Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

First and Second Order Necessary Conditions of Optimality for Impulsive Control Problems

Aram Arutyunov
Differential Equations and
Functional Analysis Dept.
Peoples Friendship Univ. of Russia
6, Mikluka-Maklai St., Moscow
117198, Russia
arutun@orc.ru

Vladimir Dykhta
Department of Mathematics
Baikal State University of
Economy and Law
11, Lenin St., Irkutsk
664003, Russia
dykhta@isea.ru

Fernando Pereira
Institute for Systems and Robotics
Faculdade de Engenharia da
Universidade do Porto
Rua Dr. Roberto Frias, 4200-465
Porto, Portugal
flp@fe.up.pt

***Abstract*— We present first and second order necessary conditions of optimality for a general class of nonlinear measure driven dynamic control systems subject to both equality and inequality endpoint state constraints. An important feature of our result is that the conditions remain informative even for abnormal control processes. Our result is obtained by decoding the necessary conditions of optimality for an abstract minimization problem with equality and inequality type constraints and constraints given by convex cone.**

I. INTRODUCTION

In this article, we present first and second order necessary conditions for a general nonlinear impulsive optimal control problem which are also informative for abnormal control processes and whose derivation does not require a priori normality assumptions. These can be regarded as an extension of the result obtained in [2] since, now, the function defining the impulsive dynamics also depends on the state variable. The proof of our result consists in applying a certain a nonlinear transformation, [8], to the initial problem, so that the new one is such that the impulsive dynamics do not depend on x, in applying the first and second order necessary conditions of optimality derived in [2], and, then, in decoding these in terms of the data of the original problem.

Dynamic optimization problems arising in a variety of application areas such as finance, mechanics, resources management, and space navigation, (see [8], [9], [10], [11], [12], [13], [16], just to mention a small but representative sample of references) whose solutions might involve discontinuous trajectories have been considered over the years, motivating a significant research effort on the so-called impulsive control problem.

Although the theory of higher order necessary conditions of optimality for conventional optimal control problems is well developed (see, for example, [1], [14], [27]), only a few publications on such conditions are available for impulsive control systems, [19], [2], [25], [26], in spite of vast amount of literature addressing optimal impulsive control problems, [4], [5], [6], [7], [17], [18], [20], [21], [22], [23], [24].

We note that, while the conditions in [19], [25] become trivial, i.e. degenerate, for abnormal problems, ours remain informative. Also, our result differs substantially from these conditions as it can be seen from the fact that these follow directly from the Maximum Principle in the case the optimal trajectory is absolutely continuous, i.e., no impulses.

In [7], Legendre-Jacobi-Morse-type second order necessary conditions of optimality for time-optimal control are derived by using in an essential way an extremal principle and the notion of index of quasiextremality provided in [28].

However, the approach followed here differs substantially from all the ones in the references cited above as we regard this problem as a specific instance of a general abstract problem for which powerful second-order optimality conditions are derived.

This article is organized as follows: In the next section, we formulate the considered impulsive optimal control problem, including the hypotheses assumed on its data as well as some key definitions and preliminary concepts. In particular, we detail the adopted notion of solution to the measure differential equation. The statement of the first and second order necessary conditions of optimality for the dynamic optimization problem described in the third section, together with some critical definitions. Issues concerning abnormality, geometric interpretation and computation are also included. Also in this section we make several remarks, including a brief outline of the approach to the proof. Finally, in the fourth section, we present one example illustrating the application of these conditions. This example shows that the first and the second order optimality conditions remain informative even for abnormal points.

II. OPTIMAL CONTROL PROBLEM FORMULATION

We consider the following impulsive optimal control problem:

$$(P)\ \text{Minimize}\quad J(x_0, u, w) \tag{1}$$

$$\text{subject to}\quad dx(t) = f(t, x(t), u(t))dt + G(t, x(t))dw(t),\ t \in [t_0, t_1], \tag{2}$$

$$L_1(a) \leq 0,\ \ L_2(a) = 0, \tag{3}$$

$$dw \in \mathcal{K}, \tag{4}$$

where $J(x_0,u,w) := L_0(a)$, $a = (x(t_0), x(t_1))$, $x(t_0) = x_0$, $x(t_1) = x_1$, $t_0 < t_1$ are given, and the functions $f : [t_0,t_1] \times R^n \times R^m \to R^n$, $G : [t_0,t_1] \times R^n \to R^{n\times k}$, $L_i : R^n \times R^n \to R^{d(L_i)}$, for $i = 0,1,2$, ($d(L_i)$ denotes the dimension of the vector functions L_i, $i = 0,1,2$, $d(L_0) = 1$), satisfy the following assumptions:

(H1) Functions L_0, L_1, L_2 are C^2.

