Equations over free inverse monoids with idempotent variables

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Abstract

We introduce the notion of idempotent variables for studying equations in inverse monoids. It is proved that it is decidable in singly exponential time (DEXPTIME) whether a system of equations in idempotent variables over a free inverse monoid has a solution. Moreover the problem becomes hard for DEXPTIME, as soon as the quotient group of the free inverse monoid has rank at least two. The upper bound

is proved by a direct reduction to solve language equations with one-sided concatenation and a known complexity result by Baader and Narendran. For the lower bound we show hardness for a restricted class of language equations.

Decidability for systems of typed equations over a free inverse monoid with one irreducible variable and at least one unbalanced equation is proved with the same complexity upper-bound.

Our results improve known complexity bounds by Deis, Meakin, and Sénizergues (*Equations in free inverse monoids*, IJAC, 17:761–795, 2007). Our results also apply to larger families of equations where no decidability has been previously known. The lower bound confirms a conjecture made in the conference version of that paper which appeared at Computer Science in Russia (CSR 2015).

1 Introduction

It is decidable whether equations over free monoids and free groups are solvable. These classical results were proved by Makanin in his seminal papers [12, 13]. A first estimation of the time complexity for deciding solvability was more than triple or four times exponential, but over the years it was lowered. It went down to PSPACE by Plandowski [17, 18] for free monoids. Extending his method Gutiérrez showed that the same complexity bound applies in the setting of free groups [8]. In [9] Jeż used his "recompression technique" and achieved the best known space complexity to date: NSPACE($n \log n$). Perhaps even more importantly, he presented the simplest known proof deciding the problem WordEquations leading to an easy-under-stand algorithmic description for the set of all solutions for equations over free monoids and free groups (with rational constraints) [6]. Actually, [4] showed that the set of all solutions in reduced words over a free group is an indexed language. More precisely it is an EDT0L language.

In the present paper we study equations over inverse monoids. Inverse monoids are monoids with involution and constitute the most natural intermediate structure between monoids and groups. They are well-studied and pop-up in various applications, for example when investigating systems which are deterministic and codeterministic. Inverse monoids arise naturally as monoids of injective transformations closed under inversion. Indeed, up to isomorphism, these are all the inverse monoids, as stated in the classical Vagner-Preston representation theorem. This makes inverse monoids ubiq-

uituous in geometry, topology and other fields.

The fifties of the last century boosted the systematic study of inverse monoids. However, the word problem remained unsolved until the early seventies, when Scheiblich [20] and Munn [14] independently provided solutions for free inverse monoids. The next natural step is to consider solvability of equations, i.e., the existential theory. Rozenblat's paper [19] destroyed all hope for a general solution: solving equations in free inverse monoids is undecidable. Thus, the best we can hope is to prove decidability for particular subclasses. For almost a decade, the reference paper on this subject has been the paper of Deis, Meakin, and Sénizergues [5]. The authors considered the following *lifting problem*. The input is given by an equation over a free inverse monoid together with a solution over the free quotient. The question is whether the solution over the group can be lifted to a solution in the inverse monoid. [5] showed decidability of the lifting problem using Rabin's tree theorem. The result is an algorithm which is super-exponential (and at least doubly exponential in their specific setting). In the present paper, we achieve various improvements. Our main result lowers the complexity of the lifting problem to singly exponential time; and as soon as the input is a system of at least two equations, then the the lifting problem becomes DEXPTIME-hard. Moreover, we study equations with idempotent variables instead of lifting properties, which leads to a uniform approach and simplified the proof. It also enabled us to generalize some results concerning one-variable equations to a broader setting, thereby leading to new decidability results.

A more precise statement about the progress achieved is as follows. First, Theorem 5.1 shows that deciding solvability of systems of equations in idempotent variables over FIM(A) is DEXPTIME-complete. The upper bound improves the [5, Thm. 8]. Our proof is based on a well-known result from [2] by Baader and Narendran, while the complexity of the algorithm in [5, Thm. 8] is much higher, since the algorithm involves Rabin's Tree Theorem[1]. The lower bound, which is DEXPTIME-hardness (for systems of two equations and where the quotient group of the free inverse monoid has rank at least two) confirms a conjecture in the conference version of the present paper [7]. Second, with respect to unbalanced one-variable equations and [5,

[1]The DEXPTIME result was obtained first by the second and third author, but not published. The same improvement was discovered later independently by the two other authors; and the present paper joins both approaches. In addition, we take the opportunity to correct a mistake in [5] about some special one-variable equations, where Assumption 2 in Definition 6.5 was missing.

Thm. 13], our Theorem 6.8 admits the presence of arbitrarily many idempotent variables, and the complexity very much improved in view of Theorem 5.1. Morover, our proofs are shorter and easier to understand by a direct reduction to language equations.

2 Preliminaries and notation

Sets and finite subsets. Given a set S, we denote by 2^S_f the set of *finite* subsets of the set S.

Complexity. A function $p : \mathbb{N} \to \mathbb{N}$ is called *polynomial*, if $p(n) \in n^{\mathcal{O}(1)}$. It is *singly exponential*, if $f(n) \leq 2^{p(n)}$ where p is some polynomial. The complexity class DEXPTIME refers to problems which can be solved on deterministic Turing machines within a singly exponential time bound. A problem is encoded as a subset over the binary alphabet $\{0,1\}$. A problem P is DEXPTIME-hard, if for every problem $L \in$ DEXPTIME there exists a polynomial time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that: $w \in L \Leftrightarrow f(w) \in P$. It is called DEXPTIME-complete if it belongs to DEXPTIME and, in addition, it is DEXPTIME-hard. In a few places also refer to other complexity classes like PSPACE (=*polynomial space*) or NP (=*nondeterministic polynomial time*). The notation is standard, see for example [15]. As usual in the literature, explicit encodings of problems are omitted. Our reductions are actually "logspace" reductions. Formally, this makes the lower bound results stronger, but this is not our primary goal: so we contend with the framework of polynomial-time reductions.

Monoids and groups. A *monoid* is a nonempty set M with a binary associative operation: $(x, y) \mapsto x \cdot y$ together with a neutral element 1 satisfying $1 \cdot x = x \cdot 1 = x$ for all $x \in M$. Frequently, we write xy instead of $x \cdot y$. A group is a monoid G where for each $x \in G$ there exists some $\overline{x} \in G$ such that $x\overline{x} = 1$. If G is a group, then its *inverse* $\overline{x} = x^{-1}$ is uniquely defined.

Words and languages. An *alphabet* is a (finite) set; and an element of an alphabet is called a *letter*. The free monoid generated by an alphabet A is denoted by A^*. The elements of A^* are called *words*: these are the finite sequences of letters. The empty word is denoted by 1 as the neutral element in other monoids as well, provided the operation is written as a

multiplication. The *length* of a word u is denoted by $|u|$. We have $|u| = n$ for $u = a_1 \cdots a_n$ where $a_i \in A$. The empty word has length 0, and it is the only word with this property. A word u is a *factor* of a word v if there exist $p, q \in A^*$ such that $puq = v$. It is a a *prefix*, if $uq = v$ for some $q \in A^*$, and it is a *suffix*, if $pu = v$ for some $p \in A^*$. A *language* L over A is a subset of A^*. It is called *factor-* (resp. *prefix-*) (resp. *suffix-*) *closed*, if with every u every factor, (resp. prefix), (resp. suffix) of u belongs to L as well. We write $\mathrm{Pref}(L)$ for its prefix-closure, thus

$$\mathrm{Pref}(L) = \{u \in A^* \mid \exists v \in L : u \leq v\}.$$

Involutions. An *involution* is a mapping $^-$ such that $\overline{\overline{x}} = x$ for all elements. In particular, an involution is a bijection. The identity is an involution.

Monoids with involutions. If an involution is defined for a monoid, then we additionally require $\overline{xy} = \overline{y}\,\overline{x}$ for all its elements x, y. Every group is a monoid with involution by letting $\overline{x} = x^{-1}$.

If an alphabet is equipped with an involution, then we extend it to the free monoid A^* by

$$\overline{a_1 \cdots a_m} = \overline{a_m} \cdots \overline{a_1}.$$

When $\overline{a} = a$ for all $a \in A$, then $\overline{w}$ simply means to read the word from right-to-left. Every alphabet B (without involution) can be embedded into an alphabet A with involution without fixed points by letting $A = B \cup \overline{B}$ where $\overline{B} = \{\overline{a} \mid a \in B\}$ is a disjoint copy of B. The involution maps $a \in B$ to $\overline{a}$ and vice versa.

The identity is a morphism for monoids with involution if and only if the monoid is commutative. In particular, given a set S the set of finite subsets 2^S_f is a commutative monoid where the operation is the union. Thus, we also write $L + K$ instead of $L \cup K$. Elements of $s \in S$ are identified with singletons $\{s\} \in 2^S_f$. According to the additive notation the neutral element in 2^S_f is denoted as 0. We have $0 = \emptyset$. Actually, in our application we have $1 \in S \subseteq A^*$ and then $1 \in 2^S_f$ denotes the singleton $\{1\}$. Thus, $0 \neq 1$ in 2^S_f. There is however no risk of confusion: similar conventions are standard for $\mathbb{N}$ or $\mathbb{Z}$.

Homorphisms and morphisms. A *homomorphism* is a mapping which respects the algebraic structure, whereas the notion *morphism* refers to a

mapping which respects the involution and in addition, depending on the category, respects the algebraic structure, too. Hence, a morphism between sets with involution is just a mapping respecting the involution, whereas a morphism between monoids with involution is a monoid homomorphism respecting the involution.

