

Optimal Control Problems with Mixed and Pure State Constraints*

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Abstract

This paper provides necessary conditions of optimality for optimal control problems, in which the pathwise constraints comprise both ‘pure’ constraints on the state variable and also ‘mixed’ constraints on control and state variables. The proofs are along the lines of earlier analysis for mixed constraint problems, according to which the Clarke’s theory of ‘stratified’ necessary conditions is applied to a modified optimal control problem resulting from absorbing the mixed constraint into the dynamics; the difference here is that necessary conditions which now take account of the presence of pure state constraints are applied to the modified problem. Necessary conditions are given for a rather general formulation of the problem containing both forms of the constraints, and then these are specialized to apply to problems having special structure. While combined pure state and mixed control/state problems have been previously treated in the literature, the necessary conditions in this paper are proved under less restrictive hypotheses and for novel formulations of the constraints.

Key Words: Optimal Control, Maximum Principle, State Constraints, Nonsmooth Analysis.

AMS Subject Classifications: 49K15, 49K21

1 Introduction

The Pontryagin Maximum Principle (PMP), dating from the 1950’s, is a cornerstone of Optimal Control Theory [13]. In its original form, the PMP provides necessary conditions that a solution $x(\cdot) : [a, b] \rightarrow \mathbb{R}^n$ to a controlled differential equation (named a state trajectory) and the associated control function $u(\cdot) : [a, b] \rightarrow \mathbb{R}^m$, minimize a cost function, subject to a pathwise constraint on the control function $u(\cdot)$:

$$u(t) \in U \quad \text{for all } t \in [a, b]$$

and to a constraint on the endpoints of $x(\cdot)$:

$$(x(a), x(b)) \in E,$$

in which $U \subset \mathbb{R}^m$ and $E \subset \mathbb{R}^n \times \mathbb{R}^n$ are given sets. The PMP features an absolutely continuous function $p(\cdot)$ (the costate trajectory) that satisfies a differential equation, named the costate equation (or a related differential inclusion, if the data is non-smooth) and boundary conditions named the transversality conditions. Information is then provided about the optimal control: ‘the Hamiltonian evaluated along

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the optimal state trajectory and costate trajectory is maximized at the minimizing control function, pointwise in time’.

In subsequent research aimed at extending the PMP to take account of pathwise constraints also for state trajectories, it was early recognised that different techniques were required to deal with the two types of constraints:

1. *mixed control/state constraints:*

$$\phi^1(t, x(t), u(t)) \leq 0 \quad \text{and} \quad \phi^2(t, x(t), u(t)) = 0, \quad (1.1)$$

2. *pure state constraints:* $h(t, x(t)) \leq 0$,

involving the given functions $\phi^1(., ., .)$, $\phi^2(., ., .)$ and $h(., .)$.

In the mixed control/state constraint literature, typified by [10], [8] and [9], the aim is usually to show that the constraint can be accommodated, when a suitable constraint qualification is satisfied, by means of Lagrange multipliers, namely suitably bounded measurable functions $\zeta_1(.)$ and $\zeta_2(.)$ on $[a, b]$. The modified PMP is, once again, expressed in terms of an absolutely continuous costate trajectory $p(.)$ and takes the form of the original PMP, when the cost is modified by the addition of the term:

$$\int_a^b \phi^1(t, x(t), u(t)) \cdot \zeta_1(t) dt + \int_a^b \phi^2(t, x(t), u(t)) \cdot \zeta_2(t) dt,$$

together with ‘complementary slackness’ conditions. Implicit in the derivation of this modified PMP is the notion that, under a suitable ‘constraint qualification’ hypothesis, the mixed constraints can be absorbed into the dynamic constraint in such a manner that the hypotheses on the dynamic constraint invoked in the original PMP continue to be satisfied; application of the original PMP to the reformulated optimal control problem, with an ‘absorbed’ dynamic constraint, yields the desired, modified, necessary conditions.

It might be thought that the theory of necessary conditions for pure state constraints ‘ $h(t, x(t)) \leq 0$ ’ could simply be subsumed into that for mixed state constraints, by setting $\phi_1(t, x, u) = h(t, x)$. But this is not possible (if we are to use the proof techniques referred to above) because, when $\phi_1(t, x, u)$ does not depend on $u(.)$, the constraint qualification is violated, in consequence of which the mixed constraint cannot be used to eliminate control variable components and thereby generate an equivalent optimal control problem to which standard versions of the PMP can be applied. This is the reason why ‘pure’ state constraints have been treated separately from ‘mixed control/state constraints’, and accounts for the fundamentally different nature of modifications to the PMP that have been derived for optimal control problems with pure state constraints: the modified PMP, in this case, is formulated in terms of a ‘measure multiplier’ and a, possibly discontinuous, costate trajectory $q(.)$ that is of bounded variation. Papers treating pure state constraints and involving a discontinuous costate trajectory go back to Gamkredidze’s and Warga’s work of the 1960’s; references to the early literature appear, for example, in [17]. Necessary conditions to cover the combined occurrence of the two types of constraints has been treated by the Dubovitski-Milyutin school [12], [9], and also by Makowski and Neustadt [11].

A breakthrough in the development of new tools to tackle a variety of differently structured optimal control problems, with nonsmooth data, was the publication of Clarke’s memoir [4], the centrepiece of which was ‘stratified’ necessary conditions for optimal control problems, whose dynamic constraint took the form of a differential inclusion. The memoir introduced a new, and very useful, way of capturing the requisite Lipschitz continuity-like properties of the differential inclusion for the validation of the Euler-Lagrange inclusion, and related necessary conditions, namely the ‘bounded slope’ hypothesis. Subsequently, Clarke and Pinho [6] examined the implications of these tools for ‘mixed constraint problems’.

Necessary conditions were provided for very general formulations of mixed constraint problems, by showing that they could be reduced to the optimal control problems treated in [4], by absorbing the mixed constraints into the dynamic constraint in such a manner that the bounded slope hypothesis continued to be satisfied. The results in [6] improve on earlier-derived mixed constraint conditions, as described in detail in [6], in some respects even when attention is restricted to problems with smooth data. However the presence of pure state constraints is excluded from the necessary conditions in [6]:

‘. . . the bounded slope condition excludes unilateral state constraints . . . It is well-known that the necessary conditions of the type given . . . fail, and their appropriate extensions involve measures and costate trajectories that are discontinuous’.

The aim of this paper is to provide extensions of the necessary conditions for mixed control/state constraints problems of [6], to allow also for pure state constraints. A key tool is the set of stratified necessary conditions of [1], that generalize the main necessary conditions in Clarke’s memoir, to allow for unilateral state constraints. The proof technique for deriving necessary conditions, applying to optimal control problems with both mixed control/state and pure state constraints, is to reduce the problems to ones involving pure state constraints alone, by absorbing the mixed constraints into the dynamic constraint, and applying the stratified pure-state-constraint necessary conditions of [1].

The necessary conditions for combined mixed control/state and pure state constraints of this paper reduce (in important respects) to the main necessary conditions in [6] and [5] for mixed constraints alone (both in a general setting and when the mixed constraint has explicit representations including that considered in [5]), following removal of the pure state constraint. [6, Section 8] gives details of the respects in which the necessary conditions in [6] improve on earlier necessary conditions, regarding hypotheses on the mixed state constraint data. These improvements are all the more in evidence in this paper, since the framework is broadened to allow also pure state constraints. On the other hand, this paper extends the earlier necessary conditions involving both mixed constraints and pure state constraints in [9] and [11], by allowing non-smooth data, by adopting a very general formulation of the mixed constraints and by permitting general endpoint constraints. The respects in which this paper improves on the more recent paper [2], which also aims to introduce state constraints into the necessary conditions of [6], are discussed in Section 2 in detail below.

Finally we mention that the ‘stratified’ necessary conditions in this paper are expressed in terms of an arbitrary radius multifunction $R(t)$, in place of the balls $\dot{x}(t) + r(t)\text{int } \mathbb{B}$ involving the radius function $r(t)$, as in [6]. This extra degree of generality in these necessary conditions for the general formulation of the optimal control problem of Section 2, simplifies the application of these conditions to take account of more structured formulations of the problem in Section 3, as compared with the analysis in [5].

Notation: The Euclidean norm of a vector $x \in \mathbb{R}^n$ is $|x|$. $\mathbb{B}$ indicates the open unit ball in $\mathbb{R}^n$ and the distance function of a point $x \in \mathbb{R}^n$ from a set $A \subset \mathbb{R}^n$ is defined as

$$d_A(x) := \inf\{|x - y| : y \in A\}.$$

The convex hull of the set A is written $\text{co } A$. Given a multifunction $F(\cdot) : \mathbb{R}^n \rightsquigarrow \mathbb{R}^k$, we denote by $\text{Gr } F(\cdot)$ the graph of $F(\cdot)$.

$W^{1,1}([a, b]; \mathbb{R}^n)$ denotes the space of absolutely continuous functions $x : [a, b] \rightarrow \mathbb{R}^n$, equipped with the norm

$$\|x\|_{W^{1,1}} := |x(a)| + \int_a^b |\dot{x}(t)| dt.$$

We write $W^{1,1}$ in place of $W^{1,1}([a, b]; \mathbb{R}^n)$ when no ambiguity arises. $NBV^+[a, b]$ denotes the space of increasing, real-valued functions $\mu(\cdot)$ on $[a, b]$ of bounded variation, vanishing at the point a and right

continuous on (a, b) . The total variation of a function $\mu(\cdot) \in NBV^+[a, b]$ is written $\|\mu\|_{T.V.}$. As is well known, each point $\mu(\cdot) \in NBV^+[a, b]$ defines a Borel measure on $[a, b]$. This associated measure is also denoted μ .

We make use of some constructs from nonsmooth analysis, described in detail, for example, in [7, 17]: given a closed set $E \subset \mathbb{R}^n$ and $x \in E$, the *proximal normal cone of E at x* is

$$N_E^P(x) := \{\zeta \in \mathbb{R}^n : \exists \epsilon > 0 \text{ and } M > 0 \text{ s.t. } \zeta \cdot (y - x) \leq M|x - y|^2 \text{ for all } y \in E \cap x + \epsilon\mathbb{B}\}.$$

The *limiting normal cone of E at x* is

$$N_E^L(x) := \left\{ \lim_{i \rightarrow \infty} \zeta_i : \zeta_i \in N_E^P(x_i) \text{ and } x_i \in E \text{ for all } i, \text{ and } x_i \rightarrow x \right\}.$$

The *Clarke normal cone of E at x* is $N_E^C(x) := co N_E^L(x)$. If E is convex, the two normal cones coincide with the normal cone of convex analysis, and are written $N_E^C(x)$. More generally, a set E is said to be *regular* at $x \in E$ if $N_E^L(x) = N_E^C(x)$.

