Alternatives to a Chart for Individuals: Moving Sums and Moving Maxima

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Abstract

In some production processes it is not possible to take more than one observation at each sampling point to control the process. In such cases it has been usual to consider a control chart for individuals to monitor the mean of the process and a moving range chart to monitor the process variability. In this paper we purpose two alternatives to the chart for individuals: the moving maxima chart and the moving sum chart. These charts were implemented with fixed and variable sampling intervals and their performance was evaluated in terms of the *Average Time to Signal*, an indicator which takes into account the structure of dependence between the observed values of the control statistic. In this study we compare the performance of the different charts to detect shifts in the mean and/or standard deviation of a normal process and to detect shifts in the scale parameter of a uniparametric exponential process, under the assumption that the observations from the process are independent.

Keywords: Average Time to Signal, Fixed Sampling Interval, Shewhart Control Charts, Statistical Quality Control, Variable Sampling Interval.

1 Introduction

In a production process, even if it has been carefully planned and maintained, there is always some variability that can be due to a random or a deterministic cause. When there is only the variability inherent to the production process, which is unavoidable, we say that the process is under statistical control. If there is another kind of variability, associated to deterministic causes, we say that the process is out of statistical control. The main purpose of the statistical quality control is to distinguish these two kinds of variability.

Control charts are the basic tools in statistical process control. They are widely used in industry and allow to distinguish the two kinds of variability referred above, but their efficiency depends on the sample size taken to inspection. Some generic results about control charts for monitoring a process can be found in Montgomery (1996).

In some production processes, particularly in the chemical industry, it is not feasible to take more than one observation at each sampling point for monitoring the process. In such cases it has been usual to consider a control chart for individuals to monitor the mean of the process and a moving range chart to monitor the process variability. However, some authors have questioned this practice. Among them, we refer Nelson (1982,1990), Roes et al. (1993) and Rigdon et al. (1994). For example, Nelson (1982,1990) argued on the one hand that the X chart contains all the available information, and on other hand that the MR chart has a difficult interpretation because the plotting moving ranges are correlated. Another comparative study of the X and MR charts can be found in Amin and Ethridge (1998). They concluded that the X chart alone is nearly as effective as the combined procedure X-MR for monitoring a normal process, which does not justify the use of the joint procedure X-MR.

In this paper we propose two alternatives to the chart for individuals (X) to monitor both the mean and the standard deviation of the process: the moving maxima chart (MM) and the moving sum chart (MS). These charts were implemented with fixed sampling intervals (FSI) and with variable sampling intervals (VSI), and we also have considered 2-dependent and 3-dependent structures for the control statistic. If we generally denote the control statistic associated to the different charts at time t by W_t, $t \geq 1$, we say that we are in a presence of a *k-dependent structure*, $k \geq 1$, if we have

$$W_t \ and \ W_{t+i} \ are \ dependent \ for \ i < k$$

$$W_t \ and \ W_{t+i} \ are \ independent \ for \ i \geq k, \ \forall t \geq 1.$$

In the particular case of $k = 1$, the variables W_t are independents, $\forall t$.

Suppose that X represents a process quality variable whose distribution is $F(.)$. In a *k-dependent structure*, $k \geq 1$, the X chart plots the value of the observation X_t taken from the process against time t, the MM^k chart plots in each sampling point t the value of the statistic M_t^k defined by

$$M_t^k = Max(X_t, X_{t-1}, ..., X_{t-k+1}), \quad t \geq 1, \tag{1}$$

and the MS^k chart plots the value of the statistic S_t^k defined by

$$S_t^k = X_t + X_{t-1} + ... + X_{t-k+1}, \quad t \geq 1, \tag{2}$$

where $X_{-k+2}, X_{-k+3}, ..., X_0$ represent available observations taken from the process before the beginning of the chart plotting.

When we use a FSI control chart, the values W_t are observed at regular time intervals of length d, and will be compared with the lower and the upper control limits of the chart, denoted by LCL and UCL. When we use a VSI control chart, the interval of time between consecutive observations is not fixed but depends on the last observed statistic value, taking into account whether it falls near or far from the control limits. In our study we consider only two sampling intervals and the region $C = [LCL, UCL]$ will be divided into two subregions C_1 and C_2, associated to the smallest and to the largest sampling intervals, d_1 and d_2, respectively. These subregions are defined by $C_1 = [LCL, LWL[\cup]UWL, UCL]$ and $C_2 = [LWL, UWL]$, where LWL and UWL denotes the lower and the upper warning limits of the chart.

The performance of these charts are evaluated by means of the *Average Time to Signal*. In this paper we compare the performance of the X, MM^k and $MS^k, k = 2, 3$, charts to detect shifts in the mean and/or standard deviation of a normal process and to detect shifts in the scale parameter of a uniparametric exponential process, under the assumption that the observations from the process are independent.

2 Measures of chart performance

The ability of a control chart to detect process changes can be measured by the expected length of time the chart takes to generate a signal, say, by ATS (*Average Time to Signal*).

To compare the different charts they must have the same in-control ATS, which must be large because it is associated to the frequency of false alarms, and the out-of-control ATS must be small so that the change is quickly detected. The most efficient chart is the one with the smallest out-of-control ATS for the shifts we want to detect.

If we represent by TS and NSS the random variables *Time to Signal* and *Number of Samples to Signal* respectively, and using the fact that $ATS = E(TS)$, the ATS for an FSI chart can be expressed in the form

$$ATS = dANSS, \tag{3}$$

where d denotes the sampling interval and $ANSS$ denotes the average number of samples taken before the chart signals. For a VSI chart the ATS will be expressed in the form

$$ATS = E\left(\sum_{t=1}^{NSS} D_{t-1}\right) = E(D)ANSS,$$

where D_t denotes the sampling interval that precedes the $t+1^{th}$ sample to be taken, $ANSS$ denotes the average number of samples taken before the chart signals and $E(D)$ denotes the average sampling interval.

