# A new notion of vertex independence and rank for finite graphs

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ABSTRACT

A new notion of vertex independence and rank for a finite graph $G$ is introduced. The independence of vertices is based on the boolean independence of columns of a natural boolean matrix associated to $G$. Rank is the cardinality of the largest set of independent columns. Some basic properties and some more advanced theorems are proved. Geometric properties of the graph are related to its rank and independent sets.

## Contents

# 1 Introduction

The background and prehistory for this paper goes something like the following. In 2006 Zur Izakhian [10] defined the notion of independence for columns (rows) of as matrix with coefficients in a supertropical semiring. Restricting this concept to the superboolean semiring $\mathbb{SB}$ (see Subsection 2.4), and then to the subset of boolean matrices (equals matrices with coefficients 0 and 1), we obtain the notion of independence of columns (rows) of a boolean matrix. This notion has several equivalent formulations (see Subsection 2.4 of this paper and references there), one involving permanent, another being the following: if $M$ is an $m \times n$ boolean matrix, then a subset $J$ of columns of $M$ is *independent* if and only if there exists a subset $I$ of rows of $M$ with $|I| = |J| = k$ and the $k \times k$ submatrix $M[I, J]$ can be put into upper triangular form (1's on the diagonal, 0's strictly above it, and 0's or 1's below it) by independently permuting the rows and columns of $M[I, J]$.

This is the notion of independence for columns of a boolean matrix we will use in this paper. In 2008 the first author suggested that this idea would have application in many branches of Mathematics and especially in Combinatorial Mathematics. In this paper we apply it to the vertices of a finite graph. For other applications of this notion to lattices, posets and matroids by Izhakian and the first author, see [11, 12, 13].

If $M$ is an $m \times n$ boolean matrix with column space $C$, then the set $\mathcal{H}$ of independent subsets of $C$ satisfies the following axioms (see [11, 12]):

(H) $\mathcal{H}$ is nonempty and closed under taking subsets (making it a *hereditary collection*);

(PR) for all nonempty $J, \{p\} \in \mathcal{H}$, there exists some $x \in J$ such that $(J \setminus \{x\}) \cup \{p\} \in \mathcal{H}$ (the *point replacement* property).

Hereditary collections arising from some boolean matrix $M$ as above are said to be *boolean representable*. A very interesting question is which hereditary collections have boolean representations, a question which the authors will address in a near future paper [18]. The elementary properties of such boolean representable collections were considered in [11, 12, 13] and it was shown in [12] that all matroids have boolean representations.

In this paper we restrict our atention to finite graphs (with no loops and no multiple edges), see Subsection 2.2. However, there are several ways to define such a graph by a boolean matrix. The one chosen in matroid theory by Whitney [20] and related to the Levi graph is to attach the boolean matrix $M(G)$ to the graph $G = (V, E)$, where $V$ is the set of vertices and $E$ the set of edges considered as 2-sets of $V$, with $M(G)$ the $|V| \times |E|$ boolean matrix defined by $M(G)(v, e) = 1$ if $v$ lies in $e$, and 0 otherwise. Now whether we consider the columns of $M(G)$ as independent in our boolean sense or in the usual vector space sense (over the field $\mathbb{Z}_2$), we obtain the same independent sets which form a matroid called a graphical matroid, see [11, 16, 17].

So this viewpoint has been extensively worked out [16, 17], and mainly following Tutte's suggestions, ideas from graphs like connectedness ($n$-connected) can be extended to matroids, etc.

A perhaps more obvious way to associate a boolean matrix to a graph $G = (V, E)$ is via the $|V| \times |V|$ boolean adjacency matrix (see Subsection 2.2) $A_G = (a_{ij})$, where $a_{ij} = 1$ if $\{i, j\}$ is an edge of $G$, and 0 otherwise. So $A_G$ can be an arbitrary symmetric square boolean matrix with 0's in the main diagonal (see also [2]). However, in this paper we choose $A_G^c$ which is $A_G$ with 0 and 1 interchanged. This approach is indicated from the lattice/poset case [13], and that finite boolean modules (equals semilattices) have dual spaces which separate points and the dual space is reversing the order, see [19, Chapter 9.1 and 9.2].

Also if $A_G$ were used, then $K_n$ (the complete graph on $n$ vertices) and its complement $\overline{K_n}$ would have sets of 2 or less vertices being the independent sets or only the empty set being independent respectively, clearly not a good choice.

Thus our new notion of independence of a subset of vertices $X \subseteq V$ of a graph $G = (V, E)$ is that the columns corresponding to $X$ in $A_G^c$ are boolean independent. Note that, by using $A_G^c$, all subsets of vertices of $K_n$ are independent. This is termed *c-independence* for vertices of $G$, and the cardinality of the largest independent set of vertices ia termed *c-rank*, denoted c-rk . Note that we work with the superboolean semiring $\mathbb{SB}$, for representation by matrices over $GF(2)$ the reader can be referred to a recent paper by Brijder and Traldi [2].

As we mentioned before, Whitney associated to each finite graph $G = (V, E)$ a (graphical) matroid [20]. In this paper we more or less reverse this procedure and treat each graph $G$ as given "like a matroid" in the following manner. The graph $G$ has the boolean representation $A_G^c = M$. Each boolean representation $M$, see [12, 13, 18], gives rise to the *lattice of flats* (see Subsection 2.2) of $M$. This corresponds to the idea in matroid theory of the geometric lattice of flats of a matroid (see [16]). Given the boolean matrix $M$ with column space $C$, the lattice of flats of $M$ consists of the subsets of $C$ corresponding to where the rows of $M$ are zero, closed under all intersections (see Subsection 2.2). Then Theorem 3.1 yields that the independent subsets of $C$ with respect to $M$ are the partial transversals of the partition of successive differences for some maximal chain of the lattice of flats. This relates to earlier ideas of Bjorner and Ziegler [1].

If $L$ is the geometric lattice of flats of a matroid $P = (C, \mathcal{H})$, then taking the boolean representation $M_L$ corresponding to $L$ and restricted to the atom generators ($M_L$ is $A_L^c$ – where $A_L$ is the $L$ incidence matrix – restricted to the atom rows $C$, then transposed so considered as columns), then the lattice of flats of the boolean matrix $M_L$ as described before is the same as the geometric lattice of usual flats of the matroid (see [12]). Thus this approach truly generalizes the matroid approach.

When applying this "boolean combinatorics" approach to some standard field of Mathematics (e.g. finite graph theory), usually the notion of rank is well known, the notion of independence is new, and the approach tends quickly to some well developed subfield of the subject under study. Somehow geometry is also supposed to show up in this approach: see below!

Enough of the general background. In this paper the boolean representation for a graph is $A^c_G$ and the notions of c-independence and c-rank are taken with respect to $A^c_G$. The lattice of flats for the graph $G = (V, E)$ can be realised by closing $\{\mathrm{St}(v) \mid v \in V\}$ under all intersections, where $\mathrm{St}(v) = \{v' \in V \mid \{v, v'\} \in E\}$. This and other preliminaries are done in Section 2. The c-rank and how to calculate the c-independent subsets of vertices are discussed in Section 3. It is proved that the c-rank is the height of the lattice of flats and c-independent subsets can be calculated by Theorem 3.1(iv)-(v).

In Section 4 we characterize graphs of low c-rank. Section 5 is devoted to the interesting case of sober connected graphs of c-rank 3 (we call a graph *sober* if the mapping St is injective). In Section 6, our new notions acquire a distinctive geometric flavor in connection with Levi graphs and partial euclidean geometries. Geo is defined in the appropriate context and Geo of the Petersen graph is computed to be the Desargues configuration, see Example 6.3. Section 7 collects results concerning cubic graphs, including characterizations of the graphs whose lattice of flats satisfies the most famous lattice-theoretic properties. A variation of the concept of c-rank appropriate to deal with minors is discussed in Section 8. Finally, Section 9 relates a graph and its complement graph in the context of our new notions.

# 2 Preliminaries

## 2.1 Posets and lattices

Our lattice and poset terminology is more or less standard (see [7, 8, 15, 19]). For ease of exposition we assume all posets, lattices and graphs to be finite, although many of the results admit generalizations to the infinite case.

Given a finite poset $(P, \leq)$ and $p, q \in P$, we say that $p$ *covers* $q$ if $p > q$ but there is no $r \in P$ such that $p > r > q$. It is standard to represent finite posets by means of their *Hasse diagram*: in this directed graph, the vertices are the elements of $P$ and $(p, q)$ is an edge when $p$ covers $q$. Note that a chain in $(P, \leq)$ is maximal if and only if it corresponds to some path in the Hasse diagram connecting a maximal element to a minimal element.

The *height* of $(P, \leq)$ is defined by

$$\mathrm{ht}\, P = \max\{k \in \mathbb{N} \mid p_0 < p_1 < \ldots < p_k \text{ is a chain in } P\}.$$

Equivalently, $\mathrm{ht}\, P$ is the maximum length of a path in the Hasse diagram of $P$.

We say that $(P, \leq)$ is a lattice if, for all $p, q \in P$, there exist

$$p \vee q = \min\{x \in P \mid x \geq p, q\},$$
$$p \wedge q = \max\{x \in P \mid x \leq p, q\}.$$

If only the first (respectively the second) of these conditions is satisfied, we talk of a $\vee$-*semilattice* (respectively $\wedge$-*semilattice*). We say that $P' \subseteq P$ constitutes a *sublattice* of $(P, \leq)$ if $p \vee q, p \wedge q \in P'$ for all $p, q \in P'$. Note there need be no relation between the top (bottom) of $P'$ and of $P$. Every point is a sublattice.

A lattice $(L, \leq)$ is said to be *distributive* if

$$p \wedge (q \vee r) = (p \wedge r) \vee (p \wedge r)$$

holds for all $p, q, r \in L$, a condition which is equivalent to its dual. We shall say that $(L, \leq)$ is *modular* if there is no sublattice of the form

$$
\begin{tikzcd}
 & a \arrow[dl] \arrow[ddr] & \\
b \arrow[d] & & \\
c \arrow[dr] & & d \arrow[dl] \\
 & e &
\end{tikzcd}
$$

If we only exclude such sublattices when $d$ covers $e$, the lattice is *semimodular*. It is well known that every distributive lattice is modular, and modular implies of course semimodular.

An *atom* of $L$ is an element covering the minimum element 0. A semimodular latice is called *geometric* if every element is a join of atoms (0 being the join of the empty set). Finally, $L$ satisfies the *Jordan-Dedekind condition* if all the maximal chains in $L$ have the same length.

## 2.2 Graphs

Throughout this paper, graphs are finite, undirected, and have neither loops nor multiple edges. Formally, a (finite) graph is an ordered pair $G = (V, E)$, where $V$ is a (finite) set (the set of vertices) and $E \subseteq \{X \in 2^V : |X| = 2\}$ (the set of edges). In other words, the edges are 2-*subsets* of $V$ (an $n$-subset is a subset with $n$ elements). We assume the reader to be familiar with the basic concepts of graph theory (see e.g. [6]).

Clearly, $(2^V, \subseteq)$ is a distributive lattice with $X \wedge Y = X \cap Y$ and $X \vee Y = X \cup Y$. If $X \subseteq V$ has $k$ elements, we say it is a $k$-*subset* of $V$.

Given $\mathcal{S} \subseteq 2^V$, it is easy to see that

$$\widehat{\mathcal{S}} = \{\cap S \mid S \subseteq \mathcal{S}\}$$

is the $\wedge$-subsemilattice of $(2^V, \subseteq)$ generated by $\mathcal{S}$. Note that $\cap\mathcal{S} = \min\widehat{\mathcal{S}}$, and also $V = \cap\emptyset = \max\widehat{\mathcal{S}}$. In fact, $(\widehat{\mathcal{S}}, \subseteq)$ is itself a lattice with

$$P \vee Q = \cap\{X \in \mathcal{S} \mid P \cup Q \subseteq X\}.$$

However, $(\widehat{\mathcal{S}}, \subseteq)$ is not in general a sublattice of $(2^V, \subseteq)$ since $P \vee Q$ (in $(\widehat{\mathcal{S}}, \subseteq)$) needs not be $P \cup Q$ (see [7, 19]).

Note that

$$\mathrm{ht}\, \widehat{\mathcal{S}} \leq |\mathcal{S}| \tag{1}$$

since any chain in $\widehat{\mathcal{S}}$ is necessarily of the form

$$V \supseteq X_1 \supset X_1 \cap X_2 \supset \ldots \supset X_1 \cap \ldots \cap X_k$$

for distinct $X_1, \ldots, X_k \in \mathcal{S}$.

Finally, we say that $\{y_1, \ldots, y_k\}$ is a *transversal* of the partition of the successive differences for the chain $X_0 \supset \ldots \supset X_k$ in $\widehat{\mathcal{S}}$ if $y_i \in X_{i-1} \setminus X_i$ for $i = 1, \ldots, k$. A subset of a transversal is a *partial transversal*.

Given $v \in V$, the *star* of $v$ is defined by

$$\mathrm{St}(v) = \{w \in V \mid w \text{ is adjacent to } v \text{ in } G\}.$$

More generally, given $W \subseteq V$, we write

$$\mathrm{St}(W) = \cap_{w \in W} \mathrm{St}(w).$$

Note that $\mathrm{St}(\emptyset) = V$. Let $\mathcal{S}_W = \{\mathrm{St}(w) \mid w \in W\}$. It is immediate that

$$\widehat{\mathcal{S}_W} = \{\mathrm{St}(W') \mid W' \subseteq W\}. \tag{2}$$

We call a subset of the form $\mathrm{St}(W)$ $(W \in V)$ a *flat* and say that $\widehat{\mathcal{S}_V}$ is the *lattice of flats* of $G$, also denoted by $\mathrm{Fl}\, G$. We believe this to be a new concept for graphs.

Note that, for a connected graph $G = (V, E)$, we can define a metric $d$ on $V$ by

$$d(v, w) = \text{ length of the shortest path connecting } v \text{ and } w \text{ (counting edges).}$$

The *diameter* of $G$, denoted by $\mathrm{diam}\, G$, is the maximum value in the image of $d$.

Given a finite graph $G$, the *girth* of $G$, denoted by $\mathrm{gth}\, G$, is the length of the shortest cycle in $G$ (assumed to be $\infty$ is $G$ is acyclic). Note that $\mathrm{gth}\, G \geq 3$ for any finite graph.

We shall use the notation

$$\hat{n} = \{1, \ldots, n\}$$

throughout the paper. Assume now that $V = \hat{n}$. The *adjacency matrix* of $G = (V, E)$ is the $n \times n$ boolean matrix $A_G = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } \{i, j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

The matrix $A_G^c$ is obtained by interchanging 0 and 1 all over $A_G$. If the graph is clear from the context, we shall write just $A$ and $A^c$.

### 2.3 Matroids

Let $V$ be a set and let $X \subseteq 2^V$. We say that $X$ is a *hereditary collection* if $X$ is closed under taking subsets. The hereditary collection is said to be a *matroid* if the following condition (the *exchange property*) holds:

(EP) For all $I, J \in X$ with $|I| = |J| + 1$, there exists some $i \in I \setminus J$ such that $J \cup \{i\} \in X$.

There are many other equivalent definitions of matroid. For details, the reader is referred to [16].

## 2.4 Superboolean matrices

Following [11], we shall view boolean matrices as matrices over the *superboolean semiring* $\mathbb{SB} = \{0, 1, 1^\nu\}$, where addition and multiplication are described respectively by

$$\begin{array}{c|ccc} + & 0 & 1 & 1^\nu \\ \hline 0 & 0 & 1 & 1^\nu \\ 1 & 1 & 1^\nu & 1^\nu \\ 1^\nu & 1^\nu & 1^\nu & 1^\nu \end{array} \qquad\qquad \begin{array}{c|ccc} \cdot & 0 & 1 & 1^\nu \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1^\nu \\ 1^\nu & 0 & 1^\nu & 1^\nu \end{array}$$

We denote by $\mathcal{M}_n(\mathbb{SB})$ the set of all $n \times n$ matrices with entries in $\mathbb{SB}$. Note that $n \times n$ boolean matrices are *not* a subsemiring of $\mathcal{M}_n(\mathbb{SB})$ since $1 + 1 = 1^\nu$.

Next we present definitions of independency and rank appropriate to the context of superboolean matrices, introduced in [10] (see also [11]).

We say that vectors $C_1, \ldots, C_m \in \mathbb{SB}^n$ are *dependent* if $\lambda_1 C_1 + \ldots \lambda_m C_m \in \{0, 1^\nu\}$ for some $\lambda_1, \ldots, \lambda_m \in \{0, 1\}$ not all zero. Otherwise, they are said to be *independent*.

Let $S_n$ denote the symmetric group on $\hat{n}$. The *permanent* of a matrix $M \in \mathcal{M}_n(\mathbb{SB})$ (a positive version of the determinant) is defined by

$$\operatorname{Per} M = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i, i\sigma}.$$

Recall that addition and multiplication take place in the semiring $\mathbb{SB}$ defined above.

Given $I, J \subseteq \hat{n}$, we denote by $M[I, J]$ the submatrix of $M$ with entries $m_{ij}$ $(i \in I, j \in J)$. In particular, $M[\hat{n}, j]$ denotes the $j$th column vector of $M$ for each $j \in \hat{n}$.

**Proposition 2.1** [10, Th. 2.10], [11, Lemma 3.2] *The following conditions are equivalent for every $M \in \mathcal{M}_n(\mathbb{SB})$:*

*(i) the column vectors $M[\hat{n}, 1], \ldots, M[\hat{n}, n]$ are independent;*

*(ii)* $\operatorname{Per} M = 1$*;*

*(iii) $M$ can be transformed into some lower triangular matrix of the form*

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ ? & 1 & 0 & \ldots & 0 \\ ? & ? & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \ldots & 1 \end{pmatrix} \tag{3}$$

*by permuting rows and permuting columns independently.*

A square matrix satisfying the above (equivalent) conditions is said to be *nonsingular*.

Given (equipotent) $I, J \subseteq \hat{n}$, we say that $I$ is a *witness* for $J$ in $M$ if $M[I, J]$ is nonsingular.

**Proposition 2.2** [10, Th. 3.11] *The following conditions are equivalent for all $M \in \mathcal{M}_n(\mathbb{SB})$ and $J \subseteq \{1, \ldots, n\}$:*

*(i) the column vectors $M[\hat{n}, j]$ $(j \in J)$ are independent;*

*(ii) $J$ has a witness in $M$.*

The subsets of independent column vectors of a given superboolean matrix, which include the empty subset and are closed for subsets, constitute an important example of a hereditary collection. Hereditary collections which have boolean representations, which include matroids as a very important particular case, were discussed in [11, 12, 13] and will be also the object of a future paper by the present authors, seeking necessary and sufficient conditions.