The *Clarke tangent cone of E at x* is the polar set of the limiting normal cone, i.e.

$$T_E^C(x) := \{\eta \in \mathbb{R}^n \mid \langle \eta, \zeta \rangle \leq 0 \text{ for all } \zeta \in N_E^L(x)\}.$$

Given a lower semicontinuous function $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and a point $x \in \text{dom } f(\cdot) := \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$, the *proximal subdifferential of $f(\cdot)$ at x* is the set

$$\partial_P f(x) := \left\{ \zeta \in \mathbb{R}^n : \exists \sigma > 0 \text{ and } \epsilon > 0 \text{ such that } \forall y \in x + \epsilon\mathbb{B} \right. \\ \left. f(y) - f(x) \geq \langle \zeta, y - x \rangle - \sigma|y - x|^2 \right\}.$$

The *limiting subdifferential of $f(\cdot)$ at x* is

$$\partial_L f(x) := \left\{ \lim_{i \rightarrow \infty} \zeta_i : \zeta_i \in \partial_P f(x_i), x_i \rightarrow x, f(x_i) \rightarrow f(x) \right\}.$$

In the case $f(\cdot)$ is Lipschitz continuous on a neighborhood of x , we define Clarke generalized gradients $\partial_C f(x) := co \partial_L f(x)$. We say that $f(\cdot)$ is *strictly differentiable at x* if $f(\cdot)$ is Fréchet differentiable at x , Lipschitz continuous on a neighborhood of x and $\{\nabla f(x)\} = \partial_C f(x)$.

2 Necessary Conditions for a General Problem

We consider the following optimal control problem:

$$(P) \left\{ \begin{array}{l} \text{Minimize } J(x(\cdot), u(\cdot)) := \ell(x(a), x(b)) + \int_a^b \Lambda(t, x(t), u(t)) dt \\ \text{over processes } (x(\cdot), u(\cdot)) \text{ such that} \\ \quad \dot{x}(t) = f(t, x(t), u(t)), \quad \text{a.e. } t \in [a, b], \\ \quad u(t) = (v(t), w(t)), \quad w(t) \in W(t), \quad \text{a.e. } t \in [a, b], \\ \quad (x(t), v(t)) \in S(t, w(t)), \quad \text{a.e. } t \in [a, b], \\ \quad h(t, x(t)) \leq 0, \quad \text{for all } t \in [a, b], \\ \quad (x(a), x(b)) \in E. \end{array} \right.$$

the data for which comprises integers $n > 0$, $m > 0$, $m_v \geq 0$ and $m_w \geq 0$ such that $m = m_v + m_w$, functions $\ell(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, $\Lambda(\cdot, \cdot, \cdot) : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$, $f(\cdot, \cdot, \cdot) : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $h(\cdot, \cdot) : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$, and sets $S \subset \mathbb{R}^{1+n+m}$, $E \subset \mathbb{R}^n \times \mathbb{R}^n$ and $W \subset \mathbb{R}^{1+m_w}$. (Interpret

$W(t) := \{w : (t, w) \in W\}$, $S(t, w) := \{(x, v) : (t, x, (v, w)) \in S\}$, etc.)

A *process* is a pair of functions $(x(\cdot), u(\cdot))$, in which $x(\cdot)$ is a $W^{1,1}([a, b]; \mathbb{R}^n)$ function and $u(\cdot) = (v(\cdot), w(\cdot)) : [a, b] \rightarrow \mathbb{R}^m$ is a measurable function, satisfying $\dot{x}(t) = f(t, x(t), u(t))$ and $w(t) \in W(t)$ a.e.. A process $(x(\cdot), u(\cdot))$ in which $x(\cdot)$ and $u(\cdot)$ satisfy the constraints of (P) , and for which $t \rightarrow \Lambda(t, x(t), u(t))$ is integrable, is called a *feasible process*.

As in [4], we consider local minimizers for problem (P) , with respect to a given ‘radius multifunction’ $R(\cdot)$ in the following sense:

Definition 2.1

- (a): a multifunction $R(\cdot) : [a, b] \rightsquigarrow \mathbb{R}^m$ is called a *radius multifunction* if $R(t)$ is a non-empty, open, convex set for each t and there exists $r_0 > 0$ such that $r_0\mathbb{B} \subset R(t)$ a.e..
- (b): Given a radius multifunction $R(\cdot) : [a, b] \rightsquigarrow \mathbb{R}^m$, a feasible process $(\bar{x}(\cdot), \bar{u}(\cdot))$ for (P) is said to be a $W^{1,1}$ local minimizer w.r.t. $R(\cdot)$ if there exists $\epsilon > 0$ such that

$$J(x(\cdot), u(\cdot)) \geq J(\bar{x}(\cdot), \bar{u}(\cdot))$$

over all the feasible processes $(x(\cdot), u(\cdot))$ satisfying

$$\|x - \bar{x}\|_{W^{1,1}} \leq \epsilon \text{ and } u(t) \in R(t) \text{ a.e..}$$

If $R(t) \equiv \mathbb{R}^m$, we simply say that ‘ $(\bar{x}(\cdot), \bar{u}(\cdot))$ is a $W^{1,1}$ local minimizer’.

We shall invoke the following hypotheses, in which $(\bar{x}(\cdot), (\bar{u}(\cdot) = (\bar{v}(\cdot), \bar{w}(\cdot)))$ is a given feasible process. For some $\epsilon > 0$:

(H1): $\ell(\cdot, \cdot)$ is Lipschitz continuous on a neighborhood of $(\bar{x}(a), \bar{x}(b))$. E is closed. W and S are $\mathcal{L} \times \mathcal{B}^{m'}$ measurable subsets of $[a, b] \times \mathbb{R}^{m_w}$ and $[a, b] \times \mathbb{R}^{m+n}$ respectively, where $\mathcal{L} \times \mathcal{B}^{m'}$ denotes the product σ -algebra of $\mathcal{L}$ and $\mathcal{B}^{m'}$, in which $\mathcal{L}$ denotes the Lebesgue subsets of $[a, b]$ and $\mathcal{B}^{m'}$ the Borel subsets of $\mathbb{R}^{m'}$ for $m' = m_w$ or $m + n$.

(H2): $h(\cdot, \cdot)$ is upper semicontinuous and there exists $k_h > 0$ such that

$$|h(t, x') - h(t, x)| \leq k_h |x' - x|,$$

for all t in $[a, b]$ and all $x', x \in \bar{x}(t) + \epsilon\mathbb{B}$.

(H3): for each $(x, v) \in \mathbb{R}^n \times \mathbb{R}^{m_v}$, $f(\cdot, x, (v, \cdot))$ and $\Lambda(\cdot, x, (v, \cdot))$ are $\mathcal{L} \times \mathcal{B}^{m_w}$ measurable. For a.e. $t \in [a, b]$ and every $w \in W(t)$ the functions $(x, v) \mapsto f(t, x, v, w)$ and $(x, v) \mapsto \Lambda(t, x, v, w)$ are Lipschitz continuous on a neighborhood of

$$((\bar{x}(t) + \epsilon\mathbb{B}) \times R(t, w)) \cap S(t, w),$$

where $R(t, w) := \{v : (v, w) \in R(t)\}$. The Lipschitz constants, which may depend on t and w , can be chosen to be $\mathcal{L} \times \mathcal{B}^{m_w}$ measurable functions. We denote these Lipschitz constants by the symbols $k_x^f(t, w)$, $k_v^f(t, w)$, $k_x^\Lambda(t, w)$, and $k_v^\Lambda(t, w)$, thus:

$$\begin{aligned} |\Lambda(t, x_1, v_1, w) - \Lambda(t, x_2, v_2, w)| &\leq k_x^\Lambda(t, w)|x_1 - x_2| + k_v^\Lambda(t, w)|v_1 - v_2|, \\ |f(t, x_1, v_1, w) - f(t, x_2, v_2, w)| &\leq k_x^f(t, w)|x_1 - x_2| + k_v^f(t, w)|v_1 - v_2| \end{aligned}$$

for any (x_1, v_1) and (x_2, v_2) in a neighbourhood of $((\bar{x}(t) + \epsilon\mathbb{B}) \times R(t, w)) \cap S(t, w)$ and $w \in W(t)$, a.e. $t \in [a, b]$.

(BS): ('Bounded Slope Condition') The set $S(t, w)$ is closed for each $w \in W(t)$, a.e. $t \in [a, b]$. There exists a measurable function $k_S(\cdot)$ such that, given any $w \in W(t)$ and $(x, v) \in ((\bar{x}(t) + \epsilon \mathbb{B}) \times R(t, w)) \cap S(t, w)$ we have

$$(\alpha, \beta) \in N_{S(t, w)}^P(x, v) \implies |\alpha| \leq k_S(t)|\beta|, \quad \text{a.e. } t \in [a, b],$$

and $k_S(t) \geq k_0 > 0$, a.e., for some $k_0 > 0$.

There follows a set of necessary conditions for a feasible process to be a $W^{1,1}$ local minimizer w.r.t. $R(\cdot)$, a given radius multifunction. For $\lambda^0 \geq 0$ define the Hamiltonian:

$$H_{\lambda^0}(t, x, v, w, p) := p \cdot f(t, x, v, w) - \lambda^0 \Lambda(t, x, v, w).$$

Theorem 2.2 (General Necessary Conditions) *Let $R(\cdot)$ be a radius multifunction and let $(\bar{x}(\cdot), \bar{u}(\cdot)) = (\bar{v}(\cdot), \bar{w}(\cdot))$ be a $W^{1,1}$ local minimizer w.r.t. $R(\cdot)$. Assume hypotheses (H1)-(H3) and (BS) are satisfied. Assume also that*

$$k_x^f(t, \bar{w}(t)), \quad k_x^\Lambda(t, \bar{w}(t)), \quad k_S(t)[k_u^f(t, \bar{w}(t)) + k_u^\Lambda(t, \bar{w}(t))] \text{ are integrable,} \quad (2.1)$$

and

$$\exists \eta > 0 \text{ such that } \bar{u}(t) + \eta k_S(t) \mathbb{B} \subset R(t), \text{ for a.e. } t \in [a, b]. \quad (2.2)$$