(H2) The function f is twice differentiable w.r.t. x, u for almost all $t \in [t_0,t_1]$, and, together with the first and second order derivatives are measurable w.r.t. t and bounded on any bounded subset.

(H3) The matrix function $G \in C^2$.

(H4) The matrix G satisfies the so called Frobenius condition, i.e.,

$$G^i_x(t,x)G^j(t,x) - G^j_x(t,x)G^i(t,x) \equiv 0 \; , \qquad (5)$$

where G^i is i^{th} column of G.

We denote by dw the k-dimensional Borel measure associated with the function of bounded variation $w(t)$ right continuous on $(t_0,t_1]$, and define the cone $\mathcal{K}$ by

$$\mathcal{K} = \{dw \in C^*([t_0,t_1];R^k) : \forall\phi \in C([t_0,t_1]) \text{ s.t. } \phi(t) \in K^0\ \forall t,\ \int_B \phi(t)dw \geq 0\ \forall \text{ Borel } B \subset [t_0,t_1]\}.$$

Here K is a given convex, closed, pointed cone from R^k, and K^0 is its dual.

The pair (u,w) is called admissible control if $u \in L^m_\infty$, $w \in BV^k$ such that $dw \in \mathcal{K}$.

Definition 1. Given any given admissible control (u,w) and initial condition x_0, there exists a unique trajectory, $x(\cdot)$, which is a right continuous function of bounded variation on $(t_0,t_1]$, such that

$$x(t) = x_0 + \int_{t_0}^t f(\theta,x(\theta),u(\theta))d\theta + \int_{t_0}^t G(\theta,x(\theta))dw_c(\theta) + \sum_{s_i\leq t}\int_0^1 G(s_i,z^i(\tau))c_i d\tau,$$

$$x(t_0) = x_0.$$

Here, dw_c and $dw_a(t) := \sum_{s_i<t} c^i\delta_{s_i(t)}$, (being $c^i \in K$, $s_i \in [t_0,t_1]$ the jump times, and δ_s the Dirac measure at s) represent, respectively, the continuous and the atomic parts of the measure μ.

Notice that, given $(H2)$, the uniqueness is guaranteed by the Frobenius condition, $(H4)$. This ensures the robustness of the dynamic system (2) with respect to the approximation of generalized control dw by conventional controls $v(\cdot) \in L^k_\infty([t_0,t_1];K)$ (see [4], [5], [7], [8]), and it implies that the solution (6) belongs to the closure of the set of absolutely continuous solutions of equation (2) corresponding to $(u,w) \in L_\infty \times AC$.

An admissible control process is a triple (x_0,u,w), where (u,w) is an admissible control and the corresponding trajectory satisfies the given endpoint constraints.

Our problem can now be clearly stated: minimize J over the set of admissible control processes.

Definition 2. We say that the admissible process (x_0^*,u^*,w^*) is a local minimizer of the problem (P) if there exists $\varepsilon > 0$ and, for any finite-dimensional subspace $R \subset L^m_\infty[t_0,t_1]$, $\varepsilon_R > 0$ such that the process (x_0^*,u^*,w^*) yields a minimum to the problem (1)-(3) with the additional constraints $\|a - a^*\| < \varepsilon$, $\|dw - dw^*\|_{C^*([t_0,t_1];R^k)} < \varepsilon$, $\|u - u^*\|_{L^m_\infty[t_0,t_1]} < \varepsilon_R$, $u(\cdot) \in R$.

Notice that the defined type of local minima is finite dimensional in u and weak in dw.

For the sake of simplicity of the arguments, we assume that admissible process and corresponding trajectory investigated for minimum of problem (P) satisfies:

(H5) The measure dw^* has the form

$$dw^*(t) = v^*(t)dt + \sum_{s\in S^*} c^s\delta_s(t) \qquad (6)$$

where $v^*(t) = \dot{w}^*(t)$ a.e. with respect to the Lebesgue measure on $[t_0,t_1]$, $S^* \subset [t_0,t_1]$ is the set of jump times of $w^*(\cdot)$, assumed to be finite, and $c^s = [w^*(s)] := w^*(s^+) - w^*(s^-)$, i.e., the function $w^*(\cdot)$ has no singular continuous part and has a finite number of jump times.