**Proposition 2.3** [10, Th. 3.11] *The following are equal for a given $M \in \mathcal{M}_n(\mathbb{SB})$:*

*(i) the maximum number of independent column vectors in $M$;*

*(ii) the maximum number of independent row vectors in $M$;*

*(iii) the maximum size of a subset $J \subseteq \hat{n}$ having a witness in $M$;*

*(iv) the maximum size of a nonsingular submatrix of $M$.*

The *rank* of a matrix $M \in \mathcal{M}_n(\mathbb{SB})$, denoted by $\operatorname{rk} M$, is the number described above. A row of $M$ is called an *n-marker* if it has one entry 1 and all the remaining entries are 0. The following remark follows from Proposition 2.1:

**Corollary 2.4** [11, Cor. 3.4] *If $M \in \mathcal{M}_n(\mathbb{SB})$ is nonsingular, then it has an $n$-marker.*

## 3 The c-rank of a graph

In this section, we assume that $G = (V, E)$ denotes a finite graph with $V = \hat{n}$.

The following result prepares the ground for an important connection between matrix rank and the height of the lattice of flats as defined in Subsection 2.2, and will acquire great relevance in the study of independence. This relates to earlier work by Bjorner and Ziegler [1].

**Theorem 3.1** *Given a finite graph $G$, the following conditions are equivalent for every $J \subseteq \hat{n}$:*

*(i) the column vectors $A^c[\hat{n}, j]$ $(j \in J)$ are independent;*

*(ii) $J$ has a witness in $A^c$;*

*(iii) $\operatorname{ht} \widehat{\mathcal{S}_J} = |J|$;*

*(iv) $J$ is a transversal of the partition of successive differences for some chain of $\operatorname{Fl} G$;*

*(v) $J$ is a partial transversal of the partition of successive differences for some maximal chain of $\operatorname{Fl} G$.*

**Proof**. We may assume that $J$ is nonempty.

(i) $\Leftrightarrow$ (ii). By Proposition 2.2.

(ii) $\Rightarrow$ (iii). Let $I$ be a witness for $J$ in $A^c$. Permuting rows and columns if necessary, we may assume that $A^c[I,J]$ is of the form (3). Write $k = |J|$ and let $i_1,\ldots,i_k$ and $j_1,\ldots,j_k$ indicate the new ordering of rows and columns in the reordered matrix. Then the reordered $A[I,J]$ is of the form

$$\begin{pmatrix} 0 & 1 & 1 & \ldots & 1 \\ ? & 0 & 1 & \ldots & 1 \\ ? & ? & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \ldots & 0 \end{pmatrix} \tag{4}$$

and so

$$\mathrm{St}(j_r,\ldots,j_k) \cap I = \{i_1,\ldots,i_{r-1}\}$$

for $r = 1,\ldots,k$. Since $j_k \notin \mathrm{St} j_k$, we get

$$V = \mathrm{St}(\emptyset) \supset \mathrm{St} j_k \supset \mathrm{St}(j_{k-1},j_k) \supset \ldots \supset \mathrm{St}(j_1,\ldots,j_k) \tag{5}$$

and so $\mathrm{ht}\,\widehat{\mathcal{S}_J} \geq k$. Hence $\mathrm{ht}\,\widehat{\mathcal{S}_J} = k$ by (1).

(iii) $\Rightarrow$ (ii). In view of (1), it is easy to see that we must have necessarily a chain of the form (5), where $J = \{j_1,\ldots,j_k\}$ and the $j_r$ are all distinct. For $r = 1,\ldots,k-1$, take $i_r \in \mathrm{St}(j_{r+1},\ldots,j_k)\backslash\mathrm{St}(j_r)$, and also $i_k = j_k$. With the rows (respectively columns) ordered by $i_1,\ldots,i_k$ (respectively $j_1,\ldots,j_k$), the matrix $A^c[I,J]$ is now of the form (3) and so $I$ is a witness for $J$ in $A^c$.

(ii) $\Rightarrow$ (iv). Let $I = \{i_1,\ldots,i_k\}$ be a witness for $J = \{i_1,\ldots,j_k\}$ in $A^c$. Similarly to the proof of (ii) $\Rightarrow$ (iii), we may assume that $A[I,J]$ is of the form (4) and so

$$\mathrm{St}(i_1,\ldots,i_r) \cap J = \{j_{r+1},\ldots,j_k\}$$

for $r = 1,\ldots,k$. Hence

$$V = \mathrm{St}(\emptyset) \supset \mathrm{St}(i_1) \supset \mathrm{St}(i_1,i_2) \supset \ldots \supset \mathrm{St}(i_1,\ldots,i_k) \tag{6}$$

is a chain in $\mathrm{Fl}\,G$. Since $j_r \in \mathrm{St}(i_1,\ldots,i_{r-1}) \setminus \mathrm{St}(i_r)$, then $J$ is a transversal for (6).

(iv) $\Rightarrow$ (ii). Assume that $J = \{i_1,\ldots,j_k\}$ is a transversal for a chain

$$\mathrm{St}(X_0) \supset \mathrm{St}(X_1) \supset \ldots \supset \mathrm{St}(X_k)$$

in $\mathrm{Fl}\,G$. We may assume that $j_r \in \mathrm{St}(X_{r-1}) \setminus \mathrm{St}(X_r)$ for $r = 1,\ldots,k$. Then, for each $r$, there exists $i_r \in X_r$ such that $j_r \notin \mathrm{St}(i_r)$. However, if $s < r$, then $j_r \in \mathrm{St}(X_{r-1}) \subseteq \mathrm{St}(X_s) \subseteq \mathrm{St}(i_s)$ and it follows easily that, with the rows (respectively columns) ordered by $i_1,\ldots,i_k$ (respectively $j_1,\ldots,j_k$), the matrix $A^c[I,J]$ is now of the form (3) and so $I$ is a witness for $J$ in $A^c$.

(iv) $\Rightarrow$ (v). Since a partial transversal of a maximal chain is a transversal for some subchain of the original chain.

(v) $\Rightarrow$ (iv). Since every chain can be refined to get a maximal chain. $\square$

To simplify terminology, we say that the vertices $j_1, \ldots, j_k \in \hat{n}$ are *c-independent* if the column vectors $A^c[\hat{n}, j_1], \ldots, A^c[\hat{n}, j_k]$ are independent.

**Remark 3.2** *Let $G$ be a finite graph and let $j_1, j_2 \in \hat{n}$. Then:*

*(i) $j_1$ is c-independent;*

*(ii) $j_1, j_2$ are c-independent if and only if $\mathrm{St}(j_1) \neq \mathrm{St}(j_2)$.*

*In particular, $j_1, j_2$ are c-independent if they are adjacent.*

**Proof**. (i) This follows from every column vector in $A^c$ being nonzero due to the absence of loops in $G$.

(ii) Since every column vector in $A^c$ is nonzero, it follows from Theorem 3.1 that $j_1, j_2$ are c-independent if and only if $A^c[\hat{n}, j_1]$ and $A^c[\hat{n}, j_2]$ are distinct, i.e. $\mathrm{St}(j_1) \neq \mathrm{St}(j_2)$. □

**Theorem 3.3** *Let $G = (V, E)$ be a finite graph. Then* $\mathrm{rk}\, A^c = \mathrm{ht}\, \mathrm{Fl}\, G$.

**Proof**. Let $k = \mathrm{rk}\, A^c$. Then there exists some $J \subseteq \hat{n}$ such that $|J| = k$ and the column vectors $A^c[\hat{n}, j]$ $(j \in J)$ are independent. Hence $\mathrm{ht}\, \widehat{\mathcal{S}_J} = k$ by Theorem 3.1. Since $\mathrm{ht}\, \widehat{\mathcal{S}_J} \leq \mathrm{ht}\, \widehat{\mathcal{S}_V}$ by (2), it follows that $\mathrm{rk}\, A^c \leq \mathrm{ht}\, \mathrm{Fl}\, G$.

Assume now that $\mathrm{ht}\, \mathrm{Fl}\, G = \ell$. Then there exists a (maximal) chain

$$V = \mathrm{St}(\emptyset) \supset \mathrm{St}(V_\ell) \supset \ldots \supset \mathrm{St}(V_1) \tag{7}$$

for some $V_1, \ldots, V_\ell \subseteq V$. We claim that there exist $j_1, \ldots, j_\ell \in V$ such that

$$\mathrm{St}(j_r, \ldots, j_\ell) = \mathrm{St}(V_r) \tag{8}$$

for $r = 1, \ldots, \ell$. Indeed, since $\mathrm{St}(V_{r+1}) \supset \mathrm{St}(V_r)$, we can take $j_r \in V_r$ such that $\mathrm{St}(V_{r+1}) \not\subseteq \mathrm{St}(j_r)$. Writing $V_{\ell+1} = \emptyset$, we proceed by induction on $r = \ell, \ldots, 1$: assume that (8) holds for $r + 1$. Hence

$$\mathrm{St}(V_{r+1}) = \mathrm{St}(j_{r+1}, \ldots, j_\ell) \supset \mathrm{St}(j_r, \ldots, j_\ell) = \mathrm{St}(j_r) \cap \mathrm{St}(j_{r+1}, \ldots, j_\ell) \supseteq \mathrm{St}(V_r)$$

and so $\mathrm{St}(V_r) = \mathrm{St}(j_r, \ldots, j_\ell)$ by the maximality of the length of the chain (7). Thus (8) holds.

Take $J = \{j_1, \ldots, j_\ell\}$. Since $\mathrm{ht}\, \widehat{\mathcal{S}_J} \leq \mathrm{ht}\, \widehat{\mathcal{S}_V} = \ell$, it follows from (7) and (8) that $\mathrm{ht}\, \widehat{\mathcal{S}_J} = \ell = |J|$ and so the column vectors $A^c[\hat{n}, j]$ $(j \in J)$ are independent by Theorem 3.1. Thus $\mathrm{ht}\, \mathrm{Fl}\, G = \ell \leq \mathrm{rk}\, A^c$ and so $\mathrm{rk}\, A^c = \mathrm{ht}\, \mathrm{Fl}\, G$. □

We say that the above number is the *c-rank* of the graph $G$ and we denote it by $\text{c-rk}\, G$. Note that, in view of Theorem 3.1, $\text{c-rk}\, G$ is also the maximum size of a (partial) transversal of the partition of successive differences of a (maximal) chain of $\mathrm{Fl}\, G$.

We present now some straightforward properties of the c-rank of a graph. Let $\mathrm{maxdeg}\, G$ (respectively $\mathrm{mindeg}\, G$) denote the maximum (respectively minimum) degree of a vertex in $G$.

**Proposition 3.4** *Let $G$ be a finite graph. Then* $\text{c-rk}\, G \leq \mathrm{maxdeg}\, G + 1$.

**Proof**. Since in a chain of the form (7), we have necessarily $|\mathrm{St}(V_\ell)| \leq \mathrm{maxdeg}\, G$. □

**Proposition 3.5** *Let $G$ be a finite graph with connected components $G_1, \ldots, G_m$. Then* $\text{c-rk}\, G = max\{\ \text{c-rk}\, G_1,\ \ldots,\ \text{c-rk}\, G_m\ \}$.

**Proof**. Since in any chain of the form (7), the $V_r$ and the $\text{St}(V_r)$ must necessarily be taken in one same connected component. □

In view of this result, we may focus our attention, from now on, on *connected* graphs.

We say that $G' = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$ (up to isomorphism!). If $V' \subseteq V$ and $E' = E \cap 2^{V'}$, we say that $G'$ is a *restriction* of $G$.

Given graphs $G = (V, E)$ and $G' = (V', E')$, a *morphism* $\varphi : G \to G'$ is a mapping $\varphi : V \to V'$ such that $v\varphi$ — $w\varphi$ is an edge of $G'$ whenever $v$ — $w$ is an edge of $G$. We say that $\varphi$ is a *retraction* if $G'$ is a restriction of $G$ and $\varphi|_{V'}$ is the identity mapping.

**Proposition 3.6** *Let $G = (V, E)$, $G' = (V', E')$ be finite graphs.*

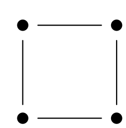
*(i) If $G'$ is a restriction of $G$, then* $\text{c-rk}\, G' \leq \text{c-rk}\, G$.

*(ii) If $G'$ is a complete subgraph of $G$, then* $\text{c-rk}\, G \geq \text{c-rk}\, G' = |V'|$.

**Proof**. (i) If $G'$ is a restriction of $G$, then any (nonsingular) submatrix of $A^c_{G'}$ is also a (nonsingular) submatrix of $A^c_G$.

(ii) A complete subgraph of $G$ is necessarily a restriction, hence $\text{c-rk}\, G \geq \text{c-rk}\, G'$ by part (i). The equality $\text{c-rk}\, G' = |V'|$ follows from the following fact: if $K_n$ denotes the complete graph on $n$ vertices, then $A^c_{K_n}$ is the identity matrix. □

Note that $\text{c-rk}\, G' \leq \text{c-rk}\, G$ may not hold if $G'$ is a mere subgraph of $G$. For instance, it is easy to check that the square



has c-rank 2, but after removing an edge the c-rank increases (cf. Proposition 4.2).

We introduce now a concept that will ease the discussion of c-rank in many circumstances. We call a finite graph $G = (V, E)$ *sober* if the star mapping $\text{St} : V \to 2^v$ is injective. The following remark is immediate from Remark 3.2:

**Remark 3.7** *The following conditions are equivalent for a finite connected graph $G$:*

*(i) $G$ is sober;*

*(ii) all 2-subsets of vertices of $G$ are independent.*

**Proposition 3.8** *Let $G = (V, E)$ be a finite connected graph. Then $G$ admits a retraction onto a sober connected restriction $G' = (V', E')$ such that* $\text{Fl}\, G \cong \text{Fl}\, G'$.

**Proof**. Let $V'$ be a cross-section for the star mapping $\text{St} : V \to 2^V$ of $G$ and let $G'$ be the restriction of $G$ induced by $V'$. It is straightforward that $G'$ is isomorphic to the graph having as vertices the equivalence classes of $V$ induced by St and edges $X$ — $Y$ whenever $x$ — $y$ is an edge of $G$ for some $x \in X$ and $y \in Y$.

For every $v \in V$, let $v' \in V'$ be the unique vertex in $V'$ such that $\mathrm{St}(v') = \mathrm{St}(v)$. We claim that, for all $v, w \in V$,

$$\{v, w\} \in E \Leftrightarrow \{v', w'\} \in E'. \tag{9}$$

Indeed, if $v$ — $w$ is an edge in $G$, then so is $v$ — $w'$ and therefore $v'$ — $w'$.

Conversely, assume that $\{v', w'\} \in E' \subseteq E$. Then we successively get $\{v, w'\} \in E$ and $\{v, w\} \in E$, hence (9) holds. Thus $\varphi : V \to V'$ defined by $v\varphi = v'$ is a graph morphism from $G$ to the restriction $G'$, indeed a retraction.

Moreover, any path $v_1$ — $\ldots$ — $v_n$ in $G$ induces a path $v'_1$ — $\ldots$ — $v'_n$ in $G'$ and so $G'$ is connected.

Let $\mathrm{St}' : V' \to 2^{V'}$ denote the star mapping of $G$. Suppose that $v, w \in V$ are such that $\mathrm{St}'(v') = \mathrm{St}'(w')$. It follows from (9) that

$$\mathrm{St}(v) = \{z \in V \mid z' \in \mathrm{St}'(v')\} = \{z \in V \mid z' \in \mathrm{St}'(w')\} = \mathrm{St}(w),$$

hence $v' = w'$ and so $G'$ is sober.

We claim that

$$\begin{aligned} \theta : (\mathrm{Fl}\, G, \subseteq) &\to (\mathrm{Fl}\, G', \subseteq) \\ \mathrm{St}(W) &\mapsto \mathrm{St}'(W') \end{aligned}$$

is an isomorphism of posets (and therefore of lattices).

It is immediate that $\theta$ is surjective and preserves order. It remains to show that $\theta$ is well defined and injective.

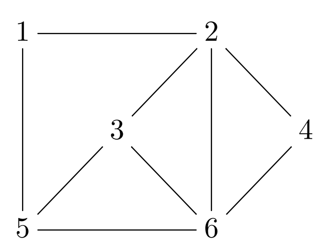
For every $W \subseteq V$, it follows from (9) that

$$\mathrm{St}(W) = \cap_{w \in W} \mathrm{St}(w) = \cap_{w \in W} \{x \in V \mid x' \in \mathrm{St}'(w')\} = \{x \in V \mid x' \in \mathrm{St}'(W')\},$$

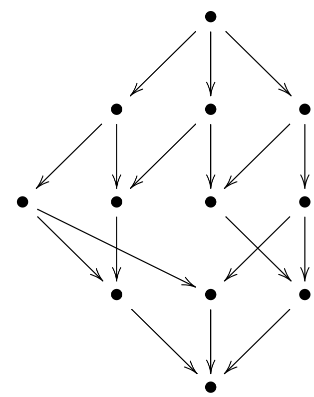
$$\mathrm{St}'(W') = \{x \in V \mid x' \in \cap_{w \in W} \mathrm{St}'(w')\}' = \{x \in V \mid x \in \cap_{w \in W} \mathrm{St}(w)\}' = (\mathrm{St}(W))'.$$

Therefore $\mathrm{St}(W) = \mathrm{St}(Z) \Leftrightarrow \mathrm{St}'(W') = \mathrm{St}'(Z')$ holds for all $W, Z \subseteq V$ and so $\theta$ is an isomorphism. $\square$

However, the restriction in Proposition 3.8 does not need to be unique (up to isomorphism). For instance, the graph




is itself sober and connected (and has mindeg 2), and so it is the restriction obtained by removing vertex 1. It is easy to check that the star lattices of both graphs are isomorphic and of the form:



It is easy to characterize sober trees. Recall that a vertex of degree 1 is called a *leaf*.

**Proposition 3.9** *A tree $T = (V, E)$ is sober if and only if no two leafs are at distance 2 from each other.*

**Proof**. Indeed, assume that $v, w \in V$ are distinct. If $\mathrm{St}(v) = \mathrm{St}(w)$ and has more than one element, then $T$ would admit a square and would not be a tree, hence $\mathrm{St}(v) = \mathrm{St}(w)$ can only occur if both $v$ and $w$ are leafs, in which case $\mathrm{St}(v) = \mathrm{St}(w)$ is equivalent to $d(v, w) = 2$. $\square$

We establish next an inductive relation that may prove useful in the computation of the c-rank. Given a graph $G = (V, E)$, and $J \subseteq V$, write $\text{c-rk}_G J = \mathrm{rk}\, A^c[V, J]$. Note that $\text{c-rk}\, G = \text{c-rk}_G V$.

