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MATHEMATICAL ASPECTS OF A NEW SPHERICAL MUSICAL INSTRUMENT: *SPHEREHARMONIC*

MARIA MANNONE and TAKASHI YOSHINO

Name: Maria Mannone (b. Palermo, 1985)
Profession: Postdoctoral researcher
Fields of interest: Mathematics and Music, Quantum Mechanics and Quantum Computing
Affiliation: University of Palermo and Ca' Foscari University of Venice, Italy
E-mail: maria.mannone@unive.it
Homepage: mariamannone.com
Major publications:
Mannone M., Seidita V., & Chella A. (2022). Categories, Quantum Computing, and Swarm Robotics: A Case Study. Mathematics, 10(3), 372.
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Name: Takashi Yoshino (b. Ichikawa, 1966)
Profession: Professor
Fields of interest: Application of Geometry to Natural Sciences, Technology and Arts
Affiliation: Toyo University
E-mail: tyoshino@toyo.jp
Homepage: takashiyoshino.random-walk.org
Major publications and/or exhibitions:
Yoshino, T., Matsuoka, A., & Kishimoto, N. (2020). Origami Reconstruction of the Cortical Shell Structures of Radiolarian genus *Pantanellium* from Planar Graphs. Revue de Micropaléontologie, 67, 100423.
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Abstract: *We consider the mathematical properties of a new musical instrument referred to as a SphereHarmonic. The SphereHarmonic is designed based on the basic divisions of a* temari*, that is, a spherical surface with congruent triangles defined by great circles. Each triangle on the sphere corresponds to a tone, and each vertex, which is connected to multiple triangles, represents a chord composed of the corresponding tones. Different chords are created by different combinations of the triangles, which are obtained by rotating the hemispheres. We describe four types of SphereHarmonics characterized by their divisions: Sn, C6, C8, and C10. These four SphereHarmonics have different mathematical characteristics, including the numbers of triangles, the degrees of the vertices, and the numbers of axes. We summarize these properties and also calculate the combinations for the possible vertices.*

Keywords: SphereHarmonic; Temari; Musical Instruments; Spherical Surface; Rotational Symmetry

INTRODUCTION

Musical instruments with various interesting shapes have been proposed recently. Historically, the shape of a musical instrument was strongly related to how it produces tones. For example, the piano uses hammers to produce vibrations in long steel wires of different lengths; on the other hand, the violin has strings of equal length and is played with a bow. However, with the development of electronic sound wave generators (e.g., MIDI systems), the shape of a musical instrument can be designed without considering the physical mechanism by which it would produce sound. In this manner, it is possible to recognize a musical instrument simply as a device that generates chords, that is, combinations of tones. As an example, the *CubeHarmonic* (Mannone *et al*., 2019) is a musical instrument based on the design of a Rubik's cube. It is also possible to design a new type of musical device by using a sphere. *SphereHarmonic* is designed based on the geometry of the basic divisions of a *temari*, a Japanese craftwork ball developed in the Edo period (Suess, 2012). The *temari* is spherical in shape, and its surface is stitched colorfully and symmetrically. The stitching starts from one of the basic divisions of the spherical surface to form congruent triangles. The *SphereHarmonic* uses these divisions to generate chords. In this manuscript, we describe the mathematical aspects of the *SphereHarmonic*. The remainder of the paper contains three sections: a description of the basic concepts of the *SphereHarmonic*, the combinational properties of the *SphereHarmonic*, and concluding remarks.

BASIC CONCEPT OF THE *SPHEREHARMONIC*

The *SphereHarmonic* uses the rotational symmetry of the basic divisions of a *temari*. The divisions are classified into two types: simple divisions (Sn) and combination divisions (C6, C8, and C10). Figure 1 shows the basic divisions; the congruent triangles are colored in alternating white and black. The simplest divisions are the divisions obtained by using latitude lines and the equator of the sphere. The even number n denotes the number of divisions at the poles. Two examples, S10 and S12, are shown in Figure 1. Next, the combination divisions correspond to the Platonic polyhedrons: tetrahedrons for C6, cubes and octahedrons for C8, and dodecahedrons and icosahedrons for C10. The divisions are obtained by a two-step process: first, the polyhedron is divided into congruent triangles using the centroids of the faces and midpoints of the edges; second, triangles are projected from the common centroids of the polyhedron and the sphere to the spherical surface. Note that the duality of Platonic polyhedrons causes the variation of the combination divisions to be three. As a result, we have four types of *SphereHarmonics* which differ in the division of the surface.

