Corrected-Hill versus Partially Reduced-Bias Value-at-Risk Estimation*

M. Ivette Gomes
CEAUL, FCUL, Universidade de Lisboa

Frederico Caeiro
CMA, FCT, Universidade Nova de Lisboa

Fernanda Figueiredo
Faculdade de Economia, Universidade do Porto and CEAUL

Lígia Henriques-Rodrigues
Universidade de São Paulo, IME, Brasil, and CEAUL

Dinis Pestana
CEAUL, FCUL, Universidade de Lisboa

Abstract

The *value-at-risk* (VaR) at a small level q, $0 < q < 1$, is the size of the loss that occurs with a probability q. Semi-parametric *partially reduced-bias* (PRB) VaR-estimation procedures based on the mean-of-order-p of a set of k quotients of upper order statistics, with p any real number, are put forward. After the study of their asymptotic behaviour, these PRB VaR-estimators are altogether compared with the classical ones for finite samples, through a large-scale Monte-Carlo simulation study. A brief application to financial log-returns is provided, as well as some final remarks.

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1 Introduction, preliminaries and scope of the article

Let us consider the common notation $(X_1, \ldots, X_n)$ for an available sample of either *independent, identically distributed* (IID) or possibly weakly dependent and stationary *random variables* (RVs), from an

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underlying *cumulative distribution function* (CDF) F. Let us denote by $(X_{1:n} \leq \cdots \leq X_{n:n})$ the sample of associated ascending order statistics. The main theoretical result in the field of *extreme value theory* (EVT) is due to Gnedenko (1943): If there exist attraction coefficients (a_n, b_n), with $a_n > 0$ and $b_n \in \mathbb{R}$, such that the sequence of linearly normalized maxima, $\{(X_{n:n} - b_n)/a_n\}_{n\geq 1}$, converges to a non-degenerate RV, such an RV is compulsory of the type of a general *extreme value* (EV) CDF,

$$\mathrm{EV}_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}),\ 1+\xi x > 0, & \text{if} \quad \xi \neq 0, \\ \exp(-\exp(-x)),\ x \in \mathbb{R}, & \text{if} \quad \xi = 0. \end{cases} \tag{1.1}$$

The CDF F is then said to be in the max-domain of attraction of EV_ξ and the notation $F \in \mathcal{D}_\mathcal{M}(\mathrm{EV}_\xi)$ is used. The parameter ξ is the so-called *extreme value index* (EVI), one of the crucial parameters in the field of statistical EVT.

For heavy or Paretian right tails, i.e. for $F \in \mathcal{D}_\mathcal{M}(\mathrm{EV}_\xi)$, $\xi > 0$, our interest lies in the semi-parametric estimation of the *value-at-risk* (VaR) at the level q, the size of the loss that occurred with a small probability q. We are thus dealing with a high quantile of $F(\cdot)$, with probability $1-q$,

$$\chi_{1-q} \equiv \mathrm{VaR}_q := F^{\leftarrow}(1-q), \tag{1.2}$$

with $F^{\leftarrow}(y) = \inf\{x : F(x) \geq y\}$ denoting the generalized inverse function of F. As usual, let us denote by $U(t)$ the reciprocal right tail quantile function, i.e. the generalized inverse function of $1/(1-F)$. We thus use the notation

$$U(t) := \big(1/(1-F)\big)^{\leftarrow}(t) = F^{\leftarrow}(1-1/t), \quad t \geq 1.$$

For small values of q, i.e. when $q = q_n \to 0$ as $n \to \infty$, being often $nq_n \leq 1$, we want to extrapolate beyond the sample, estimating the parameter $\mathrm{VaR}_q = U(1/q)$, possibly working in the whole $\mathcal{D}_\mathcal{M}(\mathrm{EV}_{\xi>0}) =: \mathcal{D}^+_\mathcal{M}$. To work in $\mathcal{D}^+_\mathcal{M}$ is equivalent to say that $U \in \mathcal{R}_\xi$ (de Haan, 1984), where $\mathcal{R}_a$ denotes the class of regularly varying functions at infinity with an index of regular variation equal to a, any real number (Seneta, 1978; Bingham *et al.*, 1987). Slightly more restrictively, and with the usual notation $a(t) \sim b(t)$ meaning that $a(t)/b(t) \to 1$, as $t \to \infty$, it is often assumed that $U(t) \sim Ct^\xi$, as

$t \to \infty$. We are then working in Hall-Welsh class of models (Hall and Welsh, 1985), where, as $t \to \infty$,

$$U(t) = Ct^{\xi}\big(1 + A(t)/\rho + o\,(t^{\rho})\big), \quad A(t) = \xi\ \beta\ t^{\rho}, \quad C,\ \xi > 0,\ \beta \neq 0. \tag{1.3}$$

The class in (1.3) is a wide class of models, which contains most of the heavy-tailed parents useful in applications, like the EV_ξ, in (1.1), if $\xi > 0$, the associated *generalized Pareto* (GP), given by $\mathrm{GP}_\xi(x) = 1 + \ln \mathrm{EV}_\xi(x),\ x \geq 0$, and the Student-$t_\nu$ parents, $\nu > 0$.

Weissman (1978) proposed the semi-parametric VaR_q-estimators,

$$\mathrm{Q}^{(q)}_{\hat{\xi}}(k) := X_{n-k:n}\ r_n^{\hat{\xi}}, \quad r_n \equiv r_n(k;q) := \frac{k}{nq}, \quad 1 \leq k < n, \tag{1.4}$$

where $\hat{\xi}$ can be any consistent estimator for ξ and Q stands for quantile. For $\xi > 0$, the classical EVI-estimator, usually the one which is used in (1.4) for a semi-parametric quantile estimation, is the Hill (H) estimator $\hat{\xi} = \hat{\xi}(k) =: \mathrm{H}(k)$ (Hill, 1975), with the functional expression,

$$\mathrm{H}(k) := \frac{1}{k}\sum_{i=1}^{k} \ln \frac{X_{n-i+1:n}}{X_{n-k:n}}, \quad 1 \leq k < n. \tag{1.5}$$

If we plug in (1.4) the H EVI-estimator, $\mathrm{H}(k)$, we get the so-called Weissman-Hill quantile or VaR_q-estimator, with the obvious notation, $\mathrm{Q}^{(q)}_{\mathrm{H}}(k)$.

The H EVI-estimators in (1.5) can often have a high asymptotic bias, and bias reduction has recently been a vivid topic of research in the area of statistical EVT (see the recent overviews by Beirlant *et al.*, 2012, and Gomes and Guillou, 2015). Working just for technical simplicity in the particular class of models in (1.3), the asymptotic distributional representation of $\mathrm{H}(k)$, given by

$$\mathrm{H}(k) \ \stackrel{d}{=}\ \xi\left(1 + \frac{\mathcal{N}(0,1)}{\sqrt{k}} + \frac{\beta(n/k)^{\rho}}{1-\rho}\right) + o_{\mathbb{P}}\big((n/k)^{\rho}\big),$$

with $\mathcal{N}(0,1)$ standing for a standard normal RV, led Caeiro *et al.* (2005) to directly remove the dominant component of the bias of the H EVI-estimators, considering the *reduced-bias* (RB) *corrected-Hill* (CH) EVI-estimators,

$$\mathrm{CH}(k) \equiv \mathrm{CH}_{\hat{\beta},\hat{\rho}}(k) := \mathrm{H}(k)\Big(1 - \frac{\hat{\beta}}{1-\hat{\rho}}\,\Big(\frac{n}{k}\Big)^{\hat{\rho}}\Big), \quad 1 \leq k < n, \tag{1.6}$$

which can be *minimum-variance reduced-bias* (MVRB) EVI-estimators for adequate second-order parameters' estimators, $(\hat{\beta}, \hat{\rho})$. If we plug in (1.4) the CH EVI-estimator, CH(k), in (1.6), we get the so-called CH quantile or VaR_q-estimator, with the obvious notation, $\mathrm{Q}^{(q)}_{\mathrm{CH}}(k)$, introduced and studied in Gomes and Pestana (2007), where an adequate algorithm for the (β, ρ)-estimation can be found.

Note next that we can write

$$\mathrm{H}(k) = \sum_{i=1}^{k} \ln \left(\frac{X_{n-i+1:n}}{X_{n-k:n}} \right)^{1/k} = \ln \left(\prod_{i=1}^{k} \frac{X_{n-i+1:n}}{X_{n-k:n}} \right)^{1/k}, \quad 1 \le k < n.$$

The H EVI-estimator is thus the logarithm of the geometric mean (or mean-of-order-0) of

$$\underline{\mathbb{U}} := \{U_{ik} := X_{n-i+1:n}/X_{n-k:n},\ 1 \le i \le k < n\}. \tag{1.7}$$

More generally, Brilhante *et al.* (2013), and almost at the same time and independently, Paulauskas and Vaičiulis (2013) and Beran *et al.* (2014), considered as basic statistics the mean-of-order-p (MO_p) of $\underline{\mathbb{U}}$, in (1.7), with $p \ge 0$. Those same statistics were used in Gomes and Caeiro (2014) and Caeiro *et al.* (2016) for any real p. We are thus thinking on the class of EVI-estimators,

$$\mathrm{H}_p(k) := \begin{cases} \frac{1 - A_p^{-p}(k)}{p}, & \text{if} \quad p < 1/\xi,\ p \ne 0, \\ \ln A_0(k), & \text{if} \quad p = 0, \end{cases} \qquad A_p(k) = \begin{cases} \left(\frac{1}{k} \sum_{i=1}^{k} U_{ik}^p \right)^{1/p}, & \text{if} \quad p \ne 0, \\ \left(\prod_{i=1}^{k} U_{ik} \right)^{1/k}, & \text{if} \quad p = 0, \end{cases} \tag{1.8}$$

with $\mathrm{H}_0(k) \equiv \mathrm{H}(k)$, given in (1.5) (see also Paulauskas and Vaičiulis, 2017). The class of MO_p EVI-estimators in (1.8) depends now on this tuning parameter $p \in \mathbb{R}$, and was shown to be consistent for any $p < 1/\xi$, whenever $k = k_n$ is an intermediate sequence, i.e.

$$k = k_n,\ 1 \le k < n, \quad k_n \to \infty \quad \text{and} \quad k_n = o(n),\ \text{as}\ n \to \infty. \tag{1.9}$$

If we plug in (1.4) the MO_p EVI-estimator, $\mathrm{H}_p(k)$, in (1.8), we get the so-called MO_p quantile or VaR_q-estimator, with the obvious notation, $\mathrm{Q}^{(q)}_{\mathrm{H}_p}(k)$ $\left[\mathrm{Q}^{(q)}_{\mathrm{H}_0}(k) \equiv \mathrm{Q}^{(q)}_{\mathrm{H}}(k)\right]$, studied asymptotically and for finite samples in Gomes *et al.* (2015b).

