Synchrony-breaking bifurcations in networks

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Resumo

Sistemas de células acopladas associados a uma rede são sistemas dinâmicos que respeitam a estrutura da rede. Um dos objectivos da teoria de células acopladas é relacionar propriedades das redes e dos seus sistemas de células acopladas. Um exemplo significativo é a correspondência biunívoca entre as colorações balanceadas no conjunto de células da rede e os espaços de sincronia que são invariantes para qualquer sistema de células acopladas associados a esta rede. A restrição de qualquer sistema de células acopladas a um espaço de sincronia é também um sistema de células acopladas associado a uma rede menor. Esta rede menor é designada de rede quociente e a rede original é dita ser um levantamento da rede quociente. Estudamos as bifurcações que quebram a sincronia com co-dimensão um de pontos de equilíbrio em sistemas de células acopladas num ponto de bifurcação com sincronia total e o problema do levantamento de bifurcações. Dada uma rede, uma das suas redes quociente e um problema de bifurcação em ambas as redes, o problema do levantamento de bifurcações questiona se todos os ramos de bifurcação associados à rede são levantados dos ramos de bifurcação associados à rede quociente. Restringimos a nossa atenção a dois tipos de redes: redes regulares e redes homogéneas com entradas assimétricas.

Dada uma rede homogénea com entradas assimétricas, a sua rede fundamental revela as suas simetrias ocultas. Métodos para o estudo das bifurcações em redes fundamentais estão disponíveis na literatura. Damos uma caracterização das redes fundamentais. Provamos a existência de ramos de bifurcação de pontos de equilíbrio com sincronia maximal e submaximal para problemas de bifurcação em redes regulares. Caracterizamos os ramos de bifurcação de pontos de equilíbrio para problemas de bifurcação em redes homogéneas com entradas assimétricas com uma condição de bifurcação dada por a valência da rede e estudamos o problema do levantamento de bifurcações para estes problemas de bifurcação. Descrevemos os ramos de bifurcação de pontos de equilíbrio para problemas de bifurcação em redes de transição adiante e abordamos o respectivo problema do levantamento de bifurcações.

Para caracterizar as redes fundamentais, estudamos a relação entre redes e as suas redes fundamentais. Damos condições para uma rede ser uma rede quociente da sua rede fundamental e para a rede fundamental ser uma sub-

rede da rede. Além disso, relacionamos propriedades gráficas das redes e das suas redes fundamentais. Essas propriedades são o tamanho dos ciclos numa rede e a distância máxima entre as células e os ciclos da rede.

No estudo de bifurcações com co-dimensão um de pontos de equilíbrio que quebram a sincronia em sistemas de células acopladas associados a redes regulares num ponto de equilíbrio com sincronia total, é conhecido um resultado semelhante ao Equivariant Branching Lemma que mostra a existência genérica de ramos de bifurcação com sincronia axial. Para além de estendermos este resultado para ramos de bifurcação com sincronia máxima, também damos condições necessárias e suficiente para a existência de ramos de bifurcação com sincronia submáxima. Estas condições apenas dependem da estrutura da rede. Ademais, apresentamos exemplos de redes que mostram que a estrutura do reticulado dos espaços de sincronia de uma rede não é suficiente para determinar quais são os espaços de sincronia que suportam um ramo de bifurcação.

Para qualquer rede homogénea com entradas assimétricas, a matriz Jacobiana de qualquer sistema de células acopladas num ponto de equilíbrio com sincronia total tem soma das linhas constante. Esta constante é designada por valência da rede. Para caracterizar os ramos de bifurcação de pontos de equilíbrio em problemas de bifurcação com uma condição de bifurcação dada por a valência da rede, provamos um resultado semelhante ao Teorema de Perron–Frobenius. Este resultado mostra que a dimensão do espaço próprio associado à valência é igual ao número de fontes na rede. No estudo do problema do levantamento de bifurcações para os problemas de bifurcação mencionados anteriormente, verificamos que o aumento do número de fontes numa rede em relaçao a uma das suas redes quocientes é uma condição necessária, mas nem sempre suficiente, para a existência de um ramo de bifurcação na rede que não seja levantado da rede quociente. Apesar disto, existe uma classe de redes e suas redes quocientes tais que qualquer ramo de bifurcação associado à rede é levantado da sua rede quociente se e só se a rede e a sua rede quociente têm o mesmo número de fontes.

Redes de transição adiante são redes homogéneas com entradas assimétricas onde o conjunto de células pode ser particionado em camadas. A restrição de um sistema de transição adiante a um dos seus espaços de sincronia pode, ou não, ser um sistema de transição adiante. Estudamos os levantamentos de redes de transição adiante que têm uma estrutura de transição adiante e definimos dois tipos desses levantamentos designados por levantamentos que criam novas camadas e levantamentos dentro de uma camada. Mostramos que os levantamentos de transição adiante conexos para trás podem ser decompostos nestes dois tipos de levantamentos. No estudo de bifurcações com co-dimensão um de pontos de equilíbrio em sistemas de transição adiante num ponto de equilíbrio com sincronia total, existem duas condições de bifurcação possíveis. Dizemos que uma das condições de bifurcação é dada por a dinâmica interna e a outra é dada por a valência da rede. Descrevemos

os ramos de bifurcação para problemas de bifurcação com qualquer uma das duas condições de bifurcação. Por último, analisamos o problema do levantamento de bifurcações em sistemas de transição adiante para os dois tipos de levantamentos. Para a maioria dos levantamentos de transição adiante, existe um ramo de bifurcação associada à rede que não é levantado da rede quociente se e só se a dimensão do espaço central do problema de bifurcação aumenta para a rede. Para levantamentos dentro de uma camada, mostramos que o problema do levantamento de bifurcações depende genericamente do sistema de transição adiante escolhido. Em particular, damos condições sobre sistemas de transição adiante, tais que, tomando a condição de bifurcação dada por a dinâmica interna, alguns levantamentos podem ter ou não ramos de bifurcação que não sejam levantados da rede quociente.

Palavras-chave: Rede de células acopladas, Sistema de células acopladas, Rede fundamental, Conectividade da rede, Rede circular, Bifurcação de pontos de equilíbrio, Bifurcação com quebra de sincronia, Problema do levantamento de bifurcações, Rede de transição adiante.

Abstract

Coupled cell systems associated to a network are dynamical systems that respect the structure of that network. One of the goals in coupled cell theory is to relate properties of the network and of their coupled cell systems. A thriving example is the one–to–one correspondence between balanced colorings on the set of cells of a network and synchrony subspaces which are flow-invariant subspaces for any coupled cell system. The restriction of any coupled cell system to a synchrony subspace is again a coupled cell system but associated with a smaller network. This smaller network is called a quotient network and the original network is said to be a lift of the smaller network. We study codimension-one steady-state synchrony-breaking bifurcations on coupled cell systems and the lifting bifurcation problem. Given a lift network, one of its quotient networks and a bifurcation problem on both networks, the lifting bifurcation problem asks if all the bifurcation branches associated with the lift network are lifted from the bifurcation branches associated with the quotient network. We restrict our attention to two types of networks: regular networks and homogenous networks with asymmetric inputs.

Given a homogenous network with asymmetric inputs, its fundamental network reveals the hidden symmetries of the given network. Bifurcation methods are available in the literature for fundamental networks. We give a characterization of fundamental networks. We prove the existence of steady-state bifurcation branches with maximal and submaximal synchrony in bifurcation problems on regular networks. We characterize the steady-state bifurcation branches in bifurcation problems with a bifurcation condition given by the network valency on homogeneous networks with asymmetric inputs and we study the lifting bifurcation problem for those bifurcation problems. We describe the steady-state bifurcation branches in bifurcation problems on feed-forward networks and we address the respective lifting bifurcation problem.

In order to characterize fundamental networks, we study the relationship between networks and their fundamental networks. We give conditions for a network to be a quotient network of its fundamental network and for a fundamental network to be a subnetwork of a network. Furthermore, we relate architectural properties of networks and their fundamental networks.

Those properties are the size of cycles in a network and the maximal distance between the cells and the cycles of a network.

In the study of codimension-one steady-state synchrony-breaking bifurcations of coupled cell systems associated to regular networks at a full-synchrony equilibrium, it is known a similar result to the Equivariant Branching Lemma showing the generic existence of bifurcation branches with axial synchrony. We extend this result to the existence of bifurcation branches with maximal synchrony. We also give necessary and sufficient conditions for the existence of bifurcation branches with submaximal synchrony. Those conditions only depend on the network structure. Furthermore, we give examples showing that the lattice structure of the synchrony subspaces is not sufficient to determine which synchrony subspaces support a bifurcation branch.

For any homogeneous network with asymmetric inputs, the Jacobian matrix of any coupled cell system at a full-synchrony equilibrium has constant row-sum. This constant is called the network valency. In order to characterize the steady-state bifurcation branches in bifurcation problems with a bifurcation condition given by the network valency, we prove a result similar to the Perron–Frobenius Theorem. This results shows that the dimension of the eigenspace associated with the network valency is equal to the number of source components in the network. In the study of the lifting bifurcation problem for the previous stated bifurcation problem, we see that increasing the number of source components in a lift network is a necessary, but not always sufficient, condition for the existence of a bifurcation branch on the lift network which is not lifted from the bifurcation branches on the quotient network. Nevertheless, there exists a class of lift networks and quotient networks such that every bifurcation branch associated with the lift network is lifted from the quotient network if and only if the two networks have the same number of source components.

Feed-forward networks are homogeneous networks with asymmetric inputs where the set of cells can be partition into layers. The restriction of a feed-forward system to one of its synchrony subspaces may be, or not, a feed-forward system. We focus on lifts of feed-forward networks that have a feed-forward structure and we define two types of those lifts called lifts that create new layers and lifts inside a layer. We show that a backward connected feed-forward lift of a feed-forward network can be decomposed using those two types of lifts. In the study of codimension-one steady-state bifurcations of feed-forward systems at a full-synchrony equilibrium, there are two different bifurcation conditions. One bifurcation condition is said to be given by the internal dynamics and the other is said to be given by the network valency. We describe the bifurcation branches for the bifurcation problems given by any of the two bifurcation conditions. In the study of the lifting bifurcation problem on feed-forward systems, we consider the two types of lifts. For most feed-forward lifts, there exists a bifurcation branch

associated with the lift network not lifted from the quotient network if and only if the dimension of center subspace of the bifurcation problem increases for the lift network. For lifts inside a layer, we show that the lifting bifurcation problem does depend on the chosen feed-forward system. In particular, we give conditions on feed-forward systems, such that, taking the bifurcation condition given by the internal dynamics, certain lifts may have or not branches of solutions that are not lifted from the quotient network.

Keywords: Coupled cell network, Coupled cell system, Fundamental Network, Network Connectivity, Ring Network, Steady-state bifurcation, Synchrony-breaking bifurcation, Lifting bifurcation problem, Feed-forward network.

Contents

1. Introduction

This thesis focus on the study of synchrony–breaking bifurcations in coupled cell systems. In the present introduction, we give an overview of the definitions and terminology that are relevant for the work developed here. We also describe the current state of the art for bifurcations in coupled cell systems and some related areas. In the last part of this introduction, we describe the main contributions of the four articles that constitute the thesis and how it is organized.

1.1 Networks and coupled cell systems

We follow the formalism proposed by Golubitsky, Stewart, and collaborators in [107, 76] for (coupled cell) networks and coupled cell systems. A network is a directed graph where the cells and edges are labeled with types. The directed graph can include self–loops and multiple edges from the same output cell to the same input cell. Moreover, the type of cells and edges satisfy the following conditions. If two edges have the same type then their starting cells have the same type and their targeting cells also have the same type. If two cells have the same type then there exists a bijection between the input edges of each cell respecting the edges' types. This condition is also called input equivalence in [107, 76] and we include it here in the definition of network. An adjacency matrix is a square matrix of dimension equal to the number of cells where its entries represent the number of connections. Networks are defined by adjacency matrices corresponding to each type of edge. Figure 1.1 is an example of a network. In general, different arrows represent the different types of edges.

Coupled cell systems associated to a network are dynamical systems that respect the network structure in the following way. First, a phase space is assigned to each cell such that cells with the same type have the same phase space. The network phase space is the product of the cells phase spaces. A vector field on the network phase space is admissible for a network if the component of the vector field corresponding to any cell depends on the state of that cell and on the state of the cells with edges directed to it. Moreover, the admissible vector fields respect the cells and edges types. The components of an admissible vector field corresponding to cells of the same

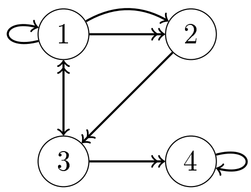

Figure 1.1: A homogeneous network with four cells of the same type and two types of edges. Different arrowheads correspond to different edge types. Each cell receives exactly one input of each type.

type are determined by the same function. If a cell receives two edges of the same type, then the component of the vector field corresponding to that cell does not change by interchanging the state variable of the starting cells of those two edges. A coupled cell system is a dynamical system given by an admissible vector field. Discrete coupled cell systems can be defined in a similar way, but they are not treated in this text.

For illustration purposes, a coupled cell system associated with the network in Figure 1.1 has the following form

$$\begin{cases} \dot{x_1} = f(x_1, x_1, x_3) \\ \dot{x_2} = f(x_2, x_1, x_1) \\ \dot{x_3} = f(x_3, x_1, x_2) \\ \dot{x_4} = f(x_4, x_4, x_3) \end{cases}, \tag{1.1}$$

where x_i belongs to the phase space of cell i and $f(x_i, x_j, x_k)$ belongs to the tangent space of the cell phase space at the point x_i for $1 \leq i, j, k \leq 4$.

Coupled cell systems have been used to describe real–world phenomena. In applications, the studied objects and their dependencies can be naturally represented using, respectively, the cells and edges of a network. Studied applications include binocular rivalry [46, 47, 45], coupled oscillators [83], homeostasis [74] and locomotion patterns [32, 96]. An overview of networks, coupled cell systems, and their applications can be found in [103, 70, 71]. Definitions of networks and coupled cell systems have appeared in the literature using combinatorial [51], categorical [39] and functional [26, 27] approaches. In [10], an extension of networks and coupled cell systems to weighted directed graphs is presented.

1.2 Balanced colorings, quotient networks and lift networks

One of the main goals of the coupled cell networks theory is to connect properties of a network and dynamical properties of the associated coupled cell systems. A striking example of that is the existence of subspaces which are flow–invariant for every coupled cell system associated with a given network. Those subspaces are called polydiagonal and are given by the equality of some cell coordinates. Each polydiagonal subspace corresponds to a coloring on the set of cells of the network where two cells share the same color only if we have equality of the corresponding coordinates in that subspace. In the other direction, each coloring of cells defines a unique polydiagonal subspace. In [76], the authors prove that a polydiagonal subspace is flow–invariant for any coupled cell system if and only if the corresponding coloring is balanced. Those flow–invariant polydiagonal subspaces are called synchrony subspaces. A coloring of cells is balanced if given any two cells with the same color there exists a bijection between their input edges preserving the type of edges and the color of the corresponding starting cells. A relevant fact is that, the restriction of a coupled cell system associated with a network to a synchrony subspace is a coupled cell system associated with a new network called the quotient network. The quotient network is a smaller network obtained by merging cells with the same color. In the other direction, splitting some cells of the network leads to a larger network. In this case, the larger network is said to be a lift of the original network when the splitting satisfies the rules for the original network to be a quotient network of the lift. Trivially, any solution of a coupled cell system associated with the quotient network is lifted to a solution of a coupled cell system associated to the lift network.

Recall the network presented in Figure 1.1 and the form of the respective coupled cell systems (1.1). Note that $\{x_1 = x_4\}$ is flow–invariant for any system of the form (1.1) and so it is a synchrony subspace. The restriction of (1.1) to $\{x_1 = x_4\}$ is equivalent to the following system

$$\begin{cases} \dot{x_1} = f(x_1, x_1, x_3) \\ \dot{x_2} = f(x_2, x_1, x_1) \\ \dot{x_3} = f(x_3, x_1, x_2) \end{cases} . \tag{1.2}$$

The coloring associated to $\{x_1 = x_4\}$ is given by the following three classes: $\{1, 4\}$, $\{2\}$ and $\{3\}$. Cells 1 and 4 receive an edge with one arrowhead from themselves and an edge with two arrowheads from cell 3. Then this coloring is balanced and the quotient network is obtained by merging cells 1 and 4. See Figure 1.2. Furthermore, any coupled cell system associated with the quotient network has the form given in (1.2). The network in Figure 1.1 is said to be a lift of the network in Figure 1.2.

Balanced colorings and synchrony subspaces have received much attention. The set of synchrony subspaces associated with a given network forms a finite lattice, partially ordered by the inclusion. Since each synchrony subspace corresponds to a unique balanced coloring, the set of balanced colorings is a lattice taking the refinement relation, [104]. Different authors have developed methods to calculate those lattices, see [18, 80, 82, 6, 90, 3].

Aguiar and coworkers have studied the impact on the set of balanced col-

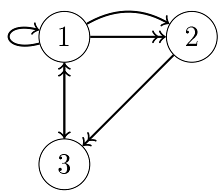

Figure 1.2: Quotient network of the network in Figure 1.1 obtained by merging cells 1 and 4. In the quotient network, the merged cell is labeled by 1. Note that in the network of Figure 1.1 cells 1 and 4 receive an edge with one arrowhead from themselves, so the merged cell receives an edge with one arrowhead from itself. Also cells 1 and 4 receive an edge with two arrowheads from cell 3, and so the merged cell in the quotient network receives an edge with two arrowheads from cell 3.

orings by some network operations such as join and coalescence [17], products [7], addition and removing of cells and edges in a network [14] and the complement of a network [9]. Also, the lattice of synchrony subspaces associated with networks having a prescribed structure is known. This is the case for grid networks with coupling to the nearest neighbors [112, 19, 20, 21, 42, 43] and feed–forward networks, [11].

There exists an intrinsic relation between robust patterns of synchrony of hyperbolic bounded solutions of coupled cell systems and balanced colorings of the network's cells, [108, 109, 65, 72]. The synchrony pattern of a solution is the lower-dimensional polydiagonal subspace that contains this solution. Given a hyperbolic equilibrium point or a hyperbolic periodic orbit of a coupled cell system, any small perturbation of that coupled cell system has a unique perturbed bounded solution associated to the given solution. Consider perturbations of a coupled cell system that preserve the relation with the same network. A pattern is robust if any perturbed solution of such small perturbations of the coupled cell system has the same synchrony pattern. In [108, 109, 65, 72], the authors prove that a hyperbolic equilibrium point or a hyperbolic periodic orbit of a coupled cell system has a robust pattern of synchrony if and only if the corresponding coloring is balanced.

1.3 Network fibrations and symmetries

A network fibration from a network to another is given by a cell function from the set of cells of the former network to the set of cells of the latter and an edge function from the set of edges of the former network to the set of edges of the latter. The cell and edge functions preserve the source cells and target cells of edges, [31], that is the source (target) cell of an edge is sent by the cell function to the source (respectively, target) cell of the edge image by the edge function. Moreover, the cell and edge functions respect the types of cells and edges, respectively. Each network fibration induces a phase space function from the phase space of the latter network to the phase space of the former one. Furthermore, the phase space function conjugates coupled cell systems associated to those networks [38].

It is known that the symmetries of a network have an impact at the admissible vector fields and the corresponding coupled cell systems. The symmetries of a network correspond to the bijective network fibrations from a network to itself. The set of phase space functions given by those network fibrations forms a group action that commutes with any coupled cell system. A dynamical system is called equivariant, under a group action, if it commutes with that group action. The vast literature on equivariant dynamical systems provides tools to understand coupled cell systems, see for example [110, 67, 75, 69, 34, 52], and it is a source of inspiration for their study. For example, it is known for equivariant dynamical systems that the spatiotemporal symmetries of periodic solutions are described by cyclic groups, see [69]. For coupled cell systems, it is known that the robust phase–shifts of a hyperbolic periodic orbit are given by the symmetries of a quotient network, [66, 72]. Nevertheless, network symmetries do not explain every feature of the dynamics of coupled cell systems. The natural flow–invariant subspaces of an equivariant system are the fixed point subspaces defined by the subgroups of the given group. Recall that the fixed point subspace defined by a subgroup is the set of points fixed by the action of every element of that subgroup. For network symmetries, the fixed point subspaces are polydiagonal subspaces. So every fixed point subspace associated to a subgroup of the network symmetries is a synchrony subspace. However, for generic networks there can be other synchrony subspaces that are not fixed point subspaces of the network symmetries. In [62, 22, 23, 24], the authors present examples of such networks.

In general, there are network fibrations from a network to itself that are not bijections but that also have an impact at the dynamical properties of the coupled cell systems associated to that network. In [92], the authors studied homogeneous networks with asymmetric inputs, i.e., networks where every cell has the same type and each cell receives exactly one edge of each type. For any such network, they prove that there exists a network, called fundamental network, that reveals the hidden partial and full symmetries of the original network. For illustration purposes, the fundamental network of the network in Figure 1.2 is given in Figure 1.3. The fundamental network is essentially the Cayley graph of a semigroup. The fundamental network has a large degree of symmetries. This large degree of symmetries can improve the understanding of the corresponding coupled cell systems.

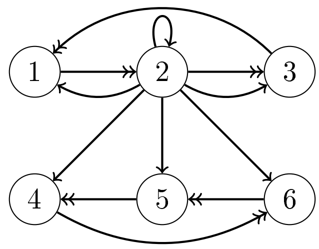

Figure 1.3: Fundamental network of the network in Figure 1.2. The fundamental network is a homogenous network with the same asymmetric inputs.

1.4 Coupled cell systems

Before we head to the main topic of this work, synchrony–breaking bifurcations in coupled cell networks, we overview two topics of research in networks and coupled cell systems.

Two non–identical networks can have equivalent coupled cell systems and, in that case, the two networks are said to be dynamical equivalent. In [44], the authors prove that the dynamical equivalence between two networks only depends on the equivalence of the linear coupled cell systems of both networks. A different approach, more combinatorial and based on the inputs and outputs sets of the networks, to study the dynamical equivalence between networks is presented in [1, 2]. Two dynamical equivalent networks have the same number of cells but the number of edges can vary. The network with fewer edges inside a class of dynamical equivalent networks is called the minimal network. In [5], the authors describe and give a method to obtain the minimal network.

Coupled cell systems display some particular properties such as robust heteroclinic networks, see for example [4]. Roughly speaking, a heteroclinic network is a collection of equilibrium points (or other invariant sets) in a dynamical system and trajectories of that dynamical system connecting those equilibrium points. In general, heteroclinic networks do not persist under small perturbations. However, heteroclinic networks can be robust for equivariant and coupled cell systems. For example, methods to construct heteroclinic networks using networks are presented in [53, 25, 54]. Given two networks and respective coupled cell systems supporting robust heteroclinic networks, the product of those two networks admits coupled cell systems that robustly support the product of the given heteroclinic networks [8].

1.5 Bifurcation theory

Given a manifold X and a smooth family of vector fields on that manifold, $F : X \times \mathbb{R}^p \to TX$, which depends on the parameter $\lambda \in \mathbb{R}^p$, consider the following differential system

$$\dot{x} = F(x, \lambda), \tag{1.3}$$

where $x \in X$, $\lambda \in \mathbb{R}^p$ and $F(x, \lambda) \in T_x X$. A codimension–p bifurcation on (1.3) occurs at $\lambda_0 \in \mathbb{R}^p$ if for any neighborhood U of λ_0 there exists $\lambda_1 \in U$ such that the system (1.3) at λ_0 is not topologically equivalent to the system (1.3) at λ_1. Two dynamical systems are topologically equivalent if there exists a homeomorphism (a continuous function with a continuous inverse function) mapping the trajectories of one dynamical system into trajectories of the other system.

Here, we consider local bifurcations at a point (x_0, λ_0) and local topological equivalence which is the restriction of the topological equivalence to a sufficiently small neighborhood of x_0 in X. Using a local coordinate chart, we assume that $X = \mathbb{R}^n$ for some $n > 0$, (x_0, λ_0) is the origin and

$$F : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n.$$

We say that the system (1.3) is (locally) structurally stable at (x_0, λ_0) if there is no local bifurcation at the point (x_0, λ_0), i.e., the system (1.3) at λ_1 is locally topologically equivalent to the system (1.3) at λ_0 for any λ_1 in a sufficiently small neighborhood of λ_0.

If x_0 is a regular point of $F(x, \lambda_0)$, i.e., $F(x_0, \lambda_0) \neq 0$, then the system (1.3) is structurally stable at (x_0, λ_0), by the Tubular Flow Theorem [95, Theorem 1.1]. Suppose that x_0 is a singular point of $F(x, \lambda_0)$, i.e., $F(x_0, \lambda_0) = 0$ and x_0 is an equilibrium point of (1.3) at λ_0. We say that the equilibrium point x_0 is hyperbolic if the real part of every eigenvalue of $DF(x_0, \lambda_0)$ is not zero, where $DF(x_0, \lambda_0)$ is the Jacobian matrix of F with respect to x at the origin. By the Hartman–Grobman Theorem [95, Theorem 4.1], we know that the system (1.3) is structurally stable at (x_0, λ_0), for any hyperbolic equilibrium point x_0 of (1.3) at λ_0. In order to have a local bifurcation, we need to have a non–hyperbolic equilibrium point x_0 of (1.3) at λ_0.

Suppose now that x_0 is a non–hyperbolic equilibrium point of (1.3) at λ_0 and the parameter space is one dimensional, $p = 1$. For generic vector fields $F(x, \lambda)$, we know that the eigenvalues of $DF(x_0, \lambda_0)$ are simple [33]. So, in general, $DF(x_0, \lambda_0)$ has a zero eigenvalue or a conjugated pair of imaginary eigenvalues. If $DF(x_0, \lambda_0)$ has a zero eigenvalue, a generic system (1.3) has a saddle–node (or fold) bifurcation at (x_0, λ_0), i.e., in one side of $\lambda = 0$ there are two equilibria points that merge at $\lambda = 0$ into x_0 and there is no equilibrium on the other side of $\lambda = 0$. See for example [111, Section 3.2]. If $DF(x_0, \lambda_0)$ has a pair of conjugated imaginary eigenvalues, then $DF(x_0, \lambda_0)$ is invertible. It follows from the Implicit Function Theorem that there exist a neighborhood U of $\lambda = 0$ and a function $z : U \to X$ such that $z(0) = 0$ and z is the unique solution of

$$F(x, \lambda) = 0 \tag{1.4}$$

in a sufficient small neighborhood of the origin. Hence the number of equilibrium points does not change when we vary the parameter λ. However, a periodic orbit appears when we cross the bifurcation parameter $\lambda = 0$. This is a well–known bifurcation called Hopf bifurcation, see, e.g., [111, Section 3.2], [84, Section 3.5]. We say that the system (1.3) at (x_0, λ_0) has a steady–state bifurcation if the number of equilibrium points in the system (1.3) changes for any small neighborhood of (x_0, λ_0). Clearly, if the number of equilibrium points changes then the two systems are not topologically equivalent. A steady–state bifurcation can only happen if $DF(x_0, \lambda_0)$ is not invertible. Understanding steady–state bifurcations in the system (1.3) at (x_0, λ_0) is given by the solutions of the equation (1.4).

We briefly describe some well–known methods in bifurcation theory. The Lyapunov–Schmidt reduction reduces the dimension of the steady–state bifurcation equation given by (1.4), [67, Chapter VII]. Using the Lyapunov–Schmidt reduction, we know that the solutions of (1.4) are in a one–to–one correspondence with the solutions of a new equation,

$$g(y, \lambda) = 0, \tag{1.5}$$

where $g : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^k$, and k is the dimension of the kernel of $DF(0, 0)$. The center manifold reduction also reduces the bifurcation problem to the study of a flow–invariant manifold that includes all trajectories which are close to x_0 at any time. This flow–invariant manifold is called the center manifold and it is tangent to the center subspace of $DF(0, 0)$, the subspace generated by the generalized eigenvectors of the eigenvalues with zero real part. After reducing the bifurcation problem using one of the previous methods, it is desirable to determine the generic vector fields up to topological equivalence. Normal forms [91] and singularity theory [60] are two different methods to obtain the generic vector fields. In normal forms, the Taylor series of a vector field is simplified up to some (arbitrary) degree using coordinate changes close to the identity. Choosing a class of coordinate changes and an appropriate Lie bracket, we can define and calculate the normal form, up to some degree, of a vector field. Next, we can truncate the vector field to that degree and study the truncated system. However, it is not trivial to see, and it is not always the case, that the truncated system is topologically equivalent to the original system. In singularity theory, the vector fields are classified according to their type of singularity in such a way that the vector fields of the first type are the most common, the vector fields of the second type are the most common vector fields which are not of the first type, and so on. In the beginning of this section, we looked at regular points which are the most common points of a generic vector field, then we look to hyperbolic equilibrium points that are the most common non–regular points of a generic vector field. The same idea is used in singularity theory and the vector fields are studied according to their type of singularity. In this way, we can determine the generic vector fields and how bifurcations can appear.

1.6 Equivariant bifurcation

As noted before, the equivariant theory is a source of inspiration for the study of coupled cell systems. In this section, we recall some well–known results and techniques for bifurcations in equivariant systems. In the next section, we see that some of those results and techniques have been adapted to the study of bifurcations in coupled cell systems.

Given a finite group action on a manifold, the symmetries of a point are given by the elements of the group fixing that point. The set of those elements is a subgroup of the given group and it is called an isotropy subgroup. A point has full symmetry if its isotropy subgroup is the complete group. Up to conjugacy, the set of isotropy subgroups is a lattice, partially ordered by inclusion. For any dynamical system equivariant under a group action, the set of points fixed by a subgroup is a flow–invariant subspace which is called the fixed point subspace given by that subgroup. The set of fixed point subspaces, up to conjugacy, of the isotropy subgroups is a lattice partially ordered by inclusion.

Usually, the bifurcation point of an equivariant bifurcation is assumed to have the full symmetry. One of the main questions in equivariant bifurcation theory asks about the symmetries of the new bounded solutions, such as equilibrium points and periodic orbits. Whenever the new solutions have less symmetry than the bifurcation point, the bifurcation is said to be symmetry–breaking. Those bifurcations have been extensively studied, see e.g. [75, 50, 34, 52].

One of the first results in symmetry–breaking bifurcations is the Equivariant Branching Lemma [35, 36]. Assume that there is a unique point with full isotropy, which is the origin and that this point is the bifurcation point. For isotropy subgroups with one–dimensional fixed point subspace, there are generic bifurcations problems in equivariant systems that have solutions with the symmetry of those isotropy subgroups. A non–trivial isotropy subgroup is said to be maximal if it is not contained in any other isotropy subgroup except the full group. The Equivariant Branching Lemma also holds for maximal isotropy subgroups which have odd dimensional fixed point subspaces. Moreover, it is known that some non–maximal isotropy subspaces support symmetry–breaking bifurcations, [55, 58, 86, 85]. For generic Hopf bifurcation in equivariant systems, there is an analogous result to the Equivariant Branching Lemma leading to the appearance of a symmetry–breaking periodic orbit, [68].

The two reduction methods described above, Lyapunov–Schmidt reduction and center manifold reduction, have adaptations to equivariant systems. The reduction obtained by those methods can preserve the symmetries of the original system, [110, 34]. So it is usually assumed that one of those two methods have been applied before the study of equivariant bifurcations. Singularities of equivariant systems and equivariant unfolding theory have been studied, see e.g. [28, 67, 75]. The unfolding theory describes the generic small perturbations of a dynamical system and it helps the understanding of the possible bifurcations.

Bierstone [29] and Field [49] define equivalent notions of transversality for equivariant systems. A function is transverse to a submanifold at a point if its evaluation at that point does not belong to that submanifold or the linearization of the function together with the tangent submanifold generate the tangent manifold at that point. In the space of functions without symmetries, the functions transverse to a submanifold are dense. This leads to the fact that the set of solutions of (1.4) is a manifold. However, this is not the case for equivariant functions. For example, it is well–known that the generic bifurcations of equivariant systems with $\mathbb{Z}_2$–symmetry is a pitchfork bifurcation. In this case, the set of solutions of (1.4) is not a manifold. Therefore the usual concept of transversality cannot be applied to equivariant functions. The set of equivariant functions is a finitely generated module over the set of invariant functions. This fact is essential in both definitions of transversality for equivariant systems. Those definitions lead to density results on the set of equivariant functions similar to the ones established for generic functions. Later, equivariant transversality prove to be useful in the study of equivariant bifurcations [56, 57, 52].

Another line of research in equivariant systems is the equivariant degree theory. Roughly speaking, the degree theory estimates the number of solutions given by some equation. As in the case of transversality, the usual degree theory cannot be applied to equivariant systems. An adaptation of degree theory to equivariant functions can be found in [78, 79]. The estimation of the number of solutions is useful in bifurcations, when proving the existence of bifurcation branches on steady–state bifurcations.

1.7 Coupled cell bifurcations

We describe now some of the work that has been done concerning bifurcations in coupled cell systems. As most of the work in the literature, we focus on homogeneous networks, i.e., networks where there is only one type of cells.

Given a network and a choice of phase space, the synchrony of a point is given by the balanced coloring corresponding to the lower-dimensional synchrony subspace that contains that point. For homogeneous networks, the diagonal subspace is always a synchrony subspace called the full–synchrony subspace and the points on the diagonal are said to have full–synchrony. Assume from now on that the bifurcation point has full–synchrony and that the bifurcation problem has codimension–one. In bifurcations of coupled cell systems at a full–synchrony point, new bounded solutions (equilibrium points or periodic orbits) can have less synchrony than the bifurcation point, i.e., they belong to a synchrony subspace bigger than the full–synchrony subspace. Analogously to equivariant bifurcations, this phenomenon is called a synchrony–breaking bifurcation. It is clear that the symmetries of a network have direct consequences in the bifurcations of coupled cell systems, [63].

There exists an analogous result to the Equivariant Branching Lemma for steady–state bifurcations in coupled cell systems with a bifurcation condition given by a simple eigenvalue [61]. This result proves the existence of synchrony–breaking bifurcations on coupled cell systems. The structure of the network imposes restrictions on the bifurcation problems. In particular, it can force the degeneracy of a steady–state bifurcation problem in coupled cell systems to be arbitrary high [106, 105].

The structure of a network can also impose that the Jacobian matrix of a generic coupled cell system at a full–synchrony equilibrium has a nilpotent part. This leads to surprising features of the bifurcations occuring on coupled cell systems. In particular, the bifurcation branches of a bifurcation problem on coupled cell systems with a nilpotent part have an exceptional growth rate, [62, 48, 64, 97, 59, 77]. This is the case for the bifurcations in coupled cell systems associated with the feed–forward networks. The set of cells in a feed–forward network can be partition into disjoint subsets: the edges targeting a cell in a layer start in the previous layer, except for cells in the first layer which only receive self–loops. For homogenous feed–forward networks, the Jacobian matrix of any coupled cell system at a full–synchrony equilibrium has a nilpotent part. The rank of the nilpotent part increases with the number of layers. Furthermore, the exceptional growth rate of the bifurcation branches increases with the number of layers.

The exhaustive study of regular networks with three cells and two inputs edges for each cell can be found in [87]. A homogenous network is said to be regular if there is exactly one type of cells and one type of edges. The authors enumerate all such networks and study the steady–state and Hopf bifurcations of the corresponding coupled cell systems. They find features of equivariant bifurcation, such as transcritical and pitchfork bifurcations. For some networks, those features can be explained by the network symmetries. However, this is not always the case. In [81], the author studies network synchrony–breaking bifurcation when the adjacency matrix of the network only has simple eigenvalues. Using the lattice of synchrony subspaces, the author is able to characterize the synchrony subspaces that support a synchrony–breaking bifurcation.

Methods of bifurcation theory have been adapted to bifurcations in coupled cell systems, in particular, for homogeneous networks with asymmetric inputs, see [98, 99, 93]. A homogeneous network has asymmetric inputs if every cell receives exactly one edge of each type. The edges of those networks can then be represented using functions between the set of cells with one function for each type of edges. The coupled cell systems form a Lie algebra, whenever their representative functions define a monoid, [99]. In this case, normal forms of coupled cell systems can be defined and computed.

In [98], the authors prove that the Lyapunov–Schmidt reduction preserves the symmetries of the network including the partial symmetries. They prove this result for fundamental networks such that their representative functions define a monoid. Furthermore, the center manifold reduction of coupled cell systems associated with those networks is studied in [93]. They show that the center manifold reduction can be made such that it also preserves the partial and total symmetries of the network.

Since coupled cell systems respect a prescribed network structure, there are some changes of coordinates that destroy the relationship between the coupled cell system and the network. In order to study singularities in coupled cell systems, it is vital to recognize the changes of coordinates that preserve this relation. In [73], the authors describe those changes of coordinates for heterogeneous networks – a network is said to be heterogeneous if each cell and each edge have a different type. Using the symmetries of a network and/or of its quotient networks, a degree theory for coupled cell systems is presented in [100].

Given a (lift) network and a quotient network of it, any solution of a quotient coupled cell system lifts to a solution of the respective coupled cell system associated to the (lifted) network. In the same way, the bifurcation branches of a bifurcation problem in a quotient coupled cell system lift to bifurcation branches of the respective bifurcation problem in the lifted network. However, there are networks such that some bifurcation branches associated with the lift network are not lifted from the branches associated to the quotient network. In [12, 13], the authors gave such examples. The lifting bifurcation problem asks if every bifurcation branch associated to a lift network is lifted from a bifurcation branch associated to the quotient network. This problem is related to the classification of synchrony subspaces that support a synchrony–breaking bifurcation in the following sense. If there exists a synchrony subspace that contains every bifurcation branch associated to the lift network, then every branch associated to the lift network is lifted from a branch associated to the quotient network associated to that synchrony subspace.

If the dimensions of the center subspaces associated with the bifurcation problem in the lift network and the quotient network are the same, then every branch on the lift network is lifted from the quotient network. This observation motivates the comparison between the spectrum of coupled cell systems associated with the quotient and the lift networks. In [40], the authors address this issue for a class of networks and lifts. Moreover, the lifting bifurcation problem is studied in [89, 94]. In [41], the authors study the decomposition of network lifts. In some cases there exists an intermediate network between a lift network and a quotient network such that this intermediate network is a quotient network of the lift network and a lift network of the quotient network. In those cases, the lift network is obtained by the composition of the lift from the quotient network to the intermediate network and the lift from the intermediate network to the lift network. A lift is said to be direct if there is no such intermediate network. In [41], the authors characterize the direct lifts and how to decompose network lifts. Moreover, they show how this decomposition can be used to compare spectrums.

1.8 Outline and main contributions

Here, we give an outline and the main contributions of this thesis which consists of four research articles.

The main focus of this thesis is the study of codimension–one steady–state synchrony breaking bifurcations on coupled cell systems. In particular, we aim to understand the relations between the network structure and those bifurcations, and the lifting bifurcation problem. We restrict our attention to homogeneous networks.

For regular networks, we study codimension–one steady–state bifurcations in maximal and submaximal synchrony subspaces. This leads to a generalization of one of the results obtained in [61]. For homogeneous networks with asymmetric inputs, there exists the concept of fundamental network. Moreover, bifurcation methods for applying to those networks can be found in the literature [98, 99, 93]. We give a characterization of fundamental networks. Moreover, we study the codimension–one steady–state bifurcations and the lifting bifurcation problem on coupled cell systems corresponding to homogeneous networks with asymmetric inputs and the bifurcation condition given by the network valency. We also study the codimension–one steady–state bifurcation and the lifting bifurcation problem on feed–forward systems.

This thesis is organized as follows. In Chapter 2, the accepted version for publication of the paper "Characterization of fundamental networks" [15] is reproduced. Chapter 3 is the reproduction of the published paper "Synchrony branching lemma for regular networks" [101]. Chapter 4 is the submitted version of the article "The steady–state lifting bifurcation problem associated with the valency on networks" [16]. In Chapter 5, we reproduce the submitted version of the paper "The lifting bifurcation problem on feed–forward networks" [102]. Finally, Chapter 6 includes a short discussion of the results obtained in this work and some directions of research.

Characterization of fundamental networks

We consider homogeneous networks with asymmetric inputs. In [92], the authors define the fundamental network of a given network.

In this paper, we study the relations between networks and their fundamental networks. We define two properties on networks: backward connectivity and transitivity. A network is backward connected if there exists a cell that has a backward walk to any other cell. A network is transitive if there is a cell such that any cell is the evaluation at the former cell of a self–fibration. We show that a backward connected network is a quotient network of its fundamental network. We also prove that the fundamental network of a transitive network is a subnetwork of it. Hence, a network is fundamental if and only if it is backward connected and transitive (Theorem 2.5.16). Moreover, the construction of fundamental networks preserves the quotient relation between networks (Proposition 2.5.2). The construction of fundamental networks transforms the subnetwork relation into the quotient relation (Proposition 2.5.6).

Two important properties of networks are the size of their cycles and the distance of their cells to a cycle. In [59, 89], the authors study the relation between bifurcations on coupled cell systems and those two properties. For a homogeneous network with asymmetric inputs, we consider the networks given by the restriction of that network to each type of edges. The maximal distances between any cell and a cycle in a restricted network and in the restriction of its fundamental network are equal (Proposition 2.7.2). Moreover, the size of the cycles on the restricted fundamental network is given by the least common multiple of the size of some cycles on the restricted network (Proposition 2.7.4).

Synchrony branching lemma for regular networks

We consider regular networks, i.e., homogenenous networks with one type of cells and one type of edges. We assume that the cells phase space is the real line and study codimension–one steady–state bifurcations on coupled cell systems at a full–synchrony equilibrium. The Jacobian matrix of such a coupled cell system at a full–synchrony equilibrium is determined by the adjacency matrix of the network. Thus the eigenvalues of the Jacobian matrix are characterized directly from the eigenvalues of the adjacency matrix, including their multiplicities. For axial synchrony subspaces, Golubitsky and Lauterbach prove a result analogous to the Equivariant Branching Lemma in [61]. A synchrony subspace is axial for an eigenvalue if the intersection of that synchrony subspace and the correspoding eigenspace is one dimensional. They prove that the bifurcation of coupled cell systems with a bifurcation condition given by such eigenvalue leads to a bifurcation branch inside that axial synchrony subspace. For a network such that the eigenvalues of its adjacency matrix are all simple, Kamei gives a classification of the synchrony subspaces which support a synchrony–breaking bifurcation [81]. In order to obtain this result, the author uses the previous result and the lattice structure of the synchrony subspaces.

We generalize the previous result of Golubitsky and Lauterbach to maximal synchrony subspaces where the bifurcation condition is given by a semi–simple eigenvalue. A synchrony subspace is maximal for an eigenvalue if the intersection of that synchrony subspace with the respective eigenspace is not trivial and the intersection of any synchrony subspace contained in that synchrony subspace with that eigenspace is trivial. We prove that a maximal synchrony subspace supports a synchrony–breaking bifurcation if the previous intersection has odd dimension (Theorem 3.4.1). We proceed to the study of synchrony–breaking bifurcations for submaximal synchrony subspaces. A non–maximal synchrony subspace is submaximal for an eigenvalue if the intersection of that synchrony subspace with the respective eigenspace is not trivial and the intersection of that eigenspace with any smaller synchrony subspace is at most one dimensional. When the intersection of a submaximal synchrony subspace with the eigenspace is two dimensional, we give necessary and sufficient conditions for the existence of a bifurcation branch with exactly that submaximal synchrony (Theorem 3.5.2). Those conditions only depend on the network. Moreover, we present examples of networks with the same lattice structure but different synchrony–breaking subspaces.

In order to prove the previous results, we use a well–known technique of coordinates changes called blow–up. For the existence of bifurcation branches in maximal synchrony subspaces, we use a standard argument in degree theory. In the study of submaximal synchrony subspaces, we obtain the mentioned conditions by direct computations.

The steady–state lifting bifurcation problem associated with the valency on networks

We consider homogeneous networks with asymmetric inputs and we assume that the cells phase space is the real line. For such networks, the Jacobian matrix of any coupled cell system at a full–synchrony equilibrium has constant row–sum. This constant is called the valency of the network and it is an eigenvalue of the Jacobian matrix. We study codimension–one steady–state bifurcations on coupled cell systems at a full–synchrony equilibrium where the bifurcation condition is given by the network valency.

In graph theory, the spectrum of adjacency matrices is an active area of study, see e.g. [30, 37, 88]. A graph is said to be regular if the number of edges targeting each cell is constant. The number of edges targeting a cell is called the valency of the graph. It is well–known that the valency of a regular graph is the adjacency matrix eigenvalue with bigger norm, Perron–Frobenious Theorem [30]. Moreover, the eigenspace associated to the graph valency is also described in that result.

In the first part of this paper, we describe the bifurcation branches of coupled cell systems when the bifurcation condition is given by the network valency (Proposition 4.5.7). In order to do that, we prove a similar result to the Perron–Frobenious Theorem which describes the eigenspace associated to the network valency of the Jacobian matrix of coupled cell systems at a full–synchrony equilibrium. The dimension of this eigenspace is equal to the number of source components in the network. A source component is a strongly connected component of the network whose cells receive edges only from cells in this component. Using the description of the eigenspace associated to the network valency, we describe the bifurcation branches of coupled cell systems for the bifurcation condition given by the network valency.

In the second part, we study the lifting bifurcation problem of coupled cell systems with a bifurcation condition given by the network valency. Using the previous description of the bifurcation branches, we prove the following results. If the lift network and the quotient network have the same number of source components, then every bifurcation branch associated to the lift network is lifted from a bifurcation branch associated to the quotient network (Proposition 4.6.1). If the quotient network has one source component and the lift network has more, then there exists a bifurcation branch for the lift network which is not lifted from the bifurcation branches for the quotient network. We present two examples of lift networks that have more source components than their quotient networks and such that every bifurcation branch associated to the lift networks is lifted from the respective quotient network (Examples 4.6.3 and 4.6.4). In those examples, the lift network is not backward connected or the quotient network is not transitive. Supposing that the lift is backward connected and the quotient network is transitive, we prove that every bifurcation branch for the lift network is lifted from the quotient network if and only if the two networks have the same number of source components (Theorem 4.6.5).

The lifting bifurcation problem on feed–forward networks

We consider homogeneous networks with asymmetric inputs and a feed–forward structure. In feed–forward networks, the set of cells can be partitioned into layers such that every edge targeting a cell in a layer starts in a cell of the previous layer, excluding the cells in the first layer that only have self–loops. We assume again that every cell phase space is the real line. We study codimension–one steady–state bifurcations on feed–forward systems at a full–synchrony equilibrium and the respective lifting bifurcation problem.

The quotient network of a feed–forward network may be, or not, a feed–forward network. We introduce two types of lifts in feed–forward networks such that the lift network is a feed–forward network: A lift that creates new layers is the lift obtained by splitting the first layer into a fixed number of consecutive layers; A lift inside a layer is given by splitting some cells inside that layer. Assuming that the lift network of a feed–forward network is again a feed–forward network and backward connected, we show that the lift network can be obtained by the composition of those two lifts (Proposition 5.3.10).

The feed–forward structure of the network forces the Jacobian matrix of a feed–forward system at a full–synchrony equilibrium to have a lower triangular block form upon appropriate ordering of cells. Thus the Jacobian matrix has only two eigenvalues that we call the valency and the internal dynamics. We study the codimension–one steady–state bifurcations on feed–forward systems at a full–synchrony equilibrium where the bifurcation condition is given by the valency or the internal dynamics. In [48, 64, 97], the authors study bifurcations on a certain class of feed–forward systems where the bifurcation condition is given by the internal dynamics. We extend those studies to include any feed–forward network. In particular, we give a full characterization of the bifurcation branches using their growth rates (square–root–orders) and their slopes (Proposition 5.6.6).

In the last part, we address the lifting bifurcation problem when the lift and quotient networks are feed–forward networks. Given the decomposition of feed–forward lifts into lifts that create new layers and lifts inside a layer, we restrict our analysis to those two types of lifts. The dimension of the center subspace does not increase in the following two cases: (*i*) when the bifurcation condition given by the valency, for lifts that create new layers and lifts inside a layer, except the first layer; (*ii*) when the bifurcation condition given by the internal dynamics, for lifts inside the first layer. In those cases, it is immediate that the bifurcation branches for the lift network are lifted from the quotient network. For backward connected lifts inside the first layer and feed–forward systems with a bifurcation condition given by the valency, there exists a bifurcation branch for the lift network which is not lifted from the quotient network (Proposition 5.7.6). For lifts that create new layers and feed–forward systems with a bifurcation condition given by the internal dynamics, there exists a bifurcation branch for the lift network which is not lifted from the quotient network (Proposition 5.7.10). More surprising is the fact that the lifting bifurcation problem where the bifurcation condition is given by the internal dynamics and the lift is inside a layer depends on the chosen feed–forward system. In this case, we give a condition on the feed–forward systems for any bifurcation branch associated with the lift network be lifted from the quotient network (Propositions 5.7.18 and 5.7.20). Moreover, we also obtain conditions for a bifurcation branch associated to the lift network not to be lifted from the quotient network (Proposition 5.7.16).

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2. Characterization of Fundamental Networks

This chapter consists of a joint article in press, Proceedings of the Royal Society of Edinburgh: Section A Mathematics.

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Abstract

In the framework of coupled cell systems, a coupled cell network describes graphically the dynamical dependencies between individual dynamical systems, the cells. The fundamental network of a network reveals the hidden symmetries of that network. Subspaces defined by equalities of coordinates which are flow-invariant for any coupled cell system consistent with a network structure are called the network synchrony subspaces. Moreover, for every synchrony subspace, each network admissible system restricted to that subspace is a dynamical system consistent with a smaller network, called a quotient network. We characterize networks such that: the network is a subnetwork of its fundamental network, and the network is a fundamental network. Moreover, we prove that the fundamental network construction preserves the quotient relation and it transforms the subnetwork relation into the quotient relation. The size of cycles in a network and the distance of a cell to a cycle are two important properties concerning the description of the network architecture. In this paper, we relate these two architectural properties in a network and its fundamental network.

Keywords: Fundamental Networks, Network Connectivity, Rings.

2010 Mathematics subject classification: 05C25; 05C60; 05C40

2.1 Introduction

Coupled cell networks describe influences between cells. A network is represented by a graph where each cell and each edge have a specific type. A cell type defines the nature of a cell, and an edge type defines the nature of the influence. A dynamical system that respects the network structure is a coupled cell system admissible by the network. Stewart, Golubitsky and Pivato [13], and Golubitsky, Stewart and Török [6] formalized the concepts of coupled cell network and coupled cell system. They showed that there exists an intrinsic relation between coupled cell systems and coupled cell networks, proving in particular, that robust patterns of synchrony of cells are in one-to-one correspondence to balanced colorings of cells in the network – see [13, theorem 6.5]. Coupled cell networks and coupled cell systems have been addressed, for example, from the bifurcation point of view, [1, 5, 7, 8].