Then there exists a multiplier set $(p(\cdot), \mu(\cdot), \lambda^0) \in W^{1,1} \times NBV^+[a, b] \times \mathbb{R}^+$ and a μ -integrable function $m(\cdot) : [a, b] \rightarrow \mathbb{R}^n$ such that

$$(a): \lambda^0 + \|p(\cdot)\|_{L^\infty} + \|\mu\|_{T.V.} = 1,$$

$$(b): (-\dot{p}(t), 0) \in \partial_C H_{\lambda^0}(t, \cdot, \cdot, \bar{w}(t), q(t))(\bar{x}(t), \bar{v}(t)) - N_{S(t, \bar{w}(t))}^C(\bar{x}(t), \bar{v}(t)) \quad \text{a.e.},$$

$$(c): m(t) \in \text{co} \partial_x^> h(t, \bar{x}(t)) \quad \mu\text{-a.e. and } \text{supp}\{\mu\} \subset \{t : h(t, \bar{x}(t)) = 0\},$$

$$(d): (q(a), -q(b)) \in \lambda^0 \partial_L \ell(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)).$$

$$(e): \text{For any } u = (v, w) \in R(t) \cap S(t, \bar{x}(t)) \text{ such that } w \in W(t),$$

$$H_{\lambda^0}(t, \bar{x}(t), \bar{u}(t), q(t)) \geq H_{\lambda^0}(t, \bar{x}(t), u, q(t)) \quad \text{a.e. } t \in [a, b].$$

$$\text{Here} \quad q(t) := \begin{cases} p(t) + \int_{[a, t]} m(s) \mu(ds) & \text{if } t \in (a, b] \\ p(a) & \text{for } t = a \end{cases} \quad (2.3)$$

and the 'hybrid' generalized gradient $\partial_x^> h(t, x)$ is the set

$$\partial_x^> h(t, x) := \left\{ \lim_{(t_i, x_i) \rightarrow (t, x)} \nabla h(t_i, x_i) : \nabla h(t_i, x_i) \text{ exists and } h(t_i, x_i) > 0 \text{ for each } i \right\}.$$

A proof of Thm. 2.2 is given in Section 4. These conditions are expressed in terms of the function $q(\cdot)$ that is right continuous on (a, b) . The assertions of the theorem remain the same if $q(\cdot)$ is replaced by the function $q'(\cdot)$, which coincides with $q(\cdot)$ at the end-times, and which, at interior points $t \in (a, b)$ is left continuous and given by $q'(t) = p(t) + \int_{[a, t]} m(s) \mu(ds)$, as in [17].

Comments.

- (i): The requirement that $k_S(\cdot)$ in (BS) satisfies $k_S(t) \geq k_0$, for some $k_0 > 0$, is essential. The minorization condition might seem superfluous because, if it is violated, we can always arrange that (BS) is satisfied by addition of a positive constant. But this ignores the fact that $k_S(\cdot)$ is also required to satisfy the condition $\bar{u}(t) + \eta k_S(t) \mathbb{B} \subset R(t)$, and addition of a positive constant to $k_S(\cdot)$ might result in violation of this latter condition. A simple counter example can be constructed along the lines of [1, Ex. 1] illustrating that the assertions of Th. 2.2 are in general false if the minorization condition is omitted.

- (ii): In the formulation (P) of the optimal control problem, we interpret ‘ $(x(t), v(t)) \in S(t, w(t))$ a.e.’ as a mixed control/state constraint, as in the previous work of Clarke and Pinho [6]. This description of the constraint reduces to the standard mixed constraint description (1.1), in the form of a set of functional inequality and equality constraints, when the data does not depend on w , so that we can identify u with the control component v alone, and $S(t)$ (no longer dependent on w) is chosen to be

$$S(t) := \{(x, u) : \phi^1(t, x, u) \leq 0 \text{ and } \phi^2(t, x, u) = 0\}. \quad (2.4)$$

While the special case (2.4) covers most applications involving mixed control/state constraints, the more general description ‘ $(x(t), v(t)) \in S(t, w(t))$ ’ is a useful starting point for formulating hypotheses under which necessary conditions of optimality may be derived for a wide variety of problems.

- (iii): The pure state constraint description ‘ $h(t, x(t)) \leq 0$ ’ of problem (P) , involving the function $h(t, x)$ which is upper semicontinuous and uniformly Lipschitz w.r.t. the x variable, has been widely employed since its introduction in [3], because of its versatility. We recall that it subsumes constraints imposed on a given closed subset $I \subset [a, b]$ of the time interval, of the form

$$x(t) \in A(t) \quad \text{for all } t \in I,$$

in which $A(\cdot) : [a, b] \rightarrow \mathbb{R}^n$ is a given upper semicontinuous multifunction, taking values closed subsets. In this case we choose

$$h(t, x) := \begin{cases} d_{A(t)}(x) & \text{if } t \in I \\ -1 & \text{otherwise.} \end{cases}$$

- (iv): Following [9] and [6], we have partitioned the control variable as $u = (v, w)$; the components v and w will each have a different status, regarding the hypotheses that are imposed on them for the derivation of necessary conditions. The purpose here is to capture within a single framework, and thereby achieve greater generality, problems involving a simple set inclusion constraint on the control ‘ $w(t) \in W(t)$ ’, problems in which the control/state variables (x, v) are required to satisfy, say, pathwise functional equality and inequality constraints, and a combination of such problems. A version of the theorem, valid for the case $m_w = 0$ is obtained by eliminating, in an obvious way, all reference to w variable in the theorem statement and accompanying hypotheses. Likewise, a version valid for the case $m_v = 0$ is obtained by removing all reference to the v variable.
- (v): Thm. 2.2 extends [6, Thm. 3.2], to allow for the presence of additional, pure state constraints. Notice that, as in [6], Thm. 2.2 incorporates information about the dependence of the Hamiltonian on the control, along a minimizing state trajectory and costate trajectory both in the form of the extended costate inclusion (c.f. [14]) and the Weierstrass condition (conditions (b) and (e) of the theorem statement). Earlier necessary conditions for problem (P) , involving both pure and mixed constraints, for problems with nonsmooth data and for which ‘bounded slope’ type hypotheses on the mixed constraint are invoked, appear in [2]. Our necessary conditions, embodied in Thm. 2.2, broadly confirm the assertions of [2, Thm. 2] (the convergence analysis in [2], justifying the costate inclusion, is incomplete) and extend them in numerous respects. The conditions in this paper allow for a more general description of the mixed constraint (not just one involving mixed equality and inequality constraints as in [2]), for partitioned control variables $u = (v, w)$ in which very much weaker hypotheses are imposed on the data regarding w dependence and for ‘stratification’ (i.e. minimizers w.r.t. a radius multifunction). In our applications of the necessary conditions in the next section, we also allow non-compact control constraint sets.

3 Necessary Condition for Mixed Constraints in Explicit Form

Thm. 2.2 of the previous section provides necessary conditions for optimal control problems involving mixed state and control constraints and pure state constraints, when the mixed constraint is captured

by the condition

$$(x(t), u(t)) \in S(t) ,$$

in which $S(\cdot)$ is a given multifunction. Thm. 2.2 can be used as a starting point for the derivation of necessary conditions, in which the set $S(\cdot)$ is of the form

$$S(t) := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m : \phi(t, x, u) \in \Phi(t)\}, \quad (3.1)$$

in which $\phi : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^\kappa$ and $\Phi(\cdot) : [a, b] \rightsquigarrow \mathbb{R}^\kappa$ is a Lebesgue measurable multifunction that takes values closed sets. The most widely considered special case of (3.1) (functional inequality and equality constraints) is that when, for some integers $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ such that $\kappa_1 + \kappa_2 = \kappa$, $\phi(\cdot, \cdot, \cdot)$ is partitioned as $\phi(\cdot, \cdot, \cdot) = (\phi^1(\cdot, \cdot, \cdot), \phi^2(\cdot, \cdot, \cdot))$ into $\mathbb{R}^{\kappa_1}$ and $\mathbb{R}^{\kappa_2}$ valued functions respectively and

$$\Phi(t) \equiv \overbrace{(-\infty, 0] \times \dots \times (-\infty, 0]}^{\kappa_1} \times \overbrace{\{0\} \times \dots \times \{0\}}^{\kappa_2}. \quad (3.2)$$

The idea is to express the conditions directly in terms of $\phi(\cdot, \cdot)$ and $\Phi(\cdot)$, under hypotheses that generalize earlier conditions governing the u -dependence of $\phi(x, u)$ (classical rank conditions or, more generally, Mangasarian-Fromowitz type conditions on the gradients of this function, and ‘surjectivity conditions’ originating in the work of Schwarzkopf [15].) We consider henceforth the variant of problem (P) , labelled (P_S) , in which we no longer distinguish block components v and w of the control variable u , and when we impose both a mixed constraint of the type (3.1), an implicit control constraint $u \in U(t)$ (in which $U(\cdot) : [a, b] \rightsquigarrow \mathbb{R}^m$), and a pure state constraint:

$$(P_S) \begin{cases} \text{Minimize } \ell(x(a), x(b)) + \int_a^b \Lambda(t, x(t), u(t)) dt \\ \text{over processes } (x(\cdot), u(\cdot)) \text{ satisfying} \\ \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e.,} \\ \phi(t, x(t), u(t)) \in \Phi(t) \text{ and } u(t) \in U(t) \text{ a.e.,} \\ h(t, x(t)) \leq 0 \text{ for all } t \in [a, b], \\ (x(a), x(b)) \in E. \end{cases}$$

The following theorem provides necessary conditions for optimal control problems involving mixed constraints formulated as (3.1), under hypotheses on the mixed constraint data, expressed directly in terms of $\phi(\cdot, \cdot, \cdot)$ and $\Phi(\cdot)$, namely:

(H3)': $\Phi(\cdot)$ is a Lebesgue measurable multifunction taking values non-empty, closed subsets of $\mathbb{R}^\kappa$. $U(\cdot)$ has $\mathcal{L} \times \mathcal{B}^m$ measurable graph and takes values closed sets. $f(t, \cdot, \cdot)$, $\Lambda(t, \cdot, \cdot)$ and $\phi(t, \cdot, \cdot)$ are Lipschitz continuous on a neighbourhood of $(\bar{x}(t) + \epsilon \mathbb{B} \times R(t)) \cap S(t)$, for a.e. $t \in [a, b]$,

in which $(\bar{x}(\cdot), \bar{u}(\cdot))$ is the process of interest, and (for a given radius multifunction $R(\cdot)$), the following constraint qualification is imposed, in place of the ‘bounded slope’ condition (BS):

(CQ): There exists a measurable function $M(\cdot) : [a, b] \rightarrow \mathbb{R}$ such that, for every $x \in \bar{x}(t) + \epsilon \mathbb{B}$ and $u \in R(t) \cap U(t)$ satisfying $\phi(t, x, u) \in \Phi(t)$,

$$\lambda \in N_{\Phi(t)}^L(\phi(t, x, u)), (\alpha, \beta) \in \partial_L (\lambda \cdot \phi)(t, \cdot, \cdot)(x, u) + \{0\} \times N_{U(t)}^L(u) \implies |\lambda| \leq M(t)|\beta| \quad a.e..$$

Theorem 3.1 (Explicit Mixed Constraints Conditions I) *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a $W^{1,1}$ local minimizer for (P_S) , w.r.t. some radius multifunction $R(\cdot)$. Assume that, for some $\epsilon > 0$, hypotheses (H1), (H2), (H3') and (CQ) are satisfied. Assume also*

$$k_x^f(t), k_x^\Lambda(t), M(t)k_x^\phi(t)[k_u^f(t) + k_u^\Lambda(t)] \text{ are integrable}$$

and there exists $\eta > 0$ such that $\bar{u}(t) + \eta M(t)k_x^\phi(t)\mathbb{B} \subset R(t)$ a.e..