Free groups. Let A be an alphabet with involution. It defines a quotient group $F(A)$ by adding defining relations $a\overline{a} = 1$ for all $a \in A$. If we can write A as a disjoint union $B \cup \{\overline{a} \mid a \in B\}$, then $F(A)$ is nothing but the free group $\mathrm{FG}(B)$ in the standard meaning. In general, $F(A)$ is a free product of a free group with cyclic groups of order 2. Although our primary interest is the usual free group $\mathrm{FG}(B)$, the notation $F(A)$ is more convenient for us. Moreover, various results hold for $F(A)$ and without changing the proofs. Last but not least, $F(A)$ is the "free group" with respect to the category of sets with involution: every morphism of A to a group G extends uniquely to a morphism from $F(A)$ to G.

As a set (with involution) we can identify $F(A)$ with the subset of reduced words in A^*. As usual, a word is called *reduced* if it does not contain any factor $a\overline{a}$ where $a \in A$. Observe that this embedding of $F(A)$ into A^* is indeed compatible with the involution. In the following we let $\pi : A^* \to F(A)$ be the canonical morphism from A^* onto $F(A)$. It is well-known (and easy to see) that every word $u \in A^*$ can be transformed into a unique reduced word $\widehat{u}$ by successively erasing factors of the form $a\overline{a}$ where $a \in A$. This leads to the assertion

$$\forall u, v \in A^* : \pi(u) = \pi(v) \;\Leftrightarrow\; \widehat{u} = \widehat{v}.$$

We systematically identify the set $F(A)$ with the subset of reduced words in A^*. Concepts such as length, factor, prefix, and prefix-closure are inherited from free monoids to free groups via reduced words. For the same reason, it makes sense to write $\widehat{u} = \pi(u)$, for $u \in A^*$, because $\pi(u) \in F(A)$ is identified with $\widehat{u} \in A^*$. If $L \subseteq A^*$ is prefix-closed, then $\widehat{L} = \{\widehat{u} \mid u \in L\} \subseteq F(A)$ is prefix-closed, too (Lemma 5.3). We have $\widehat{L} \subseteq \widehat{A^*} = F(A) \subseteq A^*$.

Free inverse monoids A monoid M is said to be *inverse* if for every $x \in M$ there exists a unique element $\overline{x} \in M$ satisfying $x\overline{x}x = x$ and $\overline{x}x\overline{x} = \overline{x}$. Clearly, $\overline{\overline{x}} = x$ by uniqueness of $\overline{x}$ and, hence, M is a set with involution. The mapping $x \mapsto \overline{x}$ is also called an *inversion*. Idempotents commute in inverse monoids (see e.g., [16]), hence the subset $E(M) = \{e \in M \mid e^2 = e\}$

is a commutative submonoid. Since necessarily $\overline{e} = e$ for $e \in E(M)$ one easily deduces that $\overline{xy} = \overline{y}\,\overline{x}$ for all $x, y \in M$. As a consequence, an inverse monoid is a monoid with involution.

In the literature the notation $\overline{x} = x^{-1}$ is also used for elements of inverse monoids, just as for groups (which constitute a proper subclass of inverse monoids). By default, the involution on an inverse monoid (and hence in every group) is supposed to be given by its inversion. We proceed now to describe Scheiblich's construction of the free inverse monoids $\mathrm{FIM}(A)$ where A is an alphabet with involution.

The elements of $\mathrm{FIM}(A)$ are pairs (P, g), where the second component is a group element $g \in F(A)$ and the first component is a finite prefix-closed subset P of $F(A)$ such that $g \in P$. In other terms, this means that P is a finite connected subset of the Cayley graph of $F(A)$ (over A) such that $1, g \in P$. Formally, we let

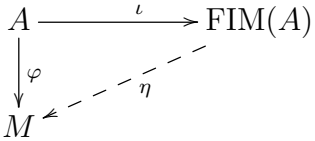
$$\mathrm{FIM}(A) = \{(P, g) \mid |P| < \infty \wedge g \in P = \mathrm{Pref}(P) \subseteq F(A)\}.$$

The multiplication on $\mathrm{FIM}(A)$ is defined through

$$(P, g)(Q, h) = (P \cup gQ, gh).$$

It is easy to see that $\mathrm{FIM}(A)$ is a monoid with identity $(\{1\}, 1)$ and every (P, g) has a unique inverse $(g^{-1}P, g^{-1})$, hence $\mathrm{FIM}(A)$ is an inverse monoid.

Let $\psi : A^* \to \mathrm{FIM}(A)$ be the homomorphism of monoids defined by $\psi(a) = (\{1, a\}, a)$. Then we have $\psi(\overline{a}) = (\{1, \overline{a}\}, \overline{a}) = \overline{(\{1, a\}, a)}$ and ψ is a morphism of monoids with involution. We obtain the universal property of being free with respect to sets with involution: let M be an inverse monoid and $\varphi : A \to M$ a morphism of sets with involution, then there is exactly one morphism $\eta : \mathrm{FIM}(A) \to M$ of monoids with involution such that $\Phi(a) = \varphi(a)$ for all $a \in A$. In other words, let $\iota = \psi|_A$ and $\varphi : A \to M$ be any mapping respecting the involution where M is an inverse monoid. Then there exists a unique morphism of inverse monoids $\eta : \mathrm{FIM}(A) \to M$ such that the following diagram commutes.



In particular, $\pi : A^* \to F(A)$ factorizes through η. The monoid $\mathrm{FIM}(A)$ is, up to isomorphism, uniquely defined by this universal property: $\mathrm{FIM}(A)$ is

a *free inverse monoid* in the category of sets with involution. If A can be written as a disjoint union $A = B \cup \{\overline{a} \mid a \in B\}$, then $\mathrm{FIM}(A)$ is the *free inverse monoid* over B in the category of sets (without involution).

The following diagram summarizes our notation.

$$\begin{array}{ccc} A^* & \xrightarrow{\quad\psi\quad} & \mathrm{FIM}(A) \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \eta} & \\ F(A) & = & \{\widehat{w} \mid w \in A^*\} \subseteq A^* \text{ as sets} \end{array}$$

3 Language equations

Henceforth, A, and B denote alphabets of constants and Ω denotes an alphabet of variables. The alphabets are finite and disjoint. We assume that $B \subseteq A$ and that $A \cup \Omega$ is a set with involution. However, for technical reasons we require that $X = \overline{X}$ for all variables. We use $a, b, c, \ldots$ to denote letters of A, whereas variables are denoted by capital letters $X, Y, Z \ldots$.

Our complexity results for solving certain equations over free inverse monoids rely on a paper of Baader and Narendran [2]. The paper shows that the satisfiability problem of language equations with one-sided concatenation is DEXPTIME-complete for free monoids. As we need the corresponding result for free groups as well, we define the notion of language equation and its solutions in a more general framework.

A *system of language equations* $\mathcal{S}$ (with one-sided concatenation) has the form

$$L_k + \sum_{i \in I_k} u_{ki} X_i = K_k + \sum_{j \in J_k} u_{kj} X_j \quad \text{for } 1 \leq k \leq n. \tag{1}$$

Here, $n \in \mathbb{N}$ and I_k, J_k are finite (disjoint) index sets, L_k, K_k are finite subsets of A^*, $u_{ki}, u_{kj} \in A^*$ are words, and $X_i, X_j \in \Omega$.

If L_k, K_k are subsets of B^* and $u_{ki}, u_{kj} \in B^*$, then we say that $\mathcal{S}$ is a system with coefficients over B.

The *size* of $\mathcal{S}$ is defined as

$$\|\mathcal{S}\| = |A \cup \Omega| + \sum_{k=1}^{n} |I_k| + |J_k| + \sum_{u \in L_k \cup K_k} |u| + \sum_{i \in I_k} |u_{ki}| + \sum_{j \in J_k} |u_{kj}| \,.$$

Example 3.1 *Recall that a word u is identified with the singleton $\{u\} \subseteq A^*$. Consider $A = \{a, \overline{a}, b, \overline{b}\}$ and*

$$a\overline{a} + a\overline{a}X + b\overline{b}Y = b\overline{b} + a\overline{a}Y + b\overline{b}X. \tag{2}$$

It is a system in one equation and its size is 22.

The notion of solution depends on the context. In our paper we use solutions in finite subsets of free groups and free monoids. Let M denote either the free monoid A^* or the group $F(A)$. In particular, we have inclusions of sets with involution $A \subseteq M \subseteq A^*$; and A generates M as a monoid.

A *solution* of $\mathcal{S}$ in (1) is a mapping $\sigma : \Omega \to 2_f^{A^*}$ such that

$$L_k + \sum_{i \in I_k} u_{ki}\sigma(X_i) = K_k + \sum_{j \in J_k} u_{kj}\sigma(X_j)$$

becomes an identity in 2_f^M for all $1 \leq k \leq n$. Thus, a solution substitutes each $X \in \Omega$ by some finite subset $\sigma(X)$ of A^*, but the interpretation is in M. Of course, for $M = F(A)$ we can demand that each $\sigma(X)$ must be a finite subset in reduced words: $\sigma(X) \subseteq F(A)$.

