Dependent Control Statistics: time of the first passage over a given threshold

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Abstract. In many inspection processes it is not feasible to take more than one observation at each sampling point to control the process. In such cases it is common to use a control statistic that accumulates the information of past data in order to improve the performance of the monitoring scheme. The most important control schemes performance measures are related with the time of the first passage of the control statistic over a given threshold, and are usually determined by Monte Carlo simulations, because analytical expressions are, in general, very difficult or even impossible, to obtain. In this paper we present some distributional results that allow us to obtain one of the most important control chart performance measure, the average run length (i.e., the average number of samples taken from the process until the first time the control statistic over pass the control limits). We consider two specific control schemes for monitoring industrial processes based on k-dependent statistics, the moving maxima and the moving sum control charts, and under the assumption of independent observations from normal or exponential processes, we provide analytical expressions for the average run length.

AMS 2000 subject classification: 62P30; 60E05;

Keywords: Statistical Process Control; Control Charts; Average Run Length; Dependent Control Statistics.

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1 Introduction

In many production processes, in particular in the chemistry industry, it is not possible to take more than one observation at each time of sampling. In such cases a common practice is to consider a control chart for individuals (X) to monitor the process mean value, and a moving range (MR) chart to monitor the process variability. However, taking in account that the performance of the monitoring scheme is positively correlated with the sample size taken to inspection, the drawback of having available only one new observation at each time of sampling could be compensated using a control statistic that accumulates the information of past data. Many practitioners began to use the moving average (MA), the cumulative sum ($CUSUM$) and the exponentially weighted moving average ($EWMA$) charts, as an alternative to the X chart for the individual observations. General details about control charts may be found, for instance, in Ryan (2000) and in Montgomery (2005).

The most common control schemes performance measures are related with the time of the first passage of the control statistic over a given threshold. These measures are usually determined by Monte Carlo simulations, because analytical expressions are in general difficult, or even impossible, to be derived. For instance, we note that consecutive values of the $CUSUM$ and of the $EWMA$ control statistics share observations of different samples, and thus, to evaluate their performance, we have to consider the structure of the data process together with the structure of the dependence between consecutive values of the control statistic. Specifically speaking, the ability of a control chart to detect changes in the process parameters is analyzed in terms of the distribution of the run length (RL) variable. This variable represents the number of samples taken until the chart signals, i.e., until the control statistic over pass the control limits of the chart; in practice, we usually compute its average run length, the ARL, and eventually, its standard deviation.

To compare different charts they must have the same in-control ARL, which is in general a large, pre-fixed value, because it is the expected number of samples to the occurrence of a false alarm; the out-of-control ARL must be small so that the change is quickly detected; consequently, the most efficient chart is the one with the smallest out-of-control ARL for the shifts we want to detect. For specific details about performance measures see, for instance, Nelson (1982), Rigdon *et. al* (1994), Amin and Ethridge (1998), Roberts (2000) and Reynolds and Stoumbos (2001).

The importance of the ARL as a performance measure, but also to set the control limits of the chart in order to obtain a control scheme with specific properties, lead us to find analytical expressions for the ARL of the charts usually used in practice. In this paper we present, in section 2, some generic expressions that allow us to obtain the ARL of a control chart based on a k-dependent statistic, and to determine the control limits in order to have an unbiased control chart. To motivate the use of the k-dependent moving maxima (MM^k) and moving sum (MS^k) control charts, we advance in section 3 with some distributional properties about these control statistics; we also present, for $k \leq 3$, explicit expressions for the probabilities that are used in the computation of the ARL of the MM^k and of the MS^k charts, implemented under the assumption of independent observations from normal or exponential processes.

2 Control chart based on k-dependent statistics: average run length and control limits

Let us generally denote the control statistic associated to a given control chart at time t by W_t, $t \geq 1$. We say that we are in a presence of a k-dependent structure, with $k \geq 1$, if for all $t \geq 1$ we have

$$\begin{aligned}&W_t \text{ and } W_{t+i} \text{ are dependent for } i < k,\\ &W_t \text{ and } W_{t+i} \text{ are independent for } i \geq k.\end{aligned}$$

In the particular case of $k = 1$, the variables W_t are independent for every t.

Most of the parametric control charts are used to detect changes in one or more control parameters relatively to pre-fixed targets in both directions (i.e., are two–sided control charts), and have the following decision rule: at each sampling point time t, the values of the control statistic W_t are compared with the lower and the upper control limits of the chart, here denoted by LCL and UCL. Whenever W_t falls outside the interval $C = [LCL, UCL]$ the chart signals, and the process is supposed to be out-of-control; otherwise, the process is considered to be in-control.