Then there exists a multiplier set $(p(\cdot), \mu, \lambda_0) \in W^{1,1} \times NBV^+ \times \mathbb{R}^+$ and a bounded μ -measurable function $m(\cdot) : [a, b] \rightarrow \mathbb{R}^n$ for which all the assertions of Thm. 2.2 are valid, when (P_S) is interpreted as a special case of (P) , in which the w component of the control is absent (write u for v) and $S(t) := \{(x, u) \mid (\phi(t, x, u), u) \in \Phi(t) \times U(t)\}$.

If in addition $\phi(t, \cdot, \cdot)$ is strictly differentiable at $(\bar{x}(t), \bar{u}(t))$ and $U(t)$ and $\Phi(t)$ are regular at $\bar{u}(t)$ and at $\phi(t, \bar{x}(t), \bar{u}(t))$ respectively, a.e. $t \in [a, b]$, then the costate inclusion condition (b) in Thm. 2.2 can be replaced by

$$(-\dot{p}(t), 0) \in \partial_C \{q(t) \cdot f(t, \cdot, \cdot) - \lambda^0 \Lambda(t, \cdot, \cdot) - \langle \lambda(t), \phi(t, \cdot, \cdot) \rangle\}(\bar{x}(t), \bar{u}(t)) - \{0\} \times N_{U(t)}^C(\bar{u}(t)) \text{ a.e.}, \quad (3.3)$$

in which $\lambda(\cdot) : [a, b] \rightarrow \mathbb{R}^\kappa$ is some measurable function that satisfies

$$\lambda(t) \in N_{\Phi(t)}^C(\phi(t, \bar{x}(t), \bar{u}(t))) \text{ a.e.} \quad (3.4)$$

Furthermore, for $M(\cdot)$ as in (CQ),

$$|\lambda(t)| \leq M(t) \{|q(t)|k_u^f(t) + \lambda_0 k_u^\Lambda(t)\} \text{ a.e.}$$

A proof of Thm. 3.1 is given in Section 5.

Comments.

- (i): As we have earlier observed, the standard formulation of mixed constraints, in the form of a combination of functional equality and inequality constraints, is captured by choosing $\phi(\cdot, \cdot, \cdot)$ and $\Phi(\cdot)$ according to (3.1) and (3.2). In this case, the necessary conditions of Thm. 3.1 take a familiar form, involving a multiplier function $\lambda(\cdot) = (\lambda^1(\cdot), \lambda^2(\cdot))$, partitioned into multiplier functions associated with the inequality constraint, $\lambda^1(\cdot)$, and with the equality constraint, $\lambda^2(\cdot)$. If the data is smooth, condition (3.3) becomes the costate equation with added multiplier terms

$$-\dot{p}(t) = q(t) \cdot f_x(t, \bar{x}(t), \bar{u}(t)) - \lambda^1(t) \cdot \phi_x^1(t, \bar{x}(t), \bar{u}(t)) - \lambda^2(t) \cdot \phi_x^2(t, \bar{x}(t), \bar{u}(t))$$

(as, for example, in [8]) and (3.4) is simply the complementary slackness condition, relating to the inequality constraint multiplier $\lambda^1(\cdot)$:

$$\lambda_i^1(t) \geq 0 \text{ and } \phi_i^1(t, \bar{x}(t), \bar{u}(t)) < 0 \implies \lambda_i^1(t) = 0 \text{ for } i = 1, \dots, \kappa_1, \text{ a.e.}$$

- (ii): Necessary conditions for nonsmooth optimal control problems with mixed constraints formulated as ' $\phi(t, x(t), u(t)) \in \Phi(t)$ ' and in which constraint qualification (CQ) is invoked, are given in [6]. Thm. 2.2 extends [6, Thm. 4.3] to allow also for the presence of a pure state constraint ($h(t, x(t)) \leq 0$).

Typically in the earlier mixed constraints literature, when $\Phi(\cdot)$ is given by (3.2), necessary conditions are derived under hypotheses on gradients of the function $\phi(t, x, u)$ w.r.t. the x and u variables, which ensure satisfaction of hypothesis (CQ). An exception are necessary conditions due to Schwarzkopf [15], [16] in which hypotheses imposed on the gradients of ϕ are replaced by a convexity hypothesis on the generalized velocity sets involved, together with a surjectivity hypothesis on the mixed constraint functional $\phi(\cdot, \cdot, \cdot)$.

The following theorem provides necessary conditions for optimal control problem (P_S) , involving both a mixed constraint and an implicit control constraint under regularity hypotheses on the data (labelled (H3'')) that replace (H3')), and under hypotheses imposed on the mixed constraint data (labelled (SC) 'Schwarzkopf Condition'), which retain the character of the hypotheses in [15], namely

- (H3'') (a) the function $(t, u) \mapsto (f(t, x, u), \phi(t, x, u))$ is $\mathcal{L} \times \mathcal{B}^m$ measurable for each x , $U(\cdot)$ has $\mathcal{L} \times \mathcal{B}^m$ measurable graph. $\Phi(\cdot)$ is a Lebesgue measurable multifunction taking values non-empty, closed subsets of $\mathbb{R}^k$,

- (b) there exists a function $k_f(\cdot) \in L^1$ such that, for every $u \in U(t)$ the function $f(t, \cdot, u)$ is Lipschitz continuous with Lipschitz constant $k_f(t)$ on $\bar{x}(t) + \epsilon\mathbb{B}$, a.e.,
- (c) there exists a constant k_ϕ such that for a.e. t , every $u \in U(t)$, the function $\phi(t, \cdot, u)$ is Lipschitz continuous with Lipschitz constant k_ϕ on $\bar{x}(t) + \epsilon\mathbb{B}$, and this function is strictly differentiable at $\bar{x}(t)$,

and

(SC) There exists an integrable function $r(\cdot)$, and a constant $\delta > 0$ such that the multifunction

$$U_r(t) := \{u \in U(t) : |f(t, \bar{x}(t), u)| \leq r(t) \text{ a.e. } \}$$

satisfies

$$\phi(t, \bar{x}(t), \bar{u}(t)) + \delta\mathbb{B} \subset \phi(t, \bar{x}(t), U_r(t)) - \mathcal{T}_t^\eta \text{ a.e.} \quad (3.5)$$

where, for each t , $\mathcal{T}_t^\eta$ is a subset of $\mathbb{R}^\kappa$ satisfying

$$\mathcal{T}_t^\eta \subset \bigcap_{z \in S_t^\eta} \left(T_{\Phi(t)}^C(z) \right)$$

in which

$$S_t^\eta := \{z \in \Phi(t) \mid |z - \phi(t, \bar{x}(t), \bar{u}(t))| \leq \eta \times (1 + d_r(t))\}$$

for some $\eta > 0$. Here $d_r(\cdot)$ is the (possibly infinite valued) function

$$d_r(t) := 2 \times \sup\{|\phi(t, \bar{x}(t), u)| \mid u \in U_r(t)\}. \quad (3.6)$$

(Notice that a possible choice of $\mathcal{T}_t^\eta$ in (SC) is $\mathcal{T}_t^\eta = \{0\}$. But other choices may be possible, such as that discussed in Comment (i) below, that make condition (3.5) less restrictive.)

Define the extended Hamiltonian by

$$H_{\lambda^0}^{(E)}(t, x, u, p, \lambda) = H_{\lambda^0}(t, x, u, p) - \langle \lambda, \phi(t, x, u) \rangle.$$

Theorem 3.2 (Explicit Mixed Constraints Conditions II) *Let $(\bar{x}(t), \bar{u}(t))$ be a $W^{1,1}$ local minimizer for (P_S) . Assume, for some $\epsilon > 0$, (H1), (H2), $(H3'')$ and (SC) are satisfied. Assume also*

(C): $\{(f(t, x, u), \Lambda(t, x, u), \phi(t, x, u)) : u \in U(t)\}$ is convex, for all $x \in \bar{x}(t) + \epsilon\mathbb{B}$, a.e. $t \in [S, T]$.

Then there exists a multiplier set $(p(\cdot), \mu, \lambda(\cdot), \lambda^0)$, in which $(p(\cdot), \mu, \lambda^0) \in W^{1,1} \times NBV^+ \times \mathbb{R}^+$ and $\lambda : [a, b] \rightarrow \mathbb{R}^\kappa$ is an integrable function, such that

$$(a) \lambda^0 + \|p\|_{L^\infty} + \|\mu\|_{TV} = 1.$$

$$(b) -\dot{p}(t) \in \partial_C H_{\lambda^0}^{(E)}(t, \cdot, \bar{u}(t), q(t), \lambda(t))(\bar{x}(t)) \text{ a.e.}$$

$$(c) m(t) \in \text{co } \partial_x^> h(t, \bar{x}(t)) \text{ } \mu\text{-a.e. and } \text{supp}\{\mu\} \subset \{t : h(t, \bar{x}(t)) = 0\}.$$

$$(d) (q(a), -q(b)) \in \lambda^0 \partial_L \ell(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)).$$

$$(e) H_{\lambda^0}^{(E)}(t, \bar{x}(t), \bar{u}(t), q(t), \lambda(t)) \geq H_{\lambda^0}^{(E)}(t, \bar{x}(t), u, q(t), \lambda(t)) \text{ for all } u \in U(t) \text{ a.e.,}$$

in which $q(\cdot)$ is defined by (2.3) and $\lambda(t) \in N_{\Phi(t)}^C(\phi(t, \bar{x}(t), \bar{u}(t)))$ a.e..