Theorem 3.2 ([2], Thm. 6.1 and Thm. 7.6) *The following problem can be solved in* DEXPTIME*; and it is* DEXPTIME*-complete for $|B| \geq 2$.*

Input. *A system $\mathcal{S}$ of language equations with coefficients over B.*

Question. *Does $\mathcal{S}$ have a solution in the free monoid B^*?*

Remark 3.3 *[2] states Theorem 3.2 for a single equation. However, this covers the general case. Indeed, assume that a system $\mathcal{S}$ of language equations over the free monoid A^* has n equations. Without restriction we have $|A| \geq 2$. Choose n pairwise different words $p_1, \ldots p_n \in A^*$ of equal length (say $\lceil \log_2 n \rceil$); and for $1 \leq k \leq n$ replace the k-th equation $L_k + \sum_{i \in I_k} u_{ki}X_i = K_k + \sum_{j \in J_k} u_{kj}X_j$ by*

$$p_kL_k + \sum_{i \in I_k} p_ku_{ki}X_i = p_kK_k + \sum_{j \in J_k} p_ku_{kj}X_j.$$

Summing all left-hand sides and all right-hand sides yields a single equation

$$\sum_{k=1}^{n}(p_kL_k + \sum_{i \in I_k} p_ku_{ki}X_i) = \sum_{k=1}^{n}(p_kK_k + \sum_{j \in J_k} p_ku_{kj}X_j). \tag{3}$$

The reduction works since $\{p_1, \ldots, p_n\}$ is a prefix code. Note that the transformation of the system $\mathcal{S}$ to Equation (3) preserves the set of solutions.

Consider again Equation (2): $a\overline{a} + a\overline{a}X + b\overline{b}Y = b\overline{b} + a\overline{a}Y + b\overline{b}X$. Over $M = F(A)$ the equation becomes trivial: it states $1 + X + Y = 1 + Y + X$ which is a tautology. Hence every substitution in finite subsets of A^* is a solution over M. However, for $M = A^*$ the structure of solutions is more restricted. In the spirit of Remark 3.3 we see that Equation (2) encodes over A^* a system of two equations: $1 + X = Y$ and $Y = 1 + X$. Hence, the set of solutions over A^* is the set of mappings $\sigma : \Omega \to 2_f^{A^*}$ such that $\sigma(X) = \sigma(Y)$ and and $1 \in \sigma(X) \cap \sigma(Y)$.

4 Typed equations over free inverse monoids

An *equation* over FIM(A) is a pair (U, V) of words over $A \cup \mathcal{X}$, sometimes written as $U = V$. Here A is an alphabet of *constants* and $\mathcal{X}$ is a set of variables. Variables $X \in \mathcal{X}$ represent elements in FIM(A) and therefore $\mathcal{X}$ is an alphabet with involution, too. Without restriction we may assume $X \neq \overline{X}$ for all $X \in \mathcal{X}$. A *solution* σ of $U = V$ is a mapping $\sigma : \mathcal{X} \to A^*$ such that $\sigma(\overline{X}) = \overline{\sigma(X)}$ for all $X \in \mathcal{X}$ and such that the replacement of variables by the substituted words in U and in V give the same element in FIM(A), i.e., $\psi(\sigma(U)) = \psi(\sigma(V))$ in FIM(A), where σ is extended to a morphism $\sigma : (A \cup \mathcal{X})^* \to A^*$ leaving the constants invariant. Clearly, we may specify σ also by a mapping from $\mathcal{X}$ to FIM(A). For the following it is convenient to have two more types of variables which are used to represent specific elements in FIM(A). We let Ω be a set of *idempotent variables* and Γ be a set of *reduced variables*. Both sets are endowed with an involution. We let $\overline{Z} = Z$ for idempotent variables and $\overline{x} \neq x$ for all reduced variables. Thus, idempotent variables are the only variables which are self-involuting; and variables in Γ or $\mathcal{X}$ are not self-involuting. We also insist that A, $\mathcal{X}$, Ω, and Γ are pairwise disjoint. A *typed equation* over FIM(A) is a pair (U, V) of words over $A \cup \Omega \cup \Gamma$. A *system of typed equation* is a collection $\mathcal{S}$ of typed equations; and a *solution* σ of $\mathcal{S}$ is given by a mapping respecting the involution from $\Omega \cup \Gamma$ to A^*, which is extended to a morphism $\sigma : (A \cup \Omega \cup \Gamma)^* \to A^*$ respecting the involution and letting the letters of A invariant, such that the following conditions hold.

1. $\psi(\sigma(Z))$ is idempotent for all $Z \in \Omega$.
2. $\sigma(x)$ is a reduced word for all $x \in \Gamma$.

3. We have $\psi(\sigma(U)) = \psi(\sigma(V))$ for all $(U, V) \in \mathcal{S}$.

Lemma 4.1 *Let (U, V) be an (untyped) equation over* FIM(A). *For each $X, \overline{X} \in \mathcal{X}$ choose a fresh idempotent variable $Z_X \in \Omega$ and fresh reduced variables $x_X, \overline{x}_X \in \Gamma$. Let τ be the word-substitution (i.e. monoid homomorphism) which replaces each $X, \overline{X} \in \Omega$ by $Z_X x_X$ and $\overline{x}_X Z_X$ respectively. If σ is a solution of (U, V) then a solution σ' for $(\tau(U), \tau(V))$ can be defined as follows. For $\sigma(X) = (P, g)$, where g is represented by a reduced word, we let $\sigma'(Z_X) = (P, 1)$ and $\sigma'(x_X) = (Pref(g), g)$.*
Conversely, if σ' solves $(\tau(U), \tau(V))$ with $\sigma'(Z_X) = (P, 1)$ and $\sigma'(x_X) = (Pref(g), g)$ then $\sigma(X) = (P \cup Pref(g), g)$ defines a solution for (U, V).

Proof. Trivial. □

By Lemma 4.1 we can reduce the satisfiability of equations in FIM(A) to satisfiability of typed equations. The framework of typed equations is more general; and it fits better to our formalism. Let (U, V) be a typed equation, by the *underlying group equation* we mean the pair $(\pi(U), \pi(V))$ which is obtained by erasing all idempotent variables. Clearly, if (U, V) is satisfiable then $(\pi(U), \pi(V))$ must be solvable in the free group $F(A)$. This leads to the idea of *lifting* a solution of a group equation to a solution of (U, V) in FIM(A). It has been known by [5] that it is decidable whether a lifting is possible. The following result improves decidability by giving a deterministic exponential time bound.

Theorem 4.2 *The following problem is in* DEXPTIME.

Input. *A system $\mathcal{S}$ of equations over* FIM(A) *and a solution $\gamma : \Gamma \to F(A)$ of the system $\pi(\mathcal{S})$ of underlying group equations.*

Question. *Does $\mathcal{S}$ have a solution $\sigma : \mathcal{X} \to$* FIM(A) *such that $\gamma = \eta \circ \sigma$?*

Moreover, the problem becomes DEXPTIME*-hard as soon as A contains four pairwise different letters $\alpha, \overline{\alpha}, \beta$, and $\overline{\beta}$, the system has two equations, and γ is the trivial mapping $\gamma(x) = 1$ for all $x \in \Gamma$.*

Proof. For the upper bound we proceed as follows. Due to Lemma 4.1 we first transform the system into a new system with variables in $\Omega \cup \Gamma$. Next we replace every reduced variable $x \in \Gamma$ by $(\mathrm{Pref}(\sigma'(x)), \sigma'(x))$. Since the solution is part of the input this increases the size of $\mathcal{S}$ at most quadratic.

We obtain a system of equations in idempotent variables and we apply Theorem 5.1 in Section 5 below. Actually, Theorem 5.1 shows also the lower bound, because fixing $\gamma : \Gamma \to F(A)$ to be the trivial mapping means that every lifting $\sigma : \mathcal{X} \to \mathrm{FIM}(A)$ turns $\sigma(X)$ into an idempotent. Thus, fixing $\gamma : \Gamma \to F(A)$ to be the trivial mapping, leads directly to the framework of idempotent variables. □

The next result combines Theorem 4.2 and a known complexity result for systems of equations over free groups [6].

Corollary 4.3 *Let $\mathcal{S}$ be a system of equations over the free inverse monoid* $\mathrm{FIM}(A)$ *and $\pi(\mathcal{S})$ the system of underlying group equations.*

1. *On input $\mathcal{S}$ it can be decided in polynomial space whether the system $\pi(\mathcal{S})$ of group equations has at most finitely many solutions. If so, then every solution has at most doubly exponential length.*

2. *On input $\mathcal{S}$ and the promise that $\pi(\mathcal{S})$ has at most finitely many solutions it can be decided in deterministic triple exponential time whether $\mathcal{S}$ has a solution.*

Proof. The statement 1 follows from [6]. In particular, the size of the set of all solutions is at most triple exponential. Since the square of a triple exponential function is triple exponential again, the statement 2 follows from Theorem 4.2. □

5 Solving equations in idempotent variables

Theorem 5.1 is the main result of the paper. We split its proof into two sections. Subsection 5.1 shows the containment in DEXPTIME. Subsection 5.2 shows DEXPTIME-hardness for systems with two equations. Theorem 5.1 improves via Theorem 4.2 the result [5, Thm. 8], which was derived from Rabin's Tree Theorem leading to a super-exponential complexity. It improves the main result of [7] since it also shows the conjecture that solving equations in idempotent variables is DEXPTIME-complete.

Theorem 5.1 *The following problem can be decided in* DEXPTIME.

Input. *A system $\mathcal{S}$ of equations in idempotent variables (i.e., without any reduced variable).*

Question. *Does $\mathcal{S}$ have a solution in* FIM(A)*?*

Moreover, if A contains four pairwise different letters $\alpha, \overline{\alpha}, \beta, \overline{\beta}$, then the problem is DEXPTIME*-hard for systems with two equations.*

5.1 Upper-bound: containment in DEXPTIME

This section proves the upper bound mentioned in Theorem 5.1. We begin with the following lemma.