We recall also that, for $X \subseteq V$, the graph $G - X$ is obtained from $G$ by removing all the vertices in $X$ and all the edges adjacent to them.

**Theorem 3.10** *Let $G = (V, E)$ be a finite graph and $m \geq 2$. Then the following conditions are equivalent:*

*(i)* $\text{c-rk}\, G \geq m$.

*(ii) There exist $v, w \in V$ such that:*

- $\mathrm{St}(v) \neq \mathrm{St}(w)$;
- $\text{c-rk}_{G-\{v,w\}}(\mathrm{St}(v) \cap \mathrm{St}(w)) \geq m - 2$.

**Proof**. (i) $\Rightarrow$ (ii). If $\text{c-rk}\, G \geq m$, then by Proposition 2.2 there exist $I, J \subseteq V$ such that $A^c[I, J]$ is nonsingular and $|J| = m$. In view of Proposition 2.1, we may reorder the rows (respectively columns) of $A^c[I, J]$ by $i_1, \ldots, i_m$ (respectively $j_1, \ldots, j_m$) to get a matrix of the form (3). Since $j_2 \in \mathrm{St}(i_1) \setminus \mathrm{St}(i_2)$, we have $\mathrm{St}(i_1) \neq \mathrm{St}(i_2)$. On the other hand, $i_1, i_2 \notin \{i_3, \ldots, i_m\} \cup \{j_3, \ldots, j_m\}$ since $i_1, i_2 \in \mathrm{St}(j_3, \ldots, j_m)$, hence $\{i_3, \ldots, i_m\}$ is a witness for $\{j_3, \ldots, j_m\}$ in $G \setminus \{i_1, i_2\}$. Therefore $j_3, \ldots, j_m \in \mathrm{St}(i_1) \cap \mathrm{St}(i_2)$ are c-independent in $G \setminus \{i_1, i_2\}$ and so condition (ii) holds.

(ii) $\Rightarrow$ (i). Since $\text{c-rk}_{G-\{v,w\}}(\mathrm{St}(v) \cap \mathrm{St}(w)) \geq m - 2$, there exist distinct $j_3, \ldots, j_m \in \mathrm{St}(v) \cap \mathrm{St}(w)$ and $i_3, \ldots, i_m \in V \setminus \{v, w\}$ such that $A^c[i_3, \ldots, i_m; j_3, \ldots, j_m]$ is nonsingular. Reordering rows and columns if necessary, we may assume that $A^c[i_3, \ldots, i_m; j_3, \ldots, j_m]$ is of the form (3). Since $\mathrm{St}(v) \neq \mathrm{St}(w)$, we may assume that there exists some $j_2 \in \mathrm{St}(v) \setminus \mathrm{St}(w)$ and take $i_1 = j_1 = v$ and $i_2 = w$. It is straightforward to check that $A^c[i_1, \ldots, i_m; j_1, \ldots, j_m]$ is of the form (3), hence condition (i) holds for $J = \{j_1, \ldots, j_m\}$. $\square$

# 4 Low c-rank

We start analyzing the sober cases and go as far as characterizing c-rank 4. In view of Propositions 3.5 and 3.8, in the discussion of c-rank $\geq 3$ we pay special attention to the case of sober connected graphs.

**Proposition 4.1** *Let $G = (V, E)$ be a finite graph. Then:*

*(i)* $\text{c-rk}\, G = 0$ *if and only if* $V = \emptyset$.

*(ii)* $\text{c-rk}\, G = 1$ *if and only if* $V \neq \emptyset$ *and* $E = \emptyset$.

**Proof**. Clearly, $\text{c-rk}\, G \geq 0$ under all circumstances and the empty graph has c-rank 0. On the other hand, if $V \neq \emptyset$, then $A^c$ has at least one 1 in the diagonal, yielding $\text{c-rk}\, G \geq 1$. This proves (i). Moreover, if $E \neq \emptyset$, it follows from Remark 3.2(ii) that $\text{c-rk}\, G \geq 2$, thus (ii) holds as well. $\square$

We recall that a graph $G = (V, E)$ is called *bipartite* if $V$ admits a nontrivial partition $V = V_1 \cup V_2$ such that
$$E \subseteq \{\{v_1, v_2\} \mid v_1 \in V_1, v_2 \in V_2\}.$$
If this inclusion can be made to be an equality, the graph is said to be *complete bipartite*.

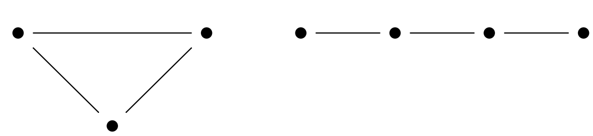
**Proposition 4.2** *Let $G = (V, E)$ be a finite graph with $E \neq \emptyset$. Then the following conditions are equivalent:*

*(i)* $\text{c-rk}\, G = 2$;

*(ii) $G$ has no subgraph $v_1$ — $v_2$ — $v_3$ with* $\mathrm{St}(v_1) \neq \mathrm{St}(v_3)$;

*(iii) $G$ is a disjoint union of complete bipartite graphs;*

*(iv) $G$ has no restrictions of the following forms:*



**Proof**. (i) $\Rightarrow$ (ii). Suppose that $G$ has a subgraph $v_1$ — $v_2$ — $v_3$ with $\mathrm{St}(v_1) \neq \mathrm{St}(v_3)$. We may assume that there exists some $w \in \mathrm{St}(v_1) \setminus \mathrm{St}(v_3)$. Consider the chain

$$V = \mathrm{St}(\emptyset) \supset \mathrm{St}(v_1) \supset \mathrm{St}(v_1, v_3) \supset \mathrm{St}(v_1, v_2, v_3)$$

(taking respectively $v_1, w, v_2$ to show that the inclusions are strict). Thus $\text{c-rk}\, G \geq 3$.

(ii) $\Rightarrow$ (iii). Since (ii) holds, any path

$$v_1 \text{ --- } v_2 \text{ --- } \ldots \text{ --- } v_k$$

must satisfy $\mathrm{St}(v_1) = \mathrm{St}(v_3) = \ldots$ and $\mathrm{St}(v_2) = \mathrm{St}(v_4) = \ldots$ Since we may also assume $G$ to be connected, it follows that we can take a pair of adjacent edges $(w_1, w_2)$ and partition $V = V_1 \cup V_2$ by

$$V_1 = \{v \in V \mid \mathrm{St}(v) = \mathrm{St}(w_1)\}, \quad V_2 = \{v \in V \mid \mathrm{St}(v) = \mathrm{St}(w_2)\}.$$

Since $w_2 \in \mathrm{St}(w_1)$, we have $w_2 \in \mathrm{St}(v)$ for every $v \in V_1$. Hence $V_1 \subseteq \mathrm{St}(w_2)$ and so $V_1 \subseteq \mathrm{St}(v)$ for every $v \in V_2$. On the other hand, if $v, v' \in V_1$ are adjacent, then $v \in \mathrm{St}(v') = \mathrm{St}(v)$, a contradiction. Similarly, no two vertices in $V_2$ can be adjacent. Thus $G$ is complete bipartite.

(iii) $\Rightarrow$ (iv). It is well-known that no bipartite graph admits cycles of odd length. Suppose that $G = (V, E)$ is bipartite complete (with respect to the partition $V = V_1 \cup V_2$) and has a restriction of the form

$$v_1 \text{ --- } v_2 \text{ --- } v_3 \text{ --- } v_4. \tag{10}$$

Then $\{v_1, v_3\}$ and $\{v_2, v_4\}$ belong to the different sides of the partition and so there exists an edge $v_1 \text{ --- } v_4$ in $G$, contradicting (10) being a restriction. Therefore $G$ can have no restriction of the form (10) either.

(iv) $\Rightarrow$ (i). Suppose that $\text{c-rk}\, G \geq 3$. After reordering, $A^c$ has a submatrix of the form

$$\begin{array}{c|ccc} i_1 & 1 & 0 & 0 \\ i_2 & ? & 1 & 0 \\ i_3 & ? & ? & 1 \\ \hline & j_1 & j_2 & j_3 \end{array}$$

and so $A$ has a submatrix of the form

$$\begin{array}{c|ccc} i_1 & 0 & 1 & 1 \\ i_2 & ? & 0 & 1 \\ i_3 & ? & ? & 0 \\ \hline & j_1 & j_2 & j_3 \end{array}$$

Then $j_2 \text{ --- } i_1 \text{ --- } j_3$ is a subgraph with 3 distinct vertices. We may assume that the triangle $K_3$ is not a restriction of $G$. Since $i_2 \text{ --- } j_3$ is an edge, it follows that $i_2 \neq j_2$. Hence

$$j_2 \text{ --- } i_1 \text{ --- } j_3 \text{ --- } i_2 \tag{11}$$

is a subgraph of $G$ with 4 distinct vertices. Since there is no edge $i_2 \text{ --- } j_2$ and $K_3$ is not a restriction of $G$, then (11) is a restriction of $G$ and so (iv) fails as required. $\square$

**Proposition 4.3** *Let $G$ be a finite graph. Then the following conditions are equivalent:*

*(i)* $\text{c-rk}\, G \geq 4$*;*

*(ii) G has a subgraph*

$$\begin{tikzcd} v_1 \arrow[r, dash] \arrow[d, dash] & v_2 \arrow[d, dash] \\ v_4 \arrow[r, dash] & v_3 \end{tikzcd}$$

*with* $\mathrm{St}(v_1) \neq \mathrm{St}(v_3)$ *and* $\mathrm{St}(v_2) \neq \mathrm{St}(v_4)$.

**Proof**. Write $G = (V, E)$.

(i) $\Rightarrow$ (ii). Suppose that $\text{c-rk}\, G \geq 4$. After reordering, $A$ has a submatrix of the form

$$\begin{array}{c|cccc} i_1 & 0 & 1 & 1 & 1 \\ i_2 & ? & 0 & 1 & 1 \\ i_3 & ? & ? & 0 & 1 \\ i_4 & ? & ? & ? & 0 \\ \hline & j_1 & j_2 & j_3 & j_4 \end{array}$$

Then we have edges

$$\begin{tikzcd} i_1 \arrow[r, dash] \arrow[d, dash] & j_3 \arrow[d, dash] \\ j_4 \arrow[r, dash] & i_2 \end{tikzcd}$$

in $G$. Since $i_3 \in \mathrm{St}(j_4)\backslash\mathrm{St}(j_3)$ and $j_2 \in \mathrm{St}(i_1)\backslash\mathrm{St}(i_2)$, the vertices $i_1, i_2, j_3, j_4$ are all distinct and (ii) holds.

(ii) $\Rightarrow$ (i). If (ii) holds, then we may assume out of symmetry that there exist some $w \in \mathrm{St}(v_1) \setminus \mathrm{St}(v_3)$ and $z \in \mathrm{St}(v_2) \setminus \mathrm{St}(v_4)$. Consider the chain

$$V \supset \mathrm{St}(v_1) \supset \mathrm{St}(v_1, v_3) \supset \mathrm{St}(v_1, v_3, z) \supset \emptyset$$

(taking respectively $v_1, w, v_4, v_2$ to show that the inclusions are strict). Thus $\text{c-rk}\, G \geq 4$. $\square$

Now Propositions 4.2 and 4.3 combined provide a characterization of c-rank 3.

**Proposition 4.4** *Let G be a finite graph. Then the following conditions are equivalent:*

*(i)* $\text{c-rk}\, G \geq 5$*;*

*(ii) G has a subgraph*

$$\begin{tikzcd} & v_1 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\ v_2 \arrow[dr, dash] & v_3 \arrow[d, dash] & v_4 \arrow[dl, dash] \\ & v_5 & \end{tikzcd}$$

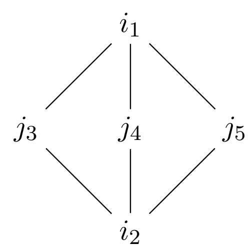
*with* $\mathrm{St}(v_1) \neq \mathrm{St}(v_5)$, $\mathrm{St}(v_2) \neq \mathrm{St}(v_3)$ *and* $\mathrm{St}(v_2) \cap \mathrm{St}(v_3) \not\subseteq \mathrm{St}(v_4)$.

**Proof**. Write $G = (V, E)$.

(i) $\Rightarrow$ (ii). Suppose that $\text{c-rk}\, G \geq 5$. After reordering, $A$ has a submatrix of the form

$$\begin{array}{c|ccccc}
i_1 & 0 & 1 & 1 & 1 & 1 \\
i_2 & ? & 0 & 1 & 1 & 1 \\
i_3 & ? & ? & 0 & 1 & 1 \\
i_4 & ? & ? & ? & 0 & 1 \\
i_5 & ? & ? & ? & ? & 0 \\
\hline
 & j_1 & j_2 & j_3 & j_4 & j_5
\end{array}$$

Then




is a subgraph of $G$ with 5 distinct vertices. Now $j_2 \in \text{St}(i_1) \setminus \text{St}(i_2)$, $i_4 \in \text{St}(j_5) \setminus \text{St}(j_4)$ and $i_3 \in (\text{St}(j_4) \cap \text{St}(j_5)) \setminus \text{St}(j_3)$ and so (ii) holds.

(ii) $\Rightarrow$ (i). If (ii) holds, then we may assume out of symmetry that there exist some $w_1 \in \text{St}(v_1) \setminus \text{St}(v_5)$, $w_2 \in \text{St}(v_2) \setminus \text{St}(v_3)$ and $w_3 \in (\text{St}(v_2) \cap \text{St}(v_3)) \setminus \text{St}(v_4)$. Consider the chain

$$V \supset \text{St}(v_1) \supset \text{St}(v_1, v_5) \supset \text{St}(v_1, v_5, w_3) \supset \text{St}(v_1, v_5, w_3, w_2) \supset \emptyset$$

(taking respectively $v_1, w_1, v_4, v_3, v_2$ to show that the inclusions are strict). Thus $\text{c-rk}\, G \geq 5$. $\square$

Now Propositions 4.3 and 4.4 combined provide a characterization of c-rank 4.

We can now use the previous results to give a complete characterization of sober connected graphs with low c-rank (in view of (iv), see [4]):

**Corollary 4.5** *Let $G = (V, E)$ be a finite sober connected graph. Then:*

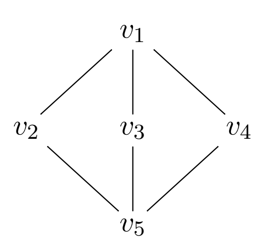
*(i)* $\text{c-rk}\, G = 0$ *if and only if $G$ is the empty graph;*

*(ii)* $\text{c-rk}\, G = 1$ *if and only if $G \cong K_1$;*

*(iii)* $\text{c-rk}\, G = 2$ *if and only if $G \cong K_2$;*

*(iv)* $\text{c-rk}\, G = 3$ *if and only if $|E| \geq 2$ and $G$ has no squares;*

*(v)* $\text{c-rk}\, G = 4$ *if and only if $G$ has a square but no subgraph*



*with* $\mathrm{St}(v_2) \cap \mathrm{St}(v_3) \not\subseteq \mathrm{St}(v_4)$.

*(vi)* $\text{c-rk}\, G \geq 5$ *if and only if* $G$ *has a subgraph of the above form.*

**Proof**. (i) and (ii) follow immediately from Proposition 4.1. Since Since sober connected nontrivial complete bipartite graphs can have only one edge, (iii) follows from Proposition 4.2.

Now part (iii) implies that $\text{c-rk}\, G \geq 3$ if and only if $|E| \geq 2$, and so (iv) follows from Proposition 4.3.

Finally, Propositions 4.3 and 4.4 yield (v) and (vi). □

## 5 The c-independent subsets in c-rank 3

We shall denote by SC$n$ the class of all finite sober connected graphs of c-rank $n$. Throughout this section, all graphs are in SC3. In view of Corollary 4.5(iv), these graphs have no squares (for such graphs with few vertices, see [4]).

The following lemma collects some elementary facts involving this class of graphs. We recall that a graph is called *cubic* if all vertices have degree 3.

**Lemma 5.1** *Let* $G$ *be a finite connected graph.*

*(i) If* $G$ *is cubic and* $\mathrm{gth}\, G \geq 5$, *then* $G \in \mathrm{SC}\,3$.

*(ii) If* $G = (V, E) \in \mathrm{SC}\,3$, *then* $|\mathrm{St}(v) \cap \mathrm{St}(w)| \leq 1$ *holds for all distinct vertices* $v, w$ *of* $G$.

**Proof**. (i) If $G$ is non sober, then $G$ would contain a square, contradicting $\mathrm{gth}\, G \geq 5$. Hence $G$ is sober. The claim now follows from Corollary 4.5.

(ii) Suppose that $|\mathrm{St}(v, w)| > 1$ for distinct vertices $v, w$ of $G = (V, E)$. Since $G$ is sober, we may assume that $\mathrm{St}(v, w) \subset \mathrm{St}(v)$. Let $a, b \in \mathrm{St}(v, w)$ be distinct. Since $G$ is sober, we may assume that there exists some $c \in \mathrm{St}(b) \setminus \mathrm{St}(a)$. Hence

$$\mathrm{St}(v) \supset \mathrm{St}(v, w) \supset \mathrm{St}(v, w, c) \supset \mathrm{St}(v, w, c, a)$$

is a chain in $\mathrm{Fl}\, G$, contradicting $\text{c-rk}\, G = 3$. □

By c-rank, the c-independent subsets of a graph $G = (V, E)$ in SC3 can have at most 3 elements. However, as it will become clear soon enough, the c-independent subsets of $V$ do not constitute a matroid. Our first result associates a matroid to $G$: we define $\mathrm{Mat}\, G$ to contain:

- all the $i$-subsets of $V$ for $i \leq 2$;
- all the 3-subsets $W$ of $V$ such that

$$\forall v \in V \; W \not\subseteq \mathrm{St}(v).$$

Note that the latter condition is equivalent to $\mathrm{St}(W) = \emptyset$.

**Proposition 5.2** *Let* $G \in \mathrm{SC}\,3$. *Then* $\mathrm{Mat}\, G$ *is a matroid.*

**Proof**. Let $I, J \in \mathrm{Mat}\, G$. Without loss of generality, we may assume that $|I| = 3$ and $|J| = 2$. Write $I = \{i_1, i_2, i_3\}$ and $J = \{j_1, j_2\}$. Suppose that $\{j_1, j_2, i_k\} \notin \mathrm{Mat}\, G$ for $k = 1, 2, 3$. Then there exists some $v_k \in \mathrm{St}(j_1, j_2, i_k) \subseteq \mathrm{St}(j_1, j_2)$. By Lemma 5.1(ii), we get $v_1 = v_2 = v_3$ and so $I \subseteq \mathrm{St}(v_1)$, contradicting $I \in \mathrm{Mat}\, G$. Therefore $\{j_1, j_2, i_k\} \in \mathrm{Mat}\, G$ for some $k \in \hat{3}$ and so $\mathrm{Mat}\, G$ is a matroid. □

We identify next the c-independent subsets of vertices for graphs in SC3. We say that a 3-subset $P \subseteq V$ is a *potential line* if $|P \cap \mathrm{St}(v)| \leq 1$ for every $v \in V$.