If the ‘convexity’ hypothesis (C) is replaced by

(C+L): There exists $r_0 > 0$ such that, for a.e. $t \in [S, T]$, $\Phi(t)$ and $U(t)$ are closed, convex sets, $U(t) \in r_0\mathbb{B}$ and $\phi(t, x, \cdot)$ is an affine function, i.e.

$$\phi(t, x, u) := \theta^0(t, x) + \sum_{i=1}^m u^i \theta^i(t, x), \text{ for some functions } \theta^i(t, x) : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^\kappa, i = 1, \dots, m.$$

Then the above assertions remain valid, except that (b) is replaced by (3.3) and the ‘extended’ Weierstrass condition (e) is replaced by the weaker Weierstrass condition of Thm. 2.2, namely

$$H_{\lambda^0}(t, \bar{x}(t), \bar{u}(t), q(t)) \geq H_{\lambda^0}(t, \bar{x}(t), u, q(t)) \quad \text{a.e. } t \in [a, b].$$

A proof of Thm. 3.1 is given in Section 5.

Comments

- (i): Necessary conditions for non - smooth optimal control problems with mixed constraints, under Schwartzkopf type surjectivity hypotheses, are given in [5]. Thm. 3.2 extends [5, Thm. 3.2] to allow also for pure state constraints. Even when the pure state constraint is absent, Thm. 3.2 refines [5, Thm. 3.2], because the condition (3.5) in hypothesis (SC) now involves the set $\mathcal{T}_t^\eta$. In the most common case when $\Phi(\cdot)$ is given by (3.2) (mixed equality and inequality constraints), we can choose

$$\mathcal{T}_t^\eta = \overbrace{(-\infty, 0] \times \dots \times (-\infty, 0]}^{\kappa_1} \times \overbrace{\{0\} \times \dots \times \{0\}}^{\kappa_2}.$$

With this choice, condition (3.5) becomes:

$$\begin{aligned} (\phi^1, \phi^2)(t, \bar{x}(t), \bar{u}(t)) + \delta \times (\mathbb{B} \times \mathbb{B}) \subset \\ (\phi^1, \phi^2)(t, \bar{x}(t), U_r(t)) + ([0, +\infty) \times \dots \times [0, +\infty) \times (\{0\} \times \dots \times \{0\})). \end{aligned} \quad (3.7)$$

In the special case when the equality constraint ($\phi^2(t, x(t), u(t)) = 0$) is absent, condition (3.7) is simply the Slater type condition: there exists $\delta > 0$ and a measurable function $u(\cdot)$ such that $u(t) \in U_r(t)$ and $\phi_i^1(t, \bar{x}(t), u(t)) \leq -\delta$ a.e..

Notice that the analogous hypothesis in [5, Thm. 3.1] can be interpreted as (SC) when we take $\mathcal{T}_t^\eta = \{0\}$, that is, it replaces (3.7) by the more restrictive hypothesis

$$(\phi^1, \phi^2)(t, \bar{x}(t), \bar{u}(t)) + \delta \times (\mathbb{B} \times \mathbb{B}) \subset (\phi^1, \phi^2)(t, \bar{x}(t), U_r(t)).$$

The possibility of invoking less restrictive hypotheses, expressed in terms of the tangent cone of $\Phi(t)$ is discussed in [5, Section 3] under the heading ‘tangential covering conditions’.

- (ii): Thms. 2.2, 3.1 and 3.2 provide three sets of necessary conditions that result from bringing together the ‘pure state constraints’ necessary conditions of [1] and the techniques of [5] for treating mixed state constraints by absorbing them into the dynamic constraint. Similar extensions can be achieved for all the sets of necessary conditions in [5], and elsewhere, for different formulations of optimal control problems involving mixed constraints, to allow for the presence of state constraints, including differential algebraic formulations and calculus of variations formulations.
- (iii) Thm. 3.2 under hypothesis (C+L) partially answers an open question raised in [5, pag. 10] regarding the validity of Thm. 3.2 without convexity hypotheses on the extended velocity set.

4 Proof of Thm. 2.2

The proof is an adaptation of the proof of [6, Thm. 2.1], to allow for the presence of state constraints and for the more general radius multifunction here considered. The idea is to reformulate Problem (P) as a differential inclusion optimal control problem with state constraints by absorbing the mixed constraint into the dynamics. We apply known necessary conditions to the reformulated problem, and express these necessary conditions directly in terms of the data for the original problem (P). The main difference is that we make use of the necessary conditions in [1, Corollary 2.2] for ‘pure’ state constrained problems, in place of the necessary conditions for pure state constraint free problems employed in [6].

Observe at the outset that we can, without loss of generality, restrict attention to the case when the integral cost is absent, since the full statement can be derived from the special case when $L(.,.,.) \equiv 0$ by means of standard state augmentation techniques. We can modify the Lipschitz constants to ensure that $k_x^f(.,w)$ is positive valued and $k_S(t) \geq 1$ a.e., without violating the hypotheses. We can assume, finally, that

(A1): For every $t \in [a, b]$, $W(t)$ is a finite set.

(A2): There exist an integrable function $C(.): [a, b] \rightarrow \mathbb{R}$ such that, for a.e. $t \in [a, b]$ and $w \in W(t)$:

$$(x, v) \in (\bar{x}(t) + \epsilon \mathbb{B}) \times R(t, w) \cap S(t, w) \implies \\ |k_x^f(t, w) - k_x^f(t, \bar{w}(t))| + k_S(t) |k_v^f(t, w) - k_v^f(t, \bar{w}(t))| + |f(t, x, (v, w)) - \dot{\bar{x}}(t)| \leq C(t).$$

Reductions to special cases, involving hypotheses (A1) and (A2), introduced in [3], are now standard in the derivation of necessary conditions (See [3, p. 201 et seq.] or [17, Lemma 6.3.1]). Justification for using them depends on introducing finite, inner approximations of the sets $W(t)$ and robustness properties of the necessary conditions under limit taking: the ‘state constraint’ necessary conditions here considered have the required robustness properties for this procedure to apply.

Let $(\bar{x}(.), \bar{u}(.))$ be a $W^{1,1}$ local minimizer for (P) w.r.t. the radius multifunction $R(.)$. Define

$$k(t) := \max_{w \in W(t)} k_x^f(t, w) + k_S(t) \times \max_{w \in W(t)} k_u^f(t, w) \quad \text{and} \quad c(t) := k(t)/k_S(t), \quad (4.1)$$

which are integrable by (2.1) and (A2) and since $k_S(t) \geq 1$, and let $F(.,.)$ be the multifunction

$$F(t, x) := \{(f(t, x, u), r(t, u)) : u = (v, w), (x, v) \in S(t, w), w \in W(t)\}. \quad (4.2)$$

in which $r(t, u) := c(t)(u - \bar{u}(t))$.

We remark that $F(t, x)$ is non-empty for all $x \in \bar{x}(t) + \epsilon \mathbb{B}$, a.e. $t \in [a, b]$ following, if required, a reduction in the size of $\epsilon > 0$. This is because we can deduce from the bounded slope condition (BS) and (2.2), with the help of [4, Thm. 3.5.2.], that $S(t, x, w)$ is non-empty for all $w \in W(t)$, $x \in \bar{x}(t) + \epsilon' \mathbb{B}$, a.e. $t \in [a, b]$ (for some suitably small $\epsilon' > 0$).

Consider the following optimal control problem in which the dynamics are modelled as a differential inclusion:

$$(P') \quad \begin{cases} \text{Minimize} & \ell(x(a), x(b)) \\ \text{over} & (x(.), z(.)) \in W^{1,1} \text{ satisfying} \\ & (\dot{x}(t), \dot{z}(t)) \in F(t, x(t)) \text{ a.e. } t \in [a, b], \\ & h(t, x(t)) \leq 0 \text{ for all } t \in [a, b], \\ & (x(a), x(b)) \in E, \quad z(a) = 0. \end{cases}$$

With the help of measurable select theory, we can show that the feasible F -trajectory $(\bar{x}(t), \bar{z}(t) \equiv 0)$ is a $W^{1,1}$ local minimizer w.r.t. the radius multifunction:

$$R_F(t) := \mathbb{R}^n \times c(t) (R(t) - \bar{u}(t)) .$$

The hypotheses are satisfied under which the necessary conditions of [1, Corollary 2.2] are valid, w.r.t. the $W^{1,1}$ local minimizer $(\bar{x}(.), \bar{z}(.)) \equiv 0$. The key conditions to check, regarding the required properties of $F(.,.)$, are the *bounded slope* condition (with reference to the radius multifunction $R_F(.)$), which is here conveniently expressed in the form: for some possibly readjusted $\epsilon > 0$ and some $k(.) \in L^1$ such that for any $x \in \bar{x}(t) + \epsilon \mathbb{B}$ and $u \in F(t, x) \cap R_F(t)$,

$$(\alpha, (\beta, \tau)) \in N_{F(t, x)}^P(x, f(t, x, u), r(t, u)) \implies |\alpha| \leq k(t)(|\beta| + |\tau|), \text{ a.e.} \quad (4.3)$$

and the compatibility condition: there exists $\eta > 0$ such that

$$(\dot{x}(t), 0) + \eta k(t) \mathbb{B} \subset R_F(t) \quad \text{a.e.} \quad (4.4)$$

Condition (4.4) is satisfied when $k(\cdot)$ is chosen according to (4.1), and since $(\bar{u}(t) + \eta k_S(t) \mathbb{B}) \subset R(t)$, which permit us to conclude

$$(\dot{x}(t), 0) + \eta k(t) \mathbb{B} \in (\dot{x}(t), 0) + \eta (\mathbb{R}^n \times c(t) k_S(t) \mathbb{B}) \subset \mathbb{R}^n \times c(t) (R(t) - \bar{u}(t)) = R_F(t) \quad \text{a.e.}$$

as required. Condition (4.3) is also satisfied, in view of the following lemma:

Lemma 4.1 *Fix $w \in W(t)$ and $(x, v) \in (\bar{x}(t) + \epsilon \mathbb{B}) \times (R(t, w) \cap S(t, w))$ and take an element $(\alpha, \beta, \tau) \in N_{GrF(t, \cdot)}^L(x, f(t, x, v, w), r(t, v, w))$. Then,*

$$|\alpha| \leq k(t)(|\beta| + |\tau|) \quad \text{a.e.}$$

Furthermore

$$(\alpha, 0) \in \partial_L \{ -\langle \beta, f(t, \cdot, \cdot, w) \rangle - \langle \tau, r(t, \cdot, w) \rangle + 2k(t)[|\beta| + |\tau|] d_{S(t, w)}(\cdot, \cdot) \} (x, v).$$