Just like happens with the H EVI-estimators, the MO_p EVI-estimators in (1.8) can often have a high asymptotic bias. Brilhante *et al.* (2014) noticed that for $p \geq 0$, there is an optimal value

$$p \equiv p_{\mathrm{M}} = \varphi_\rho/\xi, \quad \text{with} \quad \varphi_\rho = 1 - \rho/2 - \sqrt{(1-\rho/2)^2 - 1/2}, \tag{1.10}$$

which maximises the asymptotic efficiency of the class of EVI-estimators in (1.8) with respect to the H EVI-estimator. And the same result holds if we more generally consider any real p. It is thus sensible to consider the optimal RV, $\mathrm{H}^*(k) := \mathrm{H}_{p_{\mathrm{M}}}(k)$, with $\mathrm{H}_p(k)$ and p_{M} given in (1.8) and (1.10), respectively. The asymptotic behaviour of $\mathrm{H}^*(k)$ has led Gomes *et al.* (2015a) to introduce a *partially reduced-bias* (PRB) class of MO_p EVI-estimators based on $\mathrm{H}_p(k)$, in (1.8), with the functional expression

$$\mathrm{PRB}_p(k) \equiv \mathrm{PRB}_p(k; \hat\beta, \hat\rho) := \mathrm{H}_p(k)\Big(1 - \frac{\hat\beta(1-\varphi_{\hat\rho})}{1-\hat\rho-\varphi_{\hat\rho}}\Big(\frac{n}{k}\Big)^{\hat\rho}\Big), \quad 1 \leq k < n, \tag{1.11}$$

still dependent on a tuning parameter p and with φ_ρ defined in (1.10). On the basis of a large-scale simulation study, it was shown in the aforementioned article that the PRB EVI-estimators, in (1.11), are able to outperform the CH EVI-estimators, in (1.6), for a large variety of models. Moreover, just as done in Gomes *et al.* (2016a), we also consider

$$\mathrm{PRB}^*(k) := \mathrm{PRB}_{\hat p^*_{\mathrm{M}}}(k; \hat\beta, \hat\rho), \quad \text{where} \quad \hat p^*_{\mathrm{M}} = \varphi_{\hat\rho}/\xi^*, \quad \xi^* = \mathrm{CH}\big(\hat k_{0|\mathrm{H}}\big), \tag{1.12}$$

with $\lfloor x \rfloor$ denoting the integer part of x and

$$\hat k_{0|\mathrm{H}} := \min\Big(n-1, \Big\lfloor \Big((1-\hat\rho)^2 n^{-2\hat\rho}/\big(-2\hat\rho\hat\beta^2\big)\Big)^{1/(1-2\hat\rho)}\Big\rfloor + 1\Big),$$

the k-estimate of $k_{0|\mathrm{H}} := \arg\min_k \mathrm{MSE}(\mathrm{H}(k))$ suggested in Hall (1982). And we provide the option in (1.12) because we are sure that $\mathrm{CH}\big(\hat k_{0|\mathrm{H}}\big)$ outperforms $\mathrm{H}\big(\hat k_{0|\mathrm{H}}\big)$.

It is thus sensible to work with the VaR_q-estimators $\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}(k)$ and the particular case $\mathrm{Q}^{(q)}_{\mathrm{PRB}^*}(k)$ with the obvious functional forms

$$\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}(k) := X_{n-k:n}\left(\frac{k}{nq}\right)^{\mathrm{PRB}_p(k)}, \quad \mathrm{Q}^{(q)}_{\mathrm{PRB}^*}(k) := X_{n-k:n}\left(\frac{k}{nq}\right)^{\mathrm{PRB}^*(k)}, \quad 1 \leq k < n, \tag{1.13}$$

and where $\mathrm{PRB}_p(k)$ and $\mathrm{PRB}^*(k)$ have been respectively given in (1.11) and (1.12). The asymptotic behaviour of the classes of EVI and VaR-estimators under study is discussed in Section 2. The smallscale simulation performed in Gomes *et al.* (2015c) led us to enlarge such a Monte-Carlo simulation, as described in Section 3. Such a simulation shows indeed the potentiality of the VaR_q semi-parametric estimators in (1.13), particularly when we consider the dependence on the tuning parameter p. In Section 4 and relying on a simple sample path stability criterion, we provide an application to a financial data set. Finally, in Section 5 we provide a few general conclusions.

2 Asymptotic behaviour of EVI and VaR-estimators

In Section 2.1 we refer the asymptotic behaviour of the EVI-estimators under consideration. A parallel exposition is performed in Section 2.2 for the VaR-estimators.

2.1 The EVI-estimators

Just as proved in Brilhante *et al.* (2013) and Gomes and Caeiro (2014), the result obtained in de Haan and Peng (1998) for the H EVI-estimator in (1.5), i.e. for $p = 0$ in (1.8), can be generalized for any adequate real p, as indicated below. In Hall-Welsh class of models in (1.3), for intermediate k-values, i.e. if (1.9) holds, for $p < 1/(2\xi)$, and with $\mathrm{H}_p(k)$ given in (1.8),

$$\sqrt{k}\ \left(\mathrm{H}_p(k)-\xi\right)\ \stackrel{d}{=}\ \frac{\xi(1-p\xi)}{\sqrt{1-2p\xi}}\,\mathcal{N}(0,1)+\frac{\xi\beta\sqrt{k}\ (n/k)^{\rho}(1-p\xi)}{1-\rho-p\xi}(1+o_{\mathbb{P}}(1)), \tag{2.1}$$

where the bias $\xi\beta\sqrt{k}\ (n/k)^{\rho}(1-p\xi)/(1-\rho-p\xi)$ can be small, moderate or large, i.e. go to zero, a constant or infinity, as $n \to \infty$. Straightforwardly from (2.1), if we further assume that $\sqrt{k}\ A(n/k) \to \lambda_{_A}$, finite, there is a non-null bias if $\lambda_{_A} \neq 0$, i.e.

$$\sqrt{k}\ \left(\mathrm{H}_p(k)-\xi\right)\ \underset{n\to\infty}{\overset{d}{\longrightarrow}}\ \frac{\xi(1-p\xi)}{\sqrt{1-2p\xi}}\,\mathcal{N}(0,1)+\frac{\lambda_{_A}(1-p\xi)}{1-\rho-p\xi}. \tag{2.2}$$

For the same type of levels and the CH EVI-estimator in (1.6), if we work with a consistent estimator $(\hat\beta,\hat\rho)$ of (β,ρ), which additionally satisfies the condition, $\hat\rho-\rho=o_{\mathbb{P}}(1/\ln n)$, Theorem 3.1 in Caeiro *et al.* (2005), enable us to say that for all finite $\lambda_{_A}$,

$$\sqrt{k}\ \left(\mathrm{CH}(k)-\xi\right)\ \underset{n\to\infty}{\overset{d}{\longrightarrow}}\ \xi\,\mathcal{N}(0,1).$$

For the EVI-estimators in (1.11), Theorem 2 in Gomes *et al.* (2015a) enable us to guarantee that

$$\sqrt{k}\,\big(\mathrm{PRB}_p(k)-\xi\big)\ \underset{n\to\infty}{\overset{d}{\longrightarrow}}\ \frac{\xi(1-p\xi)}{\sqrt{1-2p\xi}}\,\mathcal{N}(0,1)+\frac{\lambda_A(p\xi-\varphi_\rho)}{(1-\rho-p\xi)(1-\rho-\varphi_\rho)},\tag{2.3}$$

with a null mean value only if $p\xi=\varphi_\rho$. For recent references on several second-order parameters' estimation procedures, see Caeiro *et al.* (2016), where an asymptotic comparison at optimal levels of the CH and PRB_p classes of EVI-estimators is performed.

As can be seen above, the best value of p in PRB_p depends on ξ and ρ, being given by $p=\varphi_\rho/\xi$. Let us consider $\hat{p}$, a consistent estimator of p. We can then state the following:

Proposition 2.1. *Let $\hat{p}$ be a consistent estimator of $p=\varphi_\rho/\xi$. In* Hall-Welsh *class of models in* (1.3)*, for intermediate k-values such that $\sqrt{k}\,A(n/k)\to\lambda_A$, finite, let us consider* $\mathrm{PRB}_{\hat{p}}(k)\equiv\mathrm{PRB}_{\hat{p}}(k;\hat{\beta},\hat{\rho})$*, with* $\mathrm{PRB}_p(k)$ *defined in* (1.11)*. Let us further assume that $(\hat{\beta},\hat{\rho})$ is a consistent estimator of (β,ρ), such that $\hat{\rho}-\rho=o_{\mathbb{P}}(1/\ln n)$. Then,*

$$\sqrt{k}\,\big(\mathrm{PRB}_{\hat{p}}(k)-\xi\big)\ \underset{n\to\infty}{\overset{d}{\longrightarrow}}\ \frac{\xi(1-\varphi_\rho)}{\sqrt{1-2\varphi_\rho}}\,\mathcal{N}(0,1),\tag{2.4}$$

and the same normal limit holds for $\sqrt{k}\big(\mathrm{PRB}^*(k)-\xi\big)$, with $\mathrm{PRB}^*(k)$ *given in* (1.12)*.*

Proof. If p is replaced by $\hat{p}$, a consistent estimator of p, i.e. if $\hat{p}-p=o_{\mathbb{P}}(1)$, and with E denoting either H or PRB, the use of the δ-method enables us to get

$$\mathrm{E}_{\hat{p}}(k)\ \overset{d}{=}\ \mathrm{E}_p(k)+(\hat{p}-p)\,\frac{\partial \mathrm{E}_p(k)}{\partial p}(1+o_{\mathbb{P}}(1)).$$

and consequently,

$$\sqrt{k}\,(\mathrm{E}_{\hat{p}}(k)-\xi)\ \overset{d}{=}\ \sqrt{k}\,(\mathrm{E}_p(k)-\xi)+\sqrt{k}\left((\hat{p}-p)\,\frac{\partial \mathrm{E}_p(k)}{\partial p}\right)(1+o_{\mathbb{P}}(1)).\tag{2.5}$$

Note next that, with $S_p(k):=A_p^{-p}(k)=\left(\frac{1}{k}\sum_{i=1}^k U_{ik}^p\right)^{-1}$, $\mathrm{H}_p(k)=(1-S_p(k))/p$, and

$$\frac{\partial \mathrm{H}_p(k)}{\partial p}=\frac{S_p(k)-1-p\partial S_p(k)/\partial p}{p^2}.\tag{2.6}$$