Moreover, since (x_0^*,u^*,w^*) is investigated for local minimum only, i.e., in the sense of definition 2, then we can assume, without any loss of generality, that all endpoint inequality constraints are active at the optimal trajectory x^*, i.e.,

$$L_1(a^*) = 0 \quad \text{where} \quad a^* = (x^*(t_0), x^*(t_1)). \qquad (7)$$

III. Necessary Conditions of Optimality

In order to state the necessary conditions of optimality for problem (P), we need to introduce the following fundamental auxiliary concepts: local maximum principle, critical cone, and quadratic form.

Local maximum principle. Let $\psi \in R^n$, $\lambda = (\lambda_0,\lambda_1,\lambda_2) \in R^1 \times R^{d(L_1)} \times R^{d(L_2)}$ and define the Pontryagin function $H = H_0 + H_1$, and the endpoint Lagrangian l^λ, respectively by:

$$\begin{aligned} H_0(t,x,\psi,u) &= \langle\psi, f(t,x,u)\rangle, \\ H_1(t,x,\psi,v) &= \langle\psi, G(t,x)v\rangle, \qquad \text{and} \\ l^\lambda(a) &= \lambda_0 L_0(a) + \langle\lambda_1, L_1(a)\rangle + \langle\lambda_2, L_2(a)\rangle. \end{aligned}$$

Definition 3. We say that a process (x_0^*,u^*,w^*) satisfies the Euler-Lagrange conditions or the local Maximum Principle if there exists $\lambda \neq 0$, such that

$$\lambda_0 \geq 0,\ \lambda_1 \geq 0,\ \langle\lambda_1, L_1(a^*)\rangle = 0 \qquad (8)$$

and the vector function ψ, solution, in the integral sense of (6)[1], to the adjoint system

$$\begin{cases} -d\psi(t) = H_{0x}(t)dt + H_{x_v}(t)dw^*(t), \\ -\psi(t_1) = l^\lambda_{x_1}(a^*) \end{cases} \tag{11}$$

which satisfy the following conditions:

$$\begin{aligned} \psi(t_0) &= l^\lambda_{x_0}(a^*) && && (12)\\ H_u(t) &= 0 && dt\text{-a.e.} && (13)\\ \langle H_v(t), v\rangle &\le 0 && \forall (t,v) \in [t_0,t_1]\times K, && (14)\\ \langle H_v(t), \omega^*(t)\rangle &= 0 && dw^* - \text{a.e.}, && (15) \end{aligned}$$

where $\omega^*(t) = \frac{dw^*(t)}{d|w^*(t)|}$ is the Radon Nicodym derivative of the measure dw^* with respect to its total variation measure. Remark that any adjoint trajectory $\psi(t)$ and the function $H(t)$ depend on λ due to the transversality condition (12).

The notation above needs some explanation. When some arguments of a given function are missing, i.e., $H(t)$, $H(t,u)$, or $H_v(t)$, this means that the function is considered being evaluated along the examined (reference) process. This notation is also adopted for other functions in similar contexts. The $\dot{}$ over the function label means the total derivative with respect to time. An argument variable appearing in sub index means that a partial derivative is being considered, e.g., $H_{x_v}(t) = \frac{\partial^2 H}{\partial v \partial x}(t)$.

Let $\Lambda = \Lambda(x_0^*, u^*, w^*)$ be the set of all Lagrange multipliers λ satisfying the local maximum principle and normalized by the condition $\|\lambda\| = 1$. The following result is well known (see [1]):

Theorem 4. $\Lambda \neq \emptyset$ is a first order necessary condition for a weak local minimum for problem (P).

However, we shall prove here that it is also necessary for the local minimum in the sense of definition 2. Remark that *the local maximum principle holds without the assumption* $(H5)$.

Critical cone. In order to ensure a compact statement of the second order conditions, we shall use the total derivative w.r.t. time along the solution to the following ordinary differential system[2]

$$\begin{cases} \dot x &= F(t,x,u,v) \\ -\dot\psi &= H_x(t,x,\psi,u,v) \\ \dot w &= v, \qquad v(t) \in K \end{cases} \tag{16}$$

where $F(t,x,u,v) = f(t,x,u) + G(t,x)v$. Under Frobenius condition, this derivative does not depend on v, but in any other cases, we always put $v^*(t) = \dot w^*(t)$ (6). Denote by $BV^n(S^*)$ the set of n-dimensional vector functions of bounded variation whose jump times are supported on S^*. It is clear that $x^*(\cdot)$, $\psi(\cdot)$, and $w^*(\cdot)$ are in $BV(S^*)$ spaces of the corresponding dimension.