This lemma is proved in [6]. (The proof of [6, Prop. 9.1] covers the case when the w -control is absent and the analysis on [6, pp. 23-24] extends the estimates to allow for w -dependent data). Notice we have taken advantage of the supplementary hypothesis ' $k_S(t) \geq 1$ ' to simplify the estimates in [6, Prop. 9.1]. We deduce from [1, Corollary 2.2] the following information: there exist an absolutely continuous function $p(\cdot) \in W^{1,1}$, a constant p_0 , $\lambda^0 \geq 0$, $\mu(\cdot) \in NBV^+[a, b]$ and a μ -integrable function $m(\cdot)$, such that

$$(i): \lambda^0 + |p_0| + \|p\|_{L^\infty} + \|\mu\|_{T.V.} = 1,$$

$$(ii): \dot{p}(t) \in co \left\{ \eta : (\eta, q(t), p_0) \in N_{GrF(t, \cdot)}^L(\bar{x}(t), \dot{x}(t), 0) \right\} \quad \text{a.e.},$$

$$(iii): (q(a), -q(b)) \in \lambda^0 \partial_L \ell(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)),$$

$$(iv): p_0 = 0.$$

$$(v): \text{For any } (e_1, e_2) \in F(t, \bar{x}(t)) \cap R_F(t), \text{ a.e. } t \in [a, b],$$

$$\langle q(t), \dot{x}(t) \rangle \geq \langle q(t), e_1 \rangle + \langle p_0, e_2 \rangle$$

$$(vi): m(t) \in co \partial_x^> h(t, \bar{x}(t)) \quad \mu\text{-a.e. and } \text{supp}\{\mu\} \subset \{t : h(t, \bar{x}(t)) = 0\},$$

in which $q(\cdot)$ is as defined in (2.3). By condition (ii) and Lemma 4.1,

$$(\eta, 0) \in \partial_L \{ -\langle q(t), f(t, \cdot, \cdot, \bar{w}(t)) \rangle + 2k(t)|q(t)| d_{S(t, \bar{w}(t))}(\cdot, \cdot) \} (\bar{x}(t), \bar{v}(t)).$$

Reviewing these conditions, we see that the proof of the theorem will be complete if we can show that condition (ii) implies the costate inclusion (b) and (v) implies the Weierstrass condition (e). But (v) tells

$$\langle q(t), \dot{x}(t) \rangle \geq \langle q(t), f(t, \bar{x}(t), u) \rangle$$

for a.e. t and all $u = (w, v)$ such that $(\bar{x}(t), v) \in S(t, w)$, $w \in W(t)$ and

$$(f(t, \bar{x}(t), u), r(t, u)) \in R_F(t) = \mathbb{R}^n \times c(t)(R(t) - \bar{u}(t)).$$

This is precisely condition (e) expressed in terms of the radius multifunction $R(\cdot)$, since this last condition can be equivalently stated ' $c(t)(u - \bar{u}(t)) \in c(t)(R(t) - \bar{u}(t))$ ', i.e. ' $u \in R(t)$ ' since $c(t) > 0$, a.e..

Finally, we look at the implications of (ii). Using the facts that $\partial_C = co \partial_L$ and, given two Lipschitz functions f and g , we have $\partial_C(-f) = -\partial_C f$ and $\partial_C(f + g) \subset \partial_C f + \partial_C g$ we conclude

$$(-\dot{p}(t), 0) \in \partial_C \{ \langle q(t), f(t, \cdot, \cdot, \bar{w}(t)) \rangle \} (\bar{x}(t), \bar{v}(t)) - \partial_C \{ 2k(t)|q(t)| d_{S(t, \bar{w}(t))}(\cdot, \cdot) \} (\bar{x}(t), \bar{v}(t)), \quad (4.5)$$

which implies condition (b) in the theorem statement, since the Clarke generalized gradient is positively homogeneous and, for fixed t , $\partial_C d_{S(t, \bar{w}(t))}(\bar{x}(t), \bar{v}(t)) \subset N_{S(t, \bar{w}(t))}^C(\bar{x}(t), \bar{v}(t))$.

5 Proof of Theorems 3.1 and 3.2

Proof of Theorem 3.1.

Problem (P_S) is a special case of (P) , in which the w -component of the control variable is absent (we identify v with u) and $S(t) = \{(x, u) \mid \tilde{\phi}(t, x, u) \in \tilde{\Phi}(t)\}$, in which

$$\tilde{\phi}(t, x, u) = (\phi(t, x, u), u) \text{ and } \tilde{\Phi}(t) = \Phi(t) \times U(t).$$

Proving the first part of the theorem requires us to confirm that the data for the special case satisfies the hypotheses of Thm. 2.2, with reference to the W^{11} local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ and radius multifunction $R(\cdot)$. We attend only to the verification of the key ‘bounded slope’ hypothesis (BS) of Thm. 2.2, with the above identification of $S(t)$. A straightforward refinement of the analysis in [6, Proof of Prop. 4.2] (to taken account of the special structure of $\tilde{\phi}(\cdot, \cdot, \cdot)$ and $\tilde{\Phi}(\cdot)$) permits us to deduce the following: under the hypotheses of Thm. 3.1, for a.e. t , and $(x, u) \in S(t)$ such that $x \in \bar{x}(t) + \epsilon\mathbb{B}$, $u \in R(t)$, the relation $(\alpha, \beta) \in N_{S(t)}^P(x, u)$ implies

$$|\alpha| \leq M(t)k_x^\phi(t)|\beta|, \quad (5.1)$$

where $M(t)$ is as in hypothesis (CQ); furthermore, for a.e. $t \in [a, b]$ there exists $\lambda \in N_{\Phi(t)}^L(\phi(t, x, u))$ such that

$$(\alpha, \beta) \in \partial_L \langle \lambda, \phi(t, \cdot, \cdot) \rangle(x, u) + \{0\} \times N_{U(t)}^L(u). \quad (5.2)$$

We deduce from property (5.1) that (BS) is indeed satisfied, when we identify $k_S(t) = M(t)k_x^\phi(t)$.

Now suppose that $\phi(t, \cdot, \cdot)$ is strictly differentiable at $(\bar{x}(t), \bar{u}(t))$ and that $U(t)$ and $\Phi(t)$ are regular at $\bar{u}(t)$ and at $\phi(t, \bar{x}(t), \bar{u}(t))$. Making use of relation (5.2) which, at the point $(x, u) = (\bar{x}(t), \bar{u}(t))$, becomes (by strict differentiability and regularity)

$$(\alpha, \beta) \in \nabla \langle \lambda, \phi(t, \cdot, \cdot) \rangle(x, u) + \{0\} \times N_{U(t)}^C(u),$$

and also the sum rule for Clarke generalized gradients, we see that the costate inclusion (b) of Thm. 2.2 can be replaced by

$$(-\dot{p}(t), 0) \in \partial_C \{q(t) \cdot f(t, \cdot, \cdot) - \lambda^0 \Lambda(t, \cdot, \cdot) - \langle \lambda(t), \phi(t, \cdot, \cdot) \rangle\}(\bar{x}(t), \bar{u}(t)) - \{0\} \times N_{U(t)}^C(\bar{u}(t)),$$

for some function $\lambda(\cdot)$, a.e. t . (The function $\lambda(\cdot)$ can be chosen to be measurable.) We have arrived at the costate inclusion of Thm. 3.1. Finally note that, any $(\zeta_1, \zeta_2) \in \partial_C \{q(t) \cdot f(t, \cdot, \cdot) - \lambda^0 \Lambda(t, \cdot, \cdot)\}(\bar{x}(t), \bar{u}(t))$ satisfies $|\zeta_2| \leq |q(t)|k_u^f(t) + \lambda_0 k_u^\Lambda(t)$. Examination of the adjoint inclusion in combination with hypothesis (CQ) yield the estimate

$$|\lambda(t)| \leq M(t) (|q(t)|k_u^f(t) + \lambda_0 k_u^\Lambda(t)), \quad a.e..$$

Note that the regularity hypothesis of the set $U(t)$ and $\Phi(t)$ at $\bar{u}(t)$ and at $\phi(t, \bar{x}(t), \bar{u}(t))$, a.e., is essential here to invoke hypothesis (CQ).

Proof of Theorem 3.2.

We first confirm the assertions of the theorem excluding the final assertion. Accordingly assume (C), in addition to the preceding hypotheses in the theorem statment. Following [5], we obtain, as a corollary to Thm. 2.2, necessary conditions for a special case of (P) , in which the data is affine w.r.t. the control variable. These necessary conditions are used, subsequently, to derive necessary conditions for problem (P_S) under the Schwartzkopf type surjectivity hypothesis in place of a constraint qualification such as (CQ), governing the mixed constraint data. The difference is that our analysis takes as starting point the state constrained necessary conditions of Thm. 2.2 of this paper in place of the state constraint-free

conditions of [5, Thm. 3.2]. The special case of (P) that we need to consider is:

$$(P'') \quad \begin{cases} \text{Minimize } \ell(x(a), x(b)), \\ \dot{x}(t) = \sum_{i=0}^N c^i(t) g^i(t, x(t)), \quad \text{a.e. } t \in [a, b], \\ \sum_{i=0}^N c^i(t) \theta^i(t, x(t)) \in \Phi(t), \quad c \in \Sigma, \quad \text{a.e. } t \in [a, b], \\ h(t, x(t)) \leq 0, \quad \text{for all } t \in [a, b], \\ (x(a), x(b)) \in E. \end{cases}$$

Here $g^i(.,.) : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\theta^i(.,.) : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^\kappa$ (for $i = 0, \dots, N$), $h(.,.) : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$ are given functions, and $\Phi \subset \mathbb{R} \times \mathbb{R}^\kappa$, $\Sigma \subset \mathbb{R}^{N+1}$, $E \subset \mathbb{R}^n \times \mathbb{R}^n$ are given sets.

To fit (P'') to the framework of problem (P) (when the control variable u is not partitioned, i.e. $u = (v, w)$ comprises only the variable v , which we write here as c), we take the dynamic constraint to be $\dot{x}(t) = f(t, x(t), c(t))$ with

$$f(t, x, c) = \sum_{i=0}^N c^i g^i(t, x)$$

and the sets $S(t)$ in the ‘mixed constraint’ $(x(t), c(t)) \in S(t)$ to be

$$S(t) = \{(x, c) \mid (\phi(t, x, c), c) \in \Phi(t) \times \Sigma\},$$

in which

$$\phi(t, x, c) = \sum_{i=0}^N c^i \theta^i(t, x).$$

Lemma 5.1 *Let $(\bar{x}(.), \bar{c}(.))$ be a $W^{1,1}$ local minimizer for (P'') w.r.t. the radius multifunction $R(t) = \bar{c}(t) + R\mathbb{B}$ a.e., for some $R > 0$. Assume that, for some $\epsilon > 0$,*

(L1): $\ell(.,.)$ is Lipschitz continuous on a neighborhood of $(\bar{x}(a), \bar{x}(b))$. The function $h(.,.)$ satisfies hypothesis (H2). E is closed. Σ is a compact, convex set.