Further note that
$$\frac{\partial S_p(k)}{\partial p} = -\left(\frac{1}{k}\sum_{i=1}^{k} U_{ik}^{p} \ln U_{ik}\right) \Big/ \left(\frac{1}{k}\sum_{i=1}^{k} U_{ik}^{p}\right)^2.$$
Moreover, with $Y_i,\ i \geq 1$, independent unit Pareto RVs (with CDF $F_{Y}(y) = 1 - 1/y,\ y \geq 1$), we can write, for any real p,

$$U_{ik}^{p} \stackrel{d}{=} Y_{k-i+1:k}^{\xi p}(1 + o_{\mathbb{P}}(1)) \quad \text{and} \quad U_{ik}^{p} \ln U_{ik} \stackrel{d}{=} \xi Y_{k-i+1:k}^{\xi p} \ln Y_{k-i+1:k}(1 + o_{\mathbb{P}}(1)),$$

with the $o_{\mathbb{P}}(1)$ uniformly in $i,\ 1 \leq i \leq k$ (see, Caeiro *et al.*, 2016, for details). Since $\mathbb{E}(Y^a) = 1/(1-a)$ and $\mathbb{E}(Y^a \ln Y) = 1/(1-a)^2$ if $a < 1$, the law of large numbers enables us to say that if $p < 1/\xi$,

$$A_p(k) \underset{n\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} \left(\frac{1}{1-\xi p}\right)^{1/p}, \quad S_p(k) \underset{n\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} 1 - \xi p \quad \text{and} \quad \frac{\partial S_p(k)}{\partial p} \underset{n\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} -\xi.$$

Consequently, and from (2.6),
$$\frac{\partial \mathrm{H}_p(k)}{\partial p} \underset{n\to\infty}{\overset{\mathbb{P}}{\longrightarrow}} 0.$$
A similar result holds trivially for $\partial \mathrm{PRB}_p(k)/\partial p$, with $\mathrm{PRB}_p(k) = \mathrm{H}_p(k)(1 + o_{\mathbb{P}}(1))$ given in (1.11). In Hall-Welsh class of models in (1.3), and with a proof close to the one that led to the asymptotic distributional representations in (2.1), among similar ones for other EVI-estimators, it is thus possible to show that there exists σ_{E} and b_{E} such that

$$\frac{\partial \mathrm{E}_p(k)}{\partial p} \stackrel{d}{=} \frac{\sigma_{\mathrm{E}}}{\sqrt{k}}\,\mathcal{N}(0,1) + b_{\mathrm{E}}(n/k)^{\rho}(1 + o_{\mathbb{P}}(1)),$$

i.e. $\partial \mathrm{E}_p(k)/\partial p = O_{\mathbb{P}}\big(1/\sqrt{k}\big) + O_{\mathbb{P}}(A(n/k)) = o_{\mathbb{P}}(1)$, whenever k is intermediate, i.e. (1.9) holds. Consequently, $\partial \mathrm{E}_p(k)/\partial p = O_{\mathbb{P}}\big(1/\sqrt{k}\big)$ whenever $\sqrt{k}\ A(n/k) \to \lambda_{A}$, finite. Under the aforementioned conditions, with $\sqrt{k}A(n/k) \to \lambda_{A}$, finite, and relying on (2.5), both (2.2) and (2.3) hold, with p replaced by $\hat{p}$, and in particular (2.4) and the remaining of the proposition hold if $\hat{p} = \hat{p}^*_{\mathrm{M}}$, the estimator of φ_ρ/ξ given in (1.12). $\square$

2.2 Extreme quantile or VaR-estimators

Under condition (1.3), the asymptotic behaviour of $Q_{\mathrm{H}}^{(q)}(k)$ is well-known (Weissman, 1978):

$$\frac{\sqrt{k}}{\ln r_n}\,\frac{Q_{\mathrm{H}}^{(q)}(k) - \mathrm{VaR}_q}{\mathrm{VaR}_q} \underset{n\to\infty}{\overset{d}{\longrightarrow}} \xi\,\mathcal{N}(0,1) + \frac{\lambda_{A}}{1-\rho},$$

provided that $\lim_{n\to\infty} \sqrt{k}\ A(n/k) = \lambda_{_A} \in \mathbb{R}$, finite, with r_n defined in (1.4), $A(\cdot)$ the function in (1.3), $q = q_n \to 0$ and $\ln nq_n = o(\sqrt{k})$. Under these same conditions, and an adequate estimation of (β, ρ), as stated in Theorem 5.1 (Gomes and Pestana, 2007),

$$\frac{\sqrt{k}}{\ln r_n}\ \frac{Q^{(q)}_{\rm CH}(k) - {\rm VaR}_q}{{\rm VaR}_q} \ \underset{n\to\infty}{\overset{d}{\longrightarrow}}\ \xi\mathcal{N}(0,1).$$

If we consider the classes of ${\rm VaR}_q$-estimators $Q^{(q)}_{{\rm PRB}_p}(k)$ and $Q^{(q)}_{{\rm PRB}^*}(k)$, in (1.13), adaptations of the results in Gomes and Figueiredo (2006), Gomes and Pestana (2007) and Caeiro and Gomes (2009) enable us to state:

Theorem 2.1. *In* Hall-Welsh *class of models in* (1.3)*, for intermediate k, i.e. k-values such that* (1.9) *holds, if $\sqrt{k}\ A(n/k) \to \lambda_{_A}$, finite, possibly non-null, and whenever*

$$q = q_n \to 0, \quad \ln(n\ q_n) = o\big(\sqrt{k}\big), \quad nq_n = o\big(\sqrt{k}\big), \tag{2.7}$$

let us further consistently estimate the vector of second-order parameters (β,ρ), through $(\hat\beta,\hat\rho)$, and in a way such that $\hat\rho - \rho = o_{_\mathbb{P}}(1/\ln n)$, as $n \to \infty$. Then, we can guarantee that

$$\frac{\sqrt{k}}{\ln r_n}\ \frac{Q^{(q)}_{{\rm PRB}_p}(k) - {\rm VaR}_q}{{\rm VaR}_q} \ \underset{n\to\infty}{\overset{d}{\longrightarrow}}\ \frac{\xi(1-p\xi)}{\sqrt{1-2p\xi}}\ \mathcal{N}(0,1) + \frac{\lambda_{_A}(p\xi - \varphi_\rho)}{(1-\rho-p\xi)(1-\rho-\varphi_\rho)} \tag{2.8}$$

for any real $p < 1/(2\xi)$, and

$$\frac{\sqrt{k}}{\ln r_n}\ \frac{Q^{(q)}_{{\rm PRB}^*}(k) - {\rm VaR}_q}{{\rm VaR}_q} \ \underset{n\to\infty}{\overset{d}{\longrightarrow}}\ \frac{\xi(1-\varphi_\rho)}{\sqrt{1-2\varphi_\rho}}\ \mathcal{N}(0,1), \tag{2.9}$$

with ${\rm VaR}_q$, r_n and $\Big(Q^{(q)}_{{\rm PRB}_p}(k), Q^{(q)}_{{\rm PRB}^*}(k)\Big)$, respectively given in (1.2)*,* (1.4) *and* (1.13).

Proof. Note first that under the validity of (2.7), $\ln r_n = o\big(\sqrt{k}\big)$ and $r_n \to \infty$. The use of the δ-method enables us to write for any EVI-estimator $\hat\xi$,

$$r_n^{\hat\xi} \overset{d}{=} r_n^{\xi} + r_n^{\xi}\ln r_n\ \big(\hat\xi - \xi\big)(1 + o_{_\mathbb{P}}(1)).$$

Then, since

$${\rm VaR}_q = U(1/q) = U(nr_n/k) = U(n/k)r_n^{\xi}\left(1 - A(n/k)(1+o(1))/\rho\right),$$

with $A(\cdot)$ given in (1.3), and

$$X_{n-k:n} \stackrel{d}{=} U(n/k)\big(1 + \xi \; B_k/\sqrt{k} + o_{\mathbb{P}}(A(n/k))\big),$$

with B_k a sequence of standard normal RVs (de Haan and Ferreira, 2006), we can write

$$\begin{aligned} \frac{\mathrm{VaR}_q}{X_{n-k:n}} &\stackrel{d}{=} r_n^{\xi}\big(1 - A(n/k)/\rho\big)\big(1 - \xi B_k/\sqrt{k} + o_{\mathbb{P}}(A(n/k)\big) \\ &\stackrel{d}{=} r_n^{\xi}\big(1 - A(n/k)/\rho - \xi B_k/\sqrt{k} + o_{\mathbb{P}}(A(n/k)\big). \end{aligned}$$

Consequently,

$$\begin{aligned} Q_{\hat{\xi}}^{(q)} - \mathrm{VaR}_q &= X_{n-k:n}\left(r_n^{\hat{\xi}(k)} - \frac{\mathrm{VaR}_q}{X_{n-k:n}}\right) \\ &\stackrel{d}{=} U(n/k) r_n^{\xi}\left(\ln r_n(\hat{\xi} - \xi)(1 + o_{\mathbb{P}}(1)) + \frac{\xi B_k}{\sqrt{k}} + \frac{A(n/k)}{\rho} + o_{\mathbb{P}}(A(n/k))\right), \end{aligned}$$

and

$$\frac{Q_{\hat{\xi}}^{(q)} - \mathrm{VaR}_q}{\mathrm{VaR}_q} \stackrel{d}{=} \ln r_n(\hat{\xi} - \xi)(1 + o_{\mathbb{P}}(1)) + \frac{\xi B_k}{\sqrt{k}} + \frac{A(n/k)}{\rho} + o_{\mathbb{P}}(A(n/k)). \tag{2.10}$$

Since $\ln r_n \to \infty$, as $n \to \infty$, and $\hat{\xi} - \xi = O_{\mathbb{P}}\big(1/\sqrt{k}\big)$, the dominant term in (2.10) is, thus, of the order of $\ln r_n/\sqrt{k}$, which must converge to zero, and, just as mentioned above this is true due to condition (2.7). Consequently, if we choose an EVI-estimator such that

$$\sqrt{k}\left(\hat{\xi} - \xi\right) \underset{n\to\infty}{\stackrel{d}{\longrightarrow}} \sigma \, \mathcal{N}(0,1) + b\lambda_{_A},$$

whenever $\sqrt{k}A(n/k) \to \lambda_{_A}$, finite, we can further write

$$\frac{\sqrt{k}\left(Q_{\hat{\xi}}^{(q)}(k) - \mathrm{VaR}_q\right)}{\ln r_n \; \mathrm{VaR}_q} \underset{n\to\infty}{\stackrel{d}{\longrightarrow}} \sigma \, \mathcal{N}(0,1) + b\lambda_{_A},$$

and the results in the theorem follow, i.e. (2.8) and (2.9) hold. □

3 Monte-Carlo simulation experiments

We have implemented large-scale multi-sample Monte-Carlo simulation experiments of size 5000×20, for the classes of VaR-estimators, $\mathrm{Q}_{\mathrm{PRB}_p}^{(q)}(k)$ and $\mathrm{Q}_{\mathrm{PRB}^*}^{(q)}(k)$, in (1.13). We have considered sample sizes