In this study we will denote by C^k the cartesian product of an interval C iterated k times, $C \times ... \times C$, and the probabilities associated to the control and warning intervals, C and C_2, by

$$\begin{aligned} p_i &= P\left(\left(W_t, W_{t+1}, ..., W_{t+i-1}\right) \in C^i\right), \\ p_{vi} &= P\left(\left(W_t, W_{t+1}, ..., W_{t+i-2}\right) \in C^{i-1} \cap W_{t+i-1} \in C_2\right), \; t \geq 1, \;\; 1 \leq i \leq k. \end{aligned} \tag{4}$$

where $p_0 = 1$ and C^0 represents the sampling space, by convention.

For a *k-dependent structure*, $k \geq 1$, the distribution of the random variable NSS is given by

$$f(n) = P(NSS = n) = \begin{cases} p_{n-1} - p_n, & 1 \leq n \leq k-1 \wedge k \geq 2 \\ (p_{k-1} - p_k)\left(\frac{p_k}{p_{k-1}}\right)^{n-k}, & n \geq k, \end{cases} \tag{5}$$

and the $ANSS$ is expressed by

$$ANSS = \sum_{n=1}^{\infty} nP(NSS = n) = \sum_{i=0}^{k-2} p_i + \frac{p_{k-1}^2}{p_{k-1} - p_k}, \tag{6}$$

where $\sum_{i=l}^{L} p_i = 0$ for $L < l$.

In our study, as we have considered only two sampling intervals d_1 and d_2, for the implementation of the VSI charts, the average sampling interval is defined by $E(D) = (1-\rho)d_1 + \rho d_2$, where ρ denotes the proportion of time that the longer sampling interval is used and the ATS is defined by

$$ATS = \left((1-\rho)d_1 + \rho d_2\right) ANSS, \quad \rho = \frac{p_{vk}}{p_k}, \quad k \geq 1. \tag{7}$$

In the following, we mention some distributional properties of the control statistics M_t^k and S_t^k, $k \geq 1$, needed to compute the probabilities that appear in the expressions of $ANSS$ and ATS.

Suppose that $X_{-k+2}, X_{-k+3}, ..., X_0, X_1, ...$ are random variables, independent and identically distributed whose distribution is $F(.)$. The control statistic $M_t^k = Max(X_t, X_{t-1}, ..., X_{t-k+1})$, $t \geq 1, k \geq 1$, can be expressed in the form

$$M_t^k = \begin{cases} X_t, & k = 1 \\ Max(M_t^{k-1}, X_t), & k > 1, \quad t \geq 1, \end{cases}$$

and the cumulative distribution function (*cdf*) is given by

$$F_{M_t^k}(m) = F^k(m), \quad k > 1, \quad t \geq 1.$$

The joint *cdf* of $(M_t^k, M_{t+1}^k, ..., M_{t+r-1}^k)$, $1 \leq r \leq k$, $t \geq 1$, is given by

$$F_{1,2,...,r}^k(m_1, m_2, ..., m_r) = \begin{cases} F^k(m_1), \quad r = 1 \\ \prod_{i=1}^{r-1} \left(F(\min_{1 \leq j \leq i} m_j) F(\min_{k-i+1 \leq j \leq k} m_j) \right) F^{k-r+1}(\min_{1 \leq j \leq k} m_j), \ 2 \leq r \leq k, \end{cases}$$

for all admissible combinations $(m_1, m_2, ..., m_r)$. To derive this distribution we can use the below outline:

X_1	X_2	...	X_{r-1}	X_r	...	X_k						m_1
	X_2	...	X_{r-1}	X_r	...	X_k	X_{k+1}					m_2
		...	...	...	...	...	...	...				
			X_{r-1}	X_r	...	X_k	X_{k+1}	...	X_{k+r-2}			m_{r-1}
				X_r	...	X_k	X_{k+1}	...	X_{k+r-2}	X_{k+r-1}		m_r
1	2	...	$r-1$	r		r	$r-1$	...	2	1		

where the values of the last line represent the number of variables of the correspondent column.

For admissible values $(m_1, m_2, ..., m_r)$, to hold the condition

$$M_1^k \leq m_1 \cap M_2^k \leq m_2 \cap ... \cap M_r^k \leq m_r$$

we have to hold the conditions

$$\begin{array}{l} X_1 \leq m_1 \cap X_{k+r-1} \leq m_r \\ X_2 \leq m_1 \cap X_2 \leq m_2 \cap X_{k+r-2} \leq m_r \cap X_{k+r-2} \leq m_{r-1} \\ .... \\ X_{r-1} \leq m_1 \cap ... \cap X_{r-1} \leq m_{r-1} \cap X_{k+1} \leq m_r \cap ... \cap X_{k+1} \leq m_2 \\ X_r \leq m_1 \cap ... \cap X_r \leq m_r \\ .... \\ X_k \leq m_1 \cap ... \cap X_k \leq m_r. \end{array}$$

As we have independent and identically distributed X_t variables, whose distribution is $F(.)$,

$$\begin{aligned} F_{1,2,...,r}^k(m_1, m_2, ..., m_r) &= F(m_1)F(m_r)F(\min(m_1, m_2))F(\min(m_r, m_{r-1}))... \\ &...F(\min(m_1, ..., m_{r-1}))F(\min(m_r, ..., m_2))F^{k-r+1}(\min(m_1, ..., m_r)). \end{aligned}$$