**Theorem 5.3** *Let $G = (V, E)$ be a graph in* SC 3 *and let $W \subseteq V$. Then the following conditions are equivalent:*

*(i) $W$ is c-independent;*

*(ii) $|W| \leq 2$ or*

$$|W| = 3,\ \mathrm{St}(W) = \emptyset \textit{ and } W \textit{ is not a potential line.}$$

**Proof**. Since $G$ is sober, and by Remark 3.2, $W$ is c-independent if $|W| \leq 2$. On the other hand, since $\text{c-rk}\, G = 3$, then $V$ has no c-independent 4-subsets. Therefore we may assume that $|W| = 3$. Write $W = \{w_1, w_2, w_3\}$.

Assume that $W$ is independent. By Theorem 3.1, we may assume that

$$V \supset \mathrm{St}(w_1) \supset \mathrm{St}(w_1, w_2) \supset \mathrm{St}(w_1, w_2, w_3) \tag{12}$$

is a chain in $\mathrm{Fl}\, G$. Since $\mathrm{St}(w_1, w_2, w_3) \neq \emptyset$ would allow us to adjoin the empty set to the chain and contradict c-rank 3, then $\mathrm{St}(W) = \emptyset$. On the other hand, for $v \in \mathrm{St}(w_1, w_2)$, we get $|W \cap \mathrm{St}(v)| \geq 2$ and so $W$ is not a potential line either.

Conversely, assume that $\mathrm{St}(W) = \emptyset$ and $W$ is not a potential line. Then $|W \cap \mathrm{St}(v)| \geq 2$ for some in $v \in V$. We may assume that $w_1, w_2 \in \mathrm{St}(v)$. Furthermore, since $G$ is sober, we may also assume that $\mathrm{St}(w_1) \supset \mathrm{St}(w_1, w_2)$. (12) is a chain in $\mathrm{Fl}\, G$ and so $W$ is independent by Theorem 3.1. □

Now Proposition 5.2 and Theorem 5.3 yield:

**Corollary 5.4** *Let $G = (V, E)$ be a graph in* SC 3*. If $G$ has no potential lines, then the set of all c-independent subsets of $V$ constitutes a matroid.*

In view of this result, it is only natural to enquire which graphs in the above class have no potential lines. It turns out that diameter makes the difference:

**Proposition 5.5** *Let $G$ be a graph in* SC 3.

*(i) If* $\mathrm{diam}\, G < 3$*, then $G$ has no potential lines.*

*(ii) If* $\mathrm{diam}\, G > 5$*, then $G$ has potential lines.*

*(iii) If* $\mathrm{diam}\, G \in \{3, 4, 5\}$*, then both cases may occur.*

**Proof**. (i) First, we note that if $P \subseteq V$ is a potential line and $p, q \in P$ are distinct, then $d(p, q) \neq 2$ (if $p$ — $v$ — $q$ is a path in $G$, then $|P \cap \mathrm{St}(v)| \geq 2$), and if $d(p, q) = 1$, then the edge $p$ — $q$ can lie in no triangle. Hence, if $\mathrm{diam}\, G < 3$ and $P = \{p_1, p_2, p_3\}$ is a potential line, then $d(p_1,p_2) = d(p_1,p_3) = d(p_2,p_3) = 1$ immediately gets us into a contradiction. Thus (i) holds.

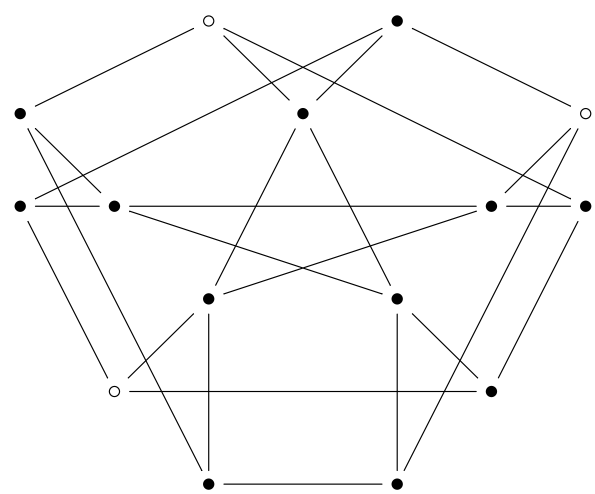
(ii) Assume that $\operatorname{diam} G > 5$. Then $G$ has a *geodesic* (path of minimum length connecting the extreme vertices) of length 6, say

$$v_0 — v_1 — v_2 — v_3 — v_4 — v_5 — v_6$$

Since $d(v_0,v_3) = d(v_3,v_6) = 3$, it follows that $\{v_0,v_3,v_6\}$ is a potential line of $G$.

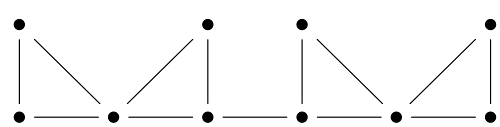
(iii) It is enough to show that there exist in SC3 a graph $G_3$ with diameter 3 and potential lines, and a graph $G_5$ with diameter 5 and no potential lines.

We can take $G_3$ to be the cubic graph

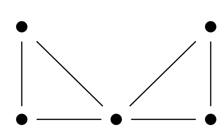


Since $\operatorname{gth} G = 5$, it follows from Lemma 5.1(ii) that $G \in$ SC3. Straightforward checking shows that $\operatorname{diam} G = 3$ and $G$ has potential lines such as the one defined by the hollow circles.

On the other hand, we can take $G_5$ to be the graph

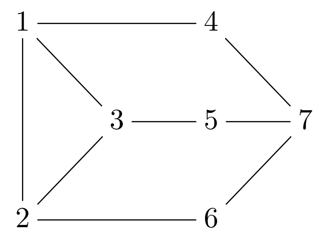


It follows easily from Corollary 4.5(iv) that $G \in$ SC3, and it is immediate that $\operatorname{diam} G = 5$. Suppose that $G$ has a potential line $P$. Then at least two points of $P$ would have to fit into a subgraph of the form

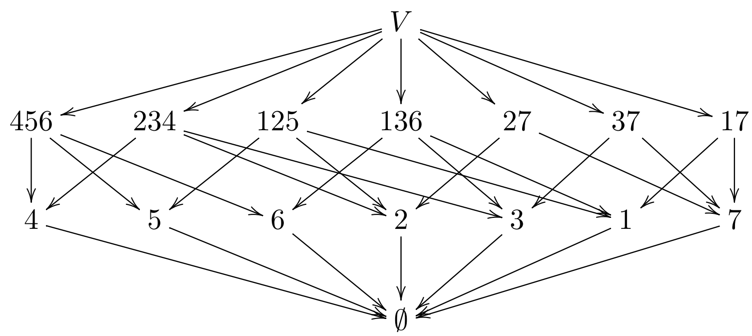


leading at once to a contradiction. Therefore $G$ has no potential lines as claimed. □

**Example 5.6** *Let $G = (V, E)$ be the graph*




(see [4]). By Corollary 4.5(iv), we have $G \in \mathrm{SC}3$. The lattice of flats of $G$ can be depicted as




It is straightforward to check that $G$ has no potential lines and $\mathrm{Mat}\, G$ contains all the $i$-subsets of $V$ for $i \leq 3$ except the flats 125, 136, 234 and 456. In view of Theorem 5.3, these are precisely the c-independent subsets of $V$. See further remarks after Corollary 6.6 relating to the Fano plane.

If we restrict our attention to cubic graphs, the range is a bit reduced. A list of all cubic graphs up to 12 vertices can be found in [27], where the handy LCF notation is explained and used.

**Corollary 5.7** *Let $G$ be a cubic graph in* $\mathrm{SC}\,3$.

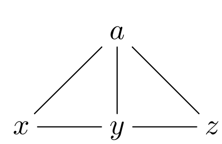
*(i) If* $\mathrm{diam} < 3$*, then $G$ has no potential lines.*

*(ii) If* $\mathrm{diam}\, G > 3$*, then $G$ has potential lines.*

*(iii) If* $\mathrm{diam}\, G = 3$*, then both cases may occur.*

**Proof**. (i) By Proposition 5.5(i).

(ii) Suppose now that $\mathrm{diam}\, G > 3$. Let $a, b \in V$ be such that $d(a, b) = 4$, and write $\mathrm{St}(a) = \{x, y, z\}$. Clearly, $b$ is at distance $\geq 3$ from $x$, $y$ or $z$. To prevent $\{a, b, x\}$ from being a potential line, $a$ — $x$ must lie in some triangle. If we try to avoid other potential lines, also $a$ — $y$ and $a$ — $z$ must lie in triangles. Now it is easy to see that at least two of the vertices $x, y, z$ must be connected through edges. Without loss of generality, we may assume that



is a subgraph of $G$. But then we have a square in a sober graph, contradicting c-rank 3 in view of Corollary 4.5(iv). Thus (ii) holds.

(iii) The example $G_3$ in the proof of Proposition 5.5(iii) is cubic, belongs to SC3, has diameter 3 and has potential lines.

However, the Heawood graph [24] is cubic, bipartite, has diameter 3 and girth 6 (and so is in SC3, see Proposition 7.1(iii) in next section). Suppose that $P = \{a, b, c\}$ is a potential line of the Heawood graph. Then the distance between any two distinct vertices in $P$ cannot be 2, and so must be 1 or 3 in view of the diameter being 3. Thus we obtain a cycle of odd length in the graph, contradicting the fact of being bipartite. Therefore the Heawood graph has no potential lines. □

# 6 The Levi graph and partial euclidean geometries

Given a finite graph $G = (V, E)$ we can consider $V$ as "points" and $E$ as "lines", where $v$ is on $e$ ($v \in e$) if and only if $e$ is incident to $v$, and so $(V, E)$ gives *some sort of geometry* (see [3, 5]). So the Levi viewpoint for "lines" in a graph is different from our view of taking $\mathrm{St}(v)$ as lines. I this section, we benefit from this other approach and introduce right away the concept of partial euclidean geometry.

Let $P$ be a finite nonempty set and let $\mathcal{L}$ be a nonempty subset of $2^P$. We shall always assume that $P \cap 2^P = \emptyset$. We say that $(P, \mathcal{L})$ is a *partial euclidean geometry* (abbreviated to PEG) if the following axioms are satisfied:

(G1) $P \subseteq \cup\mathcal{L}$;

(G2) if $L, L' \in \mathcal{L}$ are distinct, then $|L \cap L'| \leq 1$;

(G3) $|L| \geq 2$ for every $L \in \mathcal{L}$.

The elements of $P$ are called *points* and the elements of $\mathcal{L}$ are called *lines*. Given $p \in P$, we denote by $\mathcal{L}(p)$ the set of all lines containing $p$.

The concept of PEG is an abstract combinatorial generalization of the following geometric situation:

Consider a finite set of lines $\mathcal{L}$ in the euclidean space $\mathbb{R}^n$. Consider also a finite subset $P$ of $\cup\mathcal{L} \subset \mathbb{R}^n$ such that:

- if $L, L' \in \mathcal{L}$ are distinct, then $|L \cap L'| \leq 1$;
- if $L, L' \in \mathcal{L}$ and $L \cap L' = \{p\}$, then $p \in P$;
- $|L \cap P| \geq 2$ for every $L \in \mathcal{L}$.

Representing each $L \in \mathcal{L}$ by $L \cap P$, it follows that $(\mathcal{L}, P)$ constitutes a PEG. It is well known that not all PEG's can be represented over an euclidean space (nor any field) (see [9, Section 2.6]).

Using Coxeter's notation (see [5]), we say that the PEG $(P, \mathcal{L})$ is an $(m_c, n_d)$ *configuration* if:

- there are $m$ points and $n$ lines;

- each point belongs to $c$ lines;
- each line contains $d$ points.

Hence $cm = dn$, which equals the number of 1's in the (boolean) *incidence matrix* of $(P, \mathcal{L})$, where rows are labelled by points and columns by lines.

An important example is provided by the famous *Desargues configuration*. A simple way of defining it is by taking points as 2-subsets of $\hat{5}$ and lines as 3-subsets of $\hat{5}$ (identifying $\{a,b,c\}$ with $\{\{a,b\},\{a,c\},\{b,c\}\}$). For a geometric representation, see e.g. [22]. It is clear that the Desargues configuration is a $(10_3, 10_3)$ configuration. It has many interesting properties, such as being self-dual (by exchanging points and lines, we get an isomorphic configuration), and the automorphism group acts transitively on both vertices and edges. And it is of course related to the famous Desargues' Theorem [22]. Notice that, for every point $p$, there are exactly 3 points noncolinear with $p$ (i.e., not belonging to some line simultaneously with $p$), and that these 3 points constitute a line!

Now, for every $G = (V, E) \in \mathrm{SC3}$, let

$$\mathcal{L}_G = \{W \in \mathrm{Fl}\, G \setminus \{V\} : |W| \geq 2\}$$

and let $\mathrm{Geo}\, G = (V, \mathcal{L}_G)$.

**Proposition 6.1** *If* $G \in \mathrm{SC3}$*, then* $\mathrm{Geo}\, G$ *is a PEG.*

**Proof**. Let $v \in V$. Since $\text{c-rk}\, G = 3$, then $\mathrm{St}(v) \neq \emptyset$. If all the elements of $\mathrm{St}(v)$ have degree 1, then $G$ sober implies that $v$ has also degree 1 and so $G \cong K_2$, contradicting Corollary 4.5(iv). Hence there exists some $w \in \mathrm{St}(v)$ with degree $\geq 2$ and so $v \in \mathrm{St}(w) \in L_G$. Thus $\mathrm{Geo}\, G$ satisfies axiom (G1) (and also $\mathcal{L}_G \neq \emptyset$).

Finally, (G2) follows from Lemma 5.1(ii) and (G3) holds trivially. Therefore $\mathrm{Geo}\, G$ is a PEG. $\square$

Note that $\mathrm{Mat}\, G$ consists of all subsets of $V$ with at most 2 elements plus all 3-subsets which are contained in no line of $\mathrm{Geo}\, G$.

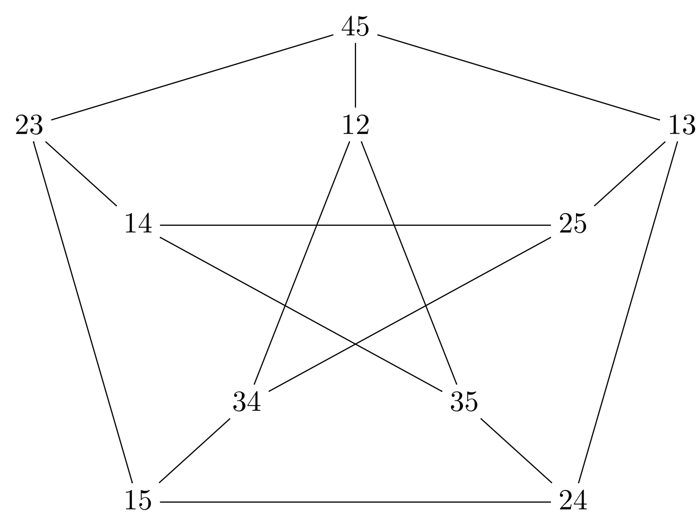
**Corollary 6.2** *If* $G \in \mathrm{SC3}$ *is cubic with* $n$ *vertices, then* $\mathrm{Geo}\, G$ *is an* $(n_3, n_3)$ *configuration.*

**Proof**. Indeed, in this case the lines are of the form $\mathrm{St}(v)$, for any $v \in V$. $\square$

**Example 6.3** *If* $G$ *is the Petersen graph, then* $\mathrm{Geo}\, G$ *is the Desargues configuration.*

Indeed, let $G = (V, E)$ denote the Petersen graph, where the vertices are described as the

2-subsets (written in the form $ij$) of $\hat{5}$ and $ij$ — $kl$ is an edge if and only if $\{i,j\} \cap \{k,l\} = \emptyset$:




Since the graph has girth 5, it follows easily that $\mathrm{Geo}\, G = (V, \mathcal{L})$ for $\mathcal{L} = \{\mathrm{St}(v) \mid v \in V\}$, which coincides precisely with our previous description of the Desargues configuration.

We say that a PEG $\mathcal{G} = (P, \mathcal{L})$ is *connected* if there is no nontrivial partition $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ such that $(\cup\mathcal{L}_1) \cap (\cup\mathcal{L}_2) = \emptyset$. Note that this is equivalent to the usual geometric concept of connectedness if our PEG has an euclidean geometric realization through real lines and real points.

**Proposition 6.4** *Let $G$ be a graph in* SC3 *with* $\mathrm{mindeg}\, \mathcal{G} \geq 2$*. Then the following conditions are equivalent:*

*(i)* $\mathrm{Geo}\, G$ *is connected;*

*(ii)* $G$ *is not bipartite.*

**Proof**. By definition, $\mathrm{Geo}\, G$ is disconnected if and only if there exists a nontrivial partition $\mathcal{L}_G = \mathcal{L}_1 \cup \mathcal{L}_2$ such that $(\cup\mathcal{L}_1) \cap (\cup\mathcal{L}_2) = \emptyset$. In view of Proposition 6.1 and (G1), this supposes a nontrivial partition $V = V_1 \cup V_2$ with $\cup\mathcal{L}_1 = V_1$ and $\cup\mathcal{L}_2 = V_2$.

If $G$ is bipartite with respect to a partition $V = V_1 \cup V_2$, then we take

$$\mathcal{L}_1 = \{\mathrm{St}(v) \mid v \in V_2\}, \quad \mathcal{L}_2 = \{\mathrm{St}(v) \mid v \in V_1\}.$$

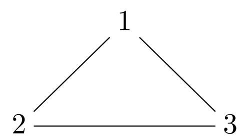
Since $\mathrm{mindeg}\, G \geq 2$, and by Proposition 6.1, this shows that $\mathrm{Geo}\, G$ is disconnected.