$n = 100(100)500$, $1000(1000)5000$, from the following models:

(1) The EV_ξ model, with CDF $F(x) = \mathrm{EV}_\xi(x)$, in (1.1), $\xi = 0.1, 0.25, 0.5, 1$ ($\rho = -\xi$);

(2) The associated GP_ξ model, with CDF $F(x) = \mathrm{GP}_\xi(x) = 1 + \ln \mathrm{EV}_\xi(x) = 1 - \big(1+\xi x\big)^{-1/\xi}$, $x \geq 0$, $1 + \xi x > 0$ ($\rho = -\xi$), and the same values of ξ, as in (1);

(3) The Burr(ξ, ρ) model, with CDF, $F(x) = 1 - (1 + x^{-\rho/\xi})^{1/\rho},\ x \geq 0$, for the aforementioned values of ξ and $\rho = -0.25, -0.5, -1$;

(4) The Student-t_ν model with $\nu = 2, 3, 4$ ($\xi = 1/\nu;\ \rho = -2/\nu$).

For details on multi-sample simulation, see Gomes and Oliveira (2001), among others.

3.1 Mean values and MSE patterns as functions of k/n

For each value of n and for each of the aforementioned models, we have first simulated the mean value (E) and root MSE (RMSE) of the VaR-estimators under consideration, as functions of the sample fraction, k/n, used in the estimation. Just as an illustration, we present Figures 1–4, associated with $\mathrm{EV}_{0.1}$, $\mathrm{GP}_{0.1}$, Burr$(0.5, -0.25)$ and Student t_4 parents. In these figures, we show, for $n = 1000$, $q = 1/n$, and on the basis of the first $N = 5000$ runs, the simulated patterns of normalized mean value and RMSE of $\mathrm{Q}^{(q)}_{\hat\xi}(k)/\mathrm{VaR}_q$, with $\mathrm{Q}^{(q)}_{\hat\xi}(k)$ defined in (1.4), respectively denoted $\mathrm{E}^N_Q[\cdot]{:=}\mathrm{E}_Q[\cdot]/\mathrm{VaR}_q$ and $\mathrm{RMSE}^N_Q[\cdot]{:=}\mathrm{RMSE}_Q[\cdot]/\mathrm{VaR}_q$. For the EVI-estimation, we have considered $\mathrm{PRB}_p(k)$, in (1.11), for a wide region of non-negative values of p, $p = p_\ell = \ell/(16\xi)$, $\ell = 1(1)15$, representing only some of these ℓ-values, and $\mathrm{PRB}^*(k)$, in (1.12). We have further considered the H and CH VaR-estimators.

Similar results have been obtained for other values of q and for the other simulated underlying parents, mainly in the sense that for $|\rho| < 0.5$, even PRB^*, not optimally chosen among the PRB_p class of VaR-estimators, outperforms the MVRB CH class of VaR-estimators, regarding minimal RMSE, as can be further seen in the following section.

3.2 Behaviour at optimal levels

We have further computed the Weissman-Hill VaR-estimator $\mathrm{Q}^{(q)}_{\mathrm{H}}(k) \equiv \mathrm{Q}^{(q)}_{\mathrm{H}_0}(k)$ at the simulated value of $k^{(q)}_{0|\mathrm{H}_0} := \arg\min_k \mathrm{RMSE}\big(\mathrm{Q}^{(q)}_{\mathrm{H}_0}(k)\big)$, the simulated optimal k in the sense of minimum RMSE, again with $\mathrm{Q}^{(q)}_{\hat\xi}(k)$ defined in (1.4). Such a value provides an indication of the best possible performance

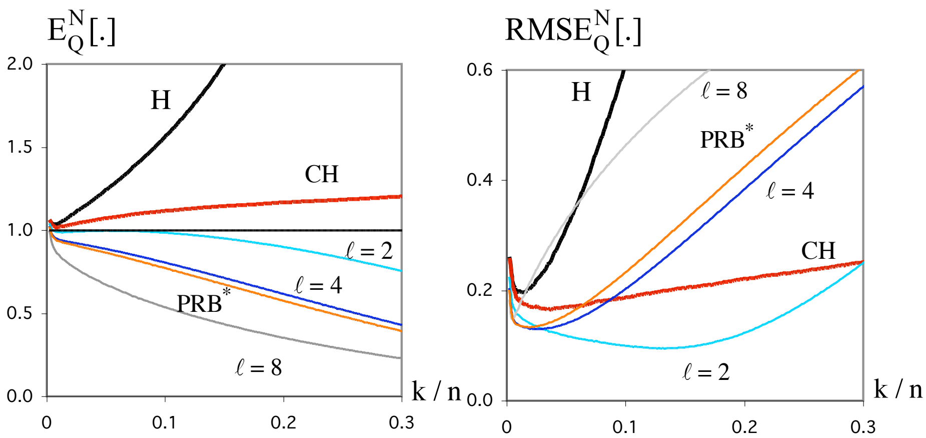


Figure 1: Normalized mean values (*left*) and RMSEs (*right*), for an underlying $EV_{0.1}$ parent

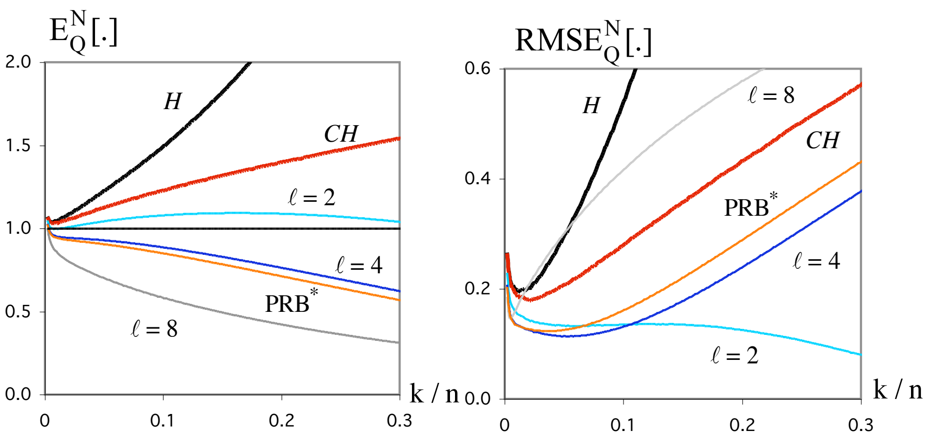


Figure 2: Normalized mean values (*left*) and RMSEs (*right*), for an underlying $GP_{0.1}$

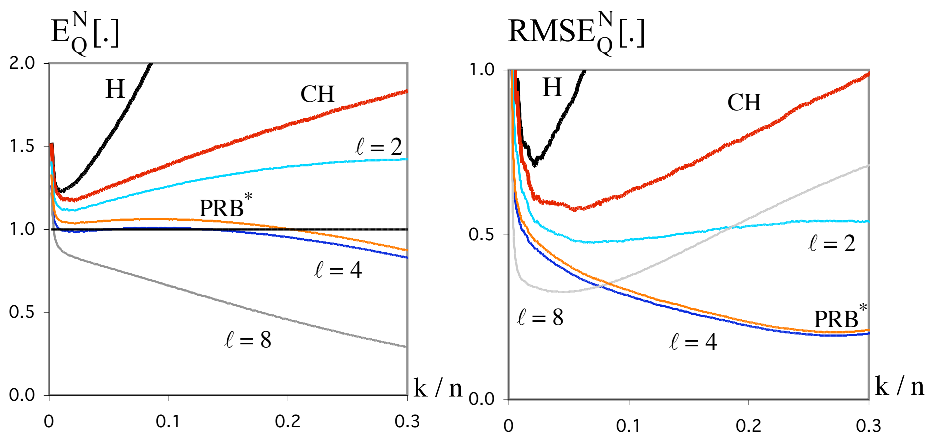


Figure 3: Normalized mean values (*left*) and RMSEs (*right*), for an underlying Burr parent with $\xi = 0.5$ and $\rho = -0.25$

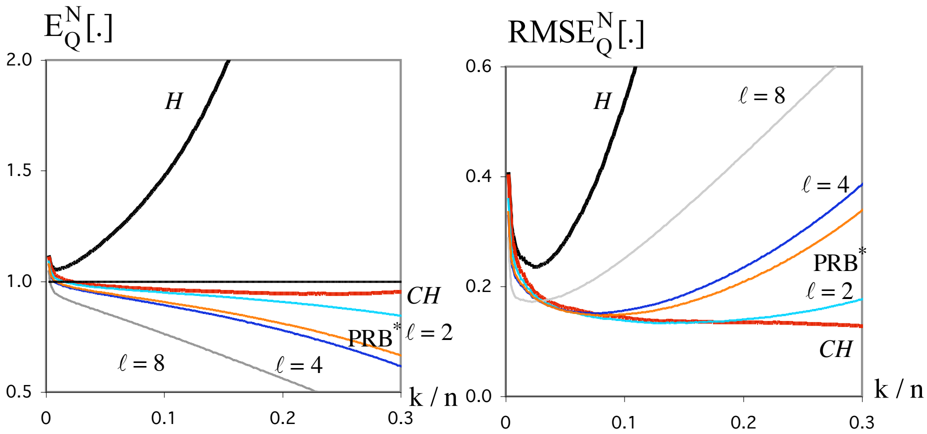


Figure 4: Normalized mean values (*left*) and RMSEs (*right*), for an underlying Student parent with 4 degrees of freedom ($\xi = 0.25$, $\rho = -0.5$)

of the Weissman-Hill VaR-estimator, difficult to achieve in practice. Such an estimator is denoted by Q_{00}. We have also computed Q_{p0} and Q_0^*, the estimators in (1.13) at optimal levels, and the simulated indicators,

$$\mathrm{REFF}_{p|0} := \frac{\mathrm{RMSE}\,(\mathrm{Q}_{00})}{\mathrm{RMSE}\,(\mathrm{Q}_{p0})}, \qquad \mathrm{REFF}_0^* := \frac{\mathrm{RMSE}\,(\mathrm{Q}_{00})}{\mathrm{RMSE}\,(\mathrm{Q}_0^*)}. \tag{3.1}$$

A similar REFF-indicator, $\mathrm{REFF}_{\mathrm{CH}|0}$ has also been computed for the VaR-estimator based on CH EVI-estimators, in (1.6).