We note that the variable $(M_t^k, M_{t+1}^k, ..., M_{t+r-1}^k)$ has a singular distribution with a probability mass function for some $(m_1, m_2, ..., m_r)$. Taking into account these distributions, for the MM^2 chart we have

$$
\begin{aligned}
p_1 &= F^2(UCL) - F^2(LCL),\\
p_2 &= F^3(LCL) + F^3(UCL) - 2F(UCL)F^2(LCL),\\
p_{v2} &= F^2(LCL)F(LWL) + F^2(UWL)F(UCL) - F^2(LCL)F(UWL) - F^2(LWL)F(UCL),
\end{aligned}
\tag{8}
$$

and for the MM^3 chart we have

$$
\begin{aligned}
p_1 &= A^3 - B^3,\\
p_2 &= A^4 + B^4 - 2AB^3,\\
p_3 &= A^5 + 2AB^4 - 3A^2B^3,\\
p_{v3} &= B^4C - B^4D + B^3D^2 - B^3C^2 + AB^3D - AB^3C + A^2C^3 - A^2D^3,
\end{aligned}
\tag{9}
$$

where $A = F(UCL)$, $B = F\ (LCL)$, $C = F(UWL)$ and $D = F(LWL)$. General results about order statistics can be found in David (1980).

The control statistic $S_t^k = X_t + X_{t-1} + ... + X_{t-k+1}$, $t \geq 1, k \geq 1$, can be expressed in the form

$$
S_t^k = \begin{cases} X_t, & k = 1 \\ S_t^{k-1} + X_t, & k > 1, \quad t \geq 1, \end{cases}
$$

and the distributions of the random variables S_t^k and $(S_t^k, S_{t+1}^k, ..., S_{t+r-1}^k)$, $1 \leq r \leq k$, $t \geq 1$, can be easily derived only for some particular models.$F(.)$, like the normal and the exponential models.

If the random variables X_t, are independent and identically distributed as a normal with mean μ and variance σ^2, the statistic S_t^k, $t \geq 1, k \geq 1$, has also a normal distribution with mean $k\mu$ and variance $k\sigma^2$, and the random vector $(S_t^k, S_{t+1}^k, ..., S_{t+r-1}^k)$, $1 \leq r \leq k$, $t \geq 1$, has a multivariate normal distribution with vector of means $\underline{\mu}_{r\times 1} = \begin{pmatrix} k\mu \\ ... \\ k\mu \end{pmatrix}$ and covariance matrix $\Sigma_r = [\sigma_{ij}]_{r\times r}$, where $\sigma_{ij} = \sigma^2(k - (j - i)), 1 \leq i \leq j \leq r$.

.

The joint probability density function (pdf) of $(S_t^k, S_{t+1}^k, ..., S_{t+r-1}^k)$, $1 \leq r \leq k$, $t \geq 1$, is given by

$$
f_{1,2,...r}(s_1, s_2, ..., s_r) = \frac{\sqrt{|\Sigma_r^{-1}|}}{(2\pi)^{\frac{r}{2}}} exp\left\{ -\frac{1}{2} \sum_{i=1}^{r} \sum_{j=1}^{r} \sigma^{ij}(s_i - k\mu)(s_j - k\mu) \right\},
$$

where $\Sigma_r = [\sigma_{ij}]$, $\Sigma_r^{-1} = [\sigma^{ij}]$ e $|\Sigma_r^{-1}| = \frac{1}{|\Sigma_r|}$.

Taking in account these distributions, for the MS^2 chart we have

$$\begin{aligned} p_1 &= \Phi\left(\tfrac{LSC-2\mu}{\sqrt{2}\sigma}\right) - \Phi\left(\tfrac{LIC-2\mu}{\sqrt{2}\sigma}\right), \\ p_2 &= \int_{LIC}^{LSC} \frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(s_1-2\mu)^2}{4\sigma^2}} \left[\Phi\left(\frac{LSC-(\mu+\frac{s_1}{2})}{\sqrt{\frac{3}{2}}\sigma}\right) - \Phi\left(\frac{LIC-(\mu+\frac{s_1}{2})}{\sqrt{\frac{3}{2}}\sigma}\right)\right] ds_1, \\ p_{v2} &= \int_{LIC}^{LSC} \frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(s_1-2\mu)^2}{4\sigma^2}} \left[\Phi\left(\frac{LSV-(\mu+\frac{s_1}{2})}{\sqrt{\frac{3}{2}}\sigma}\right) - \Phi\left(\frac{LIV-(\mu+\frac{s_1}{2})}{\sqrt{\frac{3}{2}}\sigma}\right)\right] ds_1, \end{aligned} \tag{10}$$

and for the MS^3 chart we have

$$\begin{aligned} p_1 &= \Phi\left(\tfrac{LSC-3\mu}{\sqrt{3}\sigma}\right) - \Phi\left(\tfrac{LIC-3\mu}{\sqrt{3}\sigma}\right), \\ p_2 &= \int_{LIC}^{LSC} \frac{1}{2\sigma\sqrt{\pi}} e^{-\frac{(s_1-2\mu)^2}{4\sigma^2}} \left[\Phi\left(\frac{LSC-(\mu+\frac{2s_1}{3})}{\sqrt{\frac{5}{3}}\sigma}\right) - \Phi\left(\frac{LIC-(\mu+\frac{2s_1}{3})}{\sqrt{\frac{5}{3}}\sigma}\right)\right] ds_1, \\ p_3 &= \int_{LIC}^{LSC}\int_{LIC}^{LSC} f_{S_1,S_2}(s_1,s_2) \left[\Phi\left(\frac{LSC-(\frac{6\mu-s_1+4s_2}{5})}{\sqrt{\frac{8}{5}}\sigma}\right) - \Phi\left(\frac{LIC-(\frac{6\mu-s_1+4s_2}{5})}{\sqrt{\frac{8}{5}}\sigma}\right)\right] ds_2 ds_1, \\ p_{v3} &= \int_{LIC}^{LSC}\int_{LIC}^{LSC} f_{S_1,S_2}(s_1,s_2) \left[\Phi\left(\frac{LSV-(\frac{6\mu-s_1+4s_2}{5})}{\sqrt{\frac{8}{5}}\sigma}\right) - \Phi\left(\frac{LIV-(\frac{6\mu-s_1+4s_2}{5})}{\sqrt{\frac{8}{5}}\sigma}\right)\right] ds_2 ds_1, \end{aligned} \tag{11}$$

where Φ denotes the *cdf* of a standard normal.

