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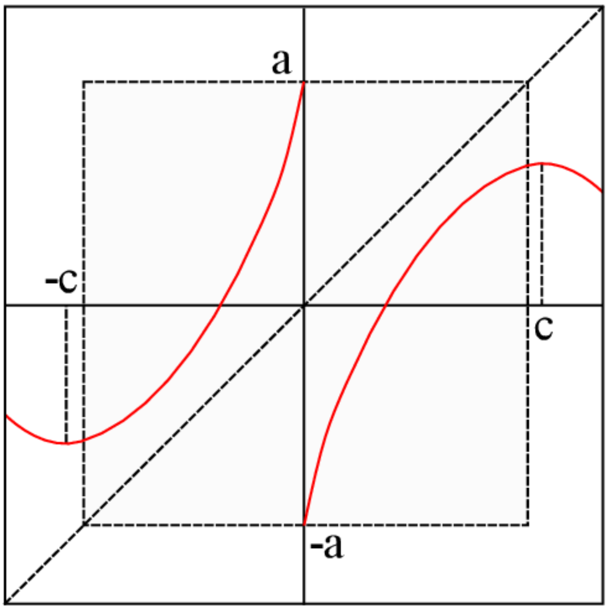

Statistical stability for Luzzatto-Viana maps

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Statistical stability for Luzzato-Viana maps

Estabilidade estatística para transformaçẽs de Luzzatto-Viana

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A distância entre o sonho e a conquista

Chama-se atitude.

Nick Vujicic

To my loves Kelly and Daniel,

To my parents Neude and João

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Abstract

We consider a one-parameter family of one-dimensional maps introduced by Luzzatto and Viana in 2000, which combines two critical points and a point of discontinuity (singular point). The concept of statistical stability was presented by Alves and Viana in 2000, referring to certain classes of dynamical systems with physical measures and meaning continuous variation of these measures with respect to the parameter, in the weak* topology, under small modifications of the law that governs the system. In chaotic dynamical systems, the physical measures are generally absolutely continuous with respect to the Lebesgue measure and are associated with the existence of positive Lyapunov exponents. In this thesis we will use the approach developed by Freitas for the quadratic family of maps and the approach developed by Alves and Soufi for the Rovella family, based on the Benedicks-Carleson techniques and an abstract result obtained by Alves for non-uniformly expanding maps. We are going to show that for each parameter the family of maps has non-uniform expanding behavior and slow recurrence to the critical/singular set.

Keywords: non-uniform expansion, slow recurrence, statistical stability.

Resumo

Consideramos uma família de transformações unidimensionais a um parâmetro introduzida por Luzzatto e Viana em 2000, que combina dois pontos críticos e um ponto de descontinuidade (ponto singular). O conceito de estabilidade estatística foi apresentado por Alves e Viana em 2000, referindo-se à certas classes de sistemas dinâmicos com medidas físicas e significando variação contínua dessas medidas em relação ao parâmetro, na topologia fraca *, sob pequenas modificações de a lei que rege o sistema. Em sistemas dinâmicos caóticos, as medidas físicas são geralmente absolutamente contínuas com relação à medida de Lebesgue e estão associadas à existência de expoentes positivos de Lyapunov. Nesta tese usaremos a abordagem desenvolvida por Freitas para a família quadrática de transformações e a abordagem desenvolvida por Alves e Soufi para a família Rovella, baseada nas técnicas de Benedicks-Carleson e um resultado abstrato obtido por Alves para transformações em expansão não uniforme. Mostraremos que, para cada parâmetro, a família de transformações possui comportamento de expansão não uniforme e recorrência lenta para o conjunto crítico/singular.

Palavras-chave: expansão não uniforme, recorrência lenta, estabilidade estatística.

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Introduction

The theory of dynamical systems has as one of its main goals to describe the asymptotic behavior of systems that evolve over time. Among the many targeted systems that have motivated the development of the area are included the solar system, the climate, billiards, financial markets or population dynamics. Some of the tools used in the study of dynamical systems are Functional Analysis, Differential Geometry, and Measure and Integration and Probability Theory.

Another main goal in Dynamical Systems is about stability. The first fundamental concept, *structural stability*, occurs in the hyperbolic context and was formulated by Andronov and Pontryagin. In this concept is required that the global structure of orbits is kept unchanged under small perturbations on the dynamical system: there is a homeomorphism of the ambient manifold sending trajectories of the initial system to trajectories of the perturbed system, preserving the direction of time. In the early 1960s, Smale introduced the notion of *uniformly hyperbolic* (or *Axiom A*) system, having as one of his main objectives to characterize structural stability. Such a characterization was conjectured by Palis and Smale: a diffeomorphisms (or flow) is structurally stable if and only if it is uniformly hyperbolic and satisfies the so-called strong transversality condition.

Structural stability is a restrictive notion, not suitable for systems out of the uniformly hyperbolic setting. Weak notions of stability, still with a topological approach, were proposed during the 1960s and 1970s, but all restrictive.

In the last decades, mathematicians has been giving an emphasis to express the stability in terms of the statistical properties of the system. Concept of *statistical stability* was introduced by Alves and Viana in [AV02], referring to certain classes of dynamical systems with physical measures and meaning the continuous variation of theses measures under small modifications of the law that governs the system.

In chaotic dynamics the physical measures are usually absolutely continuous with respect to Lebesgue measure (w.r.t. Leb., for short) and are associated to the existence of positive Lyapunov exponents. In general, the existence of such Lyapunov exponents is difficult to prove when the system has critical or singular points. In the case of one-dimensional transformations, statistical stability was proved by Freitas in [Fr05] for the quadratic family restricted to Benedicks-Carleson parameters. In higher dimensions, statistical stability has been proved for the family of Viana maps introduced in [Vi97], after the existence of physical measures for such systems have been proved in [Al00]. Also, statistical stability was proved by Alves, Carvalho and Freitas for Hénon maps of the Benedicks-Carleson type in [ACF10]. Recently, statistical stability was proved by Alves and Soufi for certain family of one-dimensional maps associated to conservative Lorenz attractors in the Rovella parameters, see [AS12].

In this current work we are proving that there is *statistical stability* for a classe of maps that we call *Luzzatto-Viana maps*, as in [LV00], whose combine singular dynamics (discontinuities with infinitive derivative) with critical dynamical (critical points) in certain parameters with positive Lyapunov exponents. In that sense, these transformations are an extension of the one-dimensional family associated to the contracting Lorenz attractor.

Our strategy is based in a mixture of approaches from [Fr05] and [AS12] and we show that the tail set (the set of points that, up to a given time n, do not satisfy either the non-uniform expansion or the slow recurrence) decays exponentially very fast to 0, as time goes to ∞. As a consequence, we obtain the continuous variation of the physical measure in the weak* topology. In addition, we consider the space L^1-norm of the densities of the measures and we use Theorem A in [Al04] for having the continuous variation of the physical measure on this spaces, it is equivalent to say that the family of maps is strongly statistical stable.

Our main difficulty is the presence of critical points within domain of map. We will assume parameters close to bifurcation parameter ($a > c$) which does not have periodic attractor orbit such that our approach can be used. Luzzatto and Viana showed that family has a persistence of non-uniform expansivity even after the critical points enters on domain of map. Actually, there is a bifurcation at parameter $a = c$ which is the point of the transition from uniform expanding dynamics ($a < c$) to non-uniformly expanding dynamics for $a \sim c$ ($a > c$). In some sense, we use this fact.

The proof of our main result will be organized as follows: in section 2.2 we announce a pair of fundamental lemmas which asserts that the family will satisfy the non-uniformly expanding and slow recurrence to the critical/singular set. Chapter 2 we describe briefly the family of maps in question, just highlighting the properties that will be used on the our statements. Then, on chapter 3 we complete the proof of main theorem A.

Chapter 1

Background and statement of results

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1.1 Preliminaries

> It is through science that we prove, but through intuition that we discover.
>
> Henri Poincaré

In this section we present some classical results from Ergodic Theory, Probability Theory and Dynamical Systems that are useful for a good understanding of this text. The results found here are standard and their deep details and proofs can be found in several books in these subjects. We just mention [Al03], [Lu17], [Mu53], [Wa00], [OV14], [AL06]. Moreover, we introduce a recent concept of stability for some classes of dynamical systems called *Statistical Stability*, see [AV02]. In the next section we draw up the strategy which we are following in the later chapters. Section 1.2 is focused in the presentation of results without details, including our main theorem A, which we shall prove in 3.

We begin with assumption that X is a metric space, $\mathcal{B}(X)$ is a σ-algebra of subsets of X and μ is a probability measure on $\mathcal{B}(X)$. The *support* of a Borel measure μ is defined as

$$\text{supp}(\mu) = \{x \in X : \mu(U) > 0 \text{ for each neighbourhood } U \text{ of } x\}.$$

Given $p \geq 1$, we define $L^p(\mu)$ as the set of those functions $\varphi : X \to \mathbb{C}$ such that $|\varphi|^p$ is integrable, identifying two functions that coincide μ almost everywhere (a.e). Then,

$$\|\varphi\|_p = \Big(\int_X |\varphi|^p\Big)^{\frac{1}{p}}$$

defines a norm in $L^p(\mu)$. If $p = 1$, we have the L^1-norm given by $\|\varphi\|_{L^1} = \int_X |\varphi|$.

We define the space $L^\infty(\mu)$ as the set of those measurable functions φ for which there is $C > 0$ such that $|\varphi(x)| \leq C$, identifying two functions that coincide μ a.e. on X. Then,

$$\varphi \mapsto \|\varphi\|_\infty \equiv \inf\{C \geq 0 : |\varphi(x)\}| \leq C \text{ a.e. on } X\},$$

defines a norm on $L^\infty(\mu)$.

Consider the space $L^p(\mu)$ endowed with the norm $\|.\|_p$ above a Banach space for $1 \leq p \leq \infty$. Let μ and ν be finite measures defined on the same σ-algebra $\mathcal{B}(X)$, we say that ν is *absolutely continuous* with respect to μ, and write $\nu \ll \mu$, if $\nu(A) = 0$ whenever $\mu(A) = 0$ for each $\mathcal{B}(X)$. The measures μ and ν are said to be the *equivalent* if both of situation occurs $\mu \ll \nu$ and $\nu \ll \mu$.

Theorem 1.1.1 (Radon-Nykodim)**.** *The measure ν is absolute continuous with respect to μ if and only if there is $\varphi : X \mapsto \mathbb{R}$ non-negative and integrable with respect* *to μ such that*

$$\nu(A) = \int_A \varphi d\mu, \ \textit{for each} \ A \ \in \mathcal{B}(X).$$

The function given by previous theorem is called the Radon-Nykodim derivative of ν with respect to μ and denoted by $\frac{d\nu}{d\mu}$. We say that $f : X \to X$ is a *measurable transformation* if $f^{-1}(A) \in \mathcal{B}(X)$ for each $A \in \mathcal{B}(X)$. The measure μ is said to be invariant by f (or f preserves μ) if $\mu(f^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}(X)$. We can associate to a measurable transformation f and a measure μ a new measure that we denote by $f_*\mu$ called the "push-forward" of the measure μ by f, and is define as

$$f_*\mu(A) = \mu(f^{-1}(A))$$

for each $A \in \mathcal{B}(X)$. Also, note that μ is invariant by f if and only if $f_*\mu = \mu$.

We denote by $\mathbb{P}(X)$ the space of probability measures defined on the Borel σ-algebra of X. Now, we may introduce the weak* topology on $\mathbb{P}(X)$ as following: a sequence $(\mu)_n \in \mathbb{P}(X)$ converges to $\mu \in \mathbb{P}(X)$ if and only if

$$\int \varphi d\mu_n \to \int \varphi d\mu, \text{ for each continuous function } \varphi : X \to \mathbb{R}.$$

Let X be a compact metric space, and let $C(X) = \{\varphi | \varphi : X \to X\}$ be a separable set, where φ are continuous and bounded functions, so we may find a sequence $(\psi_n)_n$ dense in $C(X)$. The function

$$d_{\mathbb{P}(\mu,\nu)} = \sum_{k=1}^{\infty} \frac{1}{2^n} | \int \psi_n d\mu - \int \psi_n d\nu |$$

defines a metric on $\mathbb{P}(X)$ which gives the weak* topology.

Let μ be an invariant measure by $f : X \to X$. We say that μ is an ergodic measure if the phase space cannot be decomposed into invariant regions that are relevant in terms of the measure μ, i.e if $A \in \mathcal{B}(X)$ satisfies $f^{-1}(A) = A$, then $\mu(A)\mu(X \setminus A) = 0$.

Theorem 1.1.2 (Poincaré Recurrence Theorem)**.** *Assume that f preserves a probability measure μ. If A is a measurable set, then for almost every $x \in A$, there are infinitely many $n \in \mathbb{N}$ for which $f^n(x) \in A$.*

Theorem 1.1.3 (Birkhoff)**.** *Let $f : X \to X$ preserve a probability measure μ. Given any $\varphi \in L^1(\mu)$ there exists $\varphi^* \in L^1(\mu)$ with $\varphi^* \circ f = \varphi^*$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^n(x) = \varphi^*(x)$$

for μ almost every $x \in X$. Moreover, if f is ergodic, then $\varphi^* = \int \varphi d\mu$ almost everywhere.

This previous result give us information about asymptotic frequency of the typical orbits that visits A, that is

$$\lim_{n\to\infty} \frac{1}{n} \#\{0 \leq j < n : f^j(x) \in A\}. \tag{1.1}$$

Taking φ as the characteristic function of a measurable set A, we can deduce from Birkhoff's theorem that the limit (1.1) there exist for μ almost every $x \in M$. Moreover, if μ is ergodic, then (1.1) is equal to $\mu(A)$ which means that the frequency of visits in A coincides with the proportion that A occupies in the phase space.

Assume that $f : X \to X$ preserves a measure μ, we say that μ is *ergodic* if $\mu(A) = 0$ or $\mu(M \setminus A) = 0$ whenever $A \in \mathcal{B}(X)$ satisfies $f^{-1}(A) = A$. Also, we can see that $f^{-1}(A) = A$ implies that $f(A) \subset A$ and $f(M \setminus A) \subset (M \setminus A)$, it means that the space cannot be decomposed into two parts which are relevant (positive measure) that do not interact.

The basin $B(\mu)$ of an f-invariant Borel probability measure μ is the set of points $x \in M$ for which

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu, \ \forall \varphi : M \to \mathbb{R} \text{ continuous function.} \tag{1.2}$$

It means that the averages of Dirac measures over the orbit of x converge in the weak* topology (two measures are close to each other if they assign close by integrals to each continuous function) to the measure μ,

$$\frac{1}{n}\sum_{j=0}^{n} \delta_{f^j(x)} \xrightarrow{w*} \mu.$$

Let M be a compact connected Riemannian manifold, an *invariant probability measure* μ is called an *physical measure* for $f : M \to M$ if, for *positive* Lebesgue measure set of points $x \in M$ such satisfies (1.2).

We can deduce from the Birkhoff's theorem that every *absolutely continuous* (w.r.t. Leb.) measure *ergodic* probability measure is a physical measure. In this present text we shall refer to *ergodic absolutely continuous (w.r.t. Leb.) invariant probability measures* as a.c.i.p. measures.

We are interested in the continuous variation of the a.c.i.p. measures of dynamical systems. On the other hand, the notion of convergences theses measures as a function of systems is not easy to see. This problem will be precisely formulated in the next chapters. Following the notion of *statistical stability* introduced by Alves and Viana [AV02]. Assume that each $f \in \mathcal{F}$ admits a *unique* a.c.i.p. measure μ_f. We say that f is *statistically stable* if, the map $\mathcal{F} \ni g \mapsto \mu_g$, associating to each g its a.c.i.p. measure μ_g, is continuous at f in the weak* topology. Regarding the continuity in the space, we may consider weak* topology or even strong topology given by the L^1-norm in the space of densities (if they exist). In other words, we have continuous variation of the a.c.i.p. measures as a function of the dynamical system. This definition is guaranteeing that if a system is *statistically stable*, then the times averages of continuous functions are only lightly affected when system is perturbed.