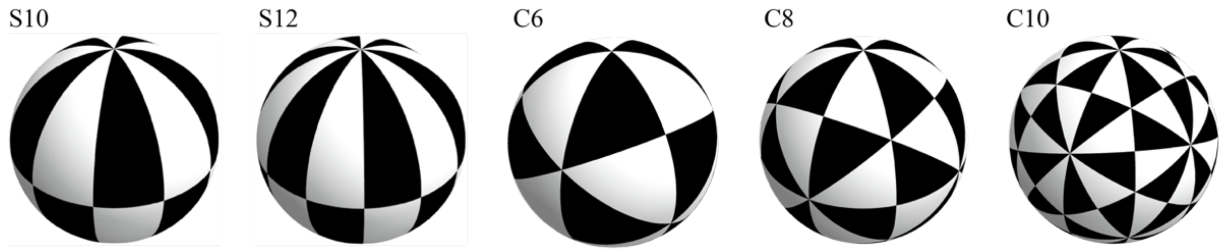

Figure 1 Computer-generated images of *temari* basic divisions.

The *SphereHarmonic* can be rotated along each great circle that divides its surface. Each spherical triangle on the surface represents a tone. The locations of the spherical triangles can be adjusted by rotations of the hemispheres. Each vertex represents a combination of tones, that is, a chord. Selecting a vertex corresponds to selecting a chord of the instrument. In other words, the combinations of the tones can be changed by the rotations of the hemispheres. Sequential rotations of hemispheres and vertex selection generate sequential changes of combinations of tones. Although the method of generating different combinations is simple, the combinational and geometrical features are not so simple.

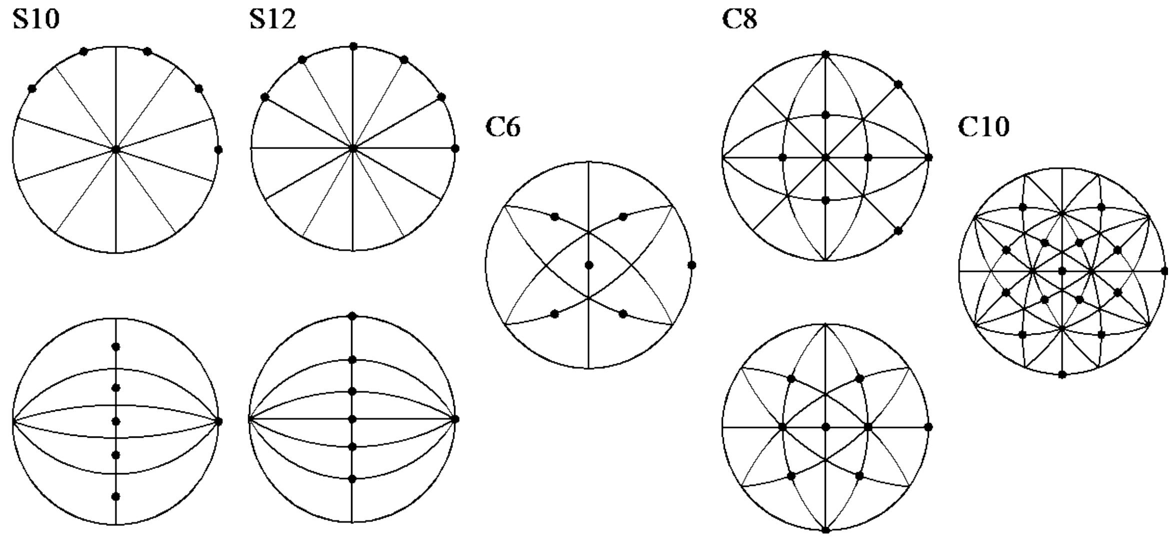

Figure 2 Stereographical projections of the lower hemispheres of the *SphereHarmonic.*

Stereographic projection (e.g., Feeman, 2002) is used to visualize the spherical surface with the details as follows. We presuppose that all spheres considered in this manuscript are unit spheres and that their centres are located at the origin in Cartesian coordinates. Figure 2 shows the stereographic projections of the basic divisions. The projection represents the surface of the lower hemisphere in three-dimensional Euclidean space. Projections of the upper hemisphere are also used in order to illustrate the whole surface of the sphere.

The great circles are characterized by unit vectors perpendicular to the circles; we refer to these unit vectors as axes. For example, the outermost circles of the projections in Figure 2 represent great circles dividing the spheres into two hemispheres: the upper for $z > 0$ and the lower for $z \leq 0$. The coordinates of the axes are taken in order for the z-coordinate of all axes to be greater than or equal to 0. Besides the description using upper and lower hemispheres, we also introduce another description of the hemispheres based on the axes. We refer to the volume close to an axis (the inner product of the axis vector and the position vector is positive) as the northern hemisphere of the axis and the rest as the southern hemisphere of that same axis.