Lemma 5.2 *There is a polynomial time algorithm for the following computation.*

Input. *A finite alphabet with involution A and an equation*

$$u_0 X_1 u_1 \cdots X_g u_g = v_0 Y_1 v_1 \cdots Y_d v_d, \tag{4}$$

where the X_i's and Y_j's are idempotent variables and such that the identity $u_0 \cdots u_g = v_0 \cdots v_d$ holds in the group $F(A)$.

Output. *A language equation which is solvable in nonempty, finite, prefix-closed subsets of $F(A)$ if and only if Equation (4) is solvable in* FIM(A).

Proof. Define for $0 \leq i \leq g$ and $0 \leq j \leq d$ the words

$$p_i = u_0 \cdots u_i, \quad q_j = v_0 \cdots v_j \in A^*.$$

In every inverse monoid we have $pZ = pZ\overline{p}p$ for every idempotent Z and every element p. Since $p_{i-1}u_i = p_i$ and $q_{j-1}v_j = p_j$, the equation in idempotent variables (4) can be rewritten as:

$$p_0 X_1 \overline{p_0} \cdots p_i X_{i+1} \overline{p_i} \cdots p_{g-1} X_g \overline{p_{g-1}} \cdot p_g \overline{p_g} p_g = q_0 Y_1 \overline{q_0} \cdots q_j Y_{j+1} \overline{q_j} \cdots q_{d-1} Y_d \overline{q_{d-1}} \cdot q_d \overline{q_d} q_d. \tag{5}$$

By hypotheses we have $p_g = q_d$ in $F(A)$. Moreover, idempotents commute. Hence, Equation (5) is equivalent in FIM(A) with the following equation

$$p_g \overline{p_g} \cdot \prod_{i=0}^{g-1} p_i X_{i+1} \, \overline{p_i} = q_d \overline{q_d} \cdot \prod_{j=0}^{d-1} q_j Y_{j+1} \, \overline{q_j}. \tag{6}$$

Each value of X_i, resp. Y_j, in FIM(A) has the form $(P_i, 1)$, resp. $(Q_j, 1)$, for non-empty, prefix-closed, and finite subsets P_i and Q_j of $F(A)$.

Recall that $\widehat{u}$ refers to the reduced word $\pi(u) \in F(A) \subseteq A^*$. Define $L = \{\widehat{p} \mid p \in \mathrm{Pref}(p_g)\}$ and $K = \{\widehat{q} \mid q \in \mathrm{Pref}(q_d)\}$. Then the output of the algorithm is the language equation where the solutions X_i's and Y_j's are required to be nonempty, finite, prefix-closed subsets of $F(A)$:

$$L + \sum_{i=0}^{g-1} \widehat{p}_i X_{i+1} = K + \sum_{j=0}^{d-1} \widehat{q}_j Y_{j+1}. \tag{7}$$

It follows from the construction and Scheiblich' s presentation of free inverse monoids that $\sigma(X_i) = (P_i, 1)$ and $\sigma(Y_j) = (Q_j, 1)$ solves Equation (4) in FIM(A) if and only if $\sigma'(X_i) = P_i$ and $\sigma'(Y_j) = Q_j$. Hence, the lemma. □

We also make use of the following easy observation.

Lemma 5.3 *Let $P \subseteq A^*$ be prefix-closed and $\widehat{P} = \{\widehat{p} \mid p \in P\}$ the corresponding set of reduced words. Then $\widehat{P}$ is prefix-closed.*

Proof. Let $p \in P$ and $\widehat{p} \in \widehat{P}$ its reduced form. We have to show that every prefix of $\widehat{p}$ belongs to $\widehat{P}$. For $p = 1$ this is trivial. Hence, let $p = qa$ with $a \in A$ and $\widehat{q}$ the reduced form of q. We have $q \in P$ and, by induction, every prefix of $\widehat{q}$ belongs to $\widehat{P}$. Now, if $\widehat{p}$ is a prefix of $\widehat{q}$, we are done. In the other case we have $\widehat{p} = \widehat{q}a$. Since $\widehat{q}, \widehat{p} \in \widehat{P}$ we are done again. □

Let us finish to prove that solving equations in idempotent variables over the free inverse monoid belongs to DEXPTIME.

The input in Theorem 5.1 is given by a system $\mathcal{S}$ of equations in idempotent variables over a free inverse monoid FIM(A). Every equation $(U, V) \in \mathcal{S}$ can be written as in Equation (4). That is:

$$u_0 X_1 u_1 \cdots X_g u_g = v_0 Y_1 v_1 \cdots Y_d v_d. \tag{8}$$

In linear time we check that all equations $u_0 \cdots u_g = v_0 \cdots v_d$ hold in the group $F(A)$. If one of these equalities is violated then $\mathcal{S}$ is not solvable and we can stop.

By Lemma 5.2 it suffices to give a DEXPTIME algorithm for solving systems of language equations over $F(A)$ of the form (7) where the solutions

X_i's and Y_j's are required to be nonempty, finite, prefix-closed subsets of $F(A)$. Thus we may assume that we start with a system $\mathcal{S}$ where every equation can be written as

$$L + \sum_{i \in I} u_i X_i = K + \sum_{j \in J} u_j Y_j, \tag{9}$$

where $u_i \in L$ and $u_j \in K$ and $L \cup K$ consist of reduced words, only. We say that a solution $\sigma : \Omega \to 2^{A^*}$ is *strong* if $\sigma(X)$ consists of reduced words, only. That is $\sigma(X) = \pi(\{u \in A^* \mid u \in \sigma(X)\})$. Clearly, $\mathcal{S}$ has a solution in $F(A)$ if and only if it has a strong solution.

Next, we transform in deterministic polynomial time the system $\mathcal{S}$ into a system $\mathcal{S}_0$ where the equations have a simple syntactic form. We begin by introducing a fresh variable X_0 and an equation $X_0 = 1$. In a second phase, we replace each equation of type as in (9) by two equations using a fresh variable X_E and, since each $u_k \in L_K = \mathrm{Pref}(L_K)$ as well as $X_0 = 1$, we may define these equations as follows:

$$X_E = \sum_{u \in L_I} (uX_0 + \mathrm{Pref}(u)) + \sum_{i \in I} (u_i X_i + \mathrm{Pref}(u_i)),$$
$$X_E = \sum_{v \in L_J} (vX_0 + \mathrm{Pref}(v)) + \sum_{j \in J} (u_j X_j + \mathrm{Pref}(u_i)).$$

Thus, there is an equation of the form $X_0 = 1$ and a bunch of equations which have the form

$$X = \sum_{k \in K} (u_k X_k + \mathrm{Pref}(u_k)) \text{ with } K \neq \emptyset.$$

With the help of polynomially many additional fresh variables, it is now obvious that we can transform $\mathcal{S}$ (with respect to satisfiability) into an equivalent system $\mathcal{S}_0$ containing only three types of equations:

1. $X = 1$,
2. $X = Y + Z$,
3. $X = uY + \mathrm{Pref}(u)$, where u is a reduced word.
4. $X = 1 + X$ for all X.

The last type of equations $X = 1 + X$ makes sure that every solution is in nonempty sets containing the empty word. (This allows to ignore the restriction that $\sigma(X) \neq \emptyset$.) Phrased differently, without restriction $\mathcal{S}$ is of the form $\mathcal{S}_0$ at the very beginning. At this point we start a nondeterministic polynomial time reduction. This means, if S has a solution then at least one outcome of the nondeterministic procedure yields a solvable system $\mathcal{S}'$ of language equations. If none of the possible outcomes is solvable then S is not solvable. During this procedure we are going to mark some equations and this forces us to define the notion of solution for systems with marked equations. A *(strong) solution* is defined as a mapping σ such that each $\sigma(X)$ is given by a prefix-closed set of (reduced) words in A^* such that all equations hold as language equations over $F(A)$, but all marked equations hold as language equations over A^* as well. (Thus, we have a stronger condition for marked equations.) We can think of an "evolution" of language equations over $F(A)$ to language equations over the free monoid A^*, and in the middle during the evolution we have a mixture of both interpretations.

Initially we mark all equations of type $X = 1$, $X = 1+X$, and $X = Y+Z$. This is possible because we may start with a strong solution in nonempty, prefix-closed and finite sets, if $\mathcal{S}$ is solvable.

Now we proceed in rounds until all equations are marked. We start a round, if some of the equations $X = uY + \mathrm{Pref}(u)$ is not yet marked. If $u = 1$ is the empty word we simply mark that equation, too. Hence we may assume $u \neq 1$ and we may write $u = va$ with $a \in A$. Nondeterministically we guess whether there exists a strong solution σ such that $\overline{a} \in \sigma(Y)$.

If our guess is "$\overline{a} \notin \sigma(Y)$", then we mark the equation $X = vaY + \mathrm{Pref}(va)$. If the guess is true then marking is correct because then vaw is reduced for all $w \in \sigma(Y)$. Whether or not $\overline{a} \notin \sigma(Y)$ is true, marking an equation never introduces new solutions. Thus, a wrong guess does not transform an unsatisfiable system into a satisfiable one. Hence, it is enough to consider the other case that the guess is "$\overline{a} \in \sigma(Y)$" for some strong solution σ. In this case we introduce two fresh variables Y', Y'' and a new marked equation

$$Y = Y' + \overline{a}Y'' + \mathrm{Pref}(\overline{a}).$$

If $\overline{a} \in \sigma(Y)$ is correct then we can extend the strong solution so that $\overline{a} \notin \sigma(Y')$. If $\overline{a} \in \sigma(Y)$ is false then, again, this step does not introduce any new solution.