Recently, Rink and Sanders [11, 12] and Nijholt, Rink and Sanders [9] developed some dynamical techniques for homogenous networks with asymmetric inputs, i.e., networks where all cells have the same type and each cell receives only one edge of each type. When the network has a semi-group structure, Rink and Sanders in [12] have calculated normal forms of coupled cell systems and in [11] have used the hidden symmetries of the network to derive Lyapunov-Schmidt reduction that preserves hidden symmetries. In [9], Nijholt, Rink and Sanders have introduced the concept of fundamental network which reveals the hidden symmetries of a network. A fundamental network is a Cayley Graph of a semi-group. The dynamics associated to a fundamental network can be studied using the revealed hidden symmetries. Moreover, the dynamics associated to a network can be derived from the dynamics associated to its fundamental network, [11, theorem 10.1].

The one-to-one correspondence between balanced colorings and synchrony subspaces leads to the definition of quotient network, such that every dynamics associated to a quotient network is the restriction to a synchrony subspace of the dynamics associated to the original network. A subnetwork of a given network is a network whose set of cells is a subset of the cells of the given network and the respective incoming edges, such that the cells are not influenced by any cell outside the subnetwork. Thus, the dynamics associated to the cells in a subnetwork is independent of the dynamics associated to the other cells. DeVille and Lerman [4] highlighted the concepts of quotient network and subnetwork using network fibrations, i.e., functions between networks that respect their structure. In particular, they showed that every surjective network fibration defines a quotient network and every injective network fibration defines a subnetwork (§ 2.4).

In this work, we will focus on the relation between a homogenous network and its fundamental network. The work is divided in two independent parts. In the first part, we show that the fundamental network construction preserves the quotient network relation and transforms the subnetwork relation into the quotient network relation (§ 2.5). We reformulate the characterization done by Nijholt et al. [9] of which networks are a quotient of its fundamental network (§ 2.5.1). Moreover, we characterize the networks such that: the network is a subnetwork of its fundamental network (§ 2.5.2), and the network is a fundamental network (§ 2.5.3). In order to do that, we introduce the properties of backward connectivity and transitivity for a cell. The backward connectedness for a cell means that we can reach that cell from any other cell in the network. This signifies that the dynamics associated to that cell is, directly or indirectly, affected by the dynamics associated to every other cell in the network. The transitivity for a cell corresponds to the existence of network fibrations pointing that cell to any other cell. This property is similar to the vertex-transitivity used in the characterization of Cayley-Graphs of groups [2, §16]. The vertex-transitivity is the ability of interchanging any two nodes using a bijective fibration, which reveals the symmetries of a graph.

In the second part, we relate the architecture of a network and of its fundamental network. In particular, we study two concepts of a network's architecture: cycles in the network and the distance of cells to a cycle (§ 2.6). We denote by rings the cycles in the network involving only one edge type, and by depth the maximal distance of any cell to a ring. Ring networks have been studied, for example, in Ganbat [5] and Moreira [8]. We start by looking at networks having a group structure (§ 2.7). Then we show that a network and its fundamental network have equal depth (§ 2.7.1), and that the size of the rings in a fundamental network is a (least common) multiple of the size of some network rings (§ 2.7.2). Last, we describe the architecture of the fundamental networks of networks that have only one edge type.

The text is organized as follows. Sections 2.2, 2.3 and 2.4 review the concepts of coupled cell networks, fundamental networks and network fibrations, respectively. Section 2.5 characterizes fundamental networks. Section 2.6 defines rings and depth of a network. Finally, § 2.7 relates rings and depth of a network and its fundamental network.

2.2 Coupled cell networks

In this section, we recall a few facts concerning coupled cell networks following [6, 13]. We also introduce the notion of backward connected network.

A *directed graph* is a tuple $G = (C, E, s, t)$, where $c \in C$ is a cell and $e \in E$ is a directed edge from the source cell, $s(e)$, to the target cell, $t(e)$. We assume that the set of cells and the set of edges are finite. The *input set* of a cell c, denoted by $I(c)$, is the set of edges that target c. Following [6, definition 2.1.] and imposing that cells of the same type are input equivalent we define (coupled cell) network.

Definition 2.2.1. A *(coupled cell) network* $N = (G, \sim_C, \sim_E)$ is a directed

graph, G, together with two equivalence relations: one on the set of cells, $\sim_C$, and another on the set of edges, $\sim_E$. The *cell type* of a cell is its $\sim_C$-equivalence class and the *edge type* of an edge is its $\sim_E$-equivalence class. It is assumed that:
(i) edges of the same type have source cells of the same type and target cells of the same type;
(ii) cells of the same type are input equivalent. That is, if two cells have the same cell type, then there is an edge type preserving bijection between their input sets. $\diamondsuit$

We say that a network is a *homogeneous network* whenever there is only one cell type. A network is a *homogeneous network with asymmetric inputs* if each cell receives exactly one edge of each edge type. We will focus our interest in homogeneous networks with asymmetric inputs.

In [11], Rink and Sanders pointed out that a homogeneous network with asymmetric inputs can be represented by a set of functions $\sigma_i : C \to C$, for each edge type i, such that there is an edge with type i from $\sigma_i(c)$ to c. We write $\sigma = [a_1 \ \dots \ a_n]$ for the function $\sigma : \{1, \dots, n\} \to \{1, \dots, m\}$ such that $\sigma(j) = a_j$, for $j = 1, \dots, n$. For examples of homogeneous networks with asymmetric inputs see figure 2.1, where distinct edge types are represented by different symbols.

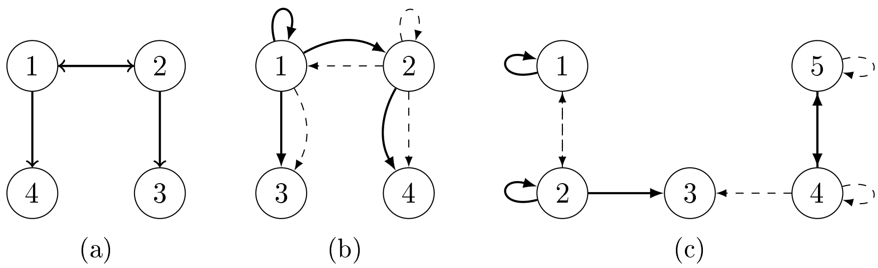


Figure 2.1: Homogeneous networks with asymmetric inputs: (a) network with one edge type represented by the function $\sigma_1 = [2\ 1\ 2\ 1]$; (b) network with two edge types, where the solid edges are represented by $\sigma_1 = [1\ 1\ 1\ 2]$ and the dashed edges are represented by $\sigma_2 = [2\ 2\ 1\ 2]$; (c) network represented by the functions $\sigma_1 = [1\ 2\ 2\ 5\ 4]$, for solid edges, and $\sigma_2 = [2\ 1\ 4\ 4\ 5]$, for dashed edges. The network (c) is backward connected, and the networks (a) and (b) are not.

A *directed path* in a network N is a sequence $(c_0, c_1, \dots, c_{m-1}, c_m)$ of cells in N such that for every $j = 1, \dots, m$ there is an edge in N from c_{j-1} to c_j.

Remark 2.2.2. Compositions of representative functions define directed paths in the network. Let N be a homogenous network with asymmetric inputs represented by the functions $(\sigma_i)_{i=1}^k$. There exists a directed path from cell c to cell d if and only if there are $1 \leq j_1, \ldots, j_m \leq k$ such that

$$\sigma_{j_m} \circ \cdots \circ \sigma_{j_1}(d) = c. \qquad \Diamond$$

Definition 2.2.3. We say that a network N is *backward connected for a cell c* if for any cell $c' \neq c$ there exists a directed path between c' and c. The network N is *backward connected* if it is backward connected for some cell. $\Diamond$

Example 2.2.4. Consider the networks in figure 2.1. For the network in figure 2.1(a), there is no directed path from cell 4 to cells 1, 2 and 3, neither from cell 3 to cells 1, 2 and 4. Thus the network is not backward connected. Similarly, we see that the network in figure 2.1(b) is not backward connected. Now, consider the network in figure 2.1(c), for each cell 1, 2, 4 and 5 there is a directed path to cell 3 starting at that cell. Thus, the network is backward connected for cell 3. $\Diamond$

Following [9], the input network for a cell of a network contains the cells that affect, directly or indirectly, that cell. The *input network* for $c \in C$, denoted by $N_{(c)}$, is the network with set of cells $C_{(c)}$ and set of edges $E_{(c)}$, where

$$C_{(c)} = \{c\} \cup \left\{c' \in C \mid \text{exists a directed path in } N \text{ from } c' \text{ to } c\right\},$$

$$E_{(c)} = \left\{e \in E \mid t(e) \in C_{(c)}\right\}.$$

Observe that every input network for a cell is backward connected for that cell. See figure 2.2 for an example.

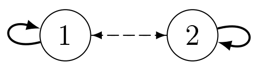


Figure 2.2: Input network of the network in figure 2.1(c) for cell 1 (and for cell 2). It is backward connected for cell 1 (and for cell 2).

2.3 Fundamental networks

In this section, we recall the definition of fundamental network of a homogenous network with asymmetric inputs introduced by Nijholt et al. [9]. We present some examples of fundamental networks and remark that every fundamental network is backward connected. The identity function in C is denoted by Id_C, and we omit the subscript when it is clear from the context.

Definition 2.3.1 ([9, definition 6.2]). Let N be a homogeneous network with asymmetric inputs represented by the functions $(\sigma_i : C \to C)_{i=1}^k$. The *fundamental network* of N is the network $\tilde{N}$ where the set of cells, $\tilde{C}$, is the semi-group generated by Id and $(\sigma_i)_{i=1}^k$, and $\tilde{N}$ is represented by the functions

$$\left(\tilde{\sigma_i} : \tilde{C} \to \tilde{C}\right)_{i=1}^k,$$

defined by $\tilde{\sigma_i}(\tilde{c}) = \sigma_i \circ \tilde{c}$, for $\tilde{c} \in \tilde{C}$ and $i = 1, \ldots, k$. ◊

Example 2.3.2. Consider the network in figure 2.1(a). This network is represented by the function $\sigma_1 = [2\ 1\ 2\ 1]$. Note that $\sigma_1^3 = \sigma_1$, and the semi-group generated by σ_1 and Id is

$$\tilde{C} = \left\{Id, \sigma_1, \sigma_1^2\right\}.$$

The representative function, $\tilde{\sigma_1}$, of the fundamental network is obtained from the compositions of σ_1 with each element of $\tilde{C}$: $\tilde{\sigma}(\sigma_1^2) = \sigma_1$ and $\tilde{\sigma}(\sigma_1^j) = \sigma^{j+1}$, when $j = 0, 1$. The fundamental network is represented graphically in figure 2.3(a). ◊

Figure 2.3 displays the fundamental networks of the networks in figures 2.1 and 2.2. Note that all the fundamental networks in figure 2.3 are backward connected for Id.

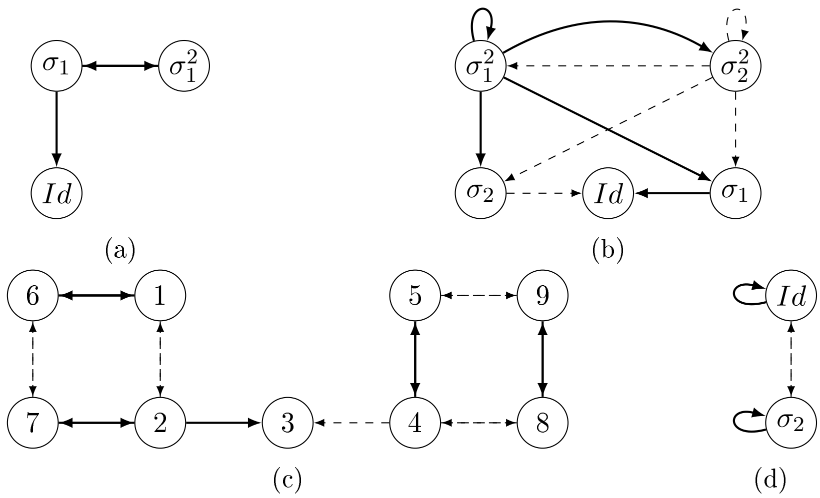


Figure 2.3: Fundamental networks of the networks in figure 2.1(a), (b), (c) and figure 2.2, respectively. The cells $1, \ldots, 9$ in (c) correspond to the functions $\sigma_2 \circ \sigma_1$, σ_1, Id, σ_2, $\sigma_1 \circ \sigma_2$, $\sigma_2 \circ \sigma_1^2$, σ_1^2, σ_2^2, $\sigma_1 \circ \sigma_2^2$, respectively. In § 2.4, we see that the fundamental network in: (a) is a quotient network and a subnetwork of the network in figure 2.1(a); (b) is neither a lift nor a quotient network of the network in figure 2.1(b); (c) is a lift of the network in figure 2.1(c); (d) is equal to the network in figure 2.2.

Proposition 2.3.3. *Every fundamental network of a homogenous network with asymmetric inputs is backward connected for Id.*

Proof. Let N be a homogenous network with asymmetric inputs represented by $(\sigma_i)_{i=1}^k$ and $\tilde{N}$ its fundamental network. If $\tilde{c} \in \tilde{C}$, then $\tilde{c} = \sigma_{l_1} \circ \cdots \circ \sigma_{l_m}$, where $1 \leq l_i \leq k$, and

$$\widetilde{\sigma_{l_1}} \circ \cdots \circ \widetilde{\sigma_{l_m}}(Id) = \sigma_{l_1} \circ \cdots \circ \sigma_{l_m} \circ Id = \tilde{c}.$$

Hence $\tilde{N}$ is backward connected for Id. □

2.4 Network fibrations

In this section, we recall the definition and some properties of network fibrations. We introduce a notion of transitivity and we recall the definitions of quotient network and subnetwork. Moreover, we highlight the relations of quotient network and subnetwork with surjective and injective network fibrations, respectively.

Roughly speaking, a graph fibration is a function between graphs that preserves the orientation of the edges and the number of input edges. Precisely, let $G = (C, E, s, t)$ and $G' = (C', E', s', t')$ be two graphs. A function $\varphi : G \to G'$ is a *graph fibration* if $\varphi(s(e)) = s'(\varphi(e))$, $\varphi(t(e)) = t'(\varphi(e))$ and $\varphi|_{I(c)} : I(c) \to I(\varphi(c))$ is a bijection, for every $c \in C$ and $e \in E$.

A network fibration between networks is then defined as a graph fibration preserving the cell types and the edge types:

Definition 2.4.1 ([4, definition 4.1.1])**.** Consider two networks $N = (G, \sim_C, \sim_E)$ and $N' = (G', \sim_C, \sim_E)$. A *network fibration* $\varphi : N \to N'$ is a graph fibration between G and G' such that $c \sim_C \varphi(c)$ and $e \sim_E \varphi(e)$.

We say that N and N' are *isomorphic*, if there is a bijective network fibration between N and N'. ◊

We do not distinguish isomorphic networks and we will say that two networks are the same if they are isomorphic.

Example 2.4.2. Let N be the network in figure 2.1(a). Denote an edge of N with source s and target t by (s, t). Consider the function $\varphi : N \to N$ such that $\varphi(1) = 1$, $\varphi(2) = \varphi(4) = 2$ and $\varphi(3) = 3$, and $\varphi((1, 2)) = \varphi((1, 4)) = (1, 2)$, $\varphi((2, 1)) = (2, 1)$ and $\varphi((2, 3)) = (2, 3)$. The function φ is a network fibration. ◊

In the case of homogeneous networks with asymmetric inputs, the network fibrations are characterized by the following property.

Proposition 2.4.3 ([9, proposition 5.3])**.** *Let N and N' be homogeneous networks with asymmetric inputs with sets of cells C and C', and represented by the functions $(\sigma_i)_{i=1}^k$ and $(\sigma'_i)_{i=1}^k$, respectively. The function $\varphi : N \to N'$ is a network fibration if and only if*

$$\varphi|_C \circ \sigma_i = \sigma'_i \circ \varphi|_C\,, \qquad i = 1, \ldots, k.$$

Example 2.4.4. Recall the network N in figure 2.1(a) represented by the function $\sigma_1 = [2\ 1\ 2\ 1]$. Consider the network fibration, given in example 2.4.2, $\varphi : N \to N$ such that $\varphi = [1\ 2\ 3\ 2]$. Observe that $\varphi \circ \sigma_1 = [2\ 1\ 2\ 1] = \sigma_1 \circ \varphi$. $\diamondsuit$

A network fibration from a network which is backward connected for a cell c is uniquely determined by the evaluation of the network fibration at cell c.

Proposition 2.4.5. *Let A be a homogeneous network with asymmetric inputs and $\phi : A \to B$ a network fibration. If A is backward connected for c, then the network fibration is uniquely determined by $\phi(c)$.*

Proof. Let A be a homogeneous network with asymmetric inputs and $\phi : A \to B$ a network fibration. Then B is a homogeneous network with asymmetric inputs and has the same edge types of A. Suppose that A and B are represented by the functions $\left(\sigma_i^1\right)_{i=1}^k$ and $\left(\sigma_i^2\right)_{i=1}^k$, respectively, and A is backward connected for c. Then for every cell $d \neq c$ in A there are $\sigma_{i_1}^1, \dots, \sigma_{i_m}^1$ with $1 \leq i_1, \dots, i_m \leq k$ such that $d = \sigma_{i_1}^1 \circ \cdots \circ \sigma_{i_m}^1(c)$. By proposition 2.4.3, we know that $\phi \circ \sigma_i^1 = \sigma_i^2 \circ \phi$, for $1 \leq i \leq k$. Then, for every cell $d \neq c$ in A,

$$\phi(d) = \phi \circ \sigma_{i_1}^1 \circ \cdots \circ \sigma_{i_m}^1(c) = \sigma_{i_1}^2 \circ \cdots \circ \sigma_{i_m}^2 \circ \phi(c).$$ $\square$

In the context of graphs, vertex-transitivity is the ability of interchanging two cells of a graph using a bijective graph fibration. The vertex-transitivity reveals symmetries in a graph and it was used in the characterization of Cayley graphs of groups, see [2, §16]. Here, we introduce a weaker version of transitivity that will play a similar role in the characterization of fundamental networks.

Definition 2.4.6. Let N be a homogenous network with asymmetric inputs and c a cell in N. We say that N is *transitive for c* if for every cell d in N, there is a network fibration $\phi_d : N \to N$ such that $\phi_d(c) = d$. We call the network N *transitive*, if it is transitive for some cell. $\diamondsuit$

Example 2.4.7. Consider the networks in figure 2.1. For the network in figure 2.1(a), we have the following four network fibrations from the network to itself: $\phi_1 = [1\ 2\ 1\ 2]$, $\phi_2 = [2\ 1\ 2\ 1]$, $\phi_3 = [1\ 2\ 3\ 4]$, and $\phi_4 = [2\ 1\ 4\ 3]$. Then the network is transitive for cell 3 (and for cell 4). For the network in figure 2.1(b), there is only one network fibration from the network to itself, the identity network fibration. Thus the network is not transitive. $\diamondsuit$

2.4.1 Surjective network fibrations

We recall now the definition of quotient networks using balanced colorings [6, 13] and establish then their relation with surjective network fibrations, [3, 4].

A *coloring* on the set of cells of a network defines an equivalence relation on those cells. Following [6, 13], a coloring is *balanced* if for any two cells with the same color there is an edge type preserving bijection between the corresponding input sets which also preserves the color of the source cells.

Each balanced coloring defines a quotient network, see [6, §5]. The *quotient network* of a network with respect to a given balanced coloring $\bowtie$, is the network where the set of equivalence classes of the coloring, $[c]_{\bowtie}$, is the set of cells and there is an edge of type i from $[c]_{\bowtie}$ to $[c']_{\bowtie}$, for each edge of type i from a cell in the class $[c]_{\bowtie}$ to c'. We say that a network L is a *lift* of N, if N is a quotient network of L.

Example 2.4.8. Let N be the network in figure 2.1(a) and $\tilde{N}$ its fundamental network displayed in figure 2.3(a). The coloring on the set of cells of N with classes $\{1,3\}$, $\{2\}$ and $\{4\}$ is balanced because cells 1 and 3 receive, each, an edge from cell 2. The quotient network of N with respect to this balanced coloring is $\tilde{N}$. Hence the fundamental network is a quotient network. ♢

Example 2.4.9. The network in figure 2.1(c) is a quotient network of its fundamental network displayed in figure 2.3(c) with respect to the balanced coloring with classes $\{1,6\}$, $\{2,7\}$, $\{4,8\}$, $\{5,9\}$ and $\{3\}$. In this case, the fundamental network is a lift. ♢

The balanced colorings are uniquely determined by surjective network fibrations, see [3, theorem 2], [4, remark 4.3.3] or [13, theorem 8.3].

Proposition 2.4.10 ([3, theorem 2])**.** *A network Q is a quotient network of a network N if and only if there is a surjective network fibration from N to Q.*

For completeness, we sketch the proof here. If Q is a quotient network of a network N, consider the associated balanced coloring. The function from N to Q that projects each cell into its equivalence class is a surjective network fibration. On the other hand, given a surjective network fibration from N to Q, consider the coloring such that two cells have the same color, when their evaluation by the network fibration is equal. This coloring is balanced, and the quotient network of N with respect to this coloring is equal to Q.

Example 2.4.11. Let N be the network in figure 2.1(c) and $\tilde{N}$ its fundamental network displayed in figure 2.3(c). The network fibration from $\tilde{N}$ to N given by $\varphi = [1\ 2\ 3\ 4\ 5\ 1\ 2\ 4\ 5]$ is surjective and N is a quotient network of $\tilde{N}$. ♢

Example 2.4.12. There is no surjective fibration from the network in figure 2.1(b) to its fundamental network displayed in figure 2.3(b), neither a surjective fibration from the fundamental network to the network. Hence, in this case, the fundamental network is neither a lift nor a quotient network of the network. ♢

2.4.2 Injective network fibrations

We consider now subnetworks and their relation with injective network fibrations. We follow [4, §5.2].

Definition 2.4.13. Let N and S be two networks with sets of cells and edges, respectively, C and E, and C' and E'. Then S is a *subnetwork* of N, if $C' \subseteq C$, $E' \subseteq E$ and for every $c' \in C'$ and every edge $e \in E$ with the target cell $t(e) = c'$, we have that $e \in E'$ and the source cell $s(e) \in C'$. ♢

Example 2.4.14. Consider the network in figure 2.1(a) and its fundamental network displayed in figure 2.3(a). The fundamental network is a subnetwork. ♢

Remark 2.4.15. Let N be a network with set of cells C.
(i) For every cell $c \in C$, the input network $N_{(c)}$ is a subnetwork of N.
(ii) The union of subnetworks of N is a subnetwork of N. ♢

Example 2.4.16. Let N be the network in figure 2.1(c). The restriction of N to the set of cells $\{1, 2, 4, 5\}$ is a subnetwork of N. That restriction corresponds to the union of the input networks for the cells 1, 2, 4 and 5. ♢

Proposition 2.4.17 ([4, §5.2])**.** *A network N' is a subnetwork of N if and only if there is an injective network fibration from N' to N.*

For completeness, we sketch the proof here. If N' is a subnetwork of N, then the embedding of N' in N is an injective network fibration. If $\varphi : N' \to N$ is an injective network fibration, then N' is equal to $\varphi(N')$ which is a subnetwork of N.

2.5 Fundamental networks and network fibrations

In this section, we recall some results presented by Nijholt et al. in [9]. We show then that the fundamental network construction preserves the quotient network relation. Moreover, we see that the fundamental network construction does not preserve the subnetwork relation, but it transforms the subnetwork relation in the quotient network relation.

Theorem 2.5.1 ([9, theorem 6.4 & remark 6.9 & lemma 7.1])**.** *Let N be a homogeneous network with asymmetric inputs and $\tilde{N}$ its fundamental network with sets of cells C and $\tilde{C}$, respectively. For every $c \in C$, there is a network fibration, $\varphi_c : \tilde{N} \to N$ given by*

$$\varphi_c(\tilde{c}) = \tilde{c}(c), \quad \tilde{c} \in \tilde{C}.$$

The image of φ_c is the input network $N_{(c)}$. Every network fibration from $\tilde{N}$ to N is equal to φ_c for some $c \in C$. The network $\tilde{N}$ and its fundamental $\tilde{\tilde{N}}$ are equal.

We prove next that the fundamental network construction preserves the quotient network relation.

Proposition 2.5.2. *Let N be a homogeneous network with asymmetric inputs. If Q is a quotient network of N, then $\tilde{Q}$ is a quotient network of $\tilde{N}$.*

Proof. Let N be a homogeneous network with asymmetric inputs and Q a quotient network of N. By proposition 2.4.10, there exists a surjective network fibration $\phi : N \to Q$ and Q is a homogeneous network with asymmetric inputs. Suppose that N and Q are represented by $(\sigma_l)_{l=1}^k$ and $(\gamma_l)_{l=1}^k$, respectively. By proposition 2.4.3,

$$\phi \circ \sigma_i = \gamma_i \circ \phi, \qquad i = 1, \ldots, k.$$

Define the function $\tilde{\phi} : \tilde{N} \to \tilde{Q}$ such that $\tilde{\phi}(Id_N) = Id_Q$ and for every cell σ in $\tilde{N}$ such that $\sigma = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}$ for some $1 \leq i_1, \ldots, i_m \leq k$, then $\tilde{\phi}$ is given by $\tilde{\phi}(\sigma) = \gamma_{i_1} \circ \cdots \circ \gamma_{i_m}$. As we show next, $\tilde{\phi}$ is well-defined and is a surjective network fibration.

Suppose that $\sigma = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m} = \sigma_{j_1} \circ \cdots \circ \sigma_{j_{m'}}$, where $1 \leq i_1, \ldots, i_m \leq k$ and $1 \leq j_1, \ldots, j_{m'} \leq k$. Note that

$$\gamma_{i_1} \circ \cdots \circ \gamma_{i_m} \circ \phi = \phi \circ \sigma = \gamma_{j_1} \circ \cdots \circ \gamma_{j_{m'}} \circ \phi.$$

Then $\gamma_{i_1} \circ \cdots \circ \gamma_{i_m}$ and $\gamma_{j_1} \circ \cdots \circ \gamma_{j_{m'}}$ are equal in the range of ϕ. Because ϕ is surjective, we have that $\gamma_{i_1} \circ \cdots \circ \gamma_{i_m} = \gamma_{j_1} \circ \cdots \circ \gamma_{j_{m'}}$. Thus the definition of $\tilde{\phi}$ does not depend on the choice of $i_1, \ldots, i_m$. Moreover, $\tilde{\phi}$ is defined for every cell in $\tilde{N}$. Hence, $\tilde{\phi}$ is well-defined.

By definition $\tilde{\phi}(Id_N) = Id_Q$. Let $\gamma \neq Id_Q$ be a cell in $\tilde{Q}$. Then there are $1 \leq i_1, \ldots, i_m \leq k$ such that $\gamma = \gamma_{i_1} \circ \cdots \circ \gamma_{i_m} = \tilde{\phi}(\sigma_{i_1} \circ \cdots \circ \sigma_{i_m})$. Thus $\tilde{\phi}$ is surjective.

From proposition 2.4.3, the function $\tilde{\phi}$ is a network fibration if and only if $\tilde{\phi} \circ \tilde{\sigma}_i = \tilde{\gamma}_i \circ \tilde{\phi}$, for every $i = 1, \ldots, k$. Let $\sigma \neq Id_N$ be a cell in $\tilde{N}$. Then there are $1 \leq i_1, \ldots, i_m \leq k$ such that $\sigma = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}$. For $1 \leq i \leq k$, we have that $\tilde{\phi} \circ \tilde{\sigma}_i(Id_N) = \tilde{\gamma}_i \circ \tilde{\phi}(Id_N)$ and

$$\begin{aligned}\tilde{\phi} \circ \tilde{\sigma}_i(\sigma) &= \tilde{\phi}(\sigma_i \circ \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}) = \gamma_i \circ \gamma_{i_1} \circ \cdots \circ \gamma_{i_m} \\ &= \tilde{\gamma}_i(\gamma_{i_1} \circ \cdots \circ \gamma_{i_m}) = \tilde{\gamma}_i \circ \tilde{\phi}(\sigma_{i_1} \circ \cdots \circ \sigma_{i_m}) = \tilde{\gamma}_i \circ \tilde{\phi}(\sigma).\end{aligned}$$

Hence $\tilde{\phi}$ is a surjective network fibration. By proposition 2.4.10, $\tilde{Q}$ is a quotient network of $\tilde{N}$. □

Using that $\tilde{N} = \tilde{\tilde{N}}$ (theorem 2.5.1) and proposition 2.5.2, we have the following.

Corollary 2.5.3. *If N is a quotient network of L and L is a quotient network of $\tilde{N}$, then $\tilde{N} = \tilde{L}$.*

Remark 2.5.4. From the proof of proposition 2.5.2, it also follows that if $\phi : N \to Q$ is a surjective network fibration, then there exists a surjective network fibration $\tilde{\phi} : \tilde{N} \to \tilde{Q}$ such that for every cell c in N the following diagram is commutative

$$\begin{array}{ccc} \tilde{N} & \xrightarrow{\tilde{\phi}} & \tilde{Q} \\ {\scriptstyle \varphi_c^N}\downarrow & & \downarrow{\scriptstyle \varphi_{\phi(c)}^Q} \\ N & \xrightarrow[\phi]{} & Q \end{array}$$

where $\varphi_{\phi(c)}^Q$ and φ_c^N are given by theorem 2.5.1. ♢

The next example illustrates the fact that S being a subnetwork of N does not imply the same relation between the corresponding fundamental networks. In fact, we see that the existence of an injective network fibration $\phi : S \to N$ does not imply the existence of a network fibration $\tilde{\phi} : \tilde{S} \to \tilde{N}$.

Example 2.5.5. Let N be the network in figure 2.1(c) and S the network in figure 2.2. The corresponding fundamental networks, $\tilde{N}$ and $\tilde{S}$, are given in figure 2.3(c) and (d). There is an injective network fibration from S to N, since S is a subnetwork of N. However there is no injective network fibration from $\tilde{S}$ to $\tilde{N}$, because $\tilde{S}$ is not a subnetwork of $\tilde{N}$. In fact, it can easily be seen that there is no network fibration from $\tilde{S}$ to $\tilde{N}$, as $\tilde{S}$ has self loops and $\tilde{N}$ has none. ♢

In the following proposition, we show that the fundamental network construction transforms the subnetwork relation into the quotient network relation.

Proposition 2.5.6. *Let N be a homogeneous network with asymmetric inputs. If S is a subnetwork of N, then $\tilde{S}$ is a quotient network of $\tilde{N}$.*

Proof. Let N be a homogeneous network with asymmetric inputs and S a subnetwork of N. Suppose that N is represented by the functions $(\sigma_i)_{i=1}^k$. Then S is represented by the functions $(\sigma_i|_S)_{i=1}^k$.

Consider the function $\tilde{\phi} : \tilde{N} \to \tilde{S}$ such that $\tilde{\phi}(\sigma) = \sigma|_S$. This function is surjective, because if $\gamma = \sigma_{i_1}|_S \circ \cdots \circ \sigma_{i_m}|_S$, then $\gamma = (\sigma_{i_1} \circ \cdots \circ \sigma_{i_m})|_S$. For every cell σ in $\tilde{N}$, we have that

$$\tilde{\phi} \circ \tilde{\sigma}_i(\sigma) = \tilde{\phi}(\sigma_i \circ \sigma) = (\sigma_i \circ \sigma)|_S = \sigma_i|_S \circ \sigma|_S = \widetilde{\sigma_i|_S} \circ \tilde{\phi}(\sigma).$$

Hence $\tilde{\phi}$ is a surjective network fibration. By proposition 2.4.10, it follows that $\tilde{S}$ is a quotient network of $\tilde{N}$. □

2.5.1 Fundamental networks and lifts

In this section, we give a characterization of the fundamental networks that are lifts of the original network, in terms of the network connectivity, using the results in [9]. We point out that Nijholt et al. in [9] consider that N' is a quotient network of N when there is a network fibration from N to N' which need not to be surjective. We also give a necessary condition for a network to be a lift of its fundamental network.

Proposition 2.5.7. *Let N be a homogeneous network with asymmetric inputs and $\tilde{N}$ its fundamental network. Then $\tilde{N}$ is a lift of N if and only if N is backward connected.*

Proof. Let N be a homogeneous network with asymmetric inputs and $\tilde{N}$ its fundamental network. By proposition 2.4.10, the fundamental network $\tilde{N}$ is a lift of N if and only if there is a surjective network fibration from $\tilde{N}$ to N.

Using theorem 2.5.1 and the network fibrations defined there, we have that every network fibration from $\tilde{N}$ to N is equal to φ_c, for some cell c. Moreover, φ_c is surjective if and only if $N_{(c)} = N$. Note that $N_{(c)} = N$ if and only if N is backward connected for c. Hence $\tilde{N}$ is a lift of N if and only if N is backward connected. □

It follows from propositions 2.5.6 and 2.5.7 that a fundamental network is a lift of every backward connected subnetwork of the original network.

Corollary 2.5.8. *Let N be a homogenous network with asymmetric inputs, $\tilde{N}$ its fundamental network and B a backward connected subnetwork of N. Then $\tilde{N}$ is a lift of B.*

In the next result, we give a necessary condition for a network to be a lift of its fundamental network.

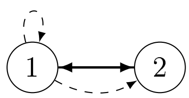
Proposition 2.5.9. *Let N be a homogenous network with asymmetric inputs and $\tilde{N}$ its fundamental network. If N is a lift of $\tilde{N}$, then N is transitive.*

Proof. Let N be a homogenous network with asymmetric inputs and $\tilde{N}$ its fundamental network. Suppose that N is a lift of $\tilde{N}$. By proposition 2.4.10, there exists a surjective network fibration $\psi : N \to \tilde{N}$. Let c be a cell in N such $\psi(c) = Id_N$. Consider the network fibration, given in theorem 2.5.1, $\varphi_d : \tilde{N} \to N$, for every cell d in N. Note that $\varphi_d \circ \psi(c) = \varphi_d(Id_N) = d$, for every cell d in N. Hence N is transitive for c. □

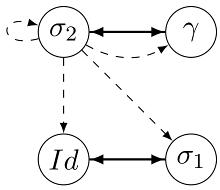
2.5.2 Fundamental networks and subnetworks

In this section, we give a necessary and sufficient condition for a network to be a subnetwork of its fundamental network. Moreover, we give a sufficient condition for a fundamental network to be a subnetwork of the original network. We start with two examples.

Example 2.5.10. (i) The network in figure 2.1(c) is not a subnetwork of its fundamental network, figure 2.3(c). (ii) The network in figure 2.4(a) is a subnetwork of its fundamental network, figure 2.4(b). ◊



(a)



(b) $\gamma = \sigma_1 \circ \sigma_2$

Figure 2.4: (a) Homogeneous network with asymmetric inputs represented by $\sigma_1 = [2\ 1]$ and $\sigma_2 = [1\ 1]$; (b) Fundamental network of the network (a).

In the next proposition, we give necessary and sufficient conditions for the existence of a network fibration from a network to its fundamental network.

Proposition 2.5.11. *Let N be a homogeneous network with asymmetric inputs and $\tilde{N}$ its fundamental network with sets of cells C and $\tilde{C}$, respectively. Suppose that N is backward connected for $c \in C$.*
(i) If $\varphi : N \to \tilde{N}$ is a network fibration, then $\sigma' \circ \varphi(c) = \sigma'' \circ \varphi(c)$, for every $\sigma', \sigma'' \in \tilde{C}$ such that $\sigma'(c) = \sigma''(c)$.
(ii) If there is $\sigma \in \tilde{C}$ such that $\sigma' \circ \sigma = \sigma'' \circ \sigma$, for every $\sigma', \sigma'' \in \tilde{C}$ such that $\sigma'(c) = \sigma''(c)$, then there is a network fibration $\varphi : N \to \tilde{N}$ such that $\varphi(c) = \sigma$.

Proof. Let N be a homogeneous network with asymmetric inputs and $\tilde{N}$ its fundamental network with sets of cells C and $\tilde{C}$, and represented by $(\sigma_i)_{i=1}^k$ and $(\tilde{\sigma_i})_{i=1}^k$, respectively. Suppose that N is backward connected for $c \in C$.

In order to prove (i), suppose that $\varphi : N \to \tilde{N}$ is a network fibration. By proposition 2.4.3, $\varphi \circ \sigma_i = \tilde{\sigma_i} \circ \varphi = \sigma_i \circ \varphi$, for every $1 \leq i \leq k$. So for every $\sigma \in \tilde{C}$, we have that

$$\varphi \circ \sigma = \sigma \circ \varphi.$$

Let $\sigma', \sigma'' \in \tilde{C}$ such that $\sigma'(c) = \sigma''(c)$. Then

$$\sigma' \circ \varphi(c) = \varphi \circ \sigma'(c) = \varphi \circ \sigma''(c) = \sigma'' \circ \varphi(c).$$

To prove (ii), suppose that there is $\sigma \in \tilde{C}$ such that $\sigma' \circ \sigma = \sigma'' \circ \sigma$, for every $\sigma', \sigma'' \in \tilde{C}$ such that $\sigma'(c) = \sigma''(c)$. Define $\varphi : N \to \tilde{N}$ given by $\varphi(c) = \sigma$ and $\varphi(c') = \sigma' \circ \sigma$, where $c' = \sigma'(c)$. This function is defined for every cell in N, because N is backward connected for c. And it is well defined, because if $c' = \sigma'(c) = \sigma''(c)$, then $\varphi(c') = \sigma' \circ \sigma = \sigma'' \circ \sigma$.

We just need to see that φ is a network fibration. Using proposition 2.4.3, we check that $\varphi \circ \sigma_i = \tilde{\sigma_i} \circ \varphi$, for every $1 \leq i \leq k$. Because N is backward connected, for every cell d of N, there is $\sigma' \in \tilde{C}$ such that $d = \sigma'(c)$ and

$$\varphi \circ \sigma_i(d) = \varphi(\sigma_i \circ \sigma'(c)) = \sigma_i \circ \sigma' \circ \sigma = \tilde{\sigma_i}(\sigma' \circ \sigma) = \tilde{\sigma_i} \circ \varphi(\sigma'(c)) = \tilde{\sigma_i} \circ \varphi(d),$$

for every $1 \leq i \leq k$. Hence $\varphi \circ \sigma_i = \tilde{\sigma_i} \circ \varphi$ and φ is a network fibration. □

Recalling proposition 2.4.17 and restricting the network fibration of proposition 2.5.11 to an injective network fibration, we obtain the characterization of the networks that are subnetworks of its fundamental network.

Corollary 2.5.12. *Let N be a homogeneous network with asymmetric inputs backward connected for a cell c and $\tilde{N}$ its fundamental network. Then N is a subnetwork of $\tilde{N}$ if and only if there is $\sigma \in \tilde{C}$ such that for every $\sigma', \sigma'' \in \tilde{C}$, the following condition is satisfied:*

$$\sigma' \circ \sigma = \sigma'' \circ \sigma \Leftrightarrow \sigma'(c) = \sigma''(c).$$

Proof. Let N be a homogeneous network with asymmetric inputs backward connected for a cell c and $\tilde{N}$ its fundamental network. Recall, by proposition 2.4.17, that N is a subnetwork of $\tilde{N}$ if and only if there is an injective network fibration from N to $\tilde{N}$. By proposition 2.5.11, there is a network fibration φ from N to $\tilde{N}$ such that $\varphi(c) = \sigma \in \tilde{C}$ if and only if

$$\sigma' \circ \sigma = \sigma'' \circ \sigma \Leftarrow \sigma'(c) = \sigma''(c), \qquad \sigma', \sigma'' \in \tilde{C}.$$

A network fibration $\varphi : N \to \tilde{N}$, such that $\varphi(c) = \sigma \in \tilde{C}$, is injective if and only if

$$\sigma' \circ \sigma = \varphi(\sigma'(c)) = \varphi(\sigma''(c)) = \sigma'' \circ \sigma \Rightarrow \sigma'(c) = \sigma''(c), \qquad \sigma', \sigma'' \in \tilde{C}.$$

Combining the previous conditions, we obtain the result. □

Example 2.5.13. Consider the network in figure 2.4(a) represented by $\sigma_1 = [2\ 1]$ and $\sigma_2 = [1\ 1]$. The network is backward connected for the cell 1 and $\sigma' \circ \sigma_2 = \sigma'' \circ \sigma_2$ if and only if $\sigma'(1) = \sigma''(1)$. By the previous corollary, the network is a subnetwork of its fundamental network. ◊

We show now that if a network is transitive, then its fundamental network is a subnetwork of the network. This result will be used in the following section to characterize fundamental networks.

Proposition 2.5.14. *Let N be a homogenous network with asymmetric inputs and $\tilde{N}$ its fundamental network. If N is transitive, then $\tilde{N}$ is a subnetwork of N.*

Proof. Let N be a homogenous network with asymmetric inputs and $\tilde{N}$ its fundamental network. Denote the network fibration, given in theorem 2.5.1, by $\varphi_d : \tilde{N} \to N$, for every cell d in N. Suppose that N is transitive for a cell

c. Then for every cell d in N there is a network fibration $\psi_d : N \to N$ such that $\psi_d(c) = d$. In order to prove that $\tilde{N}$ is a subnetwork of N, we show that φ_c is an injective network fibration.

Note that $\psi_d \circ \varphi_c(Id) = \psi_d(c) = d = \varphi_d(Id)$. By propositions 2.3.3 and 2.4.5, we have that $\psi_d \circ \varphi_c = \varphi_d$. If $\varphi_c(\gamma_1) = \varphi_c(\gamma_2)$, then for every cell d in N

$$\gamma_1(d) = \varphi_d(\gamma_1) = \psi_d \circ \varphi_c(\gamma_1) = \psi_d \circ \varphi_c(\gamma_2) = \varphi_d(\gamma_2) = \gamma_2(d),$$

and $\gamma_1 = \gamma_2$. Hence φ_c is an injective network fibration. By proposition 2.4.17, $\tilde{N}$ is a subnetwork of N. □

From propositions 2.5.9 and 2.5.14, we have the following result.

Corollary 2.5.15. *Let N be a homogenous network with asymmetric inputs and $\tilde{N}$ its fundamental network. If N is a lift of $\tilde{N}$, then $\tilde{N}$ is a subnetwork of N.*

2.5.3 Networks which are fundamental networks

Using theorem 2.5.1 and the results obtained in the previous sections, we can now characterize the networks that are fundamental networks, in terms of transitivity and backward connectedness.

Theorem 2.5.16. *Let N be a homogeneous network with asymmetric inputs. The network N is a fundamental network if and only if there are cells c and d such that N is backward connected for c and transitive for d.*

Proof. Let N be a homogeneous network with asymmetric inputs.

Suppose that N is a fundamental network. Then N is equal to $\tilde{N}$ and there is a bijective network fibration $\psi : \tilde{N} \to N$. From proposition 2.3.3, we know that $\tilde{N}$ is backward connected for Id. By theorem 2.5.1, we have for every cell σ in $\tilde{N}$ that there is a network fibration $\phi_\sigma : \tilde{\tilde{N}} = \tilde{N} \to \tilde{N}$ such that $\phi_\sigma(\gamma) = \gamma \circ \sigma$. In particular $\phi_\sigma(Id) = \sigma$, and $\tilde{N}$ is transitive for Id. Hence, N is backward connected for $\psi(Id)$ and it is transitive for $\psi(Id)$.

Suppose that there are cells c and d in N such that N is backward connected for c and transitive for d. Then $\tilde{N}$ is a subnetwork of N, by proposition 2.5.14, and $\tilde{N}$ is a lift of N, by proposition 2.5.7. So $|\tilde{N}| \leq |N| \leq |\tilde{N}|$. The network fibration $\varphi_c : \tilde{N} \to N$, given by theorem 2.5.1, is a bijection, since it is surjective by the proof of proposition 2.5.7, and it is injective because $|\tilde{N}| = |N|$. Thus N and $\tilde{N}$ are equal. □

2.6 Architecture of networks: rings and depth

In this section, we introduce the definitions of ring and depth of a homogenous network with asymmetric inputs. We start by recalling the definitions of connected and strongly connected component. We finish by describing how we can obtain the rings and the depth of a homogenous network with asymmetric inputs using the representative functions of the network.

We say that there is an *undirected path* in a network connecting the sequence of cells $(c_0, c_1, \ldots, c_{k-1}, c_k)$, if for every $j = 1, \ldots, k$ there is an edge from c_{j-1} to c_j or an edge from c_j to c_{j-1}. A directed path $(c_0, c_1, \ldots, c_{m-1}, c_k)$ is called a *cycle*, if $c_0 = c_k$.

Definition 2.6.1. Let N be a network. A subset Y of cells in N is called *connected* if for every two different cells in Y there is an undirected path between them.

We say that Y is a *connected component* of N, if Y is a maximal connected subset of cells, in the sense that if $Y \cup \{c\}$ is connected then $c \in Y$. ♢

We can partition the set of cells of a network into its connected components.

Definition 2.6.2. Let N be a network with set of cells C and a subset $X \subseteq C$.
(i) The subset X is *strongly connected*, if for every two different cells $c_1, c_2 \in X$ there are directed paths from c_1 to c_2 and from c_2 to c_1.
(ii) The subset X is a *strongly connected component* of N, if X is a maximal strongly connected subset of cells.
(iii) The subset X is a *source* of N, if X is a strongly connected component that does not receive any edge with source cell outside X, i.e., $s(I(X)) \subseteq X$. ♢

Let N be a homogeneous network with asymmetric inputs and i an edge type of N. Denote by N_i the network with the same cells of N and only the edges of type i of N. Let $C_i^1, \ldots, C_i^m$ be the partition of the set of cells of the network N_i into its connected components. For each connected component, the topology of N_i is the union of a unique source component and feed-forward networks starting at some cell of the source component. See figure 2.5 for an example and see [5, proposition 2.3] for details. For each $j = 1, \ldots, m$, we call the source of N_i in C_i^j a *ring* and denote it by R_i^j. Since the cells in the network N_i have only one input, every cycle in N_i connects every cell in a ring.

Definition 2.6.3. Let N be a homogeneous network with asymmetric inputs and i an edge type of N. Let $C_i^1, \ldots, C_i^m$ be the connected components of N_i. For each connected component, C_i^j, of N_i, we define the *depth* of N_i in C_i^j by

$$\mathrm{Depth}_i^j(N) := \max\{\min\{|(r,c)| : r \in R_i^j\} : c \in C_i^j\},$$

where $|(r,c)|$ is 0, if $r = c$, or the number of edges in the shortest directed path in N_i from r to c. And the *depth* of N_i is

$$\mathrm{Depth}_i(N) := \max_{j=1,\ldots,m}\{\mathrm{Depth}_i^j(N)\}. \qquad ♢$$

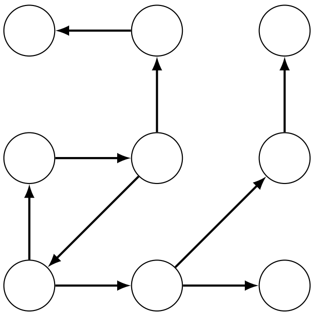


Figure 2.5: Union of a ring and feed-forward networks starting at the ring.

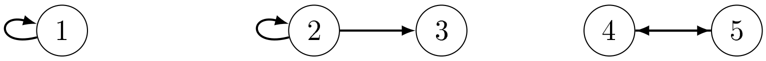


Figure 2.6: The restriction of the network in figure 2.1(c) to the solid edges has three connected components and its depth is 1. On the left, the ring is $\{1\}$ and the depth is 0. On the center, the ring is $\{2\}$ and the depth is 1. On the right, the ring is $\{4, 5\}$ and the depth is 0.

Example 2.6.4. Let N be the network in figure 2.1(c). Consider the restriction N_1 to the solid edges represented in figure 2.6. The network N_1 has three connected components, $C_1^1 = \{1\}$, $C_1^2 = \{2, 3\}$ and $C_1^3 = \{4, 5\}$. The rings of N_i are: $R_1^1 = \{1\}$ in C_1^1; $R_1^2 = \{2\}$ in C_1^2; and $R_1^3 = \{4, 5\}$ in C_1^3. The depth of N_1: in C_1^1 is 0; in C_1^2 is 1; and in C_1^3 is 0. So the depth of N_1 is 1.

Let $\tilde{N}$ be the fundamental network of N represented in figure 2.3(c). Consider the restriction $\tilde{N}_1$ to the solid edges. The network $\tilde{N}_1$ has four connected components. Each of the connected components has a ring of size 2. And the depth of $\tilde{N}_1$ is 1. Note that the size of any ring in $\tilde{N}_1$ is a multiple of the size of some rings in N_1 and the depth of N_1 is equal to the depth of $\tilde{N}_1$. In the next section, we formalize and prove these observations for networks with asymmetric inputs. $\diamondsuit$

We describe now the rings and the depth of a network using representative functions. This is derived from the following facts: every representative function, σ_i, is semi-periodic, i.e., there exist $a \geq 0$ and $b > 0$ such that $\sigma_i^a = \sigma_i^{a+b}$; if $\sigma_i^a = \sigma_i^{a+b}$, then there is a cycle for every cell in the range of σ_i^a; and the distance of a cell c to a ring R is equal to the minimum $p \geq 0$ such that $\sigma_i^p(c) \in R$.