Remark 3.1. *The indicators in* (3.1) *have been conceived so that an indicator higher than one means a better performance than the one of the* Weissman-Hill VaR-*estimator. Consequently, the higher these indicators are, the better the associated* VaR-*estimators perform, compared to* Q_{00}.

As an illustration of the results obtained for the REFF-indicators of the different VaR-estimators under consideration, we present Tables 1–4. In the first row, we provide the RMSE of Q_{00}, denoted by RMSE_{00}, so that we can easily recover the RMSE of all other estimators. The subsequent rows provide the REFF-indicators of the VaR-estimators under study, considering two different groups of VaR-estimators, (CH, PRB*) and PRB_p, for a few values of p. In each group, the highest REFF-indicator is written in **bold**. The highest REFF-indicator among them all is further double underlined. REFF-indicators smaller than $\mathrm{REFF}_{\mathrm{CH}|0}$ are written in *italic*. Note that for PRB_p, and due to the interesting and reliable behaviour of the EVI-estimators $\mathrm{H}_p(k)$ in (1.8) for large p, in a situation where we can no longer guarantee asymptotic normality (see Brilhante *et al.*, 2013), we have decided to consider not only the region $1 \le \ell \le 7$, where we can guarantee asymptotic normality, but also the region $8 \leq \ell \leq 15$, where only consistency can be guaranteed. For the mean values of the normalized VaR-estimators at optimal levels, see Tables 5–8. We present there, for the same values of n as before, the simulated mean values at optimal levels of the normalized VaR-estimators under study. Now, and among all estimators considered, the one providing the smallest squared bias is $\underline{\underline{\text{double underlined}}}$, and written in **bold**. Information on 95% confidence intervals (CIs), computed on the basis of the 20 replicates with 5000 runs each, is also provided.

Table 1: Simulated RMSE of Q_{00}, $q = 1/n$ (first row) and REFF-indicators of $Q^{(q)}_{CH|0}$, $Q^{(q)}_{PRB^*|0}$ and $Q^{(q)}_{PRB_{p_\ell}|0}$, for $p_\ell = \ell/(16\xi)$, $\ell = 1, 2, 4, 6, 10, 14$, for $EV_{0.1}$ parents, together with 95% CIs

EV_ξ parent, $\xi = 0.1$ ($\rho = -0.1$)						
n	100	200	500	1000	2000	5000
$RMSE_{00}$	0.329 ± 0.0036	0.273 ± 0.0027	0.225 ± 0.0016	0.200 ± 0.0014	0.179 ± 0.0011	0.157 ± 0.0008
CH	1.287 ± 0.0154	1.323 ± 0.0147	1.552 ± 0.0123	1.202 ± 0.0051	1.123 ± 0.0051	1.073 ± 0.0041
PRB*	$\mathbf{1.487} \pm 0.0181$	$\mathbf{1.490} \pm 0.0119$	$\mathbf{1.572} \pm 0.0101$	$\mathbf{1.496} \pm 0.0123$	$\mathbf{1.635} \pm 0.0109$	$\mathbf{2.276} \pm 0.0188$
$\ell = 1$	1.379 ± 0.0172	1.421 ± 0.0122	$\mathit{1.532} \pm 0.0135$	1.660 ± 0.0121	2.101 ± 0.0231	3.601 ± 0.0412
$\ell = 2$	1.479 ± 0.0162	1.465 ± 0.0128	$\underline{\underline{\mathbf{1.618}}} \pm 0.0134$	$\underline{\underline{\mathbf{2.078}}} \pm 0.0188$	$\underline{\underline{\mathbf{2.904}}} \pm 0.0241$	$\underline{\underline{\mathbf{4.441}}} \pm 0.0463$
$\ell = 4$	$\underline{\underline{\mathbf{1.505}}} \pm 0.0179$	$\underline{\underline{\mathbf{1.493}}} \pm 0.0114$	$\mathit{1.482} \pm 0.0101$	1.536 ± 0.0120	1.767 ± 0.0120	2.678 ± 0.0225
$\ell = 6$	1.381 ± 0.0173	1.432 ± 0.0132	$\mathit{1.431} \pm 0.0090$	1.419 ± 0.0109	1.407 ± 0.0093	1.446 ± 0.0104
$\ell = 10$	$\mathit{1.169} \pm 0.0143$	$\mathit{1.208} \pm 0.0124$	$\mathit{1.243} \pm 0.0094$	1.259 ± 0.0099	1.259 ± 0.0081	1.239 ± 0.0070
$\ell = 14$	$\mathit{1.061} \pm 0.0127$	$\mathit{1.071} \pm 0.0112$	$\mathit{1.085} \pm 0.0085$	$\mathit{1.096} \pm 0.0087$	$\mathit{1.100} \pm 0.0076$	1.092 ± 0.0066

Table 2: Simulated RMSE of Q_{00}, $q = 1/n$ (first row) and REFF-indicators of $Q^{(q)}_{CH|0}$, $Q^{(q)}_{PRB^*|0}$ and $Q^{(q)}_{PRB_{p_\ell}|0}$, for $p_\ell = \ell/(16\xi)$, $\ell = 1, 2, 4, 6, 10, 14$, for $GP_{0.1}$ parents, together with 95% CIs

GP_ξ parent, $\xi = 0.1$ ($\rho = -0.1$)						
n	100	200	500	1000	2000	5000
$RMSE_{00}$	0.320 ± 0.0024	0.269 ± 0.0023	0.224 ± 0.0020	0.199 ± 0.0013	0.179 ± 0.0010	0.157 ± 0.0008
CH	1.442 ± 0.0131	1.220 ± 0.0109	1.117 ± 0.0050	1.079 ± 0.0032	1.059 ± 0.0032	1.038 ± 0.0022
PRB*	$\mathbf{1.581} \pm 0.0107$	$\mathbf{1.542} \pm 0.0108$	$\mathbf{1.537} \pm 0.0150$	$\mathbf{1.621} \pm 0.0105$	$\mathbf{1.938} \pm 0.0121$	$\mathbf{3.058} \pm 0.0240$
$\ell = 1$	$\underline{\underline{\mathbf{1.752}}} \pm 0.116$	$\underline{\underline{\mathbf{1.801}}} \pm 0.0159$	2.241 ± 0.0262	3.223 ± 0.0254	4.301 ± 0.0275	6.021 ± 0.0477
$\ell = 2$	1.653 ± 0.0114	1.750 ± 0.0150	$\underline{\underline{\mathbf{2.387}}} \pm 0.0251$	$\underline{\underline{\mathbf{3.323}}} \pm 0.0243$	$\underline{\underline{\mathbf{4.600}}} \pm 0.0266$	$\underline{\underline{\mathbf{7.010}}} \pm 0.0468$
$\ell = 4$	1.593 ± 0.0100	1.556 ± 0.0119	1.577 ± 0.0152	1.751 ± 0.0114	2.251 ± 0.0130	3.751 ± 0.0281
$\ell = 6$	1.481 ± 0.0113	1.477 ± 0.0107	1.452 ± 0.0145	1.433 ± 0.0089	1.430 ± 0.0102	1.518 ± 0.0131
$\ell = 10$	$\mathit{1.226} \pm 0.0105$	1.240 ± 0.0111	1.263 ± 0.0129	1.267 ± 0.0087	1.267 ± 0.0086	1.239 ± 0.0098
$\ell = 14$	$\mathit{1.094} \pm 0.0096$	$\mathit{1.089} \pm 0.0103$	$\mathit{1.099} \pm 0.0111$	1.102 ± 0.0078	1.106 ± 0.0074	1.093 ± 0.0087

Table 3: Simulated RMSE of Q_{00}, $q = 1/n$ (first row) and REFF-indicators of $\mathrm{Q}^{(q)}_{\mathrm{CH}|0}$, $\mathrm{Q}^{(q)}_{\mathrm{PRB}^*|0}$ and $\mathrm{Q}^{(q)}_{\mathrm{PRB}_{p_\ell}|0}$, for $p_\ell = \ell/(16\xi)$, $\ell = 1, 2, 4, 6, 10, 14$, for Burr$(0.5, -0.25)$ parents, together with 95% CIs

Burr $(0.5, -0.25)$ parent						
n	100	200	500	1000	2000	5000
RMSE_{00}	1.452 ± 0.0290	1.123 ± 0.0207	0.880 ± 0.0137	0.758 ± 0.0070	0.658 ± 0.0052	0.560 ± 0.0047
CH	2.970 ± 0.0653	2.310 ± 0.0493	1.448 ± 0.0216	1.287 ± 0.0062	1.199 ± 0.0062	1.133 ± 0.0057
PRB*	$\mathbf{3.284} \pm 0.0681$	$\mathbf{2.945} \pm 0.0506$	$\mathbf{3.109} \pm 0.0473$	$\mathbf{3.711} \pm 0.0389$	$\mathbf{4.739} \pm 0.0332$	$\mathbf{6.911} \pm 0.0604$
$\ell = 1$	$\underline{\underline{\mathbf{3.421}}} \pm 0.0723$	3.094 ± 0.0547	3.536 ± 0.0512	4.565 ± 0.0499	6.283 ± 0.0593	7.102 ± 0.0614
$\ell = 2$	3.357 ± 0.0714	$\underline{\underline{\mathbf{3.243}}} \pm 0.0556$	$\underline{\underline{\mathbf{3.963}}} \pm 0.0609$	$\underline{\underline{\mathbf{5.419}}} \pm 0.0588$	$\underline{\underline{\mathbf{7.826}}} \pm 0.0697$	$\underline{\underline{\mathbf{7.292}}} \pm 0.0636$
$\ell = 4$	3.284 ± 0.0681	2.945 ± 0.0506	3.109 ± 0.0473	3.711 ± 0.0389	4.739 ± 0.0332	6.911 ± 0.0604
$\ell = 6$	3.303 ± 0.0658	2.829 ± 0.0510	2.603 ± 0.0378	2.696 ± 0.0273	3.028 ± 0.0307	4.070 ± 0.0786
$\ell = 10$	3.209 ± 0.0624	2.753 ± 0.0454	2.409 ± 0.0378	2.219 ± 0.0210	2.045 ± 0.0176	1.925 ± 0.0203
$\ell = 14$	3.055 ± 0.0587	2.605 ± 0.0397	2.254 ± 0.0368	2.065 ± 0.0180	1.886 ± 0.0162	1.692 ± 0.0168