Araújo, Luzzatto and Viana proved in [ALV09] that there exist a finite number of a.c.i.p. measures for Luzzatto - Viana maps. We will show that is actually these measures is *unique* and following [Al04], [Fr05] and [AS12] we are going to show that measures *vary continuously* as function of dynamical system.

1.2 Statement of results

In what follows we shall consider parameters $a \in [c + \rho\varepsilon, c + \varepsilon]$ for each we have exponential growth of the derivative on critical orbit $(f_a^n(\pm c) = (\Phi_c))$ and on singular orbit $(f_a^n(0) = (\Phi_0))$, and *does not exists attractor periodic orbits for that parameters.* More precisely, we consider points $x \in \Delta_r^{\pm c}$, where we have loss of expansivity occurring when trajectories passing close to the critical point $\pm c$, but as has been shown in [LV00, Section 2.3] there is a recovery of expansion. Here, in particular we will focus in the approach only around one critical point $+c$. On the other point $-c$ is treated similarly, and also the same way can be apply to the origin.In particular, we need of the uniqueness of a.c.i.p. measure, it results from transitivity of dynamics on all parameters $[-a, a]$. The next result gives the transitivity, whose proof will be given later in Section 3.2.

Theorem 1.2.1. *For all parameter $a \in [c, c + \varepsilon]$, we have f_a is transitive.*

The notion of *hyperbolic times* has been introduced by Alves on the study of the ergodic properties for Viana maps in [Al00]. This notion has been extended by Alves, Bonatti and Viana for general class of maps see[ABV00]. Roughly speaking, hyperbolic times are iterates of a certain point at which some uniform backward contraction holds, thus it implies in uniformly bounded distortion on some small neighborhood of that point. It is very important in the study of the statistical properties of many classes of dynamical systems. An consequence of the hyperbolic times and from previous theorem we have the following result.

Theorem 1.2.2. *For each parameter $a \in [c, c+\varepsilon]$, f_a admits an unique a.c.i.p. measure.*

The proof follows directly from [ABV00]. In fact, since the dynamics is invariant in any subset of $[-a, a]$ as the basins of measures contains full measure (w.r.t. Leb.) onto an open set, from 1.2.1 we may see that there is at least a common point in the basins of both measures, so these measures must coincide. It will be extremely needed to get our main Theorem.

Alves gave sufficient abstract conditions to get the *statistical stability* for *non-uniformly expanding* maps with *slow recurrence* to the critical/singular set $\mathcal{C}$. We are going to show that the family described in the next Section (2.1) satisfies these conditions. Notice that the non-degeneracy condition on critical/singular set needed in [Al04, Section 1.2] is satisfied in our case, because f is a local diffeomorphism. Now, we are going to give some definitions, lemmas, remarks useful to obtain the main result in this present thesis. We will assume throughout this present section that f is a C^2 piecewise expanding map with bounded distortion.

Definition 1.2.3. *We say that f_a is* non-uniformly expanding (NUE) *if there is a $b > 0$ such that for Lebesgue almost every $x \in [-a, a]$,*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log |f_a'(f_a^i(x))| > b. \tag{1.3}$$

Definition 1.2.4. *We say that f_a has* slow recurrence (SR) to the critical/singular set *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for the Lebesgue almost every $x \in [-a, a]$,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta(f_a^i(x), \mathcal{C}) \leq \varepsilon, \tag{1.4}$$

where the δ-truncated distance, defined as

$$d_\delta(x, y) = \begin{cases} |x - y| & \textit{if} \quad |x - y| \leq \delta, \\ 1 & \textit{if} \quad |x - y| > \delta. \end{cases}$$

We define *the expansion time function* (ETF) as follows

$$\mathcal{E}^a(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{i=0}^{n-1} \log |f_a'(f_a^i(x))| > b, \forall n \geq N \right\}, \tag{1.5}$$

which one is finite and defined almost everywhere x in $[-a, a]$, provided (1.3) holds. Fixing $\varepsilon > 0$ and choosing $\delta > 0$ conveniently, we define the *recurrence time function* (RTF)

$$\mathcal{R}^a_{\varepsilon,\delta}(x) = \min \left\{ N \geq 1 : \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta(f_a^i(x), \mathcal{C}) < \varepsilon, \ \forall n \geq N \right\},$$

which is defined and finite almost everywhere x in I as long as (1.4) holds almost everywhere $x \in [-a, a]$.

We define the *tail set* at the time $n \in \mathbb{N}$, as

$$\Gamma^a_n(\varepsilon, \delta) = \{x \in [-a, a] : \mathcal{E}^a(x) > n \text{ or } \mathcal{R}^a_{\varepsilon,\delta}(x) > n\} \tag{1.6}$$

Luzzatto and Viana showed that there is a set of parameters with positive Lebesgue measure $\mathcal{A} \subset [c + \rho\varepsilon, c + \varepsilon]$ (which we will call *Luzzatto-Viana parameters set*) that for each $a \in \mathcal{A}$, we have $\lim_{\varepsilon \to 0} \dfrac{Leb(\mathcal{A} \cap [c, c+\varepsilon])}{Leb([c, c+\varepsilon])} = 1$, where Leb is a Lebesgue measure on $\mathbb{R}$ and $\rho = 2^{-\lambda}(0 < \lambda < \frac{1}{2})$.

Remark 1.2.5. The *slow recurrence condition* as in [Al04, Remark 3.8], is not needed in all its strength: it is enough that (1.4) holds for some $\varepsilon > 0$ and conveniently chosen $\delta > 0$ only depending on the order λ and b. For this reason we may drop the dependence of the tail set on ε and δ in the notation. Also, we may choose the constants b in (1.3) and ε, δ in (1.4) with uniformly dependence on the set of parameters $\mathcal{A}$.

Theorem A (Main Theorem)**.** *Each f_a, with $a \in \mathcal{A}$, is non-uniformly expanding and has slow recurrence to the critical/singular set. Moreover, there are $C > 0$ and $\tau > 0$ such that for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$,*

$$|\Gamma_a^n| \leq Ce^{-\tau n}.$$

The proof is contained in Section 3.1. We point out that it will be divided in two main parts as follows. For each $a \in \mathcal{A}$,

(i_1) f_a has non-uniformly expanding behavior to the critical/singular set.

(i_2) f_a has slow recurrence to the critical/singular set.

Then,

$$|\Gamma_n^a| \leq \text{const } e^{-\tau n}, \text{ for some } \tau > 0 \text{ and const} > 0.$$

As a consequence of main Theorem A we obtain the following result.

Corollary B. *The function $\mathcal{A} \ni a \mapsto d\mu_a/dm$ is continuous in L^1-norm. In other words, $\mathcal{F}$ is strongly statistically stable.*

The proof follows immediately as consequence from main Theorem A which shows the assumptions satisfying exactly as in [Al04, Theorem A].

Notice that the continuous dependence of the a.c.i.p. measure is obtained only on the Luzzatto-Viana parameters. We believe that, similarly to the smooth case, there is no statistical stability in the *full set of parameters*, see [Th01], [K18]. It will be seen in the future study.

Steps (i_1) and (i_2) allows us to believe that there is a version of [AOT06, Theorem B] for a map with discontinuity and singularity, then we could say that the *entropy* of the a.c.i.p. measure of f_a varies continuously with $a \in \mathcal{A}$. But, for now we will focus on guarantee of the sufficient conditions to have the complete proof of our main result.

Remark 1.2.6. Notice that, the constants b in (1.3), ε, δ in (1.4), and $\alpha, \sigma, \varepsilon, \gamma$ from (BA) and (EG) can be chosen uniformly on $\mathcal{A}$. We will discuss more about this on the section 3.3.

In [Al04], Alves proved exponential decay of the tail set and using inducing schemes and Young tower several other features of a.c.i.p. measures associated to the family can be deduced. We are going to formulate our statistical properties considering the space of *Hölder continuous functions* with Hölder constant γ, for some $\gamma > 0$. The space of functions $\varphi : I \to \mathbb{R}$ with finite Hölder norm

$$\|\varphi\| \equiv \|\varphi\|_\infty + \sup_{y_1 \neq y_2} \frac{|\varphi(y_1) - \varphi(y_2)|}{|y_1 - y_2|^\gamma}.$$

Exponential decay of correlations for the system (f_a, μ_a) for Hölder observables against $L^\infty(\mu_a)$ has already been obtained in for a similar family (see for more details [DHL06]) on a subset of parameters in $\mathcal{A}$ for which some strong form of mixing condition holds. Here we obtain exponential decay of correlations for the same observables in the whole set of parameters $\mathcal{A}$ for Luzzatto-Viana family as in [LV00]. These papers are a little difference with respect to order of degree at singularity. Moreover, we obtain the Large Deviations and other properties as consequence from other works and our result, as follows:

Corollary C. *For all $a \in \mathcal{A}$, the a.c.i.p. measure μ_a satisfies:*

(1) exponential Decay of Correlations for Hölder against $L^\infty(\mu_a)$ observables.
(2) exponential Large Deviations for Hölder observables.
(3) the Central Limit Theorem, the vector-valued Almost Sure Invariance Principle, the Local Limit Theorem and the Berry-Esseen Theorem for Hölder observables.

See the references in the following for precise mathematical formulations of the statements in Corollary C. Theorem A and [Go06, Theorem 3.1] implies that each f_a with $a \in \mathcal{A}$ has a Young tower with exponential tail of recurrence times. Then, the exponential *Decay of Correlations* and the Central Limit Theorem follow from [Yo99, Theorems 3 and 4] and [Yo99].The exponential *Large Deviations* follows from [MN08, Theorem 2.1] and [MN09]. Vector-valued *Almost Sure Invariance Principle* follows from [MN05] [MN09, Theorem 2.9] and finally, the *Local Limit Theorem* and *the Berry-Esseen Theorem* follow from [Go05, Theorems 1.2 and 1.3].

Chapter 2

Maps with critical and singular points

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> It can be argued that the mathematics behind these images [of the orbit diagram for quadratic functions and the Mandelbrot set] is even prettier than the pictures themselves.
>
> Robert L. Devaney

We assume that the dynamics of Lorenz flows is well understood, specifically in the interaction between singular behavior (trajectories near equilibrium) and critical behavior (near folding regions). This problem was inspired by observations made by other researchers on the study of the geometric Lorenz flow for non-classical parameter values.

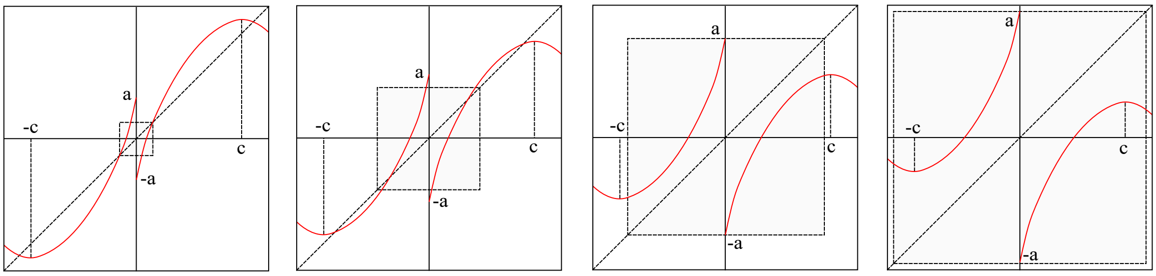

Figure 2.1: Lorenz-like families with criticalities and singularity

Here we present Luzzatto-Viana maps, $\mathcal{F} = \{f_a\}_{a \in \mathcal{A}}$, as follows:

$$f_a(x) = \begin{cases} f(x) - a, & \text{if } x > 0 \\ -f(-x) + a, & \text{if } x < 0, \end{cases} \tag{2.1}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is smooth maps and satisfies,

(S_1) $f(x) = \psi(x^\lambda)$ for all $x > 0$, where $0 < \lambda < 1/2$ and ψ is a smooth map defined on $\mathbb{R}$ with $\psi(0) = 0$ and $\psi'(0) \neq 0$;

(S_2) there exists some constant $c > 0$ such that $f'(c) = 0$;

(S_3) $f''(x) < 0$ for all $x > 0$;

(S_4) $0 < f_a(x_{\sqrt{2}}) < f_a(a) < x_{\sqrt{2}}$ for all $x_{\sqrt{2}} < a \leq c$, where for some values of parameter a we may have $[-c, c]$ within $[-a, a]$. We are specially interested in the dynamic behavior on the neighbourhood of the critical points $\pm c$. We refer by $x_{\sqrt{2}}$ to the unique point in $(0, c)$ such that $f_a'(x_{\sqrt{2}}) = \sqrt{2}$;

(S_5) $|(f_c^2)'(x)| > 2$ for all $x \in [-c, c] \setminus \{0\}$ such that $|f_c(x)| \in [x_{\sqrt{2}}, c]$.

Notice that, the last inequality in (S_4) implies that there exists a unique $x \in [-a, a]$ for given any y with $|y| \in [x_{\sqrt{2}}, a)$ such that $f_a(x) = y$ and the first inequality we can deduce $|x| < x_{\sqrt{2}}$. Also, from (S_5) if $f_c(x) = x_{\sqrt{2}}$ and $f_c(x)$ is close to c, then x is close to zero, thus $|(f_c^2)'(x)| \approx |x|^{2\lambda-1} \approx \infty$. Luzzatto and Viana proved that $\mathcal{F}$ is uniformly expanding for all parameters a up to c, it could be deduced from (S_1)-(S_5). From [LV00, Proposition 1.1] we have that given any $a \in [a_1, c]$, the interval $[-a, a]$ is f-invariant and $f\big|_{[-a,a]}$ is transitive. In 1.2.1 we proved that dynamics is transitive for parameters sufficiently near to critical point. Also, they proved at the bifurcation parameter $c = a$ the uniform expanding is impossible due to the presence of the critical point in the domain of the map f, and the nonuniform expansivity persists (in some sense), also the form of dynamics remains even after such bifurcation.

2.1 Luzzatto-Viana parameters

For each $j \geq 0$, fix some small $\varepsilon > 0$. Let $c_j = c_j(a) = f_a^j(\pm c)$, $I = [-a, a]$, denote the *dynamic distance* by $d(c_j) = \min\{|c_j|, |c_j \pm c|\}$. Let $* = 0, \pm c$ and define the *host interval*, $I^{*+}_{r,k}$ as the interval which receives some orbit and it is the union of two adjacent intervals, $I^{*+}_{r-1,k} \cup I^{*+}_{r,k} \cap I^{*+}_{r+1,k}$. Considering the parameters $a \in \mathcal{A}$ for which have exponential growth of the derivative of $f_a^j(\pm c)$. Let x in the host interval $I_r^{\pm c}$ for some $|r| \geq r_s$ (it is a radius of a small neighbourhood of origin).

Supposing that there are $n \geq \frac{r}{\alpha}$, $(r > 1)$ and for all $1 \leq j \leq n-1$ it follows,

$$d(c_j) \geq \varepsilon^{\gamma} e^{-\alpha j}, \tag{BA}$$

$$|(f_a^j)'(f_a(\pm c))| \geq e^{\sigma j}, \tag{EG}$$

with $\delta, \alpha, \sigma > 0$.