If the random variables X_t are independent and identically distributed as an exponential with scale parameter δ, $f(x) = \frac{1}{\delta} e^{-\frac{x}{\delta}}, x \geq 0$, the *pdf* of the statistic $S_t^2 = X_{t-1} + X_t$, $t \geq 1$, is given by

$$f_{S_t^2}(s) = \frac{s}{\delta^2} e^{-\frac{s}{\delta}},\ s \geq 0,$$

and the joint *pdf* of (S_t^2, S_{t+1}^2), $t \geq 1$, is given by

$$f_{S_t^2, S_{t+1}^2}(s_1, s_2) = \frac{1}{\delta^2} e^{-\frac{1}{\delta}(s_1+s_2)} (e^{\frac{\min(s_1,s_2)}{\delta}} - 1).$$

Taking into account these distributions, for the MS^2 we have

$$\begin{aligned} p_1 &= \left(\tfrac{LCL}{\delta}+1\right)B - \left(\tfrac{UCL}{\delta}+1\right)A, \\ p_2 &= 2\left(\tfrac{LCL}{\delta} - \tfrac{UCL}{\delta}\right)A + (B-A)(2+A-B), \\ p_{v2} &= \left(\tfrac{LCL}{\delta} - A + B - 2\right)(C-D) + \tfrac{LWL}{\delta}(A+D) - \tfrac{UWL}{\delta}(A+C), \end{aligned} \tag{12}$$

where $A = e^{-\frac{UCL}{\delta}}$, $B = e^{-\frac{LCL}{\delta}}$, $C = e^{-\frac{UWL}{\delta}}$ and $D = e^{-\frac{LWL}{\delta}}$.

The *pdf* of the statistic $S_t^3 = X_{t-2} + X_{t-1} + X_t$, $t \geq 1$, is given by

$$f_{S_t^3}(s) = \frac{s^2}{2\delta^3} e^{-\frac{s}{\delta}}, \; s \geq 0,$$

the joint *pdf* of (S_t^3, S_{t+1}^3), $t \geq 1$, is given by

$$f_{S_t^3, S_{t+1}^3}(s_1, s_2) = \frac{1}{\delta^3} e^{-\frac{1}{\delta}(s_1+s_2)} (\min(s_1, s_2) e^{\frac{\min(s_1,s_2)}{\delta}} - \delta e^{\frac{\min(s_1,s_2)}{\delta}} + \delta),$$

and the joint *pdf* of $(S_t^3, S_{t+1}^3, S_{t+2}^3)$, $t \geq 1$, is given by

$$f_{S_t^3, S_{t+1}^3, S_{t+2}^3}(s_1, s_2, s_3) = \begin{cases} -\frac{1}{\delta^3} e^{-\frac{s_1+s_3}{\delta}} (\frac{s_1}{\delta} + 1) + \frac{1}{\delta^3} e^{-\frac{s_3}{\delta}}, 0 < s_1 < s_2 < s_3 \\ -\frac{1}{\delta^3} e^{-\frac{s_1+s_3}{\delta}} (\frac{s_2}{\delta} + 1 - e^{\frac{s_2}{\delta}}), 0 < s_2 < s_1 < s_3 \vee 0 < s_2 < s_3 < s_1 \\ -\frac{1}{\delta^3} e^{-\frac{s_1+s_3}{\delta}} (\frac{s_3}{\delta} + 1) + \frac{1}{\delta^3} e^{-\frac{s_1}{\delta}}, \\ \quad \max\{0, 2s_2 - s_1\} < s_3 < s_2 < s_1 \vee \max\{0, s_2 - s_1\} < s_3 < s_2 < s_1 < 2s_2 - s_3 \\ -\frac{1}{\delta^3} e^{-\frac{s_1+s_3}{\delta}} (\frac{s_1 - s_2 + s_3}{\delta} + 1) + \frac{1}{\delta^3} e^{-\frac{s_2}{\delta}}, \\ \quad 0 < s_1 < s_3 < s_2 < s_1 + s_3 \vee 0 < s_3 < s_1 < s_2 < s_1 + s_3 \\ 0, \quad \textit{outros } (s_1, s_2, s_3) \end{cases}$$

The expressions for the probabilities that appear in the $ANSS$ and ATS for the MS^3 chart are complicated and we don't refer them here.

In this study the control limits of the different charts were determined such that

$$\begin{aligned} &ANSS_{in-control} = 500 \\ &ANSS_{in-control} \geq ANSS_{out-of-ccontrol}, \end{aligned} \tag{13}$$

and the warning limits such that

$$\begin{aligned} &ATS_{in-control} = 500 \\ &ATS_{in-control} \geq ATS_{out-of-control}. \end{aligned} \tag{14}$$

If we denote the shift magnitude that we want to detect by Δ, and if we assume under control $\Delta = \Delta_0$, for a *k-dependent structure*, $k \geq 1$, the conditions defined in 13 and 14 are equivalent to the conditions

$$\begin{cases} \left[\sum_{i=0}^{k-2} p_i + \frac{p_{k-1}^2}{p_{k-1}-p_k}\right]_{\Delta=\Delta_0} = \left[\left(d_1 + \frac{p_{vk}}{p_k}(d_2 - d_1)\right) ANSS\right]_{\Delta=\Delta_0} = 500 \\ \left[\sum_{i=0}^{k-2} \frac{\partial p_i}{\partial \Delta} + p_{k-1}\frac{(p_{k-1}-2p_k)\frac{\partial p_{k-1}}{\partial \Delta}+p_{k-1}\frac{\partial p_k}{\partial \Delta}}{(p_{k-1}-p_k)^2}\right]_{\Delta=\Delta_0} = [p_k \frac{\partial p_{vk}}{\partial \Delta} - p_{vk}\frac{\partial p_k}{\partial \Delta}]_{\Delta=\Delta_0} = 0, \end{cases} \tag{15}$$

where $\sum_{i=l}^{L} p_i = 0$ for $L < l$.