Conversely, assume that $\mathrm{Geo}\, G$ is disconnected. Hence there exists a nontrivial partition $V = V_1 \cup V_2$ with $\cup\mathcal{L}_1 = V_1$ and $\cup\mathcal{L}_2 = V_2$. Suppose that $\mathrm{St}(v) \subseteq V_1$ for some $v \in V_1$. Since $G$ is connected, it follows easily from an induction argument that $\mathrm{St}(w) \subseteq V_1$ for any $w \in V_1$, contradicting $V_2 \neq \emptyset$. Hence $\mathrm{St}(v) \subseteq V_2$ for every $v \in V_1$. By symmetry, we also have $\mathrm{St}(v) \subseteq V_1$ for every $v \in V_2$. Therefore $G$ is bipartite. $\square$

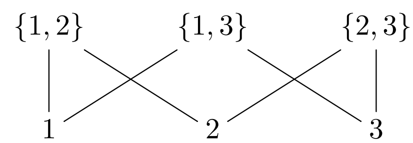
As we mentioned in the beginning of the section, we can view graphs as a particular case of PEG's, when we assume lines to have exactly two points. Note that the concept of connectedness for PEG's coincides with the usual concept of connectedness for graphs when we view graphs as PEG's.

Given a PEG $\mathcal{G} = (P, \mathcal{L})$, we define the *Levi graph* of $\mathcal{G}$ [5] by $\mathrm{Levi}\,\mathcal{G} = (P \cup \mathcal{L}, E)$, where $E$ consists of the edges of the form $p —— L$, for all $L \in \mathcal{L}$ and $p \in L$.

Viewing $K_3$ as a PEG, we have




$K_3$




$\mathrm{Levi}\,K_3$

If $G$ is a graph, its Levi graph is in fact a *subdivision* of $G$. A simple way of picturing it is by introducing a new vertex at the midpoint of every edge (breaking thus the original edge into two). Obviously, the new vertices represent the edges where they originated.

Among configurations, famous examples include the *Desargues graph* [21] as the Levi graph of the Desargues configuration and the *Heawood graph* [24] as the Levi graph of the Fano plane [23].

The following results collects some elementary properties of the Levi graph of a PEG (configuration) (see [3, 5]). Proofs are immediate.

**Proposition 6.5** *Let $\mathcal{G} = (P, \mathcal{L})$ be a PEG. Then:*

*(i)* $\mathrm{Levi}\,\mathcal{G}$ *is bipartite with respect to the partition $P \cup \mathcal{L}$;*

*(ii) the degree of $p \in P$ in* $\mathrm{Levi}\,\mathcal{G}$ *is the number of lines containing $p$;*

*(iii) the degree of $L \in \mathcal{L}$ in* $\mathrm{Levi}\,\mathcal{G}$ *is $|L|$;*

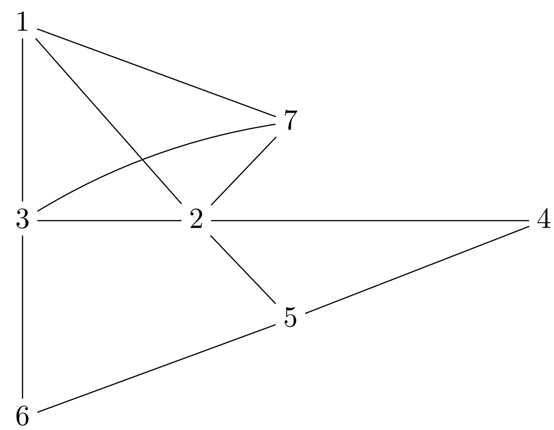
*(iv)* $\mathrm{Levi}\,\mathcal{G}$ *has $|P| + |\mathcal{L}|$ vertices and* $\dfrac{\sum_{p \in P} |\mathcal{L}(p)| \;+\; \sum_{L \in \mathcal{L}} |L|}{2}$ *edges.*

We define $\mathrm{mindeg}\,\mathcal{G}$ to be $\mathrm{mindeg}\,\mathrm{Levi}\,G$.

**Corollary 6.6** *Let $\mathcal{G} = (P, \mathcal{L})$ be an $(m_c, n_d)$ configuration. Then* $\mathrm{Levi}\,\mathcal{G}$ *has $m+n$ vertices and $cm = dn$ edges.*

In particular, the Levi graph of the Desargues configuration, which is a $(10_3, 10_3)$ configuration, has 20 vertices and 30 edges.

Going back to the graph in Example 5.6, it is easy to check that $\mathrm{Geo}\,G$ has $V = \hat{7}$ as set of points and lines $\mathrm{St}(v)$ for $v \in V$. The following picture shows that $\mathrm{Geo}\,G$ is somehow part of the Fano plane [23]:




Moreover, Levi Geo $\mathcal{G}$ can be obtained as follows: we make the Hasse diagram of $\mathrm{Fl}\, G$ into a graph (the *Hasse graph* of $\mathrm{Fl}\, G$) by taking as vertices all flats, and letting $x \to y$ be an edge whenever $x$ covers $y$ in $\mathrm{Fl}\, G$ or vice-versa; removing the vertices $V$ and $\emptyset$, we get the *restricted Hasse graph* of $\mathrm{Fl}\, G$, which is then isomorphic to Levi Geo $\mathcal{G}$. This is just a particular case of Proposition 6.12.

We discuss next girth and connectedness.

**Proposition 6.7** *Let $\mathcal{G} = (P, \mathcal{L})$ be a PEG. Then*

*(i)* $\mathrm{gth}\, \mathrm{Levi}\, \mathcal{G} \geq 6$ *and is not odd;*

*(ii)* $\mathrm{Levi}\, \mathcal{G}$ *is connected if and only if* $\mathcal{G}$ *is connected.*

**Proof**. (i) Since Levi $\mathcal{G}$ is bipartite by Proposition 6.5(i), it has no cycles of odd length. Therefore it is enough to exclude existence of squares in Levi $\mathcal{G}$. Suppose that

$$\begin{tikzcd}[arrows=dash] p \arrow[r] \arrow[d] & L \arrow[d] \\ L' \arrow[r] & p' \end{tikzcd}$$

is a square in Levi $\mathcal{G}$. Then $|L \cap L'| \geq 2$, contradicting (G2). Therefore $\mathrm{gth}\, \mathrm{Levi}\, \mathcal{G} \geq 6$.

(ii) Suppose that $\mathcal{G}$ is not connected. Then there is a nontrivial partition $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ such that $(\cup \mathcal{L}_1) \cap (\cup \mathcal{L}_2) = \emptyset$. Suppose that $L \text{ --- } p \text{ --- } L'$ is a path in Levi $\mathcal{G}$. Since $p \in L \cap L'$ and $(\cup \mathcal{L}_1) \cap (\cup \mathcal{L}_2) = \emptyset$, then $L$ and $L'$ must belong to the same side of the partition. Hence the connected component of a line in Levi $\mathcal{G}$ does not contain the lines in the other side of the partition, and so Levi $\mathcal{G}$ is not connected.

Conversely, suppose that Levi $\mathcal{G}$ is not connected. Let $\mathcal{L}_1$ be the set of all lines in a fixed connected component of Levi $\mathcal{G}$ and let $\mathcal{L}_2 = \mathcal{L} \setminus \mathcal{L}_1$. Suppose that $p \in (\cup \mathcal{L}_1) \cap (\cup \mathcal{L}_2)$. Then there exist $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_2$ such that $p \in L_1 \cap L_2$. Hence we have a path $L_1 \text{ --- } p \text{ --- } L_2$ in Levi $\mathcal{G}$ and so $L_1$ and $L_2$ belong to the same connected component, a contradiction. Thus $(\cup \mathcal{L}_1) \cap (\cup \mathcal{L}_2) = \emptyset$ and so $\mathcal{G}$ is not connected. $\square$

Note that, if $G$ is a graph, the cycles of $\mathrm{Levi}\, G$ are of the form

$$v_0 — \{v_0, v_1\} — v_1 — \{v_1, v_2\} — \ldots — \{v_n, v_0\} — v_0,$$

whenever

$$v_0 — v_1 — \ldots — v_n — v_0$$

is a cycle in $G$. Thus

$$\mathrm{gth}\,\mathrm{Levi}\, G = 2\mathrm{gth}\, G.$$

**Proposition 6.8** *The following conditions are equivalent for a PEG $\mathcal{G} = (P, \mathcal{L})$:*

*(i)* $\mathrm{Levi}\,\mathcal{G}$ *is sober;*

*(ii) the mapping $P \to 2^{\mathcal{L}} : p \mapsto \mathcal{L}(p)$ is one-to-one;*

*(iii) for all distinct points $p, p' \in P$, there exists some line $L \in \mathcal{L}$ containing just one of them.*

**Proof**. We start by computing the stars of $\mathrm{Levi}\,\mathcal{G}$. For $p \in P$ and $L \in \mathcal{L}$, we have $\mathrm{St}(p) = \mathcal{L}(p)$ and $\mathrm{St}(L) = L$ (recall that $L$ is a set of points!). By axioms (G1) and (G3), we have repectively $\mathrm{St}(p) \neq \emptyset$ and $\mathrm{St}(L) \neq \emptyset$. Since $P \cap \mathcal{L} = \emptyset$, we must have always $\mathrm{St}(p) \neq \mathrm{St}(L)$. On the other hand, the restriction $\mathrm{St}|_{\mathcal{L}}$ is always one-to-one, hence $\mathrm{Levi}\,\mathcal{G}$ is sober if and only if $\mathrm{St}|_{\mathcal{P}}$ is one-to-one, which is equivalent to (ii). The equivalence of (ii) and (iii) is trivial. $\square$

If $G$ is a graph, the above conditions are equivalent to saying that no union of connected components of $G$ has exactly two vertices.

We call a PEG satisfying the conditions of Proposition 6.8 *sober*. In view of axiom (G2), we immediately obtain:

**Corollary 6.9** *If $\mathcal{G}$ is a PEG and* $\mathrm{mindeg}\,\mathcal{G} \geq 2$*, then $\mathcal{G}$ is sober. In particular, if $\mathcal{G}$ is an $(m_c, n_d)$ configuration with $c \geq 2$, then $\mathcal{G}$ is sober.*

This provides us with infinitely many examples of graphs in SC3 with girth $\geq 6$:

**Corollary 6.10** *Let $\mathcal{G}$ be a PEG.*

*(i) If $\mathcal{G}$ is sober and connected, then* $\mathrm{Levi}\,\mathcal{G} \in \mathrm{SC3}$.

*(ii) If* $\mathrm{mindeg}\,\mathcal{G} \geq 2$*, then* $\mathrm{mindeg}\,\mathrm{Levi}\,\mathcal{G} \geq 2$.

**Proof**. (i) Since $\mathcal{G}$ is sober, so is $\mathrm{Levi}\,\mathcal{G}$. By Proposition 6.7, $\mathrm{Levi}\,\mathcal{G}$ is connected and has girth $\geq 6$. Thus $\mathrm{Levi}\,\mathcal{G}$ has c-rank 3 by Corollary 4.5(iv).

(ii) By Proposition 6.5, in view of $\mathrm{mindeg}\,\mathcal{G} \geq 2$ and (G3). $\square$

Note that, given a non bipartite cubic graph $C$ in SC3 with $n$ vertices (so $n \geq 10$), it follows from Proposition 6.4 and Corollaries 6.2 and 6.9 that $\mathrm{Geo}\, C$ is a sober connected $(n_3, n_3)$ configuration. Hence, by Proposition 6.5 and Corollary 6.10, $\mathrm{Levi}\,\mathrm{Geo}\, C$ is now a bipartite cubic graph in SC3, so one can generate cubics this way. This does not iterate because $\mathrm{Geo}\,\mathrm{Levi}\,\mathrm{Geo}\, C$ does not stay connected.

Given a graph $G = (V, E)$, we say that the vertex $v \in V$ is *closed* if $\{v\} = \mathrm{St}(W)$ for some $W \subseteq V$, i.e. $\{v\} \in \mathrm{Fl}\, G$. Note that this is also equivalent to the equality $\{v\} = \mathrm{St}(\mathrm{St}(v))$, since $\mathrm{St}(v)$ is clearly the greatest subset $W$ of $V$ such that $v \in \mathrm{St}(W)$. We say that $G$ is *closed* if all its vertices are closed.

By taking $G$ to be the graph $1 — 2 — 3 — 4$, and omitting brackets/commas in the representation of the flats, we can see that $\mathrm{Fl}\, G = \{1234, 13, 24, 2, 3, \emptyset\}$ and so 2 and 3 are closed while 1 and 4 are not.

We can now prove the following (see [4] in view of (ii)):

**Lemma 6.11** *Let $G = (V, E)$ be a finite graph satisfying one of the following two conditions:*

*(i) $G$ is sober and cubic;*

*(ii)* $\mathrm{mindeg}\, G \geq 2$ *and $G$ has no squares.*

*Then $G$ is closed.*

**Proof**. Let $v \in V$. Clearly, $v \in \mathrm{St}(\mathrm{St}(v))$. Suppose that $v \neq w \in \mathrm{St}(\mathrm{St}(v))$. Then $\mathrm{St}(v) \subseteq \mathrm{St}(w)$.

If $G$ is cubic, this implies $\mathrm{St}(v) = \mathrm{St}(w)$ and $G$ would not be sober. Therefore (i) implies $\{v\} = \mathrm{St}(\mathrm{St}(v))$.

On the other hand, if (ii) holds, then by taking distinct $a, b \in \mathrm{St}(v)$ we would get a square

$$\begin{array}{ccc} v & — & a \\ | & & | \\ b & — & w \end{array}$$

a contradiction. Therefore we also get $\{v\} = \mathrm{St}(\mathrm{St}(v))$ in this case. $\square$

We can now prove the following result:

**Proposition 6.12** *Let $G \in$* SC3 *have* $\mathrm{mindeg}\, G \geq 2$. *Then* $\mathrm{Levi}\,\mathrm{Geo}\, G$ *is isomorphic to the restricted Hasse graph of* $\mathrm{Fl}\, G$.

**Proof**. Write $G = (V, E)$. By Proposition 6.1, we have $\mathrm{Geo}\, G = (V, \{\mathrm{St}(v) \mid v \in V\})$ and so the vertex set of $\mathrm{Levi}\,\mathrm{Geo}\,\mathcal{G}$ is $V \cup \{\mathrm{St}(v) \mid v \in V\}$. On the other hand, by Lemmas 5.1(ii) and 6.11(ii), the restricted Hasse graph $G'$ of $\mathrm{Fl}\, G$ has

$$\{\{v\} \mid v \in V\} \cup \{\mathrm{St}(v) \mid v \in V\}$$

as vertex set, yielding an obvious bijection to the vertex set of $\mathrm{Levi}\,\mathrm{Geo}\,\mathcal{G}$.

Now the edges of $\mathrm{Levi}\,\mathrm{Geo}\,\mathcal{G}$ are of the form $w — \mathrm{St}(v)$ whenever $w \in \mathrm{St}(v)$ $(v \in V)$, and this is equivalent to say that $\mathrm{St}(v)$ covers $\{w\}$ in $\mathrm{Fl}\, G$. Therefore the two graphs are isomorphic. $\square$

We proceed now to analyse the lattice of flats of the Levi graph of a connected PEG with mindeg $\geq 2$.

**Theorem 6.13** *Let $\mathcal{G} = (P, \mathcal{L})$ be a PEG and let* $\mathrm{Levi}\,\mathcal{G} = (P \cup \mathcal{L}, E)$. *If $\mathcal{G}$ is connected and* $\mathrm{mindeg}\,\mathcal{G} \geq 2$, *then:*

*(i)* $\mathrm{Levi}\,\mathcal{G}$ *is closed;*

*(ii)* $\mathrm{Flats}\,\mathrm{Levi}\,\mathcal{G} = \{P \cup \mathcal{L}, \emptyset\} \cup \{\{x\} \mid x \in P \cup \mathcal{L}\} \cup \{L \mid L \in \mathcal{L}\} \cup \{\mathcal{L}_p \mid p \in P\}$*;*

*(iii)* $\mathrm{Flats}\,\mathrm{Levi}\,\mathcal{G}$ *satisfies the Jordan-Dedekind condition.*

**Proof**. (i) By Lemma 6.11 and Proposition 6.7(i).

(ii) Given $p \in P$ and $L \in \mathcal{L}$, we have $\mathrm{St}(p) = \mathcal{L}(p)$ and $\mathrm{St}(L) = L$. Moreover, $\mathrm{St}(p, L) = \emptyset$. Now, given $p' \in P \setminus \{p\}$, we have $\mathrm{St}(p, p') = \{L\}$ if $p, p' \in L \in \mathcal{L}$ (note that $L$ is then unique by (G3)), otherwise empty. Finally, if $L' \in \mathcal{L} \setminus \{L\}$, we have in view of (G2) $\mathrm{St}(L, L') = \{p\}$ if $L \cap L' = \{p\}$, otherwise empty. Note that we get all $\{L\}$ by (G3) and (G2), and we get all $\{p\}$ by (G1) and (G2). This proves (ii).

(iii) Since $\mathrm{mindeg}\,\mathcal{G} \geq 2$, it follows easily from parts (i) and (ii) that the maximal chains of $\mathrm{Fl}\,\mathrm{Levi}\,G$ are all of the form

$$\emptyset \subset \{p\} \subset L \subset P \cup \mathcal{L}$$

or

$$\emptyset \subset \{L\} \subset \mathcal{L}(p) \subset P \cup \mathcal{L}$$

for some $p \in L \in \mathcal{L}$. Therefore all maximal chains have length 3. $\square$

We can now compute the c-independent subsets of $\mathrm{Levi}\,\mathcal{G}$ for this same class of PEG's:

**Corollary 6.14** *Let* $\mathcal{G} = (P, \mathcal{L})$ *be a PEG and let* $\mathrm{Levi}\,\mathcal{G} = (P \cup \mathcal{L}, E)$*. If* $\mathcal{G}$ *is sober connected and* $\mathrm{mindeg}\,\mathcal{G} \geq 2$*, then* $W \subseteq P \cup \mathcal{L}$ *is c-independent if and only if it satisfies one of the following conditions:*

*(i)* $|W| \leq 2$*;*

*(ii)* $|W| = 3$ *and* $|W \cap L| = 2$ *for some* $L \in \mathcal{L}$*;*

*(iii)* $|W| = 3$ *and* $|W \cap \mathcal{L}(p)| = 2$ *for some* $p \in P$*.*

**Proof**. By Theorems 5.3 and 6.13, $W$ is c-independent if and only if $|W| \leq 2$ or

$$|W| = 3,\ \mathrm{St}(W) = \emptyset \text{ and } W \text{ is not a potential line.} \tag{13}$$

Thus we only need to show that the join of conditions (ii) and (iii) is equivalent to (13).