Finally, we replace the equation $X = vaY + \mathrm{Pref}(va)$ by the following

three equations, the first two of them are marked and the variables X', X'' are fresh

$$\begin{aligned} X &= X' + X'' && \text{(marked)},\\ X' &= vaY' + \mathrm{Pref}(va) && \text{(marked)},\\ X'' &= vY'' + \mathrm{Pref}(v). \end{aligned}$$

If the guess "$\overline{a} \in \sigma(Y)$" was correct, then the new system has a strong solution. If the new system has any solution then the old system has a solution because $X'' = vY'' + \mathrm{Pref}(v)$ is unmarked as long as $v \neq 1$. After polynomial many rounds all equations are marked. This defines the new system $\mathcal{S}'$. If $\mathcal{S}'$ has a solution σ' then the restriction of σ' to the original variables is also a solution of the original system $\mathcal{S}$. If all our guesses were correct with respect to a strong solution σ of $\mathcal{S}$ then $\mathcal{S}'$ has a strong solution σ' such that σ is the restriction of σ' to the original variables. Hence, $\mathcal{S}$ has a solution if and only if $\mathcal{S}'$ has a solution.

It is therefore enough to consider the system $\mathcal{S}'$ of language equations over A^*. All the equations are still of one of the types above. Let σ be any mapping from variables in $\mathcal{S}'$ to finite languages of A^*, i.e., $\sigma(X) \subseteq A^*$ denotes an arbitrary finite language for all variables. Then we have the following implications.

- $\sigma(X)) = 1$ implies $\mathrm{Pref}(\sigma(X)) = 1$.
- $\sigma(X) = \sigma(Y) + \sigma(Z)$ implies $\mathrm{Pref}(\sigma(X)) = \mathrm{Pref}(\sigma(Y)) + \mathrm{Pref}(\sigma(Z))$.
- $\sigma(X) = u\sigma(Y) + \mathrm{Pref}(u)$ implies $\mathrm{Pref}(\sigma(X)) = u\,\mathrm{Pref}(\sigma(Y)) + \mathrm{Pref}(u)$.
- $\sigma(X) = 1 + \sigma(X) \Leftrightarrow 1 \in \sigma(X) \Leftrightarrow 1 \in \mathrm{Pref}(\sigma(X))$.

Thus, the system $\mathcal{S}'$ of language equations over A^* has a solution if and only if $\mathcal{S}'$ has a language solution in nonempty, finite, and prefix-closed sets.

In order to finish the proof, let us briefly repeat what we have done so far. The input has been a system $\mathcal{S}$ of equations over $\mathrm{FIM}(A)$ in idempotent variables. If $\mathcal{S}$ has a solution then it has a strong solution and making all guesses correct we end up with a system $\mathcal{S}'$ of language equations over A^* which has a strong solution in finite and prefix-closed sets. Conversely, consider some system $\mathcal{S}'$ which is obtained by the nondeterministic choices. (Note that the number of different systems $\mathcal{S}'$ is bounded by a singly exponential function and DEXPTIME is enough time to calculate a list containing all $\mathcal{S}'$.) Assume

that $\mathcal{S}'$ has a solution σ' in finite subsets of A^*. Due to the syntactic structure of $\mathcal{S}'$ there is also a solution σ in nonempty and prefix-closed subsets of A^*. This is due to the three implications above. Using Lemma 5.3, σ solves $\mathcal{S}$ as a system of language equations over the group $F(A)$ in nonempty and prefix-closed subsets of reduced words. Hence, σ solves the original system over the free inverse monoid $\mathrm{FIM}(A)$. Since the square of a singly exponential function is singly exponential, it is enough to apply Theorem 3.2. □

5.2 Proof of the lower bound in Theorem 5.1

Throughout this section we work over a two letter alphabet $B = \{\alpha, \beta\}$ which is embedded in the alphabet $A = \{\alpha, \overline{\alpha}, \beta, \overline{\beta}\}$ with involution without fixed points. Thus, $|A| = 4$.

We show that the problem of solving a system of equations in idempotent variables over $\mathrm{FIM}(A)$ is DEXPTIME-hard, even if we restrict input to systems with two equations. The first part of this lower bound proof is about a surgery on language equations. It is the main ingredient, although free inverse monoids do not appear in that part.

5.2.1 Surgery: from solutions in B^* to solutions in $F(A)$

This section contains a sequence of transformations for language equations. We say that systems $\mathcal{S}$ and $\mathcal{S}'$ of language equations are *sat-equivalent*, provided $\mathcal{S}$ is solvable if and only if $\mathcal{S}'$ is solvable.

We consider the equations

$$L + \sum_{i \in I} u_i X_i = K + \sum_{j \in J} u_j X_j \tag{10}$$

with coefficients over B and where variables represent finite subsets of B^*. Clearly, if there is a solution over the free monoid A^*, then there is also a solution over B^*, because the coefficients are over B. Hence, for sat-equivalent it is enough to consider solutions in finite subsets of A^*.

With the help of a fresh variable X_0 each equation as in (10) can be replaced by the following system:

$$\begin{gathered} X_0 = 1, \\ \sum_{u \in L} u X_0 + \sum_{i \in I} u_i X_i = \sum_{v \in K} v X_0 + \sum_{j \in J} u_j X_j. \end{gathered}$$

If a term uX appears in a system with $|u| \geq 2$, then we write $u = av$ with $a \in B$ and we introduce a fresh variable $[vX]$. We replace uX everywhere by $a[vX]$; and we add the equation $[vX] = vX$. We can repeat the process until all terms uX satisfy $|u| \leq 1$. The transformation produces a sat-equivalent system of quadratic size in the original system. With the help of more fresh variables we can proceed to have the following form

$$\begin{aligned} X_0 &= 1, \\ X_{1k} &= a_{2k}X_{2k} + a_{3k}X_{3k} \quad \text{for } 1 \leq k \leq n. \end{aligned}$$

Here, $n \in \mathbb{N}$, $a_{ik} \in B^*$ have length at most 1, and $X_{ik} \in \Omega$.

Next, it is convenient to allow the sign $\leq$ in addition to $=$ in the notation of equations. More formally, $L \leq R$ denotes the short hand of the language equation $L + R = R$. Vice versa we can identify $L = R$ with the system

$$\begin{aligned} L &\leq R, \\ R &\leq L. \end{aligned}$$

For example, by letting $a = 1$ the equation $X = bY + cZ$ is equivalent to the following system where all equations are written in a uniform way.

$$\begin{aligned} aX &\leq bY + cZ, \\ bY &\leq aX + aX, \\ cZ &\leq aX + aX. \end{aligned}$$

The transformations above show that on input $\mathcal{S}$ we can produce in polynomial time a sat-equivalent system $\mathcal{S}_1$ which can be written as:

$$X_0 = 1, \tag{11}$$

$$a_{1k}X_{1k} \leq a_{2k}X_{2k} + a_{3k}X_{3k} \quad \text{for } 1 \leq k \leq n. \tag{12}$$

As above, $n \in \mathbb{N}$, $a_{ik} \in B^*$ have length at most 1, and $X_{ik} \in \Omega$.

The next transformation yields a sat-equivalent system which has a solution if and only if it has a solution in nonempty and prefix-closed sets.

For this we choose some letter $d \in B$ and we transform $\mathcal{S}_1$ into a system $\mathcal{S}_2$ as follows. We replace the equation $X_0 = 1$ in (11) by:

$$X_0 = 1 + d. \tag{13}$$

Moreover, we replace each equation of type $a_{1k}X_{1k} \leq a_{2k}X_{2k} + a_{3k}X_{3k}$ in (12) by:

$$a_{1k}X_{1k} \leq a_{1k} + a_{2k}X_{2k} + a_{3k}X_{3k} \tag{14}$$

Lemma 5.4 *The systems $\mathcal{S}_1$ and $\mathcal{S}_2$ are sat-equivalent. Moreover, if $\mathcal{S}_2$ has any solution, then it has a solution in nonempty prefix-closed sets.*

Proof. Let $\sigma : \Omega \to 2_f^{B^*}$ be any solution of $\mathcal{S}_1$. Then

$$\sigma'(X) = 1 + \mathrm{Pref}(\sigma(X)d)$$

defines a solution of $\mathcal{S}_2$ in nonempty prefix-closed sets. Thus, it is enough to show that if $\mathcal{S}_2$ is solvable, then $\mathcal{S}_1$ is solvable, too.

To this end, let σ' be any solution of $\mathcal{S}_2$. Define

$$\sigma(X) = \{u \in B^* \mid ud \in \sigma'(X)\}.$$

(Note that $\sigma(X)$ might be empty.) Now, $\sigma'(X_0) = \{1, d\}$ implies $\sigma(X_0) = \{1\}$. It remains to show that $a\sigma'(X) \subseteq \{a\} \cup b\sigma'(Y) \cup c\sigma'(Z)$ implies $a\sigma(X) \subseteq b\sigma(Y) \cup c\sigma(Z)$. This is straightforward. Indeed, let $u \in \sigma(X)$, hence $ud \in \sigma'(X)$. Since $aud \neq a$ we must have $aud \in b\sigma'(Y) \cup c\sigma'(Z)$. By symmetry, we may assume $aud \in b\sigma'(Y)$. Thus, $aud = bvd$ with $vd \in \sigma'(Y)$. This implies $v \in \sigma(Y)$. Therefore, $au \in b\sigma(Y)$. Hence, the result. □

The system $\mathcal{S}_2$ does not suffice for our purpose. We need a system where we can control that all solutions σ and all variables X satisfy $\sigma(X) \subseteq \{\alpha, \beta\}^*$. The crucial observation is as follows: let $L_1, \ldots, L_n \subseteq B^*$ be finite subsets. Then their union is finite, and so is the factor-closure of their union

$$K = \{v \in B^* \mid \exists u, w \in B^* : uvw \in \cup_{i=1}^n L_i\}.$$

Factor-closed languages are prefix and suffix-closed; and for suffix-closed languages we can control its alphabet by the following condition

$$K \subseteq \{1\} \cup \bigcup_{a \in B} aK. \tag{15}$$

More precisely, for every language $K \subseteq C^*$ where $B \subseteq C$ we have that K satisfies (15) if and only if both, K is suffix-closed and $K \subseteq B^*$.