Following [LV00], we will be choosing $\sigma_0 > 0$, $\sigma > 0$ and $\gamma > 1$ such that $0 < 2\sigma <$

$\sigma_0 < \log \sqrt{2}$, and $\iota > 0$ such that $1 < \gamma + \delta + \iota < \frac{1}{2\lambda}$. Moreover, take δ small with respect to λ and ι small with respect to δ. Thus, we observe that $\gamma + \delta + \iota < \frac{1}{\lambda} - 1$, assuming $0 < \alpha < \beta$ is small just depending on the previous constants. From (S_1)-(S_3), it implies that there exist $\eta_1, \eta_2 > 0$, such that

$$\eta_1 = \lim_{x \to 0} \frac{|f(x)|}{|x|^\lambda} > 0 \quad \text{and} \quad \eta_2 = \lim_{x \to 0} \frac{|f(x) - f(c)|}{|x - c|^2} > 0.$$

For each $i = 1, 2$, we fix the constants $\eta_i^- = \eta_i - \upsilon$ and $\eta_i^+ = \eta_i + \upsilon$, where υ is some small positive number, so that

(M_1) for all $x \neq 0$ close to 0,

$$\eta_1^- |x|^\lambda \le f_a(x) + a \le \eta_1^+ |x|^\lambda, \quad \text{if} x > 0,$$
$$-\eta_1^+ |x|^\lambda \le f_a(x) - a \le -\eta_1^- |x|^\lambda, \ \text{if} x < 0,$$
$$\text{and} \quad \eta_1^- \lambda |x|^{\lambda - 1} \le |f_a'(x)| \le \eta_1^+ \lambda |x|^{\lambda-1};$$

(M_2) for all x close to the $+c$,

$$\eta_2^- (x-c)^2 \le |f_a(x) - f_a(c)| \le \eta_2^+ (x-c)^2,$$
$$\text{and} \quad 2\eta_2^- |x - c| \le |f_a'(x)| \le 2\eta_2^+ |x - c|,$$

(M_3) for all x close to $-c$,

$$\eta_2^- (x+c)^2 \le |f_a(x) - f_a(-c)| \le \eta_2^+ (x+c)^2,$$
$$\text{and} \quad 2\eta_2^- |x + c| \le |f_a'(x)| \le 2\eta_2^+ |x + c|.$$

We are going to introduce the regions close to the *critical* and *singular* points. Recall that $\mathcal{C}$ is the critical/singular set and $* = \pm c, 0$ denotes an element in $\mathcal{C}$. Given any $x \in [-a, a]$, following Benedicks-Carleson techniques (see [BC85]) the orbit of x will be split into *free periods*, *returns* and *bound periods*, which occurs in this order. For which we introduce the following neighbourhoods around the $*$ and the meaning of the technical concept by splitting the orbit of x.

We fixe a small number $\varepsilon > 0$ and denote for each ε^γ-neighbourhood by Δ_r^{-c}, Δ_r^0 and Δ_r^{+c}, where

$$\Delta_r^{-c} = (-\varepsilon^\gamma e^{-r+1} - c,\ \varepsilon^\gamma e^{-r+1} - c), \tag{2.2}$$

$$\Delta_r^{0} = (-\varepsilon^\gamma e^{-r+1},\ \varepsilon^\gamma e^{-r+1}), \tag{2.3}$$

$$\Delta_r^{+c} = (-\varepsilon^\gamma e^{-r+1} + c,\ \varepsilon^\gamma e^{-r+1} + c). \tag{2.4}$$

We consider $\Delta_r^\varepsilon = \Delta_r^{-c} \cup \Delta_r^0 \cup \Delta_r^+$. In that follows we consider for $r \geq 1$ and define the partitions of Δ_r^{-c}, Δ_r^0 and Δ_r^+ by writing $I_r = [\varepsilon^\gamma e^{-r}, \varepsilon^\gamma e^{-r+1})$ and

$$\Delta_r^{-c} = \{-c\} \cup \bigcup_{|r|\geq 1} I_r^{-c}, \qquad \Delta_r^0 = \{0\} \cup \bigcup_{|r|\geq 1} I_r^0 \quad \text{and} \quad \Delta_r^+ = \{+c\} \cup \bigcup_{|r|\geq 1} I_r^{+c}, \tag{2.5}$$

where $I_r^0 = I_r$, $\quad I_r^{\pm c} = I_r^0 \pm c$ and $I_{-r}^0 = -I_r$ (for $r \leq -1$).

Also, we denote by $I_r^{\pm c} = I_r^0 \pm c$ (just a translation of the I_r^0) and $I_r^+ = I_{r-1} \cup I_r \cup I_{r+1}$ for $|r| \geq 1$. In addition, we assume through this text $\varepsilon > 0$ is small enough so that $\Delta_r^{-c}, \Delta_r^0$ and Δ_r^+ are contained in the regions which the conditions (M_1)-(M_3) remain true.

Following [LV00, Sections 2.2 and 2.3], we consider the smaller neighbourhood around the origin such contains the preimages of the points near to the *critical neighbourhood*, $f_a^{-1}(\Delta_r^{\pm c}) \subset \Delta_{r_s}^0$, where $r_s = [(\gamma + 2\delta + \iota) \log \varepsilon^{-1}]$ with $\Delta_{r_s}^0 = \{0\} \bigcup_{|r|\geq r_s+1} I_r^0$.

It will be useful for obtaining an increasing derivative growth, which was in decrease because of the presence of the *critical points* in the domain of map f.

For each I_r, we consider the collection of r^2 equal length intervals $I_{r,1}, I_{r,2}, \ldots\ldots, I_{r,r^2}$, whose union is I_r. Also, these $I_{r,k}$ are as follows: if $k > j$, then $dist(I_{r,k}, *) < dist(I_{r,j}, *)$. By $I^+_{r,k}$ we denote the *union* of $I_{r,k}$ with two adjacent intervals of the same type (that is $I^+_{r,k} = I_{r-1,k} \cup I_{r,k} \cup I_{r+1,k}$).

The *free periods* correspond to periods of time in which the orbit never enters the region Δ^ε_r and is not in a *bound period.* During these periods the orbit of x experiment an exponential growth of its derivative $|(f^n_a)'(x)|$, provided we take parameters close to $\pm c$ as in Lemma 2.1.2, for the parameter value close to *origin* the exponential growth is natural. However, it is inevitable that the orbit for almost every point $x \in [-a, a]$ makes a return to the critical region $\Delta^{\pm c}_r = (-\delta \pm c, \delta \pm c)$ or to the singular region $\Delta^0_r = (-\delta, \delta)$, where $\delta = \varepsilon^\gamma e^{-r+1}$.

We say that $n \in \mathbb{N}$ is a *return time* of the orbit of x to the critical/singular region if $f^n_a(x) \in \Delta^\varepsilon_r$. Every free period of x ends with a return to the critical/singular region. We say that the return had a *depth* of $|r| \in \mathbb{N}$, if $|r| = [-dist_\delta(f^j_a(x), *)]$, which is equivalent to say that $f^j_a(x) \in I^*_{|r|}$. Once in the Δ^ε_r, the orbit of x initiates a binding with the critical/singular point. Here, without loss of generality, we are considering a binding period equal to zero when the orbit enters the neighbourhood of origin, as in [Fr05] and [AS12].

Following the Benedicks and Carleson's ideas (see [BC85]), also as in [Fr05] and [AS12], we shall consider two types of returns: *essential* or *inessential.* We need to distinguish each type introducing a sequence of partitions of $[-a, a]$ into intervals that will be defined later.

The *bound period* is a period after the occurrence of the return time (essential or not), during which the orbit of x is bounded by orbit of the critical/singular point.

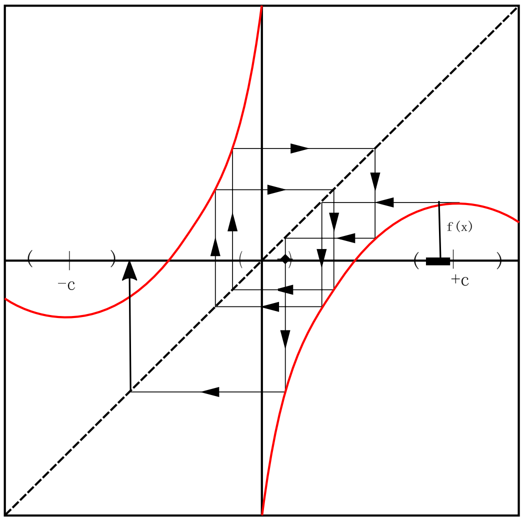

Figure 2.2: Dynamics of x under real map f on partition $\mathcal{P}_n$

Let us define more formally what it is. Assuming that the constant α in (BA) is small and β is a small number, it can be given as $\beta = \lambda\alpha$.

Defining this way, we may say about the iterates as follows:

- every iterate $j = 1, ..., n$ is called as either a *free* iterate or a *bound* iterate for $\omega \in \mathcal{P}_n$,
- the last free iterate before a bound iterate is called either an *essential return* or an *inessential return*,
- associated to each essential and inessential return there is a positive integer called the *return depth*.

We need to consider some cases depending on the *length* and *position* of the interval $\omega_n = f_a^n(\omega)$ and on whether n is a free or a bound iterate for ω. We are going to explain a litle bit about the times.

Escape times: if n is a free time for ω and $|\omega| \geq \delta$ we say that ω has escaped and that ω_n is an *escape interval.*

Free times: if n is a free time for ω and $|\omega| < \delta$ we distinguish the following cases:

(1) if $\omega_n \cap \Delta_r^\varepsilon = \emptyset$, we basically do nothing, we do not subdivide the interval ω and define $n+1$ to be again a *free* iterate for ω.

(2) if $\omega_n \cap \Delta_r^\varepsilon \neq \emptyset$, but the interval ω_n does not covers completely some interval $I_{r,k}$ we do not subdivide ω further at this moment, but wee add some information which we may say that n is an *inessential* return time with *return depth* $r := \min\{r : I_{r,k} \cap \omega_n \neq \emptyset\}$. In addition, we define all iterates $j = n+1, \ldots, n+p$ as *bound iterates* for ω, where $p = p(r)$ is the binding period associated to the return depth r as defined below.

(3) if $\omega_n \cap \Delta_r^\varepsilon \neq \emptyset$ and ω_n covers completely at least an interval $I_{r,k}$ we subdivide ω into subintervals $\omega_{r,k}$ in such a way that each $\omega_{r,k}$ stisfies $I_{r,k} \subset f_a^n(\omega_{r,k}) \subset I_{r,k}^+$. We say that $\omega_{r,k}$ is an *essential return* at time n, with retun depth r and we define the corresponding *binding* period as in the previous case.

Bound times: if n is a bound time for ω we also basically do nothing. According to the construction above, n is an iterate of some binding period in $[\upsilon+1, \upsilon+p]$ associated to a previous essential or inessential return at time υ. So, if $n < \upsilon + p$ we say that $n+1$ is still a bound iterate, if $n = \upsilon + p$ the $n+1$ is a free iterate. The biding period is important to ensure exponential growth of derivative on critical orbit.

Definition 2.1.1. *(Bound period) Given $x \in I_r^{\pm c}$, let $p(x)$ be the largest integer such that for $1 \leq i \leq p$,*

$$
\begin{aligned}
|f_a^i(x) - f_a^i(+c)| \leq& \varepsilon^\gamma e^{-\beta i}, \ \textit{if} \ \ r \geq 1, \\
|f_a^i(x) - f_a^i(-c)| \leq& \varepsilon^\gamma e^{-\beta i}, \ \textit{if} \ \ r < -1.
\end{aligned}
$$

The time interval $1, ..., p(x) - 1$ is called the bound period for x.

Since $|(f_a^n)'(x)| = \prod_{j=0}^{n-1} |f_a'(f_a^j(x))|$, the returns provide very few factors in the derivative of the orbit of x, but after the bound period not only have we recovered from the loss on the growth of the derivative caused by the return that originated from the bound period, we even have some exponential growth again.

We are going to use estimates from [LV00], which given the dynamics is essentially expanding outside of Δ_r^ε. The small derivative at the point.s near to $\Delta_r^{\pm c}$ is compensated by the large derivative at their preimages which are situated very close to the singular point. The loss of expansivity occurring after that the critical points enters in the domain of the map. Note that any piece of orbit that does not enters in Δ_r^ε has an exponential growth of the derivative. During this present text, we use σ_1 instead of e^{σ_0} (as used in [LV00, Lemma 2.3]). We define $r_c = r_c(\varepsilon) \geq 1$ by the condition $f_{c+\varepsilon}^{-1}(c) \in I_{-r_c}^0$. Observe that

$$e^{-1}\left(\frac{\varepsilon}{\eta^+}\right)^{\frac{1}{\lambda}} \leq \varepsilon^\gamma e^{-r_c} \leq \left(\frac{\varepsilon}{\eta^-}\right)^{\frac{1}{\lambda}}, \text{ by the first part in [LV00, Lemma 2.1].} \tag{2.6}$$

Moreover, the second part from that same lemma gives

$$f_a^{-1}(\Delta_r^{+c}) \subset I_{-r_c+1}^0 \cup I_{-r_c}^0 \cup I_{-r_c-1}^0, \quad \text{for every } a \in [c+\rho\varepsilon, c+\varepsilon] \tag{2.7}$$

The next lemma will estimate the accumulated derivative of the points which pass very close to the either discontinuity or the critical points.

Lemma 2.1.2. *There are $\sigma_1 > 1$ and $\varepsilon_0,\ \sigma_0 > 0$ with $\varepsilon \in\ (0,\ \varepsilon_0)$, such that for any $a \in [c+\rho\varepsilon, c+\varepsilon]$ and $x \in [-a, a]$,*

(i) if $x, f_a(x), ..., f_a^{n-1}(x) \notin \Delta_r^{\pm c}$, then $|(f_a^n)'(x)| \geq \min\{\sigma_1, |f_a'(x)|\}\ \sigma_1^{(n-1)}$,

(ii) if $x, f_a(x), ..., f_a^{n-1}(x) \notin \Delta_r^{\pm c}$, and $f_a^n(x) \in \Delta_r^{\pm c}$, then $|(f_a^n)'(x)| \geq \sigma_1^n$.

Proof. This result has essentially been obtained in [LV00, Lemmas 2.2 and 2.3]. Here we follow the main steps of that proof in order to use some properties that we need. We denote $x_j = f_a^j(x)$, for $j \in [0,\ n-1]$. We claim that given any $j \geq 1$

$$\text{either } |f_a'(x_j)| \geq \sigma_1 \text{ or } |(f_a^2)'(x_{j-1})| \geq \sigma_1^2. \tag{2.8}$$

This is true if $|x_j| \leq x_{\sqrt{2}}$, because we may have $|f_a'(x_j)| = |f_a'(x_j)| > \sigma_1$. Now, we consider $x_j \geq x_{\sqrt{2}}$, the case $x_j \leq -x_{\sqrt{2}}$ is entirely analogous.

If $x_{j-1} \in \Delta_r^0$, then $|(f_a^2)'(x_{j-1})| \geq \sigma_1^2$, by the first part in [LV00, Lemma 2.2]. Therefore, we may suppose that $x_{j-1} \notin \Delta_r^0$, that is $|x_{j-1}| \geq \varepsilon^\gamma$. Then, recall the conditions (M_1), (M_2), (M_3) and $c - f_c(x_{j-1}) \geq \eta_1^- \varepsilon^{\gamma\lambda}$ so that we have $|f'(f_c(x_{j-1}))| \geq 2\eta_1^-\eta_2^-\varepsilon^{\gamma\lambda}$. In addition, using $f_a(x_{j-1}) - f_c(x_{j-1}) = a - c \leq \varepsilon$, we get

$$\begin{aligned}\frac{|(f_a^2)'(x_{j-1})|}{|(f_c^2)'(x_{j-1})|} =& \frac{|f'(f_a(x_{j-1}))|}{|f'(f_c)(x_{j-1})|} \\ \geq& 1 - \frac{|f'(f_a(x_{j-1}))| - |f'(f_c(x_{j-1}))|}{|f'(f_c(x_{j-1}))|} \\ \geq& 1 - k_0\varepsilon^{1-\lambda\gamma},\end{aligned}$$

where $k_0 = \frac{k}{2\eta_1^-\eta_2^-}$, with k a Lipschitz constant for f' on $\{x \geq x_{\sqrt{2}} - \varepsilon_0\}$ (with ε_0 a small constant, we take for simplicity $\varepsilon \leq \varepsilon_0$). Since $1 - \lambda\gamma > 0$, the left side term is larger that $\sigma_1^2/2$ if ε is small enough and then the claim follows from (S_5). Moreover, the first statement in the Lemma is a consequence of our claim (2.8).