The small black circles in Figure 2 represent the locations of axes of rotations. Some axes are not overlapped with axes for Sn (for $n = 4k \pm 2$) and C6. The axes are also characterized by the minimum amount of rotation because the divisions of the surface after the rotation of a hemisphere must be the same as before. All axes of the combination divisions C6 and C10 are characterized as π. On the other hand, there are two types of axes for the divisions of Sn and those of C8: $2\pi/n$ and π for Sn, and $\pi/2$ and π for C8. Figure 2 illustrates the projection patterns obtained by setting all types of axes to (0, 0, 1).

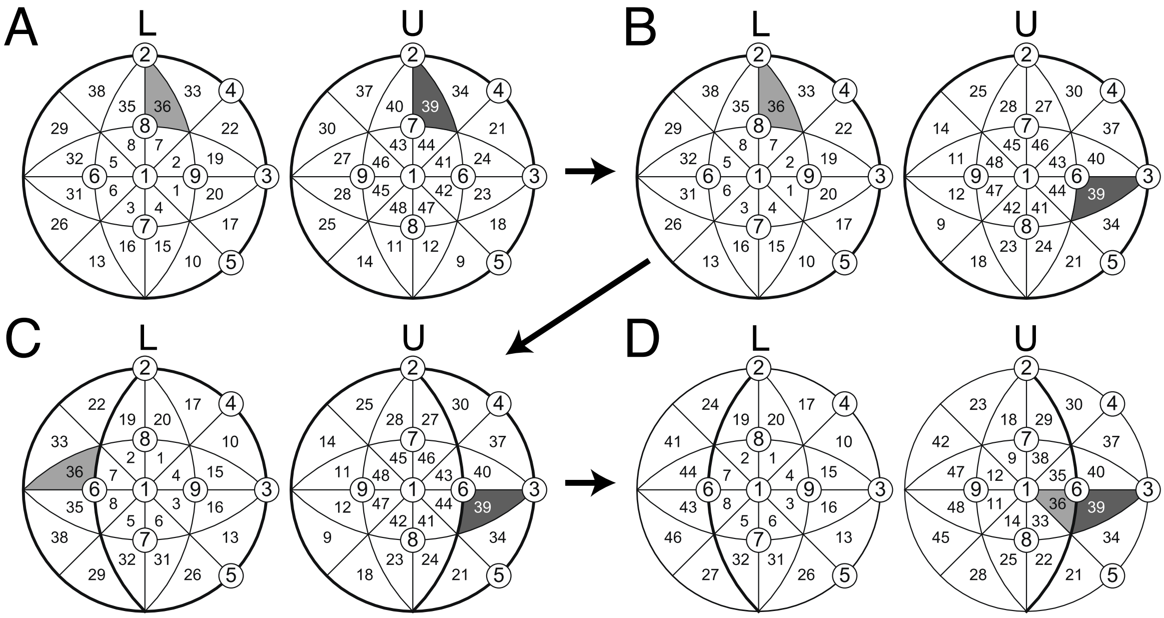

Figure 3 Representation of sequential rotations of the hemispheres of C8.

Figure 3 illustrates the sequential rotations of the hemispheres of C8 from the initial state (Figure 3A) to the final state (Figure 3D) via the intermediate states (Figs. 3B and 3C) as an example. The left and right projections for each state represent the lower and upper hemispheres, respectively. The congruent triangles and the axes of rotations are numbered for convenience. Two spherical triangles numbered 36 and 39 are filled in with light and dark gray to track their positions during the states of rotation. The rotation of the northern hemisphere of axis 1 is illustrated in Figures 3A and 3B. Similarly, the rotation of the southern hemisphere of axis 1 and that of the northern hemisphere of axis 9 is shown in Figs. 3B and 3C and Figs. 3C and 3D, respectively. Thick curves denote the great circles determined by the axes of rotations—for example, the curves containing axis 6 in Figures 3C and 3D represent the great circle perpendicular to axis 9. Note that the amount of rotation of axes 1 and 9 is different—$\pi/2$ for axis 1 and π for axis 9—because of the difference in the axis type.

For the combination divisions, the triangles have parity. This means that black triangles can never take the positions of white triangles, and vice versa (see Figure 1). This is because the black and white triangles have different orders of angles, although they are congruent. This property restricts the combination of triangles obtained from the rotations of the hemispheres. The triangles in Figure 3 were numbered accounting for parity, and the locations of even/odd numbers did not change after the rotations. It should be noted that the triangles of the simple divisions do not have this property because they are all isosceles triangles.