In this study, the FSI charts have a sampling interval $d = 1$ and the VSI charts have sampling intervals $d_1 = 0.1$ and $d_2 = 1.9$.

3 Comparison of the charts in a normal model.

Suppose that X represents a process quality variable, normally distributed and that we want to monitor the mean μ of the process and that we assume $\sigma = \sigma_0$. When the process is in-control, we assume $\mu = \mu_0$ and out-of-control we assume $\mu = \mu_0 + \delta\sigma_0$, $\delta \neq 0$.

In table 1, we have the control and warning limits of the different charts, implemented to detect shifts in the mean μ and/or standard deviation of the process. Regardless of the mean value we want to monitor, the parameters of the control charts are determined considering $\mu_0 = 0$ and $\sigma_0 = 1$, without loss of generality.

	LCL	UCL	LWL	UWL
X	$\mu_0 - 3.0903\sigma_0$	$\mu_0 + 3.0903\sigma_0$	$\mu_0 - 0.6729\sigma_0$	$\mu_0 + 0.6729\sigma_0$
MM^2	$\mu_0 - 1.8865\sigma_0$	$\mu_0 + 3.0499\sigma_0$	$\mu_0 - 0.0379\sigma_0$	$\mu_0 + 1.0643\sigma_0$
MM^3	$\mu_0 - 1.3096\sigma_0$	$\mu_0 + 3.0327\sigma_0$	$\mu_0 + 0.2686\sigma_0$	$\mu_0 + 1.2606\sigma_0$
MS^2	$2\mu_0 - 4.3487\sigma_0$	$2\mu_0 + 4.3487\sigma_0$	$2\mu_0 - 0.9498\sigma_0$	$2\mu_0 + 0.9498\sigma_0$
MS^3	$3\mu_0 - 5.8134\sigma_0$	$3\mu_0 + 5.8134\sigma_0$	$3\mu_0 - 1.1659\sigma_0$	$3\mu_0 + 1.1659\sigma_0$

Table 1: Control and warning limits

In table 2, we have the $ATS(\delta)$ for some magnitudes of the shift $\delta = \frac{\mu_1 - \mu_0}{\sigma_0}$ in μ that allow us to compare the efficiency of the charts. This table shows that all VSI charts work much better than the correspondent FSI charts. We also conclude that the MS chart performs better than the others, with significant reductions in $ATS(\delta)$, and that the X chart has the worst performance. These conclusions are easily shown in figure 1.

Following, for the charts with best performance, MM and MS charts, we analyzed the effect of increasing the order of dependence in the structure of these charts. Comparing the $ATS(\delta)$

δ	X FSI	X VSI	MM^2 FSI	MM^2 VSI	MS^2 FSI	MS^2 VSI
-3	2.2	0.3	1.6	0.2	1.2	0.1
-2.5	3.6	0.7	2.5	0.4	1.7	0.2
-2	7.3	2.1	4.7	1.1	3.2	0.6
-1.5	17.9	8.2	10.7	4.0	7.8	2.2
-1	54.6	38.0	33.5	20.3	25.6	13.2
-0.5	201.6	183.6	153.1	133.8	124.8	104.3
0	500.0	500.0	500.0	500.0	500.0	500.0
0.5	201.6	183.6	182.2	158.2	124.8	104.3
1	54.6	38.0	48.5	28.0	25.6	13.2
1.5	17.9	8.2	15.6	4.9	7.8	2.2
2	7.3	2.1	6.0	1.0	3.2	0.6
2.5	3.6	0.7	2.7	0.3	1.7	0.2
3	2.2	0.3	1.6	0.2	1.2	0.1

Table 2: Average Time to Signal in a 2-dependent structure, ATS(δ)

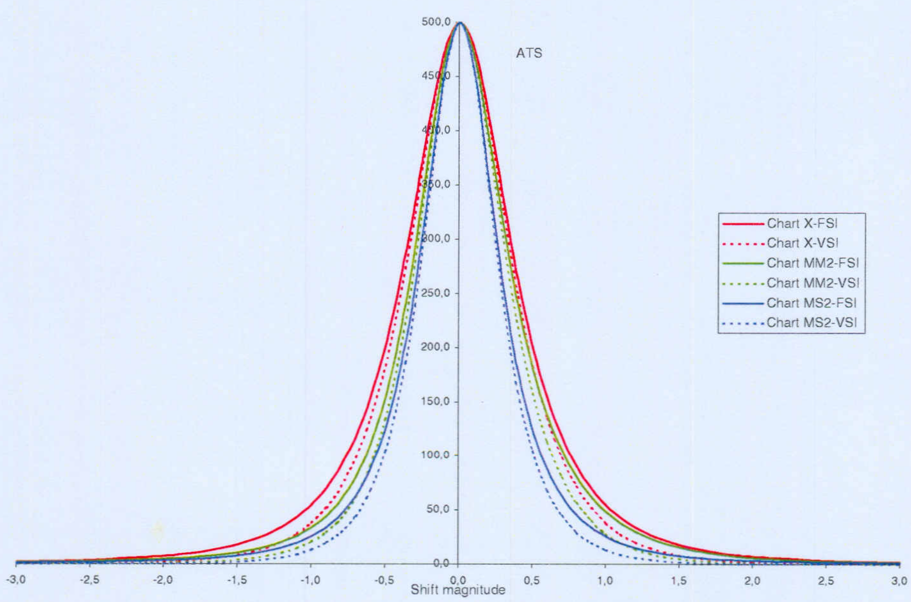

Figure 1: Average Time to Signal in a 2-dependent structure, ATS(δ)