Definition 5. A variation $(\delta x_0, \delta u, \delta w) \in R^n \times L^m_\infty \times BV^k(S^*)$ is called critical if the corresponding state trajectory variation, $\delta x \in BV^n(S^*)$, satisfies the following conditions:

$$\langle L_{i_a}(a^*), \delta a\rangle + \langle L_{i x_1}(a^*), G(t_1)\delta w_1\rangle \begin{cases} \le 0, \ i = 0,1, \\ = 0, \ i = 2 \end{cases} \tag{17}$$

$$\delta a = (\delta x(t_0), \delta x(t_1)), \qquad \delta w_1 = \delta w(t_1) \tag{18}$$

$$\dot{\delta x} = F_x(t)\delta x + F_u(t)\delta u - (\dot H_v)^T_\psi(t)\delta w, \quad t \notin S^* \tag{19}$$

$$d(\delta w) \in \mathcal{K} + \mathrm{Lin}\{dw^*\}, \qquad \delta w(t_0) = 0 \tag{20}$$

$$\delta x(s) = \delta q(1; s, c), \qquad \forall s \in S^* \tag{21}$$

where $\delta q(\tau; s, c) := \delta q^s(\tau)$ is a solution to the system

$$\begin{cases} \frac{d(\delta q^s)}{d\tau} = H_{1\psi x}(s, z^s(\tau), c)\delta q^s \\ \delta q^{t_0}(0) = \delta x_0 \\ \delta q^s(0) = \delta x(s^-), \quad s > t_0 \end{cases} \tag{22}$$

and the function $z^s(\tau)$ is the solution to the limiting system

$$\frac{dz^i}{d\tau} = G(s_i, z^i)c_i, \qquad z^i(0) = x(s_i^-) \tag{23}$$

when $s_i = s$, $x(s^-) = x^*(s^-)$ (recall that $c = [w^*(s)] := w^*(s) - w^*(s^-)$) underlying the solution concept (6).

Denote by $\mathcal{K}_{cr}$ the cone of all critical variations.

Quadratic form. For any $\lambda \in \Lambda$ define the quadratic form

$$\Omega^\lambda(\delta x_0, \delta u, \delta w) = \delta a^T l^\lambda_{aa}(a^*)\delta a + Q_1^\lambda(\delta a, \delta w_1) - \int_{t_0}^{t_1} Q^\lambda(\delta x, \delta u, \delta w)(t)dt \tag{24}$$

[1] i.e.,

$$\begin{cases} \psi(t) = -l^\lambda_{x_1}(a^*) + \int_t^{t_1} H_{0x}(\theta)d\theta + \int_t^{t_1} H_{x_v}(\theta)dw_c^*(\theta) \\ \qquad + \sum_{s_i > t}(\psi(s_i) - q(0; s_i, c^i)) \quad t \in [t_0, t_1), \\ \psi(t_1) = -l^\lambda_{x_1}(a^*). \end{cases} \tag{9}$$

Here, functions $q^i(\tau) = q^i(\tau; s_i, c^i)$ are solutions to the adjoint limiting system

$$-\frac{dq^i}{d\tau} = H_{x_v}(s_i, z^i(\tau), q^i(\tau))c^i, \qquad q^i(1) = \psi(s_i) \tag{10}$$

with the corresponding solution $z^i(\tau)$ to system (23) when $x(s^-) = x^*(s^-)$.

[2] For example, $(\dot H_v)_x = \frac{\partial}{\partial x}\left(\frac{d}{dt}\frac{\partial H}{\partial v}\right)\Big|_{(x^*(t), u^*(t), w^*(t))}$.