In the following calculations, we assume that all pairs of triangles having the same parity can be transposed by finite numbers of rotational manipulations. We have not proven this assumption; however, it seems to be acceptable intuitively.

RESULTS

The number of combinations or the variation of chords depends on the type of division. In the following, we considered the combinations that appeared in the *SphereHarmonic* by using the basic properties of the divisions summarized in Table 1. The numbers of congruent triangles are $2n$ for Sn, 24 for C6, 48 for C8, and 120 for C10. The degrees of the vertices correspond to the numbers of tones a chord consists of. The basic divisions have two or three types of vertices: 4 and n for Sn; 4 and 6 for C6; 4, 6, and 8 for C6; and 4, 6, and 10 for C10.

Table 1. Basic properties of the *SphereHarmonic*

Divisions	Number of Triangles	Number of Vertices		Degrees of Vertices	Number of vertex combinations	Number of Axes		Amount of Rotation
Sn	$2n$	$n+2$	2	n	$\binom{2n}{n}$	$n/2+1$	1	$2\pi/n$
			n	4	$\binom{2n}{4}$		$n/2$	π
C6	24	14	6	4	4,356	6		π
			4	6	48,400			
C8	48	26	12	4	76,176	9	6	π
			8	6	4,096,576		3	$\pi/2$
			6	8	112,911, 876			
C10	120	62	30	4	3, 132, 900	15		π
			20	6	1,171,008,400			
			12	10	29,828,113,326,144			

The numbers of vertex combinations were obtained, assuming that all triangles had different numbers/tones. For the cases involving simple divisions, the numbers were straightforwardly obtained by selecting n from $2n$ because of the lack of parity; they were $\binom{2n}{n}$ for degree n vertices and $\binom{2n}{4}$ for degree 4. On the other hand, the numbers for the combination divisions were calculated considering the parity as the products of numbers of combinations for white and black triangles. The results were $\binom{12}{2}^2 = 4{,}356$ for degree 4 vertices and $\binom{12}{3}^2 = 48{,}400$ for degree 6 vertices in the case of C6. Similar calculations were carried out for C8 and C10. The results are also summarized in Table 1.

It was complicated to obtain the variations of configurations for the whole surface. For the simple divisions Sn, the estimated number can be obtained by [the number of ways to select n tones from $2n$] multiplied by [the number of circular permutations of the upper hemisphere] multiplied by [the number of circular permutations of the lower hemisphere] multiplied by [the number of rotations of the upper hemisphere]: $\binom{2n}{n}(n-1)!\,(n-1)!\,n = (2n)!/n$. However, we were not able to estimate the numbers for the combination divisions.

Another simple problem is listing the configurations that cannot be used to change the combinations of tones, that is, to generate different chords. The simplest but trivial solution is to allocate the same tone to all triangles. The next simplest solutions for the combination divisions were allocating one tone to black triangles and another to white ones. However, note that this latter solution does not apply in the cases involving simple divisions because the triangles lack parity.

CONCLUSION

The mathematics of a new type of musical instrument, the *SphereHarmonic*, was discussed. This instrument is designed according to the basic divisions of a *temari* ball. Individual musical tones are allocated to the triangles, and a chord can be generated by selecting a vertex, which selects the tones that are connected to that vertex simultaneously. The variations of the chords are numerous enough to play musical pieces if we allocate different tones to all triangles. For the simple division Sn, the number of possible variations is $(2n)!/n$. The estimation of the numbers of the combinations is a complicated problem that may be fruitful for further study.

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Maria MANNONE

Maria Mannone is a theoretical physicist, and she is currently a postdoctoral researcher at the Department of Engineering, University of Palermo, Palermo, Italy. She is also a subject expert at Dipartimento di Scienze Ambientali, Informatica e Statistica (DAIS) and the European Centre for Living Technology (ECLT), Ca' Foscari University of Venice, Venezia, Italy. She is a composer, musician, and improviser. Her interests are quantum computing and interdisciplinary research between mathematics and music.

Takashi YOSHINO

Takashi Yoshino is a professor at the Department of Mechanical Engineering, Toyo University. His fields of interest are the geometry of traditional Japanese craftworks (Temari and Origami), 3DCG, non-Euclidean geometry, hyperspace, and the skeletal structures of marine plankton (e.g., radiolarians). He is a member of the Society of Science on Form (domestic in Japan) and the Physical Society Japan.