Lemma 2.6.5. *Let N be a homogeneous network with asymmetric inputs represented by the functions $(\sigma_i)_{i=1}^k$ and C the set of cells of N. Fix $1 \leq i \leq k$ and denote the connected components of N_i by $C_i^1, \ldots, C_i^m$, and the corresponding rings by $R_i^1, \ldots, R_i^m$.*

(i) If $\sigma_i^a = \sigma_i^{a+b}$ *for some* $a \geq 0$ *and* $b > 0$*, then* $R_i^j = \sigma_i^a(C_i^j)$ *for* $1 \leq j \leq m$.
(ii)

$$\mathrm{Depth}_i(N) = \min \left\{ p \in \mathbb{N}_0 : \sigma_i^p(C) \subseteq \bigcup_{j=1}^{m} R_i^j \right\}.$$

Example 2.6.6. Consider the example 2.6.4. Let N be the network in figure 2.1(c), N_1 its restriction to the solid edges represented by the function $\sigma_1 = [1\ 2\ 2\ 5\ 4]$ and $C_1^1 = \{1\}$, $C_1^2 = \{2,3\}$ and $C_1^3 = \{4,5\}$ the connected components of N_1 appearing in figure 2.6. Note that $\sigma_1 = \sigma_1^3$. By lemma 2.6.5, the rings of N_1 are $R_1^1 = \sigma_1(C_1^1) = \{1\}$, $R_1^2 = \sigma_1(C_1^2) = \{2\}$, and $R_1^3 = \sigma_1(C_1^3) = \{4,5\}$. Moreover, $\sigma_1^k(C_1^1) \subseteq R_1^1 \cup R_1^2 \cup R_1^3$ if and only if $k \geq 1$. Hence $\mathrm{Depth}_1(N) = 1$. ◊

2.7 Architecture of fundamental networks

We start this section by studying the connectivity of fundamental networks for which the semi-group generated by their representative functions is in fact a group.

Proposition 2.7.1. *Let* N *be a homogenous network with asymmetric inputs and* $\tilde{N}$ *its fundamental network.*
(a) The following statements are equivalent:
(i) $\tilde{N}$ *is strongly connected.*
(ii) $\tilde{C}$ *is a group.*
(iii) The representative functions of N *are bijections (i.e., permutations).*
(b) If N *is connected and* $\tilde{N}$ *is strongly connected, then* N *is strongly connected.*

Proof. Let N be a homogenous network with asymmetric inputs and $\tilde{N}$ its fundamental network with sets of cells C and $\tilde{C}$, respectively.

If $\tilde{N}$ is strongly connected, then there is a directed path between every pair of cells in $\tilde{C}$, in particular, between Id and $\sigma \in \tilde{C}$. Thus

$$\forall_{\sigma \in \tilde{C}}\ \exists_{\sigma' \in \tilde{C}}\ : \sigma' \circ \sigma = Id,$$

where σ' is a directed path from Id to σ. Conversely, if $\tilde{C}$ is a group, then there is a directed path between every pair of cells in $\tilde{C}$. Thus (i) is equivalent to (ii).

Any representative function is invertible if and only if it is a bijection. And every permutation has a finite order. Hence the statements (ii) and (iii) are equivalent.

Now, to prove (b), suppose that N is connected and $\tilde{N}$ is strongly connected. Then $\tilde{C}$ is a group and for every representative function σ of N, there exists σ^{-1}. Note that σ^{-1} is not always a representative function, but it is a composition of representative functions, by definition of $\tilde{C}$. We refer to σ^{-1} has the inverse path of the connection σ. Moreover, for every two cells c and d there exists an undirected path from c to d, because N is connected. From this undirected path it is possible to get a directed path in N from c to d by considering for each connection in the undirected path either the connection itself or its inverse path. □

2.7.1 Depth of fundamental networks

In example 2.6.4, we presented a network such that the depth of the network is equal to the depth of its fundamental network. We prove now that this property is valid for every homogenous network with asymmetric inputs. Moreover, we use this fact to show that an adjacency matrix of a network is non-singular if and only if the correspondent adjacency matrix of its fundamental network is non-singular.

Proposition 2.7.2. *Let N be a homogeneous network with asymmetric inputs represented by the functions $(\sigma_i)_{i=1}^k$ and $\tilde{N}$ its fundamental network. Then*

$$\mathrm{Depth}_i(N) = \mathrm{Depth}_i(\tilde{N}),$$

where $i = 1, \dots, k$.

Proof. Let N be a homogeneous network with asymmetric inputs represented by $(\sigma_i)_{i=1}^k$, C its set of cells and $\tilde{N}$ its fundamental network. Fix $1 \leq i \leq k$. Denote the connected components of N_i by $C_i^1, \dots, C_i^m$ and the corresponding rings by $R_i^1, \dots, R_i^m$. Denote the connected components of $\tilde{N}_i$ by $\tilde{C}_i^1, \dots, \tilde{C}_i^{\tilde{m}}$ and the corresponding rings by $\tilde{R}_i^1, \dots, \tilde{R}_i^{\tilde{m}}$. Let $p_i = \mathrm{Depth}_i(N)$ and $\tilde{p}_i = \mathrm{Depth}_i(\tilde{N})$.

By lemma 2.6.5 (ii), we have that $\sigma_i^{p_i}(C) \subseteq R_i^1 \cup \dots \cup R_i^m$. For each connected component C_i^j of N_i, since cycles of N_i in C_i^j start in a cell of R_i^j and travel by the other cells in R_i^j to reach the initial point, we have that $\sigma_i^{p_i}(c) = \sigma_i^{p_i+r}(c)$, for every $c \in C_i^j$, if and only if r is a multiple of $|R_i^j|$. Therefore $\sigma_i^{p_i} = \sigma_i^{p_i+r}$, if $r = \mathrm{l.c.m.}\{|R_i^1|, \dots, |R_i^k|\}$ where l. c. m. is the least common multiple.

Note that $\tilde{\sigma}_i^{p_i} = \tilde{\sigma}_i^{p_i+r}$, because $\tilde{\sigma}_i^{p_i}(\sigma) = \sigma_i^{p_i} \circ \sigma = \sigma_i^{p_i+r} \circ \sigma = \tilde{\sigma}_i^{p_i+r}(\sigma)$. By lemma 2.6.5 (i),

$$\bigcup_{j=1}^{\tilde{m}} \tilde{R}_i^j = \bigcup_{j=1}^{\tilde{m}} \tilde{\sigma}_i^{p_i}(\tilde{C}_i^j) = \tilde{\sigma}_i^{p_i}(\tilde{C})$$

Hence $\tilde{p}_i \leq p_i$, by lemma 2.6.5 (ii).

From $\tilde{\sigma}_i^{p_i} = \tilde{\sigma}_i^{p_i+r}$, we also know that $\{\sigma_i^{p_i}, \dots, \sigma_i^{p_i+r-1}\}$ is a ring of $\tilde{N}_i$, because $(\sigma_i^{p_i}, \dots, \sigma_i^{p_i+r-1}, \sigma_i^{p_i+r} = \sigma_i^{p_i})$ is a cycle in $\tilde{N}_i$. The directed path $Id = \sigma_i^0, \sigma_i^1, \dots, \sigma_i^{p_i-1}, \sigma_i^{p_i}$ is the shortest directed path in $\tilde{N}_i$ from Id to a cell in this ring. Then we have that $\tilde{p}_i \geq p_i$ and thus conclude that $\tilde{p}_i = p_i$. □

A network can be represented by its *adjacency matrices* A_i, one for each edge type i. More precisely, if the network has n cells, say $C = \{1, \dots, n\}$, then the matrix A_i is an $n \times n$ matrix, where the entry $(A_i)_{c\,c'}$ denotes the number of edges of type i from c' to c.

Corollary 2.7.3. *Let N be a homogeneous network with asymmetric inputs and $\tilde{N}$ its fundamental network. Denote by A_i the adjacency matrix of N and $\tilde{A}_i$ the adjacency matrix of $\tilde{N}$, for an edge type i.*

Then A_i is non-singular if and only if $\tilde{A}_i$ is non-singular.

Proof. The eigenvalues of the adjacency matrix of a homogeneous network with asymmetric inputs for an edge of type i, A_i, are $1, w_j, w_j^2, \dots, w_j^{r_j-1}$ where $r_j = |R_i^j|$, $w_j = \exp^{2\pi\imath/r_j}$, R_i^j is the ring of type i of N in C_i^j and $C_i^1, \dots, C_i^m$ are the connected components of N_i, and 0 if $\mathrm{Depth}_i(N) \neq 0$. Hence A_i is non-singular if and only if $\mathrm{Depth}_i(N) = 0$ if and only if $\mathrm{Depth}_i(\tilde{N}) = 0$ if and only if $\tilde{A}_i$ is non-singular. □

2.7.2 Rings of fundamental networks

We consider now the relation between the size of the rings in a network and of those in its fundamental network. Specifically, we show that the size of a ring in a fundamental network is a (least common) multiple of some ring's sizes in the network. Moreover we use this result to fully describe the fundamental network of a network with only one edge type.

Proposition 2.7.4. *Let N be a homogeneous network with asymmetric inputs represented by the functions $(\sigma_i)_{i=1}^k$, C the set of cells of N and $\tilde{N}$ its fundamental network. Fix $1 \leq i \leq k$. Let $C_i^1, \dots, C_i^m$ be the connected components of N_i and $R_i^1, \dots, R_i^m$ the corresponding rings. Analogously, let $\tilde{C}_i^1, \dots, \tilde{C}_i^{\tilde{m}}$ be the connected components of $\tilde{N}_i$ and $\tilde{R}_i^1, \dots, \tilde{R}_i^{\tilde{m}}$ the corresponding rings. If $1 \leq j \leq \tilde{m}$ and $\gamma \in \tilde{C}_i^j$, then*

$$|\tilde{R}_i^j| = \mathrm{l.c.m.}\left\{|R_i^{j'}| : C_i^{j'} \cap \gamma(C) \neq \emptyset\right\}.$$

Moreover, there exists $1 \leq j \leq \tilde{m}$ such that $|\tilde{R}_i^j| = \mathrm{l.c.m.}\left\{|R_i^1|, \dots, |R_i^m|\right\}$.

Proof. Let N be a homogeneous network with set of cells C and asymmetric inputs represented by the functions $(\sigma_i)_{i=1}^k$. Let $\tilde{N}$ be its fundamental network. Fix $1 \leq i \leq k$. Let $C_i^1, \dots, C_i^m$ be the connected components of N_i and $R_i^1, \dots, R_i^m$ the corresponding rings. Analogously, let $\tilde{C}_i^1, \dots, \tilde{C}_i^{\tilde{m}}$ be the connected components of $\tilde{N}_i$ and $\tilde{R}_i^1, \dots, \tilde{R}_i^{\tilde{m}}$ the corresponding rings. Let $p_i = \mathrm{Depth}_i(N) = \mathrm{Depth}_i(\tilde{N})$. Choose j and γ such that $1 \leq j \leq \tilde{m}$ and $\gamma \in \tilde{C}_i^j$. Define $J = \{j' : \gamma(C) \cap C_i^{j'} \neq \emptyset\}$, $r^\gamma = \mathrm{l.c.m.}\{|R_i^{j'}| : j' \in J|\}$ and $C^\gamma = \cup_{j' \in J} C_i^{j'}$.

By lemma 2.6.5,

$$\sigma_i^{p_i}(C^\gamma) = \bigcup_{j' \in J} R_i^{j'}.$$

Note that $\sigma_i^{p_i}|_{C^\gamma} = \sigma_i^{p_i+r^\gamma}\Big|_{C^\gamma}$, because r^γ is a multiple of $|R_i^{j'}|$, for every $j' \in J$. Then $\tilde{\sigma}_i^{p_i} \circ \gamma = \sigma_i^{p_i} \circ \gamma = \sigma_i^{p_i+r^\gamma} \circ \gamma = \tilde{\sigma}_i^{p_i+r^\gamma} \circ \gamma$ and $(\sigma_i^{p_i} \circ \gamma, \ldots, \sigma_i^{p_i+r^\gamma} \circ \gamma)$ is a cycle in $\tilde{N}_i$. Since $\gamma \in \tilde{C}_i^j$, we have that $\sigma_i^{p_i} \circ \gamma, \ldots, \sigma_i^{p_i+r^\gamma-1} \circ \gamma \in \tilde{C}_i^j$ and so the ring of $\tilde{N}_i$ in $\tilde{C}_i^j$ is $\tilde{R}_i^j = \{\sigma_i^{p_i+1} \circ \gamma, \ldots, \sigma_i^{p_i+r^\gamma} \circ \gamma\}$. This cycle does not repeat cells, because r^γ is the least common multiple. Thus

$$|\tilde{R}_i^j| = r^\gamma = \text{l.c.m.}\left\{|R_i^{j'}| : C_i^{j'} \cap \gamma(C) \neq \emptyset\right\}.$$

The second part of the result follows from taking $\gamma = Id_C$. □

Propositions 2.7.2 and 2.7.4 can be used to describe the fundamental network of a homogenous network with only one edge type.

Definition 2.7.5 ([8, definition 3.1.], [5, definition 2.4])**.** Let N be a homogeneous network with asymmetric inputs that has only one edge type. We say that N is a *loop-chain* with sizes $l \geq 1$ and $p \geq 0$, if N has $l+p$ cells, it has a unique source component with l cells and the depth of N is p. ♢

Loop-chains are studied in great detail in Nijholt, Rink and Sanders [10] where they are called generalized feed forward networks.

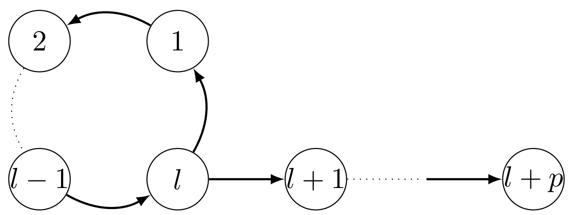


Figure 2.7: The fundamental network of a homogenous network with asymmetric inputs N having only one edge type is a loop-chain with sizes l and p, where l is the least common multiple of all ring's sizes in N and p is the depth of N.

Corollary 2.7.6. *Let N be a homogeneous network with asymmetric inputs and only one edge type. If l is the least common multiple of the size of all the rings in N and p is the depth of N, then the fundamental network of N is a loop-chain with sizes l and p.*

Proof. Let N be a homogeneous network with asymmetric inputs that has only one edge type, l the least common multiple of the size of all the rings in N, p the depth of N and $\tilde{N}$ its fundamental network.

We know by proposition 2.3.3 that $\tilde{N}$ is backward connected and so $\tilde{N}$ has only one connected component. The size of the ring of that connected component is equal to the least common multiple of the sizes of rings in N, see proposition 2.7.4. By proposition 2.7.2, we also know that $\text{Depth}(N) = \text{Depth}(\tilde{N})$. Then $\tilde{N}$ has at least the loop-chain with sizes l and p described in figure 2.7.

Next, we prove that $\tilde{N}$ has only $l + p$ cells. Suppose that there exists more than $l+p$ cells. Then there is a cell $j > l+p$ that receives an edge from the cells $1, \dots, l + p$, because $\tilde{N}$ has only one connected component and the first $l + p$ cells already receive an edge from the first $l + p$ cells. If j receives an edge from the cells $1, \dots, l + p - 1$, then $\tilde{N}$ is not backward connected. If j receives an edge from the cell $l + p$, then $\text{Depth}(\tilde{N}) > p$. Hence $\tilde{N}$ is a loop-chain with sizes l and p described in figure 2.7. □

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3. Synchrony branching lemma for regular networks

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Abstract

Coupled cell systems are dynamical systems associated to a network and synchrony subspaces, given by balanced colorings of the network, are invariant subspaces for every coupled cell systems associated to that network. Golubitsky and Lauterbach (SIAM J. Applied Dynamical Systems, 8 (1) 2009, 40–75) prove an analogue of the Equivariant Branching Lemma in the context of regular networks. We generalize this result proving the generic existence of steady-state bifurcation branches for regular networks with maximal synchrony. We also give necessary and sufficient conditions for the existence of steady-state bifurcation branches with some submaximal synchrony. Those conditions only depend on the network structure, but the lattice structure of the balanced colorings is not sufficient to decide which synchrony subspaces support a steady-state bifurcation branch.

Keywords: Coupled cell systems, Steady-state bifurcations, Synchrony-breaking bifurcations.

2010 Mathematics subject classification: 37G10, 34D06, 34C23

3.1 Introduction

Coupled cell networks describe influences between cells and can be represented by graphs. A dynamical system that respects a network structure is called a coupled cell system associated to the network. In [18] and [8], the authors formalize the concepts of (coupled cell) network and coupled cell system. They also show that there exists an intrinsic relation between coupled cell systems and networks, proving that a polydiagonal subspace given by a coloring of the network is an invariant subspace for any coupled cell system if and only if the coloring is balanced. Here, a coloring is balanced if any two cells with the same color receive, for each color, the same number of inputs starting in cells with that color. And a polydiagonal subspace given by a balanced coloring is called a synchrony subspace. Given a balanced coloring, they define the corresponding quotient network by merging cells with equal color. Moreover, the restriction of a coupled cell system to a synchrony subspace is a coupled cell system for the quotient. We will focus on regular networks, where all cells and edges are identical, and each regular network can be represented by an adjacency matrix. The adjacency matrix of a quotient network is given by the restriction of the original network adjacency matrix to the corresponding synchrony subspace, [1].

Equivariant theory is the study of dynamical systems that commute with an action of a group in the phase space and isotropy subgroups are the subgroups that fix some point of the phase space, see e.g. [5]. For each isotropy subgroup, the set of fixed points forms an invariant subspace for every equivariant dynamical system and it is called the fixed point subspace. One goal of equivariant bifurcation theory is to characterize which isotropy subgroups support a bifurcation. The Equivariant Branching Lemma [3] is one of the first important results about the existence of symmetry-breaking steady-state bifurcation branches for isotropy subgroups that have one dimensional fixed point subspaces. This result was extended for isotropy subgroups that have odd dimensional fixed point subspaces, see e.g. [10, 4, 2], and it was analyzed for isotropy subgroups that have two dimensional fixed point subspaces, [11]. This topic is a large source of inspiration to the study of synchrony-breaking bifurcations on networks, where one of the key questions concerns the characterization of synchrony subspaces which support (generically) steady-state bifurcation branches.

A similar result to the first version of the Equivariant Branching Lemma for regular networks has been already stated, see [19, Theorem 2.1], [6, Theorem 6.3] and [9, Corollary 3.1.], we call this result the Synchrony Branching Lemma. The eigenvalues of the Jacobian of a coupled cell system associated to a regular network at a full synchronous solution are related to the eigenvalues of its adjacency matrix and this relation preserves multiplicities, [12]. So, we can use the eigenvalue structure of the network adjacency matrix to tabulate the possible local codimension-one (steady-state or Hopf) synchrony-breaking bifurcations that can occur for the coupled cell systems associated to a network. We say that an eigenvalue of the adjacency matrix belongs to a balanced coloring, if it has an eigenvector in the synchrony subspace given by that coloring. Fixing an eigenvalue, we say that a coloring is maximal if the eigenvalue belongs to that coloring and it does not belong to any lower dimensional synchrony subspace. The Synchrony Branching Lemma states that every synchrony subspace given by a maximal coloring with a simple eigenvalue (algebraic multiplicity 1) generic supports a bifurcation branch. In [17], the authors study the degeneracy of steady-state bifurcation problems for regular networks and simple eigenvalues. They give conditions on the network structure for the degeneracy of steady-state bifurcation problems and they also present examples of regular networks that have generic highly degenerate steady-state bifurcation problems. In [9], it is given a characterization of the synchrony subspaces which support a synchrony-breaking bifurcation using the lattice structure of balanced colorings, for regular networks that only have simple eigenvalues.

In this manuscript, we generalize the Synchrony Branching Lemma for semisimple eigenvalues (the algebraic and geometric multiplicity are equal). We prove that a synchrony subspace given by a maximal coloring generically supports a bifurcation branch, if the semisimple eigenvalue has odd multiplicity, Theorem 3.4.1. This follows from the application of the Lyapunov-Schmidt Reduction [7] and a blow-up technique also used in equivariant bifurcation, see e.g. [10], which transforms the bifurcation problem into a problem of finding the equilibria of a vector field on a sphere. When the semisimple eigenvalue has odd multiplicity, the sphere is even dimensional and there exists at least one equilibrium point, Poincaré-Hopf theorem [14]. In the way, we prove that the degeneracy of a bifurcation problem associated to a semisimple eigenvalue only depends on the network structure, Lemma 3.3.2. Next, we focus on semisimple eigenvalues with multiplicity 2. If a coloring is maximal and has even degeneracy, then its synchrony subspace supports a bifurcation branch, Theorem 3.5.1. We also give necessary and sufficient conditions for the existence of bifurcation branches on synchrony subspaces given by submaximal colorings, i.e., the eigenvalue belongs to the submaximal coloring and it must have multiplicity 0 or 1 in any synchrony subspace strictly included in the synchrony subspace given by the submaximal coloring, Theorem 3.5.2. Those conditions only depend on the network structure. We give examples of networks where the previous results apply, including two networks that have the same synchrony lattice structure but do not have the same type of synchrony-breaking bifurcations, Examples 3.3.8 and 3.5.5. Despite we do not present an explicit network, we show how a network can have a semisimple eigenvalue with multiplicity 2 and do not support a bifurcation branch, see Example 3.4.3.

This text is organized as follows: in section 3.2, we review some concepts and results of networks, coupled cell systems and steady-state bifurcations on networks, focusing on regular networks. Choosing a semisimple eigenvalue of the adjacency matrix and considering a generic coupled cell system with a bifurcation condition associated to the eigenvalue, in section 3.3, we apply the Lyapunov-Schmidt Reduction and a blow-up technique to the coupled cell system reducing the bifurcation problem to the problem of finding zeros of a vector field on a sphere. Next, we prove the existence of a bifurcation branch, if the eigenvalue has odd multiplicity, section 3.4. Last, we study the existence of bifurcation branches, when the eigenvalue has multiplicity 2, section 3.5.

3.2 Settings

In this section, we recall some facts about networks and coupled cell systems, following [18, 8], and steady-state bifurcations on coupled cell systems.

3.2.1 Regular networks

A *directed graph* is a tuple $G = (C, E, s, t)$, where $c \in C$ is a cell and $e \in E$ is a directed edge from the source cell, $s(e)$, to the target cell, $t(e)$. We assume that the sets of cells and edges are finite. The *input set* of a cell c, $I(c)$, is the set of edges that target c.

Definition 3.2.1. A *regular network* is a directed graph N such that the cardinality of the input set of a cell is the same for all cells. The *valency* ϑ of N is the number of edges that target each cell. We denote the number of cells in N by $|N|$. ♢

See Figure 3.1 for two examples of regular networks with valency 2.

A regular network can be represented by its *adjacency matrix* A, where A is a $|N| \times |N|$ matrix and the entry $(A)_{c\,c'}$ is the number of edges from c' to c.

(a) Network #29 of [9] (b) Network #51 of [9]

Figure 3.1: Regular networks with valency 2.

Definition 3.2.2. A *coloring* of the cells of a network N is an equivalence relation on the set of cells of N. The coloring is *balanced* for N if for any two cells of N with the same color there is a bijection between the input sets of the two cells preserving the color of the source cells. ♢

Any regular network N has two trivial balanced colorings: the *full synchronous coloring*, $\bowtie_0$, and the *full asynchronous coloring* $\bowtie_=$. The full synchronous coloring has only one class, i.e., $c \bowtie_0 c'$ for every cells c, c' in N, and the full asynchronous coloring has $|N|$ classes, i.e., $c \bowtie_= c'$ if and only if $c = c'$.

Each balanced coloring defines a quotient network, [8, Section 5].

Definition 3.2.3. The *quotient network* of a regular network N with respect to a given balanced coloring $\bowtie$ is the network where the equivalence classes of the coloring, $[c]_\bowtie$, are the cells and there is an edge from $[c]_\bowtie$ to $[c']_\bowtie$, for each edge from a cell in the class $[c]_\bowtie$ to c'. We denote the quotient network by $N/\bowtie$. We also say that a network L is a *lift* of N, if N is a quotient of L with respect to some balanced coloring of L. ♢

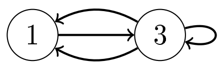

Figure 3.2: Quotient network of the network in Figure 3.1(a) associated to the balanced coloring with classes $\{1, 2\}$ and $\{3, 4\}$.

See Figure 3.2 for an example of a quotient network.

The set of balanced colorings forms a complete lattice, see [16, 9]. Denote by Λ_N the set of balanced colorings for N. For every $\bowtie_1, \bowtie_2 \in \Lambda_N$, we say that $\bowtie_1$ is a *refinement* of $\bowtie_2$, and we write $\bowtie_1 \prec \bowtie_2$, if $\bowtie_1 \neq \bowtie_2$ and $c \bowtie_1 d$ implies $c \bowtie_2 d$ for every cells c, d of N. We denote by $\preceq$ the relation of refinement or equal. The pair $(\Lambda_N, \preceq)$ forms a lattice.

Now, we introduce the definition of μ-maximal and μ-submaximal colorings, where μ is an eigenvalue of the network adjacency matrix.

Definition 3.2.4. Let N be a regular network, $\bowtie$ a balanced coloring of N and μ an eigenvalue of the adjacency matrix associated to N. We say that $\bowtie$ is a *μ-maximal coloring* if for every $\bowtie'$ such that $\bowtie \prec \bowtie'$ we have that μ is not an eigenvalue of the adjacency matrix associated to $N/\bowtie'$. ♢

Definition 3.2.5. Let N be a regular network, $\bowtie$ a balanced coloring of N and μ an eigenvalue of the adjacency matrix associated to N with multiplicity $m > 1$. We say that $\bowtie$ is a *μ-submaximal coloring of type j* if there are j balanced colorings $\bowtie_1, \dots, \bowtie_j$ all distinct such that: (i) $\bowtie \prec \bowtie_i$, $\bowtie_i \npreceq \bowtie_{i'}$ and μ is an eigenvalue with multiplicity 1 of the adjacency matrix associated to $N/\bowtie_i$ for $i, i' = 1, \dots, j$ with $i \neq i'$; (ii) for any other balanced coloring $\bowtie'$ such that $\bowtie \prec \bowtie'$ we have that μ is not an eigenvalue of the adjacency matrix associated to $N/\bowtie'$ or $\bowtie_i \preceq \bowtie'$ for some $i = 1, \dots, j$. We say that $\bowtie_1, \dots, \bowtie_j$ are the *μ-simple components* of $\bowtie$. ♢

Example 3.2.6. We consider the network #51 of [9] (Figure 3.1(b)), which we denote by N_{51} and it has the following adjacency matrix:

$$A_{51} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of A_{51} are: the network valency 2, -2 and 0 with multiplicity 1, 1 and 2, respectively. The network N_{51} has four non-trivial balanced colorings $\bowtie_1 = \{\{1,2\},\{3,4\}\}$, $\bowtie_2 = \{\{3\},\{1,2,4\}\}$, $\bowtie_3 = \{\{1,2\},\{3\},\{4\}\}$ and $\bowtie_4 = \{\{1\},\{2\},\{3,4\}\}$. The balanced colorings of N_{51} and the eigenvalues of the adjacency matrix corresponding to the quotient networks of N_{51} associated to each balanced coloring are annotated in Figure 3.3.

The balanced coloring $\bowtie_0$ is 2-maximal. The balanced coloring $\bowtie_1$ is (-2)-maximal. The balanced colorings $\bowtie_2$ and $\bowtie_4$ are 0-maximal. And the balanced coloring $\bowtie_=$ is 0-submaximal of type 2 with 0-simple components $\bowtie_3$ and $\bowtie_4$. ♢

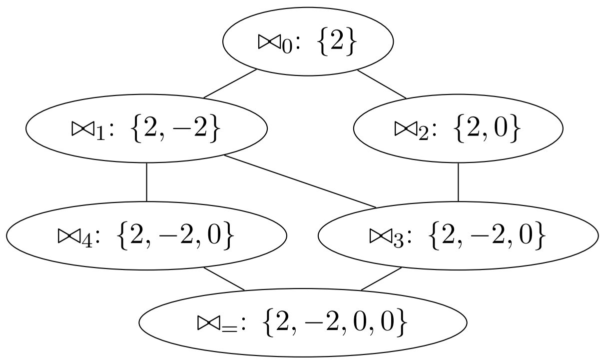

Figure 3.3: Balanced colorings of N_{51} (Figure 3.1(b)) and the eigenvalues of the adjacency matrices associated to the corresponding quotient networks.

3.2.2 Coupled cell systems

In order to associate dynamics to a network, following [18, 8], we specify a phase space for the network and describe vector fields that are admissible for the network.

Let N be a regular network with valency ϑ and represented by the adjacency matrix A. We correspond to each cell c a coordinate x_c and assume that $x_c \in \mathbb{R}$. The *network phase space* is the product of the phase space of the cells, i.e., $\mathbb{R}^{|N|}$.

A vector field $F : \mathbb{R}^{|N|} \to \mathbb{R}^{|N|}$ is *admissible* for a regular network N if:

1. The dynamics of cell c depends only on its internal state and on the state of its input cells, $s(I(c))$. Thus there is a function $f : \mathbb{R}\times\mathbb{R}^{\vartheta} \to \mathbb{R}$ such that for every cell c

$$(F(x))_c = f(x_c, x_{s(I(c))}),$$

where $x_{s(I(c))} = (x_{s(e)})_{e\in I(c)}$;

2. The state of the input cells have equal effect on the dynamics. That is, the function f is S_{ϑ}-invariant, where S_{ϑ} is the group of permutations in $\{1, \dots, \vartheta\}$. For every $\sigma \in S_{\vartheta}$

$$f(\sigma(x_0, x_1, \dots, x_{\vartheta})) = f(x_0, x_1, \dots, x_{\vartheta}),$$

where $\sigma(x_0, x_1, \dots, x_{\vartheta}) = (x_0, x_{\sigma(1)}, \dots, x_{\sigma(\vartheta)})$.

A *coupled cell system* associated to a regular network N is a dynamical system defined by an admissible vector field $F : \mathbb{R}^{|N|} \to \mathbb{R}^{|N|}$

$$\dot{x} = F(x), \quad x \in \mathbb{R}^{|N|}.$$

Let $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \to \mathbb{R}$ be a S_{ϑ}-invariant function. We denote by f^N the admissible vector field for N defined by f and given by the previous formulas. Observe that every admissible vector field for N is equal to f^N for some S_{ϑ}-invariant function f. We say that a function $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \to \mathbb{R}$ is *regular* if f is S_{ϑ}-invariant and $(0, 0, \dots, 0)$ is an isolated zero of f. In this case $0 \in \mathbb{R}^{|N|}$ is an equilibrium point of the coupled cell system defined by f^N.

For differentiable admissible vector fields f^N, its Jacobian at the origin can be represented in terms of the adjacency matrix of N, see [12]. We denote by J_f^N the Jacobian of f^N at the origin $0 \in \mathbb{R}^{|N|}$. We have that

$$J_f^N = (Df^N)_0 = f_0 Id + f_1 A,$$

where Id is the $|N| \times |N|$ identity matrix,

$$f_0 = \frac{\partial f}{\partial x_0}(0, 0, \dots, 0),\ f_1 = \frac{\partial f}{\partial x_1}(0, 0, \dots, 0) = \cdots = \frac{\partial f}{\partial x_{\vartheta}}(0, 0, \dots, 0). \quad (3.1)$$

For every eigenvalue μ of A with algebraic multiplicity m_a and geometric multiplicity m_g, we have that $f_0 + \mu f_1$ is an eigenvalue of J_f^N with the same multiplicities m_a and m_g. See [12, Proposition 3.1]. Moreover, the kernel of J_f^N can be described using the eigenvectors of A. Denote by $v_1, \dots, v_j$ a set of linear independent eigenvectors of A and $\mu_1, \dots, \mu_j$ its corresponding eigenvalues, where j is equal to the sum of all geometric multiplicities. Then

$$\ker(J_f^N) = \{v : J_f^N v = 0\} = \mathrm{Span}(\{v_i : f_0 + \mu_i f_1 = 0\}),$$

where Span denotes the linear subspace spanned by the vectors.

Definition 3.2.7. A *polydiagonal subspace* of $\mathbb{R}^{|N|}$ is a subspace defined by the equality of certain cell coordinates. Let $\bowtie$ be a coloring in N. The polydiagonal subspace associated to $\bowtie$ is defined by

$$\Delta_{\bowtie} = \{x : x_c = x_d \Leftarrow c \bowtie d\} \subseteq \mathbb{R}^{|N|}.$$

Each polydiagonal subspace defines an unique coloring of the cells. A subset $K \subseteq V$ is *invariant* under a map $g : V \to V$, if $g(K) \subseteq K$. A *synchrony subspace* of a network is an invariant polydiagonal subspace for any vector field admissible for the network. $\Diamond$

We have that the synchrony subspaces and balanced colorings are in one-to-one correspondence.

Theorem 3.2.8 ([8, Theorem 4.3])**.** *Let $\bowtie$ be a coloring of cells in a network N. Then $\Delta_{\bowtie}$ is a synchrony subspace of N if and only if $\bowtie$ is balanced.*

The restriction of an admissible vector field to a synchrony subspace $\Delta_{\bowtie}$ is an admissible vector field for the quotient network associated to the balanced coloring $\bowtie$. Moreover, any admissible vector field for the quotient network lifts to an admissible vector field for the network.

Theorem 3.2.9 ([8, Theorem 5.2])**.** *Let N be a regular network with valency ϑ, $\bowtie$ a balanced coloring of N and $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \to \mathbb{R}$ a S_{ϑ}-invariant function. If Q is the quotient network of N associated to $\bowtie$, then*
(i) The restriction of f^N to $\Delta_{\bowtie}$ is the admissible vector field f^Q for Q.
(ii) The admissible vector field f^Q for Q lifts to the admissible vector field f^N for N.

In particular, the previous result means that if $x_Q(t) \in \mathbb{R}^{|Q|}$ is a solution to $\dot{x}_Q(t) = f^Q(x_Q(t))$, then $x_N(t)$ is a solution to $\dot{x}_N(t) = f^N(x_N(t))$, where $(x_N(t))_c = (x_Q(t))_{[c]_{\bowtie}}$ for each cell c in N. And we say that $x_Q(t)$ is *lifted* to $x_N(t)$.

3.2.3 Steady-state bifurcation on regular networks

Let N be a regular network with valency ϑ and represented by the adjacency matrix A. We consider a family of regular functions $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \times \mathbb{R} \to \mathbb{R}$ depending on a parameter λ, i.e., for each $\lambda \in \mathbb{R}$, the function $f(x_0, x_1, \ldots, x_{\vartheta}, \lambda)$ is regular. In this work, we assume that this function is smooth in some neighborhood of the origin. Consider the coupled cell system

$$\dot{x} = f^N(x, \lambda), \tag{3.2}$$

where $f^N : \mathbb{R}^{|N|} \times \mathbb{R} \to \mathbb{R}^{|N|}$ is given for each cell c in N by

$$(f^N(x, \lambda))_c = f(x_c, x_{s(I(c))}, \lambda).$$

Since we are assuming that f is regular, the origin $0 \in \mathbb{R}^{|N|}$ is an equilibrium point of the coupled cell system for every $\lambda \in \mathbb{R}$.

We are interested in studying the steady-state bifurcations of (3.2) occurring from the origin $x = 0$ at $\lambda = 0$. A local steady-state bifurcation at $\lambda = 0$ near the origin can only occur if J_f^N has a zero eigenvalue. Thus we will assume that one of the eigenvalues of J_f^N is zero, say, $f_0 + \mu f_1 = 0$ for some eigenvalue μ of A, where f_0 and f_1 are defined in (3.1). Observe that, under the generic hypothesis on f, $f_1 \neq 0$, μ is the unique eigenvalue satisfying $f_0 + \mu f_1 = 0$. In this case, we say that f is a *regular function with a bifurcation condition associated to* μ.

Definition 3.2.10. Let N be a regular network with valency ϑ and represented by the adjacency matrix A, and $f : \mathbb{R}\times\mathbb{R}^{\vartheta}\times\mathbb{R} \to \mathbb{R}$ a regular function. We say that a differentiable function $b = (b_N, b_\lambda) : [0, \delta[\to \mathbb{R}^{|N|} \times \mathbb{R}$ is an *equilibrium branch of* f *on* N if $b(0) = (0, \dots, 0, 0)$, $b_\lambda(z) \neq 0$ and

$$f^N(b_N(z), b_\lambda(z)) = 0,$$

for every $z > 0$. We say that an equilibrium branch b is *a bifurcation branch of* f *on* N if b is different from the trivial equilibrium branch of f on N, i.e., for every $z \neq 0$,

$$b_N(z) \neq 0.$$

$\diamondsuit$

Despite two different bifurcation branches can define essentially the same branch (e.g., by rescaling of the parameter), it is not a problem for our discussion about the existence of bifurcation branches (see [5, Section 4.2] for a definition of bifurcation branch that takes this aspect into account).

The trivial equilibrium branch is totally synchronized, since all cell's coordinates have the same value. Other bifurcation branches can have less synchrony depending on which synchrony subspaces they belong.

Definition 3.2.11. We say that an equilibrium branch $b : [0, \delta[\to \mathbb{R}^{|N|} \times \mathbb{R}$ has *(exact) synchrony* $\bowtie$, if $b_N([0, \delta[) \subseteq \Delta_{\bowtie}$ (and $b_N([0, \delta[) \nsubseteq \Delta_{\bowtie'}$ for every $\bowtie'$ such that $\bowtie \prec \bowtie'$). $\diamondsuit$

In the same way we lift solutions, we can lift bifurcation branches on a quotient network to the original network, see the end of section 3.2.2.

3.3 Synchrony branching lemma

In this section, we establish the two main steps in order to prove the existence of a bifurcation branch of generic regular functions on regular networks. First we apply the method of Lyapunov-Schmidt Reduction, [7, Chapter VII], to the coupled cell system and we obtain a reduced equation for the bifurcation problem. Next we apply a blow-up argument (see e.g. [10]) to transform the reduced equation into a vector field on a sphere.

Let N be a regular network with valency ϑ and represented by the adjacency matrix A, μ an eigenvalue of A and $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \times \mathbb{R} \to \mathbb{R}$ a regular function with a bifurcation condition associated to μ. Hence $\ker(J_f^N) \neq \{0\}$ and we assume the generic hypothesis that $m = \dim(\ker(J_f^N)) > 0$ is the geometric multiplicity of μ in A.

Let $v_1, \ldots, v_m$ be a basis for $\ker(J_f^N)$ (and eigenvectors of A associated to μ), $v_1^*, \ldots, v_m^*$ be a basis for $\mathrm{Range}(J_f^N)^{\perp}$. Applying the Lyapunov-Schmidt Reduction method, we get a function $g : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$ such that the solutions of $f^N(x, \lambda) = 0$ are in one-to-one correspondence with the solutions of $g(y, \lambda) = 0$. We can calculate the derivatives of g at the origin using the derivatives of f at the origin, see [7, Chapter VII §1 (d)]. Since $f(0, 0, \ldots, 0, \lambda) = 0$ for every λ, we have that

$$g(0, \lambda) = 0, \qquad \frac{\partial g_i}{\partial y_j}(0, 0) = 0, \qquad \frac{\partial g_i}{\partial \lambda}(0, 0) = 0.$$

The Taylor expansion of g at $(y, \lambda) = (0, 0)$ has the following form:

$$g(y, \lambda) = L(\lambda)y + Q_k(y) + \mathcal{O}(\|y\|^{k+1} + \|y\|^2|\lambda|),$$

where

$$L(\lambda) = \lambda Dg_\lambda + \mathcal{O}(2),$$

Dg_λ is the matrix with entries $(\partial^2 g_i/\partial\lambda\partial y_j)$ evaluated at $(y, \lambda) = (0, 0)$, Q_k has homogenous polynomial components in the variable y of smallest degree k such that Q_k does not vanish. From [7, Chapter VII §1 (d)], we know that

$$\frac{\partial^2 g_i}{\partial y_j \partial \lambda}(0, 0) = \langle v_i^*, (Df_\lambda^N)v_j\rangle = \langle v_i^*, (f_{0\lambda}Id + f_{1\lambda}A)v_j\rangle = (f_{0\lambda} + \mu f_{1\lambda})\langle v_i^*, v_j\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^{|N|}$. Therefore

$$Dg_\lambda = (f_{0\lambda} + \mu f_{1\lambda})L,$$

where

$$L = \begin{bmatrix} \langle v_1^*, v_1\rangle & \langle v_1^*, v_2\rangle & \ldots & \langle v_1^*, v_m\rangle \\ \langle v_2^*, v_1\rangle & \langle v_2^*, v_2\rangle & \ldots & \langle v_2^*, v_m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_m^*, v_1\rangle & \langle v_m^*, v_2\rangle & \ldots & \langle v_m^*, v_m\rangle \end{bmatrix}.$$

We will assume that the eigenvalue μ is *semisimple*, i.e., μ has the same algebraic and geometric multiplicity. Note that L is invertible if and only if μ is semisimple.

Since L is invertible, we can choose a basis $v_1'^*, \ldots, v_m'^*$ of $\mathrm{Range}(J_f^N)^\perp$ such that

$$\begin{bmatrix} \langle v_1'^*, v_1\rangle & \langle v_1'^*, v_2\rangle & \ldots & \langle v_1'^*, v_m\rangle \\ \langle v_2'^*, v_1\rangle & \langle v_2'^*, v_2\rangle & \ldots & \langle v_2'^*, v_m\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_m'^*, v_1\rangle & \langle v_m'^*, v_2\rangle & \ldots & \langle v_m'^*, v_m\rangle \end{bmatrix} = Id,$$

by taking $v_i'^* = \sum_{l=1}^m b_{i\,l} v_l^*$ where $b_{i\,j}$ are the entries of L^{-1} for $1 \leq i, j \leq m$.

In the following we assume that μ is semisimple and that we have chosen a basis of $\mathrm{Range}(J_f^N)^\perp$ in the Lyapunov-Schmidt Reduction such

$$g(y, \lambda) = L(\lambda)y + Q_k(y) + \mathcal{O}(\|y\|^{k+1} + \|y\|^2|\lambda|), \tag{3.3}$$

where Q_k has homogenous polynomial components in the variable y of smallest degree k such that Q_k does not vanish and

$$L(\lambda) = \lambda(f_{0\lambda} + \mu f_{1\lambda})Id + \mathcal{O}(2).$$

Definition 3.3.1. Let N be a regular network with valency ϑ and $f : \mathbb{R} \times \mathbb{R}^\vartheta \times \mathbb{R} \to \mathbb{R}$ a regular function. We denote by $k(N, f)$ the integer k in (3.3). We say that the bifurcation problem of f on N has $k - 1$ *degeneracy.* $\diamondsuit$

In [17], the authors studied the degeneracy of a bifurcation problem on regular networks associated to simple eigenvalues. They have shown that there exist bifurcation problems on regular networks with high degeneracy. We refer the reader to their work for examples of k-degenerate bifurcation problems on regular networks, with $1 \leq k \leq 5$.

Before we prove that the integer $k(N, f)$ does not depend generically on the regular function f associated to some eigenvalue, we give an explicit formula for the second derivatives of g_i with respect to y_j and y_l for $1 \leq i, j, l \leq m$.

Since $f : \mathbb{R} \times \mathbb{R}^\vartheta \times \mathbb{R} \to \mathbb{R}$ is a regular function, it has the following Taylor expansion at the origin:

$$\begin{aligned} f(x_0, x_1, \ldots, x_\vartheta, \lambda) = {} & f_0 x_0 + f_1(x_1 + \cdots + x_\vartheta) + f_{0\lambda} x_0 \lambda + f_{1\lambda}(x_1 + \cdots + x_\vartheta)\lambda \\ & + \frac{f_{00}}{2} x_0^2 + f_{01} x_0 (x_1 + \cdots + x_\vartheta) + \frac{f_{11}}{2}(x_1^2 + \cdots + x_\vartheta^2) \\ & + f_{1\vartheta}(\sum_{i=1}^{\vartheta} \sum_{j>i}^{\vartheta} x_i x_j) + \mathcal{O}(3), \end{aligned}$$

where

$$f_{00} = \frac{\partial^2 f}{\partial x_0 \partial x_0}(0,0), \qquad f_{11} = \frac{\partial^2 f}{\partial x_1 \partial x_1}(0,0) = \frac{\partial^2 f}{\partial x_i \partial x_i}(0,0),$$

$$f_{01} = \frac{\partial^2 f}{\partial x_0 \partial x_1}(0,0) = \frac{\partial^2 f}{\partial x_0 \partial x_i}(0,0), \qquad f_{1\vartheta} = \frac{\partial^2 f}{\partial x_1 \partial x_\vartheta}(0,0) = \frac{\partial^2 f}{\partial x_i \partial x_j}(0,0),$$

for $i, j > 0$ and $i \neq j$.

For every cell c in N, we denote by $c_1, \ldots, c_\vartheta$ the source cells of the edges that target c (repeated, if there is more than one edge from the same cell).

$$\begin{aligned} d^2 f_c^N(v_j, v_l) &= f_{00}(v_j * v_l)_c + 2f_{01}(v_j * Av_l)_c + \sum_{a=1}^{\vartheta}\sum_{b=1}^{\vartheta} f_{ab}(v_j)_{c_a}(v_l)_{c_b} \\ &= (f_{00} + 2\mu f_{01})(v_j * v_l)_c + f_{1v}(Av_j * Av_l)_c + (f_{11} - f_{1v})\sum_{a=1}^{\vartheta}(v_j * v_l)_{c_a} \\ &= (f_{00} + 2\mu f_{01} + \mu^2 f_{1v})(v_j * v_l)_c + (f_{11} - f_{1v})(A(v_j * v_l))_c, \end{aligned}$$

where $w * z = (w_1 z_1, w_2 z_2, \ldots, w_n z_n)$, for $w, z \in \mathbb{R}^n$. So

$$d^2 f^N(v_j, v_l) = (f_{00} + 2\mu f_{01} + \mu^2 f_{1v}) v_j * v_l + (f_{11} - f_{1v}) A(v_j * v_l).$$

It follows from [7, Chapter VII §1 (d)] and the Taylor expansion of f that

$$\frac{\partial^2 g_i}{\partial y_j \partial y_l}(0,0) = (f_{00} + 2\mu f_{01} + \mu^2 f_{1\vartheta})\langle v_i^*, v_j * v_l\rangle + (f_{11} - f_{1\vartheta})\langle v_i^*, A(v_j * v_l)\rangle,$$

for $1 \leq i, j, l \leq m$. Since μ is semisimple, v_i^* is orthogonal to any generalized eigenvector associated to an eigenvalue different from μ. Writing $v_j * v_l$ in the base of generalized eigenvectors of A, we have that $\langle v_i^*, A(v_j * v_l)\rangle = \mu\langle v_i^*, v_j * v_l\rangle$. Thus, for $1 \leq i, j, l \leq m$,

$$\frac{\partial^2 g_i}{\partial y_j \partial y_l}(0,0) = (f_{00} + 2\mu f_{01} + \mu f_{11} - \mu f_{1\vartheta} + \mu^2 f_{1\vartheta})\langle v_i^*, v_j * v_l\rangle.$$

Thus, generically, the vanish of the second derivatives of g is independent of the regular function f with a bifurcation condition associated to μ. Next, we prove that the smallest integer $k(N, f)$ is, generically, the same for every regular function f with a bifurcation condition associated to μ.

Lemma 3.3.2. *Let N be a regular network with valency ϑ and adjacency matrix A, μ an eigenvalue of A and $f, f' : \mathbb{R} \times \mathbb{R}^\vartheta \times \mathbb{R} \to \mathbb{R}$ regular functions with a bifurcation condition associated to μ. Then, generically,*

$$k(N, f) = k(N, f').$$

Proof. For a given integer l, we can rearrange the terms in the Taylor expansion of f as follows

$$\begin{aligned} f(x_0, x_1, \ldots, x_\vartheta, \lambda) = &P_{l-1}(x_0, x_1, \ldots, x_\vartheta) + P_l(x_0, x_1, \ldots, x_\vartheta) \\ &+ P_{l+1}(x_0, x_1, \ldots, x_\vartheta) + R(x_0, x_1, \ldots, x_\vartheta, \lambda), \end{aligned}$$

where P_{l-1} is a polynomial of degree lower or equal to $l-1$, P_{l+1} has only terms of degree upper or equal to $l+1$, R is a function such that $R(x_0, x_1, \dots, x_\vartheta, 0) = 0$ and P_l is homogenous polynomial of degree l:

$$P_l(x_0, x_1, \dots, x_\vartheta) = \sum_{\substack{0 \le n_0, n_1, \dots, n_\vartheta \le l \\ n_0 + n_1 + \dots + n_\vartheta = l}} \frac{f^{n_0 n_1 \dots n_\vartheta}}{n_0! n_1! \dots n_\vartheta!} x_0^{n_0} x_1^{n_1} \dots x_\vartheta^{n_\vartheta},$$

$f^{n_0 n_1 \dots n_\vartheta}$ is the l-partial derivative of f at $(0, 0, \dots, 0, 0)$ with respect n_0 times to x_0, n_1 times to x_1, …, and n_ϑ times to x_ϑ. Since f is S_ϑ-invariant, P_l is also S_ϑ-invariant and it has the following form, see e.g. [13],

$$P_l(x_0, x_1, \dots, x_\vartheta) = \sum_{n_0=0}^{l} \sum_{\substack{0 \le n_1 \le \dots \le n_\vartheta \le l \\ n_0 + n_1 + \dots + n_\vartheta = l}} \frac{f^{n_0 n_1 \dots n_\vartheta}}{n_0! n_1! \dots n_\vartheta!} x_0^{n_0} \left(\sum_{\sigma \in S_\vartheta} x_{\sigma(1)}^{n_1} \dots x_{\sigma(\vartheta)}^{n_\vartheta} \right).$$

For $1 \le i, i_1, \dots, i_l \le m$, the l-th derivative of g_i with respect to $y_{i_1}, y_{i_2}, \dots, y_{i_l}$ at $(y, \lambda) = (0, \dots, 0, 0)$ is given by

$$\frac{\partial^l g_i}{\partial y_{i_1} \partial y_{i_2} \dots \partial y_{i_l}} = \sum_{n_0=0}^{l} \sum_{\substack{0 \le n_1 \le \dots \le n_\vartheta \le l \\ n_0 + n_1 + \dots + n_\vartheta = l}} \frac{f^{n_0 n_1 \dots n_\vartheta}}{n_0! n_1! \dots n_\vartheta!} \langle v_i^*, A_{n_0 n_1 \dots n_\vartheta}(v_{i_1}, \dots, v_{i_l}) \rangle,$$

where $A_{n_0 n_1 \dots n_\vartheta}(v_{i_1}, \dots, v_{i_l}) \in \mathbb{R}^{|N|}$ and it is given for every cell c in N by

$$(A_{n_0 n_1 \dots n_\vartheta}(v_{i_1}, \dots, v_{i_l}))_c = \sum_{\sigma \in S_\vartheta} \frac{\partial^l}{\partial t_1 \dots \partial t_l} \left(\prod_{b=0}^{\vartheta} (t_1 v_{i_1} + \dots + t_l v_{i_l})_{c_{\sigma(b)}}^{n_b} \right) \Bigg|_{t_i = 0},$$

where $\sigma(0) = 0$, $c_0 = c$ and $c_1, \dots, c_\vartheta$ are the source cells of the edges that target c.