δ	MM^3 FSI	MM^3 VSI	MS^3 FSI	MS^3 VSI
-3	1.3	0.2	1.0	0.1
-2.5	2.0	0.3	2.0	0.3
-2	3.6	0.8	2.8	0.3
-1.5	8.4	2.8	5.8	1.0
-1	26.6	15.2	14.7	5.5
-0.5	132.0	113.1	76.7	58.3
0	500.0	500.0	500.0	500.0
0.5	173.4	144.4	76.7	58.3
1	45.6	22.6	14.7	5.5
1.5	14.1	3.4	5.8	1.0
2	5.1	0.7	2.8	0.3
2.5	2.2	0.2	2.0	0.3
3	1.3	0.1	1.0	0.1

Table 3: Average Time to Signal in a 3-dependent structure, ATS(δ)

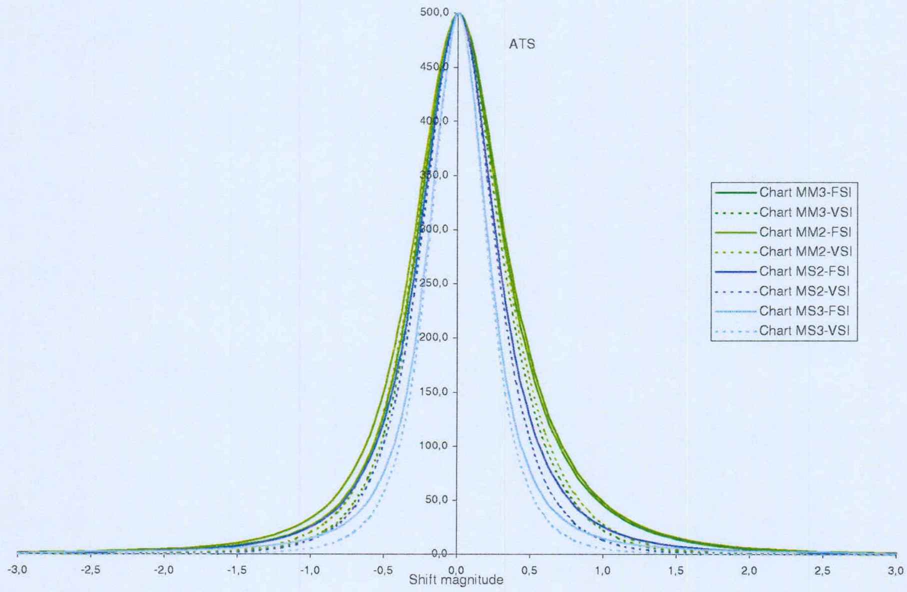

Figure 2: Average Time to Signal in 2-dependent and 3-dependent structure, ATS(δ)

values shown in tables 2 and 3 we conclude that if we consider a 3-*dependent structure* instead of a 2-*dependent structure*, we can reduce significantly the $ATS(\delta)$, as we show in figure 2.

In the following, we analyze the performance of these charts to monitor simultaneously the mean μ and the standard deviation σ of a normal process. When the process is in-control, we assume $\mu = \mu_0$ and $\sigma = \sigma_0$, and out-of-control we assume $\mu = \mu_0 + \delta\sigma_0$, $\delta \neq 0$ or $\sigma = \theta\sigma_0$, $\theta > 1$. We conclude again that all VSI charts work much better than the corresponding FSI charts, but there is not a chart that performs better than the others for all magnitudes of shifts $\delta = \frac{\mu_1 - \mu_0}{\sigma_0}$ and $\theta = \frac{\sigma_1}{\sigma_0}$, in μ and σ respectively.

Table 4 shows the FSI chart that faster detects the occurred change, for some combinations (δ, θ). We can see that the MS chart performs better than the others for many combinations (δ, θ), specially if θ is not very large. To detect large shifts in σ, the MM chart performs better to detect large increases in μ, the MS chart performs better to detect large decreases in μ and the X chart in the other cases. Table 5 shows the VSI chart, respectively, that faster detects the occurred change, for some combinations (δ, θ). Again, the MS chart performs better than the others for many combinations (δ, θ), specially if θ is not very large. To detect large shifts in σ, the MM chart performs better to detect large increases in μ and the X chart in the other cases.

Increasing the order of dependence, we obtain smaller values $ATS(\delta, \theta)$, but similar results for the type of chart we might choose, as we can see in tables 6 and 7.