Let us denote C^k the Cartesian product of the interval $C = [LCL, UCL]$

iterated k times, i.e., $C \times ... \times C$, and p_i, $0 \leq i \leq k$, the following probabilities

$$p_i = P\left((W_t, W_{t+1}, ..., W_{t+i-1}) \in C^i\right), \ t \geq 1, \quad 1 \leq i \leq k, \tag{2.1}$$

with $p_0 = 1$ and C^0 denoting the sampling space, by convention.

For a k-dependent structure, with $k \geq 1$, the distribution of the random variable RL, number of samples to signal, is given by

$$f(r) = P(RL = r) = \begin{cases} p_{r-1} - p_r & \text{if} \quad 1 \leq r \leq k-1 \wedge k \geq 2 \\ (p_{k-1} - p_k)\left(\dfrac{p_k}{p_{k-1}}\right)^{r-k} & \text{if} \quad r \geq k, \end{cases} \tag{2.2}$$

and the average number of samples to signal, ARL, is expressed by

$$ARL = \sum_{r=1}^{\infty} rP(RL = r) = \sum_{i=0}^{k-2} p_i + \frac{p_{k-1}^2}{p_{k-1} - p_k}, \tag{2.3}$$

with $\sum\limits_{i=l}^{L} p_i = 0$ for $L < l$.

As previously mentioned, the ARL is the most common performance measure of the chart, and at the same time it is used to determine its control limits in order to obtain specific properties. When the process is in-control the ARL must be equal to a pre-defined fixed value, say $ARL_{\text{in-control}} = ARL_0$, and when the process is out-of-control the ARL must be smaller than this value whenever it is possible, i.e., $ARL_{\text{out-of-control}} \leq ARL_{\text{in-control}}$. A control chart with this property is called an unbiased control chart.

If we generally denote the shift magnitude we want to detect in the control parameter by Δ, and if we assume that under control $\Delta = \Delta_0$, for a k-dependent structure the control limits of an unbiased two-sided control chart are determined such that

$$\begin{cases} \left[\sum\limits_{i=0}^{k-2} p_i + \dfrac{p_{k-1}^2}{p_{k-1} - p_k}\right]_{\Delta=\Delta_0} = ARL_0 \\ \left[\sum\limits_{i=0}^{k-2} \dfrac{\partial p_i}{\partial \Delta} + p_{k-1}\dfrac{(p_{k-1} - 2p_k)\frac{\partial p_{k-1}}{\partial \Delta} + p_{k-1}\frac{\partial p_k}{\partial \Delta}}{(p_{k-1} - p_k)^2}\right]_{\Delta=\Delta_0} = 0, \end{cases} \tag{2.4}$$

with $\sum\limits_{i=l}^{L} p_i = 0$ for $L < l$.

With little adjustments we obtain the ARL for a one-sided control chart, which has only one control limit: an upper control limit if the chart is implemented to detect increases in the control parameter, or a lower control limit if it is implemented to detect decreases; the decision rule associated to this chart is similar to the previous one. To determine the control limit of the chart in order to obtain specific properties, we only use, in this case, the first equation in (2.4). More details about other measures to evaluate the properties of a control chart can be found, for instance, in Crowder (1987), in Reynolds et al. (1988, 1990) and in Reynolds and Stoumbos (2001).

3 Moving maxima and moving sum control charts

Suppose that X represents a process quality variable whose distribution is F, and that it is not feasible to take more than one observation at each sampling point to control the process. To implement the moving maxima and the moving sum control charts in a k-dependent structure, here denoted by MM^k and MS^k, with $k \geq 1$ fixed, we must assume that in the start-up control phase and after the chart signals, the (re)implementation of the chart occurs after the process has been running for a reasonable period of time in order to have k observations available, and to assume that the control statistic has already reached a stationary distribution before the chart signals again. Moreover, we also assume that the sampling interval is sufficiently large to admit that all the observations used to implement the control charts are independent and identically distributed (i.i.d.), with cumulative distribution function (cdf) F.

The MM^k chart plots in each sampling point t the value of the statistic M_t^k, $k \geq 1$, defined by

$$M_t^k = Max(X_t, X_{t-1}, ..., X_{t-k+1}), \quad t \geq 1, \tag{3.1}$$

and the MS^k chart plots in each sampling point t the value of the statistic S_t^k, $k \geq 1$, defined by

$$S_t^k = X_t + X_{t-1} + ... + X_{t-k+1}, \quad t \geq 1, \tag{3.2}$$

where X_i, $i < t$ denotes the extra observations taken in advance from the process. For large values of k these extra observations may have some effect in the performance of the charts to detect large shifts, but for small values of k this effect is negligible. Considering $k \leq 3$ we already obtain control charts more efficient than the traditional X chart for the individual observations, and the overall benefits in terms of efficiency and difficulty of implementation of the charts for $k > 3$, may be considered insignificant.