where Q^λ and Q_1^λ are the following quadratic forms:

$$\begin{aligned} Q^\lambda(\delta x, \delta u, \delta w) = {} & \delta u^T H_{uu}^\lambda \delta u + 2\delta x^T H_{xu}^\lambda \delta u \\ & -2\delta w^T (\dot{H}_v^\lambda)_u \delta u - \delta w^T (\ddot{H}_v^\lambda)_v \delta w \\ & -2\delta w^T (\dot{H}_v^\lambda)_x \delta x + \delta x^T H_{xx}^\lambda \delta x \end{aligned} \quad (25)$$

$$\begin{aligned} Q_1^\lambda(\delta x(\cdot), \delta w_1) = {} & 2\delta x(t_0)^T l_{x_0 x_1}^\lambda (a^*) G(t_1) \delta w_1 \\ & -2\delta x(t_1)^T H_{xv}^\lambda (t_1) \delta w_1 \\ & +\delta w_1^T G^T(t_1)(L_{x_1 x_1}^\lambda (a^*) G(t_1) \\ & -H_{xv}^\lambda(t_1))\delta w_1 - \sum_{s \in S^*} [\delta x^T(s) \Psi^\lambda(s) \delta x(s) \\ & -\delta x^T(s^-)\Psi^\lambda(s^-)\delta x(s^-)]. \end{aligned} \quad (26)$$

Here $\delta x(\cdot)$ is the corresponding solution to (20), (21), (22), $\delta x(t_0^-) = \delta x_0$, $w(t_0^-) = 0$, and the t dependence in Q^λ is omitted.

The $n \times n$ matrix $\Psi^\lambda(t) \in BV^{n\times n}(S^*)$ in formula (26) is given by

$$\Psi^\lambda(t) = -Z^T(1;t) \int_0^1 Z^{-1T}(\tau, t) H_{1xx}(\tau; t) Z^{-1}(\tau; t) d\tau Z(1; t). \quad (27)$$

where $H_{1xx}(\tau;t)$ denotes $H_{1xx}(t, z^*(\tau;t), q^*(\tau;t), w^*(t))$ and the $n \times n$ matrix $Z(\tau;t)$ satisfies the linear differential equation

$$-\frac{dZ}{d\tau} = Z H_{1\psi x}(t, z^*(\tau, t), w^*(t)), \qquad Z(0;t) = I.$$

Here, $z^*(\tau;t)$, $q^*(\tau;t)$ are the solutions to the limiting systems

$$\frac{dz^*}{d\tau} = G(t, z^*) w^*(t), \; z^*(1;t) = x^*(t)$$

$$-\frac{dq^*}{d\tau} = H_{1x}(t, z^*, q^*, w^*(t)), \; q^*(1;t) = \psi(t).$$

Notice that $\Psi(t_0^-) = 0$ from the fact that $w(t_0^-) = 0$.

Main result. Consider the following modified variational equation:

$$\dot{\delta x} = F_x(t)\delta x + F_u(t)\delta u - (\dot{H}_v)_\psi(t)\pi \delta w, \quad t \notin S^* \quad (28)$$

with jump conditions (21) and (22), with

$$\delta x(0) = \delta x_0 \in R^n, \quad \delta u \in L_\infty^m, \quad \delta w \in L_\infty^k, \quad (29)$$

where π is the matrix of the orthogonal projection from R^k onto the linear subspace N defined by $N = K \cap (-K)$[3].

Define the quadratic form Ω_a^λ on $R^n \times L_\infty^m \times L_\infty^k \times R^k$ obtained from Ω^λ by formally replacing δw_1 by h.

Put $L = (L_1, L_2)$ and denote by $\mathcal{K}_\pi$ the linear subspace of $R^n \times L_\infty^m \times L_\infty^k \times R^k$ of all tuples $(\delta x_0, \delta u, \delta w, h) \in R^n \times L_\infty^m \times L_\infty^k \times R^k$ such that the corresponding solution of (28) with (21), (22), satisfies

$$L_a(a^*)\delta a + L_{x_1}(a^*)G(t_1)\pi h = 0, \quad (h \in R^k).$$

Define the linear operator $\mathcal{A} : \mathcal{K}_\pi \to R^{d(L)}$ by the formula

$$\mathcal{A}(\delta x(0), \delta u, \delta w, h) = L_{x_0}(a^*)\delta x_0 + L_{x_1}(a^*)\delta x_1 + L_{x_1}(a^*)G(t_1)\pi h,$$

where δx is the corresponding solution to (28), (29), (21), (22). Let $d = \text{codim}\,(Im\mathcal{A})$[4].

Consider the subset $\Lambda_a(x^*, u^*, w^*)$ (or Λ_a for short) of vectors $\lambda \in \Lambda(x^*, u^*, w^*)$ such that the index of the form Ω_a^λ on the subspace $\mathcal{K}_\pi$ is not greater then d. We recall that the index of a quadratic form q on a given subspace V is the maximum dimension of any subspace of V where the quadratic form is negative definite.