Table 4: Simulated RMSE of Q_{00}, $q = 1/n$ (first row) and REFF-indicators of $\mathrm{Q}^{(q)}_{\mathrm{CH}|0}$, $\mathrm{Q}^{(q)}_{\mathrm{PRB}^*|0}$ and $\mathrm{Q}^{(q)}_{\mathrm{PRB}_{p_\ell}|0}$, for $p_\ell = \ell/(16\xi)$, $\ell = 1, 2, 4, 6, 10, 14$, for Student t_4 parents ($\xi = 0.25, \rho = -0.5$), together with 95% CIs

Student t_4 parent ($\xi = 0.25,\ \rho = -0.5$)						
n	100	200	500	1000	2000	5000
RMSE_{00}	0.378 ± 0.0039	0.320 ± 0.0030	0.270 ± 0.0021	0.240 ± 0.0014	0.215 ± 0.0013	0.185 ± 0.0007
CH	1.211 ± 0.1316	1.310 ± 0.0129	$\mathbf{1.480} \pm 0.0134$	$\underline{\underline{\mathbf{1.881}}} \pm 0.0156$	1.820 ± 0.0156	1.531 ± 0.0095
PRB*	$\mathbf{1.342} \pm 0.1474$	$\mathbf{1.378} \pm 0.0127$	$\mathit{1.459} \pm 0.0111$	$\mathit{1.652} \pm 0.0102$	$\mathbf{2.014} \pm 0.0216$	$\underline{\underline{\mathbf{3.156}}} \pm 0.0313$
$\ell = 1$	1.230 ± 0.1431	1.351 ± 0.0128	$\mathit{1.432} \pm 0.0132$	$\mathit{1.800} \pm 0.0157$	2.302 ± 0.0361	1.803 ± 0.0632
$\ell = 2$	1.281 ± 0.1408	1.359 ± 0.0129	$\underline{\underline{\mathbf{1.516}}} \pm 0.0133$	$\boldsymbol{\mathit{1.844}} \pm 0.0167$	$\underline{\underline{\mathbf{2.738}}} \pm 0.0478$	1.939 ± 0.0795
$\ell = 4$	1.364 ± 0.1496	1.390 ± 0.0125	$\mathit{1.442} \pm 0.0107$	$\mathit{1.603} \pm 0.0093$	1.895 ± 0.0182	$\mathbf{2.821} \pm 0.0284$
$\ell = 6$	1.453 ± 0.1544	1.459 ± 0.0111	$\mathit{1.411} \pm 0.0112$	$\mathit{1.447} \pm 0.0084$	$\mathit{1.547} \pm 0.0117$	1.856 ± 0.0144
$\ell = 10$	$\underline{\underline{\mathbf{1.504}}} \pm 0.1025$	$\underline{\underline{\mathbf{1.535}}} \pm 0.0139$	$\mathit{1.464} \pm 0.0108$	$\mathit{1.397} \pm 0.0063$	$\mathit{1.320} \pm 0.0080$	$\mathit{1.246} \pm 0.0080$
$\ell = 14$	1.441 ± 0.0263	1.487 ± 0.0149	$\mathit{1.430} \pm 0.0100$	$\mathit{1.356} \pm 0.0064$	$\mathit{1.268} \pm 0.0079$	$\mathit{1.134} \pm 0.0076$

Table 5: Simulated mean values (at optimal levels) of $\mathrm{Q}^{(q)}_{00}$, $\mathrm{Q}^{(q)}_{\mathrm{CH}|0}$, $\mathrm{Q}^{(q)}_{\mathrm{PRB}^*|0}$ and $\mathrm{Q}^{(q)}_{\mathrm{PRB}_{p_\ell}|0}$, for $p_\ell = \ell/(16\xi)$, $\ell = 1, 2, 4, 6, 10, 14$, for $q = 1/n$ and $\mathrm{EV}_{0.1}$ underlying parents, together with 95% CIs

EV_ξ parent, $\xi = 0.1$						
n	100	200	500	1000	2000	5000
H	1.099 ± 0.0048	1.073 ± 0.0042	1.061 ± 0.0031	1.058 ± 0.0030	1.056 ± 0.0026	1.053 ± 0.0018
CH	$\underline{\underline{\mathbf{0.905}}} \pm 0.0081$	$\underline{\underline{\mathbf{0.930}}} \pm 0.0049$	$\underline{\underline{\mathbf{0.983}}} \pm 0.0073$	1.038 ± 0.0035	1.038 ± 0.0028	1.033 ± 0.0025
PRB*	0.855 ± 0.0021	0.906 ± 0.0031	0.916 ± 0.002	0.920 ± 0.0013	0.931 ± 0.0008	0.966 ± 0.0005
$\ell = 1$	0.904 ± 0.0054	0.923 ± 0.0036	0.982 ± 0.0031	$\underline{\underline{\mathbf{0.998}}} \pm 0.0012$	1.022 ± 0.0019	1.012 ± 0.0016
$\ell = 2$	0.903 ± 0.0059	0.914 ± 0.0025	0.932 ± 0.0010	0.966 ± 0.0009	$\underline{\underline{\mathbf{0.989}}} \pm 0.0007$	$\underline{\underline{\mathbf{0.997}}} \pm 0.0003$
$\ell = 4$	0.865 ± 0.0021	0.907 ± 0.0039	0.918 ± 0.0018	0.921 ± 0.0012	0.939 ± 0.0009	0.978 ± 0.0005
$\ell = 6$	0.816 ± 0.0018	0.874 ± 0.0010	0.915 ± 0.0019	0.919 ± 0.0022	0.922 ± 0.0019	0.922 ± 0.0008
$\ell = 10$	0.753 ± 0.0013	0.811 ± 0.0010	0.859 ± 0.0007	0.883 ± 0.0006	0.901 ± 0.0007	0.918 ± 0.0006
$\ell = 14$	0.718 ± 0.0011	0.774 ± 0.0009	0.821 ± 0.0007	0.847 ± 0.0005	0.866 ± 0.0006	0.886 ± 0.0006

Table 6: Simulated mean values (at optimal levels) of $\mathrm{Q}_{00}^{(q)}$, $\mathrm{Q}_{\mathrm{CH}|0}^{(q)}$, $\mathrm{Q}_{\mathrm{PRB}^*|0}^{(q)}$ and $\mathrm{Q}_{\mathrm{PRB}_{p_\ell}|0}^{(q)}$, for $p_\ell = \ell/(16\xi)$, $\ell = 1, 2, 4, 6, 10, 14$, for $q = 1/n$ and $\mathrm{GP}_{0.1}$ underlying parents, together with 95% CIs

GP_ξ parent, $\xi = 0.1$						
n	100	200	500	1000	2000	5000
H	1.085 ± 0.0030	1.070 ± 0.0041	1.064 ± 0.0036	1.058 ± 0.0035	1.054 ± 0.0033	1.053 ± 0.0030
CH	$\mathbf{0.989} \pm 0.0060$	1.057 ± 0.0053	1.060 ± 0.0038	1.038 ± 0.0028	1.036 ± 0.0030	1.032 ± 0.0029
PRB*	0.892 ± 0.0009	0.907 ± 0.0027	0.913 ± 0.0015	0.919 ± 0.0008	0.944 ± 0.0006	0.983 ± 0.0003
$\ell = 1$	0.945 ± 0.0026	$\mathbf{0.989} \pm 0.0015$	$\mathbf{1.014} \pm 0.0013$	1.024 ± 0.010	1.023 ± 0.0008	1.020 ± 0.0006
$\ell = 2$	0.900 ± 0.0028	0.921 ± 0.0016	0.968 ± 0.0012	$\mathbf{0.990} \pm 0.0008$	$\mathbf{0.997} \pm 0.0003$	$\mathbf{0.999} \pm 0.0002$
$\ell = 4$	0.897 ± 0.0048	0.905 ± 0.0024	0.911 ± 0.0012	0.928 ± 0.0009	0.959 ± 0.0006	0.991 ± 0.0003
$\ell = 6$	0.849 ± 0.0009	0.889 ± 0.0009	0.913 ± 0.0027	0.917 ± 0.0018	0.919 ± 0.0017	0.923 ± 0.0008
$\ell = 10$	0.777 ± 0.0008	0.822 ± 0.0008	0.864 ± 0.0008	0.885 ± 0.0007	0.902 ± 0.0008	0.918 ± 0.0008
$\ell = 14$	0.737 ± 0.0008	0.782 ± 0.0008	0.825 ± 0.0007	0.849 ± 0.0006	0.867 ± 0.0007	0.886 ± 0.0007

Table 7: Simulated mean values (at optimal levels) of $\mathrm{Q}_{00}^{(q)}$, $\mathrm{Q}_{\mathrm{CH}|0}^{(q)}$, $\mathrm{Q}_{\mathrm{CH}^*|0}^{(q)}$ and $\mathrm{Q}_{\mathrm{CH}_{p_\ell}|0}^{(q)}$, for $p_\ell = \ell/(16\xi)$, $\ell = 1, 2, 4, 6, 10, 14$, for $q = 1/n$ and $\mathrm{Burr}(0.5, -0.25)$ underlying parents, together with 95% CIs

$\mathrm{Burr}(0.5, -0.25)$ parent						
n	100	200	500	1000	2000	5000
H	1.516 ± 0.0220	1.419 ± 0.0128	1.343 ± 0.0195	1.303 ± 0.0123	1.271 ± 0.0100	1.246 ± 0.0088
CH	$\mathbf{0.799} \pm 0.0058$	$\mathbf{1.079} \pm 0.0041$	1.267 ± 0.0129	1.263 ± 0.0103	1.259 ± 0.0093	1.235 ± 0.0076
PRB*	0.716 ± 0.0084	0.764 ± 0.0038	0.846 ± 0.0031	0.917 ± 0.0029	0.965 ± 0.0014	0.989 ± 0.0006
$\ell = 1$	0.722 ± 0.0076	0.821 ± 0.0041	$\mathbf{0.925} \pm 0.0031$	$\mathbf{1.024} \pm 0.0033$	1.100 ± 0.0015	1.016 ± 0.0012
$\ell = 2$	0.731 ± 0.0075	0.811 ± 0.0038	0.917 ± 0.0027	0.966 ± 0.0022	$\mathbf{0.990} \pm 0.0012$	$\mathbf{0.999} \pm 0.0009$
$\ell = 4$	0.716 ± 0.0084	0.764 ± 0.0038	0.846 ± 0.0031	0.917 ± 0.0029	0.965 ± 0.0014	0.989 ± 0.0006
$\ell = 6$	0.721 ± 0.0070	0.751 ± 0.0050	0.783 ± 0.0034	0.831 ± 0.0025	0.888 ± 0.0024	0.955 ± 0.0022
$\ell = 10$	0.696 ± 0.0073	0.724 ± 0.0072	0.754 ± 0.0040	0.769 ± 0.0038	0.775 ± 0.0025	0.789 ± 0.0016
$\ell = 14$	0.646 ± 0.0020	0.703 ± 0.0061	0.726 ± 0.0082	0.745 ± 0.0056	0.756 ± 0.0061	0.769 ± 0.0047