Note that every term in the variable y of degree l in g vanish if and only if for every $1 \le i, i_1, \dots, i_l \le m$

$$\frac{\partial^l g_i}{\partial y_{i_1} \partial y_{i_2} \dots \partial y_{i_l}} = 0.$$

For each $1 \le i, i_1, \dots, i_l \le m$, regard $(\partial^l g_i)/(\partial y_{i_1} \partial y_{i_2} \dots \partial y_{i_l})$ as a polynomial function in the variables $f^{n_0 n_1 \dots n_\vartheta}$, where $0 \le n_0 \le l$ and $0 \le n_1 \le n_2 \le \dots \le n_\vartheta \le l$ such that $n_0 + n_1 + \dots + n_\vartheta = l$. We have two cases: $(\partial^l g_i)/(\partial y_{i_1} \partial y_{i_2} \dots \partial y_{i_l})$ is identically zero since

$$\langle v_i^*, A_{n_0 n_1 \dots n_\vartheta}(v_{i_1}, v_{i_2}, \dots, v_{i_l}) \rangle = 0,$$

for every $0 \le n_0 \le l$ and $0 \le n_1 \le n_2 \le \dots \le n_\vartheta \le l$; otherwise the set of regular functions such that $(\partial^l g_i)/(\partial y_{i_1} \partial y_{i_2} \dots \partial y_{i_l}) = 0$ is residual and for every generic regular function we have that

$$\frac{\partial^l g_i}{\partial y_{i_1} \partial y_{i_2} \dots \partial y_{i_l}} \ne 0.$$

Therefore, generically, given f regular every term in the variable y of degree l in g vanishes if and only if

$$\langle v_i^*, A_{n_0 n_1 \dots n_\vartheta}(v_{i_1}, v_{i_2}, \dots, v_{i_l})\rangle = 0,$$

for every $1 \leq i, i_1, \dots, i_l \leq m$, $0 \leq n_0 \leq j$ and $0 \leq n_1 \leq n_2 \leq \cdots \leq n_\vartheta \leq j$.

The second part of the previous "if and only if" does not depend on the regular function f. So for every regular functions $f, f' : \mathbb{R} \times \mathbb{R}^\vartheta \times \mathbb{R} \to \mathbb{R}$ with a bifurcation condition associated to μ, we have generically that

$$k(N, f) = k(N, f').$$

□

Following the previous lemma, we define the smallest k in (3.3) as a function of the network N and the eigenvalue μ.

Definition 3.3.3. Let N be a regular network and μ an eigenvalue of its adjacency matrix. For any generic regular function f with a bifurcation condition associated to μ, we define $k(N, \mu) = k(N, f)$. We say that the bifurcation problem associated to μ on N has $k(N, \mu) - 1$ *degeneracy.* ♢

Remark 3.3.4. Let N be a regular network, Q a quotient network of N and μ an eigenvalue of the adjacency matrix associated to Q. Then

$$k(N, \mu) \leq k(Q, \mu).$$

In fact, let f be a regular function with a bifurcation condition associated to μ. Denote by g^Q and g^N the reduced functions of f^Q and f^N, respectively, obtained by Lyapunov-Schmidt reduction. The previous inequality follows from the fact

$$g^Q = P^{-1} g^N P,$$

where $P : \mathbb{R}^{|Q|} \to \Delta \subseteq \mathbb{R}^{|N|}$ is the lift of the phase space of Q into the synchrony subspace of N associated to the quotient network and $P^{-1} : \Delta \to \mathbb{R}^{|Q|}$ is the inverse of P. ♢

In the next step we apply a blow-up argument also used in the equivariant theory, see e.g. [10]. Applying the following change of variables to the reduced function g of the Lyapunov-Schmidt reduction (3.3),

$$\begin{cases} y = \epsilon u \\ \lambda = \epsilon^{k-1}\eta \end{cases}, \tag{3.4}$$

where $u \in S^{m-1}(m-1 \text{ dimensional sphere})$, $\epsilon \in \mathbb{R}$, $\eta \in \mathbb{R}$ and $k = k(N, \mu)$, we have the following equation:

$$g(y, \lambda) = 0 \Leftrightarrow g(\epsilon u, \epsilon^{k-1}\eta) = 0 \Leftrightarrow \epsilon^k \left(\eta(f_{0\lambda} + \mu f_{1\lambda})u + Q_k(u) + \mathcal{O}(|\epsilon|)\right) = 0.$$

Let $h : S^{m-1} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$ be the function given by

$$h(u, \epsilon, \eta) = \eta(f_{0\lambda} + \mu f_{1\lambda})u + Q_k(u) + \mathcal{O}(|\epsilon|).$$

For $y \neq 0$, we have that

$$g(y, \lambda) = 0 \Leftrightarrow h(u, \epsilon, \eta) = 0. \tag{3.5}$$

Proposition 3.3.5. *Let N be a regular network with valency ϑ and adjacency matrix A, μ a semisimple eigenvalue of A with multiplicity m and $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \times \mathbb{R} \to \mathbb{R}$ a regular function with a bifurcation condition associated to μ. If b is a bifurcation branch of f on N with synchrony $\bowtie$, then generically there exists $\tilde{u} \in S^{m-1}$ and $\tilde{\eta} \in \mathbb{R}$ such that*

$$h(\tilde{u}, 0, \tilde{\eta}) = 0,$$

and

$$\tilde{u}_1 v_1 + \cdots + \tilde{u}_m v_m \in \Delta_{\bowtie},$$

where $v_1, \ldots, v_m$ is the basis of $\ker(J_f^N)$*.*

Proof. Let $b = (b_N, b_\lambda)$ be a bifurcation branch of f on N and $k = k(N, \mu)$. Consider $\tilde{b} = (\tilde{b}_y, \tilde{b}_\lambda) : [0, \delta[\to \mathbb{R}^m \times \mathbb{R}$ such that $\tilde{b}_y$ is the projection of b_N into $\ker(J_f^N)$ according to the basis $v_1, \ldots, v_m$ and $\tilde{b}_\lambda = b_\lambda$. Then

$$g(\tilde{b}_y(z), \tilde{b}_\lambda(z)) = h\left(\frac{\tilde{b}_y(z)}{\|\tilde{b}_y(z)\|}, \|\tilde{b}_y(z)\|, \frac{\tilde{b}_\lambda(z)}{\|\tilde{b}_y(z)\|^{k-1}}\right) = 0,$$

for $z \neq 0$. Note that $\tilde{b}'_y(0) \neq 0$, see e.g. [5, Lemma 4.2.1]. We can also prove by induction on j, that $\tilde{b}_\lambda^{(j)}(0) = 0$, for $1 \leq j < k-1$, where $\tilde{b}_\lambda^{(j)}(0)$ is the j-derivative of $\tilde{b}_\lambda$ at $z = 0$. Taking the limit of $z \to 0$ in the previous equation,

$$h(\tilde{u}, 0, \tilde{\eta}) = 0,$$

where

$$\tilde{u} = \frac{\tilde{b}'_y(0)}{\|\tilde{b}'_y(0)\|}, \quad \tilde{\eta} = \frac{\tilde{b}_\lambda^{(k-1)}(0)}{(k-1)!\|\tilde{b}'_y(0)\|^{k-1}}.$$

If b has synchrony $\bowtie$, then $b_N(z) \in \Delta_{\bowtie}$ and $\tilde{b}_{y_1}(z)v_1 + \cdots + \tilde{b}_{y_m}(z)v_m \in \Delta_{\bowtie}$ for every $z \geq 0$. So $\tilde{b}'_{y_1}(0)v_1 + \cdots + \tilde{b}'_{y_m}(0)v_m \in \Delta_{\bowtie}$ and

$$\tilde{u}_1 v_1 + \cdots + \tilde{u}_m v_m \in \Delta_{\bowtie}.$$

□

It follows from Proposition 3.3.5 that we can study the zeros of $h(u,0,\eta)$ to understand the bifurcation branches. For the radial component

$$\langle h(u,0,\eta),u\rangle = 0 \Leftrightarrow \eta = -\frac{\langle Q_k(u),u\rangle}{f_{0\lambda}+\mu f_{1\lambda}},$$

where it is assumed, generically, that $f_{0\lambda}+\mu f_{1\lambda}\neq 0$. Defining

$$\tilde{\eta}(u) = -\frac{\langle Q_k(u),u\rangle}{f_{0\lambda}+\mu f_{1\lambda}}, \qquad \tilde{h}(u) = h(u,0,\tilde{\eta}(u)),$$

we obtain a vector field $\tilde{h}$ on the $(m-1)$-sphere and

$$h(u,0,\eta) = 0 \Leftrightarrow \tilde{h}(u) = 0 \wedge \eta(u) = -\frac{\langle Q_k(u),u\rangle}{f_{0\lambda}+\mu f_{1\lambda}}.$$

Now, we see how to make the correspondence between a zero of the vector field $\tilde{h}$ and a bifurcation branch of f on N. For this purpose, we will assume for a zero $\tilde{u}\in S^{m-1}$ of $\tilde{h}$ that

$$\left(\frac{\partial h}{\partial(u,\eta)}\right)_{(\tilde{u},0,\tilde{\eta}(\tilde{u}))} \text{ is non-singular,} \tag{3.6}$$

where the Jacobian is calculated in the geometry of $S^{m-1}\times\mathbb{R}$.

Moreover, we say that condition (H1) holds for f and N if

$$\forall_{\tilde{u}} \;\; \tilde{h}(\tilde{u}) = 0 \Rightarrow \text{(3.6) holds for } \tilde{u}. \tag{H1}$$

Proposition 3.3.6. *Let N be a regular network with valency ϑ and adjacency matrix A, μ a semisimple eigenvalue of A with multiplicity m, $f:\mathbb{R}\times\mathbb{R}^{\vartheta}\times\mathbb{R}\to\mathbb{R}$ a regular function with a bifurcation condition associated to μ and $\tilde{u}\in S^{m-1}$ such that $\tilde{h}(\tilde{u})=0$ and (3.6) holds for $\tilde{u}$. Then generically there exists a bifurcation branch of f on N.*

Proof. We have that $h(\tilde{u},0,\tilde{\eta}(\tilde{u}))=0$, because $\tilde{h}(\tilde{u})=0$. Since (3.6) holds for $\tilde{u}$, it follows from the implicit function theorem that there exists a neighborhood $W\subseteq\mathbb{R}$ of $\epsilon=0$ and a differentiable function $(u^*,\eta^*):W\to S^{m-1}\times\mathbb{R}$ such that $\tilde{h}(u^*(\epsilon),\epsilon,\eta^*(\epsilon))=0$ and $(u^*,\eta^*)(0)=(\tilde{u},\tilde{\eta})$. Recalling (3.4) and (3.5), we have that

$$h(u^*(\epsilon),\epsilon,\eta^*(\epsilon))=0 \Leftrightarrow g(\epsilon u^*(\epsilon),\epsilon^{k-1}\eta^*(\epsilon))=0 \Leftrightarrow g(y^*(\epsilon),\lambda^*(\epsilon))=0,$$

defining $(y^*,\lambda^*):W\to\mathbb{R}^m\times\mathbb{R}$ by $y^*(\epsilon)=\epsilon u^*(\epsilon)$ and $\lambda^*(\epsilon)=\epsilon^{k-1}\eta^*(\epsilon)$.

By the Lyapunov-Schmidt reduction, there exists a differentiable function $b^*:W\to\mathbb{R}^{|N|}\times\mathbb{R}$ associated to (y^*,λ^*) such that $0\in W$, $b^*(0)=(0,0)$, $b^*_N(\epsilon)\neq 0$ and $f^N(b^*_N(\epsilon),b^*_\lambda(\epsilon))=0$, for every $\epsilon\in W\setminus\{0\}$. Since the origin is an isolated zero of $f^N(x,0)$, there exists $\delta>0$ such that $b^*_\lambda(\epsilon)\neq 0$ for $0<\epsilon<\delta$. Restricting the function b^* to $[0,\delta[$, we have that b^* is a bifurcation branch of f on N. □

Remark 3.3.7. Equation (3.6) is related to the determinacy of a bifurcation problem. When the kernel of the Jacobian is one-dimensional, the determinacy and degeneracy are equal. If $m = 1$ and $f_{0\lambda} + \mu f_{1\lambda} \neq 0$, then (3.6) holds. When $m > 1$, determinacy and degeneracy can be different, see Example 3.3.8. $\Diamond$

If condition (3.6) fails, we can apply the Lyapunov-Schmidt Reduction to the function $h(u, \epsilon, \eta)$ at $(\tilde{u}, 0, \tilde{\eta})$ and a blow-up change of coordinates to obtain a vector field on a sphere, then we look for zeros of this new reduced vector field on a sphere associated to the parameter ϵ. Since the Jacobian of h with respect to (u, η) at $(\tilde{u}, 0, \tilde{\eta}(\tilde{u}))$ is not identically null, the dimension of the problem will be further reduced and we will need to calculate derivatives of higher order of the original vector field f^N. In the new reduced vector field on a sphere, $\tilde{u}$ can continue or not to be an equilibrium. If it is not an equilibrium, then do not correspond to a bifurcation branch. If it continues to be an equilibrium and a similar condition to (3.6) holds, then we obtain a bifurcation branch. If it continues to be an equilibrium and a similar condition to (3.6) fails, we need to repeat the previous process. See e.g. [15].

If a zero $\tilde{u}$ of $\tilde{h}$ corresponds to a point in some synchrony subspace $\Delta_{\bowtie}$ $(\tilde{u}_1 v_1 + \cdots + \tilde{u}_m v_m \in \Delta_{\bowtie})$ such that $k(N/\bowtie, \mu) > k(N, \mu)$, then (3.6) fails at $\tilde{u}$. In this case, we should study the bifurcation problem of f on $N/\bowtie$, or look for zeros of $\tilde{h}$ which do not correspond to a point in $\Delta_{\bowtie}$.

For submaximal colorings, we use condition (H1a). We say that condition (H1a) holds for f and N if

$$\forall_{\tilde{u},\bowtie} \ \ \tilde{h}(\tilde{u}) = 0 \wedge \bowtie_{=} \prec \bowtie \wedge (\tilde{u}_1 v_1 + \cdots + \tilde{u}_m v_m) \notin \Delta_{\bowtie} \Rightarrow \ \text{(3.6) holds for } \tilde{u}. \tag{H1a}$$

Example 3.3.8. As in Example 3.2.6, consider again the network #51 of [9]. The eigenvalue 0 of A_{51} is semisimple and has multiplicity 2. The balanced coloring $\bowtie_{=}$ is 0-submaximal of type 2 with 0-simple components $\bowtie_3$ and $\bowtie_4$. Let $f : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be a generic regular function with a bifurcation condition associated to 0, i.e., $f_0 = 0$. We have that $\dim(\ker(J_f^{N_{51}})) = 2$. We choose a basis of $\ker(J_f^{N_{51}})$ such that $v_1 \in \Delta_{\bowtie_3}$ and $v_2 \in \Delta_{\bowtie_4}$. Let $v_1 = (0, 0, 1, 0) \in \Delta_{\bowtie_3}$, $v_2 = (-1, 1, 0, 0) \in \Delta_{\bowtie_4}$, $v_1^* = (0, 0, 1, -1)$ and $v_2^* = (-1, 1, 0, 0)/2$, where v_1^*, v_2^* is a basis of $\mathrm{Range}(J_f^{N_{51}})^{\perp}$. Then

$$h(u_1, u_2, \epsilon, \eta) = \begin{bmatrix} f_{0\lambda}\eta u_1 + \dfrac{f_{00}u_1^2}{2} \\ f_{0\lambda}\eta u_2 \end{bmatrix} + \mathcal{O}(\epsilon).$$

Moreover, $h(u_1, u_2, 0, \eta) = 0$ if and only if $(u_1, u_2, \eta) = (\pm 1, 0, \mp f_{00}/(2f_{0\lambda}))$ or $(u_1, u_2, \eta) = (0, \pm 1, 0)$. We have

$$\left(\frac{\partial h}{\partial(u, \eta)}\right)_{(\pm 1, 0, \mp \frac{f_{00}}{(2f_{0\lambda})})} = \begin{bmatrix} 0 & \pm f_{0\lambda} \\ \mp \dfrac{f_{00}}{2} & 0 \end{bmatrix},$$

$$\left(\frac{\partial h}{\partial(u,\eta)}\right)_{(0,\pm 1,0)} = \begin{bmatrix} 0 & 0 \\ 0 & \pm f_{0\lambda} \end{bmatrix}.$$

Since (3.6) holds at the zeros $(\pm 1, 0)$, then by Proposition 3.3.6 there is a bifurcation branch with exact synchrony $\bowtie_3$. However condition (3.6) fails at the zeros $(0, \pm 1)$ and the dimension of the kernel is equal to 1. We could apply the Lyapunov-Schmidt Reduction and obtain an one-dimensional bifurcation problem, then we should solve it by finding the lowest non-vanishing terms of the reduced equation. Alternatively, we note that the zero $(0, 1)$ corresponds to a point in the synchrony subspace $\Delta_{\bowtie_4}$ and that $k(N/\bowtie_4, 0) = 3$. We can also obtain the bifurcation branch of f on N with synchrony $\bowtie_4$ by studying the bifurcation problem of f on $N/\bowtie_4$. ♢

3.4 Synchrony branching lemma – odd dimensional case

Now, we prove the analogous version of the odd dimensional version of the Equivariant Branch Lemma for regular networks. Recall the notation of section 3.3.

Theorem 3.4.1. *Let N be a regular network with valency ϑ and adjacency matrix A, $\bowtie$ a balanced coloring of N and μ a semisimple eigenvalue of A. Let $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \times \mathbb{R} \to \mathbb{R}$ be a regular function with a bifurcation condition associated to μ such that $\ker(J_f^N) \cap \Delta_{\bowtie}$ has odd dimension. Assume that condition (H1) holds for f and $N/\bowtie$. Then generically there is a bifurcation branch of f on N with at least synchrony $\bowtie$. Moreover, if $\bowtie$ is μ-maximal, then the bifurcation branch has exact synchrony $\bowtie$.*

Proof. Let N be a regular network with valency ϑ and represented by the adjacency matrix A, $\bowtie$ a balanced coloring of N and μ a semisimple eigenvalue of A. Let $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \times \mathbb{R} \to \mathbb{R}$ be a regular function with a bifurcation condition associated to μ such that $\ker(J_f^N) \cap \Delta_{\bowtie}$ has odd dimension.

Denote by Q the quotient network of N with respect to $\bowtie$ which is represented by the adjacency matrix A_Q. Then μ is a semisimple eigenvalue of A_Q and $m = \dim(\ker(J_f^Q)) = \dim(\ker(J_f^N) \cap \Delta_{\bowtie})$ is odd.

We perform the calculation of section 3.3 for the network Q and the generic regular function f. Following section 3.3, let $\tilde{h}$ be the vector field in S^{m-1} obtained. Since m is odd, from the Poincaré-Hopf theorem [14], we know that there exists at least one $\tilde{u} \in S^{m-1}$ such that $\tilde{h}(\tilde{u}) = 0$. Then $h(\tilde{u}, 0, \tilde{\eta}(\tilde{u})) = 0$, where $\tilde{\eta}(\tilde{u})$ is defined in section 3.3.

Assuming that condition (H1) holds for f and $N/\bowtie$, we have that (3.6) holds for $\tilde{u}$. From Proposition 3.3.6, there exists a bifurcation branch of f on Q. Last, we can lift this bifurcation branch of f on Q to a bifurcation branch of f on N with at least synchrony $\bowtie$.

If $\bowtie$ is μ-maximal, then for every $\bowtie'$ such that $\bowtie \prec \bowtie'$ there is no bifurcation branch of f on $N/\bowtie'$. Thus the bifurcation branch has exact synchrony $\bowtie$. □

We establish the existence of a bifurcation branch, using the result above.

Example 3.4.2. Let N be the regular network given by the adjacency matrix:

$$A = \begin{bmatrix} 24 & 1 & 2 & 3 & 4 \\ 16 & 0 & 5 & 6 & 7 \\ 8 & 6 & 3 & 8 & 9 \\ 0 & 8 & 9 & 6 & 11 \\ 20 & 3 & 4 & 5 & 2 \end{bmatrix}.$$

The eigenvalues of A are: the network valency 34 with multiplicity 1; 13 with multiplicity 1; and -4 with multiplicity 3.

Let $f : \mathbb{R} \times \mathbb{R}^{34} \times \mathbb{R} \to \mathbb{R}$ be a regular function with a bifurcation condition associated to the eigenvalue $\mu = -4$. Considering the trivial balanced coloring, $\bowtie_=$, we have that $\dim(\ker^*(J_f^N) \cap \Delta_{\bowtie}) = 3$. Assume that condition (H1) holds for f and N. Then there exists a bifurcation branch of f on N, by Theorem 3.4.1.

The network N has no non-trivial balanced colorings and the unique eigenvalue of the adjacency matrix associated to $N/\bowtie_0$ is 34. So $\bowtie_=$ is (-4)-maximal and the bifurcation branch has no synchrony. ◊

In the next two examples, we do not explicitly present the networks but we assume that the networks satisfy some conditions. Since the number of cells in a network is not restricted, those conditions may be solved with as many variables as it is needed to consider. The next example shows that we cannot remove the odd dimension condition in Theorem 3.4.1.

Example 3.4.3. Let N be a network with adjacency matrix A, μ a semisimple eigenvalue of A with multiplicity 2. Let f be a generic regular function with a bifurcation condition associated to μ, (v_1, v_2) be a basis for $\ker(J_f^N)$ and (v_1^*, v_2^*) be a basis for $\mathrm{Range}(J_f^N)^\perp$. We will assume that A, (v_1, v_2) and (v_1^*, v_2^*) respect the following conditions:

$$\langle v_1^*, v_1\rangle = \langle v_2^*, v_2\rangle = 1, \qquad \langle v_2^*, v_1\rangle = \langle v_1^*, v_2\rangle = 0,$$

$$\langle v_i^*, v_1 * v_1\rangle = \langle v_i^*, v_1 * v_2\rangle = \langle v_i^*, v_2 * v_2\rangle = 0, \quad i = 1, 2,$$

$$\langle v_1^*, v_1 * v_1 * v_1)\rangle = \langle v_2^*, v_1 * v_1 * v_2)\rangle, \quad \langle v_1^*, v_2 * v_2 * v_1)\rangle = \langle v_2^*, v_2 * v_2 * v_2)\rangle,$$

$$\langle v_1^*, 3(v_1 * (A(v_1 * v_1)))\rangle = \langle v_2^*, 2v_1 * (A(v_1 * v_2)) + v_2 * (A(v_1 * v_1))\rangle,$$

$$\langle v_2^*, 3(v_2 * (A(v_2 * v_2)))\rangle = \langle v_1^*, 2v_2 * (A(v_2 * v_1)) + v_1 * (A(v_2 * v_2))\rangle,$$

$$0 \neq \langle v_1^*, v_1{*}v_1{*}v_2)\rangle = \langle v_1^*, v_2{*}v_2{*}v_2)\rangle = -\langle v_2^*, v_1{*}v_2{*}v_2)\rangle = -\langle v_2^*, v_1{*}v_1{*}v_1)\rangle,$$

$$0 \neq \langle v_1^*, 2v_1 * (A(v_1 * v_2)) + v_2 * (A(v_1 * v_1))\rangle = \langle v_1^*, 3(v_2 * (A(v_2 * v_2)))\rangle =$$
$$= -\langle v_2^*, 2v_2 * (A(v_1 * v_2)) + v_1 * (A(v_2 * v_2))\rangle = -\langle v_2^*, 3(v_1 * (A(v_1 * v_1)))\rangle.$$

Then $k(N,\mu) = 3$. Let $g = (g_1, g_2) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the reduced equation obtained by the Lyapunov-Schmidt Reduction. The previous equalities imply for every regular function f that

$$\frac{\partial^3 g_1}{\partial y_1^3} = \frac{\partial^3 g_2}{\partial y_1^2 \partial y_2}, \qquad \frac{\partial^3 g_1}{\partial y_1 \partial y_2^2} = \frac{\partial^3 g_2}{\partial y_2^3},$$

$$\alpha = \frac{\partial^3 g_1}{\partial y_1^2 \partial y_2} = \frac{\partial^3 g_1}{\partial y_2^3} = -\frac{\partial^3 g_2}{\partial y_1^3} = -\frac{\partial^3 g_2}{\partial y_1 \partial y_2^2} \neq 0$$

at $(y_1, y_2, \lambda) = (0,0,0)$. Then the vector field, $\tilde{h}$, on the 1-sphere is given by

$$\tilde{h}(u_1, u_2) = \begin{bmatrix} \alpha u_2 (u_1^2 + u_2^2)^2 \\ -\alpha u_1 (u_1^2 + u_2^2)^2 \end{bmatrix} = \alpha \begin{bmatrix} u_2 \\ -u_1 \end{bmatrix},$$

where $(u_1, u_2) \in S^1$. We have that $\tilde{h}(u_1, u_2) \neq 0$ for every $(u_1, u_2) \in S^1$. By Proposition 3.3.5, there is no bifurcation branch of f on N. ◊

In the next example, we lift the network of the previous example to a network with one more cell. This example shows that there are zeros of the vector field on the sphere which do not correspond to a bifurcation branch and the relevance of (3.6) in Proposition 3.3.6.

Example 3.4.4. Let N, A, μ, v_1, v_2, v_1^* and v_2^* as in Example 3.4.3. Assume that $\mu = 0$. Fix a cell c of N. Let $\hat{N}$ be the network with $|N|+1$ cells given by the following adjacency matrix:

$$\hat{A} = \begin{bmatrix} A & 0 \\ R & 0 \end{bmatrix},$$

where $R = (A_{c\,d})_{d \in N}$ is a $1 \times |N|$-matrix. Denote by c' the new cell of $\hat{N}$ and by $\bowtie$ the balanced coloring of $\hat{N}$ given by $c \bowtie c'$. The network N is a quotient network of $\hat{N}$ with respect to $\bowtie$. Note that 0 is a semisimple eigenvalue of $\hat{A}$ with multiplicity 3.

Let f be a generic regular function with a bifurcation condition associated to $\mu = 0$, $\hat{v}_1 = (v_1, (v_1)_c)$, $\hat{v}_2 = (v_2, (v_2)_c)$, $\hat{v}_3 = (0, \ldots, 0, 1)$, $\hat{v}_1^* = (v_1^*, 0)$, $\hat{v}_2^* = (v_2^*, 0)$ and $\hat{v}_3^* \in \mathbb{R}^{|\hat{N}|}$ such that $(\hat{v}_3^*)_c = -1$, $(\hat{v}_3^*)_{c'} = 1$ and $(\hat{v}_3^*)_d = 0$ otherwise. Then $(\hat{v}_1, \hat{v}_2, \hat{v}_3)$ is a basis of $\ker(J_f^{\hat{N}})$ and $(\hat{v}_1^*, \hat{v}_2^*, \hat{v}_3^*)$ is a basis of $\text{Range}(J_f^{\hat{N}})^\perp$. Let $\hat{g} = (\hat{g}_1, \hat{g}_2, \hat{g}_3) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ be the reduced equation obtained by the Lyapunov-Schmidt Reduction of $f^{\hat{N}}$. Note that

$$\frac{\partial^2 g_3}{\partial y_1 \partial y_3} = f_{00}(v_1)_c, \qquad \frac{\partial^2 g_3}{\partial y_2 \partial y_3} = f_{00}(v_2)_c, \qquad \frac{\partial^2 g_3}{\partial y_3^2} = f_{00}, \qquad \frac{\partial^2 g_i}{\partial y_j \partial y_l} = 0,$$

for any other i, j and l. Then $k(\hat{N}, 0) = 2$ and

$$h(u_1, u_2, u_3, 0, \eta) = \begin{bmatrix} f_{0\lambda}\eta u_1 \\ f_{0\lambda}\eta u_2 \\ f_{0\lambda}\eta u_3 + \frac{f_{00}}{2}u_3^2 + f_{00}(v_1)_c u_1 u_3 + f_{00}(v_2)_c u_2 u_3 \end{bmatrix},$$

where $(u_1, u_2, u_3) \in S^2$. We have that $h(0, 0, 1, -f_{00}/(2f_{0\lambda})) = 0$, condition (3.6) holds for $(0, 0, 1)$ and it leads to a bifurcation branch of f on $\hat{N}$.

On the other hand $h(u_1, u_2, 0, 0) = 0$, condition (3.6) does not hold for $(u_1, u_2, 0)$, where $(u_1, u_2, 0, 0) \in S^2 \times \mathbb{R}$, and it does not lead to a bifurcation branch of f on $\hat{N}$. Otherwise this bifurcation branch would be inside $\Delta_{\bowtie} \subseteq \mathbb{R}^{|\hat{N}|}$, since $\hat{v}_1, \hat{v}_2 \in \Delta_{\bowtie}$, and would lead to a bifurcation branch of f on N. But, as we saw, there is no bifurcation branch of f on N. $\diamondsuit$

3.5 Synchrony branching lemma – two dimensional case

In this section, we study the smallest case not included in the results of the previous section, i.e., when the semisimple eigenvalue μ has multiplicity $m = 2$ and $k(N, \mu)$ is even. We give conditions for the existence of bifurcation branches with maximal or submaximal synchrony.

Let N be a regular network with valency ϑ and represented by the adjacency matrix A, μ a semisimple eigenvalue of A and $f : \mathbb{R} \times \mathbb{R}^{\vartheta} \times \mathbb{R} \to \mathbb{R}$ a generic regular function with a bifurcation condition associated to μ such that $m = \dim(\ker(J_f^N)) = 2$.

Taking into account the calculations in section 3.3, when $m = \dim(\ker(J_f^N) = 2$, we have the following vector field on the 1-sphere

$$\tilde{h}(u_1, u_2) = Q_k(u_1, u_2) - \langle Q_k(u_1, u_2), (u_1, u_2)\rangle(u_1, u_2),$$

where $u_1^2 + u_2^2 = 1$, $k = k(N, \mu)$. Transforming the variables (u_1, u_2) to the angular variable θ, where $(u_1, u_2) = (\cos(\theta), \sin(\theta))$. The dynamical system $\dot{u} = \tilde{h}(u)$ is equivalent to the dynamical system $\dot{\theta} = \Theta(\theta)$ given by

$$\Theta(\theta) = \cos(\theta)q_2(\cos(\theta), \sin(\theta)) - \sin(\theta)q_1(\cos(\theta), \sin(\theta)),$$

where $Q_k(u_1, u_2) = (q_1(u_1, u_2), q_2(u_1, u_2))$. So, finding zeros of $\tilde{h}$ is equivalent to solve the equation

$$\Theta(\theta) = 0.$$

Theorem 3.5.1. *Let N be a regular network with valency ϑ and adjacency matrix A, $\bowtie$ a balanced coloring of N and μ a semisimple eigenvalue of A.*

Let $f : \mathbb{R} \times \mathbb{R}^\vartheta \times \mathbb{R} \to \mathbb{R}$ be a regular function with a bifurcation condition associated to μ and such that $\dim(\ker(J_f^N) \cap \Delta_{\bowtie}) = 2$. *Assume that condition (H1) holds for f and $N/\bowtie$. If $\bowtie$ μ-maximal and $k(N/\bowtie, \mu)$ is even, then generically there is a bifurcation branch of f on N with exact synchrony $\bowtie$.*

Proof. Let N be a regular network with valency ϑ, $\bowtie$ a balanced coloring of N and μ a semisimple eigenvalue of the adjacency matrix of N. Take $f : \mathbb{R} \times \mathbb{R}^\vartheta \times \mathbb{R} \to \mathbb{R}$ to be a regular function with a bifurcation condition associated to μ such that $\dim(\ker(J_f^N) \cap \Delta_{\bowtie}) = 2$.

Let $A_{\bowtie}$ to be the adjacency matrix of $N/\bowtie$. Then $m = \dim(\ker(J_f^{N/\bowtie})) = \dim(\ker(J_f^N) \cap \Delta_{\bowtie}) = 2$ and μ is a semisimple eigenvalue of $A_{\bowtie}$.

Performing the calculations of section 3.3 and of this section for the network $N/\bowtie$ and the regular function f, we consider the function $\Theta(\theta)$ given by

$$\Theta(\theta) = \cos(\theta) q_2(\cos(\theta), \sin(\theta)) - \sin(\theta) q_1(\cos(\theta), \sin(\theta)),$$

where $Q_k(u_1, u_2) = (q_1(u_1, u_2), q_2(u_1, u_2))$ and $k = k(N/\bowtie, \mu)$. We look for solutions of $\Theta(\theta) = 0$.

Suppose that $\bowtie$ is μ-maximal and $k(N/\bowtie, \mu)$ is even. Then $Q_k(u_1, u_2) = Q_k(-u_1, -u_2)$ and

$$\Theta(\theta + \pi) = -\Theta(\theta).$$

By the intermediate value theorem, we know that it must exists $\tilde{\theta}$ such that $\Theta(\tilde{\theta}) = 0$. Consider $\tilde{u} = (\cos(\tilde{\theta}), \sin(\tilde{\theta}))$, then $\tilde{h}(\tilde{u}) = 0$. Assuming that condition (H1) holds for f and $N/\bowtie$, we know that (3.6) holds for $\tilde{u}$. From Proposition 3.3.6, there exists a bifurcation branch of f on $N/\bowtie$. Lifting this bifurcation branch, we obtain a bifurcation branch of f on N with at least synchrony $\bowtie$. Since $\bowtie$ is μ-maximal this bifurcation branch of f on N has exact synchrony $\bowtie$. □

Returning to the beginning of this section. If $k(N, \mu) = 2$, then

$$q_i(u_1, u_2) = \beta \left(\frac{d_{i11}}{2} u_1^2 + d_{i12} u_1 u_2 + \frac{d_{i22}}{2} u_2^2 \right), \tag{3.7}$$

where $i = 1, 2$, $\beta = (f_{00} + 2\mu f_{01} + \mu f_{11} - \mu f_{1\vartheta} + \mu^2 f_{1\vartheta})$ and for $l_1, l_2 = 1, 2$

$$d_{il_1l_2} = \langle v_i^*, v_{l_1} * v_{l_2} \rangle.$$

Theorem 3.5.2. *Let N be a regular network with valency ϑ and adjacency matrix A, $\bowtie$ a balanced coloring of N and μ a semisimple eigenvalue of A. Let $f : \mathbb{R} \times \mathbb{R}^\vartheta \times \mathbb{R} \to \mathbb{R}$ be a regular function with a bifurcation condition associated to μ and such that* $\dim(\ker(J_f^N) \cap \Delta_{\bowtie}) = 2$. *Assume that condition (H1a) holds for f and $N/\bowtie$ and $k(N/\bowtie, \mu) = 2$.*

(i) If $\bowtie$ is μ-submaximal of type 1. Then generically there is a bifurcation branch of f on N with exact synchrony $\bowtie$ if and only if

$$(d_{222} - 2d_{112})^2 \geq 4d_{122}(d_{111} - 2d_{212}). \tag{3.8}$$

(ii) If $\bowtie$ is μ-submaximal of type 2. Then generically there is a bifurcation branch of f on N with exact synchrony $\bowtie$ if and only if

$$(2d_{212} - d_{111})(2d_{112} - d_{222}) \neq 0. \tag{3.9}$$

Proof. Let N be a regular network with valency ϑ and adjacency matrix A, $\bowtie$ a balanced coloring of N and μ a semisimple eigenvalue of A. Let $f : \mathbb{R} \times \mathbb{R}^\vartheta \times \mathbb{R} \to \mathbb{R}$ be a regular function with a bifurcation condition associated to μ and such that $\dim(\ker(J_f^N) \cap \Delta_\bowtie) = 2$ and $k(N/\bowtie, \mu) = 2$.

Let $A_\bowtie$ to be the adjacency matrix of $N/\bowtie$. Then $m = \dim(\ker(J_f^{N/\bowtie})) = \dim(\ker(J_f^N) \cap \Delta_\bowtie) = 2$ and μ is a semisimple eigenvalue of $A_\bowtie$.

Performing the calculations of section 3.3 and of this section for the network $N/\bowtie$ and the regular function f, we consider the function $\Theta(\theta)$ given by

$$\Theta(\theta) = \cos(\theta)q_2(\cos(\theta), \sin(\theta)) - \sin(\theta)q_1(\cos(\theta), \sin(\theta)),$$

where $Q_k(u_1, u_2) = (q_1(u_1, u_2), q_2(u_1, u_2))$ and $k = k(N/\bowtie, \mu)$. We look for solutions of $\Theta(\theta) = 0$.

(i) Suppose that $\bowtie$ is μ-submaximal of type 1 with a μ-simple component $\bowtie_1$. We denote by $\bowtie_1'$ the balanced coloring of $N/\bowtie$ that corresponds to the balanced coloring $\bowtie_1$ of N. Now, we return to the calculations of section 3.3 for the network $N/\bowtie$ and we choose a basis v_1, v_2 of $\ker(J_f^{N/\bowtie})$ such that $v_1 \in \Delta_{\bowtie_1'}$.

Note that $d_{211} = 0$, for the following reason. From Theorem 3.4.1 and $\dim(\ker(J_f^{N/\bowtie}) \cap \Delta_{\bowtie_1'}) = 1$, there exists a bifurcation branch of f on $N/\bowtie$ with synchrony $\bowtie_1'$. By 3.3.5, $\tilde{h}(\pm 1, 0) = 0$. So $d_{211} = 0$. (This can be also shown using the fact that $\frac{\partial^2 g_2}{\partial y_1 \partial y_1}(0,0) = 0$, since $\Delta_{\bowtie_1'}$ is invariant.)

Using the expansion of q_1 and q_2 presented in (3.7), we have that

$$\Theta(\theta) = \beta \left(u_1 \left(d_{212}u_1u_2 + \frac{d_{222}u_2^2}{2} \right) - u_2 \left(\frac{d_{111}u_1^2}{2} + d_{112}u_1u_2 + \frac{d_{122}u_2^2}{2} \right) \right),$$

where $u_1 = \cos(\theta)$, $u_2 = \sin(\theta)$ and $\beta = (f_{00} + 2\mu f_{01} + \mu f_{11} - \mu f_{1\vartheta} + \mu^2 f_{1\vartheta}) \neq 0$ generically. We have that $\Theta(\theta) = 0$, if $\sin(\theta) = 0$, however those zeros correspond to the known bifurcation branch of f on $N/\bowtie$ with synchrony $\bowtie_1'$. For $\sin(\theta) \neq 0$, we have that

$$\Theta(\theta) = 0 \Leftrightarrow (2d_{212} - d_{111}) \left(\frac{\cos(\theta)}{\sin(\theta)} \right)^2 + (d_{222} - 2d_{112}) \frac{\cos(\theta)}{\sin(\theta)} - d_{122} = 0.$$

Define $x = \cos(\theta)/\sin(\theta)$ and consider the equation

$$(2d_{212} - d_{111})x^2 + (d_{222} - 2d_{112})x - d_{122} = 0 \tag{3.10}$$

that has a real solution if and only if (3.8) holds.

If (3.8) holds, then there exists a solution $\tilde{x}$ of (3.10). Since the image of $]0, \pi[\ni \theta \mapsto \cos(\theta)/\sin(\theta)$ is the entire real line, there exists $\tilde{\theta}$ such that $\tilde{x} = \cos(\tilde{\theta})/\sin(\tilde{\theta})$ and $\Theta(\tilde{\theta}) = 0$. Consider $\tilde{u} = (\cos(\tilde{\theta}), \sin(\tilde{\theta}))$, then $\tilde{h}(\tilde{u}) = 0$. Assuming that (H1a) holds for f and $N/\bowtie$, we know that (3.6) holds for $\tilde{u}$. From Proposition 3.3.6, there exists a bifurcation branch of f on $N/\bowtie$. This bifurcation branch has no synchrony in $N/\bowtie$, since $\bowtie$ is μ-submaximal and $\sin(\tilde{\theta}) \neq 0$. Lifting this bifurcation branch to N, we obtain a bifurcation branch of f on N with exact synchrony $\bowtie$.

Suppose by contradiction that there exists a bifurcation branch of f on N with exact synchrony $\bowtie$ and (3.8) does not hold. We have by Proposition 3.3.5 that there exists $(u_1, u_2) \in S^1$ such that $u_2 \neq 0$ and $\tilde{h}(u_1, u_2) = 0$. So $x = u_1/u_2$ is a real solution of equation (3.10), which is an absurd since we are supposing that (3.8) does not hold. This proves (i).

(ii) Suppose that $\bowtie$ is μ-submaximal of type 2 with μ-simple components $\bowtie_1$ and $\bowtie_2$. We denote by $\bowtie_1', \bowtie_2'$ the balanced colorings of $N/\bowtie$ that corresponds to the balanced colorings $\bowtie_1, \bowtie_2$ of N, respectively. In the calculations of section 3.3 for the network $N/\bowtie$, we choose a basis v_1, v_2 of $\ker(J_f^{N/\bowtie})$ such that $v_1 \in \Delta_{\bowtie_1'}$ and $v_2 \in \Delta_{\bowtie_2'}$.

As before, we note that $d_{211} = 0$ and $d_{122} = 0$. So

$$\Theta(\theta) = \beta\left(u_1u_2\left(d_{212}u_1 + \frac{d_{222}u_2}{2}\right) - u_2u_1\left(\frac{d_{111}u_1}{2} + d_{112}u_2\right)\right),$$

where $u_1 = \cos(\theta)$ and $u_2 = \sin(\theta)$. We have that $\Theta(\theta) = 0$, if $\sin(\theta) = 0$ or $\cos(\theta) = 0$, however those zeros correspond to the known bifurcation branch of f on $N/\bowtie$ with synchrony $\bowtie_1'$ or $\bowtie_2'$. For $\cos(\theta), \sin(\theta) \neq 0$, we have that

$$\Theta(\theta) = 0 \Leftrightarrow (2d_{212} - d_{111})\cos(\theta) = (2d_{112} - d_{222})\sin(\theta).$$

has a solution such that $\cos(\theta), \sin(\theta) \neq 0$ if and only if (3.9) holds.

If (3.9) holds, let $\tilde{\theta}$ be a solution of the equation above, i.e., $\Theta(\tilde{\theta}) = 0$. Assuming that condition (H1a) holds for f and $N/\bowtie$, we know that (3.6) holds for $\tilde{u} = (\cos(\tilde{\theta}), \sin(\tilde{\theta}))$. From Proposition 3.3.6, there exists a bifurcation branch of f on $N/\bowtie$ which has no synchrony in $N/\bowtie$, since $\bowtie$ is μ-submaximal and $\cos(\theta), \sin(\theta) \neq 0$. Lifting this bifurcation branch to N, we obtain a bifurcation branch of f on N with exact synchrony $\bowtie$.

Suppose by contradiction that there exists a bifurcation branch of f on N with exact synchrony $\bowtie$ and (3.9) does not hold. We have by Proposition 3.3.5 that there exists $(u_1, u_2) \in S^1$ such that $u_1, u_2 \neq 0$ and $\tilde{h}(u_1, u_2) = 0$. So $(2d_{212} - d_{111})u_1 = (2d_{112} - d_{222})u_2$ and $u_1, u_2 \neq 0$. This is an absurd since $(2d_{212} - d_{111})(2d_{112} - d_{222}) = 0$. This proves (ii). □

Now, we study some examples where the trivial balanced coloring $\bowtie_=$ is maximal or submaximal. We see, in particular, that there are networks with similar lattice structures but different synchrony-breaking bifurcations.

Example 3.5.3. Consider the network N_1 given by the adjacency matrix

$$A_1 = \begin{bmatrix} 343 & 430 & 86 & 129 \\ 377 & 453 & 77 & 81 \\ 47 & 214 & 166 & 561 \\ 432 & 494 & 62 & 0 \end{bmatrix}.$$

The eigenvalues of A_1 are the network valency 988, -24 and -1 with multiplicity 1, 1 and 2, respectively. They are semisimple. We consider the trivial balanced coloring $\bowtie_=$ and the bifurcations associated to the eigenvalue $\mu = -1$ of A_1. The network N_1 does not have any non-trivial balanced coloring. So the balanced coloring $\bowtie_=$ is maximal and is (-1)-maximal.

Let $f : \mathbb{R} \times \mathbb{R}^{988} \times \mathbb{R} \to \mathbb{R}$ be a generic regular function with a bifurcation condition associated to -1, i.e., $f_0 - f_1 = 0$. We have that $\dim(\ker(J_f^{N_1})) = 2$. Let $v_1 = (1, -1, 1, 0)$ and $v_2 = (8, -7, 0, 2)$ be a basis of $\ker(J_f^{N_1})$. Let (v_1^*, v_2^*) be a basis of $\mathrm{Range}(J_f^{N_1})^\perp$ such that $\langle v_i^*, v_j \rangle$ is 1 if $i = j$ and 0 otherwise, for $i, j = 1, 2$. Then $d_{abc} \neq 0$ for every $a, b, c = 1, 2$ and $k(N_1, \mu) = 2$ is even.

By Theorem 3.5.1, there is a bifurcation branch of f on N_1 without synchrony, if condition (H1) holds for f and N_1. $\Diamond$

Example 3.5.4. Consider the network N_2 given by the adjacency matrix

$$A_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of A_2 are the network valency 2 and 0 with multiplicity 2 and 1, respectively. They are semisimple. We consider the trivial balanced coloring $\bowtie_=$ and bifurcations associated to the eigenvalue 2 of A_2. The network N_2 has only one non-trivial balanced coloring: $\bowtie_1$ given by the classes: $\{\{1, 2\}, \{3\}\}$. The balanced coloring $\bowtie_=$ is 2-submaximal of type 1, with 2-simple component $\bowtie_1$. See Figure 3.4.

Let $f : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be a generic regular function with a bifurcation condition associated to 2, i.e., $f_0 + 2f_1 = 0$. We have that $\dim(\ker(J_f^{N_2})) = 2$. We choose a basis of $\ker(J_f^{N_2})$ such that $v_1 \in \Delta_1$. Let $v_1 = (1, 1, 1) \in \Delta_1$ and $v_2 = (1, -1, 0)$. Let $v_1^* = (1/2, 1/2, 0)$ and $v_2^* = (1/2, -1/2, 0)$ that form a basis of $\mathrm{Range}(J_f^{N_2})^\perp$. Then

$$d_{112} = d_{211} = d_{222} = 0, \qquad d_{111} = d_{122} = d_{212} = 1.$$

Therefore $k(N_2, \mu) = 2$ and (3.8) holds.

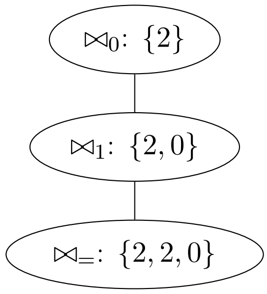

Figure 3.4: Balanced colorings of N_2 and the eigenvalues of the adjacency matrices associated to the corresponding quotient networks.

By explicit calculations, we can see that condition (H1a) holds whenever $f_{0\lambda} + 2f_{1\lambda} \neq 0$ and $f_{00} + 4f_{01} + 2f_{11} + 2f_{19} \neq 0$. Then there is a bifurcation branch of f on N_2 without synchrony, by Theorem 3.5.2 (i). $\Diamond$

Example 3.5.5. Consider the network #29 of [9] (Figure 3.1(a)) which will be denoted by N_{29} and has the adjacency matrix

$$A_{29} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The eigenvalues of A_{29} are the network valency 2, -1 and 0 with multiplicity 1, 1 and 2, respectively, which are semisimple. The network N_{29} has four non-trivial balanced colorings $\bowtie_1 = \{\{1,2\},\{3,4\}\}$, $\bowtie_2 = \{\{1\},\{2,3,4\}\}$, $\bowtie_3 = \{\{1,2\},\{3\},\{4\}\}$ and $\bowtie_4 = \{\{1\},\{2\},\{3,4\}\}$. The balanced coloring $\bowtie_=$ is 0-submaximal of type 2 with 0-simple components $\bowtie_3$ and $\bowtie_4$. See Figure 3.5.

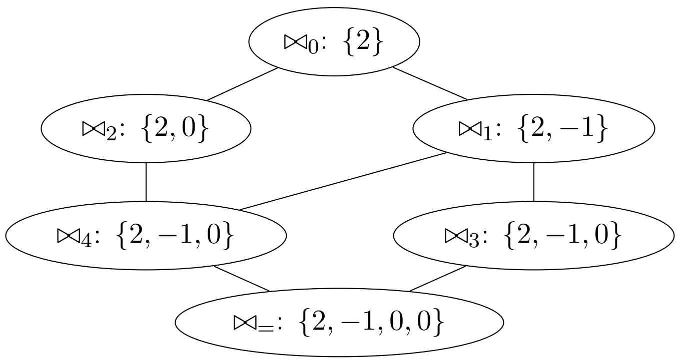

Figure 3.5: Balanced colorings of network N_{29} and the eigenvalues of the adjacency matrices associated to the corresponding quotient networks.

We consider the trivial balanced coloring $\bowtie_=$. Let $f : \mathbb{R} \times \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}$ be a generic regular function with a bifurcation condition associated to 0, i.e.,

$f_0 = 0$. We have that $\dim(\ker(J_f^{N_{29}})) = 2$. We choose a basis of $\ker(J_f^{N_{29}})$ such that $v_1 \in \Delta_{\bowtie_3}$ and $v_2 \in \Delta_{\bowtie_4}$. Let $v_1 = (1, 1, -1, 0) \in \Delta_{\bowtie_3}$ and $v_2 = (1, 0, 0, 0) \in \Delta_{\bowtie_4}$. Let $v_1^* = (0, 0, -1, 1)$ and $v_2^* = (1, -1, 0, 0)$ that form a basis of $\mathrm{Range}(J_f^{N_{29}})^{\perp}$. Then

$$d_{111} = -1, \quad d_{112} = d_{122} = d_{211} = 0, \quad d_{212} = 1, \quad d_{222} = 1.$$

Therefore $k(N_{29}, \mu) = 2$ and (3.9) holds.

By explicit calculations, we can see that condition (H1a) holds, whenever $f_{0\lambda} \neq 0$ and $f_{00} \neq 0$. Then there is a bifurcation branch of f on N_{29} without synchrony, by Theorem 3.5.2 (*ii*). $\diamondsuit$

Remark 3.5.6. Note that the networks N_{51} and N_{29} share the same lattice structure. However, the network N_{51} does not support a bifurcation branch without synchrony, Example 3.3.8, and the network N_{29} supports a bifurcation branch without synchrony, Example 3.5.5. $\diamondsuit$

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4. The Steady-state Lifting Bifurcation Problem Associated with the Valency on Networks

This chapter consists of a joint article submitted for publication.