Proposition 5.5 *There is a polynomial time algorithm which produces on an input, which is a system $\mathcal{S}$ of language equations over B^*, an output, which is a system of two language equations $\mathcal{S}'$ satisfying the following conditions.*

- *$\mathcal{S}$ and $\mathcal{S}'$ are sat-equivalent.*
- *If $\mathcal{S}'$ has any solution, then it has a solution in nonempty prefix-closed subsets of B^* and therefore a solution as a language equation over the group $F(A)$.*
- *If $\mathcal{S}'$ has a solution as system of language equations over the group $F(A)$, then $\mathcal{S}$ is solvable.*
- *The system $\mathcal{S}'$ can be written in the following syntactic form*

$$L + \sum_{i \in I} u_i X_i = K + \sum_{j \in J} v_j X_j, \tag{16}$$

$$1 + \sum_{a \in B} aZ + \sum_{X \in \Omega} X = 1 + \sum_{a \in B} aZ \tag{17}$$

where $L, K \in 2_f^{B^*}$, $u_i, v_j, X_i, X_j; Z$ denote nonempty words in B^* and variables, respectively. Moreover, $X_i \neq Z \neq X_j$ for all i, j.

Proof. We may start with the system $\mathcal{S}_2$ which satisfies Lemma 5.4. Since it is sat-equivalent to $\mathcal{S}_1$, it is also sat-equivalent to $\mathcal{S}$. The system $\mathcal{S}_2$ contains equations $X_0 \leq 1 + d$ and $1 + d \leq X_0$ for some $d \in B$ and all other equations have the form $a_{1k}X_{1k} \leq a_{2k}X_{2k} + a_{3k}X_{3k}$ where $a_{ik} \in \{1\} \cup B$ and X_{ik} are variables. By the procedure described in Remark 3.3 we transform $\mathcal{S}_2$ into a single equation $\mathcal{E}'$ which has "almost" the syntactic form as required in Equation (16), because we have

$$L + \sum_{i \in I} u_i X_i \leq K + \sum_{j \in J} v_j X_j,$$

If RHS denotes the right-hand side, then we can replace

$$L + \sum_{i \in I} u_i X_i \leq \text{RHS}$$

by

$$L + \sum_{i \in I} u_i X_i + \text{RHS} = \text{RHS}$$

Hence, a syntactic form as it is required by (16). Recall that these transformations do not change the set of solutions. Without restriction, $u_i \neq 1$ and $v_j \neq 1$ for all i, j. Moreover, we may assume that there is a variable Z which does not appear in (16). Adding Equation (17) defines $\mathcal{S}'$. The system $\mathcal{S}'$ contains two equations.

If $\mathcal{S}$ is solvable, then (16) has a solution σ in nonempty prefix-closed subsets of B^*. As Z does not appear we may assume $\sigma(Z) = \{1\}$. Redefining

$$\sigma(Z) = \{v \in B^* \mid \exists u \in B^* \, \exists X \in \Omega : uv \in \sigma(X)\}$$

yields a solution in nonempty prefix-closed subsets of B^* of the system $\mathcal{S}'$. Hence, a solution as a language equation over the group $F(A)$.

Finally, let $\sigma' : \Omega \to 2_f^{F(A)}$ a solution of $\mathcal{S}'$ as a language equation over the group $F(A)$. We claim that σ' is also a solution in the free monoid A^*. If so, then $\mathcal{S}_2$ has a solution in A^*, and this implies that $\mathcal{S}$ is solvable.

By contradiction, assume that σ' does not solve $\mathcal{S}'$ over A^*. Then there are some $b \in B$, $u \in A^*$, and $X \in \Omega$ with $\overline{b}u \in \sigma'(X) \subseteq F(A)$. We may choose b and X such that $|u|$ is maximal. Equation (17) implies $\overline{b}u = \pi(av)$ for some $a \in B$ and some reduced word $v \in \sigma(Z)$. Since $\overline{b} \neq a$, this implies $v = \overline{a}w$ and $\pi(av) = w$. Hence $\overline{a}\overline{b}u \in Z$, which contradicts that u was of maximal length. □

5.2.2 Finishing the proof of Theorem 5.1

Due to Theorem 3.2 and Proposition 5.5 we know that the problem to decide systems $\mathcal{S}'$ with two language equations in the form of Proposition 5.5 is DEXPTIME-complete. Thus, all we need to finish the proof of Theorem 5.1 is the following lemma.

Lemma 5.6 *There is polynomial time algorithm which produces on an input equation $\mathcal{S}'$ as in Proposition 5.5 a system $\mathcal{S}''$ of two equations $U_1 = V_1$ and $U_1 = V_2$ over* FIM(A) *in idempotent variables such that $\mathcal{S}'$ is solvable as a language equation if and only if $\mathcal{S}''$ is solvable over* FIM(A).

Proof. Consider the system $\mathcal{S}'$ in Proposition 5.5 and let LHS_i resp. RHS_i be the left- resp. right-hand sides of the corresponding equations,

$i = 1, 2$. Each of these terms has the form $T = L + \sum_{i \in I} u_i X_i$ which is defines a word $W(T)$ by

$$W(T) = \prod_{u \in L} u\overline{u} \cdot \prod_{i \in I} u_i X_i \overline{u_i}.$$

The ordering in the products can be chosen arbitrarily. We define $\mathcal{S}''$ by a system of two equations:

$$W(\mathrm{LHS}_1) = W(\mathrm{RHS}_1),$$
$$W(\mathrm{LHS}_2) = W(\mathrm{RHS}_2).$$

If $\mathcal{S}'$ is solvable in nonempty prefix-closed subsets of B^*, then $\mathcal{S}''$ is solvable in $\mathrm{FIM}(A)$. Conversely, if $\mathcal{S}''$ is solvable in $\mathrm{FIM}(A)$, then $\mathcal{S}'$ has a solution as system of language equations over the group $F(A)$. □

Proposition 5.5 and Lemma 5.6 conclude the proof of Theorem 5.1.

6 One-variable equations

Throughout this section we assume that the involution on A is without fixed points, i.e., $F(A)$ is equal to the free group $\mathrm{FG}(A_+)$ in the standard terminology. It is open whether we can remove this restriction.

The following notation is defined for any alphabet Σ and any nonempty word $p \in \Sigma^+$. For $u \in \Sigma^*$ we let $|u|_p$ be the number of occurrences of p as a factor in u. Formally:

$$|u|_p = |\{u' \mid u'p \leq u\}|.$$

The following equation is trivial since p may occur across the border between u and v at most $|p| - 1$ times.

$$0 \leq |uv|_p - |u|_p - |v|_p \leq |p| - 1. \tag{18}$$

Next, assuming that Σ is equipped with an involution, we define a "difference" function $\delta_p : \Sigma^* \to \mathbb{Z}$ by

$$\delta_p(u) = |u|_p - |u|_{\overline{p}}.$$

Since $\delta_p(u) = \delta_{\overline{p}}(\overline{u})$ we have $\delta_p(u) = -\delta_p(\overline{u})$, and the mapping δ_p respects the involution.

By definition, we have

$$\delta_p(uv) - \delta_p(u) - \delta_p(v) = (|uv|_p - |u|_p - |v|_p) - (|uv|_{\overline{p}} - |u|_{\overline{p}} - |v|_{\overline{p}})$$

Hence, we may use Equation (18) to conclude:

$$|\delta_p(uv) - \delta_p(u) - \delta_p(v)| \leq |p| - 1. \tag{19}$$

As we identify $F(\Sigma)$ with the subset of reduced words in Σ^*, the mapping δ_p is defined from $F(\Sigma)$ to $\mathbb{Z}$, too. The next lemma shows that its deviation from being a homomorphism can be upper bounded. The next lemma will be applied to a primitive word p, only. Let us remind that a word is defined to be primitive if it cannot be written in the form v^i for some word v with $i > 1$ and it is not empty . Every nonempty word u has a *primitive root*: it is the uniquely defined primitive word p such that $u \in p^+$.

Lemma 6.1 *Let $u_1, \ldots, u_n, p$ be reduced words with $p \neq 1$. Let w be the uniquely defined reduced word such that w is equal to $u_1 \cdots u_n$ in the group $F(\Sigma)$. Then we have:*

$$|\delta_p(w) - \delta_p(u_1) - \cdots - \delta_p(u_n)| \leq 3(|p| - 1)(n - 1). \tag{20}$$

Proof. Clearly, Equation (20) holds for $n = 1$. Hence, let $n \geq 2$. Let u be the reduced word such that $u_1 \cdots u_{n-1}$ reduces to u. By induction, we have $|\delta_p(u) - \delta_p(u_1) - \cdots - \delta_p(u_{n-1})| \leq 3(|p| - 1)(n - 2)$. Let $v = u_n$. By triangle inequality it is enough to show

$$|\delta_p(w) - \delta_p(u) - \delta_p(v)| \leq 3(|p| - 1). \tag{21}$$

To see this write $u = u'r$ and $v = \overline{r}v'$ such that $w = u'v'$.

$$\begin{aligned} \delta_p(w) - \delta_p(u) - \delta_p(v) = {} & \delta_p(w) - \delta_p(u') - \delta_p(v') \\ & + \delta_p(u') + \delta_p(r) - \delta_p(u) \\ & + \delta_p(\overline{r}) + \delta_p(v') - \delta_p(v) \end{aligned}$$

The result follows by Equation (19) and triangle inequality. □

We will apply Lemma 6.1 in the following equivalent form.

$$\begin{aligned} \delta_p(u_1) + \cdots + \delta_p(u_n) - 3(|p| - 1)(n - 1) \leq \delta_p(w) \\ \leq \delta_p(u_1) + \cdots + \delta_p(u_n) + 3(|p| - 1)(n - 1). \end{aligned} \tag{22}$$

The following lemma is easy to prove. It is however here where we use $a \neq \overline{a}$ for all $a \in A$. Let us recall that a word q is *cyclically reduced* if qq is reduced. In other words if a is the first letter of q, the last letter of q is different from $\overline{a}$.