In order to get the second part of the Lemma, we may assume that $|f_a'(x)| \leq \sigma_1$, for otherwise there is nothing to prove. Notice also that if $f_a^n(x) \in \Delta_r^{\pm c}$ then, by (2.6) and (2.7), we have that $|f_a'(f_a^{n-1})(x)| \geq k_1\varepsilon^{1-1/\lambda}$, where $k_1 = \lambda e^{2(\lambda-1)}(\eta_1^-)\ 1/\lambda$. Moreover, by hypothesis, $x \in \Delta_r^{\pm c}$ and so $|f_a'(x)| \geq \eta_1\lambda\varepsilon^{\gamma+\delta}$. It implies that, we may write $k_3 = \eta_1 \lambda k_2$, and

$$\begin{aligned} |(f_a^n)'(x)| \geq & |f_a'(x)| {\sigma_1}^{n-2} |f'(f_a^{n-1})(x)| \\ \geq & k_3 \varepsilon^{\lambda+\delta} {\sigma_1}^{n-2} \varepsilon^{1-1/\lambda} \\ \geq & k_3 \varepsilon^{\gamma+\delta+1-1/\lambda} \ {\sigma_1}^{n-2} \\ \geq & {\sigma_1}^n, \quad \varepsilon \text{ small enough.} \end{aligned}$$

□

In [LV00, Section 2.3] shows that assuming the critical trajectories satisfying *exponential expansivity* and *bounded recurrence* conditions during a convenient number of iterates depending on $|x \pm c|$ the small value of $f_a'(x)$ is automatically compensated in the subsequent iterates. In this period of time the orbit of x stays close to the *critical point* and so that has growth of the derivative.

The next Lemmas allows us to use the properties of maps with parameters close to critical/singular points.

Lemma 2.1.3. *Let $x \in I_r^{\pm c}$ for some $|r| \geq r_s$, supposing that there is $n \geq \frac{|r|}{\alpha}$ satisfying* (EG) *and* (BA)*, also there exists $C_1 = C_1(\beta - \alpha) > 0$ such that the following estimates holds,*

(a) for all $y_1, \upsilon_1 \in [f_a(x), f_a(\pm c)]$ and for all $1 \leq k \leq p$,

$$\frac{1}{C_1} \leq \frac{|(f_a^k)'(\upsilon_1)|}{|(f_a^k)'(y_1)|} \leq C_1,$$

(b) $\dfrac{2|r|}{\beta + \log C_0} - K \leq p \leq \left(2|r| + \frac{3}{2}\gamma \log \varepsilon^{-1}\right)/\sigma \leq n-1,$ *where C_0 is positive constant such that $K = (\beta + 2 + \log(\varepsilon^\gamma \eta_2^+))/\beta + \log C_0$,*

(c) $|(f_a^{p+1})'(x)| \geq \varepsilon^{2\beta\gamma/\sigma} e^{(1-2\frac{\beta}{\sigma})|r|}$.

Proof. We are going to follows the proof exactly as in [LV00, Lemma 2.4] giving a contribution in the proof of item (b). Suppose $x \in I_r^{+c}$ with $r \geq 1$, the other cases are the same way. We consider $p = p(x) \geq 1$ as in the definition of binding period, we fixe $\beta > \alpha$ and recall that the definition of biding period and (BA) ensures that the intervals $[f^i(x),\ f^i(\pm c)]$ with $1 \leq i \leq p$ do not contain the origin nor any critical point $\pm c$. Therefore,

$$f_a^i : [f_a(x), f_a(c)] \to [f_a^{i+1}, f_a^{i+1}(c)], \text{ is a diffeomorphism for all } 1 \leq \text{i} \leq \text{p}.$$

In particular, given any $y_1, \upsilon_1 \in [f_a(x), f_a(c)]$ we have $y_i, \upsilon_i \in [f_a^i(x), f_a^i(c)]$ for $1 \leq i \leq p$, where $y_i = f_a^i(y)$ and $\upsilon_i = f_a^i(z)$. By the chain rule, it follows that

$$\left|\frac{(f_a^k)'(\upsilon_1)}{(f_a^k)'(y_1)}\right| = \prod_{i=1}^{k}\left|\frac{f_a'(\upsilon_i)}{f_a'(y_i)}\right| = \prod_{i=1}^{k}\left|1 + \frac{f_a'(\upsilon_i) - f_a'(y_i)}{f_a'(y_i)}\right|.$$

We note the part (1) is done when

$$\sum_{i=1}^{k}\left|\frac{f_a'(\upsilon_i) - f_a'(y_i)}{f_a'(y_i)}\right| \tag{2.9}$$

is bounded by some constant depending only on $\beta - \alpha$. Now, by the mean value theorem there exists , for each $1 \leq i \leq k$ some $\xi_i \in [f^i(z), f^i(y)]$ such that

$$\left|\frac{f_a'(\upsilon_i) - f_a'(y_i)}{f_a'(y_i)}\right| = \left|\frac{|\upsilon_i - f_a'(y_i)| f_a''(\xi_i)}{f_a'(y_i)}\right| = \varepsilon^{\gamma} e^{-\beta i}\left|\frac{f_a''(\xi_i)}{f_a'(y_i)}\right|.$$

It is enough to show that $\left|\frac{f_a''(\xi_i)}{f_a'(y_i)}\right| \leq$ const $\varepsilon^{-\gamma} e^{\alpha i}$ to conclude that the terms of the sum (2.9) are decreasing exponentially. Therefore the entire sum is *bounded* by a constant independent of k. We fix some small constant $\varepsilon' > 0$ independent of ε and note that $|f_a''(x)|$ is bounded above and below outside $(-\varepsilon', \varepsilon')$ by some constant

$C = \sup\{|f_a''(x)| \ : \ x \notin (-\varepsilon', \varepsilon')\}$. We may assume without loss of generality that this supremum is actually achieved at ε'. On the other hand, inside $(-\varepsilon', \varepsilon')$, we have

$$|f_a''(x)| \le \eta^+ \lambda(\lambda - 1)|x|^{\lambda-2}.$$

We distinguish two cases: if $[f_a^i(x), f_a^i(c)] \cap (-\varepsilon', \varepsilon') = \emptyset$ then we have

$$|f_a'(y_i)| \ge 2\eta_2^- |y_i - c| \ge 2\eta_2^- \ \varepsilon^\gamma(\varepsilon^{-\alpha i} - e^{-\beta i}) \ge 2\eta_2^- (1 - e^{\alpha-\beta})\varepsilon^\gamma e^{-\beta i}$$

and so

$$\left|\frac{f_a''(\xi_i)}{f_a'(y_i)}\right| \le \frac{C}{2\eta_2^-(1 - e^{\alpha-\beta})}\varepsilon^{-\gamma} e^{\alpha i}.$$

If we consider that $[f_a^i(x), f_a^i(c)] \cap (-\varepsilon', \varepsilon') \neq \emptyset$, and to simplify the notation we shall suppose that $[f_a^i(x), f_a^i(c)] \subset (0, c)$. In the other case $[f_a^i(x), f_a^i(c)] \subset (-c, 0)$ is similar. Taking ε' small and since $|f_a^i(x) - f_a^i(c)| \le \varepsilon^\gamma$ we can suppose that $[f_a^i(x), f_a^i(c)]$ is contained in the neighbourhood of origin for which (M_1) and (M_2) holds, thus we have

$$|f_a'(y_i)| \ge \left|f_a'(f_a^i(c) + \varepsilon^\gamma e^{-\beta i})^{\lambda-1}\right| \ge \eta_1^- \ \lambda(f_a^i(c) + \varepsilon^\gamma e^{-\beta i})^{\lambda-1}$$

and

$$|f_a''(\xi_i)| \le \left|f_a''(f_a^i(c) - \varepsilon^\gamma e^{-\beta i})\right| \le \eta_1^+ \lambda(\lambda - 1)(f_a^i(c) - \varepsilon^\gamma e^{-\beta i})^{\lambda-2}.$$

It implies

$$\left|\frac{f_a''(\xi_i)}{f_a'(y_i)}\right| \leq \frac{\eta_1^+(\lambda-1)}{\eta_1^-}\left(\frac{f_a^i(c)-\varepsilon^\gamma e^{-\beta i}}{f_a^i(c)+\varepsilon^\gamma e^{-\beta i}}\right)^{\lambda-1}(f_a^i(c)-\varepsilon^\gamma e^{-\beta i})^{-1} \leq \text{const } (\varepsilon^\gamma e^{-\alpha i})^{-1}.$$

This follows from the fact that $|f_a^i(c)| \geq \varepsilon^\gamma e^{-\alpha i}$ and, therefore

$$(f_a^i(c)-\varepsilon^\gamma e^{-\beta i})^{-1} \leq (\varepsilon^\gamma e^{-\alpha i})^{-1}(1-e^{(\alpha-\beta)i})^{-1} \leq \text{const } (\varepsilon^\gamma e^{-\alpha i})^{-1},$$

and that $(f_a^i(c)-\varepsilon^\gamma e^{-\beta i})/(f_a^i(c)+\varepsilon^\gamma e^{-\beta i}) \leq$ const. Indeed, it is explained by the following inequalities

$$f_a^i(c)-\varepsilon^\gamma e^{-\beta i} \geq \varepsilon^\gamma e^{-\alpha i}-\varepsilon^\gamma e^{-\beta i} \geq \varepsilon^\gamma e^{-\alpha i}(1-e^{(\alpha-\beta)i}) \geq (1-e^{\alpha-\beta})\ \varepsilon^\gamma e^{-\alpha i}$$

and similarly

$$f_a^i(c)+\varepsilon^\gamma e^{-\beta i} \leq (1+e^{\alpha-\beta})\ \varepsilon^\gamma e^{-\alpha i}$$

which gives

$$\frac{f_a^i(c)-\varepsilon^\gamma e^{-\beta i}}{f_a^i(c)+\varepsilon^\gamma e^{-\beta i}} \leq \frac{(1-e^{\alpha-\beta})}{(1-e^{\alpha+\beta})} = \text{const},$$

this proves item (a).

Now, we are going to show the right side of item (b), let $q=\min\{p,n-1\}$. As $x \in I_r^c$, we have $|x-c| \geq \varepsilon^\gamma e^{-r}$ and so $|f_a(x)-f_a(c)| \geq \eta_2^-\varepsilon^{2\gamma}e^{-2r}$. Then, in view of the condition (EG) and the distortion estimate, we just proved, the mean value theorem $\eta_2^-\varepsilon^{2\gamma}e^{-2r}C_1^{-1}e^{\sigma(q-1)} \leq |f_a^q(x)-f_a^q(c)| \leq \varepsilon^\gamma e^{-\beta q}$. Thus,

$$q \leq \frac{2r+\gamma\log(1/\varepsilon)+\sigma-\log(\eta_2^-/C_1)}{\sigma+\beta} \leq \left(2r+\frac{3}{2}\gamma\log\varepsilon^{-1}\right)/\sigma,$$

as long as ε is sufficiently small. Since we take $\alpha n \geq r \geq [\delta\log\varepsilon^{-1}] \gg 1$, we find that

$q \leq (2\alpha n + 3\gamma\alpha n/2\delta)/\sigma < n$ (if α is small), so that it must be $q = p$. In this way we have $p \leq (2r + \frac{3}{2}\gamma \log \varepsilon^{-1})/\sigma < n$ which is what we wanted to prove.

In addition, we are going to prove the left hand side of item (b) showing an extra information to use in our case. So, given $i = p+1$, by previous definition follows that

$$\varepsilon^{\gamma} e^{-\beta(p+1)} < |x_{p+1} - c_{p+1}| = |f_a^{p+1}(x) - f_a^{p+1}(c)|,$$

by the mean value theorem,

$$\varepsilon^{\gamma} e^{-\beta(p+1)} < |f_a^{p+1}(x) - f_a^{p+1}(c)| = |(f_a^p)'(y)||f_a(x) - f_a(c)|.$$

Using (M_2), it follows that

$$|f_a^{p+1}(x) - f_a^{p+1}(c)| = |(f_a^p)'(y)||x - c|^2 \eta_2^+.$$

Since that $|x - c|^2 < \varepsilon^{2\gamma} e^{-2r+2}$, it implies that

$$\varepsilon^{\gamma} e^{-\beta(p+1)} < |(f_a^p)'(y)| \varepsilon^{2\gamma} e^{-2r+2} \eta_2^+.$$

Also, note that if $f' \leq C_0$, we have $(f_a^p)'(y) = f_a'(f_a^{p-1}).....f_a'(y) \leq C_0^p$, thus

$$e^{-\beta(p+1)} < C_0^p \varepsilon^\gamma e^{-2r+2} \eta_2^+ \Leftrightarrow$$

$$-\beta p - \beta \leq p \log C_0 + \log \varepsilon^\gamma + \log \eta_2^+ + 2 - 2r \Leftrightarrow$$

$$-\beta p - \beta \leq p \log C_0 + \log(\varepsilon^\gamma \eta_2^+) + 2 - 2r \Leftrightarrow$$

$$-\beta \leq p(\beta + \log C_0) + \log(\varepsilon^\gamma \eta_2^+) + 2 - 2r \Leftrightarrow$$

$$-\beta - \log(\varepsilon^\gamma \eta_2^+) - 2 + 2r \leq p(\beta + \log C_0) \Leftrightarrow$$

$$\Leftrightarrow \frac{2|r|}{\beta + \log C_0} - K \leq p, \text{ where}$$

$K = \dfrac{\beta + 2 + \log(\varepsilon^\gamma \eta_2^+)}{\beta + \log C_0}$.

Now, by definition of bound period, we have $|f_a^{p+1}(x) - f_a^{p+1}(c)| \geq \varepsilon^\gamma e^{-\beta(p+1)}$. Thus, using part (a) together with the mean value theorem, it follows

$$|(f_a^p)'(f_a(x))| \geq \frac{1}{C_1} \frac{|f_a^{p+1}(x) - f_a^{p+1}(c)|}{|f_a(x) - f_a(c)|} \geq \frac{\varepsilon^\gamma e^{-\beta(p+1)}}{C_1 \eta_2^- \ \varepsilon^{2\gamma} e^{-2r+2}} \geq \text{const } \varepsilon^{-\gamma} e^{2r-\beta p}.$$

Since $|f_a'(x)| \geq 2\eta_2^- \varepsilon^\gamma e^{-r}$, it follows

$$|(f_a^{p+1})(x)| \geq \text{const } e^{r-\beta p}. \tag{2.10}$$

Using part (b), and since $|(f_a^{p+1})'(x)| \geq \text{const } \varepsilon^{3\beta\gamma/2\sigma} e^{(1-2\beta/\sigma)r}$ we get

$$e^{r-\beta p} \geq e^{r - \frac{\beta}{\sigma}(2r + \frac{3}{2}\gamma \log \varepsilon^{-1})} \geq e^{(1-\frac{2\beta}{\sigma})r - \frac{3\beta\gamma}{2\sigma} \ \log \varepsilon^{-1}} \geq \varepsilon^{\frac{3\beta\gamma}{2\sigma}} \ e^{(1-\frac{2\beta}{\sigma})r},$$

it implies that

$$|(f_a^{p+1})'(x)| \geq \varepsilon^{2\beta\gamma/\sigma} e^{(1-2\beta/\sigma)r}.$$

This proves item (c). □

Now we estimate the length of $|f_a^n(\omega)|$. The next Lemma follows from 2.1.2 and 2.1.3 the same way as in [LV00].