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Abstract

We consider coupled cell networks whose cells receive the same number of inputs which are said to be homogeneous. The coupled cell systems associated with a network are the dynamical systems that respect the network structure. There are subspaces, determined solely by the network structure, that are flow-invariant under any such coupled cell system – the synchrony subspaces. For a homogeneous network, one of the eigenvalues of the Jacobian matrix of any coupled cell system at an equilibrium in the full-synchrony subspace corresponds to the valency of the network. In this work, we study the codimension-one steady-state bifurcations of coupled cell systems with a bifurcation condition associated with the valency. We start by giving an adaptation of the Perron–Frobenius Theorem for the eigenspace associated with the valency showing that the dimension of that eigenspace equals the number of the network source components. A network source component is a strongly connected component of the network whose cells receive inputs only from cells in the component. Each synchrony subspace determines a smaller network called quotient network. The lifting bifurcation problem addresses the issue of understanding when the bifurcation branches of a network can be lifted from one of its quotient networks. We consider the lifting bifurcation problem when the bifurcation condition is associated with the valency. We give sufficient conditions on the number of source components for the answer to the lifting bifurcation problem to be positive and prove that those conditions are necessary and sufficient for a class of networks.

Keywords: Coupled cell network, Steady-state bifurcation, Lifting bifurcation problem.

2010 Mathematics subject classification: 37G10; 34D06; 05C50

4.1 Introduction

A coupled cell network is a directed graph with labels on the cells and edges, which describe their types. A coupled cell system is a dynamical system given by an admissible vector field for the network, that is, it must respect the graph structure. More precisely, the admissible vector fields for a network determine that the dynamics of each cell is affected by its own state and the state of the cells with an input edge directed to that cell. Moreover, the admissible vector fields must respect the type of the cells and edges. In [21, 11], the authors formalize the concepts of coupled cell network and coupled cell system and enlighten about their intrinsic relation. They prove the existence of synchrony subspaces that are flow invariant for any coupled cell system. The synchrony subspaces are given by the state's equality of some cells of the network. The restriction of a network admissible vector field to a synchrony subspace is an admissible vector field for a smaller network, called a quotient network. A quotient network is obtained by merging cells that have the same state in the corresponding synchrony subspace. The original network is said to be a lift of the smaller network.

We focus on homogeneous networks with asymmetric inputs, where all cells have the same type and each cell has exactly one input of each type. These kind of networks have been studied in [17, 18, 16, 1]. For each type of input, there is an adjacency matrix that represents the inputs of that type. Moreover, the Jacobian matrix of an admissible vector field at a full-synchrony point can be expressed using the adjacency matrices of the network, and it has a constant row-sum called the valency of the network. Thus, the valency of the network is an eigenvalue of the Jacobian matrix. In this paper, we study codimension-one steady-state bifurcations for coupled cell systems of homogeneous networks where the bifurcation condition corresponds to the network valency and address the respective lifting bifurcation problem.

Bifurcation problems on coupled cell systems have been previously studied by different authors, see for example [13, 2, 12, 9, 8, 20]. These include specific network examples, classes of networks that have an additional structure such as (partial) symmetries or a feed-forward structure or even some conditions about the bifurcation condition which need to be verified in a case-by-case scenario. Here, we address a bifurcation problem that it is transverse to every homogeneous network with asymmetric inputs. We first describe the kernel of generic coupled cell systems with a bifurcation condition corresponding to the network valency. This is a technical step where we use a novel recursive argument on the number of cells of the network. After that, the standard methods of bifurcation theory are applied to describe the bifurcation branches. Next, we analyze the lifting bifurcation problem which has been studied in [2, 14, 7, 16, 19]. We observe that this problem is closely related with the study of synchrony-breaking bifurcations, since bifurcation branches which do not break the synchrony associated with a quotient network are lifted from that quotient network. Despite we give a complete description of the bifurcation branches for every generic coupled cell system with a bifurcation condition associated to the network valency, we present examples suggesting that it is not trivial to fully understand the lifting bifurcation problem. Nevertheless, we are able to give a complete answer to the lifting bifurcation problem for a class of networks. Below, we make this discussion more precise.

In order to study the bifurcation problem where the bifurcation condition corresponds to the network valency, we first give an adaptation of the Perron-Frobenius Theorem to generic coupled cell systems. For real squares matrices with non-negative entries and constant row-sum, the Perron-Frobenius Theorem proves that the row-sum is the greater eigenvalue in absolute value and it describes the eigenspace associated with that eigenvalue. As every adjacency matrix of an homogenous network has constant row-sum 1 and entries 0 or 1, this result applies to the homogenous network adjacency matrices. The Jacobian matrix of a coupled cell system at a full synchrony subspace is not in general non-negative, but it has constant row-sum equal, that we call the network valency. Then the valency of the network is an eigenvalue of the Jacobian matrix. In this paper, we describe the eigenspace corresponding to the valency of the network for generic coupled cell system in Propositions 4.5.1, 4.5.5 and 4.5.6. The dimension of this eigenspace is equal to the number of source components in the network. Every network can be partitioned into its strongly connected components. We say that a component is a source if every input targeting a cell in that component starts in a cell also inside that component. After this first step, we use well-known methods of bifurcation theory to describe the codimension-one steady-state bifurcations of generic coupled cell systems where the bifurcation condition corresponds to the network valency, see Proposition 4.5.7. In particular, we show that there exists a synchrony-breaking bifurcation branch if and only if the network has at least two source components.

Given a network and a lift network, the solutions (and the bifurcation branches) of a coupled cell system in the quotient network lift to solutions of the corresponding coupled cell system in the lift network. The lifting bifurcation problem addresses the issue of whether all bifurcation branches occurring in a coupled cell system of the lift network are lifted from the smaller network. In the last part of this paper, we study the lifting bifurcation problem of generic coupled cell systems where the bifurcation condition corresponds to the network valency. We obtain sufficient conditions, given by the number of source components in the lift and quotient network, to answer the lifting bifurcation problem, Proposition 4.6.1. More precisely, if the lift and quotient networks have the same number of source components then all the bifurcation branches on the lift network are lifted from the quotient network. On the other hand, if the quotient network has exactly one source component and the lift network has at least two source components, then there is a synchrony-breaking bifurcation branch on the lift network which is not lifted from the quotient network. This result is expected, since the number of source components equals the dimension of the kernel of the Jacobian matrix at a full synchronous point associated to a coupled cell system where the bifurcation condition corresponds to the network valency. Thereby, it would be expected that if the number of source components increases for the lift network, then some bifurcation branch on the lift network would not be lifted from the quotient network. We present, however, two examples where this does not hold (Examples 4.6.3 and 4.6.4). Despite the number of source components increases in those examples, every bifurcation branch in each lift network is lifted from the respective quotient network. In one of the examples, a condition on the partial symmetries of the quotient network, called transitive, is broken. In the other example, a condition on the connectivity of the lift network, called backward connected, is broken. Networks that are backward connected and transitive have received an extra attention in [15, 3]. Restricting the lifting bifurcation problem to transitive quotient networks and backward connected lift networks, we prove that every bifurcation branch on the lift network is lifted from the quotient network if and only if the quotient and lift network have the same number of source components, Theorem 4.6.5.

The structure of this paper is the following. In Section 4.2, we recall some notions about coupled cell networks such as quotient network, backward connected network and transitive network. In Section 4.3, we review the definition of coupled cell systems. In Section 4.4, we describe coupled cell systems having a bifurcation condition corresponding to the network valency and the respective lifting bifurcation problem. In Section 4.5, we study the codimension-one steady-state bifurcation problem for those coupled cell systems. In Section 4.6, we discuss the lifting bifurcation problem.

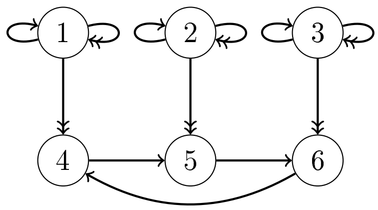

Figure 4.1: A homogeneous network with asymmetric inputs represented by $\sigma_1 = [1\ 2\ 3\ 6\ 4\ 5]$ and $\sigma_2 = [1\ 2\ 3\ 1\ 2\ 3]$. The strongly connected components of this network are $\{1\}$, $\{2\}$, $\{3\}$ and $\{4, 5, 6\}$.

4.2 Coupled cell networks

In this section, we recall definitions and results concerning coupled cell networks, connectivity of networks, balanced colorings, quotient networks and network fibrations. We follow the presentation given in [21, 11, 17, 6].

Definition 4.2.1. A *network* N is defined by a directed graph with a finite set of cells C and finite sets of directed edges divided by types $E_1, \ldots, E_k$ such that each edge $e \in E_i$ starts in a cell s and targets a cell t, where $1 \le i \le k$ and $s, t \in C$. We denote by $|N|$ the number of cells in the network N. In this work we will assume that all networks are *homogeneous with asymmetric inputs* in the sense that each cell c is target by exactly one edge of each type. ◊

Graphically, we use different symbols to distinguish the edge types. As an example, the network in Figure 4.1 has two types of edges.

Let N be a network and $E_1, \ldots, E_k$ the sets of edges in N. By relabeling the cells, we can assume that $C = \{1, \ldots, n\}$, with $n = |N|$.

Given $1 \le a_1, \ldots, a_n \le m$, we denote by $\sigma = [a_1\ \ldots\ a_n]$ the function $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $\sigma(j) = a_j$, $1 \le j \le n$. The identity function on $\{1, \ldots, n\}$ is denoted by σ_0, i.e., $\sigma_0(j) = j$, for $1 \le j \le n$.

As pointed out by Rink and Sanders [17], a homogeneous network with asymmetric inputs can be represented using a collection of functions. For each $1 \le i \le k$, consider the function $\sigma_i = [s_i(1)\ \ldots\ s_i(n)]$ such that there exists an edge $e \in E_i$ from $s_i(c)$ to c for $1 \le c \le n$. In fact a homogeneous network with asymmetric inputs is uniquely determined by the functions $(\sigma_i)_{i=1}^k$ and we say that N is *represented by* $(\sigma_i)_{i=1}^k$. See the network in Figure 4.1.

A network can be also represented by its adjacency matrices. For each $1 \le i \le k$, the $n \times n$-matrix A_i is the *adjacency matrix of type* i, if $(A_i)_{c\,\sigma_i(c)} = 1$ and $(A_i)_{c\,c'} = 0$, when $\sigma_i(c) \ne c'$. A network N is uniquely represented by its adjacency matrices $(A_i)_{i=1}^k$. We denote the identity matrix by A_0.

Given cells c and d of N, we say that c and d are *connected* if there exists a sequence of cells $c_0, c_1, \ldots, c_{l-1}, c_l$ such that $c_0 = c$, $c_l = d$ and there is an

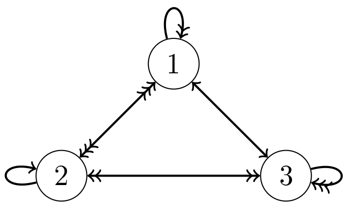

Figure 4.2: A strongly connected network represented by $\sigma_1 = [3\,2\,1]$, $\sigma_2 = [1\,3\,2]$ and $\sigma_3 = [2\,1\,3]$ which is backward connected for every cell.

edge from c_{j-1} to c_j or an edge from c_j to c_{j-1}, for every $1 \leq j \leq l$. In this work, we always consider *connected* networks, i.e., networks where every two distinct cells are connected.

Definition 4.2.2. Let N be a network. A *path* in N from the cell c to the cell d is a sequence of cells $c_0, c_1, \ldots, c_{l-1}, c_l$ such that $c_0 = c$, $c_l = d$ and there is an edge from c_{j-1} to c_j, for every $1 \leq j \leq l$. We say that cells c and d are *strongly connected*, if there are paths from c to d and from d to c. A subset B of cells is a *strongly connected component* of N, if any two distinct cells $c, d \in B$ are strongly connected and B is a maximal subset of strongly connected cells, i.e., for every strongly connected cells $c \in B$ and d we have that $d \in B$. ♢

The set of cells of a network can be partitioned into its strongly connected components, see e.g. [5, Theorem 2.4].

In the network example of Figure 4.1, the strongly connected components are $\{1\}$, $\{2\}$, $\{3\}$ and $\{4, 5, 6\}$.

Definition 4.2.3. Let N be a network. The network N is *strongly connected*, if N has exactly one strongly connected component given by the set of cells. A strongly connected component S is a *source component*, if every edge targeting a cell of S starts in a cell of S. We denote by $s(N)$ the *number of source components* of N. ♢

The network in Figure 4.2 is an example of a strongly connected network and its unique source component is the set of cells. The networks in Figure 4.1 and Figure 4.3 have three source components: $\{1\}$, $\{2\}$ and $\{3\}$.

Definition 4.2.4. A network N is *backward connected for a cell* c, if for every other cell c' there exists a path from c' to c. A network N is *backward connected* if it is backward connected for some cell. ♢

Every strongly connected network is backward connected for every cell. Figure 4.2 shows an example of a backward connected network. The network in Figure 4.1 is backward connected for the cells 4, 5 and 6. On the other hand, Figure 4.3 shows an example of a network which is not backward connected.

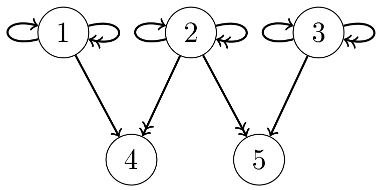

Figure 4.3: A network represented by $\sigma_1 = [1\,2\,3\,1\,3]$ and $\sigma_2 = [1\,2\,3\,2\,2]$ with three source components. This network is not backward connected, as there is no path from cell 4 to cell 5 and vice versa.

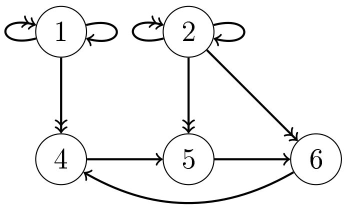

Figure 4.4: The quotient network of the network in Figure 4.1 associated with the balanced coloring $\bowtie$ such that $2 \bowtie 3$. This quotient network is backward connected, since the network in Figure 4.1 is also backward connected, but it is not transitive.

Let N be a network represented by the functions $(\sigma_i)_{i=1}^k$ such that $|N| = n$. A *coloring* of the set of cells of N is an equivalence relation on the set of cells. A coloring $\bowtie$ is *balanced* if $\sigma_i(c) \bowtie \sigma_i(c')$, for every $1 \leq i \leq k$ and $1 \leq c, c' \leq n$ such that $c \bowtie c'$. It follows from [18, Proposition 7.2] that this definition coincides with the definition of balanced coloring given in [11, Definition 4.1]. Given a subset of cells $S \subseteq \{1, \ldots, n\}$, we denote by $[S]_{\bowtie}$ the set of $\bowtie$-classes containing the cells in S, i.e., $[S]_{\bowtie} = \{[c]_{\bowtie} : c \in S\}$.

Definition 4.2.5 ([11, Section 5])**.** Let N be a network represented by the functions $(\sigma_i)_{i=1}^k$ such that $|N| = n$ and $\bowtie$ a balanced coloring in N. The *quotient network* of N associated to $\bowtie$ is the network where the set of cells are the $\bowtie$-classes and the edges are represented by the functions $(\sigma_i^{\bowtie})_{i=1}^k$ such that

$$\sigma_i^{\bowtie}([c]_{\bowtie}) = [\sigma_i(c)]_{\bowtie}, \qquad 1 \leq i \leq k, 1 \leq c \leq n.$$

We denote the quotient network by $N/\bowtie$. We also say that a network L is a *lift* of N, if N is a quotient of L for some balanced coloring in L. $\Diamond$

The networks in Figures 4.4 and 4.5 are examples of quotient networks of the networks described in Figures 4.1 and 4.3, respectively.

Remark 4.2.6. Let L be a lift of the network N. Trivially, we have the following relation between the number of source components of N and L: $s(N) \leq s(L)$. In particular, if L is backward connected, then N is also backward connected. $\Diamond$

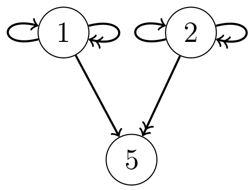

Figure 4.5: The quotient network of the network in Figure 4.3 associated with the balanced coloring $\bowtie$ such that $1 \bowtie 3$ and $4 \bowtie 5$. This network is transitive and backward connected for the cell 5.

Next, we define network fibrations following [6, Definition 2.1.4] and [15, Proposition 5.3]. A network fibration is a function between two networks which respects the edges and their types.

Definition 4.2.7. Let N and N' be the networks with the set of cells C, C' and represented by the functions $(\sigma_i)_{i=1}^k$ and $(\sigma_i')_{i=1}^k$. A function $\varphi : C \to C'$ is a *network fibration from N to N'*, if

$$\varphi \circ \sigma_i = \sigma_i' \circ \varphi, \qquad i = 1, \ldots, k.$$

We denote a network fibration from N to N' by $\varphi : N \to N'$. We say that N and N' *are equal* and write that $N = N'$ if there exists a bijective network fibration $\varphi : N \to N'$. ♢

If N is a network and $\bowtie$ a balanced coloring of the set of cells of N, the *network fibration induced by* $\bowtie$ is the function $\varphi_{\bowtie} : N \to N/\bowtie$ given for every cell c of N by

$$\varphi_{\bowtie}(c) = [c]_{\bowtie}.$$

Example 4.2.8. Let N be the network in Figure 4.5. There are three network fibrations from N to itself: $\varphi_1 = [1\,1\,1]$; $\varphi_2 = [2\,2\,2]$ and $\varphi_3 = [1\,2\,5]$.

Let L the network in Figure 4.3 and $\bowtie$ the balanced coloring in L such that $1 \bowtie 3$ and $4 \bowtie 5$. The network L is a lift of $N = L/\bowtie$ and $\varphi_{\bowtie} = [12155]$ is a network fibration from L to N, where 1, 2 and 5 are representatives of the classes $[1]_{\bowtie}$, $[2]_{\bowtie}$ and $[5]_{\bowtie}$, respectively. ♢

Remark 4.2.9. Let $\varphi : N \to N'$ be a network fibration between two networks N and N'. If S is a source component of N, then $\varphi(S)$ is a source component of N'. This follows from the fact that each path in the network N is projected by the network fibration $\varphi : N \to N'$ into a path in the network N'. ♢

The evaluation of a network fibration at each cell of a path can be determined by the evaluation of the network fibration at the end cell of that path. Since for a backward connected network there is a cell such that there is a path from every other cell to it, the network fibrations from such network are uniquely determined by their evaluation at that cell:

Remark 4.2.10. Let $\varphi : N \to N'$ be a network fibration between the networks N and N'. If N is backward connected for c, then the network fibration is uniquely determined by its evaluation at c, $\varphi(c)$. ◊

The self-fibrations of a network are an indicator of the partial symmetries of that network. Next, we define a class of networks with at least one self-fibration for each cell.

Definition 4.2.11. Let N be a network and c a cell of N. We say that N is *transitive for* c if for every cell d in N, there is a network fibration $\phi_d : N \to N$ such that $\phi_d(c) = d$. We say that N is *transitive*, if it is transitive for some cell. ◊

The network in Figure 4.5 is an example of a transitive network, since it is transitive for the cell 5, see Example 4.2.8. The network in Figure 4.4 is not transitive, because there are only three self-fibrations. In [15], the authors have defined fundamental networks. A network is fundamental if and only if it is backward connected and transitive, see [3, Theorem 5.16]. Figure 4.5 is an example of a fundamental network.

4.3 Coupled cell systems

In this section, we recall concepts and results about coupled cell systems, synchrony subspaces and conjugacies induced by network fibrations, following [21, 11, 6]. We restrict the phase space of each cell to be the one-dimensional real space, however the definitions and results are valid for any differential manifold, see [21, 11, 6].

Let N be a network represented by the functions $(\sigma_i)_{i=1}^k$ and $|N| = n$. For each cell c of the network, we associate a coordinate $x_c \in \mathbb{R}$. We say that $F : \mathbb{R}^n \to \mathbb{R}^n$ is an *admissible vector field* for N, if there is $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ such that

$$(F(x))_c = f(x_c, x_{\sigma_1(c)}, \dots, x_{\sigma_k(c)}),$$

for every cell c of N. The admissible vector fields for N are uniquely defined by such function $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. We denote by f^N the admissible vector field for N defined by f.

Let $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ be a smooth function. A *coupled cell system* associated to a network N is a system of ordinary differential equations

$$\dot{x} = f^N(x), \quad x \in \mathbb{R}^n.$$

Let $(A_i)_{i=1}^k$ be the adjacency matrices of N. The *Jacobian matrix of* f^N at the origin is

$$J_f^N := (Df^N)_0 = \sum_{i=0}^{k} f_i A_i,$$

where

$$f_i := \frac{\partial f}{\partial x_i}(0, 0 \ldots, 0),$$

for $0 \leq i \leq k$ and A_0 is the identity $n \times n$ matrix. Since $(1, \ldots, 1)$ is an eigenvalue of $A_0, \ldots, A_k$, then

$$J_f^N \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \left(\sum_{i=0}^{k} f_i \right) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

and $\sum_{i=0}^{k} f_i$ is always an eigenvalue of J_f^N that we call the *network valency.*

A *polydiagonal subspace* is a subspace of $\mathbb{R}^n$ given by the equalities of some cell coordinates. Given a coloring $\bowtie$ on the set of cells of N, the *polydiagonal subspace associated to* $\bowtie$ is

$$\Delta_{\bowtie} := \{x : c \bowtie d \Rightarrow x_c = x_d\} \subseteq \mathbb{R}^n.$$

We say that a polydiagonal subspace $\Delta \subseteq \mathbb{R}^n$ is a *synchrony subspace* of a network N if the polydiagonal subspace is invariant by any admissible vector field of N, i.e., $f^N(\Delta) \subseteq \Delta$, for every $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. There is a one-to-one correspondence between balanced colorings $\bowtie$ and synchrony subspaces $\Delta_{\bowtie}$, see [11, Theorem 4.3]. More specifically, the polydiagonal $\Delta_{\bowtie}$ is a synchrony subspace of N if and only if $\bowtie$ is a balanced coloring. For homogeneous networks, the coloring with only one color is always balanced and the corresponding synchrony subspace is called the *full-synchrony subspace*:

$$\Delta_0 := \{(x, \ldots, x) \in \mathbb{R}^n : x \in \mathbb{R}\}.$$

Since a synchrony subspace $\Delta_{\bowtie}$ is invariant by every admissible vector field f^N, the coupled cell systems of N can be restricted to $\Delta_{\bowtie}$. The restricted systems are coupled cell systems of $N/\bowtie$ given by admissibles $f^{N/\bowtie}$, see [11, Theorem 5.2].

Let N' be a network and $n' = |N'|$. Following [6], every network fibration $\varphi : N \to N'$ induces a map between the phase spaces of those networks, $P\varphi : \mathbb{R}^{n'} \to \mathbb{R}^n$ such that

$$(P\varphi(x))_c = x_{\varphi(c)}, \qquad 1 \leq c \leq n.$$

Moreover, the coupled cell systems defined by f^N and $f^{N'}$ are conjugated

$$P\varphi \circ f^{N'} = f^N \circ P\varphi.$$

Any solution $y(t) \in \mathbb{R}^{n'}$ of $\dot{y} = f^{N'}(y)$, induces a solution $x(t) = P\varphi(y(t))$ of $\dot{x} = f^N(x)$. In particular, for any balanced coloring $\bowtie$ in N, the solutions of every coupled cell system on $N/\bowtie$ *are lifted* by $P\varphi_{\bowtie}$ to solutions of the corresponding coupled cell system on N and those solutions belong to the synchrony subspace $\Delta_{\bowtie}$.

4.4 Steady-state bifurcations

In this section, we review some concepts related to steady-state bifurcations on coupled cell systems and the lifting bifurcation problem is formulated.

Let $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ be a one-parameter family of smooth functions and consider the family of coupled cell systems, depending on the parameter λ,

$$\dot{x} = f^N(x, \lambda) \qquad x \in \mathbb{R}^n, \lambda \in \mathbb{R}. \tag{4.1}$$

Assume that the origin is an *equilibrium point* of (4.1) for every $\lambda \in \mathbb{R}$, i.e. $f^N(0, 0, \ldots, 0, \lambda) = 0$ for every $\lambda \in \mathbb{R}$. If the Jacobian matrix of f^N at $(x, \lambda) = (0, 0)$, $J_f^N := (Df^N)_{(0,0)}$, is invertible, then the origin is the unique equilibrium point of (4.1) in a sufficient small neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}$. We say that a *steady-state bifurcation* occurs if there exists an equilibrium point of (4.1) different from the origin in any small neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}$. Hence a necessary condition for a steady-state bifurcation to occur is that J_f^N is non-invertible.

Recall that the network valency $\sum_{i=0}^k f_i$ is an eigenvalue of J_f^N. In this paper, we study steady-state bifurcation where the *bifurcation condition is given by the network valency.* Let $\mathcal{V}(N)$ be the set of smooth functions $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ defined by:

$$\mathcal{V}(N) := \left\{ f : \sum_{i=0}^{k} f_i = 0, \quad f(0, 0, \ldots, 0, \lambda) = 0, \quad \lambda \in \mathbb{R} \right\}.$$

Since our study is local, we recall the definition of germ. Let $U_1, U_2 \subset \mathbb{R}$ be open neighborhoods of 0. We say that two smooth functions $b_1 : U_1 \to \mathbb{R}^n$ and $b_2 : U_2 \to \mathbb{R}^n$ are *germ equivalents* if $b_1(\lambda) = b_2(\lambda)$, for every $\lambda \in U_1 \cap U_2$. Given a smooth function b, we use the term *germ* b to refer to a representative element of the equivalence class of b with respect to germ equivalence.

Let $f \in \mathcal{V}(N)$. We say that a germ $b : U \to \mathbb{R}^n$ is an *equilibrium branch of f on N*, if

$$f^N(b(\lambda), \lambda) = 0,$$

for every $\lambda \in U$. As $f(0, 0, \ldots, 0, \lambda) = 0$, we have that $x(\lambda) = (0, \ldots, 0)$ is an equilibrium branch of f on N, called *the trivial branch of f on N*. The equilibrium branches of f on N different from the trivial branch are called the *bifurcation branches of f on N*.

As usual in bifurcation theory, the study of the steady-state bifurcation problem is posed for a large class of functions called generic functions. The generic functions are defined using non-degenerated conditions. A *non-degenerated condition* is given by a polynomial p on some partial derivatives of a function evaluated at the bifurcation point. Given a function f, we denote by $p(f)$ the evaluation of the polynomial p at that function and we say that a function f satisfies the non-degenerated condition given by p, if $p(f) \neq 0$.

Given $f \in \mathcal{V}(N)$, the value of its first partial derivative with respect to x_0 at the origin, f_0, is given by its first partial derivatives with respect to x_i at the origin, f_i, for $i = 1, \ldots, k$, Also, partial derivatives of any order $l > 0$ with respect to λ at the origin, $\partial^l f/\partial\lambda^l$, are zero. Hence, we do not use non-degenerated conditions which depend on f_0 and $\partial^l f/\partial\lambda^l$ for any $l > 0$. We say that an assertion holds for *generic functions* in $\mathcal{V}(N)$, if there exists a finite number of non-degenerated conditions such that this assertion holds for any function in $\mathcal{V}(N)$ satisfying those non-degenerated conditions.

Let N be a network and L a lift of N. If f^N is a coupled cell system on N with a bifurcation condition corresponding to the network valency, then f^L is a coupled cell system on L with a bifurcation condition corresponding to the network valency. Thus

$$\mathcal{V}(N) = \mathcal{V}(L).$$

In the end of the previous section, it was stated how to lift solutions of a coupled cell system associated to N to the corresponding coupled cell system associated to L using network fibrations. In the same way, we can lift bifurcation branches of a coupled cell system to another using network fibrations.

Definition 4.4.1. Let N be a network, L a lift of N and $f \in \mathcal{V}(N)$. We say that *a bifurcation branch b of f on L is lifted from N*, if there exists a network fibration $\varphi : L \to N$ and a bifurcation branch b' of f on N such that

$$b = P\varphi(b'). \qquad \diamondsuit$$

Given a network N and a lift network L of N, the *lifting bifurcation problem* asks when every bifurcation branch of L is lifted from N.

4.5 Steady-state bifurcations associated to the valency

In this section, we study the bifurcations branches of (4.1) where $f \in \mathcal{V}(N)$ and N is a homogeneous network with asymmetric inputs. We start by describing the kernel of the Jacobian matrix J_f^N, when the network N is strongly connected. By the Perron-Frobenius Theorem ([5, Theorem 0.3]), the kernel of J_f^N is equal to the full-synchrony subspace, when the network N is strongly connected, $f \in \mathcal{V}(N)$ and $f_i > 0$ for every $1 \leq i \leq k$. We show next that this holds for generic functions $f \in \mathcal{V}(N)$.

Proposition 4.5.1. *Let N be a strongly connected network. For generic $f \in \mathcal{V}(N)$, the kernel of J_f^N is the full-synchrony subspace*

$$\ker(J_f^N) = \Delta_0.$$

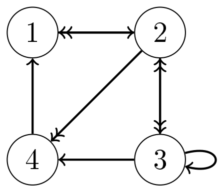

Figure 4.6: A strongly connected network.

We will prove this result by recursion on the number of cells. In order to do the recursive step, we create a new network with one cell less. Before the proof, we present a concrete example to illustrate how the recursive step is done.

Example 4.5.2. Let N be the network in Figure 4.6 and $f : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$. The network N is represented by (σ_1, σ_2), where $\sigma_1 = [4\ 1\ 3\ 3]$ and $\sigma_2 = [2\ 3\ 2\ 2]$. Recall that $\sigma_0 = [1\ 2\ 3\ 4]$ corresponds to the hidden self-dependence.

The Jacobian matrix of f^N at the origin is

$$J_f^N = \begin{bmatrix} f_0 & f_2 & 0 & f_1 \\ f_1 & f_0 & f_2 & 0 \\ 0 & f_2 & f_0 + f_1 & 0 \\ 0 & f_2 & f_1 & f_0 \end{bmatrix},$$

where each f_i is the partial derivative of f with respect to x_i at the origin. The eigenvalues of J_f^N are $f_0 + f_1 + f_2$, f_0 (twice) and $f_0 - f_2$. Assume f has a bifurcation condition associated with the valency, that is $f_0 + f_1 + f_2 = 0$ and f satisfies the non-degenerated conditions $f_1 + f_2 \neq 0$ and $f_1 + 2f_2 \neq 0$. Then the kernel of J_f^N is the eigenspace associated with $f_0 + f_1 + f_2$, that is,

$$\ker(J_f^N) = \Delta_0.$$

For concrete coupled cell systems, we can explicitly calculate the required non-degenerated conditions in Proposition 4.5.1 by computing the eigenvalues of J_f^N. Another approach that we present, goes through considering a new network with one less cell such that we can derive the kernel of the Jacobian matrix on the origianl network using the kernel of the Jacobian matrix on this new network.

Suppose that $f \in \mathcal{V}(N)$, i.e., $f_0 + f_1 + f_2 = 0$. Then the kernel of J_f^N is characterized by the following system:

$$\begin{bmatrix} f_0 & f_2 & 0 & f_1 \\ f_1 & f_0 & f_2 & 0 \\ 0 & f_2 & f_0 + f_1 & 0 \\ 0 & f_2 & f_1 & f_0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0 \Leftrightarrow \begin{cases} f_0 v_1 + f_2 v_2 + f_1 v_4 = 0 \\ f_1 v_1 + f_0 v_2 + f_2 v_3 = 0 \\ f_2 v_2 + (f_0 + f_1) v_3 = 0 \\ f_2 v_2 + f_1 v_3 + f_0 v_4 = 0 \end{cases}.$$

Assume the non-degenerated condition $f_0 = -f_1 - f_2 \neq 0$. From the last equality in the previous system, we have that

$$v_4 = -\frac{f_2 v_2 + f_1 v_3}{f_0}.$$

Replacing v_4 in the other equalities and multiplying by f_0, we obtain the system

$$\begin{cases} f_0^2 v_1 + (f_0 f_2 - f_1 f_2) v_2 - f_1^2 v_3 = 0 \\ f_0 f_1 v_1 + f_0^2 v_2 + f_0 f_2 v_3 = 0 \\ f_0 f_2 v_2 + (f_0^2 + f_0 f_1) v_3 = 0 \end{cases} . \quad (4.2)$$

As the variable v_4 does not appear in (4.2), we remove this cell from the network N. In order to apply a recursive argument on the number of cells, we find a network without cell 4 such that (4.2) defines the kernel of the Jacobian matrix for a coupled cell system of this new network with a bifurcation condition associated to the valency. We use the rules described on Table 4.1 to remove cell 4 and define the new network.

Fixing the cell $n = 4$ of N, we define the network M with 3 cells, $\{1, 2, 3\}$ and 8 edge's types, $(\gamma_{(0,1)}, \gamma_{(0,2)}, \gamma_{(1,0)}, \gamma_{(1,1)}, \gamma_{(2,1)}, \gamma_{(2,0)}, \gamma_{(2,1)}, \gamma_{(2,2)})$. The edges of M are given by the rules presented in Table 4.1. See Figure 4.7. Following the first row of the table, the cell 1 receives in N an input of type 1 from the cell 4 and the cell 4 receives in N a self-input of type 0, then the cell 1 receives in M a self-input of type $(1, 0)$. Following the second row, the cells 1 and 4 receive in N an input of type 2 from the cell 2 , then the cell 1 receives in M a self-input of type $(2, 2)$. Following the third row, the cell 1 receives in N an input of type 1 from the cell 4 and the cell 4 receives in N an input of type 1 from the cell 3, then the cell 1 receives in M an input of type $(1, 1)$ from the cell 3. Following the fourth row, the cell 1 receives in N an input of type 2 from the cell 2 and the cell 4 receives in N a self-input of type 0, then the cell 1 receives in M an input of type $(2, 0)$ from the cell 2. Doing the same for the other cells and inputs, we see that $\gamma_{(0,1)} = \gamma_{(0,2)} = \gamma_{(2,1)} = \gamma_{(2,2)} = [1\ 2\ 3]$, $\gamma_{(1,0)} = [1\ 1\ 3]$, $\gamma_{(1,1)} = [3\ 2\ 3]$, $\gamma_{(1,2)} = [2\ 2\ 3]$ and $\gamma_{(2,0)} = [2\ 3\ 2]$. Note that the network M is also strongly connected.

Looking to the system of equations (4.2), we see now that the type of inputs (i, j) adapt to that system. We find a coupled cell system g^M such that the system (4.2) corresponds to the kernel of J_g^M. We define the function $g : \mathbb{R} \times \mathbb{R}^8 \times \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x_{00}, x_{01}, x_{02}, x_{10}, x_{11}, x_{12}, x_{20}, x_{21}, x_{22}, \lambda) =$$

$$f(f(x_{00}, -x_{01}, -x_{02}, \lambda), f(x_{10}, -x_{11}, -x_{12}, \lambda), f(x_{20}, -x_{21}, -x_{22}, \lambda), \lambda),$$

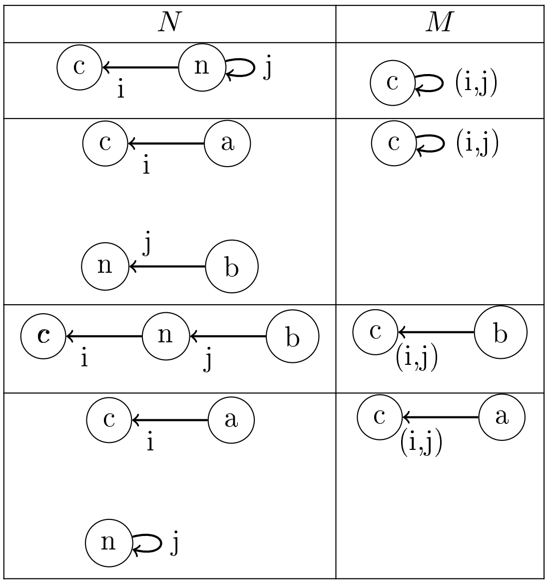

Table 4.1: Given an n-cell network N with set of cells $\{1, \ldots, n\}$ and k types of edges $\{1, \ldots, k\}$, we define the (n-1)-cell network M with set of cells $\{1, \ldots, n-1\}$ and $k^2 + 2k$ types of edges $\{(0,1), \ldots, (0,k), (1,0), (1,1), \ldots, (1,k), \ldots, (k,0), (k,1), \ldots, (k,k)\}$ using the rules given by the table. Each edge in the table is annotated with its type and the cells $a, b, c \in \{1, \ldots, n-1\}$. Given a cell $c \neq n$ in N, the left hand side of the table displays the edge of type i that targets c and the edge of type j that targets n where $0 \leq i, j \leq k$. Depending on the configuration of the left hand side of the table, we give the corresponding edge in M of type (i,j) that targets c.

where $f \in \mathcal{V}(N)$. The partial derivative of g with respect to x_{ij} at the origin is

$$\begin{cases} g_{i0} = f_i f_0, & i = 0, 1, 2 \\ g_{ij} = -f_i f_j, & i = 0, 1, 2, j = 1, 2 \end{cases}.$$

Note that g^M has a bifurcation condition associated with the valency of M:

$$\sum_{i,j=0}^{2} g_{ij} = f_0 \sum_{i=0}^{2} f_i - \sum_{j=1}^{2} f_j \sum_{i=0}^{2} f_i = 0.$$

The Jacobian matrix of J_g^M at the origin is

$$J_g^M = \begin{bmatrix} a + g_{10} & g_{12} + g_{20} & g_{11} \\ g_{10} & a + g_{11} + g_{12} & g_{20} \\ 0 & g_{20} & a + g_{10} + g_{11} + g_{12} \end{bmatrix},$$

where $a = g_{00} + g_{01} + g_{02} + g_{21} + g_{22}$. Recalling that $f \in \mathcal{V}(N)$ and so $f_0 + f_1 + f_2 = 0$,

$$J_g^M = \begin{bmatrix} f_0^2 & f_2 f_0 - f_1 f_2 & -f_1 f_1 \\ f_1 f_0 & f_0^2 & f_2 f_0 \\ 0 & f_2 f_0 & f_0^2 + f_1 f_0 \end{bmatrix}.$$

Thus $(v_1, v_2, v_3) \in \ker(J_g^M)$ if and only if (v_1, v_2, v_3) satisfies the system (4.2). We could further reduce the network M. After two reductions, we would obtain a network with only one cell where the kernel of the Jacobian matrix of a coupled cell system with a bifurcation condition associated to the valency is the full synchrony subspace. Instead, we assume that $\ker(J_g^M) = \{(x, x, x) : x \in \mathbb{R}\}$, for generic functions $g \in \mathcal{V}(M)$. Therefore,

$$(v_1, v_2, v_3, v_4) \in \ker(J_f^N) \Leftrightarrow \begin{cases} (v_1, v_2, v_3) \in \ker(J_g^M) \\ v_4 = -\dfrac{f_2 v_2 + f_1 v_3}{f_0} \end{cases} \Leftrightarrow v_1 = v_2 = v_3 = v_4.$$

Thus $\ker(J_f^N) = \Delta_0$, for generic functions $f \in \mathcal{V}(N)$. ♢

Proof of Proposition 4.5.1. Let N be a strongly connected network with n cells and represented by $(\sigma_i)_{i=1}^k$ and $f \in \mathcal{V}(N)$ generic. Recall that σ_0 is the identity function on $\{1, \dots, n\}$ and it corresponds to the hidden self-dependence. We recursively prove that

$$\ker(J_f^N) = \{(x, \dots, x) \in \mathbb{R}^n : x \in \mathbb{R}\}.$$

Suppose that the network N has one cell, $n = 1$. Since $\sum_{i=0}^{k} f_i = 0$, we have that $J_f^N = [0]$ and $\ker(J_f^N) = \mathbb{R} = \Delta_0$.

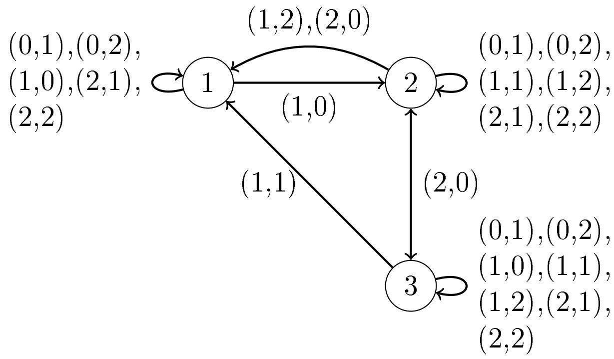

Figure 4.7: A three-cell homogeneous network with asymmetric inputs obtained from the network in Figure 4.6 by removing cell 4 and applying the rules in Table 4.1. The edges are annotated with their edge types and edges with more than one label represent multiple edges. This network is strongly connected.

Suppose that the network N has $m+1$ cells $\{1,\ldots,m,n\}$ with $n = m+1$. Since $f \in \mathcal{V}(N)$, we have that $\ker(J_f^N)$ is nontrivial. Take $v = (v_1,\ldots,v_m,v_n) \in \mathbb{R}^n$ such that $J_f^N v = 0$. Denote by $J_{c\,d}$ the (c,d) entry of J_f^N, i.e.,

$$J_{c\,d} := \sum_{\sigma_i(c)=d} f_i.$$

Thus, using this notation, we have that

$$J_f^N v = 0 \Leftrightarrow \begin{cases} J_{1\,1}v_1 + J_{1\,2}v_2 + \cdots + J_{1\,n}v_n = 0 \\ \quad\vdots \\ J_{n\,1}v_1 + J_{n\,2}v_2 + \cdots + J_{n\,n}v_n = 0 \end{cases}. \tag{4.3}$$

Since N is strongly connected, the cell n receives an edge from some other cell. Thus $J_{n\,n} \neq \sum_{i=0}^k f_i$ and so, we can generically assume on f that

$$J_{n\,n} = \sum_{\sigma_i(n)=n}^{k} f_i = -\sum_{\sigma_i(n)\neq n}^{k} f_i \neq 0.$$

Moreover,

$$v_n = -\frac{J_{n\,1}v_1 + J_{n\,2}v_2 + \cdots + J_{n\,m}v_m}{J_{n\,n}}.$$

Replacing v_n in the first m equations of the system (4.3), we obtain

$$\sum_{d=1}^{m}(J_{n\,n}J_{c\,d} - J_{c\,n}J_{n\,d})v_d = 0, \qquad 1 \leq c \leq m.$$

Let J' be the $m \times m$-matrix with entries

$$J'_{c\,d} = J_{n\,n} J_{c\,d} - J_{c\,n} J_{n\,d},$$

where $1 \leq c, d \leq m$. Next, we define a network M with m cells and a function $g \in \mathcal{V}(M)$ such that

$$J_g^M = J'.$$

In order to remove cell n from network N and define the network M, we use the rules presented in Table 4.1. The type of edges in M are (i,j) where $0 \leq i, j \leq k$ and $(i,j) \neq (0,0)$, and the edges of type (i,j) are represented by the function $\gamma_{i\,j}$. Following Table 4.1, each function $\gamma_{i\,j}$ is given by

$$\gamma_{i\,j}(c) = \begin{cases} c, & \sigma_i(c) = \sigma_j(n) = n \\ c, & \sigma_i(c) \neq n, \sigma_j(n) \neq n \\ \sigma_j(n), & \sigma_i(c) = n, \sigma_j(n) \neq n \\ \sigma_i(c), & \sigma_i(c) \neq n, \sigma_j(n) = n \end{cases},$$

where $1 \leq c \leq m$ and each case corresponds to the corresponding row in the table.

Each path in N induces a path in M by removing any transition by cell n. Therefore M is strongly connected, because N is strongly connected.

In order to use a recursive argument, we define now a function g such that $g \in \mathcal{V}(M)$ and $J' = J_g^M$. Let $g : \mathbb{R} \times \mathbb{R}^{(k+1)^2-1} \times \mathbb{R} \to \mathbb{R}$ be the function

$$g(x_{00}, \ldots, x_{0k}, x_{10}, \ldots, x_{1k}, \ldots, x_{k0}, \ldots, x_{kk}, \lambda) = \\ f(f(y_{00}, \ldots, y_{0k}, \lambda), f(y_{10}, \ldots, y_{1k}, \lambda), \ldots, f(y_{k0}, \ldots, y_{kk}, \lambda), \lambda),$$

where $f \in \mathcal{V}(N)$, $y_{ij} = \beta_j x_{ij}$, $\beta_j = 1$, if $\sigma_j(n) = n$, and $\beta_j = -1$, if $\sigma_j(n) \neq n$, for $0 \leq i, j \leq k$. For $1 \leq i, j \leq k$, we have that

$$g_{ij} = \frac{\partial g}{\partial x_{ij}}(0, 0 \ldots, 0, 0) = \beta_j f_i f_j.$$

Now, we prove that g satisfies the required conditions: $J' = J_g^M$ and $g \in \mathcal{V}(M)$. Let $1 \leq c, d \leq m$. If $c \neq d$, then

$$(J_g^M)_{c\,d} = \sum_{\gamma_{i\,j}(c)=d} g_{ij} = \sum_{\sigma_i(c)=d} \sum_{\sigma_j(n)=n} f_i f_j - \sum_{\sigma_i(c)=n} \sum_{\sigma_j(n)=d} f_i f_j = J'_{c\,d}.$$

Recall that $\sum_{\sigma_i(c)<n} f_i + \sum_{\sigma_i(c)=n} f_i = 0$ as $f \in \mathcal{V}(N)$. Then

$$\left(J_g^M\right)_{c\,c} = \sum_{\substack{\sigma_i(c)=c \\ \sigma_j(n)=n}} f_i f_j - \sum_{\substack{\sigma_i(c)=n \\ \sigma_j(n)=c}} f_i f_j + \sum_{\substack{\sigma_i(c)=n \\ \sigma_j(n)=n}} f_i f_j - \sum_{\substack{\sigma_i(c)<n \\ \sigma_j(n)<n}} f_i f_j =$$

$$= J'_{c\,c} + \sum_{\sigma_i(c)=n} f_i \sum_{\sigma_j(n)=n} f_j - \sum_{\sigma_i(c)=n} f_i \sum_{\sigma_j(n)=n} f_j = J'_{c\,c}.$$

Hence

$$J' = J_g^M.$$

Note that $g \in \mathcal{V}(M)$, since

$$\sum_{i=0}^{k}\sum_{j=0}^{k} g_{ij} = \sum_{j=0}^{k} \beta_j f_j \sum_{i=0}^{k} f_i = 0.$$

Before we apply the recursive argument, we emphasize that, when $m > 1$, the generic condition on g can be regarded as a generic condition on f:

$$\sum_{\gamma_{ij}(m)=m} g_{ij} = \sum_{\sigma_i(m)=m} f_i \sum_{\sigma_j(n)=n} f_j - \sum_{\sigma_i(m)=n} f_i \sum_{\sigma_j(n)=m} f_j \neq 0,$$

We can repeat the previous reduction to the network M and the function g. After a finite number of steps, we obtain a network with only one cell where the kernel of the Jacobian matrix is the full synchrony subspace. So we assume that $\ker(J_g^M)$ is the full-synchrony subspace and prove that $\ker(J_f^N) = \Delta_0$. We have that $v \in \ker(J_f^N)$ if and only if $(v_1, \ldots, v_m) \in \ker(J_g^M)$ and

$$v_n = -\frac{J_{n\,1}v_1 + J_{n\,2}v_2 + \cdots + J_{n\,m}v_m}{J_{n\,n}} = -\frac{\sum_{c=1}^{m} J_{n\,c}}{J_{n\,n}} v_1 = v_1,$$

because $v_1 = \cdots = v_m$. Therefore $v \in \ker(J_f^N)$ if and only if $v \in \Delta_0$. □

In the following example, we present a strongly connected network N and a degenerated function $f \in \mathcal{V}(N)$ for which the kernel of J_f^N is not the full-synchrony subspace.

Example 4.5.3. Let N be the strongly connected network represented in Figure 4.2 and $f \in \mathcal{V}(N)$ such that $f_1 = f_2 = 1$ and $f_3 = -1/2$. Then $f_0 = -3/2$,

$$J_f^N = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and

$$\Delta_0 \subsetneq \ker(J_f^N) = \{(2x, 2y, x+y) : x, y \in \mathbb{R}\}.$$ ◊

Next, we describe the codimension-one steady-state bifurcation of coupled cell systems associated to strongly connected networks where the bifurcation condition corresponds to the network valency. As shown below, this bifurcation does not break the full-synchrony. This result follows from the previous result and well-know methods in bifurcation theory.

Proposition 4.5.4. *Let N be a strongly connected network and $f \in \mathcal{V}(N)$ generic. Then, there exist a neighborhood $U \subset \mathbb{R}$ of 0 and a germ $b_f : U \to \mathbb{R}$ such that if $b : U \to \mathbb{R}^{|N|}$ is a bifurcation branch of f on N then*

$$b(\lambda) = (b_f(\lambda), \ldots, b_f(\lambda)) \in \Delta_0, \qquad \lambda \in U.$$

Proof. Let N be a strongly connected network and $f \in \mathcal{V}(N)$ generic. By Proposition 4.5.1, we know that $\ker(J_f^N) = \Delta_0$. Applying the Lyapunov-Schmidt Reduction ([10, Chapter VII]), there exists $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{|N|}$ such that

$$f^N(b, \lambda) = 0 \Leftrightarrow b = (x, \dots, x) + W(x, \lambda) \wedge f(x, x, \dots, x, \lambda) = 0.$$

If $f(x, x, \dots, x, \lambda) = 0$, then $f^N((x, \dots, x), \lambda) = 0$. By uniqueness of W, $W \equiv 0$. So $b \in \Delta_0$.

Assuming non-degenerated conditions on the function f, the equation $f(x, x, \dots, x, \lambda) = 0$ has a transcritical bifurcation, see e.g., [4]. There exist a neighborhood $U \subseteq \mathbb{R}$ of 0 and a non-zero germ $b_f : U \to \mathbb{R}$ such that

$$f(x, x, \dots, x, \lambda) = 0 \Leftrightarrow x = 0 \vee x = b_f(\lambda), \qquad \lambda \in U.$$

Moreover

$$f^N(b, \lambda) = 0 \Leftrightarrow b = (0, \dots, 0) \vee b = (b_f(\lambda), \dots, b_f(\lambda)), \qquad \lambda \in U.$$

Therefore, if $b : U \to \mathbb{R}^{|N|}$ is a bifurcation branch of f on N then $b(\lambda) = (b_f(\lambda), \dots, b_f(\lambda))$, $\lambda \in U$. □

Now, we address the same bifurcation problem assuming that the network is not necessarily strongly connected. Let N be a network and $f \in \mathcal{V}(N)$ generic. We start by describing the kernel of J_f^N. Reordering the cells in the network by its strongly connected components, we have that the eigenvalues of J_f^N are the union of the eigenvalues of J_f^B for each strongly connected component B of N. Here J_f^B is the submatrix of J_f^N with columns and rows corresponding to the cells in B. We prove now that the kernel of J_f^B is trivial, if B is not a source.