3.1 Distributional properties of the moving maxima statistic

The control statistic M_t^k, $k \geq 1$, in (3.1), can be expressed in the form

$$M_t^k = \begin{cases} X_t & \text{if} \quad k = 1 \\ Max(M_t^{k-1}, X_t) & \text{if} \quad k > 1 \end{cases}, \quad t \geq 1, \tag{3.3}$$

and the cdf of M_t^k, $k \geq 1$, is given by

$$F_{M_t^k}(m) = F^k(m), \quad t \geq 1. \tag{3.4}$$

The joint cdf of $(M_t^k, M_{t+1}^k, ..., M_{t+r-1}^k)$, $1 \leq r \leq k$, $k \geq 1$ and $t \geq 1$, is given for all admissible combinations $(m_1, ..., m_r)$, by

$$F_{1,...,r}^k(m_1, ..., m_r) =$$
$$= \begin{cases} F^k(m_1), \quad r = 1 \\ \prod_{i=1}^{r-1} \left(F(\min_{1 \leq j \leq i} m_j) F(\min_{k-i+1 \leq j \leq k} m_j) \right) F^{k-r+1}(\min_{1 \leq j \leq k} m_j), \quad r \geq 2. \end{cases} \tag{3.5}$$

To derive this distribution we take in account the variables X_t that are presented in each of the M_t^k, $t \geq 1$ variables, as well as the number of times that X_t appears in the vector $(M_t^k, M_{t+1}^k, ..., M_{t+r-1}^k)$. Thus, for admissible values $(m_1, ..., m_r)$, the condition

$$M_1^k \leq m_1 \cap M_2^k \leq m_2 \cap ... \cap M_r^k \leq m_r$$

holds if and only if

$$X_1 \leq m_1 \cap X_{k+r-1} \leq m_r$$
$$X_2 \leq m_1 \cap X_2 \leq m_2 \cap X_{k+r-2} \leq m_r \cap X_{k+r-2} \leq m_{r-1}$$
....
$$X_{r-1} \leq m_1 \cap ... \cap X_{r-1} \leq m_{r-1} \cap X_{k+1} \leq m_r \cap ... \cap X_{k+1} \leq m_2$$
$$X_r \leq m_1 \cap ... \cap X_r \leq m_r$$
....
$$X_k \leq m_1 \cap ... \cap X_k \leq m_r.$$

Working with independent and identically distributed variables, X_t, with cdf F, we get

$$\begin{aligned}&F^k_{1,...,r}(m_1,...,m_r) = \\ &\quad = F(m_1)F(m_r)F(\min(m_1,m_2))F(\min(m_r,m_{r-1}))... \\ &\qquad ...F(\min(m_1,...,m_{r-1}))F(\min(m_r,...,m_2))F^{k+r-1}(\min(m_1,...,m_r)).\end{aligned} \tag{3.6}$$

We note that the variable $(M^k_t, M^k_{t+1}, ..., M^k_{t+r-1})$ has a singular distribution with a probability mass function for some values $(m_1, ..., m_r)$. General results about order statistics can be found in David (1980).

Taking in account these distributions we compute, for $k \leq 3$, the probabilities p_i in (2.1), needed to compute the average run length of the chart. For the MM^2 chart we have

$$\begin{aligned}p_1 &= F^2(UCL) - F^2(LCL), \\ p_2 &= F^3(LCL) + F^3(UCL) - 2F(UCL)F^2(LCL),\end{aligned} \tag{3.7}$$

and for the MM^3 chart, we have

$$\begin{aligned}p_1 &= F^3(UCL) - F^3(LCL), \\ p_2 &= F^4(LCL) + F^4(UCL) - 2F(UCL)F^3(LCL), \\ p_3 &= F^5(LCL) + 2F(UCL)F^4(LCL) - 3F^2(UCL)F^3(LCL).\end{aligned} \tag{3.8}$$

3.2 Distributional properties of the moving sum statistic

The control statistic S^k_t, $k \geq 1$, in (3.2), can also be expressed in the form

$$S^k_t = \begin{cases} X_t & \text{if} \quad k = 1 \\ S^{k-1}_t + X_t & \text{if} \quad k > 1 \end{cases}, \quad t \geq 1, \tag{3.9}$$

and, for $k \geq 1$, the distributions of the random variables S^k_t and $(S^k_t, S^k_{t+1}, ..., S^k_{t+r-1})$, $1 \leq r \leq k$, and $t \geq 1$, can be easily derived only for some particular models F and some values of k. In the sequel, we present these distributions for a normal model F and $k \geq 1$, and for an exponential model F in the case of $k \leq 3$, models that are used in many applications of different areas of research.