Lemma 2.1.4. *Let υ be a return time for $\omega \in \mathcal{P}_{n-1}$ with host interval $I^*_{r,k}$ $(* = 0, \pm c)$ and $p = p(r)$. Then we have the following estimates.*

1. *assuming that $\tilde{\upsilon} \leq n-1$ is the next return time(either essential or not) for ω associated to Δ_r^0, defining free time as $q = \tilde{\upsilon} - (\upsilon + p)$ and*

   (a) *if $n = \tilde{\upsilon} + 1$ is a return to $\Delta_r^{\pm c}$, then $|f_a^{\tilde{\upsilon}}(\omega)| \geq {\sigma_1}^q \varepsilon^{1+\iota-\gamma} e^{(1-\frac{2\beta}{\sigma})|r|} |f_a^{\upsilon}(\omega)|$,*

   (b) *if $n > \tilde{\upsilon} + 1$ is a return to Δ_r^0, then $|f_a^{\tilde{\upsilon}}(\omega)| \geq {\sigma_1}^q \varepsilon^{\frac{2\gamma\beta}{\sigma}} e^{(1-\frac{2\beta}{\sigma})|r|} |f_a^{\upsilon}(\omega)|$,*

   (c) *if $n > \tilde{\upsilon}+1$ is a return to $\Delta_r^{\pm c}$, then $|f_a^{\tilde{\upsilon}}(\omega)| \geq {\sigma_1}^q\ C_2\ \varepsilon^{\frac{2\gamma\beta}{\sigma}} e^{(1-\frac{2\beta}{\sigma})|r|} |f_a^{\upsilon}(\omega)|$.*

   In addition, for a sufficiently large Λ it follows that $|f_a^{\tilde{\upsilon}}(\omega)| \geq B_1 |f_a^{\upsilon}(\omega)|$, for some constant $B_1 > 0$.

2. *assuming that $\upsilon \leq n-1$ is the last essential return time associated to $\Delta_r^{\pm c}$, $r_s \leq r \leq (\gamma + \delta + \iota) \log \varepsilon^{-1}$, putting $q = n - (\upsilon + p)$, then n is necessarialy a return to Δ_r^0 and $|f_a^n(\omega)| \geq {\sigma_1}^q \varepsilon^{\frac{-\iota}{2}} \varepsilon^{\gamma} e^{-|r_s|}$*

3. *Also, if υ is the last return time of ω up to $n-1$ associated to $\Delta_r^{\pm c}$ and $r_s \geq r \geq (\gamma + \delta + \iota) \log \varepsilon^{-1}$, besides this if*

   (a) *n is a return situation to Δ_r^0, then $|f_a^{\tilde{\upsilon}}(\omega)| \geq {\sigma_1}^q \varepsilon^{\frac{2\gamma\beta}{\sigma}} e^{(1-\frac{2\beta}{\sigma})|r|} |f_a^{\upsilon}(\omega)|$,*

   (b) *n is a return situation to Δ_r^0, and υ is the last essential return, then*

   $$|f_a^n(\omega)| \geq {\sigma_1}^q \varepsilon^{\gamma - \frac{\beta}{\sigma}} e^{-(1-\frac{\beta}{\sigma})|r|},$$

   (c) *n is a return situation to $\Delta_r^{\pm c}$, then $|f_a^{\tilde{\upsilon}}(\omega)| \geq {\sigma_1}^q\ C_2\ \varepsilon^{\frac{2\gamma\beta}{\sigma}} e^{(1-\frac{2\beta}{\sigma})|r|} |f_a^{\upsilon}(\omega)|$.*

(d) n is a return situation to $\Delta_r^{\pm c}$, and υ is the last essential return, then

$$|f_a^n(\omega)| \geq {\sigma_1}^q \; C_2 \; \varepsilon^{\gamma-\frac{\beta}{\sigma}} e^{-(1-\frac{\beta}{\sigma})|r|}.$$

In the items above the constant C_2 is equal to $\varepsilon^{1-\frac{1}{\lambda}}$

Proof. The proof follows from [LV00, Lemma 3.3] and the strategy used by in [Fr05] and [AS12]. By the mean value theorem, for some $y \in \omega \in \mathcal{P}_{n-1}$ we have

$$|f_a^{\tilde{\upsilon}(\omega)}| \geq |(f_a^q)'(f_a^{\upsilon+p})(y)||(f_a^p)'(f_a^\upsilon(y))||f_a^\upsilon(\omega)|. \tag{2.11}$$

Using the part (ii) of Lemma 2.1.2 and part (c) of Lemma 2.1.3, it follows that

$$|f_a^{\tilde{\upsilon}(\omega)}| \geq {\sigma_1}^q \varepsilon^{\frac{2\gamma\beta}{\sigma}} e^{(1-\frac{2\beta}{\sigma})|r|}|f_a^\upsilon(\omega)|, \tag{2.12}$$

this proves item (1.b). Also, since $\frac{2\beta\gamma}{\sigma} > 1 + \iota - \gamma$, from (2.12) we get

$$|f_a^{\tilde{\upsilon}(\omega)}| \geq {\sigma_1}^q \varepsilon^{1+\iota-\gamma} e^{(1-\frac{2\beta}{\sigma})|r|}|f_a^\upsilon(\omega)|, \tag{2.13}$$

it is enough to prove of item (1.a).

To obtain the proof of item (1.c) we notice that from (1.b), we just add the term $\varepsilon^{1-1/\lambda} > 0$ as in [LV00, Lemma 3.3], because n is a return to $\Delta_r^{\pm c}$. Moreover, if we take Λ big enough such that ${\sigma_1}^q \varepsilon^{\frac{2\gamma\beta}{\sigma}} e^{(1-\frac{2\beta}{\sigma})|r|} > B_1$, follows the claim in part (1), $|f_a^{\tilde{\upsilon}}(\omega)| \geq B_1 |f_a^\upsilon(\omega)|$, for some constant $B_1 > 0$.

Note that part (2) is proved in [LV00, Lemma 3.3]. Now, by the same argument using the lemmas 2.1.2 and 2.1.3 into (2.11), we get

$$|f_a^{\tilde{\upsilon}}(\omega)| \geq {\sigma_1}^q \varepsilon^{\frac{2\gamma\beta}{\sigma}} e^{(1-\frac{2\beta}{\sigma})|r|}|f_a^\upsilon(\omega)|.$$

From the host interval definition and since $\frac{2\gamma\beta}{\sigma} \geq \gamma - \frac{\beta}{\sigma}$. From the previous inequality it follows that

$$|f_a^{\tilde{\upsilon}}(\omega)| \geq {\sigma_1}^q \varepsilon^{\frac{2\gamma\beta}{\sigma}} e^{-\frac{2\beta}{\sigma}|r|} \geq {\sigma_1}^q \varepsilon^{\gamma-\frac{\beta}{\sigma}} e^{-(1\frac{\beta}{\sigma})|r|}.$$

Now, since

$$e^{-\frac{2\beta}{\sigma}|r|} = e^{-(1-\frac{\beta}{\sigma})|r|} e^{(1-\frac{\beta}{\sigma})|r|}, \tag{2.14}$$

we get that

$$|f_a^n(\omega)| \geq {\sigma_1}^q \varepsilon^{\gamma-\frac{\beta}{\sigma}} e^{-(1-\frac{\beta}{\sigma})|r|}.$$

The items (3.a), (3.b) have just been proved above and the item (3.c) follows directly from item (1.c). The last item (3.d) will be obtained using the host interval definition into (3.c), and from (2.14), it follows that

$$|f_a^n(\omega)| \geq {\sigma_1}^q \ C_2 \varepsilon^{\gamma-\frac{\beta}{\sigma}} \ e^{-(1-\frac{\beta}{\sigma})|r|},$$

this completes the proof of Lemma. □

We are going to build inductively a monotone sequence of partitions $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ of the subintervals ω_0, $\mathcal{P}_0 \prec \mathcal{P}_1 \prec \ldots \prec \mathcal{P}_{n-1} \prec \mathcal{P}_n \prec \ldots$ of the set $I = [-a, a]$ (modulo a zero Lebesgue measure set) into intervals (given $\omega \in \mathcal{P}_{n-1}$, there is $\omega' \in \mathcal{P}_{n-1}$ such that $\omega \subset \omega'$). We shall define inductively the sets $R_n(\omega) = \{\upsilon_1, \ldots, \upsilon_{\gamma(n)}\}$ which is the set of the return times of $\omega \in \mathcal{P}_n$ up to n and the set $Q_n(\omega) = \{(r_1, k_1), \ldots, (r_{\gamma(n)}, k_{\gamma(n)})\}$ which records the indices of the intervals such that $f_a^{\upsilon_i}(\omega) \subset I^+_{r_i,k_i}$, $i = 1, \ldots, \upsilon_{\gamma(n)}$. By construction, we must have for all $n \in \mathbb{N}_0$

$$\forall \omega \in \mathcal{P}_n \quad f_a^{n+1}|_\omega \quad \text{is a diffeomorphism.} \tag{2.15}$$

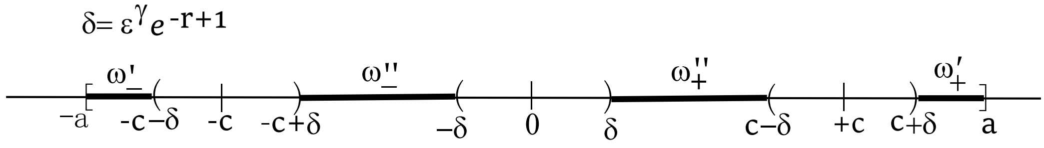

Figure 2.3: subintervals of [-a, a] out of critical/singular region

For $n = 0$, integer Λ, recall that $\delta = \varepsilon^{\gamma} e^{-r+1}$ we define

$$\mathcal{P}_0 = \{[-a, -c-\delta], [-c+\delta, -\delta], [\delta, c-\delta], [c+\delta, a]\} \bigcup \{I^*_{r,k} : |r| \geq \Lambda, 1 \leq k \leq r^2\}.$$

Note that for every $\omega \in \mathcal{P}_0$, $f|_\omega$ satisfies (2.15). Set $R_0([-a, -c-\delta]) = R_0([-c+\delta, -\delta) = R_0([\delta, c-\delta]) = R_0([c+\delta, a]) = \emptyset$, $R_0(I^0_{r,k}) = \{0\}$, $R_0(I^{+c}_{r,k}) = \{+c\}$, $R_0(I^{-c}_{r,k}) = \{-c\}$. Also, $Q_0([-a, -c-\delta]) = Q_0([-c+\delta, -\delta]) = Q_0([\delta, c-\delta]) = Q_0([c+\delta, a]) = \emptyset$ and $Q_0(I^*_{r,k}) = \{(r, k)\}$.

Assuming that $\mathcal{P}_{n-1}$ is defined and it satisfies (2.15) and R_{n-1}, Q_{n-1} are defined for each element of $\mathcal{P}_{n-1}$. Fixing $\omega \in \mathcal{P}_{n-1}$, there are three possible situations:

1. If $R_{n-1}(\omega) \neq \emptyset$ and $n \leq \upsilon_{\gamma(n-1)} + p(r_{\gamma(n-1)})$, we call n a *bound time* for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega)$, $Q_n(\omega) = Q_{n-1}(\omega)$.

2. If $R_{n-1}(\omega) = \emptyset$ or $n > \upsilon_{\gamma(n-1)} + p(r_{\gamma(n)})$ and $f^n(\omega) \cap \Delta_\Lambda \subset I_{\Lambda,a} \cup I_{-\Lambda,a}$, we call n a *free time* for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega)$, $Q_n(\omega) = Q_{n-1}(\omega)$.

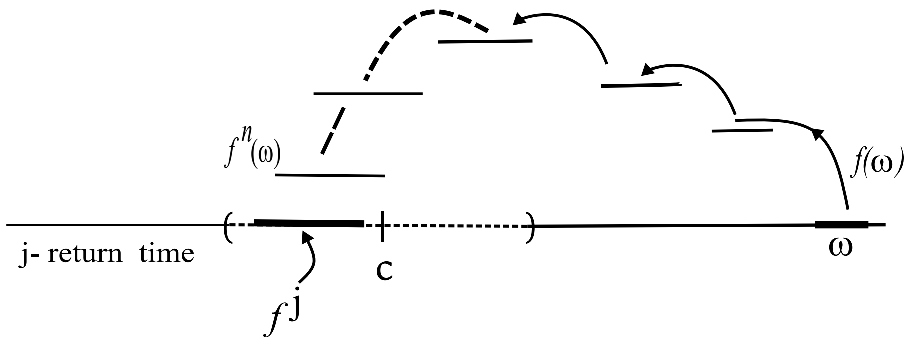

Figure 2.4: Return situation of $\omega \in \mathcal{P}_n$ in $\Delta_r^{\pm c}$

3. If the two above conditions do not hold, ω has a *free return situation* at time n. Thus, We consider the following cases:

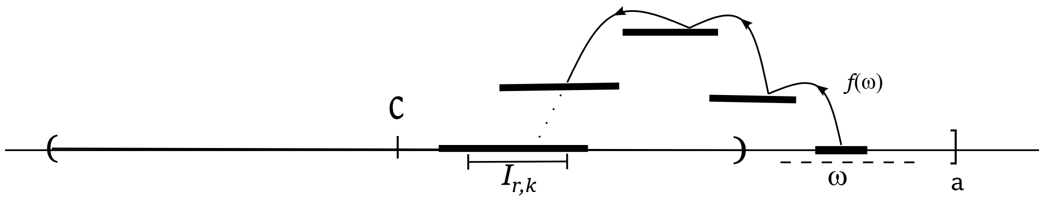

Figure 2.5: Essential return situation, $I_{r,k} \subset f_a^v(\omega)$

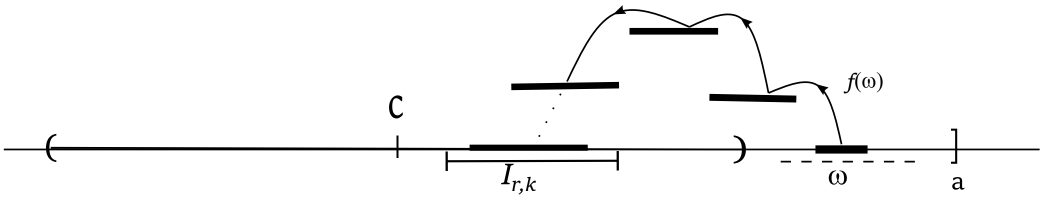

Figure 2.6: Inessential return situation, $f_a^v(\omega) \subset I_{r,k}$

(a) $f_a^n(\omega)$ does not completely cover some interval $I^*_{r,k}$. Since $f_a^n|_\omega$ is a diffeomorphism and ω is an interval, $f^n(\omega)$ is also an interval and thus is contained in some $I^+_{r,k}$, which is called the *host interval* of the return (i.e., is an interval that receive some return). We call n an *inessential return time* for ω, put $\omega \in \mathcal{P}_n$ and set $R_n(\omega) = R_{n-1}(\omega) \cup \{n\}$, $Q_n(\omega) = Q_{n-1}(\omega) \cup \{(r,k)\}$.