Proposition 4.5.5. *Let N be a network, $f \in \mathcal{V}(N)$ generic and B a strongly connected component of N which is not a source. Then* $\ker(J_f^B) = \{0\}$.

Proof. Let N be a network represented by $(\sigma_i)_{i=1}^k$, $f \in \mathcal{V}(N)$ generic and B a strongly connected component of N which is not a source. Like in the proof of Proposition 4.5.1, we use a recursive argument on the number of cells of B to prove the result. Denote by $c_1, \dots, c_n$ the cells of B and by $J_{p\,q}$ the (c_p, c_q)-entry of J_f^B. Define $\theta = (\theta_1, \dots, \theta_n)$, where

$$\theta_p := \sum_{\sigma_i(c_p) \notin B} f_i.$$

Generically on f, we can assume that $\theta \neq (0, \dots, 0)$, i.e., $\theta_p \neq 0$ for some $1 \leq p \leq n$, as B is not a source. Since $f \in \mathcal{V}(N)$,

$$\sum_{q=1}^{n} J_{p\,q} = -\theta_p, \qquad 1 \leq p \leq n.$$

Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^{|B|}$ be such that $J_f^B v = 0$.

Suppose that $n = 1$. Then $J_f^B = [-\theta_1] \neq 0$, generically, and so

$$\ker(J_f^B) = \{0\}.$$

Suppose now that $n = m + 1$. We can assume generically on f that $J_{n\,n} \neq 0$, since B is a strongly connected component. Let J' be the $m \times m$-matrix and let $\theta' = (\theta'_1, \ldots, \theta'_m)$ which are, respectively, defined by

$$J'_{p\,q} = J_{n\,n} J_{p\,q} - J_{p\,n} J_{n\,q},$$

$$\theta'_p = J_{n\,n}\theta_p - J_{p\,n}\theta_n,$$

for $1 \leq p, q \leq m$. Generically on f, we assume that $\theta' \neq (0, \ldots, 0)$, and if $m > 1$ we also assume that $J'_{m\,m} \neq 0$. Note that $(v_1, \ldots, v_m) \in \ker(J')$ and

$$\begin{aligned}\sum_{q=1}^{m} J'_{p\,q} =& J_{n\,n} \sum_{q=1}^{m} J_{p\,q} - J_{p\,n} \sum_{q=1}^{m} J_{n\,q} \\ =& J_{p\,n}(J_{n\,n} + \theta_n) - J_{n\,n}(J_{p\,n} + \theta_p) = -\theta'_p.\end{aligned}$$

As in the proof of Proposition 4.5.1, we can remove the cell n belonging to B from the network N and define a network M with a strongly connected component B' and a function $g \in \mathcal{V}(M)$ such that $B' = B \setminus \{n\}$ has m cells and $J' = J_g^{B'}$. Hence we can apply the same recursive argument and conclude that

$$\ker(J_f^B) = \{0\}.$$ □

In the next result, we describe the kernel of the Jacobian matrix of a coupled cell system with the bifurcation condition corresponding to the network valency.

Proposition 4.5.6. *Let N be a network and $f \in \mathcal{V}(N)$ generic. Then $\ker(J_f^N)$ has dimension equal to $s(N)$. Moreover, if S is a source component of N and $v \in \ker(J_f^N)$, then*

$$v_c = v_d, \qquad c, d \in S.$$

Proof. Let N be a network and $f \in \mathcal{V}(N)$ generic such that $s(N) = s$. Denote the source components of N by $S_1, \ldots, S_s$ and order the other strongly connected components by $B_1, \ldots, B_r$ such that any edge targeting a cell in B_i starts in a cell of $S_1 \cup \cdots \cup S_s \cup B_1 \cup \cdots \cup B_i$. Reordering the cells of N by its strongly connected components, we see that the matrix J_f^N has the following block form

$$J_f^N = \left[\begin{array}{cccc|cccc} J_f^{S_1} & 0 & \ldots & 0 & & & & \\ 0 & J_f^{S_2} & \ldots & 0 & & \mathbf{0} & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ 0 & 0 & \ldots & J_f^{S_s} & & & & \\ \hline & & & & J_f^{B_1} & 0 & \ldots & 0 \\ & \mathrm{R} & & & R_{21} & J_f^{B_2} & \ldots & 0 \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & R_{r1} & R_{r2} & \ldots & J_f^{B_r} \end{array}\right].$$

By Propositions 4.5.1 and 4.5.5, $\ker(J_f^{S_i})$ is one-dimensional for every $1 \leq i \leq s$ and $\ker(J_f^{B_j})$ is trivial for every $1 \leq j \leq r$. So $\ker(J_f^N)$ is $s = s(N)$-th dimensional.

Let $v \in \ker(J_f^N)$ and $1 \leq i \leq s$. Then $v_{S_i} = (v_c)_{c \in S_i} \in \ker(J_f^{S_i})$ and, it follows from Proposition 4.5.1 that $v_c = v_d$, for $c, d \in S_i$. □

Using the previous results, we describe next the codimension-one steady-state bifurcations of coupled cell systems associated to a network and imposing the bifurcation condition corresponding to the network valency.

Proposition 4.5.7. *Let N be a network and $f \in \mathcal{V}(N)$ generic. Then there are $2^{s(N)}$ equilibrium branches of f on N with the following properties:*
(i) For every equilibrium branch b, if $c, d \in S$, for some source component S, then $b_c = b_d = 0$ or $b_c = b_d = b_f$, where b_f is defined by Proposition 4.5.4.
(ii) Given two equilibrium branches b and b', if $b_S = b'_S$ for every source component S, then $b = b'$.

Proof. Let N be a network, $s = s(N)$ and $f \in \mathcal{V}(N)$ generic. Denote by $S_1, \ldots, S_s$ the source components of N and by B the set of cells not belonging to $S_1 \cup \cdots \cup S_s$.

It follows from the proof of Proposition 4.5.6 that J_f^B is invertible. By the Implicit Function Theorem, there exists $W : \mathbb{R}^{|S_1|} \times \cdots \times \mathbb{R}^{|S_s|} \times \mathbb{R} \to \mathbb{R}^{|B|}$ such that

$$f^N(x, \lambda) = 0 \Leftrightarrow \begin{cases} f^{S_i}(x_{S_i}, \lambda) = 0, & i = 1, \ldots, s \\ x_B = W(x_{S_1}, \ldots, x_{S_s}, \lambda) \end{cases}.$$

Using Proposition 4.5.4, it follows that, for each source S_i, $1 \leq i \leq s$, we can solve as:

$$f^{S_i}(x_{S_i}, \lambda) = 0 \Leftrightarrow x_{S_i} = (0, \ldots, 0) \vee x_{S_i} = (b_f(\lambda), \ldots, b_f(\lambda)),$$

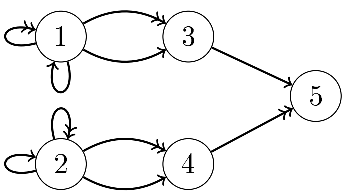

Figure 4.8: Homogeneous network with asymmetric inputs and two source components: $\{1\}$ and $\{2\}$. Given a generic coupled cell system with a bifurcation condition associated to the network valency, there are four bifurcation branches.

where $b_f : U \to \mathbb{R}$ is defined in Proposition 4.5.4 and it does not depend on the source component. Hence

$$f^N(x,\lambda) = 0 \Leftrightarrow \begin{cases} x_{S_i} = (0,\ldots,0) \vee x_{S_i} = (b_f(\lambda),\ldots,b_f(\lambda)), \quad i = 1,\ldots,s \\ x_B = W(x_{S_1},\ldots,x_{S_s},\lambda) \end{cases} .$$

For each source component we have two choices in the previous equation, then there are $2^{s(N)}$ equilibrium branches of f on N.

If b is an equilibrium branch of f on N and c, d belong to the same source component, then $b_c = b_d = 0$ or $b_c = b_d = b_f$. This proves (i).

Let b and b' be two equilibrium branches of f on N. If $b_S = b'_S$ for every source component, then $b_B = W(b_{S_1},\ldots,b_{S_s},\lambda) = W(b'_{S_1},\ldots,b'_{S_s},\lambda) = b'_B$ and $b = b'$. Proving (ii). □

Example 4.5.8. Let N be the network in Figure 4.8 and $f \in \mathcal{V}(N)$ generic. The network N has two source components. By Proposition 4.5.7, there are 4 bifurcation branches of f on N. The bifurcation branches are $(0,0,0,0,0)$, $(b_f,0,b_f,0,b_1)$, $(0,b_f,0,b_f,b_2)$ and (b_f,b_f,b_f,b_f,b_f), where b_f is defined by Proposition 4.5.4 applied to (any) of the source components of N, that is $f(b_f(\lambda),b_f(\lambda),b_f(\lambda),\lambda) = 0$, b_1 is the unique solution of $f(x,b_f(\lambda),0,\lambda) = 0$ and b_2 is the unique solution of $f(x,0,b_f(\lambda),\lambda) = 0$. ◊

Remark 4.5.9. Let N be a network and $f \in \mathcal{V}(N)$ generic. Denote by $S_1,\ldots,S_s$ the source components of N. The cells inside a source component receive every input from a cell inside that source component. Then the coloring that assigns a different color for each source component and it assigns the same color only for cells inside the same source component is balanced. The corresponding synchrony subspace is

$$\Delta_{S_1} \times \cdots \times \Delta_{S_s} \times \mathbb{R}^{|N|-(|S_1|+\cdots+|S_s|)},$$

where $\Delta_{S_i} = \{x \in \mathbb{R}^{|S_i|} : x_c = x_d, c, d \in S_i\}$ is the full synchrony subspace in the source component S_j, for $j = 1,\ldots,s$.

By Proposition 4.5.7, if b is a bifurcation branch of f on N, then

$$b \in \Delta_{S_1} \times \cdots \times \Delta_{S_s} \times \mathbb{R}^{|N|-(|S_1|+\cdots+|S_s|)}. \qquad \Diamond$$

4.6 The lifting bifurcation problem associated to the valency

In this section, we give conditions that characterize the lifting bifurcation problem for generic coupled cell systems with a bifurcation condition associated to the valency. Those conditions only depend on the number of source components. The results follow from the characterization of the bifurcation branches obtained in Section 4.5.

Proposition 4.6.1. *Let N be a network, L a lift of N and $f \in \mathcal{V}(N)$ generic. Then:*
(i) If $s(N) = s(L)$, then every bifurcation branch of f on L is lifted from N.
(ii) If $1 = s(N) < s(L)$, then there exists at least one bifurcation branch of f on L not lifted from N.

Proof. Let N be a network, L a lift of N, $\bowtie$ a balanced coloring in L such that $N = L/\bowtie$ and $f \in \mathcal{V}(N)$ generic. Denote by $\varphi_{\bowtie} : L \to N$ the network fibration induced by $\bowtie$ and by $S_1, \dots, S_{s(L)}$ the source components of L. Note that the source components of N are $\varphi_{\bowtie}(S_1), \dots, \varphi_{\bowtie}(S_{s(L)})$.
(i) Suppose that $s(N) = s(L)$. Let b be a bifurcation branch of f on L. Using Proposition 4.5.7, we define the bifurcation branch a of f on N such that

$$a_{\varphi_{\bowtie}(c)} := b_c, \qquad c \in S_i, 1 \leq i \leq s(L).$$

The bifurcation branch a is defined for each source component because the network fibration $\varphi_{\bowtie}$ sends each source component of L into a different source component of N. Therefore the bifurcation branch a is well defined. Note that

$$b_{S_i} = (P\varphi_{\bowtie}(a))_{S_i}, \qquad 1 \leq i \leq s(L),$$

where $P\varphi_{\bowtie}$ is the map between the phase spaces of N and L induced by $\varphi_{\bowtie}$. So the bifurcation branches b and $P\varphi_{\bowtie}(a)$ coincide on the source components and $b = P\varphi_{\bowtie}(a)$, by Proposition 4.5.7 (ii).
(ii) Suppose that $1 = s(N) < s(L)$. Denote by S the unique source component of N. Let b' be a bifurcation branch of f on N. By Proposition 4.5.7 (i), we know that $b'_S = (0, \dots, 0)$ or $b'_S = (b_f, \dots, b_f)$. Returning to the proof of Proposition 4.5.7, we have that $W(b'_S, \lambda) = b'_s(1, \dots, 1)$ for any $s \in S$, because $W : \mathbb{R}^{|S|} \times \mathbb{R} \to \mathbb{R}^{|N|-|S|}$ is the unique solution of the system $f^N(b'_S + W(b'_S, \lambda), \lambda) = 0$. So $b'_c = b'_d$, for any cells c and d of N and

$$b' \in \Delta_N \subseteq \mathbb{R}^{|N|},$$

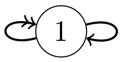

Figure 4.9: Network with a single cell and two edges types.

where Δ_N is the full synchrony subspace associated to the network N. For any network fibration $\varphi : L \to N$, $(P\varphi b')_c = b'_{\varphi(c)} = b'_{\varphi(d)} = (P\varphi b')_d$ for any cells c and d of L and

$$P\varphi b' \in \Delta_L \subseteq \mathbb{R}^{|L|},$$

where Δ_L is the full synchrony subspace associated to the network L. By Proposition 4.5.7 there exists a bifurcation branch b of f on L such that $b \notin \Delta_L$, because $s(L) > 1$. Moreover b is not lifted from N, because any bifurcation branch lifted from N belongs to the full-synchrony subspace, Δ_L. □

The previous result shows that the bifurcation branches of a generic coupled cell system, associated to a network with only one source component and the bifurcation condition correspondent to the network valency, are lifted from the trivial quotient network with a single cell and the same number of edge types. Thus those bifurcation branches do not break the full-synchrony.

Example 4.6.2. The network N in Figure 4.9 is the trivial quotient network of every network with two types of edges associated to the balanced coloring with exactly one color. Consider the lifts L_1 and L_2 of N given in Figures 4.6 and 4.8, respectively and $f \in \mathcal{V}(N)$ generic. Note that $1 = s(N) = s(L_1) < s(L_2) = 2$.

By Proposition 4.6.1 (i), the bifurcations branches of f on L_1 are lifted from N. On the other hand, by Proposition 4.6.1 (ii) there exists a bifurcation branch of f on L_2 which is not lifted from N. ◊

In Proposition 4.6.1 (ii) we assume that $1 = s(N) < s(L)$. If we change that assumption to $1 < s(N) < s(L)$, then it may happen that all the bifurcation branches of f on L are lifted from N, as illustrated in the following two examples.

Example 4.6.3. Let N be the network described in Figure 4.5 which has 2 source components, L the lift network of N described in Figure 4.3 with 3 source components and $f \in \mathcal{V}(N)$ generic. Consider the network fibrations from L to N given by: $\varphi_{1,2} = [22125]$; $\varphi_{1,3} = [12155]$; and $\varphi_{2,3} = [12252]$. Let b be a bifurcation branch of f on L. According to Proposition 4.5.7 (i), the bifurcation branch b can take one of two possible values on the coordinates of each of the cells 1, 2 and 3. Then at least one of the equalities $b_1 = b_2$, $b_1 = b_3$, $b_2 = b_3$ holds. Suppose that $b_i = b_j$, for some $1 \leq i < j \leq 3$. Let b' be the bifurcation branch of f on N such that $b'_{\varphi_{i,j}(c)} = b_c$, for $c \in \{1, 2, 3\}$. Then $b = P\varphi_{i,j} b'$ and it is lifted from N.

The networks L and N satisfy $1 < s(N) < s(L)$ and every bifurcation branch of f on L is lifted from N. $\diamondsuit$

Example 4.6.4. Let N be the network in Figure 4.4, L the lift network in Figure 4.1 and $f \in \mathcal{V}(N)$ generic. Consider the network fibrations from L to N given by: $\varphi_{1,2} = [2\,2\,1\,5\,6\,4]$; $\varphi_{1,3} = [2\,1\,2\,6\,4\,5]$; and $\varphi_{2,3} = [1\,2\,2\,4\,5\,6]$. Let b be a bifurcation branch of f on L. By Proposition 4.5.7, we know that $b_1 = b_2$, $b_2 = b_3$ or $b_1 = b_3$. Suppose that $b_i = b_j$, for some $1 \leq i < j \leq 3$. Then b is lifted from N using $\varphi_{i,j}$.

Again, we have that $1 < s(N) < s(L)$ and every bifurcation branch of f on L is lifted from N. $\diamondsuit$

In the previous examples, we saw that increasing the number of source components on the lift network is not sufficient to ensure that some bifurcation branch on the lift network is not lifted from the quotient network. The lift network in Figure 4.3 and considered in Example 4.6.3 is not backward connected. In Example 4.6.4, we consider the quotient network given by Figure 4.4 which is not transitive. The next result shows that increasing the number of source components on the lift network with respect to the quotient network is a necessary and sufficient condition for the existence of some bifurcation branch on the lift network that is not lifted from the quotient network, provided that the lift network is backward connected and the quotient network is transitive.

Theorem 4.6.5. *Let N be a transitive network, L a backward connected lift of N and $f \in \mathcal{V}(N)$ generic. Then every bifurcation branch of f on L is lifted from N if and only if $s(N) = s(L)$.*

Proof. Let N be a transitive network for the cell t in N and represented by the functions $(\sigma_i)_{i=1}^k$ and $f \in \mathcal{V}(N)$ generic. Let L be a backward connected network for the cell l, $\bowtie$ a balanced coloring in L such that $N = L/\bowtie$.

If $s(N) = s(L)$, then every bifurcation branch of f on L is lifted from N, by Proposition 4.6.1(i).

Next,we suppose that $s(N) < s(L)$ and prove that there is a bifurcation branch of f on L that is not lifted from N. Denote by $\varphi_{\bowtie} : L \to N$ the network fibration induced by $\bowtie$. Note that the network N is backward connected for the cell $l' = \varphi_{\bowtie}(l)$. Denote by $\phi_c : N \to N$ the network fibrations for each cell c in N such that $\phi_c(t) = c$.

Since N is backward connected for l', for every cell c in N there exist $1 \leq i_1, \ldots, i_m \leq k$ such that $\phi_{l'}(\sigma_{i_1} \circ \cdots \circ \sigma_{i_m}(t)) = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}(\phi_{l'}(t)) = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m}(l') = c$. Then $\phi_{l'}$ is surjective and it is also bijective, since N is finite. Applying the inverse of $\phi_{l'}$ to ϕ_c, we see that N is transitive for the cell l'. Assume that $l' = t$.

From $s(N) < s(L)$, it follows that there exist two source components S_1, S_2 of L such that $\varphi_{\bowtie}(S_1) = \varphi_{\bowtie}(S_2)$ is a source component of N. Let

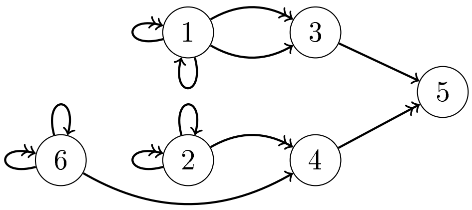

Figure 4.10: Homogeneous network with three source components: $\{1\}$, $\{2\}$ and $\{6\}$. This network is a lift of the network in Figure 4.5, taking the balanced coloring $\bowtie$ such that $1 \bowtie 3$ and $2 \bowtie 4 \bowtie 6$.

$\varphi : L \to N$ be any network fibration from L to N. By Remark 4.2.10 and $\varphi(l) = \phi_{\varphi(l)} \circ \varphi_{\bowtie}(l)$, we have that $\varphi = \phi_{\varphi(t)} \circ \varphi_{\bowtie}$. Hence $\varphi(S_1) = \varphi(S_2)$, for every network fibration $\varphi : L \to N$. If b' is a bifurcation branch of f on N, then

$$P\varphi(b')_{c_1} = P\varphi(b')_{c_2},$$

for $c_1 \in S_1$ and $c_2 \in S_2$. However, we know from Proposition 4.5.7 (i) that there exists a bifurcation branch b of f on L such that $b_{c_1} \neq b_{c_2}$, for $c_1 \in S_1$ and $c_2 \in S_2$. So b is not lifted from N. □

Example 4.6.6. Let N be the transitive network described in Figure 4.5 and $f \in \mathcal{V}(N)$ generic. Consider the lift networks L_1 and L_2 of N described in the Figures 4.8 and 4.10, respectively. Note that $s(N) = s(L_1) = 2$, $s(L_2) = 3$ and the network L_1 and L_2 are backward connected for the cell 5. Using Theorem 4.6.5, we know that every bifurcation branch of f on L_1 is lifted from N but there exists a bifurcation branch of f on L_2 that is not lifted from N. ◊

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5. The Lifting Bifurcation Problem on Feed-Forward Networks

This chapter consists of an article submitted for publication.

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Abstract

We consider feed-forward networks, that is, networks where cells can be divided into layers, such that every edge targeting a layer, excluding the first one, starts in the prior layer. A feed-forward system is a dynamical system that respects the structure of a feed-forward network. The synchrony subspaces for a network, are the subspaces defined by equalities of some cells coordinates, that are flow-invariant by all the network systems. The restriction of a network system to each of its synchrony subspaces is a system associated with a smaller network, which may be, or not, a feed-forward network. The original network is then said to be a lift of the smaller network. We show that a feed-forward lift of a feed-forward network is given by the composition of two types of lifts: lifts that create new layers and lifts inside a layer. Furthermore, we address the lifting bifurcation problem on feed-forward systems. More precisely, the comparison of the possible codimension-one local steady-state bifurcations of a feed-forward system and those of the corresponding lifts is considered. We show that for most of the feed-forward lifts, the increase of the center subspace is a sufficient condition for the existence of additional bifurcating branches of solutions, which are not lifted from the restricted system. However, when the bifurcation condition is associated with the internal dynamics and the lift occurs inside an intermediate layer, we prove that the existence of a bifurcation branch not lifted from the restricted system does depend generically on the chosen feed-forward system.

Keywords: Feed-forward networks, Steady-state bifurcations, Lifting bifurcation problem.

2010 Mathematics subject classification: 37G10; 34D06

5.1 Introduction

Coupled cell networks describe influences between cells and are represented by directed graphs with possible multiple arrows. Examples of modeling through networks include biological, computational and physical real-world applications, see e.g., [25, 4, 12, 16]. In applications, networks are commonly used to describe properties of dynamical systems formed by interacting individual dynamical systems. We consider coupled cell systems which are given by vector fields that respect the structure of the network [24, 15]. Briefly, a vector field respects the structure of a network if each cell corresponds to an individual dynamical system that depends on its own state and on the state of the cells in its input set. Examples of dynamics that can be observed in coupled cell systems include full-synchronized attractors, synchrony-breaking bifurcations and heteroclinic networks, see e.g., [4, 13, 1].

In this work, we consider networks such that all cells are identical and receive exactly one edge of each type. The different edge's types are graphically represented by different arrowheads. For example, considering the networks in Figure 5.1, every cell receives two edges of a different type.

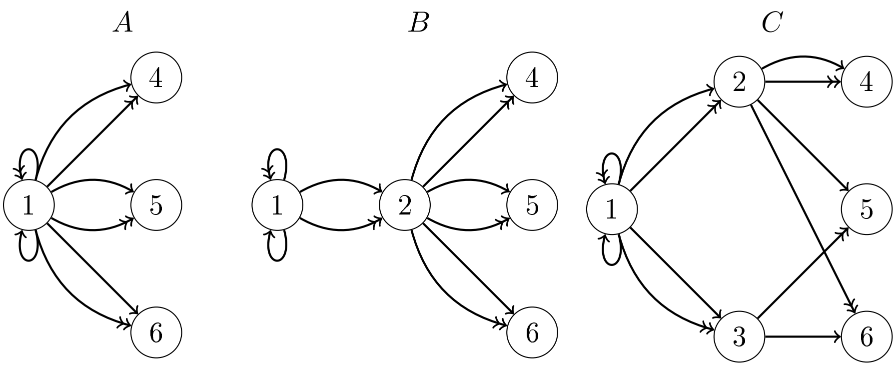

Figure 5.1: Feed-forward networks with two types of inputs that are distinguishes by their arrow heads. The network A has 2 layers and the networks B and C have 3 layers. The network A is the quotient network of B obtained by merging cells 1 and 2, that is, B is a lift of A that creates a new layer. The network B is the quotient network of C got by merging cells 2 and 3, C is a lift of B inside the second layer.

Two cells are synchronized if their dynamics agree for trajectories with the same initial condition. In [24, 15], the authors showed that there is an intrinsic relation between cells' synchronization and colorings of the network set of cells. More precisely, it is shown that, given a network, a subspace defined by equalities of some of the network cell coordinates is an invariant subspace for any coupled cell system if and only if the coloring of the cells determined by those equalities is balanced. Given a balanced coloring, the correspondent quotient network is obtained by merging cells with the same color. Moreover, the dynamics associated with the quotient network is the restriction of the dynamics on the original network to the correspondent invariant subspace. The original network is then said to be a lift of the smaller network.

Consider the networks in Figure 5.1. As the cells 2 and 3 of the network C receive their inputs from cell 1, the coloring in C that identify cells 2 and 3 is balanced. The quotient network associated to this coloring is the network B. By merging cells 2 and 3, we have that each edge of C starting at cell 2 or 3 has a corresponding edge in B starting at the merged cell. In the same way, we can see that the network A is obtained from B by merging cells 1 and 2. Hence the network C is a lift of B which is itself a lift of A.

In general real-world networks, due to their complexity, are big in size. Quotient networks are a way of reducing the size of a network and study, at least a fraction of, the dynamics associated with a big network. If a dynamical property in a lifted network follows from the study of that dynamical property on a smaller network, then the dimension of the problem can be reduced. Given a bifurcation problem on a feed-forward network. The main goal of this paper is to study the lifting bifurcation problem for feed-forward networks, i.e., investigate when every bifurcation branch in a lift feed-forward network is obtained by (lifting) a bifurcation branch in a quotient feed-forward network. As we explain below, the answer to this problem can help us, for example, to understand which synchronization patterns are broken via bifurcation. Synchrony-breaking events can be observed, for example, in biological networks, [25], and neuronal networks, [18]. In the following paragraphs, we describe the lifting bifurcation problem for feed-forward networks, our approach to it and the results obtained.

Feed-forward networks are a class of networks where the set of cells can be partitioned into disjoint sets called layers. We consider feed-forward networks where each cell in the first layer only receives inputs from itself and cells in the other layers only receive inputs from cells in the previous layer. Feed-forward networks have been used, for example, to design machine learning networks and neuronal networks, [12, 16]. In those applications, the cells emulate neurons and each edge corresponds to a unidirectional connection between two neurons. The cells in the first layer are viewed as the original information receptors. The following layers receive and transform this information and then transmit it to the next layer. In the last layer, we can find the outcome of processing the original information. This process is usually assumed to be discrete. Nevertheless some features of neuronal networks have been observed in continuous models, such as binocular rivalry [8, 9, 7]. The feed-forward structure allows us to reason layer by layer and we use this to study feed-forward networks and the associated feed-forward systems.

Three examples of feed-forward networks appear in Figure 5.1. The network C has 3 layers: $\{1\}$, $\{2,3\}$ and $\{4,5,6\}$. The networks A and B have, respectively, 2 and 3 layers.

The lift network of a feed-forward network does not need to be a feed-forward network. Our first goal is to understand which lifts of feed-forward networks also have a feed-forward structure. Previous work about balanced colorings on feed-forward networks can be found in [2]. We introduce two basic lifts on feed-forward networks, lifts inside a layer and lifts that create new layers. A lift that creates new layers is the replication of the first layer into consecutive layers. A lift inside a layer is given by the split of some cells within a layer. In Proposition 5.3.10, we prove that the lifts of feed-forward networks that also have a feed-forward structure are given by the composition of basic lifts. The definition of the basic lifts is crucial to the study of the lifting bifurcation problem, since it reduces our study to the two cases corresponding to the basic lifts.

Returning to the networks in Figure 5.1. The network C is obtained from the network B by splitting the cell 2 of B into the cells 2 and 3. That is, C is a lift of B inside the second layer. Also, the network B is a lift of the network A with one more layer. Thus the network B is a lift of A that creates a new layer. We have then that the network C is a lift of A given by the composition of a lift that creates a new layer with a lift inside a layer.

Our second goal is to address the (codimension-one steady-state) bifurcations from a full synchrony equilibrium on feed-forward systems. Previous works have addressed bifurcations on feed-forward systems with one cell per layer, [10, 5, 14, 20, 11], and have identified a surprising phenomenon of bifurcation on feed-forward systems: the layers of the feed-forward network act as amplifiers for the growth rate of the bifurcation branches. We generalize that study of steady-state bifurcations to general feed-forward networks. In this work, we assume that the phase space of each cell is real and one dimensional and that the feed-forward system has a steady-state solution with full synchrony. It follows from the feed-forward structure that the linearization of a feed-forward system at a full synchrony equilibrium has only two eigenvalues: the valency and the internal dynamics. This leads to two types of bifurcation conditions from a full synchrony equilibrium on feed-forward systems. One bifurcation condition is given by the linearization of the self input, that we call internal dynamics. The other bifurcation condition is given by the sum of all inputs' linearization, that we call valency.

In this work, we study bifurcations on feed-forward systems given by the two different bifurcation conditions. A direct application of the Implicit Function Theorem describes the bifurcation branches of a feed-forward system with a bifurcation condition associated to the valency. Furthermore, we give in Proposition 5.6.6 a full characterization of the bifurcation branches of a feed-forward system with a bifurcation condition associated to the internal dynamics in terms of their square-root-orders and slopes. In order to obtain this characterization, we follow the technique used in [20] for feed-forward networks with one cell per layer. In [20], the authors used a suitable change of coordinates to we see how the growth rate of a bifurcation branch propagates from one layer to the next. Using this change of coordinates and exploiting the layer structure, we obtain the complete description of the bifurcation branches on any feed-forward system.

As an example, we present some conclusions that follow, using our results, from the characterization of the bifurcation branches of a feed-forward system on the network C in Figure 5.1 with a bifurcation condition associated to the internal dynamics. The full characterization of the bifurcation branches is given in Example 5.6.5. There are 8 bifurcation branches with a linear growth rate and 8 bifurcation branches with a square root growth. The cells 2 and 3 of C are in synchrony for any of those bifurcation branches. Moreover, there are 16 more bifurcation branches with a square root growth if and only if the feed-forward system has the same sign for the linearizations of the first and second inputs. In the latter case, for the extra bifurcation branch solutions, cells 2 and 3 of C are not synchronized.

Last, we study our main goal, the lifting bifurcation problem for feed-forward networks. The restriction of a coupled cell system to a synchrony subspace is a coupled cell system of the correspondent quotient network. Thus any bifurcation branch for the quotient system lifts to a bifurcation branch on the lift system. The lifting bifurcation problem asks if there are more bifurcation branches on the lift system. Note that the bifurcation branches lifted from a quotient network preserve the synchrony associated to the quotient network. So any bifurcation branch not lifted from a quotient network breaks the synchrony associated to that quotient network. The lifting bifurcation problem was first raised in [3] where the authors proved that there are networks which have more bifurcation branches on some lift systems than the ones lifted from the quotient network. This problem was also studied in [17, 19]. A well-known result gives a necessary condition for the lifting bifurcation problem: There can be additional bifurcation branches on a lifted network than the ones lifted only if the center subspace of the coupled cell systems associated to the original network and the lift network have different dimensions. We refer to Corollary 5.7.3. As we show, the study of the lifting bifurcation problem for feed-forward networks reduces to its analysis on the basic lifts: lifts that create new layers and lifts inside a layer. We describe now the results that we obtain about the lifting bifurcation problem on feed-forward networks. In order to obtain those results, we use the characterization of the bifurcation branches on feed-forward systems, stated above.

Frequently, the aforementioned necessary condition for the lifting bifurcation problem is also sufficient. In Propositions 5.7.6, 5.7.10 and 5.7.13, we prove this in the following cases. For feed-forward systems with a bifurcation condition associated to the valency. For lifts that create new layers, inside the first layer and inside a layer which has only one cell in the next layer, and feed-forward systems with a bifurcation condition associated with the internal dynamics. Moreover, those cases do not depend generically on the feed-forward system.

Consider the networks in Figure 5.1 and the correspondent feed-forward systems. If the feed-forward systems have a bifurcation condition associated to the valency, then the center subspaces associated to A, B and C have dimension 1. So every bifurcation branch on the network C is lifted from A. On the other hand, if the feed-forward systems have a bifurcation condition associated to the internal dynamics then there exists a bifurcation branch on B which is not lifted from A. This follows from the fact that the network B is a lift of A that creates a new layer and the center subspaces associated to the network A and B are, respectively, one and two dimensional.

For lifts inside an intermediate layer and feed-forward systems with a bifurcation condition associated to the internal dynamics, the center subspaces of the feed-forward system and of the lift system have different dimensions. For lifts inside an intermediate layer, we show in Proposition 5.7.16 that there exists an open set of feed-forward systems with a bifurcation condition associated to the internal dynamics for which there are more bifurcation branches on the lifted network than the ones lifted from the quotient network. Remarkably, we see in Propositions 5.7.18 and 5.7.20 that, for a class of lifts inside an intermediate layer, there is also an open set of feed-forward systems with a bifurcation associated to the internal dynamics such that there are no more bifurcation branches on the lift system.

Consider the network C in Figure 5.1 and a feed-forward system with a bifurcation condition associated to the internal dynamics. Note that the feed-forward network C is a lift of the network B inside the second layer and the center subspace associated to C is bigger than the center subspace associated to B. We stated before that all bifurcation branches of the feed-forward system associated with C have synchrony between the cells 2 and 3 if and only if the linearization of the two inputs have opposite signs. We can show that a bifurcation branch of C is lifted from B only if it has synchrony between the cells 2 and 3. Therefore every bifurcation branch on the network C is lifted from B if and only if the linearization of the two inputs have the opposite signs. This condition provides the open sets of feed-forward systems mentioned in the previous paragraph.

The paper is organized as follows. In Section 5.2, we recall the definitions of coupled cell networks and feed-forward networks. Next, we study lifts of a feed-forward network which preserve the feed-forward structure (Section 5.3). Coupled cell systems and feed-forward systems are recalled in Section 5.4. Then we analyze the steady-state bifurcations in feed-forward systems with bifurcation conditions associated to the valency (Section 5.5) and the internal dynamics (Section 5.6). Finally, we study the lifting bifurcation problem for feed-forward systems with bifurcation conditions associated to the valency (Section 5.7.1) and the internal dynamics (Section 5.7.2).

5.2 Feed-forward networks

In this section, we recall a few facts concerning coupled cell networks, following [15, 21], and define feed-forward networks.

Definition 5.2.1. A *network* N is defined by a directed graph with a finite set of cells C and a finite sets of directed edges divided by types $E_1, \dots, E_k$. We assume that each cell c is target by one and only one edge of each type. We denote by $|N|$ the number of cells in the network N. ◊

Figure 5.2 displays an example of a network with 2 types of edges.

We say that the networks N and N' *are equal* and write $N = N'$ if they only differ by a relabel of cells, edges and types. In this paper, we consider connected networks, i.e., networks such that every two distinct cells have an undirected path between them.

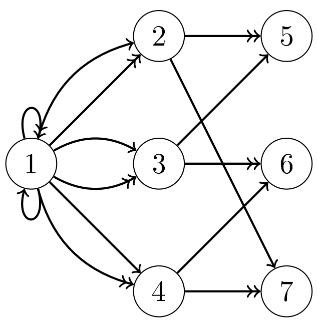

Figure 5.2: Feed-forward network with 3 layers

Following [21], we can regard each type of edge as a function from the target cell to the input cell. Let $(\sigma_i : C \to C)_{i=1}^k$ be the collection of functions such that there exists an edge $e \in E_i$ from $\sigma_i(c)$ to c, for every $c \in C$ and $1 \leq i \leq k$. We write that N is *represented by the functions* $(\sigma_i : C \to C)_{i=1}^k$.

Example 5.2.2. Consider the functions σ_1 and σ_2 defined by

$$\sigma_1(1) = \sigma_1(2) = \sigma_1(3) = \sigma_1(4) = 1, \quad \sigma_1(5) = 3, \quad \sigma_1(6) = 4, \quad \sigma_1(7) = 2,$$

$$\sigma_2(1) = \sigma_2(2) = \sigma_2(3) = \sigma_2(4) = 1, \quad \sigma_2(5) = 2, \quad \sigma_2(6) = 3, \quad \sigma_2(7) = 4.$$

The network in Figure 5.2 is represented by (σ_1, σ_2), where σ_1 represents the edges with one head and σ_2 represents the edges with two heads. ◊

In feed-forward networks, the cells can be partition into layers. Each cell in the first layer only receives inputs from itself. Cells in the other layers only receive inputs from cells in the previous layer. The network in Figure 5.2 is an example of a feed-forward network with 3 layers.

Definition 5.2.3. Let N be a network represented by the functions $(\sigma_i)_{i=1}^{k}$. We say that N is a *feed-forward network (FFN)*, if there exists a partition of the set of cells of N into subsets $C_1, \ldots, C_m$ such that $\cup_{i=1}^{k} \sigma_i(C_j) = C_{j-1}$, for every $2 \leq j \leq m$ and $\sigma_i(c) = c$, for every $c \in C_1$ and $1 \leq i \leq k$. The subset C_j is called the *jth layer* of N. $\diamondsuit$

Backward connectivity of a network is an important concept for the results obtained in this paper. Roughly speaking, a network is backward connected for some cell if there exists a directed path starting in any other cell and ending in that cell.

Definition 5.2.4. We say that a network N is *backward connected for a cell* c if for every cell c' different from c there exists a sequence of cells $c_0, c_1, \ldots, c_{l-1}, c_l$ in N such that $c' = c_0$, $c = c_l$ and there is an edge from c_{a-1} to c_a, for every $1 \leq a \leq l$. The network N is *backward connected* if it is backward connected for some cell. $\diamondsuit$

The network in Figure 5.3 is an example of a backward connected network for the cell 10. An example of a network which is not backward connected is the one pictured in Figure 5.2, as there is no directed path between the cells 5, 6 and 7. Observe that, by definition, a feed-forward network is backward connected if and only if it has only one cell in the last layer.

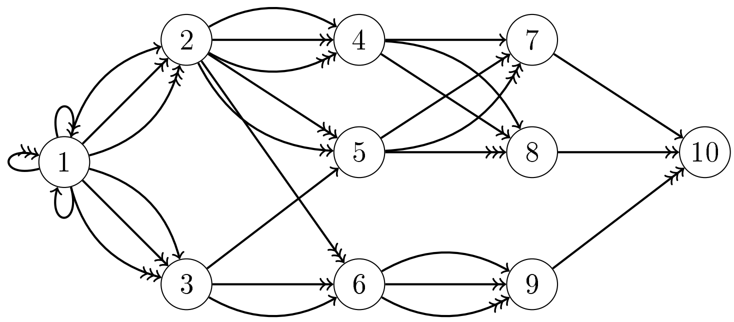

Figure 5.3: Backward connected feed-forward network with 5 layers.

5.3 Lifts of feed-forward networks

The main goal of this section is to show how the feed-forward lifts can be decomposed. We recall the notions of balanced colorings, quotient networks and lifts of networks, following [24, 15, 21, 22], with emphasis on feed-forward networks. Roughly speaking, in a balanced coloring, cells with the same color receive each type of input from cells with the same color. And the associated quotient network is obtained by merging cells with the same color.

Let N be a network represented by the functions $(\sigma_i : C \to C)_{i=1}^k$. A *coloring* of the set of cells of N is an equivalence relation on the set of cells. A coloring $\bowtie$ is *balanced* if $\sigma_i(c) \bowtie \sigma_i(c')$, for every $1 \leq i \leq k$ and $c, c' \in C$ such that $c \bowtie c'$. Given a subset of cells S in N, we denote by $[S]_\bowtie$ the set of $\bowtie$-classes of the cells in S, i.e., $[S]_\bowtie = \{[c]_\bowtie : c \in S\}$.

Definition 5.3.1 ([15, Section 5])**.** Let N be a network represented by the functions $(\sigma_i : C \to C)_{i=1}^k$ and $\bowtie$ a balanced coloring in N. The *quotient network* of N associated to $\bowtie$ is the network where the set of cells is $[C]_\bowtie$ and there is an edge of type i from $[\sigma_i(c)]_\bowtie$ to $[c]_\bowtie$ for every $1 \leq i \leq k$ and $c \in C$. We denote by $N/\bowtie$ the quotient network of N associated to $\bowtie$. We also say that a network L is a *lift* of N, if N is a quotient of L for some balanced coloring in L. $\diamondsuit$

Let N bet a network represented by the functions $(\sigma_i : C \to C)_{i=1}^k$ and $\bowtie$ a balanced coloring in N. The quotient network $N/\bowtie$ is represented by the functions $(\sigma_i^\bowtie : [C]_\bowtie \to [C]_\bowtie)_{i=1}^k$, where $\sigma_i^\bowtie$ is given by $\sigma_i^\bowtie([c]_\bowtie) = [\sigma_i(c)]_\bowtie$, for every $1 \leq i \leq k$ and $c \in C$. Note that the quotient network of a backward connected network is also backward connected.

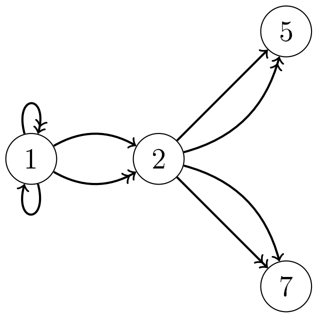

Figure 5.4: Quotient network of the network in Figure 5.2 given by merging cells 2, 3, 4, and cells 5, 6.

Example 5.3.2. Let N be the network in Figure 5.2. Consider the coloring $\bowtie$ in N given by $2 \bowtie 3 \bowtie 4$ and $5 \bowtie 6$. Note that the cells 5 and 6 receive inputs from the cells 2, 3 and 4 that have the same color. Moreover, the cells 2 and 3 receive inputs from a unique cell 1. So the coloring is balanced and the network in Figure 5.4 is the quotient of N associated to $\bowtie$. $\diamondsuit$

In [2], the authors studied and described the balanced colorings of feed-forward networks. The set of balanced colorings forms a partial order set as studied in [23] given by the refinement relation. Given two balanced colorings

$\bowtie', \bowtie$ of a network N, we say that $\bowtie'$ refines $\bowtie$ and we write $\bowtie' \preceq \bowtie$, if $c \bowtie' d$ implies that $c \bowtie d$, for every cells c and d of N. We have that if $\bowtie' \preceq \bowtie$, then $N/\bowtie$ is a quotient of $N/\bowtie'$.

If L, N and Q are networks such that L is a lift of N and N is a lift of Q, then L is a lift of Q. Moreover, we say that the lift L of Q is given by the composition of the lift N of Q and the lift L of N. In some cases, a lift can be seen as the composition of two lifts, see [6, Theorem 2.4]. In the next result, we give a sufficient condition for the existence of an intermediate quotient which only merges cells in an independent subset of cells.

Lemma 5.3.3. *Let L be a network represented by the functions $(\sigma_i : C \to C)_{i=0}^k$, $\bowtie$ a balanced coloring in L and $S \subseteq C$ such that $\sigma_i(S) \subseteq S$, for $1 \leq i \leq k$. Then, there exists a balanced coloring $\bowtie'$ in L such that $L/\bowtie'$ is a lift of $L/\bowtie$, $[C \setminus S]_{\bowtie'} = C \setminus S$ and there exists a bijection between $[S]_{\bowtie}$ and $[S]_{\bowtie'}$.*

Proof. Let L be a network represented by the functions $(\sigma_i : C \to C)_{i=0}^k$, $\bowtie$ a balanced coloring in L and $S \subseteq C$ such that $\sigma_i(S) \subseteq S$, for $1 \leq i \leq k$.

Define $\bowtie'$ as the coloring of L such that $c \bowtie' c'$ if $c \bowtie c'$ and $c, c' \in S$. Let $c, c' \in S$ such that $c \bowtie' c'$. Then $c \bowtie c'$, $\sigma_i(c) \bowtie \sigma_i(c')$ and $\sigma_i(c), \sigma_i(c') \in S$, for every $1 \leq i \leq k$. Hence $\sigma_i(c) \bowtie' \sigma_i(c')$, for every $1 \leq i \leq k$, and $\bowtie'$ is a balanced coloring of L. Note that $\bowtie' \preceq \bowtie$ and so $L/\bowtie$ is a quotient of $L/\bowtie'$.

The $\bowtie'$-class of any cell in $C \setminus S$ is singular, so $[C \setminus S]_{\bowtie'} = C \setminus S$.

Let $\alpha : [S]_{\bowtie} \to [S]_{\bowtie'}$ be given by $\alpha([c]_{\bowtie}) = [c]_{\bowtie'}$, where $c \in S$. Let $c, c' \in S$ such that $[c]_{\bowtie} = [c']_{\bowtie}$. Then $c \bowtie c'$ and $c \bowtie' c'$. So α is well-defined. Suppose that $\alpha([c]_{\bowtie}) = \alpha([c']_{\bowtie})$. Then $c \bowtie' c'$ and $c \bowtie c'$. So $[c]_{\bowtie} = [c']_{\bowtie}$ and α is injective. Let $[c]_{\bowtie'} \in [S]_{\bowtie'}$. Then $\alpha([c]_{\bowtie}) = [c]_{\bowtie'}$ and α is surjective. So α is a bijection between $[S]_{\bowtie}$ and $[S]_{\bowtie'}$. □

We exemplify the use of the previous lemma in the next example.

Example 5.3.4. Consider the network L in Figure 5.2 and the balanced coloring defined by $2 \bowtie 3 \bowtie 4$ and $5 \bowtie 6$. Every input of a cell in the subset $S = \{1, 2, 3, 5\}$ is from a cell in S. By Lemma 5.3.3, there exists a network between L and $L/\bowtie$. Let $\bowtie'$ be the balanced coloring such that $2 \bowtie' 3$. The quotient $L/\bowtie'$ is the network C in Figure 5.1. And $L/\bowtie'$ is a lift of $L/\bowtie$.

However, if we consider the subset $S' = \{1, 2, 3, 5, 6\}$, then does not exist an intermediate balanced coloring as stated in Lemma 5.3.3. The cell 6 belongs to S' but it receives an input from cell 4 which does not belong to S'. If there was such an intermediate balanced coloring, then cells 5 and 6 would have the same color. If the coloring is balanced and cells 5 and 6 have the same color, then their inputs must have the same color and cells 3 and 4 must have the same color. In Lemma 5.3.3, cells not in S' do not share the color with any other cell. If there was a balanced coloring as stated in Lemma 5.3.3, then cells 3 and 4 would not have the same color. This contradiction implies that Lemma 5.3.3 does not hold for S'. ◊

There are lift networks of feed-forward networks that are not feed-forward networks. For example, the network with exactly one cell and k types of edges is a feed-forward network and every other network with k types of edges is a lift of that network. We study feed-forward lift networks of feed-forward networks.

We define two types of basic lifts for feed-forward networks: lifts inside a layer and lifts that create new layers. A lift is inside a layer if the corresponding balanced coloring only identifies cells within some layer. And a lift that replicates the first layer is a lift that creates new layers. In Figure 5.1, we present examples of those basic lifts.

Definition 5.3.5. Let N be a feed-forward network and L a feed-forward lift of N. Denote the layers of N and L by $C_1, \dots, C_m$ and $C'_1, \dots, C'_n$, respectively.

We say that L is a *lift inside the layer* C_j, where $1 \leq j \leq m$, if $m = n$, $|C'_j| \neq |C_j|$ and $|C'_i| = |C_i|$ for every $i \neq j$.

We say that L is a *lift that creates* $n - m$ *new layers*, if $m < n$, $|C'_1| = \cdots = |C'_{n-m}| = |C_1|$ and $|C'_{n-m+j}| = |C_j|$, for every $2 \leq j \leq m$. $\diamondsuit$

The lift network in Figure 5.2 of the network in Figure 5.4 can be seen as the composition of two lifts inside a layer. The network B of Figure 5.1 is a lift inside the third layer of the network in the Figure 5.4 and the network in Figure 5.2 is a lift inside the second layer of B.

Combining the feed-forward structure and Lemma 5.3.3, we see that a lift from a feed-forward network to a feed-forward network with the same number of layers is a composition of lifts inside a layer.

Example 5.3.6. Let N be a feed-forward network and L a feed-forward lift of N. Suppose that N and L have the same number of layers and denote the layers of L by $C_1, \dots, C_m$.

Consider Lemma 5.3.3 applied to the subset $S_1 = C_1$ and to the lift L of N. We have that there exists a network Q_1 between the lift L and the network N. The network Q_1 is given by the merge of cells in the first layer of L. So Q_1 is also a feed-forward network with the same number of layers a N and L. Moreover, the number of cells in each layer of L and Q_1 coincide, except in the first layer. Hence L is a lift of Q_1 inside the first layer (or $N = Q_{m-1}$).

We can repeat the previous application of Lemma 5.3.3 to the subset $S_j = C_1 \cup \cdots \cup C_j$ and the lift Q_{j-1} of N, for each $2 \leq j \leq m-1$. Thus we obtain a sequence of networks $L = Q_0, Q_1, \dots, Q_{m-1}, Q_m = N$ such that Q_{j-1} is a lift of Q_j inside the jth layer (or $Q_{j-1} = Q_j$) for $j = 1, \dots, m-1$ and Q_{m-1} is a lift of N. Therefore the lift of N to L is the composition of lifts inside the layers and the lift of N to Q_{m-1}.

If L is backward connected, then the networks N and Q_{m-1} have only one cell in the last layer. By the previous construction, we already knew that N and Q_{m-1} are equal in every layer, except the last. Thus $N = Q_{m-1}$ and the lift of L from N is the composition of lifts inside a layer. $\diamondsuit$

When the lifted network is not backward connected, we can have lifts which do not decompose into basic lifts.