Assume $|W| = 3$. It is easy to see that $\mathrm{St}(W) \neq \emptyset$ can only occur if $W \subseteq L$ for some $L \in \mathcal{L}$ or $W \subseteq \mathcal{L}(p)$ for some $p \in P$. On the other hand, if $W$ is not a potential line, then $|W \cap \mathrm{St}(x)| \geq 2$ for some $x \in P \cup \mathcal{L}$, that is, either $|W \cap L| \geq 2$ or $|W \cap \mathcal{L}(p)| \geq 2$ for some $p \in P$, $L \in \mathcal{L}$. Since $|W| = 3$, the result follows. $\square$

Going back to the $K_3$ example at the beginning of this section, it is now easy to check that every 3-subset $W$ of $V \cup E$ is c-independent in $\mathrm{Levi}\,K_3 \cong C_6$. Indeed, since $|E_v| = 2$ for every $v \in V$, we only need to show that there exist necessarily some $w_1, w_2 \in W$ at distance 2 (in $\mathrm{Levi}\,K_3$). This is certainly true for $C_6$, hence the c-independent subsets of vertices of $\mathrm{Levi}\,K_3$ (and therefore of $C_6$!) are all the subsets with at most 3 vertices.

Another example is given by the Fano plane [23]. We have remarked before that the Heawood graph $H$ is isomorphic to the Levi graph of the Fano plane and has no potential lines. It follows from Theorem 5.3 that the c-independent subsets of $H = (V, E)$ are all subsets with at most 3 vertices except the flats $\mathrm{St}(v)$ $(v \in V)$. The reader can now check that these 463 subsets correspond to the ones given by Corollary 6.14.

Given a PEG $\mathcal{G} = (P, \mathcal{L})$, and since $\mathcal{L} \subseteq 2^P$, we can consider the lattice $\widehat{\mathcal{L}}$ defined in Subsection 2.2. We denote it by $\mathrm{Lat}\,\mathcal{G}$.

**Lemma 6.15** *Given PEG's $\mathcal{G}$ and $\mathcal{G}'$ with mindeg $\geq 2$, the following conditions are equivalent:*

*(i)* $\mathcal{G} \cong \mathcal{G}'$;

*(ii)* $\mathrm{Lat}\,\mathcal{G} \cong \mathrm{Lat}\,\mathcal{G}'$.

**Proof**. It is immediate that the structure of $\mathcal{G}$ determines the structure of $\mathrm{Lat}\,\mathcal{G}$, up to isomorphism. Conversely, we can recover the structure of $\mathcal{G}$ from $\mathrm{Lat}\,\mathcal{G}$:

Indeed, in view of (G2) and $\mathrm{mindeg}\,\mathcal{G} \geq 2$, we have

$$\mathrm{Lat}\,\mathcal{G} = \{P, \emptyset\} \cup \mathcal{L} \cup P \tag{14}$$

and so we can identify the points in $P$ with the atoms of $\mathrm{Lat}\,\mathcal{G}$ and the lines in $\mathcal{L}$ with the maximal elements of $\mathrm{Lat}\,\mathcal{G} \setminus \{P\}$. Moreover $p \in L$ if and only if the corresponding atom of $\mathrm{Lat}\,\mathcal{G}$ is below the element representing in $L$, hence $\mathrm{Lat}\,\mathcal{G}$ determines the structure of $\mathcal{G}$ up to isomorphism and the lemma follows. $\square$

If $\mathrm{mindeg}\,\mathcal{G} \geq 2$, we can also introduce the *dual* PEG $\mathcal{G}^d$ (see [5]):

**Lemma 6.16** *Let $\mathcal{G} = (P, \mathcal{L})$ be a PEG with $\mathrm{mindeg}\,\mathcal{G} \geq 2$. Then $\mathcal{G}^d = (\mathcal{L}, \{\mathcal{L}(p) \mid p \in P\})$ is also a PEG with $\mathrm{mindeg}\,\mathcal{G}^d \geq 2$. Moreover, $\mathrm{Levi}\,\mathcal{G} \cong \mathrm{Levi}\,\mathcal{G}^d$.*

**Proof**. We have $\mathcal{L} \subseteq \cup_{p \in P} \mathcal{L}(p)$ since $\mathcal{G}$ satisfies (G3). Hence $\mathcal{G}^d$ satisfies (G1). Given distinct $p, p' \in P$, we have $|\mathcal{L}(p) \cap \mathcal{L}(p')| \leq 1$ since $\mathcal{G}$ satisfies (G2). Hence also $\mathcal{G}^d$ satisfies (G2). Since $\mathrm{mindeg}\,\mathcal{G} \geq 2$ implies that $|\mathcal{L}(p)| \geq 2$ for every $p \in P$, then $\mathcal{G}^d$ satisfies (G3) and is thus a PEG.

Next, since $\mathcal{G}$ satisfies (G3), every $L \in \mathcal{L}$ belongs at least to two $\mathcal{L}(p)$ and so $\mathrm{mindeg}\,\mathcal{G}^d \geq 2$.

Finally, let $\theta : P \cup \mathcal{L} \to \mathcal{L} \cup \{\mathcal{L}(p) \mid p \in P\}$ be the bijection defined by $p\theta = \mathcal{L}(p)$ $(p \in P)$ and $L\theta = L$ $(L \in \mathcal{L})$. It is immediate that $\theta$ preserves the edges, thus $\mathrm{Levi}\,\mathcal{G} \cong \mathrm{Levi}\,\mathcal{G}^d$. $\square$

Let $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be lattices. We denote the maximum (respectively the minimum) of both lattices by 1 (respectively 0) and assume the remaining elements to be disjoint. The *coproduct* of $(X_1, \leq_1)$ and $(X_2, \leq_2)$, denoted by $(X_1, \leq_1) \sqcup (X_2, \leq_2)$, has elements $X_1 \cup X_2$ (identifying the two 0's and the two 1's) and partial order $\leq_1 \cup \leq_2$. In particular, $x_1 \wedge x_2 = 0$, $x_1 \vee x_2 = 1$ for all $x_1 \in X_1$ and $x_2 \in X_2$.

**Theorem 6.17** *Let $\mathcal{G}$ be a PEG with $\mathrm{mindeg}\,\mathcal{G} \geq 2$. Then $\mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G} \cong \mathrm{Lat}\,\mathcal{G} \sqcup \mathrm{Lat}\,\mathcal{G}^d$. Moreover, this is the unique coproduct decomposition of $\mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G}$.*

**Proof**. Write $\mathcal{G} = (P, \mathcal{L})$. The isomorphism $\mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G} \cong \mathrm{Lat}\,\mathcal{G} \sqcup \mathrm{Lat}\,\mathcal{G}^d$ follows easily from Theorem 6.13 and (14).

Suppose now that $\varphi : \mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G} \to X_1 \sqcup X_2$ is a lattice isomorphism for some nontrivial lattices $X_1, X_2$. Let $Y_i$ denote the atoms of $\mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G}$ belonging to $X_i\varphi^{-1}$ $(i = 1, 2)$. Suppose that $\{p\} \in Y_1$ with $p \in P$. If $p$ — $L$ — $p'$ is a path in Levi $\mathcal{G}$, then it follows from (G2) that $\{p\} \vee \{p'\} = L < P \cup \mathcal{L}$ and so $\{p'\} \in Y_1$. Since $\mathcal{G}$ is connected, it follows that $\{q\} \in Y_1$ for every $q \in P$. Since $X_2$ is nontrivial, then $\{L\} \in Y_2$ for some $L \in \mathcal{L}$. If $L$ — $q$ — $L'$ is a path in Levi $\mathcal{G}$, then it follows from (G2) that $\{L\} \vee \{L'\} \subseteq \mathcal{L}(q) < P \cup \mathcal{L}$ and so $\{L'\} \in Y_2$. Since $\mathcal{G}$ is connected, it follows that $\{M\} \in Y_2$ for every $M \in \mathcal{L}$. Since the atoms determine the coproduct decomposition, it follows that $X_1 \cong \mathrm{Lat}\,\mathcal{G}$ and $X_2 \cong \mathrm{Lat}\,\mathcal{G}^d$. $\square$

Now we can prove the following:

**Theorem 6.18** *Let $\mathcal{G}$ and $\mathcal{G}'$ be PEG's with* mindeg $\mathcal{G}$, mindeg $\mathcal{G}' \geq 2$. *Then the following conditions are equivalent:*

*(i)* $\mathcal{G} \cong \mathcal{G}'$ *or* $\mathcal{G}^d \cong \mathcal{G}'$;

*(ii)* $\mathrm{Levi}\,\mathcal{G} \cong \mathrm{Levi}\,\mathcal{G}'$;

*(iii)* $\mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G} \cong \mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G}'$.

**Proof**. (i) $\Rightarrow$ (ii). In view of Lemma 6.16.

(ii) $\Rightarrow$ (iii). Immediate.

(iii) $\Rightarrow$ (i). Write $\mathcal{G} = (P, \mathcal{L})$ and $\mathcal{G}' = (P', \mathcal{L}')$. Assume that $\mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G} \cong \mathrm{Fl}\,\mathrm{Levi}\,\mathcal{G}'$. By Theorem 6.17, we have $\mathrm{Lat}\,(\mathcal{G}') \cong \mathrm{Lat}\,(\mathcal{G})$ or $\mathrm{Lat}\,(\mathcal{G}') \cong \mathrm{Lat}\,(\mathcal{G}^d)$. Now (i) follows from Lemma 6.15. $\square$

The graph version is slightly simpler:

**Corollary 6.19** *Let $G$ and $G'$ be finite connected graphs with* mindeg $G$, mindeg $G' \geq 2$. *Then the following conditions are equivalent:*

*(i)* $G \cong G'$;

*(ii)* $\mathrm{Levi}\,G \cong \mathrm{Levi}\,G'$;

*(iii)* $\mathrm{Fl}\,\mathrm{Levi}\,G \cong \mathrm{Fl}\,\mathrm{Levi}\,G'$.

**Proof**. Viewing a graph $G$ as a PEG, its dual $G^d$ is a graph if and only if each vertex of $G$ has degree 2, implying $G$ to be a cycle and therefore self-dual. Now we apply Theorem 6.18. $\square$

However, we recall that $\mathrm{Fl}\,G \cong \mathrm{Fl}\,G'$ does not imply $G \cong G'$, even when mindeg $G$, mindeg $G' \geq 2$ (see the example following the proof of Proposition 3.8).

# 7 Cubic graphs

We present in this section some specific results concerning cubic graphs.

We start by some easy remarks concerning girth and c-rank of cubic graphs. For instance, note that $\mathrm{gth}\,G < \infty$ for every finite cubic graph: any acyclic graph contains necessarily vertices of degree 1.

In view of Propositions 3.4 and 4.1, we have $2 \leq \text{c-rk}\,G \leq 4$ for every cubic graph $G$. However, the following result shows that c-rank and girth are not independent for cubic graphs:

**Proposition 7.1** *Let $G = (V, E)$ be a cubic graph.*

*(i) If* $\operatorname{gth} G = 3$*, then* $\operatorname{c-rk} G = 3$ *or 4.*

*(ii) If* $\operatorname{gth} G = 4$*, then* $\operatorname{c-rk} G = 2$ *or 3 or 4.*

*(iii) If* $\operatorname{gth} G \geq 5$*, then $G$ is sober,* $\operatorname{c-rk} G = 3$ *and* $|St(v, w)| \leq 1$ *for distinct* $v, w \in V$*.*

*Moreover, all these combinations with girth $\leq 8$ can occur. If $G$ is sober and connected, only the cases with* $\operatorname{gth} G = 4$ *and* $\operatorname{c-rk} G < 4$ *are excluded.*

**Proof**. (i) By Proposition 4.2, since $G$ has a triangle.

(ii) By the comment preceding the proposition.

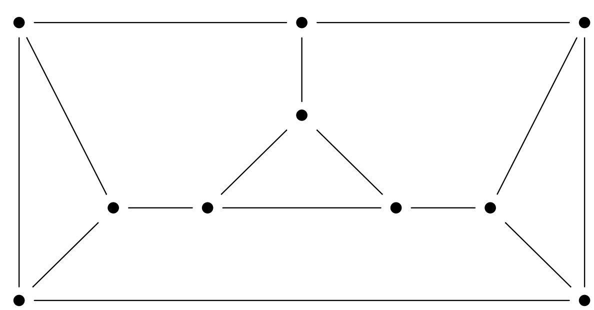
(iii) On the other hand, Since $\operatorname{gth} G \geq 5$, $G$ has a restriction of the form

$$\bullet - \bullet - \bullet - \bullet$$

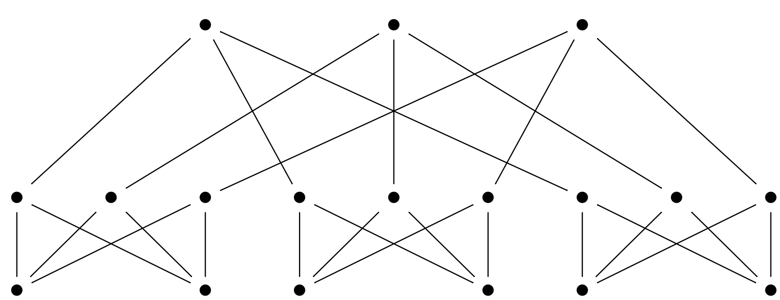
and so $\operatorname{c-rk} G > 2$ by Proposition 4.2. On the other hand, if $\operatorname{c-rk} G = 4$, then $G$ would have a square by Proposition 4.3, a contradiction. Therefore $\operatorname{c-rk} G = 3$. Since $G$ has no squares, the remaining conditions follow as well.

We present next examples to show that all these combinations with girth $\leq 8$ occur:

- c-rank 2, girth 4: the complete bipartite graph $K_{3,3}$;
- c-rank 3, girth 3:

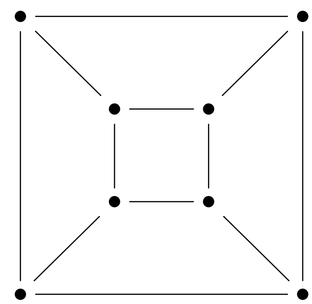


- c-rank 3, girth 4:



- c-rank 3, girth 5: the Petersen graph
- c-rank 3, girth 6: the Heawood graph [24];

- c-rank 3, girth 7: the McGee graph [25];
- c-rank 3, girth 8: the Tutte-Coxeter graph [28];
- c-rank 4, girth 3: the complete graph $K_4$;
- c-rank 4, girth 4:



Note that all these examples are sober and connected except those with $\operatorname{gth} G = 4$ and $\operatorname{c-rk} G < 4$. The reason for the exclusion of these combinations lies within Corollary 4.5: if $G$ is sober and $\operatorname{c-rk} G < 4$, then $G$ has no squares. $\square$

Note that some of the arguments used in this proof are valid also for graphs which are not cubic. For instance, if $\operatorname{gth} G \geq 5$ and all vertices of $G$ have degree $> 1$, then $G$ is necessarily sober.

It is an interesting problem to determine under which conditions the lattice of flats of a graph has certain properties.

In the following theorems, we present results for the case of connected cubic graphs. We start with a couple of useful lemmas.

**Lemma 7.2** *Let $G = (V, E)$ be a finite nonempty graph. Then:*

*(i) every atom of* $\operatorname{Fl} G$ *is of the form* $\operatorname{St}(\operatorname{St}(v))$ *for some* $v \in V$*;*

*(ii) the converse is true if $G$ is cubic.*

**Proof**. (i) Let $W$ be an atom of $\operatorname{Fl} G$. We may write $W = \operatorname{St}(X)$ for some $X \subseteq V$. Let $v \in W = \operatorname{St}(X)$. Then $X \subseteq \operatorname{St}(v)$ and so $\operatorname{St}(\operatorname{St}(v)) \subseteq \operatorname{St}(X) = W$. Since $v \in \operatorname{St}(\operatorname{St}(v))$ and $W$ is an atom, we get $\operatorname{St}(\operatorname{St}(v)) = W$.

(ii) Assume that $G$ is cubic and $W = \operatorname{St}(\operatorname{St}(v))$ for some $v \in V$. Let $u \in V$ be such that $W \cap \operatorname{St}(u) \neq \emptyset$. We must prove that $W \subseteq \operatorname{St}(u)$.

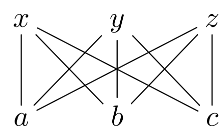
Indeed, if $x \in W \cap \operatorname{St}(u) = \operatorname{St}(\operatorname{St}(v)) \cap \operatorname{St}(u)$, then $\operatorname{St}(v) \cup \{u\} \subseteq \operatorname{St}(x)$. Since $|\operatorname{St}(v)| = |\operatorname{St}(x)| = 3$, it follows that $u \in \operatorname{St}(v)$ and so $W = \operatorname{St}(\operatorname{St}(v)) \subseteq \operatorname{St}(u)$ as required. $\square$

**Lemma 7.3** *Let $G = (V, E)$ be a finite connected cubic graph. Then the following conditions are equivalent:*

*(i)* $\operatorname{St}(v)$ *is an atom of* $\operatorname{Fl} G$ *for some* $v \in V$*;*

*(ii)* $G \cong K_{3,3}$.

**Proof**. (i) $\Rightarrow$ (ii). If $\mathrm{St}(v)$ is an atom of $\mathrm{Fl}\, G$, then, for every $u \in V$, either $\mathrm{St}(v) \subseteq \mathrm{St}(u)$ or $\mathrm{St}(v) \cap \mathrm{St}(u) = \emptyset$. Since $G$ is cubic, $\mathrm{St}(v) \subseteq \mathrm{St}(u)$ is actually equivalent to $\mathrm{St}(v) = \mathrm{St}(u)$. Writing $\mathrm{St}(v) = \{a, b, c\}$ and $\mathrm{St}(a) = \{v, x, y\}$, we can take $u$ above equal to $x$ and $y$ to obtain $\mathrm{St}(x) = \mathrm{St}(y) = \mathrm{St}(v) = \{a, b, c\}$. It follows that $G$ has a subgraph of the form




(note that $\{x, y, z\} \cap \{a, b, c\} = \emptyset$ due to the absence of loops). Since $G$ is cubic and connected, this must be the whole of $G$, which is then isomorphic to $K_{3,3}$.

(ii) $\Rightarrow$ (i). Since the lattice of flats of $K_{3,3}$ is isomorphic to $(2^{\hat{2}}, \subseteq)$. $\square$

**Theorem 7.4** *Let $G = (V, E)$ be a finite connected cubic graph. Then the following conditions are equivalent:*

*(i)* $\mathrm{Fl}\, G$ *is distributive;*

*(ii)* $\mathrm{Fl}\, G$ *is modular;*

*(iii)* $\mathrm{Fl}\, G$ *is semimodular;*

*(iv)* $\mathrm{Fl}\, G$ *is geometric;*

*(v)* $G \cong K_4$ *or* $G \cong K_{3,3}$.

**Proof**. The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (iii) are immediate. Since the lattices of flats of $K_4$ and $K_{3,3}$ are isomorphic respectively to $(2^{\hat{4}}, \subseteq)$ and $(2^{\hat{2}}, \subseteq)$, we get (v) $\Rightarrow$ (i) and (v) $\Rightarrow$ (iv). It remains to be proved that (iii) $\Rightarrow$ (v).