(b) $f_a^n(\omega)$ contains at least an interval $I^*_{r,k}$, in in this case we say that ω has an *essential return situation* at time n. Take

$$\begin{aligned}
\omega_{r,k} &= f_a^{-n}(I^*_{r,k}) \cap \omega, \quad \text{for } |r| - 1 \geq \Lambda \\
\omega'_+ &= f_a^{-n}([\delta + c, a]) \cap \omega, \\
\omega''_+ &= f_a^{-n}([\delta, c - \delta]) \cap \omega, \\
\omega'_- &= f_a^{-n}([-a, -c - \delta]) \cap \omega, \\
\omega''_- &= f_a^{-n}([-c + \delta, -\delta]) \cap \omega.
\end{aligned}$$

We have $\omega\backslash f_a^{-n}(*) = \cup\, \omega_{r,k} \cup\ \omega'_+ \cup\ \omega''_+ \cup\ \omega'_- \cup\ \omega''_-$. By the induction hypothesis $f_a^n|_\omega$ is a diffeomorphism and then each $\omega_{r,k}$ is an interval. Moreover $f_a^n(\omega_{r,k})$ covers $I^*_{r,k}$ except perhaps for the two end intervals. We join $\omega_{r,k}$ with its adjacent interval, if it does not cover $I^*_{r,k}$ entirely. We also proceed likewise when $f_a^n(\omega'_+)$ does not cover $I_{(\Lambda+c)-a,((\Lambda+c)-a)^2}$, $f_a^n(\omega''_+)$ does not cover $I_{2\Lambda-c,(2\Lambda-c)^2}$, $f_a^n(\omega'_-)$ does not cover $I_{a-(\Lambda+c),((\Lambda+c)-a)^2}$ or $f_a^n(\omega''_-)$ does not cover $I_{c-2\Lambda,(2\Lambda-c)^2}$. So we get a new decomposition of $\omega\backslash(\omega'_+\cup\omega''_+\cup\omega'_-\cup\omega''_-)$ into intervals $\omega_{r,k}$ (mod 0) such that $I^*_{r,k} \subset f_a^n(\omega_{r,k}) \subset I^{*+}_{r,k}$. Putting $\omega_{r,k} \in \mathcal{P}_n$ for all indices (r,k) such that $\omega_{r,k} \neq \emptyset$, set $R_n(\omega_{r,k}) = R_{n-1}(\omega) \cup \{n\}$ and call n an *essential return time* for $\omega_{r,k}$. The interval $I^{*+}_{r,k}$ is called the *host interval* of $\omega_{r,k}$ and $Q_n(\omega_{r,k}) = Q_{n-1}(\omega) \cup \{(r,k)\}$. In the case when $f_a^n(\omega'_+)$ covers $I_{(\Lambda+c)-a,((\Lambda+c)-a)^2}$ we say n is an *escape time* for ω'_+ and $R_n(\omega'_+) = R_{n-1}(\omega)$, $Q_n(\omega'_+) = Q_{n-1}(\omega)$. We proceed similarly for ω''_+, ω'_-, and ω''_-. In this setting we refer to ω'_+, ω''_+, $\omega'_-$and ω''_- as *escaping components*.

To finish the construction we have to verify that (2.15) holds for $\mathcal{P}_n$.Indeed, since for any interval $J \subset I$

$$\left.\begin{array}{l} f_a^n|_J \ \text{ is a diffeomorphism} \\ 0 \notin f^n(J) \end{array}\right\} \Rightarrow f_a^{n+1}|_J \ \text{ is a diffeomorphism.} \tag{2.16}$$

We must prove $* \notin f_a^n(\omega)$ for all $\omega \in \mathcal{P}_n$. So take $\omega \in \mathcal{P}_n$. If n is a free time, there is nothing to prove. If n is a return time, either essential or inessential, we have by construction $f_a^n(\omega) \subset I^{*+}_{r,k}$ for some $|r| \geq \Lambda$ and $k = 1, 2, \ldots, r^2$ and thus $0 \notin f_a^n(\omega)$, the same way we conclude that $\pm c \notin f_a^n(\omega)$. If n is a bound time, then from basic assumption (BA) and by definition of bound period we have for all $x \in \omega$

with $r_{\gamma(n-1)} > 0$ and $n - \upsilon_{\gamma(n-1)} \geq 1$,

$$\begin{aligned}
|f_a^n(x)| \geq \;& |f_a^{n-1-\upsilon_{\gamma(n-1)}}(-c)| - |f_a^n(x) - f_a^{n-1-\upsilon_{\gamma(n-1)}}(-c)| \\
\geq \;& |f_a^{n-1-\upsilon_{\gamma(n-1)}}(-c)| - |f_a^{n-\upsilon_{\gamma(n-1)}}(f_a^{\upsilon_{\gamma(n-1)}}(x)) - f_a^{n-1-\upsilon_{\gamma(n-1)}}(-c)| \\
\geq \;& e^{-\alpha(n-\upsilon_{\gamma(n-1)})} - \varepsilon^\gamma e^{-\beta(n-\upsilon_{\gamma(n-1)})} \\
\geq \;& e^{-\alpha(n-\upsilon_{\gamma(n-1)})}\left(1 - \varepsilon^\gamma e^{-(\beta-\alpha)(n-\upsilon_{\gamma(n-1)})}\right) \\
> \;& 0, \qquad\qquad \text{since } \;\varepsilon^\gamma > 0, \beta - \alpha > 0.
\end{aligned}$$

The same conclusion can be written for $\omega \in \mathcal{P}_n$ with $r_{\gamma(n-1)} < 0$. We just need to replace $-c$ by $+c$ in the calculation above, and since at the *origin* we do not have binding (by definition).

The next Lemma tells us about the escape component returns considerably large in the return situation after the escape time.

Lemma 2.1.5. *Suppose that $\omega \in \mathcal{P}_{n-1}$ is an escape component, then if n is the next return situation for ω,*

$$|f_a^n(\omega)| \geq \sigma_1{}^q \varepsilon^{\frac{-\iota}{2}} \varepsilon^\gamma e^{-|r_s|}$$

Proof. The statement follows directly from Lemma 2.1.4. If υ is a return to $\Delta_r^{\pm c}$ the result follows from part (2), on the other hand if υ is s return to Δ_r^0, then we have

$$|f_a^n(\omega)| \geq \sigma_1{}^q \varepsilon^{\gamma - \frac{\beta}{\sigma}} e^{-(1-\frac{\beta}{\sigma})|r|} \gg \sigma_1{}^q \varepsilon^{\frac{-\iota}{2}} \varepsilon^\gamma e^{-|r_s|}$$

□

Before the bounded distortion result, let us introduce a preliminary lemma. Take $\omega \subset [-a, a]$ an interval non-empty. For each $x \in \omega$, we define the distance from a point x to a critical/singular region Δ_r^ε as follows:

$$\mathfrak{D}(x) = \min\{|x|,\ ||x| - c|\}, \qquad \text{and} \qquad \mathfrak{D}(\omega) = \inf_{x\in\omega} \mathfrak{D}(x)$$

Lemma 2.1.6. *Given an interval $\omega \in \mathcal{P}_n$ on $[-a, a]$ with $|\omega| \leq \mathfrak{D}(\omega)$. Then, there exists $\tilde{C} > 0$ such that*

$$\sup_{x,y\in\omega} \left| \frac{f_a''(x)}{f_a'(y)} \right| \leq \frac{\tilde{C}}{\mathfrak{D}(\omega)}.$$

Proof. We have two cases to study. First of all, if $\omega \cap \Delta_r^\varepsilon = \emptyset$, there is nothing to check. On the other hand, if $\omega \cap \Delta_r^\varepsilon \neq \emptyset$, so the interval can be on either $\Delta_r^{\pm c}$ or Δ_r^0.

We assume that $\omega \cap \Delta_r^0 \neq \emptyset$. From (M_1), the derivatives f'' and f' are bounded above and below. In this way, it follows that, $\omega \subseteq 2\Delta_r^0$ and

$$\begin{aligned} \eta_1^- \lambda \mathfrak{D}(\omega)^{\lambda-1} \leq \quad |f_a'(y)| \quad &\leq \eta_1^+ \lambda |y|^{\lambda-1}, \quad \text{for all } y \in \omega. \\ |f_a''(x)| \quad &\leq \eta_1^+ \lambda^+ \mathfrak{D}(\omega)^{\lambda-2}, \quad \text{for all } x \in \omega \quad \text{with} \quad \lambda^+ = \lambda^2 - \lambda. \end{aligned}$$

Taking the quotient and supremum at the previous inequalities, the result follows.

Similarly, if we assume that $\omega \cap \Delta_r^{\pm c} \neq \emptyset$. We use (M_2) and (M_3) instead (M_1), the derivatives f' and f'' also are bounded above and below. We have $\omega \subseteq 2\Delta_r^{\pm c}$ and

$$\begin{aligned} |f_a'(y)| &\geq 2\eta_2^- |x - c|, \quad \text{for all} \quad y \in \omega. \\ |f_a''(x)| &\leq 2\eta_2^+, \quad \text{for all} \quad x \in \omega. \end{aligned}$$

Taking the quotient and supremum at the previous inequalities, the result follows. □

As the orbit of $x \in \omega$ will split into *free times, return times* and *bound times*, respectively, we are going to analyze the contribution of the distortion in each period of time. The next result follows from [LV00, Lemma 3.7] with some adaptation for our situation in this thesis.

Proposition 2.1.7 (Bounded distortion). *There exists a constant $B > 1$(independent of ε, n or ω) such that if $\omega \in \mathcal{P}_{n-1}$ with $f_a^n(\omega) \subset \Delta_r^\varepsilon$. Then,*

$$\left|\frac{(f_a^n)'(x)}{(f_a^n)'(y)}\right| \le B \qquad \textit{for all} \;\; x, y \in \omega$$

Proof. We consider the sets of returns time and host indices of $\omega \in \mathcal{P}_n$, as the following, respectively:

- $R_{n+1}(\omega) = \{\upsilon_0, \dots, \upsilon_{\gamma(n+1)}\}$,
- $Q_{n+1}(\omega) = \{(r_1, k_1), \dots, (r_{\gamma(n+1)}, k_{\gamma(n+1)})\}$,
- for $x, y \in \omega$. Let $\omega_i = f_a^i(\omega)$, $p_i = p(\omega_{u_i})$, $x_i = f_a^i(x)$ and $y_i = f_a^i(y)$, where p_i corresponding to the *binding periods*. Here, throughout this text, we consider $p_i = 0$, if ω_{υ_i} returns to Δ_r^0.

By the chain rule for $k = n-1$, we have

$$\left|\frac{(f_a^n)'(x)}{(f_a^n)'(y)}\right| = \prod_{j=0}^{k}\left|\frac{f_a'(x_j)}{f_a'(y_j)}\right| = \left|\prod_{j=0}^{k}\left(1 + \frac{f_a'(x_j) - f_a'(y_j)}{f_a'(y_j)}\right)\right| \le \prod_{j=0}^{k}\left(1 + \frac{|f_a'(x_j) - f_a'(y_j)|}{|f_a'(y_j)|}\right).$$

We consider $A_j = \dfrac{f_a'(x_j) - f_a'(y_j)}{f_a'(y_j)}$. We are going to show that $\sum_{j=0}^{k} |A_j|$ is *uniformly bounded.* Taking the log in both side in the previous inequality and using that $\log(1+|x|) \le |x|$, it follows that

$$\log\left|\frac{(f_a^n)'(x)}{(f_a^n)'(y)}\right| \le \log\prod_{j=0}^{k}\left(1 + \frac{|f_a'(x_j) - f_a'(y_j)|}{|f_a'(y_j)|}\right) \le \sum_{j=0}^{k}\left(\frac{|f_a'(x_j) - f_a'(y_j)|}{|f_a'(y_j)|}\right).$$

If we show the right side at the inequality above is limited by some constant $B > 0$, that is

$$\sum_{j=0}^{k}\left(\frac{|f'_a(x_j)-f'_a(y_j)|}{|f'_a(y_j)|}\right)<B.$$

Then, $\left|\dfrac{(f_a^n)'(x)}{(f_a^n)'(y)}\right|<e^B$ is *uniformly bounded.*

By the mean value theorem, we have

$$\frac{|f'_a(x_j)-f'_a(y_j)|}{|f'_a(y_j)|}\leq\frac{|f''_a(\sigma_{1j})||x_j-y_j|}{|f'_a(y_j)|}\quad\text{for some}\quad\sigma_j\in(x_j,y_j)$$

and

$$\frac{|f'_a(x_j)-f'_a(y_j)|}{|f'_a(y_j)|}\leq\sup_{\sigma_1,\sigma_2\in\omega_j}\left|\frac{f''_a(\sigma_1)}{f'_a(\sigma_1)}\right|\;|x_j-y_j|.$$

So, the proof will be complete when the sum $\sum_{j=0}^{k}\sup_{\sigma_1,\sigma_2\in\omega_j}\left|\dfrac{f''_a(\sigma_1)}{f'_a(\sigma_2)}\right|\;|x_j-y_j|$ is uniformly bounded.

Now, we are going to look at the contribution of the *free period*, *returns* and *bound period.*

If $n<\upsilon_\gamma+p_\gamma$, we split the sum $\sum_{j=0}^{n}|A_j|$ into three sums involving free period, return time and bound period. Then,

$$\sum_{j=0}^{n}|A_j|=\sum_{i=1}^{\gamma(n+1)}\left(\sum_{j=\upsilon_{i-1}+p_{i-1}}^{\upsilon_i-1}|A_j|+|A_{\upsilon_i}|+\sum_{j=\upsilon_i+1}^{\upsilon_i+p_i-1}|A_j|\right)\tag{2.17}$$

We want to show that each one of the three terms on the right-hand side can be bounded above by some uniform constant multiple of $|\omega_{\upsilon_j}|e^{|r_j|}$. It will be proved in three consecutive Lemmas as follows.

First of all, we shall study the contribution of the free period between υ_{i-1} and υ_i for the sum $\sum_{j=0}^{n}|A_j|$.

Lemma 2.1.8. *We have,* $\sum_{j=\upsilon_{i-1}+p_{i-1}}^{\upsilon_i-1}|A_j|\leq\dfrac{\tilde{C}}{\sigma_1-1}|\omega_{\upsilon_j}|e^{|r_j|}$

Proof. Since $\omega_i \subset I_r^{c^+} \subset [\varepsilon^\gamma e^{-(r+1)}, \varepsilon^\gamma e^{-(r-1)})$ we suppose, without of loss of generality that

$$|f_a^n(\omega)| \leq (\varepsilon^\gamma e^{-\alpha n})^{2\lambda}. \tag{2.18}$$

For $j \in [\upsilon_{i-1} + p_{i-1}, \upsilon_i - 1]$, we have

$$|\omega_{\upsilon_i}| \geq |f_a^{\upsilon_i - j}(x_j) - f_a^{\upsilon_i - j}(y_j)| = |(f_a^{\upsilon_i - j})'(\sigma)||x_j - y_j|,$$

for some $\sigma \in (x_j, y_j)$, by the Lemma 2.1.2, we have

$$|(f_a^{\upsilon_i - j})'(\sigma)||x_j - y_j| \geq \sigma^{\upsilon_i - j}|x_j - y_j|,$$

which implies that

$$|\omega_{\upsilon_i}|\sigma^{j-\upsilon_i} \geq \sup_{x_j, y_j \in \omega_j} |x_j - y_j|. \tag{2.19}$$

Moreover, if ω_i stays out of Δ_r^ε, then a preliminary estimative is given from 2.18, by

$$|\omega_{\upsilon_i}| \leq e^{-\sigma_0(n-1)}|f_a^n(\omega)| \leq e^{-\sigma_0(n-1)}(\varepsilon^\gamma e^{-\alpha n})^{2\lambda} \ll \varepsilon_c,$$

with $\varepsilon_c = \max\{\varepsilon_{-c}, \varepsilon_0, \varepsilon_{+c}\}$, where $(-\varepsilon_{\pm c}, \varepsilon_{\pm c})$ and $(-\varepsilon_0, \varepsilon_0)$ are some neighbourhoods of the critical points and the singular point.

As ω_i stays out of Δ_r^ε, if ε_c is the maximum of the neighbourhoods as defined above and $\varepsilon_c < \mathfrak{D}(\omega_j) := \inf_{x \in \omega_j} |x - \Delta_r^\varepsilon|$. By the previous inequalities (2.18) and (2.19), follows that

$$\mathfrak{D}(\omega_j) > \varepsilon_c \gg e^{-\sigma_0(n-1)}|f_a^n(\omega)| \geq |\omega_{\upsilon_i}| > |\omega_j|. \tag{2.20}$$

Therefore,

$$\mathfrak{D}(\omega_j) > |\omega_j|.$$

Since $\sigma > 1,\ \upsilon_i - j > 0,\ |r_i| > \Lambda$, by hypothesis of the Lemma 2.1.6 holds, so that using (2.19), we have

$$\sup_{x,y\in\omega_j}\left|\frac{f_a''(x)}{f_a'(y)}\right||x_j - y_j| \leq \frac{\tilde{C}}{\mathfrak{D}(\omega_j)}|\omega_{\upsilon_j}|\sigma^{j-\upsilon_i}. \tag{2.21}$$

So, taking the sum $\sum\limits_{j=\upsilon_i-1+p_i-1}^{\upsilon_i-1}$ in the previous inequality, results in

$$\begin{aligned}
\sum_{j=\upsilon_i-1+p_i-1}^{\upsilon_i-1}\ \sup_{x,y\in\omega_j}\left|\frac{f_a''(x)}{f_a'(y)}\right||x_j - y_j| &\leq \frac{\tilde{C}}{\mathfrak{D}(\omega_j)}|\omega_{\upsilon_j}|\sum_{j=\upsilon_i-1+p_i-1}^{\upsilon_i-1}\sigma^{j-\upsilon_i},\\
&\leq \frac{\tilde{C}}{\varepsilon_c}|\omega_{\upsilon_j}|\sum_{j=\upsilon_i-1+p_i-1}^{\upsilon_i-1}\sigma^{j-\upsilon_i},\\
&\leq \tilde{C}|\omega_{\upsilon_j}|e^{|r_j|}\sum_{j=\upsilon_i-1+p_i-1}^{\upsilon_i-1}\sigma^{j-\upsilon_i}, \qquad \frac{1}{\varepsilon_c} < e^{\Lambda} \leq e^{|r_i|},\\
&\leq \frac{\tilde{C}}{\sigma-1}|\omega_{\upsilon_i}|e^{|r_j|}.
\end{aligned}$$

□

In the next Lemma, we deal with the return times.