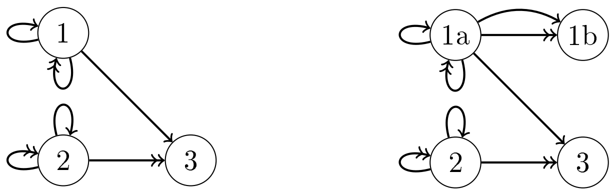

Figure 5.5: The feed-forward network on the right is a lift of the feed-forward network on the left. This lift is not given by the composition of lifts that create new layers and lifts inside layers. Note that the lift network is not backward connected.

Example 5.3.7. Let N be the feed-forward network in the left of Figure 5.5 and L the feed-forward network on the right of Figure 5.5. The network L is a lift of N, considering the coloring in L given by the classes $\{1a, 1b\}$, $\{2\}$, $\{3\}$. This lift cannot be obtained by a composition of lifts that create new layers and lifts inside the layers. Note that N and L have the same number of layers and the coloring in L given by the class $\{1b, 3\}$, $\{1a\}$, $\{2\}$ is not balanced. However, L is not backward connected. $\diamondsuit$

For backward connected lifts, we show the following. The first layer of a feed-forward quotient network is given by the merge of all the cells in a number of consecutive layers, starting in the second layer, with the cells in the first layer, eventually with the merge of some cells in the first layer. Moreover, the next layers of the feed-forward quotient network are given by merging cells in a specific layer of the lift network respecting the layers order.

Lemma 5.3.8. *Let N be a feed-forward network with layers $C_1, \ldots, C_m$ and L a feed-forward lift of N with layers $C'_1, \ldots, C'_n$ such that L is backward connected. Denote by $\bowtie$ the balanced coloring of L such that $L/\bowtie = N$. Then*

$$[C'_{n-m+1}]_{\bowtie} = \cdots = [C'_1]_{\bowtie} = C_1, \qquad [C'_{n-j}]_{\bowtie} = C_{m-j}, \qquad 0 \leq j \leq m-2.$$

Proof. Let N be a feed-forward network with layers $C_1, \ldots, C_m$ and L a feed-forward lift of N with layers $C'_1, \ldots, C'_n$ such that L is backward connected. Assume that N and L are represented by $(\sigma_i)_{i=1}^k$ and $(\sigma'_i)_{i=1}^k$, respectively. Let $\bowtie$ be a balanced coloring such that $L/\bowtie = N$. Then $n \geq m$ and $[\sigma'_i(c)]_{\bowtie} = \sigma_i([c]_{\bowtie})$, for every cell c in L and $1 \leq i \leq k$.

Since L is backward connected, we have that N is backward connected, $C'_n = \{c_1\}$, $C_m = \{[c_1]_{\bowtie}\}$ and $[C'_n]_{\bowtie} = C_m$. If $m = 1$, then N has only one cell and there is only one equivalence class of $\bowtie$. Hence $[C'_n]_{\bowtie} = \cdots = [C'_1]_{\bowtie} = C_1$.

Now, suppose that $m > 1$. Assuming that $[C'_{n+1-j}]_{\bowtie} = C_{m+1-j}$, we see that $[C'_{n-j}]_{\bowtie} = C_{m-j}$ for $j = 1, \ldots, m-1$. Let $d_1 \in C_{m-j}$ where $1 \leq j \leq m$. Then there exist $1 \leq i \leq k$ and $d_2 \in C_{m+1-j}$ such that $d_1 = \sigma_i(d_2)$. By assumption, there exists $d'_2 \in C'_{n+1-j}$ such that $d_2 = [d'_2]_{\bowtie}$, $d_1 = [\sigma'_i(d'_2)]_{\bowtie}$ and $\sigma'_i(d'_2) \in C'_{n-j}$. Thus $C_{m-j} \subseteq [C'_{n-j}]_{\bowtie}$. On the other hand, let $d'_1 \in C'_{n-j}$. Then there exist $1 \leq i \leq k$ and $d'_2 \in C'_{n+1-j}$ such that $d'_1 = \sigma'_i(d'_2)$. By assumption, we have that $[d'_2]_{\bowtie} \in C_{m+1-j}$ and $[d'_1]_{\bowtie} = [\sigma'_i(d'_2)]_{\bowtie} \in C_{m-j}$. Therefore $[C'_{n-j}]_{\bowtie} = C_{m-j}$. It follows inductively from $[C'_n]_{\bowtie} = C_m$ that

$$[C'_{n-j}]_{\bowtie} = C_{m-j} \qquad 1 \leq j \leq m-1.$$

In particular, $[C'_{n-m+1}]_{\bowtie} = C_1$ and $\sigma_i(C_1) = C_1$, for $1 \leq i \leq k$. Using the same argument, we conclude that

$$[C'_{n-m}]_{\bowtie} = \cdots = [C'_1]_{\bowtie} = C_1. \qquad \square$$

Example 5.3.9. Let L be the network in Figure 5.3 which is backward connected for the cell 10. The network L has 5 layers that we denote by C'_1, C'_2, C'_3, C'_4 and C'_5. Consider the balanced coloring $\bowtie$ in L given by $1 \bowtie 2 \bowtie 3$ and $4 \bowtie 5$. Note that $L/\bowtie$ is a feed-forward network with 4 layers. Denote the layers of $L/\bowtie$ by C_1, C_2, C_3 and C_4. Lemma 5.3.8 states that

$$[C'_1]_{\bowtie} = [C'_2]_{\bowtie} = C_1, \qquad [C'_3]_{\bowtie} = C_2, \qquad [C'_4]_{\bowtie} = C_3, \qquad [C'_5]_{\bowtie} = C_4 \quad \diamond$$

Example 5.3.7 shows that Lemma 5.3.8 does not hold if the lift network is not backward connected. The lift in Figure 5.5 merges a cell in the second layer with one in the first layer and does not merge the other cell in the second layer with a cell on the first layer. The lift network in Figure 5.5 is not backward connected and this lift does not satisfy the conclusion of the previous lemma.

Using the method described in Example 5.3.6 and Lemma 5.3.8, we state the main result of this section. This result shows how to decompose a feed-forward lift into lifts that create new layers and lifts inside a layer.

Proposition 5.3.10. *Let N be a feed-forward network and L a feed-forward lift of N such that L is backward connected. Then, the lift of N to L is the composition of a lift that creates new layers with lifts inside the layers.*

Proof. Let N be a feed-forward network with layers $C_1, \ldots, C_m$ and L a feed-forward lift of N with layers $C'_1, \ldots, C'_n$ such that L is a backward connected and L is represented by $(\sigma_i)_{i=1}^k$. Denote by $\bowtie$ the balanced coloring in L such that N is the quotient network of L associated to $\bowtie$.

Define the coloring $\bowtie_1$ in L such that $c \bowtie_1 d$ if $c \bowtie d$ and $c, d \in C'_j$ for $1 \leq j \leq n$. Let $c \bowtie_1 d$. Since $\bowtie$ is balanced, we have that $\sigma_i(c) \bowtie \sigma_i(d)$

and $\sigma_i(c), \sigma_i(d) \in C'_{j'}$, where $j' = \max\{1, j-1\}$ and $1 \leq i \leq k$. Then $\sigma_i(c) \bowtie_1 \sigma_i(d)$ for $1 \leq i \leq k$. Hence $\bowtie_1$ is balanced.

Define the network $Q_1 = L/\bowtie_1$ and the set of cells $A_j = [C'_j]_{\bowtie_1}$, for $1 \leq j \leq n$. The network Q_1 is a feed-forward network with the layers $A_1, \ldots, A_n$. Let $c \in A_1$. There exists $d \in C'_1$ such that $c = [d]_{\bowtie_1}$. Then $\sigma_i^{\bowtie_1}(c) = \sigma_i^{\bowtie_1}([d]_{\bowtie_1}) = [\sigma_i(d)]_{\bowtie_1} = [d]_{\bowtie_1} = c$, for $1 \leq i \leq k$. We have that $A_{j-1} = [C'_{j-1}]_{\bowtie_1} = [\cup_{i=1}^k \sigma_i(C'_j)]_{\bowtie_1} = \cup_{i=1}^k \sigma_i^{\bowtie_1}(A_j)$, for $j = 2, \ldots, m$. Therefore Q_1 is a feed-forward network with the layers $A_1, \ldots, A_n$.

Note that $\bowtie_1 \preceq \bowtie$. Hence Q_1 is a lift of N, if $\bowtie_1 \prec \bowtie$, and $Q_1 = N$, if $\bowtie_1 = \bowtie$. It follows from Lemma 5.3.8 that

$$|A_{n-m+j}| = |[C'_{n-m+j}]_{\bowtie_1}| = |[C'_{n-m+j}]_{\bowtie}| = |C_j|, \qquad 1 \leq j \leq m,$$

and

$$|A_{j'}| = |[C'_{j'}]_{\bowtie_1}| = |[C'_{j'}]_{\bowtie}| = |C_1|, \qquad 1 \leq j' \leq n-m.$$

Hence Q_1 is a lift of N that creates $n-m$ new layers or $Q_1 = N$.

The networks Q_1 and L have the same number of layers and L is a backward connected lift of Q_1. Following Example 5.3.6, we see that the lift of Q_1 to L is the composition of lifts inside the layers. Thus the lift of N to L is the composition of a lift that creates new layers and lifts inside the layers. □

Example 5.3.11. Let L be the feed-forward network in Figure 5.3 which is backward connected. Consider the balanced coloring $\bowtie$ in L given by $1 \bowtie 2 \bowtie 3$ and $4 \bowtie 5$. Note that $L/\bowtie$ is a feed-forward network.

Using Proposition 5.3.10, we know that L can be obtained from $L/\bowtie$ by a lift that creates a new layer and lifts inside the layers. In fact, we can see that L is given by a lift of $L/\bowtie$ that creates a new layer, then a lift on the second layer and finally a lift inside the third layer. ♢

The lift in Figure 5.5 shows that Proposition 5.3.10 does not hold if the lift is not backward connected.

In the rest of this section, we make two remarks that will be useful in the next sections. First, we see that a lift inside a layer can be further decomposed into simpler lifts. This will allow us to consider simpler lifts.

Definition 5.3.12. Let N be a network and L a lift of N. We say that L is the *split* of a cell c in N into cells $c_1, c_2, \ldots, c_l$ in L, if the coloring $\bowtie$ in L given by $c_i \bowtie c_j$, for $1 \leq i, j \leq l$, is balanced, $L/\bowtie = N$ and $[c_i]_{\bowtie} = c$. ♢

The network in Figure 5.2 is a split of the cell 2 in the network B of Figure 5.1 into the cells 2, 3 and 4. The split of the cell 2 in B into the cells 2 and 4 gives the network C of Figure 5.1. And the split of the cell 2 in C into the cells 2 and 3 return back the network in Figure 5.2. Hence the lift inside the second layer from B to the lift network in Figure 5.2 is the composition of splits of a cell into two cells. Using Lemma 5.3.3, we can easily see that this is the case for every lift inside a layer.

Remark 5.3.13. A lift inside a layer is the composition of splits of a cell into two cells. $\Diamond$

Second, we prove that each lift of a feed-forward network is given by a unique balanced coloring, if the lifted network is backward connected. This statement will be useful to understand which solutions are lifted from the quotient system to the lift system. We start by looking at an example of a lift given by more than one balanced colorings. In this example, the lift is not backward connected.

Example 5.3.14. Consider the feed-forward network N in Figure 5.4 and the feed-forward network B in Figure 5.1. Take the following three balanced colorings in B: $\bowtie_1$ given by $4 \bowtie_1 5$; $\bowtie_2$ given by $4 \bowtie_2 6$; and $\bowtie_3$ given by $5 \bowtie_3 6$. Then $N = B/\bowtie_1 = B/\bowtie_2 = B/\bowtie_3$. $\Diamond$

Using the backward connectedness of the lifted network, we have the following lemma.

Lemma 5.3.15. *Let N be a feed-forward network and L a lift of N such that L is a backward connected feed-forward network. Let $\bowtie_1, \bowtie_2$ be two balanced colorings in L. If $L/\bowtie_1 = L/\bowtie_2 = N$, then $\bowtie_1 = \bowtie_2$.*

Proof. Let N be a feed-forward network and L a lift of N such that L is a backward connected feed-forward network. Let $\bowtie_1, \bowtie_2$ be two balanced colorings in L. Denote by $C_1, \dots, C_m$ the layers of N, by $(\sigma_i^N)_{i=1}^k$ the representative functions of N, by $C'_1, \dots, C'_n$ the layers of L and by $(\sigma_i^L)_{i=1}^k$ the representative functions of L. Suppose that $L/\bowtie_1 = L/\bowtie_2 = N$.

Since L is backward connected, we know that N is backward connected and $|C_m| = |C'_n| = 1$. So for $c \in C'_n$, we have that $[c]_{\bowtie_1} = [c]_{\bowtie_2}$. Suppose that $[c]_{\bowtie_1} = [c]_{\bowtie_2}$, for every $c \in C'_j$ where $j > 1$. Let $d \in C'_{j-1}$. Then there exist $1 \leq i \leq k$ and $c \in C'_j$ such that $\sigma_i^L(c) = d$. Thus

$$[d]_{\bowtie_1} = [\sigma_i^L(c)]_{\bowtie_1} = \sigma_i^N([c]_{\bowtie_1}) = \sigma_i^N([c]_{\bowtie_2}) = [\sigma_i^L(c)]_{\bowtie_2} = [d]_{\bowtie_2}.$$

And $[d]_{\bowtie_1} = [d]_{\bowtie_2}$, for every $d \in C'_{j-1}$. By induction, $[c]_{\bowtie_1} = [c]_{\bowtie_2}$, for every $c \in C'_j$ and $1 \leq j \leq n$. Hence $\bowtie_1 = \bowtie_2$. $\square$

Example 5.3.16. Let L be the network in Figure 5.3. Consider the balanced coloring $\bowtie$ in L given by $1 \bowtie 2 \bowtie 3$ and $4 \bowtie 5$ and the quotient network $N = L/\bowtie$. Note that L is backward connected for the cell 10. By Lemma 5.3.15, we know that $\bowtie$ is the unique balanced coloring such that $N = L/\bowtie$. $\Diamond$

5.4 Feed-forward systems

Given a network, a coupled cell system admissible by that network is a system that respects the network structure. In a coupled cell system, we view each cell of the network as a dynamical system whose dynamics depends on its own state and on the state of the cells that are coupled to it. In this section, we formalize coupled cell systems associated to a network, synchrony subspaces and steady-state bifurcations, following [24, 15].

Let N be a network represented by the functions $(\sigma_i)_{i=1}^k$. For each cell c of the network, we associate a coordinate $x_c \in \mathbb{R}$. We say that $F : \mathbb{R}^{|N|} \to \mathbb{R}^{|N|}$ is an *admissible vector field* for N, if there is $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$ such that

$$(F(x))_c = f(x_c, x_{\sigma_1(c)}, \dots, x_{\sigma_k(c)}),$$

for every cell c of N. The admissible vector fields for N are defined by the functions $f : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}$. We denote by f^N the admissible vector field for N defined by f.

A *coupled cell system* associated to a network N is a system of ordinary differential equations

$$\dot{x} = f^N(x), \quad x \in \mathbb{R}^{|N|},$$

where $f^N : \mathbb{R}^{|N|} \to \mathbb{R}^{|N|}$ is an admissible vector field for N. When N is a feed-forward network, we refer to a coupled cell system associated to N as a *feed-forward system.*

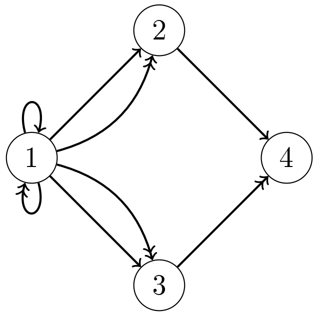

Figure 5.6: Feed-forward network with 3 layers.

Example 5.4.1. Let N be the feed-forward network in Figure 5.6. A feed-forward system associated to N has the following form

$$\begin{cases} \dot{x}_1 = f(x_1, x_1, x_1) \\ \dot{x}_2 = f(x_2, x_1, x_1) \\ \dot{x}_3 = f(x_3, x_1, x_1) \\ \dot{x}_4 = f(x_4, x_2, x_3) \end{cases},$$

where $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth function that defines the dynamics of each cell. For example the variable x_4, that corresponds to the cell 4, depends on the variable x_2 and x_3, which correspond to its input cells 2 and 3. $\diamondsuit$

In order to study steady-state bifurcation, we will assume that the system has an equilibrium at a full-synchronized equilibrium. Without loss of generality, we assume that this equilibrium is the origin.

Let $C_1, \ldots, C_m$ be the layers of N. For every feed-forward system f^N associated to N, the Jacobian matrix at the origin has the form

$$J_f^N = \begin{bmatrix} (\sum_{i=0}^k f_j)I_1 & 0 & 0 & \ldots & 0 & 0 \\ R_2 & f_0 I_2 & 0 & \ldots & 0 & 0 \\ 0 & R_3 & f_0 I_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & f_0 I_{m-1} & 0 \\ 0 & 0 & 0 & \ldots & R_m & f_0 I_m \end{bmatrix},$$

where

$$f_i := \frac{\partial f}{\partial x_i}(0, 0 \ldots, 0),$$

$0 \leq i \leq k$, I_j is the identity matrix of size $|C_j| \times |C_j|$, $j = 1, \ldots, m$, R_l is a $|C_l| \times |C_{l-1}|$-matrix, $l = 2, \ldots, m$. The eigenvalues of J_f^N are

$$\sum_{i=0}^{k} f_j \qquad \text{and} \qquad f_0.$$

The Jacobian matrix of a feed-forward system at a full synchrony equilibrium has only those two eigenvalues which we call the valency and the internal dynamics, respectively.

A *polydiagonal subspace* is a subspace of $\mathbb{R}^{|N|}$ given by the equalities of some cell coordinates. Given a coloring $\bowtie$ on the set of cell of N, the polydiagonal subspace associated to $\bowtie$ is

$$\Delta_{\bowtie} := \{x : c \bowtie d \Rightarrow x_c = x_d\} \subseteq \mathbb{R}^{|N|}.$$

Note that any polydiagonal subspace of $\mathbb{R}^{|N|}$ defines a unique coloring on the set of cells of N.

Given a function $G : \mathbb{R}^{|N|} \to \mathbb{R}^{|N|}$ and a subset $\Delta \subseteq \mathbb{R}^{|N|}$, we say that Δ is *invariant* by G if $G(\Delta) \subseteq \Delta$. A *synchrony subspace* of a network N is a polydiagonal subspace of $\mathbb{R}^{|N|}$ that is invariant by any admissible vector field of N. There is a one-to-one correspondence between balanced colorings $\bowtie$ and synchrony subspaces $\Delta_{\bowtie}$. See [15, Theorem 4.3]. More specifically, the polydiagonal $\Delta_{\bowtie}$ associated to a coloring $\bowtie$ is a synchrony subspace of N if and only if the coloring $\bowtie$ is balanced.

Since a synchrony subspace $\Delta_{\bowtie}$ is invariant by every admissible vector field f^N of N, every coupled cell system of N given by f^N can be restricted to $\Delta_{\bowtie}$. Each restricted system is a coupled cell system of $N/\bowtie$ given by $f^{N/\bowtie}$. Moreover, given a solution $y(t) \in \mathbb{R}^{|N/\bowtie|}$ of the coupled cell system of $N/\bowtie$ given by $f^{N/\bowtie}$, we have that $x(t) = (x_c(t))$, where $x_c(t) = y_{[c]_\bowtie}(t)$ is a solution of the coupled cell system of N given by f^N. See [15, Theorem 5.2].

Example 5.4.2. Consider the network N in Figure 5.6 and the general form of a feed-forward system associated to N given in Example 5.4.1. It is easy to see that the polydiagonal subspace $\Delta_1 = \{x_2 = x_3\}$ is flow-invariant for every such feed-forward system. This synchrony subspace corresponds to the balanced coloring given by $2 \bowtie 3$ and the quotient network $Q = N/\bowtie$ is the feed-forward network with 3 layers and one cell in each layer.

Let $y = (y_1, y_2, y_4) : \mathbb{R} \to \mathbb{R}^3$ be a solution of $\dot{y} = f^Q(y)$, a feed-forward system of Q. Then we can lift this solution to a solution of the corresponding feed-forward system of N, $\dot{x} = f^N(x)$. The lifted solution has the form $x(t) = (y_1(t), y_2(t), y_2(t), y_4(t))$. $\Diamond$

Now we define the classes of feed-forward systems with a steady-state bifurcation from a full synchronized equilibrium.

Let $G : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a family of smooth vector fields, $d > 0$ and the corresponding dynamical systems, depending on a parameter λ,

$$\dot{x} = G(x, \lambda). \tag{5.1}$$

Consider an equilibrium (x^*, λ^*) of (5.1), i.e., $G(x^*, \lambda^*) = 0$. The family of dynamical systems (5.1) suffers a *local bifurcation* at (x^*, λ^*) if for every neighborhoods U_x and U_λ of x^* and λ^*, respectively, there exists $\lambda_1, \lambda_2 \in U_\lambda$ such that the family (5.1) at λ_1 and λ_2 have different topological structures (different stability/number of equilibrium points or periodic orbits, etc.). A necessary condition for a local bifurcation to occur is that the Jacobian of G at (x^*, λ^*), $DG_{(x^*,\lambda^*)}$, has an eigenvalue with zero real part. We focus on steady-state bifurcations and we say that a *steady-state bifurcation* at (x^*, λ^*) occurs if the number of equilibrium points in a neighborhood of x^* changes when the parameter λ crosses λ^*. A necessary condition for the occurrence of a steady-state bifurcation at (x^*, λ^*) is that 0 is an eigenvalue of $DG_{(x^*,\lambda^*)}$.

In order to study the steady-state bifurcations of a family of coupled cell systems associated to N from a fully synchronous equilibrium at $\lambda = 0$, we consider a family of smooth functions $f : \mathbb{R} \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$ such that

$$f(0, 0, \ldots, 0, \lambda) = 0,$$

for every $\lambda \in \mathbb{R}$. We denote by $\mathcal{V}(N)$ the set of those functions. The set of functions $f \in \mathcal{V}(N)$ such that a steady-state bifurcation occurs at $(0, 0)$ for f^N is given by the union of the following sets

$$\mathcal{V}_k(N) := \{f \in \mathcal{V}(N) : \sum_{i=0}^{k} f_j = 0\}, \qquad \mathcal{V}_0(N) := \{f \in \mathcal{V}(N) : f_0 = 0\}.$$

Thus $\mathcal{V}_k(N)$ denotes the set of functions with a bifurcation condition associated with the *valency* of N and $\mathcal{V}_0(N)$ the set of functions with a bifurcation condition associated with the *internal dynamics* of the cells.

Next, we define equilibrium branches of a coupled cell system. Since our study is local, we define branches using germs. A germ is a class of functions with the same values in some neighborhood of the origin.

We say that $D \subseteq \mathbb{R}$ is a *domain* if D has one of the following forms: $]-\lambda_0, 0]$; $]-\lambda_0, \lambda_0[$; or $[0, \lambda_0[$, for some $\lambda_0 > 0$. Let D_1, D_2 be domains. We say that two smooth functions $b_1 : D_1 \to \mathbb{R}^{|N|}$ and $b_2 : D_2 \to \mathbb{R}^{|N|}$ are *germ equivalents* if there exists an open neighborhood U of 0 such that $U \cap D_1 \cap D_2 \neq \{0\}$ and $b_1(\lambda) = b_2(\lambda)$, for every $\lambda \in U \cap D_1 \cap D_2$. The previous relation is not transitive, so we consider its closer by transitivity. Given a smooth function b, we use the term *germ* b to refer to a representative element of the equivalence class of b with respect to germ equivalence.

Let D be a domain. We say that a germ $b : D \to \mathbb{R}^{|N|}$ is an *equilibrium branch of f on N*, if

$$f^N(b(\lambda), \lambda) = 0,$$

for every $\lambda \in D$. Since $f(0, 0, \ldots, 0, \lambda) = 0$ for every λ, we have that $x(\lambda) = (0, \ldots, 0)$ is an equilibrium branch of f on N, called *the trivial branch of f on N*. The equilibrium branches of f on N different from trivial branch are called the *bifurcation branches of f on N*. We define the set of equilibrium branches of f on N

$$\mathcal{B}(N, f) = \{b : D \to \mathbb{R}^{|N|} : b \text{ is an equilibrium branch of } f \text{ on } N\}.$$

5.5 Steady-state bifurcations for FFNs associated with the valency

In this section, we study the bifurcation branches on a feed-forward system with a bifurcation condition associated with the valency. This corresponds to solve the equation

$$f^N(x, \lambda) = 0$$

in a neighborhood of the origin. By the feed-forward structure, we know that the solution for each cell in the first layer is independent of the others cells. Using the Implicit Function Theorem, we see that the solution on the first layer determinates the solution on the other layers, and the dynamics in each cell of the first layer has a transcritical bifurcation.

Proposition 5.5.1. *Let N be a feed-forward network with layers $C_1, \ldots, C_m$. Let $f \in \mathcal{V}_k(N)$. Then, generically, there are $2^{|C_1|}$ equilibrium branches of f on N. Moreover, every equilibrium branch is uniquely determined by its value at the cells of the first layer C_1.*

Proof. Let N be a feed-forward network with layers $C_1, \ldots, C_m$. Let $f \in \mathcal{V}_k(N)$. Generically, assume that $f_0 \neq 0$, $\sum_{i,j=0}^{k} f_{ij} \neq 0$ and $\sum_{i=0}^{k} f_{i\lambda} \neq 0$, where f_{ij} is the second order partial derivatives of $f(x_0, x_1, \ldots, x_k, \lambda)$ at $(0, 0, \ldots, 0, 0)$ with respect to x_i and x_j, and $f_{i\lambda}$ is the second order partial derivatives of f at $(0, 0, \ldots, 0, 0)$ with respect x_i and λ, for $0 \leq i, j \leq k$.

The equilibrium branches of f on N are given by the solutions of

$$f^N(x, \lambda) = 0,$$

in a neighborhood of the origin. The Taylor expansion of f at $(0, 0, \ldots, 0, 0)$ is given by

$$f(x, x_1, \ldots, x_k, \lambda) = \sum_{i=0}^{k} f_i x_i + \sum_{i=0}^{k} f_{i\lambda} x_i \lambda + \sum_{i,j=0}^{k} \frac{f_{ij}}{2} x_i x_j + h.o.t.,$$

where $h.o.t$ denotes high order terms.

For $c \in C_1$, we have that

$$\begin{aligned} f_c^N(x, \lambda) = 0 \Leftrightarrow & f(x_c, x_c, \ldots, x_c, \lambda) = 0. \\ \Leftrightarrow & x_c \lambda \sum_{i=0}^{k} f_{i\lambda} + x_c^2 \sum_{i,j=0}^{k} \frac{f_{ij}}{2} + h.o.t. = 0 \\ \Leftrightarrow & x_c = 0 \ \vee \ \lambda \sum_{i=0}^{k} f_{i\lambda} + x_c \sum_{i,j=0}^{k} \frac{f_{ij}}{2} + h.o.t. = 0 \end{aligned}$$

Using the Implicit Function Theorem, there exist $\lambda_0 > 0$ and a germ $\beta :]-\lambda_0, \lambda_0[\to \mathbb{R}$ such that $\beta(0) = 0$ and

$$f(x_c, x_c, \ldots, x_c, \lambda) = 0 \Leftrightarrow x_c = 0 \ \vee \ x_c = \beta(\lambda), \qquad -\lambda_0 < \lambda < \lambda_0.$$

Denote by D the set of cells $C_2 \cup \cdots \cup C_m$. Since $f_0 \neq 0$, the matrix $\left[\partial f_d^N / \partial x_{d'}\right]_{d,d' \in D}$is invertible. By the Implicit Function Theorem, there exist $\lambda_0' > 0$ and $W : \mathbb{R}^{|C_1|} \times]-\lambda_0', \lambda_0'[\to \mathbb{R}^{|D|}$ such that $\lambda_0' \leq \lambda_0$ and

$$\begin{aligned} f^N(x, \lambda) = 0 \Leftrightarrow & \left(\bigwedge_{c \in C_1} f(x_c, x_c, \ldots, x_c, \lambda) = 0 \right) \ \wedge \ x_D = W(x_{C_1}, \lambda). \\ \Leftrightarrow & \left(\bigwedge_{c \in C_1} [x_c = 0 \ \vee \ x_c = \beta(\lambda)] \right) \ \wedge \ x_D = W(x_{C_1}, \lambda). \end{aligned}$$

for $-\lambda_0' < \lambda < \lambda_0'$.

Therefore any equilibrium branch is uniquely determined by its value at the cells of the first layer C_1 and each cell of C_1 has one of two possible values. So there are $2^{|C_1|}$ equilibrium branches. $\square$

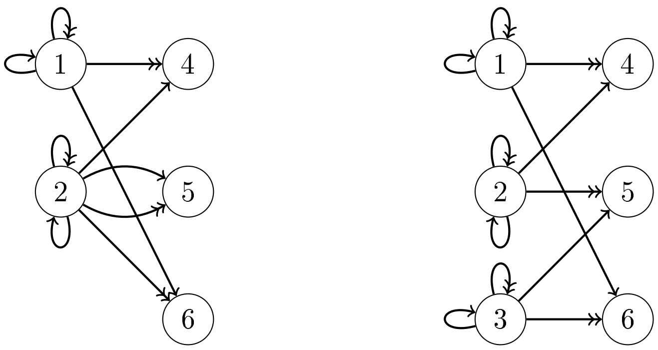

Figure 5.7: The feed-forward network on the right is a lift of the network on the left. This is a lift inside the first layer. There are different balanced colorings on the right network such that the left network is the quotient network associated to those colorings.

Example 5.5.2. Let L be the feed-forward network on the right of Figure 5.7. Consider a generic feed-forward system with a bifurcation condition associated to the valency. By Proposition 5.5.1, there are 8 equilibrium branches. The equilibrium branches must synchronize for at least two cells in the first layer. Since each cell in the first layer has two possible values and there are 3 cells in the first layer.

Consider the following colorings in L: $\bowtie_1$ given by $1 \bowtie_1 2$; $\bowtie_2$ given by $2 \bowtie_2 3$; and $\bowtie_3$ given by $1 \bowtie_3 3$. They are balanced and the network on the left of Figure 5.7 is the quotient of L by any of the colorings.

Every equilibrium branch in L corresponds to a bifurcation branch in the quotient network. There are more bifurcation branches in L, however, they are copies of bifurcation branches in the quotient network. $\Diamond$

5.6 Steady-state bifurcations for FFNs associated with the internal dynamics

Now, we study the bifurcation branches of a feed-forward system with a bifurcation condition associated to the internal dynamics. Layer by layer, we solve the equation

$$f^N(x, \lambda) = 0$$

in a neighborhood of the origin. In the first layer, there exists no bifurcation and the trivial solution is the unique solution. The inputs of each cell on the second layer are from cells in the first layer. Using the Implicit Function Theorem, we can see that the cells in the second layer have a transcritical bifurcation. Fixing a solution in the first two layers, we solve the equation for the cells in the following layer

$$f(x_c, x_{c_1}, \dots, x_{c_k}, \lambda) = 0,$$

where $c_1, \dots, c_k$ are the input cells of c which belong to the previous layer and have a fixed solution value. In order to solve this equation, we use an appropriate change of coordinates that was used in [20]. The solutions of this equation have a predictable growth-rate and slopes.

Definition 5.6.1 ([20, Definition 2.2])**.** Let $b : D \to \mathbb{R}$ be a germ and $D = [0, \lambda_0[$ or $D =]-\lambda_0, 0]$. If $b = 0$, we say that b_c *has square-root-order* -1 *and slope* 0. Otherwise, we say that b *has square-root-order* p *and slope* s and write that $b \sim \mathcal{O}(2^{-p})$, if p is the smallest non-negative integer such that there is a smooth function $b^* : [0, \lambda_0^{2^{-p}}[\to \mathbb{R}$ satisfying

$$b(\lambda) = b^*(|\lambda|^{2^{-p}}), \quad s = \lim_{\substack{|\lambda| \searrow 0 \\ \lambda \in D}} \frac{b^*(|\lambda|)}{\lambda} \neq 0. \qquad \diamondsuit$$

In previous studies of bifurcation in feed-forward networks, different authors have noticed that the layers act as amplifiers, [14, 20]. We will also see that the square-root-order of a bifurcation branch increases from a layer to the next layer.

In the next two lemmas, we determine the square-root-order and slope of a solution to $f(x, x_1, \dots, x_k, \lambda) = 0$, when the square-root-orders and slopes of the inputs $x_1, \dots, x_k$ are known. In the first lemma, we consider inputs that are defined for positive values of the parameter λ.

Lemma 5.6.2. *Let $f \in \mathcal{V}_0(N)$ generic, $y : [0, \lambda_0[\to \mathbb{R}^k$ a germ, $p_1, \dots, p_k$ and $s_1, \dots, s_k$ such that y_i has square-root-order p_i and slope s_i for $1 \leq i \leq k$. Suppose that $p := \max\{p_1, \dots, p_k\} \geq 0$ and define*

$$A := \left\{ i : y_i \sim \mathcal{O}(2^{-p}) \right\}, \qquad Z = \sum_{i \in A} \frac{f_i s_i}{f_{00}}.$$

(i) If $Z < 0$, then there exist $0 < \lambda_0^* < \lambda_0$ and germs $b^+, b^- : [0, \lambda_0^*[\to \mathbb{R}$ such that $b^{\pm}$ have square-root-order $p + 1$ and slope $\pm\sqrt{-2Z}$, and

$$f(x, y(\lambda), \lambda) = 0 \Leftrightarrow x = b^{\pm}(\lambda), \qquad 0 < \lambda < \lambda_0^*$$

(ii) If $Z > 0$, then the equation $f(x, y(\lambda), \lambda) = 0$ has only the trivial solution $(x, \lambda) = (0, 0)$.

Proof. Let $f \in \mathcal{V}_0(N)$, $y : [0, \lambda_0[\to \mathbb{R}^k$, $p_1, \dots, p_k$ and $s_1, \dots, s_k$ such that y_i has square-root-order p_i and slope s_i for $1 \leq i \leq k$. Suppose that $p := \max\{p_1, \dots, p_k\} \geq 0$ and define

$$A := \left\{ i : y_i \sim \mathcal{O}(2^{-p}) \right\}, \qquad Z = \sum_{i \in A} \frac{f_i s_i}{f_{00}}.$$

The Taylor expansion of f at the origin is

$$f(x,x_1,\ldots,x_k,\lambda)=\sum_{i=1}^{k} f_i x_i+\frac{f_{00}}{2}x^2+f_{0\lambda}x\lambda+\sum_{i=1}^{k} f_{i\lambda}x_i\lambda+ \\ +\sum_{i=1}^{k} f_{0i}x_i x+\sum_{i,j=1}^{k}\frac{f_{ij}}{2}x_i x_j+h.o.t..$$

For $\lambda \geq 0$, consider the following transformation of variables

$$\mu=\lambda^{2^{-(p+1)}}, \qquad x=\mu z, \qquad y_i(\lambda)=\lambda^{2^{-p_i}}w_i(\mu).$$

Then

$$w_i(0)=\lim_{\lambda\searrow 0}\frac{y_i(\lambda)}{\lambda^{2^{-p_i}}}=s_i, \qquad \lambda=\mu^{2^{(p+1)}}, \qquad y_i(\lambda)=\mu^{2^{(p+1-p_i)}}w_i(\mu).$$

Moreover $p-p_i=0$, if $i\in A$, and $p-p_i>0$, otherwise. Using the transformation of variables and the Taylor expansion of f, we obtain that

$$\begin{aligned} f(x,y(\lambda),\lambda)=0 \Leftrightarrow & \sum_{i=1}^{k} f_i y_i(\lambda)+\frac{f_{00}}{2}x^2+f_{0\lambda}x\lambda+\sum_{i=1}^{k} f_{i\lambda}y_i(\lambda)\lambda+ \\ & +\sum_{i=1}^{k} f_{0i}y_i(\lambda)x+\sum_{i,j=1}^{k}\frac{f_{ij}}{2}y_i(\lambda)y_j(\lambda)+h.o.t.=0 \\ & \Leftrightarrow \sum_{i\in A} f_i\mu^2 w_i(0)+\frac{f_{00}}{2}\mu^2 z^2+h.o.t.=0 \\ & \Leftrightarrow \mu^2\left(\sum_{i\in A} f_i w_i(0)+\frac{f_{00}}{2}z^2+h.o.t.\right)=0 \\ & \Leftrightarrow \mu=0 \ \vee \ \sum_{i\in A} f_i s_i+\frac{f_{00}}{2}z^2+h.o.t.=0. \end{aligned}$$

Define

$$h(z,\mu)=\sum_{i\in A} f_i s_i+f_{00}z^2/2+h.o.t..$$

If $Z<0$, we have that $h(\pm\sqrt{-2Z},0)=0$ and $h_z(\pm\sqrt{-2Z},0)\neq 0$. By the Implicit Function Theorem, there exist a neighborhood U of 0 and functions $z^+,z^-:U\to\mathbb{R}$ such that

$$h(z,\mu)=0 \Leftrightarrow z=z^{\pm}(\mu), \qquad z^{\pm}(\mu)=\pm\sqrt{-2Z}+h.o.t.$$

Let $0<\lambda_0^*<\lambda_0$ and $b^+,b^-:[0,\lambda_0^*[\to\mathbb{R}$ such that $[0,(\lambda_0^*)^{2^{-(p+1)}}[\subseteq U$ and

$$b^{\pm}(\lambda)=\mu z^{\pm}(\mu)=\pm\sqrt{-2Z}\lambda^{2^{-(p+1)}}+h.o.t.\sim\mathcal{O}(2^{-(p+1)}).$$

Then $b^{\pm}$ have square-root-order $p+1$ and slope $\pm\sqrt{-2Z}$, and

$$f(x, y(\lambda), \lambda) = 0 \Leftrightarrow \mu = 0 \ \vee \ z = z^{\pm}(\mu) \Leftrightarrow \mu z = \mu z^{\pm}(\mu) \Leftrightarrow x = b^{\pm}(\lambda).$$

This proves (i).

If $Z > 0$, then $h(z, 0)$ is always positive, when $f_{00} > 0$, or it is always negative, when $f_{00} < 0$. So there is no solution to the equation $h(z, 0) = 0$. And the equation $f(x, y(\lambda), \lambda) = 0$ has only the trivial solution $(x, \lambda) = (0, 0)$, proving (ii). □

In the second lemma, we look to the solution of $f(x, x_1, \ldots, x_k, \lambda) = 0$ when the inputs solutions $(x_1, \ldots, x_k)$ are defined for $\lambda < 0$. The proof is very similar to the previous one and it is omitted.

Lemma 5.6.3. *Let $f \in \mathcal{V}_0(N)$ generic, $y :]-\lambda_0, 0] \to \mathbb{R}^k$ a germ, $p_1, \ldots, p_k$ and $s_1, \ldots, s_k$ such that y_i has square-root-order p_i and slope s_i for $1 \le i \le k$. Suppose that $p := \max\{p_1, \ldots, p_k\} \ge 0$ and define*

$$A := \left\{i : y_i \sim \mathcal{O}(2^{-p})\right\}, \qquad Z = \sum_{i \in A} \frac{f_i s_i}{f_{00}}.$$

(i) If $Z > 0$, then there exist $\lambda_0 < \lambda_0^* < 0$ and germs $b^+, b^- :]-\lambda_0^*, 0] \to \mathbb{R}$ such that $b^{\pm}$ have square-root-order $p+1$ and slope $\mp\sqrt{2Z}$, and

$$f(x, y(\lambda), \lambda) = 0 \Leftrightarrow x = b^{\pm}(\lambda), \qquad 0 > \lambda > \lambda_0^*.$$

(ii) If $Z < 0$, then the equation $f(x, y(\lambda), \lambda) = 0$ has only the trivial solution $(x, \lambda) = (0, 0)$.

We return to the network C of Figure 5.1 and calculate some bifurcation branches of a feed-forward system associated to this network with a bifurcation condition associated with the internal dynamics.

Example 5.6.4. Let C be the feed-forward network in Figure 5.1 with 3 layers $\{1\}, \{2, 3\}, \{4, 5, 6\}$ and f^C a feed-forward system with a bifurcation condition associated with the internal dynamics. In order to study the bifurcation branches, we need to solve the equation

$$f^C(x, \lambda) = 0$$

in a neighborhood of the origin. We solve this equation layer by layer and start by the first layer. For the cell 1, we need to solve the equation

$$f(x_1, x_1, x_1, \lambda) = 0.$$

By the Implicit Function Theorem, there exists only the trivial branch

$$x_1(\lambda) = 0,$$

for λ in a neighborhood of 0. This completes the study of the first layer and we proceed to the second layer.

Fixing the unique solution on the first layer, we look to the equation in one of the cells in the second layer. Take for example the cell 2, we have the following equation

$$f(x_2, 0, 0, \lambda) = 0.$$

The study of this equation is similar to the bifurcations with a condition associated to the valency. Assuming that $f_{00} \neq 0$ and $f_{0\lambda} \neq 0$, we conclude that there are two solutions

$$x_2(\lambda) = 0 \quad \vee \quad x_2(\lambda) = b^0(\lambda) = -\frac{2f_{0\lambda}}{f_{00}}\lambda + h.o.t. \tag{5.2}$$

for λ in a neighborhood of 0. Repeat the same procedure for the other cell in the second layer. Then there are 4 solutions on the second layer

$$\begin{cases} x_2(\lambda) = 0 \quad \vee \quad x_2(\lambda) = b^0(\lambda) \\ x_3(\lambda) = 0 \quad \vee \quad x_3(\lambda) = b^0(\lambda) \end{cases}$$

Before we study the next layer, note that the branch $b^0(\lambda)$ has square-root-order 0 and slope $-2f_{0\lambda}/f_{00}$.

Now, we fix one of the solutions for the previous layer and solve the equation in the next layer using Lemmas 5.6.2 and 5.6.3. For example we fix the solution $(x_1, x_2, x_3) = (0, 0, b^0)$ and look to the equation of the cell 5

$$f(x_5, x_2, x_3, \lambda) = 0 \Leftrightarrow f(x_5, 0, b^0(\lambda), \lambda) = 0.$$

Remember that x_2 has square-root-order -1 and x_3 has square-root-order 0. By the Lemmas 5.6.2 and 5.6.3, we know that any solution has square-root-order 1. If $f_2 f_{0\lambda} > 0$, Lemma 5.6.2 implies that there are two solutions with square-root-order 1

$$x_5 = s_1^+(\lambda) \quad \vee \quad x_5 = s_1^-(\lambda),$$

where $s_1^\pm$ has slope $\pm 2\sqrt{f_2 f_{0\lambda}}/f_{00}$ and λ is restricted to positive values, $\lambda \geq 0$. If $f_2 f_{0\lambda} < 0$, we apply Lemma 5.6.3 and obtain two solutions with square-root-order 1 and slope $\pm 2\sqrt{|f_2 f_{0\lambda}|}/f_{00}$, however the solutions will be defined for negative values of λ, $\lambda \leq 0$.

We still need to solve the equations for the cell 4 and 6

$$\begin{cases} f(x_4, x_2, x_2, \lambda) = 0 \\ f(x_6, x_3, x_2, \lambda) = 0 \end{cases} \Leftrightarrow \begin{cases} f(x_4, 0, 0, \lambda) = 0 \\ f(x_6, b^0, 0, \lambda) = 0 \end{cases}$$

As before and using the Implicit Function Theorem, we note that the first equation has two solutions

$$x_4(\lambda) = 0 \quad \vee \quad x_4(\lambda) = b^0(\lambda),$$

defined in a neighborhood of 0. For the second equation, we need to use one of the previous Lemmas. If $f_1 f_{0\lambda} > 0$, we obtain two solutions with square-root-order 1

$$x_6 = s_2^+(\lambda) \quad \vee \quad x_6 = s_2^-(\lambda),$$

slopes $\pm 2\sqrt{f_1 f_{0\lambda}}/f_{00}$ which are defined for λ positive. If $f_1 f_{0\lambda} < 0$, we also have two solutions but they are defined for λ negative.

We have computed the possible solutions on each cell of the third layer when we fix a solution on the first two layers. Now, we need to patch the solutions into a solution of the network system. In order to do that we define the solution in the intersection of the domain of each cell solution. However, the solutions can be defined on different sides of $\lambda = 0$. When this occurs, it does not correspond to a solution of the whole system. For example take $f_2 f_{0\lambda} < 0$ and $f_1 f_{0\lambda} > 0$, then the solutions for x_5 are defined for λ positive and x_6 is defined for λ negative. Thus, it does not correspond to a solution of the whole system.

Fixing the solution $(x_1, x_2, x_3) = (0, 0, b^0)$. If $f_1 f_2 > 0$, we have the following solutions

$$\begin{cases} x_4 = 0 \quad \vee \quad x_4 = b^0 \\ x_5 = s_1^+(\lambda) \quad \vee \quad x_5 = s_1^-(\lambda) \\ x_6 = s_2^+(\lambda) \quad \vee \quad x_6 = s_2^-(\lambda), \end{cases}$$

where $s_i^\pm$ have square-root-order 1 and it is defined for $\lambda \geq 0 (\lambda \leq 0)$, when $f_1 f_{0\lambda} > 0 (f_1 f_{0\lambda} < 0$, respectively). If $f_1 f_2 < 0$, then there is no solution to the equation in the next layer and there is no bifurcation branch b of f on C such that $(b_1, b_2, b_3) = (0, 0, b^0)$.

Repeating this process for the other solutions on the first two layers, we find all bifurcation branches of f on C. We can check that there is no bifurcation branch of f on C without synchrony on the cells 2 and 3, if $f_1 f_2 < 0$. $\Diamond$

Finally, we represent all bifurcation branches using their growth-rates and slopes. Let N be a feed-forward network and $f \in \mathcal{V}_0(N)$. We define the function that assign for each bifurcation branch a symbol

$$\begin{aligned} \Theta : \mathcal{B}(N, f) &\to \{-1, 0, 1\} \times \mathbb{Z}^{|N|} \times \mathbb{R}^{|N|} \\ (b_c)_c &\mapsto (\delta, (p_c)_c, (s_c)_c), \end{aligned}$$

where b_c has square-root-order p_c and slope s_c, for each cell c of N, and δ indicates the domain of the branch b. If b is defined in a neighborhood of 0, then $\delta = 0$. If b is only defined for $\lambda > 0$, then $\delta = 1$. Finally, if b is only defined for $\lambda < 0$, then $\delta = 1$.

Example 5.6.5. Returning to the feed-forward network C of Figure 5.1, we give the complete description of the bifurcation branches using the symbols $(\delta, (p_c)_c, (s_c)_c)$. There is the trivial branch

$$(0, (-1, -1, -1, -1, -1, -1), (0, 0, 0, 0, 0, 0)).$$

There are 7 branches with square-root-order 0

$$(0, (-1, -1, -1, -1, -1, 0), (0, 0, 0, 0, 0, s_0)),$$
$$(0, (-1, -1, -1, -1, 0, -1), (0, 0, 0, 0, s_0, 0)),$$
$$(0, (-1, -1, -1, 0, -1, -1), (0, 0, 0, s_0, 0, 0)),$$
$$(0, (-1, -1, -1, -1, 0, 0), (0, 0, 0, 0, s_0, s_0)),$$
$$(0, (-1, -1, -1, 0, -1, 0), (0, 0, 0, s_0, 0, s_0)),$$
$$(0, (-1, -1, -1, 0, 0, -1), (0, 0, 0, s_0, s_0, 0)),$$
$$(0, (-1, -1, -1, 0, 0, 0), (0, 0, 0, s_0, s_0, s_0)),$$

where $s_0 = -2f_{0\lambda}/f_{00}$. There are 8 branches with square-root-order 1

$$(\delta, (-1, 0, 0, 1, 1, 1), (0, s_0, s_0, \pm s_1, \pm s_1, \pm s_1)),$$

where $\delta = \text{Sign}((f_1 + f_2)f_{0\lambda}$ and $s_1 = 2\sqrt{|(f_1 + f_2)f_{0\lambda}|}/f_{00}$. If $f_1 f_2 > 0$, there are 16 more branches with square-root-order 1

$$(\delta, (-1, 0, -1, 1, 1, 1), (0, s_0, 0, \pm s_1, \pm s_2, \pm s_3)),$$
$$(\delta, (-1, -1, 0, -1, 1, 1), (0, 0, s_0, 0, \pm s_3, \pm s_2)),$$
$$(\delta, (-1, -1, 0, 0, 1, 1), (0, 0, s_0, s_0, \pm s_3, \pm s_2)),$$

where $s_2 = 2\sqrt{|f_1 f_{0\lambda}|}/f_{00}$ and $s_3 = 2\sqrt{|f_2 f_{0\lambda}|}/f_{00}$. $\Diamond$

The symbols which are the image of some bifurcation branch respect some rules. If $(\delta, (p_c)_c, (s_c)_c) \in \Theta(\mathcal{B}(N, f))$, then

Ω.1 $p_c = -1 \Leftrightarrow s_c = 0$,

This follows from Definition 5.6.1.

Ω.2 $\delta = 0 \Rightarrow \forall_c\ p_c \leq 0$,

The bifurcation branch is defined on an open neighborhood of 0 if and only if $\delta = 0$. By Lemmas 5.6.2 and 5.6.3, we know that every branch with a square-root-order greater than 0 is defined only on one side of $\lambda = 0$ and $\delta \neq 0$. So $\delta = 0$ implies that $p_c \leq 0$ for every cell c.

Ω.3 $p_c = -1 \Rightarrow \forall_i\ p_{\sigma_i(c)} = -1$,

Suppose that the branch is not trivial on one input of c, i.e., $p_{\sigma_i(c)} > -1$ for some i. By Lemmas 5.6.2 and 5.6.3, we have that $p_c > -1$.

Ω.4 $p_c > -1 \Rightarrow \forall_i \ p_{\sigma_i(c)} \le p_c - 1 \ \wedge \ \exists_{i'} \ p_{\sigma_{i'}(c)} = p_c - 1$,

Suppose that the branch has a square-root-order greater than $p_c - 1$ for some input of c, i.e., $p_{\sigma_i(c)} > p_c - 1$ for some i. By Lemmas 5.6.2 and 5.6.3, we obtain the contradiction that b_c has square-root-order greater than p_c. Supposing that every input of c has square-root-order less than $p_c - 1$, Lemmas 5.6.2 and 5.6.3 lead to an absurd.