Lemma 6.2 *Let $n \in \mathbb{Z}$ and $q \in F(A)$ be a primitive and cyclically reduced word. Then we have $\delta_q(q^n) = n$.*

Proof. We may assume without loss of generality that $n > 0$. Clearly, $|q^n|_q \geq n$. Suppose that $|q^n|_q > n$. Then q is a proper factor of qq, hence we may write $q = q_1q_2 = q_2q_1$ in reduced products with $q_1, q_2 \neq 1$. It is well known (see e.g. [11]) that this contradicts the primitivity of q. Thus, $|q^n|_q = n$.

Suppose now that $\overline{q}$ is a proper factor of qq. Then we may write $q = q_1q_2$ as a reduced product with $\overline{q} = q_2q_1$ since q is cyclically reduced. Moreover, since $\overline{q} = \overline{q}_2\overline{q}_1$ we get $q_2 = \overline{q}_2$ and $q_1 = \overline{q}_1$. Hence $q_1 = q_2 = 1$ because q_1, q_2 are reduced and $a \neq \overline{a}$ for all $a \in A$. Thus, $|q^n|_{\overline{q}} = 0$ and so $\delta_q(q^n) = |q^n|_q - |q^n|_{\overline{q}} = |q^n|_q = n$. □

An (untyped) equation (U, V) is called a *one-variable equation*, if we can write $UV \in (A \cup \{X, \overline{X}\})^*$. More generally, we also consider systems of typed equations with at most one reduced variable x (and $\overline{x}$), i.e., every equation (U, V) in the system satisfies $UV \in (A \cup \Omega \cup \{x, \overline{x}\})^*$. Let us fix some more notation, we let $\Sigma = A \cup \Omega \cup \Gamma$ with $\Gamma = \{x, \overline{x}\}$. In particular, we have $\overline{X} = X$ for all $X \in \Omega$ and $\alpha \neq \overline{\alpha}$ for all $\alpha \in A \cup \Gamma$.

Definition 6.3 *Let $u, v \in \Gamma^*$. We say that (u, v) is* unbalanced *if $u \neq v$ in the free inverse monoid* FIM(Γ).

Otherwise we say that (u, v) is balanced.

Remark 6.4 *Using the well-known structure of* FIM(Γ)*, a pair (u, v) as in Definition 6.3 is balanced if and only if the following three conditions are satisfied.*

- $\delta_x(u) = \delta_x(v)$.
- $max\{\delta_x(u') \mid u' \leq u\} = max\{\delta_x(v') \mid v' \leq v\}$.
- $min\{\delta_x(u') \mid u' \leq u\} = min\{\delta_x(v') \mid v' \leq v\}$.

We extend the notion defined in Definition 6.3 to an untyped one-variable equation. In the following we let $\pi_{A,\Gamma}$ be the morphism from $(A \cup \Omega \cup \Gamma)^*$ to $F(A \cup \Gamma)$ which is induced by cancelling the symbols in Ω.

Definition 6.5 *Let (U, V) be an untyped one-variable equation with $\mathcal{X} = \{X, \overline{X}\}$. We say that (U, V) is* unbalanced *if it fulfills both conditions:*
1- (u, v) is unbalanced as a word over Γ where u (resp. v) is obtained from U (resp. V) by replacing X by x (and $\overline{X}$ by $\overline{x}$) and erasing all other symbols.
2- $\pi_{A,\Gamma}(U) \neq \pi_{A,\Gamma}(V)$ in the free group $F(A \cup \Gamma)$.

The following definition is a bit more technical, but it will lead to better results.

Definition 6.6 *Let U, V be words over $A \cup \Omega \cup \Gamma$. We say that (U, V) is* strongly unbalanced *if $\pi_{A,\Gamma}(U) \neq \pi_{A,\Gamma}(V)$ in the free group $F(A \cup \Gamma)$ and at least one of the following conditions is satisfied.*

(SU1) $\delta_x(U) \neq \delta_x(V)$.

(SU2) For all $z \in \Omega \cup \{1\}$ and all prefixes $V'z$ of V there exists some prefix $U'z$ of U such that $\delta_x(U') > \delta_x(V')$.

(SU3) For all $z \in \Omega \cup \{1\}$ and all prefixes $V'z$ of V there exists some prefix $U'z$ of U such that $\delta_{\overline{x}}(U') > \delta_{\overline{x}}(V')$.

The following result improves the complexity in the corresponding statement of [5]. (Note that the condition $\pi_{A,\Gamma}(U) \neq \pi_{A,\Gamma}(V)$ was missing in [5], but the proof is not valid without this additional requirement.)

Theorem 6.7 *The following problem can be decided in* DEXPTIME.
Input: A system $\mathcal{S}$ of one-variable equations over $\mathcal{X} = \{X, \overline{X}\}$ where at least one equation (U, V) is unbalanced according to Definition 6.5.
Question: Does $\mathcal{S}$ have a solution in FIM(A)*?*

Proof. Suppose that (U, V) is unbalanced. The pair (U, V) must then contradict one of the three conditions of Remark 6.4. Let us distinguish cases and, in each case, reduce the given unbalanced equation into a *strongly* unbalanced *typed* equation.
In all cases, we introduce a fresh idempotent variable Z, a fresh reduced variable x, and use the word-substitutions τ' (or τ) defined in Lemma 4.1: $\tau'(X) = xZ, \tau'(\overline{X}) = \overline{Z}\overline{x}, \tau(X) = Zx, \tau(\overline{X}) = \overline{x}\overline{Z}$ or the trivial substitution

$\theta(X) = x, \theta(\overline{X}) = \overline{x}$.
Case 1: $\delta_X(U) \neq \delta_X(V)$.
In this case $(\theta(U), \theta(V))$ fulfills condition $(SU1)$.
Case 2: $\max\{\delta_X(U') \mid U' \leq U\} > \max\{\delta_X(V') \mid V' \leq V\}$.
There is some prefix $U' \leq U$ such that for all prefixes $V' \leq V$ we have $\delta_X(U') > \delta_X(V')$ and, in particular, $\delta_X(U') > \delta_X(1) = 0$. We choose $\delta_X(U')$ to be maximal and, since $\delta_X(U')$ is positive, we may choose U' such that $X = \text{last}(U')$, so that $\text{last}(\tau'(U')) = Z$. Now, for every $z \in \{Z, 1\}$,

$$\begin{aligned} \delta_x(\tau'(U')) = \delta_X(U') &> \max\{\delta_X(V') \mid V' \leq V\} \\ &= \max\{\delta_x(W) \mid W \leq \tau'(V)\} \\ &\geq \max\{\delta_x(W'z) \mid W'z \leq \tau'(V)\}. \end{aligned}$$

This prefix $\tau'(U')$ shows that $(\tau'(U), \tau'(V))$ fulfills condition $(SU2)$ (this is actually a stronger requirement than asked by Definition 6.6, because this single prefix $\tau'(U')$ serves for all $W'z$).
Case 2': $\max\{\delta_X(U') \mid U' \leq U\} < \max\{\delta_X(V') \mid V' \leq V\}$.
By Case 2 the typed equation $(\tau'(V), \tau'(U))$ fulfills condition $(SU2)$.
Case 3: $\min\{\delta_X(U') \mid U' \leq U\} > \min\{\delta_X(V') \mid V' \leq V\}$.
We may assume that $\delta_X(U) = \delta_X(V) = k$. If $U = U'U''$ and $V = V'V''$, we have $\delta_{\overline{X}}(\overline{U''}) = \delta_X(U'') = k - \delta_X(U')$ and $\delta_{\overline{X}}(\overline{V''}) = \delta_X(V'') = k - \delta_X(V')$, thus $(\overline{U}, \overline{V})$ fulfills that $\max\left\{\delta_{\overline{X}}(U') \;\middle|\; U' \leq \overline{U}\right\} < \max\left\{\delta_{\overline{X}}(V') \;\middle|\; V' \leq \overline{V}\right\}$.
By a reasoning similar to that of case 2, one can show that $(\tau(\overline{V}), \tau(\overline{U}))$ fulfills condition $(SU3)$.
Case 3': $\min\{\delta_X(U') \mid U' \leq U\} < \min\{\delta_X(V') \mid V' \leq V\}$.
By Case 3 the typed equation $(\tau(\overline{U}), \tau(\overline{V}))$ fulfills condition $(SU3)$.
We have thus reduced Theorem 6.7 above to Theorem 6.8 below. □

Theorem 6.8 *The following problem can be decided in* DEXPTIME.

Input: A system $\mathcal{S}$ of typed equations with at most one reduced variable (i.e., $\Gamma = \{x, \overline{x}\}$) where at least one equation $(U, V) \in \mathcal{S}$ is strongly unbalanced.