Theorem 6 (Necessary conditions of optimality). Let the control process (x^*, u^*, w^*) be a local optimal to the problem (P). Then, $\Lambda_a \neq \emptyset$ and, for any $(\delta x_0, \delta u, \delta w) \in \mathcal{K}_{cr}$, we have

$$\max_{\lambda \in \Lambda_a} \Omega^\lambda(\delta x_0, \delta u, \delta w) \geq 0. \quad (30)$$

Note that, by definition of Λ_a, the cone $\Lambda_a \subseteq \Lambda$, and, therefore, theorem 6 is stronger than well known conditions for which cone Λ_a in (30) is replaced by Λ, [25], [26].

The proof of this result is organized into several steps as follows. First, we transform (P) into an equivalent problem whose impulsive dynamics do not depend on the state variable. Then, we apply the optimality conditions proved in [3]. Finally, the local maximum principle and the second order conditions are decoded in order to be expressed in terms of data of the original problem.

Remark 7. Second order necessary conditions of optimality are also considerable for the abnormal case (see [1], [2], [3]). For problem (P), the abnormality of admissible control process $(\delta x_0, u(\cdot), w(\cdot))$ means that the convex hull of the $\Lambda(\delta x_0, u(\cdot), w(\cdot))$ contains 0. Notice that, for the abnormal

[3]Obviously $C^*([t_0, t_1], N)$ is the maximal linear subspace contained in $\mathcal{K}$.

[4]It can be easily shown that d is equal to the dimension of the kernel of $(n + k + d(L)) \times d(L)$ block matrix operator $\begin{bmatrix} A \\ B \\ G(t_1)\pi \end{bmatrix} : R^{d(L)} \to R^{n+k+d(L)}$, where

$$A = L_{x_0}(a^*) + \Phi(t_1)L_{x_1}(a^*),$$

$$B = L_{x_1}(a^*)^T \Phi(t_1) \int_{t_0}^{t_1} \Phi^{-1}(t)\Gamma(t) \times \Gamma(t)^T \Phi^{-1}(t)^T dt \Phi(t_1)^T \frac{\partial L}{\partial x_1}(a^*),$$

being Φ a fundamental solution to the system (28), i.e.,

$$\frac{d}{dt}\Phi(t) = F_x(t)\Phi(t), dt\text{-a.e.} \quad \Phi(t_0) = I,$$

and $\Gamma(t)$ is the $n \times (m+k)$ block matrix defined by $[F_u(t) \;\; -(\dot{H}_v)_\psi(t)\pi]$ and A^T denotes the transpose of A.

case, second order conditions in which Λ is used instead of Λ_a in formula (30) become trivial (i.e., uninformative).

Remark 8. In the expression for the form Q_1^λ in the last term there appears terms which feature left limits, $\delta x(s^-)^T$, $\Psi^\lambda(s^-)$, $s \in S^*$. However, in order to compute these terms it is not necessary to extract limits from the left. For example, to compute $\Psi^\lambda(t^-)$ at some fixed point $t \in S^*$, $t > t_0$, it is sufficient to do the following. First, we must solve the following system of differential equations

$$\frac{dz^*}{d\tau} = G(t, z^*)(w^*(t) - c^t), \quad z^*(1;t) = x^*(t)$$
$$-\frac{dq^*}{d\tau} = H_{1x}(t, z^*, q^*, (w^*(t) - c^t)), \quad q^*(1;t) = \psi(t),$$

and, then, solve the matrix differential equation

$$-\frac{dZ}{d\tau} = ZH_{1\psi x}(t, z^*(\tau, t), (w^*(t) - c^t)), \quad Z(0;t) = I.$$

This gives us the function $Z(\tau, t^-)$. After this, we compute $\Psi^\lambda(t^-)$ by formula (27). So, for our purpose, we need to solve two Cauchy problems (for each atom).