If the random variables X_t are independent and normally distributed with mean value μ and variance σ^2, the statistic S_t^k, $k \geq 1$ and $t \geq 1$, has a normal distribution with mean value $k\mu$ and variance $k\sigma^2$, and the probability density function (pdf) is given by

$$f_{S_t^k}(s) = \frac{1}{\sigma\sqrt{2\pi k}} exp\left\{-\frac{1}{2k\sigma^2}(s-k\mu)\right\}, \quad s \in R. \tag{3.10}$$

The random vector $(S_t^k, S_{t+1}^k, ..., S_{t+r-1}^k)$, $1 \leq r \leq k$, $k \geq 1$ and $t \geq 1$, has a multivariate normal distribution, with vector of means $\underline{\mu}_{r\times 1} = \begin{pmatrix} k\mu \\ ... \\ k\mu \end{pmatrix}$ and covariance matrix $\Sigma_r = [\sigma_{ij}]_{r\times r}$, with $\sigma_{ij} = \sigma^2(k-(j-i))$, $1 \leq i \leq j \leq r$.

The joint pdf of $(S_t^k, S_{t+1}^k, ..., S_{t+r-1}^k)$, $1 \leq r \leq k$ and $t \geq 1$, is given by

$$\begin{aligned} &f_{1,2,...r}(s_1, s_2, ..., s_r) = \\ &= \frac{\sqrt{|\Sigma_r^{-1}|}}{(2\pi)^{\frac{r}{2}}} exp\left\{-\frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\sigma^{ij}(s_i-k\mu)(s_j-k\mu)\right\}, \end{aligned} \tag{3.11}$$

where $\Sigma_r = [\sigma_{ij}]$, $\Sigma_r^{-1} = \left[\sigma^{ij}\right]$ and $|\Sigma_r^{-1}| = \frac{1}{|\Sigma_r|}$.

For the MS^2 chart, the probabilities p_i in (2.1) are given by

$$\begin{aligned} p_1 &= \Phi\left(\tfrac{LSC-2\mu}{\sqrt{2}\sigma}\right) - \Phi\left(\tfrac{LIC-2\mu}{\sqrt{2}\sigma}\right), \\ p_2 &= \int_{LIC}^{LSC} \tfrac{1}{2\sigma\sqrt{\pi}}\, e^{-\frac{(s_1-2\mu)^2}{4\sigma^2}} \left[\Phi\left(\tfrac{LSC-(\mu+\frac{s_1}{2})}{\sqrt{\frac{3}{2}}\sigma}\right) - \Phi\left(\tfrac{LIC-(\mu+\frac{s_1}{2})}{\sqrt{\frac{3}{2}}\sigma}\right)\right] ds_1, \end{aligned} \tag{3.12}$$

and for the MS^3 chart, we have

$$\begin{aligned} p_1 &= \Phi\left(\tfrac{LSC-3\mu}{\sqrt{3}\sigma}\right) - \Phi\left(\tfrac{LIC-3\mu}{\sqrt{3}\sigma}\right), \\ p_2 &= \int_{LIC}^{LSC} \tfrac{1}{2\sigma\sqrt{\pi}}\, e^{-\frac{(s_1-2\mu)^2}{4\sigma^2}} \left[\Phi\left(\tfrac{LSC-\left(\mu+\frac{2s_1}{3}\right)}{\sqrt{\frac{5}{3}}\sigma}\right) - \Phi\left(\tfrac{LIC-\left(\mu+\frac{2s_1}{3}\right)}{\sqrt{\frac{5}{3}}\sigma}\right)\right] ds_1, \\ p_3 &= \int_{LIC}^{LSC}\int_{LIC}^{LSC} f_{S_1,S_2}(s_1,s_2) \left[\Phi\left(\frac{LSC-\frac{6\mu-s_1+4s_2}{5}}{\sqrt{\frac{8}{5}}\sigma}\right) - \Phi\left(\frac{LIC-\frac{6\mu-s_1+4s_2}{5}}{\sqrt{\frac{8}{5}}\sigma}\right)\right] ds_2 ds_1, \end{aligned} \tag{3.13}$$

where Φ denotes the cdf of a standard normal distribution.