Ω.5 $p_c = 0 \Rightarrow s_c = -\dfrac{2f_{0\lambda}}{f_{00}}$,

Assume that $p_c = 0$. By Ω.4, we know that $p_{\sigma_i(c)} = -1$ for every $1 \le i \le k$ and $b_{\sigma_i(c)} = 0$. In the running example, we saw that if a cell has trivial inputs, then there are two options. See equation (5.2). Since $p_c = 0$, we know that $b_c \neq 0$ and b_c has slope $-2f_{0\lambda}/f_{00}$. So $s_c = -2f_{0\lambda}/f_{00}$.

Ω.6 $p_c > 0 \Rightarrow s_c = \pm\sqrt{-\dfrac{2\delta}{f_{00}} \sum_{i \in A_c} f_i s_{\sigma_i(c)}}$, where $A_c = \{i : p_{\sigma_i(c)} = p_c - 1\}$.

This follows from Lemma 5.6.2, if $\delta = 1$, and Lemma 5.6.3, if $\delta = -1$.

Let $\Omega(N, f) \subseteq \{-1, 0, 1\} \times \mathbb{Z}^{|N|} \times \mathbb{R}^{|N|}$ be the set of symbols $(\delta, (p_c)_c, (s_c)_c) \in \{-1, 0, 1\} \times \mathbb{Z}^{|N|} \times \mathbb{R}^{|N|}$ satisfying Ω.1, Ω.2, Ω.3, Ω.4, Ω.5 and Ω.6. Next, we prove that Θ is a one-to-one correspondence between $\mathcal{B}(N, f)$ and $\Omega(N, f)$.

Proposition 5.6.6. *Let N be a feed-forward network and $f \in \mathcal{V}_0(N)$ generic. If $(\delta, (p_c)_c, (s_c)_c) \in \Omega(N, f)$, then there exists a unique $b \in \mathcal{B}(N, f)$ such that*

$$\Theta(b) = (\delta, (p_c)_c, (s_c)_c).$$

Proof. Let N be a feed-forward network represented by the function $(\sigma_i)_{i=1}^k$ and $f \in \mathcal{V}_0(N)$ generic.

Let $(\delta, (p_c)_c, (s_c)_c) \in \Omega(N, f)$. We construct the equilibrium branch b of f on N such that $\Theta(b) = (\delta, (p_c)_c, (s_c)_c)$. It follows from Ω.4 that $p_c = -1$ for every cell c in the first layer and $-1 \le p_c \le m - 2$, for every cell c of N.

Let c be a cell of N such that $p_c = -1$. Then $s_c = 0$, by Ω.1. Define b_c as the germ defined on an open neighborhood of 0 such that $b_c = 0$. Then b_c has square-root-order p_c and slope s_c. It follows from Ω.3 that

$$f(b_c, b_{\sigma_1(c)}, \ldots, b_{\sigma_k(c)}, \lambda) = f(0, 0, \ldots, 0, \lambda) = 0.$$

Let c be a cell of N such that $p_c = 0$. Then $s_c = -2f_{0\lambda}/f_{00}$, by Ω.5, and $p_{\sigma_i(c)} = -1$ for $1 \le i \le k$, by Ω.4. Define b_c as the germ b^0 defined in (5.2) on an open neighborhood of 0. Then b_c has square-root-order p_c, slope s_c and

$$f(b_c, b_{\sigma_1(c)}, \ldots, b_{\sigma_k(c)}, \lambda) = f(b^0, 0, \ldots, 0, \lambda) = 0.$$

The following germs are defined by induction on $p \geq 1$. Assuming that $b_{c'}$ is defined for every cell c' of N such that $p_{c'} < p$ and $b_{c'}$ has square-root-order $p_{c'}$ and slope $s_{c'}$, we define the germ b_c which has square-root-order p_c and slope s_c, for each cell c of N such that $p_c = p$. Since $p_c \leq m-2$, this process must terminate.

Let $p \geq 1$ and c be a cell of N such that $p_c = p$. By $\Omega.4$ and the assumption, $p_{\sigma_i(c)} < p$ and $b_{\sigma_i(c)}$ is defined for every $1 \leq i \leq k$. Consider the germ $y : D \to \mathbb{R}^k$ such that $y_i = b_{\sigma_i(c)}$ for every $1 \leq i \leq k$, and let b_c be the germ obtained in Lemma 5.6.2 (5.6.3), if $\delta = 1$ (-1, respectively), such that b_c has square-root-order p_c slope s_c and it is defined for positive (negative) values. It follows from $\Omega.6$ that there exists such germ and it is unique. Moreover,

$$f(b_c, b_{\sigma_1(c)}, \ldots, b_{\sigma_k(c)}, \lambda) = 0.$$

Define the germ $b = (b_c)_c : D \to \mathbb{R}^{|N|}$, where D is the intersection of the domains of each b_c. By construction $f^N(b(\lambda), \lambda) = 0$, so b is an equilibrium branch of f on N. Let $(\delta', (p'_c)_c, (s'_c)_c) := \Theta(b)$. By construction, $p'_c = p_c$ and $s'_c = s_c$, for every cell c. If $\delta = 0$, then $p_c \leq 0$ and $\delta' = 0$, by $\Omega.2$. If $\delta = \pm 1$, then there exists $p_c > 0$, by $\Omega.1$ and $\Omega.6$. If $\delta = 1$, then b_c is defined for positive values and $\delta' = 1$. Similar, if $\delta = -1$, then $\delta' = 1$. Therefore

$$\Theta(b) = (\delta, (p_c)_c, (s_c)_c).$$

We can see in each step of the construction of b that we choose the unique germ that respects the conditions of square-root-order, slope and be a solution to the equation. □

Next, we see how to use this result to find some bifurcation branches.

Let N be a feed-forward network with layers $C_1, \ldots, C_m$ and $f : \mathbb{R}^{k+1} \times \mathbb{R} \to \mathbb{R} \in \mathcal{V}_0(N)$ generic. Define

$$\tilde{\delta} = \operatorname{Sign}(f_{0\lambda} \sum_{i=1}^{k} f_i) = \frac{f_{0\lambda} \sum_{i=1}^{k} f_i}{\left| f_{0\lambda} \sum_{i=1}^{k} f_i \right|}, \quad \tilde{p}_1 = -1, \quad \tilde{s}_1 = 0, \tag{5.3}$$

and

$$\tilde{p}_j = j-2, \quad \tilde{s}_j = -\operatorname{Sign}(f_{0\lambda}) \frac{2|f_{0\lambda}|^{2^{-(j-2)}}}{f_{00}} \left| \sum_{i=1}^{k} f_i \right|^{1-2^{-(j-2)}}, \tag{5.4}$$

for $2 \leq j \leq m$. Now, for each $3 \leq r \leq m$, define $\delta^{r\pm} = \tilde{\delta}$,

$$p_c^{r\pm} = \tilde{p}_1, \quad s_c^{r\pm} = \tilde{s}_1, \qquad c \in C_1 \cup \cdots \cup C_{m-r+1},$$

$$p_c^{r\pm} = \tilde{p}_l, \quad s_c^{r\pm} = \tilde{s}_l \qquad c \in C_{m-r+l}, 2 \leq l \leq r-1,$$
$$p_c^{r\pm} = \tilde{p}_r, \quad s_c^{r\pm} = \pm\tilde{s}_r \qquad c \in C_m,$$

We also define $\delta^2 = 0$,

$$p_c^2 = -1, s_c^2 = 0, \quad c \in C_1 \cup \cdots \cup C_{m-1}, \quad p_c^2 = 0, s_c^2 = -\frac{2f_{0\lambda}}{f_{00}}, \quad c \in C_m,$$
$$\delta^1 = 0, \qquad p_c^1 = -1, \qquad s_c^1 = 0, \quad c \in C_1 \cup \cdots \cup C_m.$$

We can see that the symbols

$$(\delta^1, (p_c^1)_c, (s_c^1)_c),$$
$$(\delta^2, (p_c^2)_c, (s_c^2)_c),$$
$$(\delta^{r\pm}, (p_c^{r\pm})_c, (s_c^{r\pm})_c)$$

respect the rules $\Omega.1$, $\Omega.2$, $\Omega.3$, $\Omega.4$, $\Omega.5$ and $\Omega.6$, for $3 \leq r \leq m$. Therefore, they belong to $\Omega(N, f)$ and correspond to equilibrium branches of f on N, Proposition 5.6.6.

This means that the set $\mathcal{B}(N, f)$ contains the trivial equilibrium branch b^1, a bifurcation branch b^2 which have a square-root-order 0 and two bifurcation branches b^{r+}, b^{r-} with square-root-order $r-2$ for every $3 \leq r \leq m$. Moreover, those equilibrium branches have synchrony inside each layer. We summarize the previous in the following result.

Corollary 5.6.7. *Let N be a feed-forward network with m layers and $f \in \mathcal{V}_0(N)$ generic. For every $1 \leq r \leq m$, there exists $b \in \mathcal{B}(N, f)$ such that b has square-root order $r-2$. If $b \in \mathcal{B}(N, f)$, then b has square-root-order less or equal to $m-2$.*

In Example 5.6.5, we saw that the network C in Figure 5.1 respects the previous corollary.

Example 5.6.8. Let N be the feed-forward network C in Figure 5.1 with 3 layers. In Example 5.6.5, we saw that any feed-forward system on N with a bifurcation condition associated to the internal dynamics has equilibrium branches with square-root-order -1, 0 and 1. Moreover, every equilibrium branch has square-root-order less or equal to 1. ♢

5.7 Lifting bifurcation problem on FFNs

The bifurcation branches occurring in a quotient system are lifted to bifurcation branches occurring in a lift system. In this section, we study the lifting bifurcation problem which consists on understanding if every bifurcation branch occurring in a coupled cell system associated to a lift network is lifted from a bifurcation branch occurring in the coupled cell system associated to the quotient network.

Definition 5.7.1. Let N be a network and L a lift of N. We say that *a bifurcation branch b of f on L is lifted from N*, if there exists a balanced coloring $\bowtie$ in L such that $b \in \Delta_{\bowtie}$ and $N = L/\bowtie$. $\Diamond$

In the next proposition, we recover a well-known result about the bifurcation branches being inside a flow-invariant space which contains the center subspace. We present the proof here for completeness. Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be a linear operator from $\mathbb{R}^d$ to itself and $d > 0$. The *center subspace* of A is given by

$$\ker^*(A) = \{v \in \mathbb{R}^d : A^k v = 0 \text{ for some } k\}.$$

We denote the orthogonal complement with respect to the usual inner product of a subspace $B \subseteq \mathbb{R}^d$ by $B^\perp$.

Proposition 5.7.2. *Let $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a smooth function and $K \subseteq \mathbb{R}^d$ such that $\ker^*(DF_{(0,0)}) \subseteq K$, $F(0,0) = 0$ and $F(K, \lambda) \subseteq K$ for every $\lambda \in \mathbb{R}$. Suppose that there exists a function $x : D \to \mathbb{R}^d$ defined in a domain D such that $F(x(\lambda), \lambda) = 0$ for $\lambda \in D$. Then there exists a neighborhood U of 0 such that $x(\lambda) \in K$ for every $\lambda \in U \cap D$.*

Proof. Let $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ be a smooth function and $K \subseteq \mathbb{R}^d$ such that $\ker^*(DF_{(0,0)}) \subseteq K$, $F(0,0) = 0$ and $F(K, \lambda) \subseteq K$ for every $\lambda \in \mathbb{R}$. Note that $\mathbb{R}^d = K \oplus K^\perp$. Writing every element of $\mathbb{R}^d$ in its decomposition in K and $K^\perp$, $v = y + w$, where $y \in K$ and $w \in K^\perp$, there are $g : K \times K^\perp \times \mathbb{R} \to K$ and $h : K \times K^\perp \times \mathbb{R} \to K^\perp$ such that

$$\dot{v} = F(v, \lambda) \Leftrightarrow \begin{cases} \dot{y} = g(y, w, \lambda) \\ \dot{w} = h(y, w, \lambda) \end{cases} .$$

Hence

$$DF_{(0,0)} = \begin{bmatrix} D_y g_{(0,0)} & D_w g_{(0,0)} \\ D_y h_{(0,0)} & D_w h_{(0,0)} \end{bmatrix}.$$

Observe that $h(y, 0, \lambda) = 0$, because K is invariant. Then $D_y h_{(0,0)} = 0$ and $D_w h_{(0,0)}$ is invertible, since $\ker^*(DF_{(0,0)}) \subseteq K$. By the Implicit Function Theorem, there is $W : K \times \mathbb{R} \to K^\perp$ such that $W(0,0) = 0$ and $h(y, w, \lambda) = 0$ if and only if $w = W(y, \lambda)$.

From $h(y, 0, \lambda) = 0$, we have that $W(y, \lambda) = 0$. Therefore

$$F(y, w, \lambda) = 0 \Leftrightarrow g(y, 0, \lambda) = 0 \ \wedge \ w = 0.$$

Supposing that x is a solution to $F(v, \lambda) = 0$, we have that $x \in K$. $\square$

It follows that a necessary condition for the existence of a bifurcation branch on a lift network not lifted from the original network is that the center subspace of the coupled cell systems associated to the original network and the lift network have different dimensions.

Corollary 5.7.3. *Let N be a network, L a lift of N associated to the coloring $\bowtie$ and $f \in \mathcal{V}(N)$. If $\ker^*(J_f^N)$ and $\ker^*(J_f^L)$ have the same dimension, then every bifurcation branch of f in L belongs to $\Delta_{\bowtie}$ and it is lifted from N.*

We recall the dimension of the center subspace of J_f^N for feed-forward systems with a bifurcation condition associated to the valency and internal dynamics.

Remark 5.7.4. Let N be a feed-forward network with layers $C_1, \dots, C_m$.
(i) If $f \in \mathcal{V}_k(N)$, then the dimension of $\ker^*(J_f^N)$ is $|C_1|$.
(ii) If $f \in \mathcal{V}_0(N)$, then the dimension of $\ker^*(J_f^N)$ is $|C_2| + \cdots + |C_m|$. ♢

Example 5.7.5. Let N be the network on the left of Figure 5.7 and L the lift of N on the right of Figure 5.7.

Consider $f \in \mathcal{V}_0(N)$ with a bifurcation condition associated to the internal dynamics. The spaces $\ker^*(J_f^N)$ and $\ker^*(J_f^L)$ have the same dimension, the conclusion of Corollary 5.7.3 holds and every bifurcation branch of f on L is lifted from N.

Consider $f \in \mathcal{V}_k(N)$ with a bifurcation condition associated to the valency. Note that $\ker^*(J_f^N)$ has dimension 2 and $\ker^*(J_f^L)$ has dimension 3. We saw in Example 5.5.2 that every bifurcation branch of f on L is lifted from N. This example shows that the condition in Corollary 5.7.3 is sufficient but not necessary. ♢

5.7.1 Lifting bifurcation problem on FFNs associated with the valency

In this section, we study the lifting bifurcation problem for feed-forward systems determined by a regular function that has a bifurcation condition associated to the valency. We prove that a lift has an extra bifurcation branch if and only if the center subspace is bigger on the lift than on the quotient network.

Proposition 5.7.6. *Let N be a feed-forward network, $f \in \mathcal{V}_k(N)$ generic and L a feed-forward lift of N.*
(i) If L is a lift that creates new layers or a lift inside a layer, except the first layer, then every bifurcation branch of f on L is lifted from N.
(ii) If L is a lift inside the first layer and L is backward connected, then there is at least one bifurcation branch of f on L which is not lifted from N.

Proof. Let N be a feed-forward network, $f \in \mathcal{V}_k(N)$ generic and L a feed-forward lift of N. Denote by C_1 and by C_1' the first layer of N and L, respectively.

If L is a lift that creates new layers or a lift inside a layer, except the first, then $\ker^*(J_f^N)$ and $\ker^*(J_f^L)$ have the same dimension. Recall Remark 5.7.4. By Corollary 5.7.3, every bifurcation branch of f on L is lifted from N.

Suppose that L is a lift inside the first layer and L is backward connected. By Remark 5.3.13, we assume that L is the split of a cell $c \in C_1$ into two cells $c_1, c_2 \in C_1'$ and denote by $\bowtie$ the balanced coloring in L given by $c_1 \bowtie c_2$. By Lemma 5.3.15, $\bowtie$ is the unique balanced coloring such that $L/\bowtie = N$. By the proof of Proposition 5.5.1, we know that there exists a bifurcation branch $b \in \mathcal{B}(L, f)$ such that $b_{c_1} \neq b_{c_2}$. So $b \notin \Delta_{\bowtie}$ and it is not lifted from N. □

We give two examples where the previous result can be applied.

Example 5.7.7. Consider the networks in Figure 5.1 and $f \in \mathcal{V}_k(A)$ generic. The network B is a lift that creates a new layer from A. Moreover, the network C is a lift inside the second layer of B.

By Proposition 5.7.6 (i), every bifurcation branch of f on B is lifted from A. Again by Proposition 5.7.6 (i), every bifurcation branch of f on C is lifted from B. Thus every bifurcation branch of f on C is lifted from A. ♢

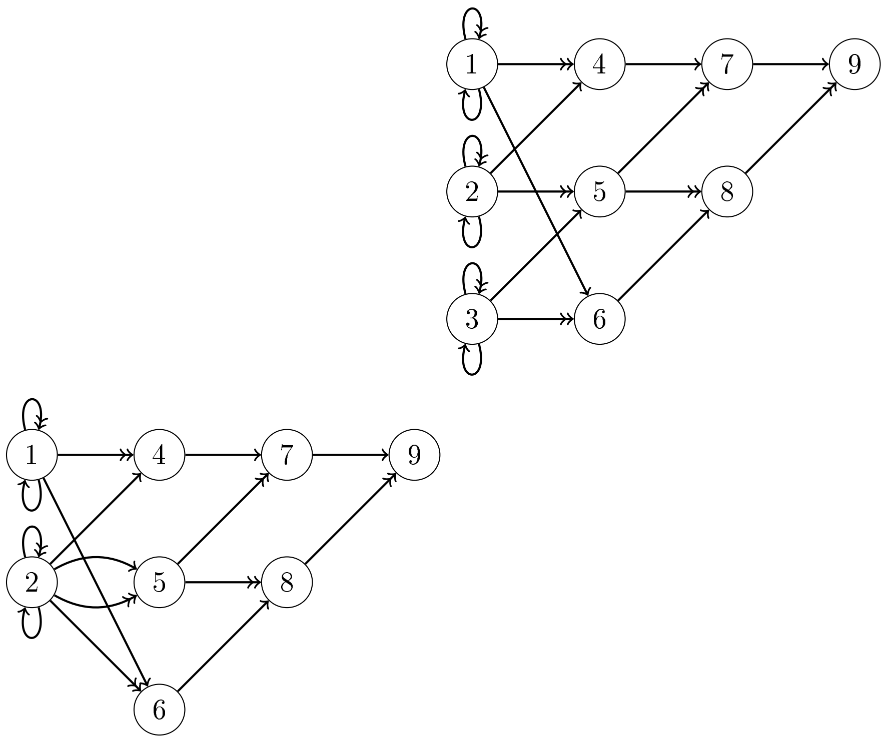

Figure 5.8: Lift inside the first layer from the first network to the second.

Example 5.7.8. Let N be the network on the left of Figure 5.8, L the lift inside the first layer described on the right of Figure 5.8 and $f \in \mathcal{V}_k(L)$.

Consider the balanced coloring $\bowtie$ in L given by $2 \bowtie 3$. By Lemma 5.3.15, this is the unique balanced coloring in L such that $L/\bowtie= N$, as L is backward connected. There exists a bifurcation branch $b \in \mathcal{B}(L, f)$ such that $b_2 \neq b_3$ and thus it is not lifted from N. This agrees with the Proposition 5.7.6 (*ii*). ◊

In Examples 5.5.2 and 5.7.5, we saw that Proposition 5.7.6 (*ii*) is not valid if the lift is not backward connected. In the above example, the lifted network has more bifurcation branches but they are copies of some bifurcation branches on the smaller network. The bifurcation branches of the smaller network can be lifted in multiple ways because there is more than one balanced coloring on the lift network that corresponds to the quotient network.

Example 5.7.9. Let N be the feed-forward network on the left of Figure 5.7, L the lift inside the first layer of N represented in the right of Figure 5.7 and $f \in \mathcal{V}_k(N)$ generic. In Example 5.5.2, we saw that L has three balanced colorings $\bowtie_1$, $\bowtie_2$ and $\bowtie_3$ such that $L/\bowtie_1= L/\bowtie_2= L/\bowtie_3= N$. Every bifurcation branch $b \in \mathcal{B}(L, f)$ is lifted from N using one of the previous colorings. ◊

5.7.2 Lifting bifurcation problem on FFNs associated with the internal dynamics

In this section, we study the lifting bifurcation problem for feed-forward systems determined by a regular function that has a bifurcation condition associated to the internal dynamics. We start by the lifts that create new layers and lifts inside the first layer. These cases do not depend on the feed-forward system and every bifurcation branch is lifted if and only if the center subspace in the lifted network is equal to the center subspace on the smaller network.

Proposition 5.7.10. *Let N be a feed-forward network, $f \in \mathcal{V}_0(N)$ generic and L a feed-forward lift of N.*
(i) If L is a lift inside the first layer, then every bifurcation branch of f on L is lifted from N.
(ii) If L is a lift that creates new layers, then there is a bifurcation branch of f on L which is not lifted from N.

Proof. Let N be a feed-forward network, $f \in \mathcal{V}_0(N)$ generic and L a feed-forward lift of N.

Suppose that L is a lift inside the first layer. Then the center subspace of J_f^N and J_f^L have the same dimension. By Corollary 5.7.3, every bifurcation branch of f on L is lifted from N.

Suppose that L is a lift that creates new layers. By Corollary 5.6.7, there exists a bifurcation branch $b \in \mathcal{B}(L, f)$ having square-root-order greater than any bifurcation branch of f on N. Hence there is a bifurcation branch of f on L which is not lifted from N. □

We apply the previous result to the networks in Figures 5.1 and 5.7.

Example 5.7.11. Let N the feed-forward network on the left of Figure 5.7 and L the lift inside the first layer of N given on the right of Figure 5.7. Consider $f \in \mathcal{V}_0(N)$. Using Proposition 5.7.10 (i), we know that every bifurcation branch of f on L is lifted from N. ◊

Example 5.7.12. Consider the networks A and B in Figure 5.1 and $f \in \mathcal{V}_0(A)$ generic. The network B is a lift that creates a new layer from A. Proposition 5.7.10 (ii) states that there exists a bifurcation branch of f on B not lifted from A. In fact, the feed-forward system f^B has a bifurcation branch with square-root-order 1 but every bifurcation branch of f on A has square-root-order less or equal to 0, because A has only 2 layers. ◊

There is one more special case that does not depend on the specific feed-forward system considered. For lifts inside a layer such that the next layer only has one cell, there is a bifurcation branch on the lift network not lifted from the original network.

Proposition 5.7.13. *Let N be a feed-forward network with layers $C_1, \dots, C_m$, $f \in \mathcal{V}_0(N)$ generic and L a feed-forward lift of N.*

If L is a lift inside C_j, $1 < j < m$ and $|C_{j+1}| = 1$, then there is a bifurcation branch of f on L which is not lifted from N.

Proof. Let N be a feed-forward network with layers $C_1, \dots, C_m$, $f \in \mathcal{V}_0(N)$ generic and L a feed-forward lift of N. Suppose that L is a lift inside C_j, $1 < j < m$ and $C_{j+1} = \{d\}$. Denote by C'_j the j-layer of L and by $(\sigma_i^L)_{i=1}^k$ the representative functions of L. By Remark 5.3.13, we assume that L is the split of a cell $c \in C_j$ into two cells $c_1, c_2 \in C'_j$ and denote by $\bowtie$ the balanced coloring in L given by $c_1 \bowtie c_2$. Since $[d]_{\bowtie} = d$ and $C_{j+1} = \{d\}$, $\bowtie$ is the unique balanced coloring such that $L/\bowtie = N$.

Using Proposition 5.6.6, we construct a bifurcation branch $b \in \mathcal{B}(L, f)$ such that $b_{c_1} \neq b_{c_2}$. Let $A = \{i : \sigma_i^L(d) = c_2\}$, $\delta = \mathrm{Sign}(f_{0\lambda} \sum_{i \in A} f_i)$ and

$$p_a = -1, \quad s_a = 0, \qquad a \in C_1 \cup \dots \cup C_{j-1} \cup C'_j \setminus \{c_2\},$$

$$p_{c_2} = 0, \quad s_{c_2} = -\frac{2 f_{0\lambda}}{f_{00}}, \quad p_d = 1, \quad s_d = -\,\mathrm{Sign}\left(\delta \sum_{i=1}^{k} f_i\right) \frac{2}{f_{00}} \left| f_{0\lambda} \sum_{i \in A} f_i \right|^{2^{-1}},$$

$$p_a = l, \quad s_a = -\,\mathrm{Sign}\left(\delta \sum_{i=1}^{k} f_i\right) \frac{2}{f_{00}} \left| \sum_{i=1}^{k} f_i \right|^{1-2^{-(l-1)}} \left| f_{0\lambda} \sum_{i \in A} f_i \right|^{2^{-l}},$$

for $a \in C_{j+l}$ and $2 \leq l \leq m - j$. We have that $(\delta, (p_c)_c, (s_c)_c) \in \Omega(L, f)$. By Proposition 5.6.6, there exists a bifurcation branch $b \in \mathcal{B}(L, f)$ such that $b_{c_1} \neq b_{c_2}$, since $p_{c_1} \neq p_{c_2}$. Thus $b \notin \Delta_{\bowtie}$ and b is not lifted from N. □

Example 5.7.14. Let L be the network in Figure 5.6 and $\bowtie$ the balanced coloring in L given by $2 \bowtie 3$. The network L is a lift inside the second layer of $L/\bowtie$ and $L/\bowtie$ has only one cell in the third layer. Taking $f \in \mathcal{V}_0(L)$, we know that there exists a bifurcation branch of f on L not lifted from $L/\bowtie$. ◊

The last layer of a backward connected feed-forward network has only one cell. So we can apply Proposition 5.7.13 for backward connected lifts inside the last but one layer.

We look next to a lift inside a layer such that the next layer has more than one cell. The example is similar to Example 5.5.2 and it exemplifies how multiple balanced colorings can allow for the existence of new bifurcation branches which are copies of bifurcation branches on the quotient network. In this case the conclusion of Proposition 5.7.13 does not hold.

Example 5.7.15. Consider the network C in Figure 5.1 and the network L in Figure 5.2. The network L is a lift inside the second layer of C. Note that the third layer of L has 3 cells.

The lift network has three balanced colorings $\bowtie_1$, $\bowtie_2$ and $\bowtie_3$ such that $C = L/\bowtie_1 = L/\bowtie_2 = L/\bowtie_3$. The balanced colorings in L are defined by $2 \bowtie_1 3$; $3 \bowtie_2 4$; and $2 \bowtie_3 4$.

Let $f \in \mathcal{V}_0(C)$ generic and $b \in \mathcal{B}(L, f)$. We know that b_2, b_3 and b_4 have square-root-order -1 or 0 and only two possible values. Thus $b_2 = b_3$, $b_2 = b_4$ or $b_3 = b_4$ and b is lifted from C. ◊

We focus on backward connected lifts, since every backward connected lift has a unique corresponding balanced coloring, see Lemma 5.3.15. Lifts that create new layers and lifts inside the first layer are covered by Proposition 5.7.10. A backward connected feed-forward network has only one cell in the last layer. So a backward connected lift cannot be a lift inside the last layer. And Proposition 5.7.13 includes backward connected lifts inside the last but one layer. The following results consider lifts inside an intermediate layer. This result depends on the specific feed-forward system consider. The first result shows that there exists an open set of feed-forward systems in $\mathcal{V}_0(N)$ such that every lift inside an intermediate layer has a bifurcation branch which is not lifted.

Proposition 5.7.16. *Let N be a feed-forward network with layers $C_1, \dots, C_m$, $f \in \mathcal{V}_0(N)$ generic and L a feed-forward lift of N such that L is backward connected and a lift inside a layer C_j, where $1 < j < m - 1$.*

If $f_i > 0$ for every $1 \leq i \leq k$ (or $f_i < 0$ for every $1 \leq i \leq k$), then there is a bifurcation branch of f on L which is not lifted from N.

Proof. Let N be a feed-forward network with layers $C_1, \dots, C_m$, $f \in \mathcal{V}_0(N)$ generic and L a feed-forward lift of N such that L is backward connected and a lift inside a layer C_j, where $1 < j < m - 1$. Denote by C'_j the j-layer of L and by $(\sigma_i^L)_{i=1}^k$ the representative functions of L. By Remark 5.3.13, we assume that L is the split of a cell $c \in C_j$ into two cells $c_1, c_2 \in C'_j$ and denote by $\bowtie$ the balanced coloring in L given by $c_1 \bowtie c_2$. By Lemma 5.3.15, $\bowtie$ is the unique balanced coloring such that $L/\bowtie = N$.

Assuming that $f_i > 0$ for every $1 \leq i \leq k$, we use Proposition 5.6.6 to construct a bifurcation branch $b \in \mathcal{B}(L, f)$ such that $b \notin \Delta_{\bowtie}$. Define $\delta = \text{Sign}(f_{0\lambda})$, $p_a = -1$ and $s_a = 0$, for $a \in C_1 \cup \cdots \cup C'_j \setminus \{c_1\}$, $p_{c_1} = 0$ and $s_{c_1} = -2f_{0\lambda}/f_{00}$. We define the value of p and s by induction in the layers $C_{j+1}, \ldots, C_m$ in the following way: for $a \in C_l$, $j < l \leq m$, if $p_{\sigma_1^L(a)} = \cdots = p_{\sigma_k^L(a)} = -1$ define $p_a = -1$ and $s_a = 0$, otherwise define $p_a = \max\{p_{\sigma_1^L(a)}, \ldots, p_{\sigma_k^L(a)}\} + 1$ and

$$s_a = -\text{Sign}(f_{00} f_{0\lambda}) \sqrt{-\frac{2\delta}{f_{00}} \sum_{i \in A(a)} f_i s_{\sigma_i(a)}},$$

where $A(a) = \{i : p_{\sigma_i^L(a)} = p_a - 1\}$.

We have that $(\delta, (p_a)_a, (s_a)_a) \in \Omega(L, f)$ and $p_{c_1} \neq p_{c_2}$. By Proposition 5.6.6, there exists $b \in \mathcal{B}(L, f)$ such that $b \notin \Delta_{\bowtie}$. Thus there is a bifurcation branch of f on L not lifted from N.

The case $f_i < 0$ for every $1 \leq i \leq k$ is analogous. □

Example 5.7.15 shows that the previous result does not always hold if the lift is not backward connected.

Example 5.7.17. Let L be the feed-forward network of Figure 5.3, $\bowtie$ the balanced coloring in L given by $2 \bowtie 3$ and N the quotient network of L associated to $\bowtie$. The network N is a feed-forward network and L is a lift inside the second layer of N. And L is backward connected for the cell 10. Take $f \in \mathcal{V}_0(N)$ generic such that $f_1 > 0$, $f_2 > 0$ and $f_3 > 0$.

Proposition 5.7.16 states that there exists a bifurcation branch of f on L not lifted from N. In Example 5.7.23, we will see that this is not true for every generic $f \in \mathcal{V}_0(N)$. In fact, there exists an open set of functions in $\mathcal{V}_0(N)$ such that every bifurcation branch of f on L is lifted from N. ◊

Finally, we give sufficient conditions on a feed-forward lift, with a bigger central subspace, and on a feed-forward system for every bifurcation branch on the lift be lifted from the quotient network. First, we look to lifts inside the second layer. By Remark 5.3.13, we assume that the lift inside the second layer is a split of two cells.

Proposition 5.7.18. *Let N be a feed-forward network with layers $C_1, \ldots, C_m$, $f \in \mathcal{V}_0(N)$ generic and L a feed-forward lift of N. Denote by C'_2 the second layer of L and by $(\sigma_i^L)_{i=1}^k$ the representative function of L. Assume that L is the split of $c \in C_2$ into $c_1, c_2 \in C'_2$ (and a lift inside C_2).*

If for every $I \subseteq C_2' \setminus \{c_1, c_2\}$ there exist $d', d'' \in C_3$ such that

$$(w_I^{d'} + w_1^{d'})(w_I^{d''} + w_1^{d''}) < 0 \ \wedge \ (w_I^{d'} + w_2^{d'})(w_I^{d''} + w_2^{d''}) < 0,$$

where $w_I^d = \sum_{\sigma_i^L(d) \in I} f_i$, $w_1^d = \sum_{\sigma_i^L(d)=c_1} f_i$ and $w_2^d = \sum_{\sigma_i^L(d)=c_2} f_i$, then every bifurcation branch of f on L is lifted from N.

Proof. Let N be a feed-forward network with layers $C_1, \dots, C_m$, $f \in \mathcal{V}_0(N)$ generic and L a feed-forward lift of N. Denote by C_2' the second layer of L and by $(\sigma_i^L)_{i=1}^k$ the representative function of L. Assume that L is the split of $c \in C_2$ into $c_1, c_2 \in C_2'$.

We prove the result by contraposition. Suppose that there exists $b \in \mathcal{B}(L, f)$ not lifted from N. Then $b_{c_1} \neq b_{c_2}$. Let $(\delta, (p_a)_a, (s_a)_a) = \Theta(b) \in \Omega(L, f)$. For every $a \in C_2'$ we have that

$$p_a \in \{-1, 0\} \qquad s_a = -(p_a + 1)\frac{2f_{0\lambda}}{f_{00}}.$$

Let $I = \{a \in C_2' \setminus \{c_1, c_2\} : p_a = 0\} \subseteq C_2' \setminus \{c_1, c_2\}$. By Ω.6, for $d \in C_3$ such that $p_d = 1$ we have that

$$s_d = \pm \frac{2}{f_{00}} \sqrt{\delta f_{0\lambda} \sum_{i \in A(d)} f_i},$$

where $A(d) = \{i : p_{\sigma_i^L(d)} = 0\}$. Then $(\sum_{i \in A(d')} f_i)(\sum_{i \in A(d'')} f_i) > 0$, if $p_{d'} = p_{d''} = 1$, $(\sum_{i \in A(d')} f_i)(\sum_{i \in A(d'')} f_i) = 0$, if $p_{d'} < 1$ or $p_{d''} < 1$, for every $d', d'' \in C_3$. Thus

$$\left(\sum_{i \in A(d')} f_i \right) \left(\sum_{i \in A(d'')} f_i \right) \geq 0,$$

for every $d', d'' \in C_2$. Since $b_{c_1} \neq b_{c_2}$, $-1 \leq p_{c_1} \neq p_{c_2} \leq 0$. If $p_{c_1} = 0$ and $p_{c_2} = -1$, then $\sum_{i \in A(d)} f_i = w_I^d + w_1^d$. If $p_{c_1} = -1$ and $p_{c_2} = 0$, then $\sum_{i \in A(d)} f_i = w_I^d + w_2^d$. So

$$(w_I^{d'} + w_1^{d'})(w_I^{d''} + w_1^{d''}) \geq 0 \ \vee \ (w_I^{d'} + w_2^{d'})(w_I^{d''} + w_2^{d''}) \geq 0,$$

for every $d', d'' \in C_3$. By contraposition, we obtain the result. □

Example 5.7.19. Consider the networks B and C in Figure 5.1. Consider $f \in \mathcal{V}_0(C)$ such that $f_1 f_2 < 0$. The lift C of B and the function f satisfy the conditions of Proposition 5.7.18. As we saw in Example 5.6.5, and accordingly with Proposition 5.7.18, every bifurcation branch of f on C is lifted from B. ◊

Last, we consider lifts inside the other intermediate layers. We give sufficient conditions on a lift network and feed-forward systems such that there is no new bifurcation branch. We will assume that the lift is given by the split of a cell into two cells which are the unique inputs cells of two other cells in the next layer.

Proposition 5.7.20. *Let N be a feed-forward network with layers $C_1, \dots, C_m$, $f \in \mathcal{V}_0(N)$ generic, L a feed-forward lift of N and $2 < j \leq m-1$. Denote by C'_j the j layer of L and by $(\sigma_i^L)_{i=1}^k$ the representative function of L. Assume that L is the split of $c \in C_j$ into $c_1, c_2 \in C'_j$ (and a lift inside C_j).*

If there exist $d', d'' \in C_{j+1}$ such that $\sigma_i^L(d'), \sigma_i^L(d') \in \{c_1, c_2\}$, for every $1 \leq i \leq k$, and

$$w_1^{d'} w_1^{d''} < 0 \ \wedge \ w_2^{d'} w_2^{d''} < 0 \ \wedge \ w_1^{d'} w_1^{d''} + w_2^{d'} w_2^{d''} < w_1^{d'} w_2^{d''} + w_1^{d''} w_2^{d'},$$

where $w_1^d = \sum_{\sigma_i^L(d)=c_1} f_i$ and $w_2^d = \sum_{\sigma_i^L(d)=c_2} f_i$, then every bifurcation branch b of f on L is lifted from N.

Proof. Let N be a feed-forward network with layers $C_1, \dots, C_m$, $f \in \mathcal{V}_0(N)$ generic, L a feed-forward lift of N and $2 < j \leq m-1$. Denote by C'_j the j-layer of L and by $(\sigma_i^L)_{i=1}^k$ the representative function of L. Assume that L is the split of $c \in C_j$ into $c_1, c_2 \in C'_j$. Suppose that there exist $d', d'' \in C_{j+1}$ such that $\sigma_i^L(d'), \sigma_i^L(d') \in \{c_1, c_2\}$, for $1 \leq i \leq k$. Let $b \in \mathcal{B}(L, f)$ be a bifurcation branch and $(\delta, (p_a)_a, (s_a)_a) = \Theta(b) \in \Omega(L, f)$ the corespondent symbol.

We know that $b \in \mathcal{B}(L, f)$ is not lifted from N if and only if $b_{c_1} \neq b_{c_2}$. We assume that $b_{c_1} \neq b_{c_2}$ and obtain a contradiction with the given conditions.

Suppose that $b_{c_1} \neq b_{c_2}$. Then $p_{c_1} = 0 \wedge p_{c_2} = -1$ or $p_{c_1} = -1 \wedge p_{c_2} = 0$ or $p_{c_1} = p_{c_2} > 0 \ \wedge \ s_{c_1} = -s_{c_2}$. We have that $w_1^{d'} w_1^{d''} \geq 0$, if $p_{c_1} = 0$ and $p_{c_2} = -1$. And $w_2^{d'} w_2^{d''} \geq 0$, if $p_{c_1} = -1$ and $p_{c_2} = 0$. If $p_{c_1} = p_{c_2} > 0$ and $s_{c_1} = -s_{c_2}$, then $p_{d'} = p_{d'} = p_{c_1} + 1$ and

$$s_{d'} = \pm\sqrt{-\frac{2\delta}{f_{00}}(w_1^{d'} - w_2^{d'}) s_{c_1}}, \qquad s_{d''} = \pm\sqrt{-\frac{2\delta}{f_{00}}(w_1^{d''} - w_2^{d''}) s_{c_1}}.$$

Thus $(w_1^{d'} - w_2^{d'})(w_1^{d''} - w_2^{d''}) > 0$. Generically, $(w_1^{d'} - w_2^{d'})(w_1^{d''} - w_2^{d''}) \neq 0$.

Therefore

$$w_1^{d'} w_1^{d''} < 0 \ \wedge \ w_2^{d'} w_2^{d''} < 0 \ \wedge \ w_1^{d'} w_1^{d''} + w_2^{d'} w_2^{d''} < w_1^{d'} w_2^{d''} + w_1^{d''} w_2^{d'},$$

implies that $b_{c_1} = b_{c_2}$ and that $b \in \mathcal{B}(L, f)$ is lifted from N. □

Example 5.7.21. Let L be the network in Figure 5.9 and $\bowtie$ the balanced coloring in L given by $4 \bowtie 5$. Consider a function $f \in \mathcal{V}_0(L)$ such that f^L

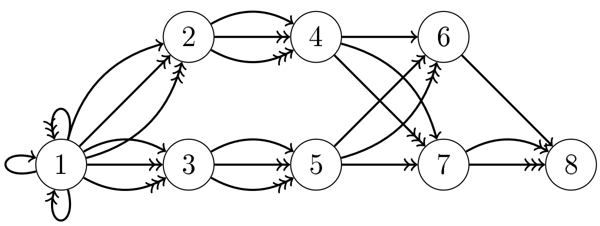

Figure 5.9: Feed-forward network with 5 layers.

has a bifurcation condition associated to the internal dynamics. Denote by N the quotient network $L/\bowtie$.

The network L is a split of a cell in N into the cells 4 and 5 and the cells 6 and 7 only receive inputs from those cells. Denote by w_c^d the sum of the linear inputs from c to d, where $c = 4, 5$ and $d = 6, 7$. We have that

$$w_4^6 = f_1 \qquad w_4^7 = f_1 + f_3 \qquad w_5^6 = f_2 + f_3 \qquad w_5^7 = f_2.$$

By Proposition 5.7.20, we know that every bifurcation branch of f on L is lifted from N if

$$f_1(f_1 + f_3) < 0 \quad \wedge \quad f_2(f_2 + f_3) < 0 \quad \wedge \quad (f_1 - f_2)^2 < f_3^2.$$

In order to see that the previous inequalities are satisfied by some function f, take $f_1 = 2$, $f_2 = 1$ and $f_3 = -3$. ♢

The previous result does not necessary hold if the splitted cells only target one cell or if the splitted cells are not the unique inputs of two cells in the next layer. We present next an example where the splitted cells are not the unique inputs and the conclusion does not hold.

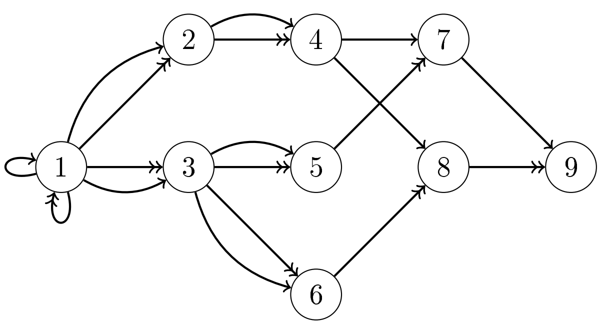

Figure 5.10: A network L with a quotient network N obtained by the balanced coloring $\bowtie$ given by $5 \bowtie 6$. If $f \in \mathcal{V}_0(N)$, then there exists a bifurcation branch of f on L not lifted from N.

Example 5.7.22. Let L be the feed-forward network of Figure 5.10, $\bowtie$ the balanced coloring in L given by $5 \bowtie 6$ and N the quotient network of L

associated to $\bowtie$. The network N is a feed-forward network and L is a split of one cell inside the third layer. Let $f \in \mathcal{V}_0(N)$ generic. We show that there exists a bifurcation branch of f on L not lifted from N.

Let $\delta = \operatorname{Sign}(f_{0\lambda}(f_1 + f_2))$, $p_1 = p_3 = p_6 = -1$, $p_2 = p_5 = 0$, $p_4 = 1$, $p_7 = p_8 = 2$, $p_9 = 3$, $s_1 = s_3 = s_6 = 0$,

$$s_2 = s_5 = -\operatorname{Sign}(f_1)\delta\frac{2|f_{0\lambda}|}{f_{00}}, \quad s_4 = -\operatorname{Sign}(f_{0\lambda})\frac{2\sqrt{|f_{0\lambda}|}}{f_{00}}\sqrt{|f_1 + f_2|},$$

$$s_7 = s_8 = -\operatorname{Sign}(f_{0\lambda})\frac{2\sqrt[4]{|f_{0\lambda}|}}{f_{00}}\sqrt{|f_1|\sqrt{|f_1 + f_2|}}$$

and

$$s_9 = \frac{2\sqrt[8]{|f_{0\lambda}|}}{f_{00}}\sqrt{|f_1 + f_2|\sqrt{|f_1|\sqrt{|f_1 + f_2|}}}$$

Note that $(\delta, (p_a)_a, (s_a)_a) \in \Omega(L, f)$. Let $b \in \mathcal{B}(L, f)$ be the bifurcation branch associated to $(\delta, (p_a)_a, (s_a)_a)$. Since L is backward connected, a bifurcation branch b on L can be lifted from N if and only if $b_5 = b_6$. However $p_5 \neq p_6$ and b is not lifted from N. $\diamondsuit$

In the next example, we see that the lifting bifurcation problem goes beyond the one layer to the next layer reasoning. As the example shows, the network structure can further restrict the possible bifurcation branches. In this case, we need to look for the next two layers to understand the possible bifurcation branches.

Example 5.7.23. Let L be the feed-forward network of Figure 5.3, $\bowtie$ the balanced coloring in L given by the class $\{2, 3\}$ and N the quotient network of L associated to $\bowtie$. The network N is a feed-forward network and L is a lift inside the second layer.

Let $f \in \mathcal{V}_0(L)$ be generic such that $f_3(f_2 + f_3) < 0$. As we show next, there is no bifurcation branch $b \in \mathcal{B}(L, f)$ such that $b_2 \neq b_3$ and every bifurcation branch of f on L is lifted from N.

Let $b \in \mathcal{B}(L, f)$. Suppose by contradiction that $b_2 \neq b_3$. We know that b_2 and b_3 are the trivial branch or the branch b^0 with square-root-order 0 and defined in (5.2). There are two options: $b_2 = b^0$ and $b_3 = 0$; or $b_2 = 0$ and $b_3 = b^0$. If $b_2 = b^0$ and $b_3 = 0$, we have that b_5 and b_6 are defined on different sides of $\lambda = 0$, like we saw in Proposition 5.7.18. Thus there is no bifurcation branch of f on L such that $b_2 = b^0$ and $b_3 = 0$. If $b_2 = 0$ and $b_3 = b^0$, then b_4 has square-root-order -1 or 0 and b_5 has square-root-order 1. Now, we look to the next layer, in particular to the cells 7 and 8. We know that b_7 and b_8 have square-root-order 2 and that the side of $\lambda = 0$ where they are defined depend on b_5. Since $f_3(f_2 + f_3) < 0$, the cell 7 receives inputs of type 2 and of type 3 from the cell 5 and the cell 8 receives an input of type 3 from cell 5, we know that b_7 and b_8 are defined on different sides of $\lambda = 0$. So there is no bifurcation branch of f on L such that $b_2 \neq b_3$. $\diamondsuit$

5.8 Discussion

The main goal of this work is to address the lifting bifurcation problem in the context of feed-forward systems. We identify two important types of lifts in feed-forward networks, lifts that create new layers and lifts inside a layer. We show that every backward connected lift is given by the composition of basic lifts. When studying codimension-one steady-state bifurcations on feed-forward systems, we identify two possible bifurcations conditions, that we call valency and internal dynamics. For both conditions, we give a complete description of the bifurcation branches. For bifurcations associated to the internal dynamics, we introduce a new symbolic set which describes all bifurcation branches. The symbols represent the growth rate and slope of the bifurcation branches and the symbolic set is given by the symbols that satisfy some given rules. Finally, we study the lifting bifurcation problem for feed-forward systems. For a fixed bifurcation condition, when the lifted network has a center subspace bigger than the center subspace associated to the smaller network, we expect the existence of new synchrony-breaking bifurcation branches. We prove this for different cases, including lifts that create new layers. From the lifting bifurcation problem point of view, it can be important when no new bifurcation branches appear and so the study of the quotient network is sufficient to understand the bifurcations on the lift network. For some lifts inside an intermediate layer, we prove that there is a class of feed-forward systems without new bifurcation branches. This depends on the correct balance between the signs of each of the input's linearization.

We stress that some of the restrictions that we impose at the class of feed-forward systems in this work can, in fact, be easily removed. If the dimension of each cell phase space is bigger than 1, then the steady-state bifurcation analysis is generically the same, using the Lyapunov-Schmidt reduction. If every layer has a different type of cells, then we only have lifts inside a layer and the steady-state bifurcation study is similar to the case of feed-forward systems with a bifurcation condition associated with the valency. Also, if we do not impose that the origin is an equilibrium for every value of the parameter, then the steady-state bifurcation with a condition associated to the internal dynamics is the same. This follows from the fact that the full-synchronized subspace does not support a bifurcation and a full-synchronized equilibrium will still exist for every value of the parameter which can be assumed to be the origin. For the steady-state bifurcations with a condition associated to the valency, we will have a fold bifurcation instead of a transcritical bifurcation. However, the lifting bifurcation problem with a condition associated to the valency will be essentially the same, because we still have two options for each cell in the first layer.

Example 5.7.23 shows that the study of the lifting bifurcation problem for feed-forward networks is not limited to the next layer reasoning. Examples that depend on the next two (or more) layers are not covered by our results on the lifting bifurcation problem. It would be interesting to have results that include that kind of examples.

We could consider the lifting bifurcation problem for feed-forward networks with a fixed number of layers. And we can ask if there exists a minimal feed-forward network such that all bifurcation branches of any feed-forward lift, with the same number of layers, are lifted from that minimal network.

We can also ask the same questions about the lifts, the bifurcation branches and the lifting bifurcation problem for feed-forward networks with intra-layer connections.

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6. Conclusions and future work

The goal of this thesis is to understand the codimension one steady-state synchrony-breaking bifurcations on coupled cell systems. For homogenous networks with asymmetric inputs, we characterize the fundamental networks in terms of their graphical properties. Methods to study the synchrony-breaking bifurcations for this class of networks are available in the literature. For regular networks, we study the synchrony-breaking bifurcations on maximal and submaximal synchrony subspaces. Here, we show that the lattice structure of the synchrony subspaces of a network is not sufficient to understand the synchrony-breaking bifurcations of the coupled cell systems associated with the network. In the last two parts of the thesis, we focus on the lifting bifurcation problem, a problem that compares the synchrony-breaking bifurcations between a network and their lifts. We study the lifting bifurcation problem where the bifurcation condition is given by the network valency. We prove that there exists a class of networks such that the dimension of the eigenspace associated with the valency is sufficient to understand this problem. Last, we address the lifting bifurcation problem for feed-forward networks. One of the main conclusions from our work, it is that the lifting bifurcation problem does depend on the chosen coupled cell system.

The study of synchrony-breaking bifurcations on coupled cell systems is an ongoing subject of research, as well as, the study of robustly supported heteroclinic networks in coupled cell systems. An interesting question is to determine the bifurcations in coupled cell systems that lead to robust heteroclinic networks. Some other directions of study in coupled cell systems are:

- Hopf bifurcations.
- Classification of singularities.
- Transversality.
- Degree theory.

We can also ask which networks are dynamical equivalent to a fundamental network.