Assume that $\mathrm{Fl}\, G$ is semimodular. Suppose that $\mathrm{diam}\, G > 2$. Let $v, w \in V$ be such that $d(v, w) > 2$. Write $\mathrm{St}(w) = \{w_1, w_2, w_3\}$. By Lemma 7.2, $\mathrm{St}(\mathrm{St}(w_j))$ is an atom of $\mathrm{Fl}\, G$ for $j = 1, 2, 3$. Since $\mathrm{diam}\, K_{3,3} = 2$, it follows from Lemma 7.3 that $\mathrm{St}(v)$ is not an atom of $\mathrm{Fl}\, G$. Write $\mathrm{St}(v) = \{v_1, v_2, v_3\}$. By Lemma 7.2, $\mathrm{St}(\mathrm{St}(v_i))$ is an atom of $\mathrm{Fl}\, G$ for $i = 1, 2, 3$.

Suppose that

$$\forall i, j \in \hat{3}\ \exists z_{ij} \in V : v_i, w_j \in \mathrm{St}(z_{ij}). \tag{15}$$

Since $d(v, w) > 2$, we must have $z_{ij} \neq v, w$. Hence $\{z_{i1}, z_{i2}, z_{i3}\} \subseteq \mathrm{St}(v_i) \setminus \{v\}$ and so

$$|\{z_{i1}, z_{i2}, z_{i3}\}| \leq 2 \ \text{ for } i = 1, 2, 3. \tag{16}$$

Similarly,

$$|\{z_{1j}, z_{2j}, z_{3j}\}| \leq 2 \ \text{ for } j = 1, 2, 3. \tag{17}$$

Let $G' = (V', E')$ be the graph such that $V' = \hat{3} \times \hat{3}$ and $(i, j)$ — $(i', j')$ is an edge if and only if $(i, j) \neq (i', j')$ and $z_{ij} = z_{i'j'}$. By (16) and (17), $G'$ has at least 6 edges. Since $G'$ has 9 vertices, there must be a pair of incident edges. Hence there exist distinct $(i, j), (i', j'), (i'', j'') \in V'$ such that $z_{ij} = z_{i'j'} = z_{i''j''}$. Thus $v_i, v_{i'}, v_{i''}, w_j, w_{j'}, w_{j''} \in \mathrm{St}(z_{ij})$ and so

$$|\{v_i, v_{i'}, v_{i''}, w_j, w_{j'}, w_{j''}\}| \leq 3.$$

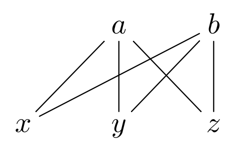
Since $\{v_i, v_{i'}, v_{i''}\} \cap \{w_j, w_{j'}, w_{j''}\} = \emptyset$ due to $d(v,w) > 2$, this contradicts $(i,j), (i',j'), (i'',j'')$ being all distinct. Therefore (15) fails and so there exist $i, j \in \hat{3}$ such that $\mathrm{St}(v_i) \cap \mathrm{St}(w_j) = \emptyset$.

Now it is easy to check that

$$\begin{tikzcd}
& V \arrow[dl] \arrow[ddr] & \\
\mathrm{St}(v) \arrow[d] & & \\
\mathrm{St}(\mathrm{St}(v_i)) \arrow[dr] & & \mathrm{St}(\mathrm{St}(w_j)) \arrow[dl] \\
& \emptyset &
\end{tikzcd}$$

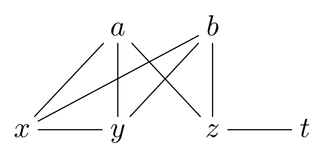
is a sublattice of $\mathrm{Fl}\,G$. On the one hand, we have $\mathrm{St}(v) \cap \mathrm{St}(\mathrm{St}(w_j)) \subseteq \mathrm{St}(v) \cap \mathrm{St}(w) = \emptyset$ since $d(v,w) > 2$. On the other hand, suppose that $\mathrm{St}(\mathrm{St}(v_i)) \cup \mathrm{St}(\mathrm{St}(w_j)) \subseteq \mathrm{St}(z)$ for some $z \in V$. Then $v_i, w_j \in \mathrm{St}(z)$, contradicting $\mathrm{St}(v_i) \cap \mathrm{St}(w_j) = \emptyset$. This proves that $G$ cannot be semimodular if $\mathrm{diam}\, G > 2$. Hence $\mathrm{diam}\, G \leq 2$.

Suppose first that $G$ is not sober. Then we have $\mathrm{St}(a) = \{x,y,z\} = \mathrm{St}(b)$ for some distinct $a, b \in V$, and so $G$ has a subgraph of the form




(18)

Supose that there exists an edge connecting two of the vertices $x, y, z$, say $x$ — $y$. Then $G$ has a subgraph of then form




and it is now clear that $d(t,y) > 2$, contradicting $\mathrm{diam}\, G \leq 2$. Hence (18) is a restriction of $G$. If there exists some $X \in \mathrm{Fl}\,G$ satisfying $\{x,y,z\} \supset X \supset \emptyset$, it is easy to check that

$$\begin{tikzcd}
& V \arrow[dl] \arrow[ddr] & \\
\{x,y,z\} \arrow[d] & & \\
X \arrow[dr] & & \{a,b\} \arrow[dl] \\
& \emptyset &
\end{tikzcd}$$

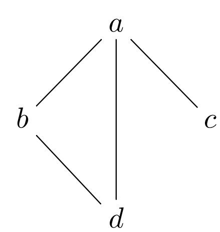
would be a sublattice of $\operatorname{Fl} G$, contradicting semimodularity. Thus $\{x, y, z\}$ is an atom and so $G \cong K_{3,3}$ by Lemma 7.3.

Therefore we may assume that $G$ is sober. Suppose first that there exists some edge $a$ — $b$ which does not lie in any triangle of $G$. Since $G$ is sober, if follows from Lemma 6.11(i) that $\{a\} = \mathrm{St}(\mathrm{St}(a))$ and $\{b\} = \mathrm{St}(\mathrm{St}(b))$. Moreover, $a \in \mathrm{St}(v)$ for some $v \in V$. We claim that

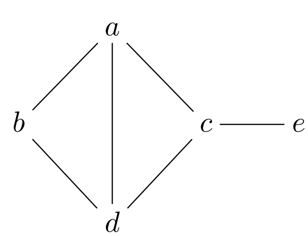
$$\begin{tikzcd}
 & V \arrow[dl] \arrow[ddr] & \\
\mathrm{St}(v) \arrow[d] & & \\
\{a\} \arrow[dr] & & \{b\} \arrow[dl] \\
 & \emptyset &
\end{tikzcd}$$

is a sublattice of $\operatorname{Fl} G$, a contradiction. Indeed, if $b \in \mathrm{St}(v)$, then $a, b, v$ would be the vertices of a triangle, contradicting our assumption, and no flat can contain $a, b$ simultaneously by the same reason. Hence every edge of $G$ must lie in some triangle.

Now $G$ must have a subgraph of the form

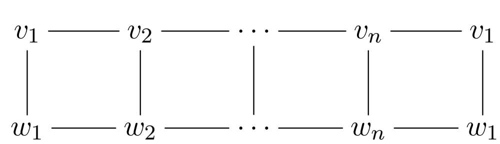



Since the edge $a$ — $c$ must lie in some triangle, we have an edge $c$ — $d$ or an edge $b$ — $c$. Without loss of generality, we may assume that $c$ — $d$ is an edge. If we have an edge $c$ — $e$ with $e \neq b$, then we have




and so $d(b, e) = 3$, a contradiction. Hence there is an edge $b$ — $c$ as well and so $G \cong K_4$. Therefore (v) holds. $\square$

For every $n \geq 3$, we define the cylindrical strip $H_n$ by



and the Möbius strip $\tilde{H}_n$ by

$$\begin{tikzcd}[arrows=dash]
v_1 \arrow[r] \arrow[d] & v_2 \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & v_n \arrow[r] \arrow[d] & w_1 \arrow[d] \\
w_1 \arrow[r] & w_2 \arrow[r] & \cdots \arrow[r] & w_n \arrow[r] & v_1
\end{tikzcd}$$

**Theorem 7.5** *Let $G = (V, E)$ be a finite connected cubic graph. Then the following conditions are equivalent:*

*(i)* $\mathrm{Fl}\, G$ *satisfies the Jordan-Dedekind condition;*

*(ii)* $\text{c-rk}\, G \leq 3$ *or ($G$ is sober and every edge of $G$ lies in some square);*

*(iii)* $\text{c-rk}\, G \leq 3$ *or* $G \cong K_4$ *or* $G \cong H_n$ *for some* $n \geq 3$ *or* $G \cong \tilde{H}_n$ *for some* $n \geq 4$.

**Proof**. (i) $\Rightarrow$ (ii). We may assume that $\text{c-rk}\, G \geq 4$. Suppose that $G$ is not sober. Then there exist $v, w \in V$ such that $\mathrm{St}(v) = \mathrm{St}(w)$. By Lemma 7.2, $\mathrm{St}(\mathrm{St}(v))$ is an atom of $\mathrm{Fl}\, G$. Since $v, w \in \mathrm{St}(\mathrm{St}(v))$, it follows that $\mathrm{Fl}\, G$ has an atom with 2 elements. Since any $X \in \mathrm{Fl}\, G \setminus \{V\}$ has at most 3 elements, then $\mathrm{Fl}\, G$ has a maximal chain with length $\leq 3$. Since $\text{c-rk}\, G \geq 4$ implies the existence of some maximal chain with length 4, $G$ fails the Jordan-Dedekind condition.

Hence we may assume also that $G$ is sober. Suppose now that $a$ — $b$ is an edge of $G$. Write $\mathrm{St}(a) = \{b, c, d\}$. If $\mathrm{St}(b, c)$ contains some other element $x \neq a$, then $a$ — $b$ belongs to the square

$$\begin{tikzcd}[arrows=dash]
a \arrow[r] \arrow[d] & b \arrow[d] \\
c \arrow[r] & x
\end{tikzcd}$$

hence we may assume that $\mathrm{St}(b, c) = \{a\}$ and so $\{a\}$ is an atom of $\mathrm{Fl}\, G$. Write $\mathrm{St}(b) = \{a, y, z\}$. If (i) holds, and since $\text{c-rk}\, G \geq 4$, the chain $\emptyset \subset \{a\} \subset \{a, y, z\} \subset V$ must admit a refinement. We may therefore assume that $\{a, y\} \in \mathrm{Fl}\, G$. It follows that $\{a, y\} = \mathrm{St}(p, q)$ for some distinct $p, q \in V$. Hence $p, q \in \mathrm{St}(a) = \{b, c, d\}$ and so $\{p, q\} \cap \{c, d\} \neq \emptyset$. Assuming that $p \in \{c, d\}$, we obtain a 4-cycle

$$\begin{tikzcd}[arrows=dash]
a \arrow[r] \arrow[d] & b \arrow[d] \\
p \arrow[r] & y
\end{tikzcd}$$

and so (ii) holds.

(ii) $\Rightarrow$ (iii). We may assume that $\text{c-rk}\, G \geq 4$, $G$ is sober and every edge of $G$ lies in some square. We consider two cases:

Case I: $\mathrm{gth}\, G = 3$.

Suppose first that $G$ has a subgraph $G_0$ the form

$$\begin{tikzcd}[arrows=dash]
a \arrow[r] \arrow[d] \arrow[dr] & b \arrow[d] \\
c \arrow[r] & d
\end{tikzcd}$$

Then the edge $a$ — $d$ must be part of a square

$$\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & d \arrow[d, dash] \\ x \arrow[r, dash] & y \end{tikzcd}$$

Since $\mathrm{St}(a)$ and $\mathrm{St}(d)$ are fully determined, we must have $\{x, y\} = \{b, c\}$ and so $G$ has a subgraph isomorphic to $K_4$. Since $G$ is connected and cubic, then $G \cong K_4$.

Hence we may assume that $G$ has no subgraph isomorphic to $G_0$ above. Take a triangle in $G$. Since every edge must belong to a square and we are excuding subgraphs isomorphic to $G_0$, then $G$ must have a subgraph of the form

$$\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & b \arrow[r, dash] \arrow[d, dash] & c \arrow[dl, dash] \\ d \arrow[r, dash] & e & \end{tikzcd}$$

The existence of an edge $c$ — $a$ or $c$ — $d$ would imply the presence of a subgraph isomorphic to $G_0$, hence we have an edge $c$ — $f$ for some new vertex $f$. Considering a square

$$\begin{tikzcd} c \arrow[r, dash] \arrow[d, dash] & f \arrow[d, dash] \\ x \arrow[r, dash] & y \end{tikzcd}$$

it follows easily that either $x = b$ and $y = a$, or $x = e$ and $y = d$. These cases yield in fact isomorphic subgraphs, hence we assume the first to get

$$\begin{tikzcd} c \arrow[rr, dash] \arrow[dr, dash] \arrow[dd, dash] & & e \arrow[dl, dash] \arrow[dd, dash] \\ & b \arrow[d, dash] & \\ f \arrow[r, dash] & a \arrow[r, dash] & d \end{tikzcd}$$

It is straightforward to check that the only square that can contain the edge $c$ — $e$ is

$$\begin{tikzcd} c \arrow[r, dash] \arrow[d, dash] & e \arrow[d, dash] \\ f \arrow[r, dash] & d \end{tikzcd}$$

hence $G$ contains a subgraph isomorphic to $H_3$ and is therefore isomorphic to $H_3$.

<u>Case II</u>: $\operatorname{gth} G = 4$.

Let $G'$ be a subgraph of $G$ of the form

$$\begin{tikzcd} v_1 \arrow[r, dash] \arrow[d, dash] & v_2 \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & v_n \arrow[d, dash] \\ w_1 \arrow[r, dash] & w_2 \arrow[r, dash] & \cdots \arrow[r, dash] & w_n \end{tikzcd}$$

with $n$ maximum. We claim that $n \geq 4$.

Indeed, suppose first that $n = 2$. Since $G$ has no triangles, then we have a subgraph

$$\begin{tikzcd} v_1 \arrow[r, dash] \arrow[d, dash] & v_2 \arrow[r, dash] \arrow[d, dash] & a \\ w_1 \arrow[r, dash] & w_2 \arrow[r, dash] & b \end{tikzcd}$$

Let

$$\begin{tikzcd} v_2 \arrow[r, dash] \arrow[d, dash] & a \arrow[d, dash] \\ x \arrow[r, dash] & y \end{tikzcd}$$

be a square containing $v_2$ — $a$. Then either $x = v_1$ or $x = w_2$. Suppose first that $x = v_1$. If $y = w_1$, then $\mathrm{St}(v_2) = \mathrm{St}(w_1)$, contradicting $G$ being sober. On the other hand, if $y$ is a new vertex $c$, we get two adjacent squares and contradict the maximality of $n$. The case $x = w_2$ is similar, hence $n > 2$ in this case.

Suppose now that $n = 3$. Since there are no triangles and $G$ is sober, we have a subgraph

$$\begin{tikzcd} v_1 \arrow[r, dash] \arrow[d, dash] & v_2 \arrow[r, dash] \arrow[d, dash] & v_3 \arrow[r, dash] \arrow[d, dash] & a \\ w_1 \arrow[r, dash] & w_2 \arrow[r, dash] & w_3 \arrow[r, dash] & b \end{tikzcd}$$

To avoid contradicting the maximality of $n$, we cannot accept an edge $a$ — $b$. Considering squares containing the edges $v_3$ — $a$ and $w_3$ — $b$, we obtain edges $v_1$ — $a$ and $w_1$ — $b$. Taking an edge $a$ — $c$, where $c$ is necessarily a new vertex, we immediately get a contradiction by trying to fit the new edge into a square. Thus $n \geq 4$.

Now if $v_n$ — $a$ is an edge, where $a$ is a new vertex, we cannot fit this edge into a square without compromising the maximality of $n$, hence we have either edges $v_1$ — $v_n$ and $w_1$ — $w_n$ (yielding $H_n$) or edges $v_1$ — $w_n$ and $w_1$ — $v_n$ (yielding $\tilde{H}_n$). Therefore (iii) holds.

(iii) $\Rightarrow$ (ii). Immediate.

(ii) $\Rightarrow$ (i). The case $\text{c-rk}\, G = 2$ being trivial, suppose first that $\text{c-rk}\, G = 3$. Since the flats $\mathrm{St}(v)$ are the maximal elements of $\mathrm{Fl}\, G \setminus \{V\}$ and no such flat is an atom of $\mathrm{Fl}\, G$ in view of Lemma 7.3, it follows that every maximal chain of $\mathrm{Fl}\, G$ must have length 3 and so $G$ satisfies the Jordan-Dedekind condition.

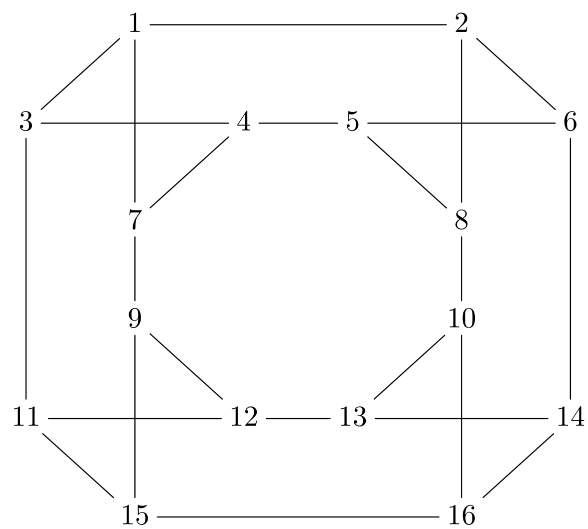
Finally, assume that $\text{c-rk}\, G > 3$. Let $v \in V$. By Lemma 6.11(i), $\{v\}$ is an atom for every $v \in V$. Now, if $\{a\} \subset \{a, b, c\} = \mathrm{St}(x)$ is a chain in $\mathrm{Fl}\, G$, then we may assume that there exists some square

$$\begin{tikzcd} a \arrow[r, dash] \arrow[d, dash] & x \arrow[d, dash] \\ y \arrow[r, dash] & b \end{tikzcd}$$

Since $G$ is sober, it follows that $\mathrm{St}(x, y) = \{a, b\}$ and so all maximal chains in $\mathrm{Fl}\, G$ must have length 4. Thus (i) holds. $\square$

It is easy to check that all graphs $H_n$ and $\tilde{H}_n$ are *vertex-transitive*: for all vertices $v$ and $w$, there exists an automorphism $\varphi$ of the graph such that $v\varphi = w$ (i.e. all vertices lie in a single automorphic orbit).