θ / δ	1	1.1	1.2	1.3	1.4	1.5	2	2.5	3	3.5	4
-3	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS
-2.5	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS
-2	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS	X
-1.5	MS	MS	MS	MS	MS	MS	MS	MS	X	X	X
-1	MS	MS	MS	MS	MS	MS	X	X	X	X	X
-0.5	MS	MS	MS	MS	MS	MS	X	X	X	X	X
0	X	X	X	X	X	X	X	X	X	X	X
0.5	MS	MS	MS	MS	MS	MS	X	X	X	X	X
1	MS	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM
1.5	MS	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM
2	MS	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM
2.5	MS	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM
3	MS	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM

Table 4: FSI chart with small ATS(δ,θ), in a 2-dependent structure

θ / δ	1	1.1	1.2	1.3	1.4	1.5	2	2.5	3	3.5	4
-3	MS	MS	MS	MS	MS	MS	X	X	X	X	X
-2.5	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
-2	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
-1.5	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
-1	MS	MS	MS	MS	MS	MS	X	X	X	X	X
-0.5	MS	MS	MS	MS	MS	MS	X	X	X	X	X
0	X	X	X	X	X	X	X	X	X	X	X
0.5	MS	MS	MS	MS	MS	MM	X	X	X	X	X
1	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM	MM
1.5	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM	MM
2	MS	MS	MS	MS	MM	MM	MM	MM	MM	MM	MM
2.5	MS	MS	MM	MM	MM	MM	MM	MM	MM	MM	MM
3	MS	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM

Table 5: VSI chart with small ATS(δ,θ), in a 2-dependent structure

θ / δ	1	1.1	1.2	1.3	1.4	1.5	2	2.5	3	3.5	4
-3	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS
-2.5	MS	MS	MS	MS	MS	MS	MS	MS	MS	X	X
-2	MS	MS	MS	MS	MS	MS	MS	MS	X	X	X
-1.5	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
-1	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
-0.5	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
0	MS	MS	MS	MS	MS	MS	X	X	X	X	X
0.5	MS	MS	MS	MS	MS	MS	MS	X	MM	MM	MM
1	MS	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM
1.5	MS	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM
2	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM	MM
2.5	MS	MS	MM	MM	MM	MM	MM	MM	MM	MM	MM
3	MS	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM

Table 6: FSI chart with small ATS(δ,θ), in a 3-dependent structure

δ \ θ	1	1.1	1.2	1.3	1.4	1.5	2	2.5	3	3.5	4
-3	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS
-2.5	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS	MS
-2	MS	MS	MS	MS	MS	MS	MS	MS	X	X	X
-1.5	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
-1	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
-0.5	MS	MS	MS	MS	MS	MS	MS	X	X	X	X
0	MS	MS	MS	MS	MS	MS	X	X	X	X	X
0.5	MS	MS	MS	MS	MS	MM	X	X	X	X	X
1	MS	MS	MS	MS	MS	MM	MM	MM	MM	MM	MM
1.5	MS	MS	MS	MS	MM	MM	MM	MM	MM	MM	MM
2	MS	MS	MS	MM	MM	MM	MM	MM	MM	MM	MM
2.5	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM
3	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM

Table 7: VSI chart with small ATS(δ,θ), in a 3-dependent structure

4 Comparison of the charts in an exponential model

Suppose now that X represents a process quality variable exponentially distributed, with *pdf* $f(x) = \frac{1}{\delta} e^{-\frac{x}{\delta}}, x \geq 0$, and we want to monitor the scale parameter δ of the process. When the process is in-control, we assume $\delta = \delta_0$ and out-of-control we assume $\delta_1 = \theta\delta_0$, with $\theta \neq 1$.

In table 8, we have the control and warning limits of the different charts, implemented to detect shifts in the parameter δ. Regardless of the value δ_0 we want to monitor, the parameters of the charts are determined considering $\delta_0 = 1$, without loss of generality.

	LCL	UCL	LWL	UWL
X	$0.00179\delta_0$	$8.46085\delta_0$	$0.43650\delta_0$	$1.91539\delta_0$
MM^2	$0.04175\delta_0$	$7.84433\delta_0$	$0.82838\delta_0$	$2.33494\delta_0$
MM^3	$0.12618\delta_0$	$7.55601\delta_0$	$1.10372\delta_0$	$2.61697\delta_0$
MS^2	$0.05972\delta_0$	$10.17838\delta_0$	$1.16560\delta_0$	$3.15923\delta_0$
MS^3	$0.23187\delta_0$	$11.78764\delta_0$	$1.46180\delta_0$	$4.34605\delta_0$

Table 8: Control and warning limits

In table 9, we have the $ATS(\theta)$ for some magnitudes of the shift $\theta = \frac{\delta_1}{\delta_0}$, which allows us to compare the efficiency of the charts. This table shows that all VSI charts work much better than the corresponding FSI charts. We also conclude that the MS chart performs better than the others, with significant reductions in $ATS(\theta)$, and the X chart has the worst performance. These conclusions are easily shown in figure 3.

θ	XFSI	XVSI	MM^2FSI	MM^2VSI	MS^2FSI	MS^2VSI
0.1	56.4	7.0	11.2	1.1	10.5	1.1
0.3	168.1	87.3	66.8	21.7	64.5	18.4
0.5	279.8	228.2	168.2	116.6	163.2	108.5
0.7	390.7	371.3	314.2	286.3	306.1	276.4
0.9	483.2	481.2	469.4	465.8	465.8	461.9
1	500.0	500.0	500.0	500.0	500.0	500.0
1.2	424.4	419.3	386.9	378.6	367.9	359.5
1.4	273.9	263.7	220.0	205.3	189.9	176.4
1.6	162.1	151.0	122.9	108.0	99.1	86.5
1.8	99.2	89.2	74.1	61.1	57.6	47.2
2	64.8	56.2	48.5	37.5	37.1	28.5
3	16.6	12.4	12.7	7.5	9.9	5.9
4	8.3	5.5	6.2	3.1	5.1	2.6
5	5.4	3.3	4.0	1.7	3.4	1.6
6	4.1	2.4	3.0	1.2	2.6	1.1
7	3.3	1.9	2.4	0.9	2.1	0.9

Table 9: Average Time to Signal in a 2-dependent structure, ATS(Δ)