(L2): The functions $g^i(.,.)$ and $\theta^i(.,.)$ are $\mathcal{L} \times \mathcal{B}^n$ measurable. There exist an integrable function $k^g(.,.)$ and a constant k^θ such that $g^i(t, .)$ and $\theta^i(t, .)$ are Lipschitz continuous on $\bar{x}(t) + \epsilon\mathbb{B}$, a.e. $t \in (a, b)$, $i = 0, \dots, N$. $t \rightarrow g^i(t, \bar{x}(t))$ is integrable and $\theta^i(t, .)$ is strictly differentiable at $\bar{x}(t)$, for a.e. $t \in [a, b]$. $\Phi(.,.)$ is a Lebesgue measurable multifunction taking values non-empty, closed subsets of $\mathbb{R}^\kappa$.

(BS'): There exists a constant $M > 0$ such that, for a.e. $t \in [a, b]$, the following condition is satisfied: given any $x \in \bar{x}(t) + \epsilon\mathbb{B}$, $c \in \Sigma \cap (\bar{c}(t) + R\mathbb{B})$ such that $\phi(t, x, c) \in \Phi(t)$, $\lambda \in N_{\Phi(t)}^C(\phi(t, x, c))$ and $\gamma \in N_\Sigma^C(c)$, we have

$$\beta = (\lambda \cdot \theta^0(t, x), \dots, \lambda \cdot \theta^N(t, x)) + \gamma \implies |\lambda| \leq M|\beta|. \quad (5.3)$$

Then there exist $p(.,.) \in W^{1,1}$, $\lambda^0 \geq 0$, $\mu(.,.) \in NBV^+[a, b]$, a μ -integrable function $m(.,.) \in \partial_x^> h(t, \bar{x}(t))$ μ -a.e. and a integrable function $\lambda : [a, b] \rightarrow \mathbb{R}^\kappa$ satisfying $\lambda(t) \in N_{\Phi(t)}^C(\phi(t, \bar{x}(t), \bar{c}(t)))$ such that

$$(i): (p(.,.), \lambda^0, \mu) \neq (0, 0, 0)$$

$$(ii): -\dot{p}(t) \in \partial_C \{ \langle q(t), f(t, ., \bar{c}(t)) \rangle - \langle \lambda(t), \phi(t, ., \bar{c}(t)) \rangle \} (\bar{x}(t)) \quad \text{a.e.},$$

$$(iii): \langle q(t), f(t, \bar{x}(t), \bar{c}(t)) \rangle - \langle \lambda(t), \phi(t, \bar{x}(t), \bar{c}(t)) \rangle \\ \geq \langle q(t), f(t, \bar{x}(t), c) \rangle - \langle \lambda(t), \phi(t, \bar{x}(t), c) \rangle \text{ for all } c \in \Sigma, \text{ a.e.},$$

$$(iv): (q(a), -q(b)) \in \lambda^0 \partial_L \ell(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)),$$

$$\text{in which } q(t) := \begin{cases} p(t) + \int_{[a,t]} m(s) \mu(ds) & \text{if } t \in (a, b] \\ p(a) & \text{for } t = a. \end{cases}$$

The lemma is proved as [5, Cor. 2.2], in the special case that there are no state constraints, by showing that the data for problem (P'') , regarded as a special case of P , satisfies the conditions of the state constraint free version of Thm. 2.2 and by applying the necessary conditions of this theorem to (P'') . The analysis in [5] transcribes directly and without alteration, when the full version of Thm. 2.2 is substituted for the state constraint-free version, to furnish a proof of the lemma.

We are now ready to prove Thm. 3.2. Consider then a $W^{1,1}$ local minimizer $(\bar{x}(t), \bar{u}(t))$ for (P_S) under the hypotheses of Thm. 3.2. We notice immediately that we can assume, without loss of generality, that $\phi(t, \bar{x}(t), \bar{u}(t)) \equiv 0$ since replacing $\phi(t, x, u)$ by $\phi(t, x, u) - \phi(t, \bar{x}(t), \bar{u}(t))$ and $\Phi(t)$ by $\Phi(t) - \phi(t, \bar{x}(t), \bar{u}(t))$ ensures that this condition is satisfied, yet has no effect on the assertions of the theorem or validity of the hypotheses under which it applies. The use of state augmentation techniques permit us to restrict attention to the case when $\Lambda(., ., .) \equiv 0$.

Write $e_i, i = 1, \dots, \kappa$, for the canonical basis vectors of R^κ . In consequence of (SC), there exist controls $u_i^+(\cdot)$ and $u_i^-(\cdot)$ (we can choose them to be measurable) such that

$$\phi(t, \bar{x}(t), u_i^+(t)) - \phi(t, \bar{x}(t), \bar{u}(t)) \in \delta e_i + \mathcal{T}_t^\eta \quad (5.4)$$

$$\phi(t, \bar{x}(t), u_i^-(t)) - \phi(t, \bar{x}(t), \bar{u}(t)) \in -\delta e_i + \mathcal{T}_t^\eta \quad (5.5)$$

$$|f(t, \bar{x}(t), u_i^+(t)) - f(t, \bar{x}(t), \bar{u}(t))| \leq r(t) + |\dot{\bar{x}}(t)|, \\ |f(t, \bar{x}(t), u_i^-(t)) - f(t, \bar{x}(t), \bar{u}(t))| \leq r(t) + |\dot{\bar{x}}(t)|,$$

for $i = 1, \dots, \kappa$, where $r(\cdot)$ is the function of hypothesis (SC). In view of hypothesis $(H3)''$, we can further arrange, by reducing the size of $\epsilon > 0$ if necessary, that

$$|\phi(t, x, u_i^+(t)) - \phi(t, \bar{x}(t), u_i^+(t))| \leq \delta/(12\kappa), |\phi(t, x, u_i^-(t)) - \phi(t, \bar{x}(t), u_i^-(t))| \leq \delta/(12\kappa), \quad (5.6)$$

and

$$|\phi(t, x, \bar{u}(t)) - \phi(t, \bar{x}(t), \bar{u}(t))| \leq \delta/(12\kappa), \quad (5.7)$$

for all $x \in \bar{x}(t) + \epsilon \mathbb{B}$, $j = 1, \dots, \kappa$, a.e.

Fix any $d > 0$ and define

$$U^d(t) := \{u \in U(t) : |(f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t))), \phi(t, \bar{x}(t), u)| \leq 1/d\} \\ h'(t, p, \lambda, u) := \langle p, f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t)) \rangle - \langle \lambda, \phi(t, \bar{x}(t), u) \rangle \\ H^d(t, p, \lambda) := \sup\{h'(t, p, \lambda, u) : u \in U^d(t)\}.$$

Let $\{(p_j, \lambda_j) \mid j = 2\kappa + 1, \dots, N_d\}$ be a collection of vectors in $\mathbb{B}$ such that

$$\mathbb{B} \subset \bigcup_j ((p_j, \lambda_j) + d^2 \mathbb{B}). \quad (5.8)$$

We can choose measurable controls $v_j(\cdot)$, $j = 2\kappa + 1, \dots, N_d$ with values in $U^d(\cdot)$ and satisfying

$$H^d(t, p_j, \lambda_j) - d < h'(t, p_j, \lambda_j, v_j(t)) \quad \text{a.e.}$$

Consider now problem (P'') when $\Phi(\cdot)$, $\ell(\cdot, \cdot)$, $h(\cdot)$ and E are as in problem (P_S) ,

$$\begin{aligned} g^0(t, x) &:= f(t, x, \bar{u}(t)), & \theta^0(t, x) &= \phi(t, x, \bar{u}(t)) \\ g^i(t, x) &:= f(t, x, u_{(i+1)/2}^+(t)) - f(t, x, \bar{u}(t)), & \theta^i(t, x) &= \phi(t, x, u_{(1+i)/2}^+(t)) - \phi(t, x, \bar{u}(t)), \\ & & & i = 1, 3, \dots, 2\kappa - 1 \\ g^i(t, x) &:= f(t, x, u_{i/2}^-(t)) - f(t, x, \bar{u}(t)), & \theta^i(t, x) &= \phi(t, x, u_{i/2}^-(t)) - \phi(t, x, \bar{u}(t)), \\ & & & i = 2, 4, \dots, 2\kappa \\ g^i(t, x) &:= f(t, x, v_i(t)) - f(t, x, \bar{u}(t)), & \theta^i(t, x) &= \phi(t, x, v_i(t)) - \phi(t, x, \bar{u}(t)), \\ & & & i = 2\kappa + 1, \dots, N_d, \end{aligned}$$

and

$$\Sigma := \left\{ c = (c_0, \dots, c_{N_d}) : c_0 = 1, c_i \geq 0, i = 1, 2, \dots, N_d \text{ and } \sum_1^{N_d} c_i \leq 1 \right\}.$$

Making use of the ‘convexity’ hypothesis (C) in Thm. 3.2 and appealing to measurable selection theory, we can show that $(\bar{x}(\cdot), \bar{c}(t) \equiv (1, 0, \dots, 0))$ is a $W^{1,1}$ local minimizer w.r.t. to the constant radius multifunction $\bar{c}(t) + R\mathbb{B}$ for any positive constant $R > 0$. Following reduction in the size of ϵ , we can choose $R > 0$ to satisfy:

$$0 < R < 1 \text{ and } \max\{\epsilon k_\phi(1 + 2R), R\} < \eta, \quad (5.9)$$

in which $\eta > 0$ is the constant of Hypothesis (SC).

Our aim now is to apply the necessary conditions of Lemma 5.1, with respect to the minimizer $(\bar{x}(\cdot), \bar{c}(\cdot))$, for radius multifunction $R(\cdot) \equiv R\mathbb{B}$, checking first that the relevant hypotheses are satisfied. Consider hypothesis (BS’). Take $(x, c) \in (\bar{x}(t) + \epsilon\mathbb{B}) \times \Sigma$ such that $\sum_{i=1}^{N_d} c_i < R$, $\sum_{i=0}^{N_d} c_i \theta^i(t, x) \in \Phi(t)$, $\lambda \in N_{\Phi(t)}^C(\phi(t, x, c))$, $\gamma \in N_\Sigma^C(c)$ and

$$\beta = (\langle \lambda, \theta^0(t, x) \rangle, \dots, \langle \lambda, \theta^{N_d}(t, x) \rangle) + \gamma. \quad (5.10)$$

We must show

$$|\lambda| \leq M|\beta|, \quad (5.11)$$

for some M independent of (x, c) and γ . Notice that γ is a vector of non-positive numbers because $\gamma \in N_\Sigma^C(c)$ and $\sum_{i=1}^{N_d} c_i < 1$. Examination of the $i = 1$ and 2 components of the $N_d + 1$ -vector relation (5.10), in the light of (5.4), (5.5), (5.6) and (5.7), yields:

$$\beta_1 = \delta\lambda_1 + \mu_1|\lambda| + \langle \lambda, \xi_1 \rangle + \gamma_1 \text{ and } \beta_2 = -\delta\lambda_1 + \mu_2|\lambda| + \langle \lambda, \xi_2 \rangle + \gamma_2, \quad (5.12)$$

for some numbers μ_i , γ_i such that $|\mu_i| \leq \delta/6\kappa$ and $\gamma_i \leq 0$ for $i = 1, 2$, and vectors $\xi_i \in \mathcal{T}_t^\eta$ for $i = 1, 2$. Notice however that, in consequence of (5.9),

$$\begin{aligned} \left| \sum_{i=0}^{N_d} c_i \theta^i(t, x) - \theta_0(t, \bar{x}(t)) \right| &\leq \epsilon k_\phi + \left(\sum_{i=1}^{N_d} c_i \right) (2\epsilon k_\phi + d_r(t)) \leq \epsilon k_\phi (1 + 2R) + R d_r(t) \\ &\leq \eta \times (1 + d_r(t)) \end{aligned}$$

in which $d_r(\cdot)$ is the function in (3.6).