IV. Example

Let $n \geq 5$, $k = n-1$, $x = \text{col}(x_1, \ldots, x_n) \in R^n$, $\zeta \in R^k$ be a given nonzero vector, and Q be a symmetric $k \times k$ matrix such that the index of each of the matrices Q and $(-Q)$ is not less than 2.[5]

Consider the problem

$$\begin{aligned} &\text{Minimize} && J(x, w) \\ &\text{subject to} && dx_i = f_i(x,t)dt + dw_i,\ i = \overline{1,k}, \quad \forall\, t \in [0,1] \\ &&& dx_n = f_n(x,t)dt + \langle Q\text{col}(x_1, \ldots, x_k), dw\rangle, \\ &&& \qquad \forall\, t \in [0,1] \\ &&& x(0) = 0, \quad x_n(1) = 0. \end{aligned}$$

Here, $J(x,w) := \langle \zeta, (x_1(1), \ldots, x_k(1))\rangle$, $K = R^k$, and $w = \text{col}(w_1, \ldots, w_k)$.

Assume that, for $i = 1, \ldots, n$, the f_i's are arbitrary given smooth functions such that $f_i(0,t) \equiv 0$, $f_{ix}(0,t) \equiv 0$, and $f_{nxx}(0,t) \equiv 0$. It can be readily seen that the Frobenius condition holds.

Let us investigate the admissible control process $(0,0,0)$ and prove that it is not a locally optimal control process.

Fix any $\lambda \in \Lambda$. From (14), we obtain, for $\psi(\cdot) = \psi^\lambda(\cdot) = (\psi_1(\cdot), \ldots \psi_n(\cdot))$, $\psi_i(t) \equiv 0$, $i = 1, \ldots, k$, and from (11), we have $\psi_n(t) \equiv \psi_{n,0} = \text{const}$.

Hence, by using (11), (12), and the fact that $\zeta \neq 0$, we obtain $\Lambda = \{\lambda : \lambda_0 = 0,\ \lambda_{2,i} = 0,\ i = \overline{1, n-1},\ \lambda_{2,n} = -\lambda_{2,n+1}\}$ and, consequently, Λ consists of only two vectors $\bar{\bar{\lambda}} = -\bar{\lambda}$ and $\bar{\lambda} = \frac{1}{\sqrt{2}}(0, \ldots, 0, 1, -1)$ and $\psi_{n,0} = \pm\frac{1}{\sqrt{2}}$.

[5] As an example, consider $k = 4$ and $Q = \text{diag}(1, 1, -1, -1)$.

Let $\delta\bar{x} = \text{col}(\delta x_1, \ldots, \delta x_k)$. It can be easily shown that $d = 1$ and

$$\Omega_a^\lambda(\delta w) = \psi_{n,0} \int_0^1 \langle Q\delta\bar{x}, \delta w\rangle dt.$$

Hence, $\Omega_a^\lambda(\delta w) = \frac{1}{2}\psi_{n,0}\langle Q\delta\bar{x}(1), \delta\bar{x}^T(1)\rangle$. This implies that, for any $\psi_{n,0} = \pm 1$ the index of the function Ω_a^λ is not less than 2. So $\Lambda_a = \emptyset$ and consequently the process $(0,0,0)$ is not optimal. Also notice that this process is abnormal and

$$\max_{\lambda \in \Lambda} \Omega^\lambda(\delta w) \geq 0,$$

$\forall\ \delta w$ (because of $\bar{\lambda}$, $\bar{\bar{\lambda}} \in \Lambda$) and the last inequality is not informative.

V. Acknowledgments

Aram Arutyunov acknowledges the support of Russian Foundation for Basic Research, grant N 02-01-00334, and the mobility grant of INVOTAN.

Vladimir Dykhta acknowledges the support of Russian Foundation for Basic Research, grant N 00-01-00869, and the mobility grant of INVOTAN.

Fernando Pereira acknowledges the support of FCT via the CorDyAL project funded under the Praxis XXI program and the mobility grant of INVOTAN.

VI. References

[1] A. V. Arutyunov, *Second order necessary conditions in optimal control problems*, Doklady Math., v. 61, N. 2, (2000), pp. 158–161.

[2] A. V. Arutyunov, *Optimality Conditions: Abnormal and Degenerate Problems*, Kluwer Academic Publishers, 2000.

[3] A. Arutyunov, V. Jacimovic, F. Lobo Pereira, *Second order necessary conditions for optimal impulsive control problems*, European Control Conference 2001, Porto, Portugal, 2001, pp. 2759–2763.

[4] B.M. Miller, *The optimality conditions in a problem of control of a system that can be described with a differential equation with measure*, Autom. Telemekh., vol. 6, 1982, pp. 752–761.

[5] S. Zavalischin and A. Sesekin, *Impulsive processes: models and applications*, Nauka, Moscow, 1991.