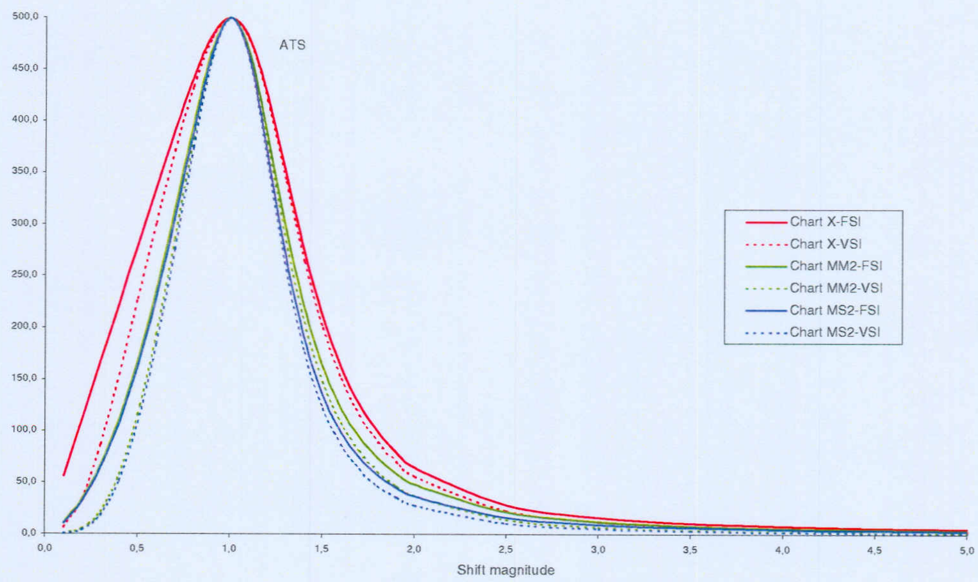

Figure 3: Average Time to Signal in a 2-dependent structure, ATS(θ)

θ	MM^3FSI	MM^3VSI	MS^3FSI	MS^3VSI
0.1	4.3	0.4	3.6	0.4
0.3	34.7	8.3	31.7	5.1
0.5	113.5	69.4	105.5	51.9
0.7	263.0	231.8	247.8	207.1
0.9	458.2	453.4	450.1	443.5
1	500.0	500.0	500.0	500.0
1.2	363.3	352.7	327.8	315.8
1.4	193.5	175.8	148.9	132.3
1.6	195.6	88.4	73.8	59.7
1.8	63.3	48.5	42.3	31.1
2	41.4	29.1	27.1	18.2
3	10.6	5.2	7.2	3.4
4	5.0	1.9	3.7	1.3
5	3.1	1.0	2.4	0.7
6	2.3	0.7	1.9	0.4
7	1.8	0.5	1.6	0.2

Table 10: Average Time to Signal in a 3-dependent structure, ATS(Δ)

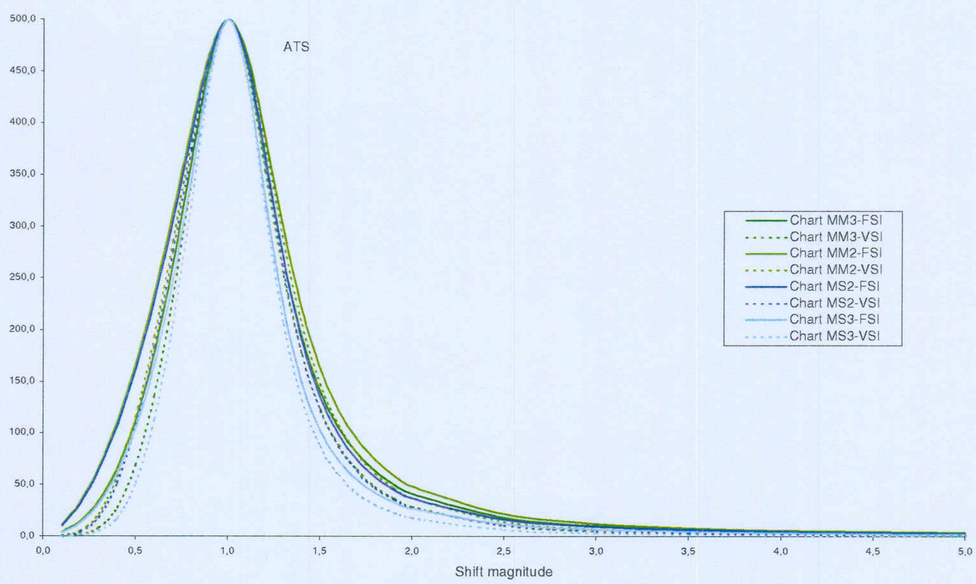

Figure 4: Average Time to Signal in 2-dependent and 3-dependent structure, ATS(θ)

This table shows that all VSI charts work much better than the corresponding FSI charts. We also conclude that the MS chart performs better than the others, with significant reductions in $ATS(\theta)$, and the X chart has the worst performance. These conclusions are easily shown in figure 3.

Following, for the charts with best performance, MM and MS charts, we analyzed the effect of increasing the order of dependence in the structure of these charts. Comparing the $ATS(\delta)$ values shown in tables 9 and 10, we conclude that if we consider a 3-*dependent structure* instead of a 2-*dependent structure*, we can reduce significantly the $ATS(\delta)$, as we show in figure 4.

5 Conclusions

Although the X chart is very easy to use, it is not always the best option to monitor a normal process. Depending on the magnitude of the shift in μ and/or σ we want to detect, the MM or the MS charts will be possible options. The same happens when the process is exponentially distributed, that is, the MM or the MS charts will give better protection against possible changes in the process relatively to the X chart. In both situations the VSI charts work much better than the corresponding FSI charts. However, the MM^3 and MS^3 charts perform even better than the MM^2 and MS^2 charts, respectively, and they are the elected ones, despite the extra computational effort needed.

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