde{B}B'$, and therefore also perpendicular to the parallel plane δ. Thus $\rho := \sigma^{\mathrm{bis}\,\overline{AB'}} \circ \sigma^{\mathrm{bis}\,\overline{A\tilde{B}}}$ is a rotation around ℓ, and again we have the desired conclusion. The second possibility is that

$$\mathfrak{m}' = \sigma^{C\tilde{B}B'} \circ \rho,$$

which would imply that $\mathfrak{m}'(D) = \sigma^{C\tilde{B}B'}(D')$. But, from $\mathfrak{m}'(D) = \sigma^\delta \circ \sigma^{C\tilde{B}B'} \circ \sigma^{\mathrm{bis}\,\overline{AB'}} \circ \sigma^{\mathrm{bis}\,\overline{A\tilde{B}}}(D)$, it is easy to check that $\mathfrak{m}'(D) = D'$. Thus we would have $D' \in \mathbf{plane}\,C\tilde{B}B' \iff \tilde{B} \in \mathbf{plane}\,CD'B' = \mathbf{plane}\,CBB'$, implying $B = \tilde{B}$, contrary to hypothesis. □

1.3. **Classification of 3-isometries.** In the next corollary we complete the classification of rigid motions in space and for this we remind the reader that, given two points A and B, the translation τ^{AB} which sends A to B can be constructed as follows: let σ' be reflection in the plane through B parallel to $\mathrm{bis}\,\overline{AB}$. Then $\tau^{AB} = \sigma' \circ \sigma^{\mathrm{bis}\,\overline{AB}}$. Recall that for points A, B and C, one has $\tau^{BC} \circ \tau^{AB} = \tau^{AC}$.

Lemma. *Let ρ be a non-trivial rotation about a line ℓ and let τ be a translation in a direction perpendicular to ℓ. Then the composition $\tau \circ \rho$ is a rotation about a line parallel to ℓ.*

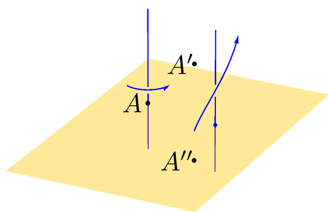
Proof. Write $\tau = \sigma_1 \circ \sigma_2$ as the composition of reflections in parallel planes π_1 and π_2, and $\rho = \sigma_3 \circ \sigma_4$ as the composition of reflections in planes π_3 and π_4 which intersect along ℓ. Note that we can take for π_3 any plane through ℓ, by changing π_4 accordingly. By hypothesis ℓ is parallel to π_1 and π_2 and thus we may take π_3 to be parallel to those two planes as well. It follows that $\sigma_3 \circ \sigma_2 \circ \sigma_1$ is a reflection σ' in plane π' parallel to σ_3 and hence $\tau \circ \rho = \sigma_4 \circ \sigma'$ is a rotation. □

Corollary 3. *Let $\mathfrak{m}$ be a rigid motion in space different from the identity.*

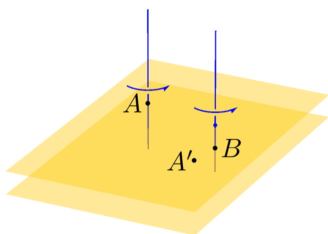
- (Mozzi-Chasles) *If $\mathfrak{m}$ is an orientation preserving isometry then $\mathfrak{m}$ is a screw displacement, that is, a rotation about a line ℓ followed (or preceded) by a (possibly trivial) translation in the direction of ℓ.*
- *If $\mathfrak{m}$ is not orientation preserving then $\mathfrak{m}$ is either an inversion, or a reflection in a plane, or a rotary reflection, or a* glide plane operation*, that is, a reflection in a plane π followed (or preceded) by a translation in π.*

Proof. In view of Theorem 2, we may assume that $\mathfrak{m}$ has no fixed point. Let us take any point A and define a motion $\mathfrak{l}$ by $\mathfrak{l}(X) = \tau^{A'A} \circ \mathfrak{m}(X)$. Note that $\mathfrak{m} = \tau^{AA'} \circ \mathfrak{l}$. Then $\mathfrak{l}(A) = A$ and, by Theorem 2, $\mathfrak{l}$ fixes a plane [4] π which contains A. Let $A'' \in \pi$ be the foot of the perpendicular to π through A'. We now consider each of the possibilities for $\mathfrak{l}$ given in Theorem 2.

(1) If $\mathfrak{l}$ is a rotation about a line through A, we may take π to be the perpendicular through A to this line, and then, by the Lemma, $\tau^{AA''} \circ \mathfrak{l}$ is a rotation about a parallel line. Hence, $\mathfrak{m} = \tau^{A''A'} \circ \tau^{AA''} \circ \mathfrak{l}$ is a screw displacement.



(2) If $\mathfrak{l}$ is a reflection σ^{π} in a plane π through A, followed by a rotation ρ^{ℓ} about a line ℓ through A perpendicular to π, then the plane $\pi' = \text{bis}\,\overline{A'A''}$ is fixed by $\mathfrak{m}$. Hence, replacing A by a point in π', we may assume $A'' = A'$.



Now, If ρ^{ℓ} is not the identity, then $\tau^{AA'} \circ \rho^{\ell}$ has a fixed point $B \in \pi$ by the Lemma, and so $\mathfrak{m} = (\tau^{AA'} \circ \rho^{\ell}) \circ \sigma^{\pi}$ also fixes B, contrary to hypothesis. We conclude that $\mathfrak{m} = \tau^{AA'} \circ \sigma^{\pi}$, which is a glide plane operation.

Note that in Case (1) $\mathfrak{m}$ is orientation preserving, while in Case (2) it is orientation reversing. □

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[4] The plane π is not unique, in general.

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[2] Palais, B, Palais, R, Rodi, S. , A Disorienting Look at Euler's Theorem on the Axis of a Rotation, *Am. Math. Mon.* **116** (2009) 892–909. URL: `https://doi.org/10.4169/000298909X477014`

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