Table 8: Simulated mean values (at optimal levels) of $\mathrm{Q}_{00}^{(q)}$, $\mathrm{Q}_{\mathrm{CH}|0}^{(q)}$, $\mathrm{Q}_{\mathrm{PRB}^*|0}^{(q)}$ and $\mathrm{Q}_{\mathrm{PRB}_{p_\ell}|0}^{(q)}$, for $p_\ell = \ell/(16\xi)$, $\ell = 1, 2, 4, 6, 10, 14$, for $q = 1/n$ and Student t_4 underlying parents, together with 95% CIs

Student t_4 parent ($\xi = 0.25$)						
n	100	200	500	1000	2000	5000
H	1.114 ± 0.0056	1.099 ± 0.0043	1.089 ± 0.0037	1.085 ± 0.0037	1.081 ± 0.0035	1.077 ± 0.0037
CH	0.905 ± 0.0351	0.903 ± 0.0053	$\mathbf{0.922} \pm 0.0030$	$\mathbf{0.978} \pm 0.0028$	1.034 ± 0.0014	1.056 ± 0.0015
PRB*	0.927 ± 0.0558	0.905 ± 0.0048	0.907 ± 0.0023	0.924 ± 0.0009	0.948 ± 0.0014	$\mathbf{0.987} \pm 0.0011$
$\ell = 1$	0.929 ± 0.0071	0.903 ± 0.0041	0.908 ± 0.0026	0.930 ± 0.0014	0.960 ± 0.0012	1.014 ± 0.0041
$\ell = 2$	$\mathbf{0.930} \pm 0.0669$	0.897 ± 0.0040	0.912 ± 0.0025	0.940 ± 0.0011	$\mathbf{0.979} \pm 0.0013$	1.049 ± 0.0057
$\ell = 4$	0.927 ± 0.0428	$\mathbf{0.908} \pm 0.0039$	0.904 ± 0.0024	0.919 ± 0.0014	0.941 ± 0.0014	0.978 ± 0.0011
$\ell = 6$	0.898 ± 0.0098	0.906 ± 0.0045	0.906 ± 0.0018	0.908 ± 0.0014	0.921 ± 0.0010	0.946 ± 0.0009
$\ell = 10$	0.839 ± 0.0016	0.890 ± 0.0046	0.899 ± 0.0019	0.905 ± 0.0024	0.904 ± 0.0017	0.907 ± 0.0012
$\ell = 14$	0.795 ± 0.0013	0.853 ± 0.0011	0.890 ± 0.0008	0.890 ± 0.0020	0.893 ± 0.0020	0.894 ± 0.0017

For a better visualization of the Tables 1–8, we present Figures 5–8.

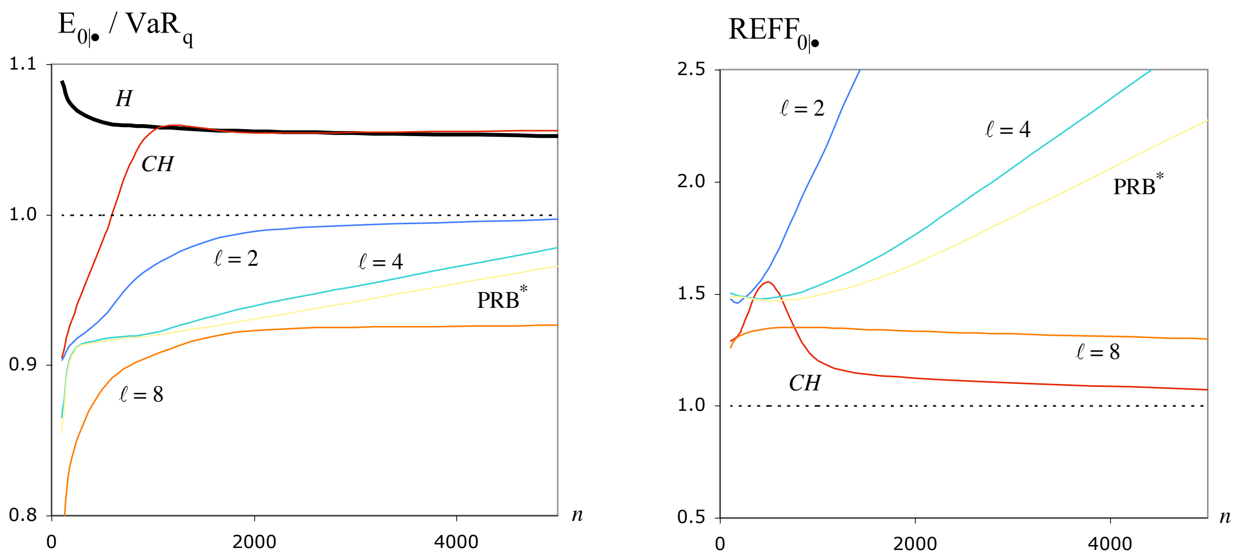


Figure 5: Normalized mean values (*left*) and REFF-indicators (*right*) of the VaR_q-estimators under study, at optimal levels, for $q = 1/n$ and $\mathrm{EV}_{0.1}$ parents

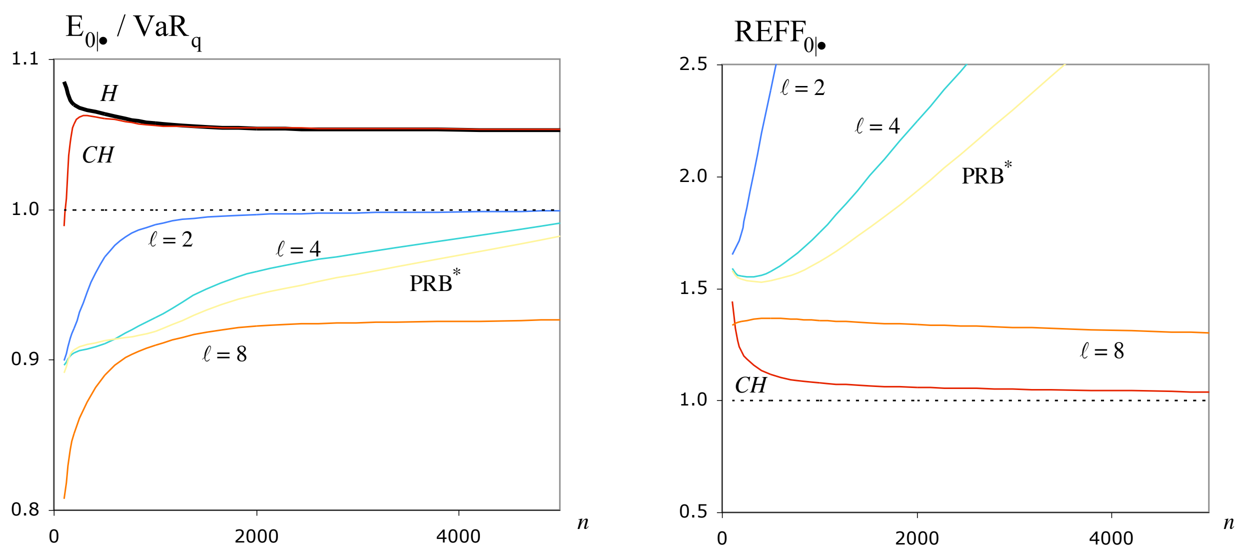
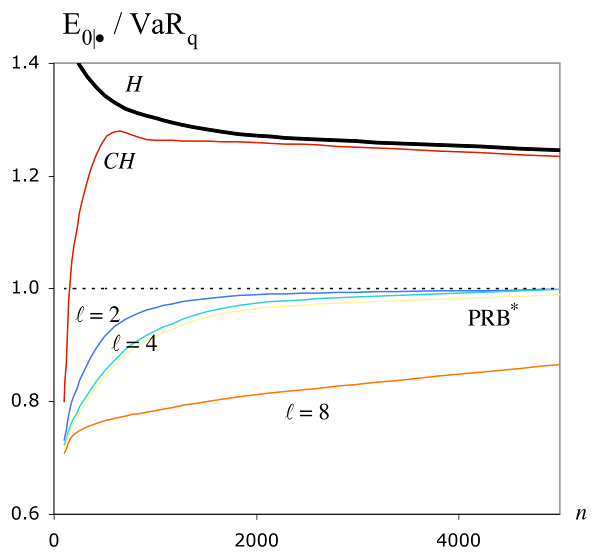


Figure 6: Normalized mean values (*left*) and REFF-indicators (*right*) of the VaR_q-estimators under study, at optimal levels, for $q = 1/n$ and $\mathrm{GP}_{0.1}$ parents



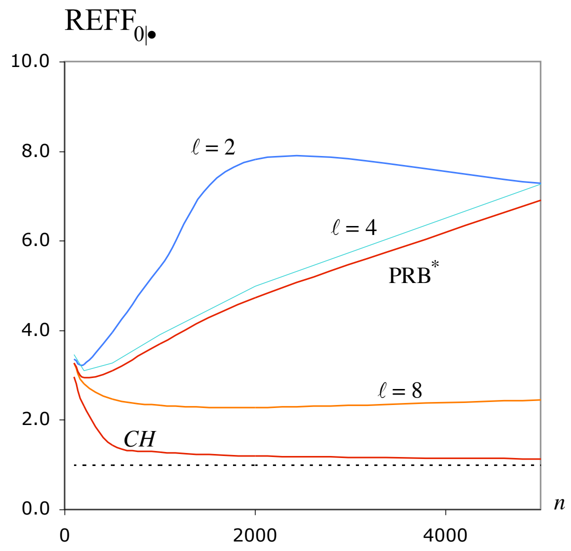
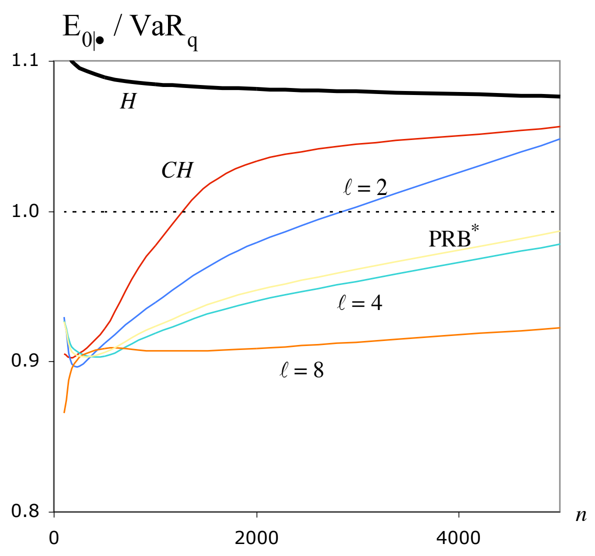