It follows from the definition of $\mathcal{T}_t^\eta$ as an intersection of tangent cones to $\Phi(t)$ over base points in the set S_t^η , and the fact that $\sum_{i=0}^{N_d} c_i \theta^i(t, x)$ lies in this set, that

$$\xi_i \in \mathcal{T}_t^\eta \subset T_{\Phi(t)}^C \left(\sum_{i=0}^{N_d} c_i \theta^i(t, x) \right) \quad \text{for } i = 1, 2.$$

Since $\lambda \in N_{\Phi(t)}^C(\sum_{i=0}^{N_d} c_i \theta^i(t, x))$, we have $\langle \lambda, \xi_i \rangle \leq 0$, for $i = 1, 2$. From (5.12) then,

$$\beta_1 = \delta \lambda_1 + \mu_1 |\lambda| + \tilde{\gamma}_1 \quad \text{and} \quad \beta_2 = -\delta \lambda_1 + \mu_2 |\lambda| + \tilde{\gamma}_2, \quad (5.13)$$

for some $\tilde{\gamma}_i \leq 0$, $i = 1, 2$. Adding the two preceding relations gives $\beta_1 + \beta_2 = (\mu_1 + \mu_2) |\lambda| + \tilde{\gamma}_1 + \tilde{\gamma}_2$. Noting that $\tilde{\gamma}_i$'s are non-positive, and using the bounds on the μ_i 's, we conclude $|\tilde{\gamma}_i| \leq 2|\beta| + (\delta/3\kappa)|\lambda|$, for $i = 1, 2$. But then, by (5.13),

$$|\lambda_i| \leq |\beta|/\delta + (1/6\kappa)|\lambda| + 2|\beta|/\delta + (1/3\kappa)|\lambda| = 3|\beta|/\delta + (1/2\kappa)|\lambda| \quad \text{for } i = 1, 2.$$

Analogous bounds can be established for $\lambda_2, \dots, \lambda_{2\kappa}$. Summing such bounds over i , yields $|\lambda| \leq (3\kappa/\delta)|\beta| + (1/2)|\lambda|$, whence $|\lambda| \leq (6\kappa/\delta)|\beta|$. We have confirmed (5.11) with $M = (6\kappa/\delta)$.

Verification of the remaining hypotheses is straightforward. Applying Lemma 5.1, we deduce immediately that, for some $(p(\cdot), \lambda^0, \mu)$ and some integrable $\lambda(\cdot)$:

$$\lambda^0 + \|p\|_{L^\infty} + \|\mu\|_{T.V.} = 1 \quad (5.14)$$

$$-\dot{p}(t) \in \partial_C \{ \langle q(t), f(t, \cdot, \bar{u}(t)) \rangle - \langle \lambda(t), \phi(t, \cdot, \bar{u}(t)) \rangle \} (\bar{x}(t)) \quad \text{a.e.} \quad (5.15)$$

$$m(t) \in \text{co } \partial_x^> h(t, \bar{x}(t)) \quad \mu\text{-a.e. and } \text{supp}\{\mu\} \subset \{t : h(t, \bar{x}(t)) = 0\}, \quad (5.16)$$

$$(q(a), -q(b)) \in \lambda^0 \partial_L \ell(\bar{x}(a), \bar{x}(b)) + N_E^L(\bar{x}(a), \bar{x}(b)). \quad (5.17)$$

in which $q(\cdot)$ is defined by (2.3). Following the analysis, in [5] we deduce from the Weierstrass condition in the Lemma that the ‘multiplier’ $\lambda(\cdot)$ is bounded according to:

$$|\lambda(t)| \leq 12 \times (\kappa^2/\delta) \times \|q(\cdot)\|_{L^\infty} \times (r(t) + |\dot{\bar{x}}(t)|), \quad (5.18)$$

The Weierstrass condition and (5.8) can also be shown to imply (see [5]):

$$\begin{aligned} & \langle q(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - \langle \lambda(t), f(t, \bar{x}(t), u) \rangle \\ & \geq -3d (12\kappa^2/\delta \times (r(t) + |\dot{\bar{x}}(t)|)(k_h + 1) + 1) \quad \text{for each } u \in U^d(t), \text{ a.e.} \end{aligned} \quad (5.19)$$

(Here, $r(\cdot)$ and δ are as in hypothesis (SC) and k_h is as in hypothesis (H2)). Reviewing (5.14) - (5.19), we see that the assertions of the Theorem have been confirmed except that the Weierstrass condition (5.19) appears in a restricted, approximate form, in which $U(t)$ is replaced by the subset $U^d(t)$. The final step is to take a sequence $d_i \downarrow 0$ and, for each i , carry out the preceding analysis, expressing the multipliers (in relations (5.14) - (5.19)) as $(p_i(\cdot), \lambda_i^0, \mu_i, \lambda_i)$. Define $q_i(\cdot)$ according to (2.3) (when $p_i(\cdot)$ replaces $p(\cdot)$, etc). Using the fact that, for a.e. t ,

$$u \in U(t) \implies u \in U^d(t) \text{ for all } i \text{ sufficiently large,}$$

we carry out a standard convergence analysis to confirm the validity of (5.14)-(5.19), now involving the ‘full’ Weierstrass condition, in terms of multipliers $(p(\cdot), \mu, \lambda(\cdot))$ and some selection of $\text{co } \partial^> h(t, \bar{x}(t))$, where the multipliers are cluster points of $p_i(\cdot)$, etc. (See, e.g. [17]). (Observe that (5.18), implying that the $\dot{p}_i(\cdot)$'s are uniformly integrably bounded, has a crucial role here.)

Finally, we need to consider the case when hypothesis (C+L) replaces (C). Let us assume then that $\phi(t, x, u)$ has the special structure $\phi(t, x, u) := \theta^0(t, x) + \sum_{k=1}^m u^i \theta^i(t, x)$, and $\Phi(t)$ and $U(t)$ are convex sets. The assertions of Thm. 3.2 will follow, in this case, from an application of Thm. 3.1 to problem (P_S) . provided, of course, the relevant hypotheses are satisfied. We attend only to the verification of (CQ), since satisfaction of the remaining hypotheses is automatic. This requires us to verify:

Claim: There exists a constant $M > 0$ such that, for every $x \in \bar{x}(t) + \epsilon \mathbb{B}$ and $u \in U(t)$ such that $\phi(t, x, u) \in \Phi(t)$, we have:

$$\lambda \in N_{\Phi(t)}^C(\phi(t, x, u)), \quad \gamma \in N_{U(t)}^C(u), \quad \beta = (\lambda \cdot \theta^1(t, x), \dots, \lambda \cdot \theta^m(t, x)) + \gamma \implies |\lambda| \leq M|\beta|.$$

for a.e. $t \in [a, b]$.

To accomplish this task, let us take any time t located in the set of full measure on which relevant conditions are satisfied, and (x, u) , (λ, γ) and β with the properties listed in the above ‘claim’. Since $\gamma \in N_{U(t)}^C(u)$ and $U(t)$ is convex,

$$\gamma \cdot (u' - u) \leq 0 \text{ for all } u' \in U(t). \quad (5.20)$$

Since $\phi(t, \bar{x}(t), \bar{u}(t)) \in \Phi(t)$, $\lambda \in N_{\Phi(t)}^C(\phi(t, x, u))$ and $\Phi(t)$ is convex,

$$\lambda \cdot (\phi(t, \bar{x}(t), \bar{u}(t)) - \phi(t, x, u)) \leq 0. \quad (5.21)$$

Since, finally, $\lambda \in N_{\Phi(t)}^C(\phi(t, x_0, u_0))$ and $\xi \in \mathcal{T}_t^\eta \subset T_{\Phi(t)}^C(\phi(t, x_0, u_0))$, we have

$$\lambda \cdot \xi \leq 0. \quad (5.22)$$

We may assume, without loss of generality, that $\lambda \neq 0$ (otherwise the claimed property is true with $M = 0$). By hypothesis (SC) (in which, we may choose $U_r(t) \equiv U(t)$ since $U(\cdot)$ is bounded) there exists $u' \in U(t)$ such that

$$\phi(t, \bar{x}(t), \bar{u}(t)) - \frac{\lambda}{|\lambda|} \delta = \phi(t, \bar{x}(t), u') - \xi. \quad (5.23)$$

By (5.23), (5.20), (5.21) and (5.22), and since $\beta = (\lambda \cdot \theta^1(t, x), \dots, \lambda \cdot \theta^m(t, x)) + \gamma$, we have

$$\begin{aligned} \beta \cdot (u' - u) &= (\lambda \cdot \theta^1(t, x), \dots, \lambda \cdot \theta^m(t, x)) \cdot (u' - u) + \gamma \cdot (u' - u) \\ &\leq \lambda \cdot \phi(t, \bar{x}(t), u') - \lambda \cdot \phi(t, \bar{x}(t), \bar{u}(t)) + k_\phi \epsilon |\lambda| \\ &\leq -|\lambda|(\delta - k_\phi \epsilon). \end{aligned}$$

Reducing the size of ϵ , if necessary, we can arrange that $\delta > k_\phi \epsilon$. (δ is as in hypothesis (SC) and k_ϕ is as in hypothesis (H3'')(c)). Since $U(t) \in r_0 \mathbb{B}$, we have from the preceding relation

$$|\lambda| \leq M|\beta|,$$

with $M = 2r_0/(\delta - k_\phi \epsilon)$. We have justified the claim, and the proof of the theorem is complete. $\square$

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