Lemma 2.1.9. *We have* $|A_j| \leq \varepsilon^{\gamma}e\tilde{C}|\omega_{\upsilon_j}|e^{|r_j|}$.

Proof. We are going to obtain the contribution of the return times. Let ω_{υ_i} be a subset on $I_{r_i}^{*+}$.

First of all, we suppose that $\omega_{\upsilon_i} \subset I_{r_i}^{0+} \subset [\varepsilon^{\gamma}e^{-|r_i|-1}, \varepsilon^{\gamma}e^{-|r_i|+2})$. From inequality (2.20), we have the hypothesis in the Lemma 2.1.6 for ω_{υ_i} which satisfies ($\mathfrak{D}(\omega_{\upsilon_i}) \geq \varepsilon^{\gamma}e^{-|r_i|-1}$). In addition, from Lemma 2.1.6, it follows that

$$\sup_{x,y\in\omega_j}\left|\frac{f_a''(x)}{f_a'(y)}\right||x_j - y_j| \leq \frac{\tilde{C}}{\mathfrak{D}(\omega_{\upsilon_i})}|\omega_{\upsilon_i}| \leq \tilde{C}e^{-\gamma}e^{|r_i|+1}|\omega_{\upsilon_i}| \leq \varepsilon^{\gamma}e\ \tilde{C}|\omega_{\upsilon_i}e^{|r_i|}.$$

As $|A_{\upsilon_i}| \leq \sup_{x,y\in\omega_j}\left|\frac{f_a''(x)}{f_a'(y)}\right| |x_j - y_j|$, the result follows,

$$|A_j| \leq \varepsilon^{\gamma}\ e\ \tilde{C}|\omega_{\upsilon_j}|e^{|r_j|}.$$

Now, we are going to calculate the contribution of the return time when $* = +c$, then the case $* = -c$ follows similarly.

Since $\omega_{\upsilon_i} \subset I_{r_i}^{+c^+} \subset [\varepsilon^{\gamma}e^{-|r_i|-1} + c, \varepsilon^{\gamma}e^{-|r_i|+2} + c)$. From the previous inequality (2.20), we have the hypothesis of the Lemma 2.1.6 for ω_{υ_i} satisfied ($\mathfrak{D}(\omega_{\upsilon_i}) \geq \varepsilon^{\gamma}e^{-|r_i|-1} + c$). In addition, from Lemma 2.1.6, it follows that

$$\sup_{x,y\in\omega_j}\left|\frac{f_a''(x)}{f_a'(y)}\right| |x_j - y_j| \leq \frac{\tilde{C}}{\mathfrak{D}(\omega_{\upsilon_i}) - c}|\omega_{\upsilon_i}| \leq \tilde{C}\varepsilon^{-\gamma}e^{|r_i|+1}|\omega_{\upsilon_i}| \leq \varepsilon^{\gamma}\ e\ \tilde{C}|\omega_{\upsilon_i}|\ e^{|r_i|}.$$

As $|A_{\upsilon_i}| \leq \sup_{x,y\in\omega_j}\left|\frac{f_a''(x)}{f_a'(y)}\right| |x_j - y_j|$. Therefore,

$$|A_{\upsilon_i}| \leq \varepsilon^{\gamma}\ e\ \tilde{C}|\omega_{\upsilon_i}|\ e^{|r_i|}.$$

The case when $* = -c$, we use the host interval $I_{r_i}^{-c^+}$ instead of $I_{r_i}^{+c^+}$ and using a similar calculation, we can conclude the proof of this lemma. □

In the final part, we must show the contribution of the bound period time. After theses Lemmas we go back to the proof of the Proposition 2.1.7.

Lemma 2.1.10. *We have,* $\sum_{j=\upsilon_i+1}^{\upsilon_i+p_i-1}|A_j| \leq \tilde{C}B'\frac{e^{\alpha-\beta}}{(1-e^{\alpha-\beta})^2}\ |\omega_{\upsilon_i}|e^{|r_j|}$

Proof. First of all, if ω_{υ_i} is a returning to Δ_r^0, the binding period is zero (by definition), there is nothing to prove. Thus, we may assume that $\omega_{\upsilon_i} \subseteq I_{r_i}^{c^+} \subset \Delta_r^c$ (We can deal similarly for Δ_r^{-c}). Now, take the interval $\tilde{\omega} = (+c, \varepsilon^{\gamma}e^{-r+2} + c]$ into $\mathcal{P}_{n-1}$. We will compute the contribution of the bound period for $j \in (\upsilon_i, \upsilon_i + p_i)$. Denote by $\tilde{\omega}_j$ the iterates of $\tilde{\omega}$, that is $\tilde{\omega}_j = f_a^j(\tilde{\omega})$. By bound period definition, we have

$$|\tilde{\omega}_j| \leq \varepsilon^\gamma e^{-\beta j}. \tag{2.22}$$

Moreover,

$$\begin{aligned}
|\tilde{\omega}_j| =& |(f_a^{j-1})'(\sigma)||\tilde{\omega}_1|, \text{ for some } \sigma \in \tilde{\omega}_1 \subset [c, f_a(\varepsilon^\gamma e^{-r+2})], \\
\geq & \frac{1}{C_1}|(f_a^{j-1})'(c)||\tilde{\omega_1}|, \quad \text{by the lemma 2.1.3.}
\end{aligned}$$

Therefore, from (2.22),

$$|(f_a^{j-1})'(c)| \leq C_1 \frac{|\tilde{\omega}_j|}{|\tilde{\omega_1}|} \leq C_1 \frac{\varepsilon^\gamma e^{-\beta_j}}{|\tilde{\omega_1}|}. \tag{2.23}$$

Now, consider $\sigma \subseteq [\varepsilon^\gamma e^{-|r_j|-1}, \varepsilon^\gamma e^{-|r_j|+2})$ and by Lemma 2.1.3, (2.23), (M_2),

$$\begin{aligned}
|f_a^j(\omega_{v_i})| &= |\omega_{v_i}||(f_a^j)'(\sigma)|, \qquad \text{for some } \sigma \in \omega_{v_i} \\
&= |\omega_{v_i}||(f_a^{j-1})' f_a((\sigma))||f_a'(\sigma)|, \\
&\leq C_1 |\omega_{v_i}||f_a'(\sigma)||(f_a^{j-1})'(c)|, \\
&\leq C_1 |\omega_{v_i}| 2\eta_2^+(\varepsilon^\gamma e^{-|r_j|+2} - c)|(f_a^{j-1})'(c), \\
&\leq C_1 |\omega_{v_i}| 2\eta_2^+(\varepsilon^\gamma e^{-|r_j|+2} - c) C_1 \varepsilon^\gamma \frac{1}{|\tilde{\omega_1}|} e^{-\beta_j}, \\
&\leq C_1^2 \, \frac{2\eta_2^+}{\eta_2^-} |\omega_{v_i}| e^{-\beta_j} e^{|r_j|-2}, \\
&\leq B' |\omega_{v_i}| e^{-\beta_j} e^{|r_j|}.
\end{aligned} \tag{2.24}$$

Since $\left[\varepsilon^\gamma e^{-|r_j|+1}, \varepsilon^\gamma e^{-|r_j|}\right] \subset \left[\varepsilon^\gamma e^{-|r_j|-1}, \varepsilon^\gamma e^{-|r_j|+2}\right)$, we have

$$|\omega_{v_i}| \leq 3 \, \varepsilon^\gamma \frac{e^{-|r_j|+1} - e^{-|r_j|}}{(|r_j| - 1)^2} = 3 \, \varepsilon^\gamma \frac{e - 1}{(|r_j| - 1)^2} \, e^{-|r_j|}. \tag{2.25}$$

Combining (2.24) and (2.25),

$$|f_a^j(\omega_{v_i})| \leq B' \frac{|\omega_{v_i}|}{|\sigma - c|}\ e^{-\beta_j}. \tag{2.26}$$

Since, $\sigma \in \omega_{v_i} \subset \left[\varepsilon^\gamma e^{-|r_j|-1},\ \varepsilon^\gamma e^{-|r_j|+2}\right)$, we have

$$|\omega_{v_i}| \leq 3\ \frac{\varepsilon^\gamma e^{-|r_j|+2} - \varepsilon^\gamma e^{-|r_j|+1}}{(|r_j|-1)^2} = 3\ \frac{e-1}{(|r_j|-1)^2}\ \varepsilon^\gamma e^{-|r_j|+1}. \tag{2.27}$$

Combining (2.26) and (2.27), it follows that

$$\begin{aligned} |f^j(\omega_{v_i})| &\leq 3\ \frac{B'}{|\sigma - c|}\ \varepsilon^\gamma \frac{e-1}{(|r_j|-1)^2}\ e^{-|r_j|} e^{-\beta_j}, \\ &\leq 3B'\varepsilon^\gamma \frac{e-1}{(|r_j|-1)^2} e^{-\beta_j}. \end{aligned} \tag{2.28}$$

If we take Λ large enough such that $3B'\varepsilon^\gamma \frac{e-1}{(\Lambda-1)^2} + 1 \leq e^{\beta-\alpha}$, then

$$3B'\varepsilon^\gamma \frac{e-1}{(|r_j|-1)^2} + 1 \leq 3B'\ \varepsilon^\gamma \frac{e-1}{(\Delta-1)^2} + 1 \leq \varepsilon^\gamma e^{\beta-\alpha} \leq \varepsilon^\gamma e^{(\beta-\alpha)j}. \tag{2.29}$$

On the other hand, for j in the bound period and $x \in \omega_{v_i}$, we have

$$\varepsilon^\gamma e^{-\beta_j} \geq |f_a^{j-1}(+c) - f_a^j(x)| \geq |f_a^{j-1}(+c)| - |f_a^j(x)| \geq \varepsilon^\gamma e^{-\alpha j} - |f_a^j(x)|,$$

which implies that

$$|f_a^j(x) \geq \varepsilon^\gamma e^{-\alpha j} - \varepsilon^\gamma e^{-\beta j},$$

so that

$$\mathfrak{D}(f_a^j(\omega_{v_i})) \geq \varepsilon^\gamma (e^{-\alpha j} - e^{-\beta j}).$$

Using (2.29), we have

$$\mathfrak{D}(f_a^j(\omega_{v_i})) \geq \varepsilon^\gamma(e^{-\alpha j} - e^{-\beta j}) \geq 3B'\varepsilon^\gamma \frac{e-1}{(|r_j|-1)^2} e^{-\beta_j} \geq |f_a^j(\omega_{v_i})|.$$

Now, using Lemma 2.1.6

$$\sup_{\sigma_1,\sigma_2 \in f^j(\omega_{v_i})} \left| \frac{f_a''(\sigma_1)}{f_a'(\sigma_2)} \right| \leq \frac{\tilde{C}}{\mathfrak{D}(f_a^j(\omega_{v_i}))}, \quad \text{from the last inequality follows that}$$

$$\frac{\tilde{C}}{\mathfrak{D}(f_a^j(\omega_{v_i}))} \leq \frac{\tilde{C}}{e^{-\alpha j} - e^{-\beta_j}}.$$

In this step, we are ready to compute the contribution of the bound period

$$\begin{aligned} \sum_{j=v_i+1}^{v_i+p_i-1} |A_j| &\leq \sum_{j=1}^{p_i-1} |f_a^j(\omega_{v_i})| \sup_{\sigma_1,\sigma_2 \in f_a^j(\omega_{v_i})} \left| \frac{f_a''(\sigma_1)}{f_a'(\sigma_2)} \right| \\ &\leq \sum_{j=1}^{p_i-1} B' \frac{|\omega_{v_i}|}{|\sigma - c|}\; e^{-\beta j}\; \frac{\tilde{C}}{e^{-\alpha j} - e^{-\beta j}}. \end{aligned}$$

From (2.26), we have

$$\begin{aligned} \sum_{j=1}^{p_i-1} B'\; e^{-\beta j}\; \frac{|\omega_{v_i}|}{|\sigma - c|}\; \frac{\tilde{C}}{e^{-\alpha j} - e^{-\beta j}} &\leq \sum_{j=1}^{p_i-1} B' \frac{|\omega_{v_i}|}{|\sigma - c|}\; e^{-\beta_j} \frac{\tilde{C}}{e^{-\alpha j} - e^{-\beta j}} \\ &\leq \tilde{C}B' \frac{|\omega_{v_i|}}{|\sigma - c|} \sum_{j=1}^{p_i-1} \frac{e^{-\beta_j}}{e^{-\alpha j} - e^{-\beta j}} \\ &\leq \tilde{C}B' \frac{e^{\alpha-\beta}}{(1-e^{\alpha-\beta})^2}\; |\omega_{v_i}| e^{|r_j|}, \end{aligned}$$

□

If we assume that $n < v_\gamma + p_\gamma$, then we split the sum $\sum_{j=0}^{n} |A_j|$ into three sums in the following order: free period, return time and bound period.

$$\begin{aligned}
\sum_{j=0}^{n} |A_j| &= \sum_{i=1}^{\gamma(n+1)} \left(\sum_{j=\upsilon_{i-1}+p_{i-1}}^{\upsilon_i-1} |A_j| + |A_{\upsilon_i}| + \sum_{j=\upsilon_i+1}^{\upsilon_i+p_i-1} |A_j| \right) \\
&\leq \sum_{i=1}^{\gamma(n+1)} \left(\frac{\tilde{C}}{\sigma-1} |\omega_{\upsilon_i}| e^{|r_i|} + \varepsilon^{\gamma}\ e\ \tilde{C} |\omega_{\upsilon_i}| e^{|r_i|} + \tilde{C}B' \frac{e^{\alpha-\beta}}{(1-e^{\alpha-\beta})^2}\ |\omega_{\upsilon_i}| e^{|r_j|} \right) \\
&\leq \tilde{C}_1 \sum_{i=1}^{\gamma(n+1)} |\omega_{\upsilon_i}| e^{|r_j|} \\
&\leq \tilde{C}_1 \sum_{R\geq\Lambda} e^{R} \sum_{i:|r_i|=R} |\omega_{\upsilon_i}|.
\end{aligned}$$

We notice that by the first part of Lemma 2.1.4, exactly as in [Fr05, Lemma 4.1], we can have for Λ large $|f_a^{\tilde{\upsilon}}(\omega)| \geq |f_a^{\upsilon}(\omega)|$. In addition, it implies that if $\{\upsilon_{i_j} : j = 1, \ldots, r\}$ is a set of returns with depth R that is in an increasing order, then

$$\sum_{i:|r_i|=R} |\omega_{\upsilon_i}| = \sum_{j=1}^{r} |\omega_{\upsilon_{i_j}}| \leq \sum_{j=1}^{r} 2^{-r+j} |\omega_{\upsilon_{i_r}}| \leq 2|\omega_{\upsilon_{i_r}}| \leq 2\frac{e^{-R}-e^{-R-1}}{R^2}.$$