Figure 7: Normalized mean values (*left*) and REFF-indicators (*right*) of the VaR_q-estimators under study, at optimal levels, for $q = 1/n$ and $\mathrm{BURR}_{0.5,-0.25}$ parents



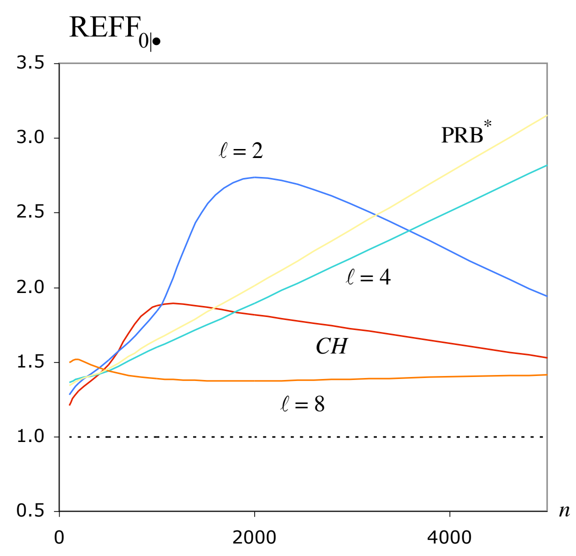


Figure 8: Normalized mean values (*left*) and REFF-indicators (*right*) of the VaR_q-estimators under study, at optimal levels, for $q = 1/n$ and Student t_4 parents

4 A case-study in the field of finance

We next exhibit the performance of the above mentioned estimators in the analysis of Euro-UK Pound daily exchange rates from January 4, 1999, till December 14, 2004. We have worked with the $n_0 = 725$ positive log-returns, and ln-VaR-estimates were considered. Being aware that the log-returns of any financial time series are not IID and that the possible presence of clustered volatility is a question of particular relevance to applied financial research (see McNeil and Frey, 2000, among others), we also know that the semi-parametric behaviour of estimators of any parameter of rare events can be generalized to weakly dependent data (see Drees, 2002, 2003, and references therein). Semi-parametric estimators of extreme event parameters, devised for IID sequences of RVs, are usually based on the tail empirical process, remaining consistent and asymptotically normal in a large class of weakly dependent data. However, although financial returns series typically exhibit little correlation, the squared returns often indicate significant correlation and persistence, an evidence of the presence of heteroscedasticity. Engle's ARCH test for detecting the presence of ARCH effects (see Engle, 1982; Box *et al.*, 1994), and the ARCH/GARCH model, a typical model for this type of empirical data, was not rejected for this log-returns data set. Such a test has also shown significant evidence on support of GARCH effects, i.e. heteroscedasticity, indicating that GARCH modeling is appropriate. In order to remove the observed stock returns heteroscedasticity, we have fitted the volatility model GARCH(1,1) to the data set, and have then applied the above mentioned estimators to the standardized log-returns, $y_t^s = y_t/\sigma_t$, where y_t are the log-returns and σ_t the standard deviation forecast. There was next no significant evidence in support of GARCH effects for the standardized return series, and we have more confidently assumed a stationary setup for the standardized log-returns.

The second-order estimates were computed at the level $k_1 = \lfloor n_0^{0.999} \rfloor = 720$ and are equal to $(\hat\rho(k_1), \hat\beta(k_1)) = (-0.673, 1.038)$. The H, CH and PRB* EVI-estimates were obtained heuristically and on the basis of a sample-path stability algorithm similar to the one presented in Gomes *et al.* (2013). The associated 95% asymptotic confidence intervals (CIs) were obtained taking into account Remarks 3.2 and 3.3. of Gomes and Pestana (2007). The sample paths of the H, CH and PRB* ln-VaR–estimators, for $q = 0.001$, together with the estimated ln-VaR$_q$, are pictured in Figure 9. The final estimate of ln-VaR$_q$ was obtained heuristically again and on the basis of sample path stability. Indeed, when we consider the three estimates with one decimal figure only and the first 50 values of k, the percentage of times that the value 1.6 appears is 28%, 54% and 64% respectively for the H, the CH and the PRB* ln-VaR-estimates. Moreover, and when considering the first 200 values of k, 87% of the PRB* ln-VaR-estimates are equal to 1.6.

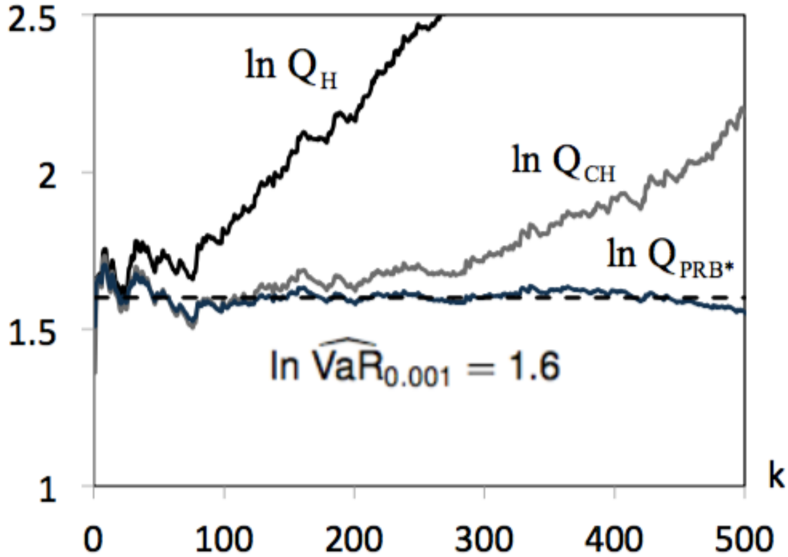


Figure 9: ln-VaR$_q$-estimates provided through the different classes of VaR-estimators under consideration, for the standardized daily log-returns on the Euro-UK Pound and $q = 0.001$

The largest value of k in the aforementioned stability regions leads then to estimates of k, denoted by $\hat{k}_0^{Q_\bullet}$, and to the computation of the asymptotic confidence intervals (CIs) for the ln-VaR$_q$ estimates as suggested in Gomes and Pestana (2007), Remark 5.3., both presented in Table 9.

Table 9: Heuristic choice of k, associated ln-VaR-estimates, asymptotic 95% CIs and respective CI size

$\bullet$	$\hat{k}_0^{Q_\bullet}$	ln-$Q_\bullet\left(\hat{k}_0^{Q_\bullet}\right)$	$(LCL_{Q_\bullet}, UCL_{Q_\bullet})$	95% CI size
H	26	1.645	(1.120, 2.016)	0.896
CH	206	1.644	(1.343, 1.944)	0.601
PRB*	500	1.553	(1.334, 1.771)	0.437

As expected, due to the larger value of $\hat{k}_0^{Q_\bullet}$, and despite the larger asymptotic variance for a similar value of k, the estimated and non-optimal PRB* VaR-estimate leads to the shortest CI.

5 Concluding remarks

— It is well-known that Weissman-Hill VaR-estimation leads to a strong over-estimation of VaR and the PRB-MO$_p$ methodology can provide a more adequate VaR-estimation, being even able to beat the MVRB VaR-estimators in a very large variety of situations.

— For all simulated models with $|\rho| < 0.5$, and regarding minimal RMSE, even the non-optimal adaptive VaR-estimator PRB*, dependent on the estimation of ξ and ρ, always beats the CH VaR-estimator. The pattern is not so clear-cut regarding bias.

— The use of $\mathrm{Q}_{\mathrm{PRB}_p}$, with an adequate value of p, always enables a reduction in RMSE regarding the the CH VaR-estimator, and consequently, regarding the Weissman-Hill VaR-estimator. Moreover, the bias is also reduced. Such a reduction in squared bias is particularly high for values of ρ close to zero.

— The reduction, both in squared bias and RMSE, frequently happens for $p < 1/(2\xi)$ $(\ell < 8)$. However, for small n $(n \leq 200)$ and Student t_4 parents, the highest efficiency is attained at $p > 1/(2\xi)$, and we thus cannot assure the asymptotic normality of the PRB VaR-estimators, being such an asymptotic behaviour under current research.

— The patterns of the estimators' sample paths are always of the same type, in the sense that for all k the VaR-estimator, $\mathrm{Q}^{(q)}_{\mathrm{PRB}_p}$, decreases as p increases.

— The choice of the tuning parameters (k, p) can be done on the basis of reliable heuristic procedures related to sample path stability, in the line of the algorithms in Gomes *et al.* (2013) and Neves *et al.* (2015), as performed in Gomes *et al.* (2015d) and in the case-study presented in Section 4. Indeed, even a non-optimal choice of p associated with any simple rule of sample stability, will lead us to an adaptive value of p, with a lot of gain in the estimation of VaR.

— We further think sensible to devise and study in the near future, and both theoretical and computationally, an algorithm of the type of the double-bootstrap algorithms in Gomes *et al.* (2011, 2012, 2015e), among others, taking into account a possible dependence among data. Indeed, double-bootstrapping procedures for sample fraction selection are quite common and reliable under IID frameworks, but suffer an important caveat for dependent data since the bootstrapped sample does not possess the same serial dependence structure as the original sample. Therefore, the estimators based on the bootstrapped samples do not share the same asymptotic behaviour as the original estimator based on an original serial dependent sample. Consequently, the optimal (k, p) that minimises the RMSE based on double-bootstrapped samples may not be optimal for the original sample. Ignoring the serial dependence in double bootstrapping can be therefore misleading. For further applications of the bootstrap methodology to the estimation of parameters of extreme events under an IID framework, see also Caeiro and Gomes (2015) and Gomes *et al.* (2016b), where R-scripts are provided.

— An application involving the model of heteroscedastic extremes in Einmahl *et al.* (2016), is feasible and under development.

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