If the random variables X_t are independent and identically distributed to an exponential random variable X with scale parameter δ, i.e., $f(x) = \frac{1}{\delta}\, e^{-\frac{x}{\delta}}, x \geq 0$, it is very difficult to obtain the previous distributions for $k > 3$. However, the pdf of the statistic S_t^2, $t \geq 1$, is given by

$$f_{S_t^2}(s) = \frac{s}{\delta^2} e^{-\frac{s}{\delta}}, \ \ s \geq 0, \tag{3.14}$$

and the joint pdf of (S_t^2, S_{t+1}^2), $t \geq 1$, is given by

$$f_{S_t^2, S_{t+1}^2}(s_1, s_2) = \frac{1}{\delta^2} e^{-\frac{1}{\delta}(s_1+s_2)} \left(e^{\frac{\min(s_1,s_2)}{\delta}} - 1 \right). \tag{3.15}$$

Taking in account these distributions, for the MS^2 we have

$$\begin{aligned}
p_1 &= \left(\tfrac{LCL}{\delta} + 1\right) e^{-\frac{LCL}{\delta}} - \left(\tfrac{UCL}{\delta} + 1\right) e^{-\frac{UCL}{\delta}}, \\
p_2 &= 2\left(\tfrac{LCL}{\delta} - \tfrac{UCL}{\delta}\right) e^{-\frac{UCL}{\delta}} + \left(e^{-\frac{LCL}{\delta}} - e^{-\frac{UCL}{\delta}}\right)\left(2 + e^{-\frac{UCL}{\delta}} - e^{-\frac{LCL}{\delta}}\right).
\end{aligned} \tag{3.16}$$

The pdf of the statistic S_t^3, $t \geq 1$, is given by

$$f_{S_t^3}(s) = \frac{s^2}{2\delta^3} e^{-\frac{s}{\delta}}, \ \ s \geq 0, \tag{3.17}$$

the joint pdf of (S_t^3, S_{t+1}^3) is given by

$$\begin{aligned}
&f_{S_t^3, S_{t+1}^3}(s_1, s_2) = \\
&\quad = \frac{1}{\delta^3} e^{-\frac{1}{\delta}(s_1+s_2)} \left(\min(s_1, s_2) e^{\frac{\min(s_1,s_2)}{\delta}} - \delta e^{\frac{\min(s_1,s_2)}{\delta}} + \delta \right),
\end{aligned} \tag{3.18}$$

and the joint pdf of $(S_t^3, S_{t+1}^3, S_{t+2}^3)$, for $t \geq 1$, is given by

$$\begin{aligned}
&f_{S_t^3, S_{t+1}^3, S_{t+2}^3}(s_1, s_2, s_3) = \\
&= \begin{cases}
-\frac{1}{\delta^3} e^{-\frac{s_1+s_3}{\delta}} \left(\frac{s_1}{\delta} + 1\right) + \frac{1}{\delta^3} e^{-\frac{s_3}{\delta}}, \ \ 0 < s_1 < s_2 < s_3 \\
-\frac{1}{\delta^3} e^{-\frac{s_1+s_3}{\delta}} \left(\frac{s_2}{\delta} + 1 - e^{\frac{s_2}{\delta}}\right), \ \ 0 < s_2 < s_1 < s_3 \ \ \vee \ \ 0 < s_2 < s_3 < s_1 \\
-\frac{1}{\delta^3} e^{-\frac{s_1+s_3}{\delta}} \left(\frac{s_3}{\delta} + 1\right) + \frac{1}{\delta^3} e^{-\frac{s_1}{\delta}}, \ \ \max\{0, 2s_2 - s_1\} < s_3 < s_2 < s_1 \\
-\frac{1}{\delta^3} e^{-\frac{s_1+s_3}{\delta}} \left(\frac{s_3}{\delta} + 1\right) + \frac{1}{\delta^3} e^{-\frac{s_1}{\delta}}, \\
\qquad \max\{0, s_2 - s_1\} < s_3 < s_2 < s_1 < 2s_2 - s_3 \\
-\frac{1}{\delta^3} e^{-\dfrac{s_1+s_3}{\delta}} \left(\frac{s_1 - s_2 + s_3}{\delta} + 1\right) + \frac{1}{\delta^3} e^{-\dfrac{s_2}{\delta}}, \\
\qquad 0 < \min\{s_1, s_3\} < \max\{s_1, s_3\} < s_2 < s_1 + s_3 \\
0, \text{otherwise.}
\end{cases}
\end{aligned} \tag{3.19}$$

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