By Proposition 3.4 and Corollary 4.5, a finite sober connected cubic graph has c-rank 4 if and only if it has a square. If it is also vertex-transitive, then every vertex must lie in some square. The next example shows that one cannot replace *edge* by *vertex* in condition (ii) of Theorem 7.5, even if we require vertex-transitivity:




Indeed, this finite sober connected cubic graph has c-rank 4 and is vertex-transitive (hence every vertex lies in some square), and yet it fails the conditions of Theorem 7.5. Note that in this case

$$V \supset \mathrm{St}(1) \supset \mathrm{St}(1,4) \supset \mathrm{St}(1,4,11) \supset \emptyset$$

and

$$V \supset \mathrm{St}(3) \supset \mathrm{St}(3,12) \supset \emptyset$$

are two maximal chains of different length.

# 8 Minors and cm-rank

Recall that a finite graph $G'$ is said to be a *minor* of a finite graph $G$ if $G'$ can be obtained (up to isomorphism) from $G$ by successive application of the following three operations:

($D_1$) *vertex-deletion*: we delete a vertex;

($D_2$) *edge-deletion*: we delete an edge;

(C) *contraction*: we delete an edge $v$ — $w$ and identify the vertices $v$ and $w$.

If $G'$ is a minor of $G$, we write $G' \leq_m G$.

It is easy to check that these operators commute with each other in the sense that

$$D_1D_2(G) \subseteq (D_2D_1 \cup D_1)(G), \quad D_2D_1(G) \subseteq D_1D_2(G),$$
$$CD_1(G) \subseteq D_1C(G), \quad D_1C(G) \subseteq (CD_1 \cup D_1^2)(G),$$
$$CD_2(G) \subseteq (D_2C \cup C)(G), \quad D_2C(G) \subseteq (CD_2 \cup CD_2^2)(G),$$

hence a minor of $G$ can in particular be obtained by applying to $G$ sequences of contractions followed by edge-deletions followed by vertex-deletions. Clearly, c-rank cannot increase by means of vertex-deletions since we are bound to get a submatrix of he original one. However, the example following Proposition 3.6 shows that c-rank can increase by means of edge-deletions. The same happens for contractions: taking the very same square as an example, which has c-rank 2, and performing a contraction, we get $K_3$ which has higher c-rank.

Thus we introduce a second rank function for finite graphs: given a finite connected graph $G$, let

$$\text{cm-rk}\, G = \max\{\text{c-rk}\, G' \mid G' \leq_m G\}.$$

Since a minor has at most as many vertices as the original graph, cm-rank is well defined. For every $m \in \mathbb{N}$, we denote by $\mathcal{G}_m$ the class of all finite graphs with cm-rank $\leq m$. Since the minor relation is transitive, $\mathcal{G}_m$ is closed for minors. In view of the Robertson-Seymour Theorem (see [6]), there exists a finite set of graphs $\mathcal{F}$ such that

$$G \in G_m \Leftrightarrow \forall F \in \mathcal{F} \quad F \not\leq_m G.$$

We can easily construct the set $\mathcal{F}$ of forbidden graphs in our case. For $m \geq 1$, let $\mathcal{F}_m$ consist of representatives of all isomorphism classes of graphs with at most $2m$ vertices and c-rank $m+1$. Let $\mathcal{F}_0$ contain a one-vertex graph.

**Proposition 8.1** *The following conditions are equivalent for every finite graph G and every $m \in \mathbb{N}$:*

*(i)* $G \in \mathcal{G}_m$*;*

*(ii)* $\forall F \in \mathcal{F}_m\ F \not\leq_m G$.

**Proof**. The case $m = 0$ holding trivially, we assume that $m \geq 1$.

(i) $\Rightarrow$ (ii). If $G$ has a minor $G' \cong F \in F_m$, then $\text{cm-rk}\, G \geq \text{c-rk}\, G' = \text{c-rk}\, F = m+1$ and so $G \notin \mathcal{G}_m$.

(ii) $\Rightarrow$ (i). If $G \notin \mathcal{G}_m$, then $G$ has a minor $G' = (V', E')$ of c-rank $> m$. Since a subgraph of a minor is itself a minor, we may assume that $\text{c-rk}\, G' = m+1$. Hence there exist $I, J \subseteq V'$ such that $|I| = |J| = m+1$ and $A^c_{G'}[I, J]$ is nonsingular. Write $I = \{i_1, \ldots, i_{m+1}\}$ and $J = \{j_1, \ldots, j_{m+1}\}$. Reordering rows and columns if necessary, we may assume that $A^c[I, J]$ is of the form (3), for the ordering $i_1 < \ldots < i_{m+1}$ and $j_1 < \ldots < j_{m+1}$. Replacing $j_1$ by $i_1$ and $i_{m+1}$ by $j_{m+1}$, the resulting matrix is still of the form (3). Let $F$ be the restriction of $G'$ induced by the vertices $\{i_1, \ldots, i_m, j_2, \ldots, j_{m+1}\}$. Up to isomorphism, we have $F \in \mathcal{G}_m$. Since $F \leq_m G' \leq_m G$, (ii) fails as required. $\square$

Next we initiate a discussion on how the computation of cm-rank relates to the matrix representation of graphs. A sequence of contractions on a graph $G = (V, E)$ determines a partition $P : V = V_1 \cup \ldots \cup V_m$ corresponding to the subsets of vertices that are eventually identified into a single one. It is immediate that the restriction of $G$ induced by each $V_i$ must be connected (we call such a partition *connected*). How do we identify a connected restriction within $A^c$? Through the following straightforward observation:

**Proposition 8.2** *The following conditions are equivalent for a finite graph $G = (V, E)$:*

*(i) $G$ is connected;*

*(ii) there exists no nontrivial partition $V = I \cup J$ such that $A[I, J]$ is the null matrix;*

*(iii) there exists no nontrivial partition $V = I \cup J$ such that all the entries in $A^c[I, J]$ are equal to 1.*

What happens to the adjacency matrix when we perform a sequence of contractions inducing the partition $P : V = V_1 \cup \ldots \cup V_m$? Let the new graph be $G/P = (V/P, E/P)$, with $V/P = \hat{m}$, where each vertex $i$ corresponds to the identification of the vertices in $V_i$. It is straightforward to check that

$$A^c_{G/P}[i,j] = \begin{cases} 0 & \text{if } i \neq j \text{ and } 0 \text{ occurs in } A^c_G[V_i, V_j] \\ 1 & \text{otherwise} \end{cases}$$

If we follow a sequence of contractions by a sequence of edge-deletions, we are entitled to replace 0s by 1s in the matrix $A^c_{G/P}$. Finally, vertex-deletions correspond to deleting rows and columns in this modified matrix, which does not increase c-rank, and can therefore be ignored in the computation of the cm-rank. We therefore obtain:

**Proposition 8.3** *Let $G = (V, E)$ be a finite graph. Then* $\text{cm-rk}\, G$ *is the maximum value of* $\text{rk}\, \widetilde{A^c_{G/P}}$ *when $P : V = V_1 \cup \ldots \cup V_m$ is a connected partition of $V$ and*

$$\widetilde{A^c_{G/P}}[i,j] = \begin{cases} 0 \textit{ or } 1 & \textit{if } i \neq j \textit{ and } 0 \textit{ occurs in } A^c_G[V_i, V_j] \\ 1 & \textit{otherwise} \end{cases}$$

# 9 The complement graph

Given a graph $G = (V, E)$, its *complement graph* $\overline{G} = (V, \overline{G})$ is the graph defined by the condition

$$\{v, w\} \in \overline{E} \quad \Leftrightarrow \quad \{v, w\} \notin E\},$$

for all distinct $v, w \in V$.

The classical idea of independence for a subset $W$ of vertices of $G$ (no edges between them) is related to our notion of c-independence by $W$ being necessarily c-independent in $\overline{G}$, but not conversely.

We can get a lower bound for $\text{c-rk}\, \overline{G}$ through the chromatic number. An *edge coloring* of a graph $G = (V, E)$ with $c$ colors is a partition $V = V_1 \cup \ldots \cup V_c$ such that no edge of $G$ connects two vertices in the same $V_j$. The *chromatic number* $c(G)$ is the minimum number $c$ of colors to edge color $G$.

**Proposition 9.1** *Let $G$ be a finite graph. Then* $\text{c-rk}\, \overline{G} \geq \frac{|V|}{c(G)}$.

**Proof**. Since $|V_j| \geq \frac{|V|}{c(G)}$ for some $j$, then $\overline{G}$ has a complete subgraph with at least $\frac{|V|}{c(G)}$ vertices and the claim follows from Proposition 3.6(ii). $\square$

An important issue consists of the study of the sum $S = \text{c-rk}\, G + \text{c-rk}\, \overline{G}$ for a graph with $n$ vertices. The examples we analyzed so far show that $S$ can be as small as $\frac{n}{2} + 2$ (taking $G = K_{\frac{n}{2},\frac{n}{2}}$ for $n$ even, then $\overline{G}$ is a disjoint union of two copies of $K_{\frac{n}{2}}$) and as large as $n + 2$ (taking $G$ to be a graph of the form

$$v_1 \longrightarrow v_2 \longrightarrow \ldots \longrightarrow v_n$$

for $n \geq 4$). The next result offers an upper bound for $S$:

**Proposition 9.2** *Let $G = (V, E)$ be a finite graph with $|V| = n$. Then $S = \text{c-rk}\, G + \text{c-rk}\, \overline{G} < \sqrt{2}n + 1$.*

**Proof**. Assume that $|E| = k$ and $\text{c-rk}\, G = m$. Then $m = \text{rk}\, A_G^c$ and the witness characterization in Proposition 2.3 yields

$$(m-1) + \ldots + 2 + 1 = \frac{m(m-1)}{2} \leq k.$$

Thus $2k \geq m^2 - m$, yielding $m \leq \frac{1+\sqrt{1+8k}}{2}$.

Similarly, since $\overline{G}$ has $\frac{n(n-1)}{2} - k$ edges, we get

$$\text{c-rk}\, \overline{G} \leq \frac{1 + \sqrt{1 + 4n^2 - 4n - 8k}}{2}.$$

Hence

$$S \leq 1 + \frac{\sqrt{1+8k} + \sqrt{1 + 4n^2 - 4n - 8k}}{2}.$$

A simple calculus exercise shows that a real-valued function of the form

$$\sqrt{1+8x} + \sqrt{1 + 8(a-x)} \quad (a > 0)$$

reaches its maximum when $x = \frac{a}{2}$. Hence

$$S \leq 1 + \sqrt{1 + 2n^2 - 2n} \leq 1 + \sqrt{2}n.$$

□

We can also note the following:

**Proposition 9.3** *Let $(G_n)_n$ be a sequence of nonisomorphic finite graphs and let $M_n = max\{\, \text{c-rk}\, G_n,\, \text{c-rk}\, \overline{G_n}\}$. Then*

$$\lim_{n\to+\infty} M_n = +\infty.$$

**Proof**. Let $R(k, k)$ denote the Ramsey number that ensures every complete graph with at least $R(k, k)$ vertices, with edges colored by two colors, to have a monocromatic complete subgraph with $k$ vertices. Let $k \in \mathbb{N}$. Since the graphs $G_n$ are nonisomorphic, there exists some $p \in \mathbb{N}$ such that all graphs $G_n$ have at least $R(k, k)$ vertices for $n > p$. In particular, either $G_n$ or $\overline{G_n}$ must contain a complete subgraph with $k$ vertices, and so $M_n \geq k$ by Proposition 3.6(ii). Therefore $\lim_{n\to+\infty} M_n = +\infty$. □

We can give another perspective of the complement graph through the *dual lattice of closed stars*. Given a graph $G = (V, E)$, the *closed star* of a vertex $v \in V$ is defined by

$$\overline{\mathrm{St}}(v) = \mathrm{St}(v) \cup \{v\}.$$

Given $\mathcal{S} \subseteq 2^V$, it is easy to see that

$$\widetilde{\mathcal{S}} = \{\cup S \mid S \subseteq \mathcal{S}\}$$

is the $\vee$-subsemilattice of $(2^V, \subseteq)$ generated by $\mathcal{S}$. Note that $\cup\mathcal{S} = \max\widetilde{\mathcal{S}}$, and also $\emptyset = \cup\emptyset = \min\widetilde{\mathcal{S}}$. Similarly to the dual case, $(\widetilde{\mathcal{S}}, \subseteq)$ is itself a lattice with

$$P \wedge Q = \cup\{X \in \mathcal{S} \mid P \cap Q \subseteq X\}.$$

In particular, we can take $\overline{\mathcal{S}}_V = \{\overline{\mathrm{St}}(v) \mid v \in V\}$ and consider the lattice $\widetilde{\overline{\mathcal{S}}_V}$, which we call the *dual lattice of closed stars* of $G$.

**Theorem 9.4** *Let $G = (V, E)$ be a finite graph. Then* $\text{c-rk}\,\overline{G} = \mathrm{ht}\,\widetilde{\overline{\mathcal{S}}_V}$.

**Proof**. We know that $\text{c-rk}\,\overline{G}$ is the maximum length $n$ of a chain of the form

$$V \supset \mathrm{St}_{\overline{G}}(v_1) \supset \mathrm{St}_{\overline{G}}(v_1, v_2) \supset \ldots \supset \mathrm{St}_{\overline{G}}(v_1, \ldots, v_n) = \emptyset. \tag{19}$$

Now

$$\mathrm{St}_{\overline{G}}(v_1, \ldots, v_i) = \mathrm{St}_{\overline{G}}(v_1) \cap \ldots \cap \mathrm{St}_{\overline{G}}(v_i) = (V \setminus \overline{\mathrm{St}}_G(v_1)) \cap \ldots \cap (V \setminus \overline{\mathrm{St}}_G(v_i))$$

and so

$$V \setminus \mathrm{St}_{\overline{G}}(v_1, \ldots, v_i) = \overline{\mathrm{St}}_G(v_1) \cup \ldots \cup \overline{\mathrm{St}}_G(v_1).$$

Passing (19) to complement, it follows that $\text{c-rk}\,\overline{G}$ is the maximum length $n$ of a chain of the form

$$\emptyset \subset \overline{\mathrm{St}}_G(v_1) \subset \overline{\mathrm{St}}_G(v_1) \cup \overline{\mathrm{St}}_G(v_2) \subset \ldots \subset \overline{\mathrm{St}}_G(v_1) \cup \ldots \cup \overline{\mathrm{St}}_G(v_n) = V,$$

which is precisely $\mathrm{ht}\,\widetilde{\overline{\mathcal{S}}_V}$. $\square$

As an example, we can now apply this result to the computation of the c-rank of the complement of the Petersen graph $P$:

**Example 9.5** $\text{c-rk}\,\overline{P} = 5$.

Write $P = (V, E)$, $\mathrm{St}(v) = \mathrm{St}_P(v)$ and $\overline{\mathrm{St}}(v) = \overline{\mathrm{St}}_P(v)$. Assume that

$$\emptyset \subset \overline{\mathrm{St}}(v_1) \subset \overline{\mathrm{St}}(v_1) \cup \overline{\mathrm{St}}(v_2) \subset \ldots \subset \overline{\mathrm{St}}(v_1) \cup \ldots \cup \overline{\mathrm{St}}(v_n) = V,$$

is a chain of maximum length in $\widetilde{\overline{\mathcal{S}}_V}$. We claim that

$$|\overline{\mathrm{St}}(v_1) \cup \overline{\mathrm{St}}(v_2) \cup \overline{\mathrm{St}}(v_3)| \geq 8. \tag{20}$$

Since $P$ is cubic, it follows from Proposition 7.1(iii) that $|\mathrm{St}(v_1) \cup \mathrm{St}(v_2)| \geq 5$. If $v_1 \text{ — } v_2$ is not an edge of $P$, then $|\overline{\mathrm{St}}(v_1) \cup \overline{\mathrm{St}}(v_2)| \geq 7$ and so (20) must hold in this case. Hence we may assume that $v_1$ and $v_2$ are adjacent in $P$. Since $P$ has no triangles, then $\mathrm{St}(v_1, v_2) = \emptyset$ and so $|\overline{\mathrm{St}}(v_1) \cup \overline{\mathrm{St}}(v_2)| = |\mathrm{St}(v_1) \cup \mathrm{St}(v_2)| = 6$. Suppose that $|\overline{\mathrm{St}}(v_1) \cup \overline{\mathrm{St}}(v_2) \cup \overline{\mathrm{St}}(v_3)| < 8$. Then $|\overline{\mathrm{St}}(v_3) \cap \overline{\mathrm{St}}(v_i)| \geq 2$ for some $i \in \{1, 2\}$. If $v_i \text{ — } v_3$ is an edge, we get a triangle in $P$; if $v_i \text{ — } v_3$ is not an edge, we get a square in $P$, a contradiction in any case since $\mathrm{gth}\,P = 5$. Therefore (20) holds and so $n \leq 5$.

It is a simple exercise to produce a chain of length 5 in $\widetilde{\overline{\mathcal{S}}_V}$, hence $\text{c-rk}\,\overline{P} = 5$.

## 10 Open questions

Here is a list of open questions, some mentioned in the preceding text:

1. Characterize those finite graphs $G$ whose lattice of flats are distributive, modular, semimodular, satisfy the Jordan-Dedekind chain condition, or are extremal lattices.

   For extremal lattices see [14]. The most important questions are for the Jordan-Dedekind chain condition and for the extremal lattices which need not satisfy the Jordan-Dedekind chain condition. See Theorems 7.4 and 7.5 for some results.

2. Which hereditary collections have Boolean representations?

   See Subsection 2.3 and [11] for definitions. All matroids have boolean representations (see [12]), but what else? This will be the topic of the future paper [18], which will also carry out a similar analysis like Section 5 but for higher c-rank.

3. When do two graphs have isomorphic lattices of flats?

4. Which matroids arise as the c-independent sets of a graph? Prove not all matroids arise this way.

5. When is $\operatorname{Geo} G$, for (cubic) $G \in \mathrm{SC3}$ realizable as lines in Euclidean space (i.e. realizable affinely)? Analyse further the map which associates to a non bipartite cubic $C \in \mathrm{SC3}$ the bipartite cubic $\operatorname{Levi} \operatorname{Geo} C$.

6. Compare the results of this paper with those of Brijder and Traldi in [2].

7. Extend the analysis of graphs in this paper to Moore graphs, generalized Petersen graphs, etc. What is $\operatorname{Mat} M$ when $M$ is the Moore mystery graph of girth 5 (which may or may not exist) on 57 vertices? See [26].

8. What is the smallest number of edges we can have in a graph with $n$ vertices to maximize (minimize) $\operatorname{c-rk} G + \operatorname{c-rk} \overline{G}$?

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