[6] A. Bressan and F. Rampazzo, *Impulsive Control Systems with Commutative Vector Fields*, J. of Optimization Theory and Applications, vol. 71, (1991), pp. 67–83.

[7] A. V. Sarychev, *Optimization of Generalized Controls in a Nonlinear Time Optimal Problem*, Differential Equations, v. 27, n 5, (1991), pp. 539–550.

[8] V.A. Dykhta and O.N. Samsonyuk, *Optimal impulse control with applications*, Nauka, Moscow, 2000.

[9] J. P. MAREC, *Optimal Space Trajectories*, Elsevier, 1979.

[10] D. F. LAWDEN, *"Optimal Trajectories for Space Navigation"*, Butterworth, London, 1663.

[11] B. BROGLIATO, *Nonsmooth Impact Mechanics: Models, Dynamics and Control*, Lect. Notes in Control and Inform. Sci., vol. 220, Springer-Verlag, 1996.

[12] C. CLARK, F. CLARKE, G. MUNRO, *The Optimal Exploitation of Renewable Stocks*, Econometrica, 47, 1979, pp. 25–47.

[13] V.F. KROTOV, V.Z. BUKREEV, V.I. GURMAN, *New variational methods in flight dynamics*, Mashinostroenie, Moscow, 1969.

[14] V.F. KROTOV, *Global methods in optimal control*, Marcel Dekker, New York, 1996.

[15] V.I. GURMAN, *The extension principle in control problems*, Nauka, Moscow, 1985.

[16] J. BAUMEISTER, *On Optimal Control of a Fishery*,in: "NOLCOS'01 - 5th IFAC Symposium on Nonlinear Control Systems, 2001.

[17] B.M. MILLER AND E. Y. RUBINOVITCH, *Impulsive control in continuous and discrete-continuous systems*, Kluwer Academic Publisher,Amesterdam, Holland, 2002.

[18] V. A. DYKHTA *Conditions of local minimum for singular modes in the system with linear control*, Automation and Remote Control, Vol. 12, 1981, pp. 5–10.

[19] V. A. DYKHTA *Variational maximum principle and quadratic optimality conditions for impulsive and singular processes*, Sib. Math. Journal, Vol. 35, 1994, pp. 70–82.

[20] V. A. DYKHTA *Necessary optimality conditions for impulsive processes under constraints on the image of the control measure*, Izvestiya Vuz. Mathematica, Vol. 12, 1996, pp. 1–9.

[21] G. KOLOKOLNIKOVA, *A Variational Maximum Principle for Discontinuous Trajectories of Unbounded Asymptotically Linear Control Systems*, J. of Differential Equations, Vol. 33, N 12, (1997), pp. 1633–1640 (transl. from Diff. Uravn., 33, N 12, pp. 1631–1638).

[22] R. B. VINTER, F. M. L. PEREIRA, *A maximum principle for optimal processes with discontinuous trajectories*, SIAM J. Control and Optim., 26 (1988), pp.205–229.

[23] F. PEREIRA, G. SILVA, *Necessary Conditions of Optimality for Vector-valued Impulsive Control Problems*, Systems Control Letters, 40, (2000), pp. 205–215.

[24] G. N. SILVA AND R. B. VINTER, *Necessary Conditions For Optimal Impulsive Control Problems*, SIAM J. Control and Optim., 35 (1997), pp.1829–1846.

[25] V. A. DYKHTA *Second order necessary optimality conditions for impulsive control problems and multiprocesses*, in: Singular solutions and perturbations in control systems. Eds. V. Gurman, B. Miller, M. Dmitriev. Pergamon, Elsevier Science Ltd., Kidlington, Oxford (UK), (1997), pp. 97–101.

[26] O. N. SUMSONUK, *Quadratic Optimality Conditions for Optimal Impulsive Control Problems*, Proceedings of the 12th Baikal International Conference, Optimization Methods and Their Applications, Irkutsk, Baikal, (2001), vol. 2, pp. 144–149.

[27] U. LEDZEWICZ AND H. SCHAETTLER, *Higher Order Conditions for Optimality*, SIAM J. Control and Optim., Vol. 37, N 1, (1998), pp. 33–53.

[28] A. A. AGRACHEV, R. V. GAMKRELIDZE, *Index of Extremality and Quasiextremality*, Russian Math. Doklady, v. 284 (1985), pp. 11–14.