Question: Does $\mathcal{S}$ have a solution in $\text{FIM}(A)$*?*

The proof of Theorem 6.8 relies on the following combinatorial observation.

Lemma 6.9 *Let (U, V) be a strongly unbalanced equation with $U, V \in (A \cup \Omega \cup \{x, \overline{x}\})^*$ and $n = max\{|U|, |V|\}$. Let $k \in \mathbb{Z}$ be an integer and σ be a solution to (U, V) such that $\sigma(x) = (Pref(p^k), p^k)$ for some nonempty cyclically reduced word $p \in A^*$. Then we have $|k| \leq 6n\,|p|$.*

Proof. Without restriction, p is primitive and $k > 1$. (Replace p by its primitive root and interchange the role of p and $\overline{p}$, if necessary.) For a word $W \in (A \cup \Omega \cup \{x, \overline{x}\})^*$ we write $\sigma(W) = (\sigma_1(W), \sigma_2(W))$ where $\sigma_1(W) \subseteq A^*$ is a prefix-closed set of reduced words and $\sigma_2(W) \in F(A)$. Choose m maximal such that $\delta_p(w) = m$ for some $w \in \sigma_1(V)$. We fix $w \in A^*$ and we observe that we have $m \geq 0$ and for every word $u \in \sigma_1(U) = \sigma_1(V)$, we have

$$\delta_p(u) \leq m \tag{23}$$

Case (SU2): (U, V) fulfills condition (SU2).
We choose a prefix V' of V of minimal length with respect to the property $w \in \sigma_1(V')$. We consider two subcases.

Subcase Ω: $\text{last}(V') \in \Omega$.

Let $z := \text{last}(V')$. The word V' thus decomposes as $V' = V''z$. Since $\sigma_1(V''z) = \sigma_1(V'') \cup \sigma_2(V'')\sigma_1(z)$, it follows from the minimality of $V''z$ that $w \in \sigma_2(V'')\sigma_1(z)$. Since $\sigma_2(Z) = 1$ for every $Z \in \Omega$ and $|V'| \leq n - 1$, it follows that w is the product of at most $n - 1$ reduced words $v_1 \dots v_t$ in $A \cup \{\sigma_2(x), \sigma_2(\overline{x})\}$ by some $z' \in \sigma_1(z)$.

For each letter a of A, $\delta_p(a) \leq 1$ and, since p is primitive, by Lemma 6.2, $\delta_p(\sigma_2(x)) = k, \delta_p(\sigma_2(\overline{x})) = -k$. We thus get

$$\sum_{i=1}^{t} \delta_p(v_i) \leq k\delta_x(V') + n - 1. \tag{24}$$

Since $w = v_1 \dots v_t z'$, we obtain the following upper bound:

$$\begin{array}{rll} m = & \delta_p(w) & \\ \leq & \sum_{i=1}^{t} \delta_p(v_i) + \delta_p(z') + 3(|p| - 1)(n - 1) & \text{by (22)} \\ \leq & k\delta_x(V') + \delta_p(z') + n - 1 + 3(|p| - 1)(n - 1) & \text{by (24)} \end{array} \tag{25}$$

By (SU2) there exists a prefix U' of U such that $\delta_x(U') > \delta_x(V')$ and $\text{last}(U') = z$. The word U' thus decomposes as $U' = U''z$. Let us define

$u := \sigma_2(U'')z'$. We remark that $u \in \sigma_1(U)$, hence it fulfills Equation (23). Using similar arguments based on Equation (22) and Lemma 6.2 we obtain:

$$k\delta_x(U') + \delta_p(z') - (n-1) - 3(|p|-1)(n-1) \leq \delta_p(u). \tag{26}$$

Combining the above inequalities we obtain:

$$\begin{array}{lcll} k \leq & k(\delta_x(U') - \delta_x(V')) & \text{since } \delta_x(U') > \delta_x(V') \\ \leq & -\delta_p(z') + (n-1) + 3(|p|-1)(n-1) + \delta_p(u) - k\delta_x(V') & \text{by (26)} \\ \leq & -\delta_p(z') + (n-1) + 3(|p|-1)(n-1) + m - k\delta_x(V') & \text{by (23)} \\ \leq & 2(n-1) + 6(|p|-1)(n-1) & \text{by (25)} \\ \leq & 6n(|p|) & (27) \end{array}$$

Subcase 1: $\text{last}(V') \notin \Omega$.
We just need to perform some adaptations to the preceding case. The word w is the product of at most n reduced words $v_1 \dots v_t$ in $A \cup \{\sigma_2(x), \sigma_2(\overline{x})\}$, and by similar methods we obtain

$$m = \delta_p(w) \leq k\delta_x(V') + n + 3(|p|-1)(n-1). \tag{28}$$

By (SU2) (where we choose $z := 1$) there exists a prefix U' of U such that $\delta_x(U') > \delta_x(V')$. Let us define $u := \sigma_2(U')$. We get

$$k\delta_x(U') - n - 3(|p|-1)(n-1) \leq \delta_p(u). \tag{29}$$

Since $u = \sigma_2(U') \in \sigma_1(U)$, here also u fulfills (23). Hence, putting (29) (23) and (28) together we obtain the desired result:

$$k \leq k(\delta_x(U') - \delta_x(V')) \leq 6(|p|-1)(n-1) + 2n \leq 6n\,|p|\,. \tag{30}$$

Case (SU3): (U, V) fulfills condition (SU3).
This case is dealt with in a similar manner.
Case (SU1): (U, V) fulfills condition (SU1).
By symmetry in U and V, we may assume without restriction $\delta_x(U) > \delta_x(V)$. Let us choose $V' := V, w := \sigma_2(V), m := \delta_p(w), U' := U, u := \sigma_2(U)$. The arguments of Case (SU2), Subcase 1, apply on these choices for V', w, m, U', u. (In fact, an argument provided by James Howie in [21] shows that in this

case the solution $\sigma(x)$ is unique). $\square$

Proof of Theorem 6.8. Let n be the size of the system $\mathcal{S}$, it is defined as

$$\|\mathcal{S}\| = \sum_{(U,V)\in\mathcal{S}} |UV| .$$

Since $\pi_{A,\Gamma}(U) \neq \pi_{A,\Gamma}(V)$ for at least one equation in the system, the set of solutions for the underlying group equations is never equal to $F(A)$. By [1, 10], the set of solutions of a one-variable free group equation is therefore a finite union of sets of the form

$$\left\{ rq^k s \;\middle|\; k \in \mathbb{Z} \right\}, \tag{31}$$

where q is cyclically reduced and both products rqs and $r\overline{q}s$ are reduced. A self-contained proof of this fact has been given in [3].

In the description above $q = 1$ is possible. Moreover, [3] shows $|rqs| \in \mathcal{O}(n)$. Hence, as we aim for DEXPTIME there is time enough to consider all possible candidates for r and s. This means we can fix r and s; and it is enough to consider a single set $S = \left\{ rq^k s \;\middle|\; k \in \mathbb{Z} \right\}$, only. Next we replace in $\mathcal{S}$ all occurrences of x by rxs (and $\overline{x}$ by $\overline{s}\,\overline{x}\,\overline{r}$). This leads to a new system which we still denote by $\mathcal{S}$ and without restriction we have $S = \left\{ q^k \;\middle|\; k \in \mathbb{Z} \right\}$. The new size m of $\mathcal{S}$ is at most quadratic in n.

Now, we check if $k = 0$ leads to a solution of $\mathcal{S}$. This means that we simply cancel x and $\overline{x}$ everywhere. We obtain a system over idempotent variables; and we can check satisfiability by Theorem 5.1. Note that this includes the case $q = 1$. Thus, henceforth we may assume that q is a primitive cyclically reduced word. By Lemma 6.9 we see that it is enough to replace S by $S' = \left\{ q^k \;\middle|\; |k| \leq 6m\,|q| \right\}$. Since $|q| \in \mathcal{O}(m)$ we obtain a cubic bound for the maximal length of words in S', this means the length of each word in S' is bounded by $\mathcal{O}(n^6)$. This is small enough to check satisfiability of the original system $\mathcal{S}$ in DEXPTIME by Theorem 5.1. $\square$

Conclusion and directions for future research

The notion of "idempotent variable" unifies the approach for studying equations in free inverse monoids. As the general situation is undecidable, progress is possible only by improving complexities in classes where decidability is

known and/or to enlarge the class of equations where decidability is possible. We achieved progress in both fields. For equations in idempotent variables we lowered the complexity down to DEXPTIME and proved that this bound is tight. Using a recent result in [6] that it is decidable in PSPACE whether an equation in free groups has only finitely many solutions, we derived a "promise result" in Corollary 4.3 with triple exponential time complexity. We don't think that this is optimal, because we believe that solving equations in free groups is in NP. But this fundamental conjecture is wide open and resisted all known techniques.

More concretely, let us resume some interesting and specific problems on equations in free inverse monoids which are open:

- Is the decision problem in Theorem 5.1 restricted to single equation in idempotent variables DEXPTIME-hard? We conjecture: yes!
- Is the (other) special kind of equations solved by Theorem 23 of [5] also solvable in DEXPTIME?
- Is it possible to remove Assumption 2 in Definition 6.5, and still maintain decidability of the system of equations? (The assumption asserts that the image of the left-hand side and right-hand side are different in the free group.)
- What happens if the underlying equation in the free group is true for all elements in the free group? This means the statement is a tautology the free group.
- What *more general* kinds of one-variable equations in the free inverse monoid are algorithmically solvable (possibly all of them)?
- Does Jeż' recompression technique apply to language equations? If yes, then this would open a new approach to tackle equations over free inverse monoids.

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