It implies that

$$\tilde{C}_1 \sum_{R\geq\Lambda} e^{R} \sum_{i:|r_i|=R} |\omega_{\upsilon_i}| \leq \tilde{C}_1 \sum_{R\geq\Lambda} e^{R}\ 2\frac{e^{-R}-e^{-R-1}}{R^2}.$$ Wherefore, we have

$$\sum_{j=0}^{n} |A_j| \leq 2\ \tilde{C}_1 \sum_{R\geq\Lambda} \frac{1-e^{-1}}{R^2}.$$

On the other side, if we assume that $n+1 \geq \upsilon_\gamma + p_\gamma$, then we must be careful with the last piece of the free period, i.e., for $j \in [\upsilon_\gamma + p_\gamma,\ n]$, as we are in free period, it follows that:

$$\begin{aligned}
|\omega_{n+1}| &= (f_a^{n+1-j})'(\sigma)\ |\omega_j|, \quad \text{for some } \sigma \in \omega_j \\
&\geq {\sigma_1}^{n+1-j}\ |\omega_j|. \qquad (2.30)
\end{aligned}$$

We consider two cases: the first one we suppose that $|\omega_{n+1}| \leq \varepsilon^{\gamma} e^{-r+2}$. Proceeding as before, we have $|\omega_j| \leq {\sigma_1}^{j-n-1}\varepsilon^{\gamma} e^{-r+2} \leq \varepsilon^{\gamma} e^{-r+2}$, since $j \leq n$,

$$\begin{aligned} |\omega_j| &\leq {\sigma_1}^{j-n}\varepsilon^{\gamma} e^{-r+2} \\ &\leq \varepsilon^{\gamma} e^{-r+2}. \end{aligned} \tag{2.31}$$

During the last *free period* ω_j is outside of $(-c-\delta, -c+\delta) \cup (-\delta, \delta) \cup (c-\delta, c+\delta)$, then

$$\mathfrak{D}(\omega_j) \geq \varepsilon^{\gamma} e^{-r+2}.$$

Consequently, the assumption in the Lemma 2.1.6 holds, i.e., $\varepsilon^{\gamma} e^{-r+2} \leq \mathfrak{D}(\omega_j)$, which implies that

$$\sup_{\sigma_1, \sigma_2 \in \omega_j} \left| \frac{f_a''(\sigma_1)}{f_a'(\sigma_2)} \right| \leq \frac{C_1}{\mathfrak{D}(\omega_j)},$$

and so that

$$\begin{aligned} \sum_{j=\upsilon_\gamma+p_\gamma}^{n} \sup_{\sigma_1, \sigma_2 \in \omega_j} \left| \frac{f_a''(\sigma_1)}{f_a'(\sigma_2)} \right| |x_j - y_j| &\leq \sum_{j=\upsilon_\gamma+p_\gamma}^{n} |\omega_j| \frac{C_1}{\mathfrak{D}(\omega_j)} \\ &\leq \sum_{j=\upsilon_\gamma+p_\gamma}^{n} \frac{C_1}{\varepsilon^{\gamma} e^{-r+2}} \, |\omega_j| \\ &\leq \sum_{j=\upsilon_\gamma+p_\gamma}^{n} \frac{C_1}{\varepsilon^{\gamma} e^{-r+2}} \varepsilon^{\gamma} e^{-r+2} \, {\sigma_1}^{j-n} \\ &= C_1 \sum_{j=\upsilon_\gamma+p_\gamma}^{n} {\sigma_1}^{j-n} \\ &= \frac{\sigma_1 C_1}{\sigma_1 - 1}. \end{aligned}$$

Now, to complete the proof we take the case where $|\omega_{n+1}| > \varepsilon^{\gamma} e^{-r+2}$. Let $q \geq \upsilon_\gamma + p_\gamma$ be the last integer such that $|\omega_q| \leq \varepsilon^{\gamma} e^{-r+2}$. Then, since $\omega_{n+1} = f_a^{n+1}(\omega) \subset$

$(c,\ \varepsilon^{\gamma}e^{-r+1}+c]$, we have $|\omega_{n+1}| \leq \varepsilon^{\gamma}e^{-r+1}$. Hence, by (2.30) follows that

$$\varepsilon^{\gamma}e^{-r+1} \geq |\omega_{n+1}| \geq {\sigma_1}^{n+1-j}|\omega_j|.$$

On the other hand, if we consider $j = q+1$, for $j \in [\upsilon_\gamma + p_\gamma,\ n]$, we get

$$|\omega_{n+1}| \geq {\sigma_1}^{n-q}|\omega_{q+1}|.$$

Therefore,

$$\varepsilon^{\gamma}e^{-r+2} \geq |\omega_{n+1}| \geq {\sigma_1}^{n-q}|\omega_{q+1}|. \tag{2.32}$$

Moreover, we noticed that if q is the last integer such that $|\omega_q| \leq \varepsilon^{\gamma}e^{-r+2}$, then we have $|\omega_{q+1}| > \varepsilon^{\gamma}e^{-r+2}$ and from (2.32),

$$\varepsilon^{\gamma}e^{-r+2} \geq |\omega_{n+1}| \geq {\sigma_1}^{n-q}|\omega_{q+1}| > \varepsilon^{\gamma}e^{-r+2}\ {\sigma_1}^{n-q}, \quad \text{and so that,} \tag{2.33}$$

$$n-q \leq \frac{1}{\ln\sigma_1}.$$

In other words, $n-q$ is bounded by above. Hence, we have $f_a^{n+1}(\omega) = f_a^{n-q}(\omega_{q+1}) \subset \Delta_r^{+c}$ holds for $j \in [0, n-q]$, $\omega_j \cap \Delta_\Lambda^{+c} = \emptyset$ $(|r| > \Lambda)$, and the parameter value a is very close to $+c$ (the similar occurs to $-c$).

So, since f is a *Misiurewicz map* with negative Schwarzian derivative, there exists a constant $B < \infty$ independent of Λ such that $\frac{(f_a^{n-q})'(x_q)}{(f_a^{n-q})'(y_q)} \leq B$, see [MV12, Proposition 6.1].

Therefore, as $n-q$ is bounded and derivatives of f_a depend continuously on a, we may take the parameter value close to $+c$ sufficiently in order to have

$$\frac{(f_a^{n-q})'(x_q)}{(f_a^{n-q})'(y_q)} \leq 2B.$$

Finally, from the last three Lemmas the bound distortion result follows.

□

2.2 Return depths

In this section we will look at the return depths which provides a reasonable explanation of the first basic idea for the proof of our main theorem: *the depth of the inessential and bound returns is smaller than the depth of the essential return preceding them (as we will show in Lemmas 2.2.1 and 2.2.2).* Also, we have the total sum of the depth of bound and inessential returns is proportional to the depth of the essential return preceding them, as we will show in Propositions 2.2.3 and 2.2.4.

The main ingredients to prove derives from (BA), (EG) and other properties of the critical and singular orbit. As it has been seen by other researchers in other cases, there are three types or returns: *essential, bound* and *inessential*, which are denoted by υ, u and $\tilde{\upsilon}$ respectively. Each essential return might be followed by some bound return and inessential return. We proceed to show that the depth of an inessential return is not greater than the depth of an essential return that precedes it. For simplicity, we are going to consider $* = +c$, $I_{r,k}^{+c^+} = [\varepsilon^\gamma e^{-(r+1)} + c, \varepsilon^\gamma e^{-(r-1)} + c)$ in the next two Lemmas. The same conclusion can be drawn for bound returns with $* = 0, -c$.

Lemma 2.2.1. *Suppose υ is an essential return time for $\omega \in \mathcal{P}_\upsilon$ with $I_{r,k}^{+c} \subset f_a^\upsilon(\omega) \subset I_{r,k}^{+c^+}$. The depth of each inessential return $\tilde{\upsilon}$ before the next essential return is not* *grater than r.*

Proof. By Lemma 2.1.4 part 1-item (c), it follows that

$$|f_a^{\tilde{v}}(\omega)| \geq B_1|f_a^{v}(\omega)| \geq B_1|I_{r,k}^{+c}|, \quad B_1 \geq 1.$$

Since $\tilde{v}$ is an inessential return time, $f_a^{\tilde{v}}(\omega) \subseteq I_{r_1,k_1}^{+c}$ for some $r_1 \geq \Lambda$ and $1 \leq k_1 \leq r_1^2$. Therefore, $|I_{r,k}^{+c}| \leq |I_{r_1,k_1}^{+c}|$, which implies that $r > r_1$. □

The same conclusion can be drawn for bound returns and for $* = 0, -c$

Lemma 2.2.2. *Suppose v is an return situation(essential or an inessential return time) for ω with $f_a^v(\omega) \subset I_{r,k}^{+c}$ and $p = p(r)$ is the bound period associated to this return. Then for $x \in \omega$, if the orbit of x returns to Δ_Λ between t and $t+p$ the depth of this bound return will not be grater than r.*

Proof. Without loss of generality, assume that $r > 0$, since we are in the bound period

$$|f_a^{j-1}(+c)| - |f^{t+j}(x)| \leq |f_a^{j-1}(+c) - f^{t+j}(x)| \leq \varepsilon^\gamma e^{-\beta j}, \quad \text{for } j = 1, ..., p-1.$$

Hence,

$$\begin{aligned} |f^{t+j}(x)| \geq & |f_a^{j-1}(+c)| - e^{-\beta j} \\ \geq & \varepsilon^\gamma e^{-\alpha j} - \varepsilon^\gamma e^{-\beta j} = \varepsilon^\gamma e^{-\alpha j}(1 - e^{(\alpha-\beta)j}), \qquad \text{(BA)} \\ \geq & \varepsilon^\gamma e^{-\alpha p}(1 - e^{\alpha-\beta}), \quad \alpha - \beta < 0. \end{aligned}$$

Following the proof in [LV00, Lemma 2.4], since $p \le \frac{4}{\sigma}|r|$, it implies that

$$\begin{aligned}|f^{t+j}(x)| \geq & \varepsilon^{\gamma} e^{-\alpha \frac{4}{\sigma}|r|}(1-e^{\alpha-\beta}) \\ \geq & e^{-5\alpha/\sigma|r|}, \qquad 1-e^{\alpha-\beta} > \varepsilon^{-\gamma}\; e^{-\alpha/\sigma|r|} \\ \geq & e^{-|r|}, \qquad \alpha < \sigma/5.\end{aligned}$$

□

In the proof of the next Lemma we shall use the *free period notion* which asserts that the fact of the time spent by either critical or singular orbits in bound period of the length $n \in \mathbb{N}$ is at most $\epsilon_0 n$ (for small ϵ_0).

Lemma 2.2.3. *There is a constant $C_3 > 0$ such that if t is a return time for $\omega \in \mathcal{P}_t$ with $I^{*+}_{r,k}$ ($* = 0, \pm c$) the host interval, p is the bound period associated with this return, and S is the sum of the depth of all bound returns between t and $t+p$ plus the depth of the return t that originated the bound period p, then $S \le C_3|r|$.*

Proof. We consider u_1 as the first time between t and $t+p$ that the orbit of $x \in \omega$ enters Δ^{ε}_{r}. Since the bound period at time t has not finished yet we may assume at time u_1 there is just one active binding to the critical point and we call u_1 a bound return of level 1. Recall that we are using the binding period to the origin equal to zero by standard definition.

Exactly at time u_1 the orbit of x establishes a new binding to the critical point which ends before $t+p$ that we denote by p_1.

Notice that all along the period from u_1 to $u_1 + p_1$ a new return may occur and its level is at least 2, because there are still at least two active bindings, one initiated at t and the another one initiated at u_1. Nevertheless new bound returns of level 1 may occur after $u_1 + p_1$. In this way we define the notion of bound return of level i at which the orbit has already initiated exactly i bindings to the critical point and all of them are still active at the moment. By active we mean that the respective bound periods have not finished yet.

Here we use free period assumption which gives that from t to $t+p$, the orbit of $x \in \omega$ can spend at most the fraction of time εp in bound periods. Now suppose n denotes the number of bound returns of level 1 at $u_1, \dots, u_n$ with depths $r_1, \dots, r_n$ and bound periods $p_1, \dots, p_n$. Then by the Lemma 2.1.3 we have,

$$\frac{1}{\beta + \log C_0} \sum_{i=1}^{n} |r_i| \leq \sum_{i=1}^{n} p_i \leq \varepsilon p \leq \varepsilon \ (2|r| + \frac{3}{2}\gamma \log \varepsilon^{-1})/\sigma.$$

Consequently,

$$\sum_{i=1}^{n} |r_i| \leq \varepsilon \ (2|r| + \frac{3}{2}\gamma \log \varepsilon^{-1})/\sigma \ (\beta + \log C_0).$$

We denote n_i the number of bound returns of level 2 within the i-th bound period of level 1 at $u_{i1}, \dots, u_{in_i}$ with depths $r_{i1}, \dots, r_{in_i}$ and bound periods $p_{i1}, \dots, p_{in_i}$. Then

$$\frac{1}{\beta + \log C_0} \sum_{i=1}^{n} |r_i| \leq \sum_{j=1}^{n} p_{ij} \leq \varepsilon p_i \leq \varepsilon \ (2|r_i| + \frac{3}{2}\gamma \log \varepsilon^{-1})/\sigma.$$

Thus,

$$\begin{aligned}\sum_{i=1}^{n} \sum_{j=1}^{n_i} |r_{i,j}| &\leq \sum_{j=1}^{n} \varepsilon \ (2 + \frac{3}{2}\gamma \log \varepsilon^{-1}) \ \frac{|r_i|}{\sigma} \ (\beta + \log C_0) \\ &\leq \Big(\varepsilon(2 + \frac{3\gamma}{2} \log \varepsilon^{-1})/\sigma \ (\beta + \log C_0)\Big)^2 |r|.\end{aligned}$$

By induction we have

$$S \leq \sum_{i=0}^{\infty} \left(\alpha(2 + \frac{3\gamma}{2} \log \varepsilon^{-1})/\sigma \ (\beta + \log C_0)\right)^i |r| \leq C_3|r|.$$

As the depth t that originated from the bound period is precisely $|r|$, it is enough to take $C_3 = \tilde{C_3} + 1$ for some $\tilde{C_3} > 0$, $\alpha(2 + \frac{3\gamma}{2}\log\varepsilon^{-1})/\sigma\ (\beta + \log C_0) < 1$ and $C_3 = \big(1 - \alpha(2 + \frac{3\gamma}{2}\log\varepsilon^{-1})/\sigma\ (\beta + \log C_0)\big)^{-1}$. We get, $S \leq C_3|r|$, where $C_3 = \text{const}(\alpha, \gamma, \beta, C_0)$,

$$C_3 = \Big(1 - \alpha\big(2 + \frac{3\gamma}{2}\ \log\varepsilon^{-1}\big)/\sigma\ (\beta + \log C_0)\Big)^{-1}.$$

□

Lemma 2.2.4. *Let t be an essential return time for $\omega \in \mathcal{P}_t$ with host interval $I^{*+}_{r,k}$ ($* = 0, \pm c$), S is the sum of the depths of all free inessential returns before the next essential return situation and p denotes the associated bound period. Then $S \leq C_4|r|$, with constant $C_4 > 0$.*

Proof. Suppose that n is the number of inessential returns before the next essential return situation of ω which time occurrence $\upsilon_1, \ldots, \upsilon_n$, with respective depths $r_1, \ldots, r_n$ and with respective bound periods $p_1, \ldots, p_n$. Moreover, we denote by $\tilde{\upsilon} = \upsilon_{j+1}$ the next essential return situation. Let $\omega_j = f^{\upsilon_j}(\omega)$ for $j = 1, \ldots, n+1$.

We obtain some cases from Lemma 2.1.4, which we describe below

1. $|\omega_1| \geq {\sigma_1}^{q_0}\varepsilon^{\gamma - \frac{\beta}{\sigma}}e^{-(1-\frac{\beta}{\sigma})\ |r|}$ and $\dfrac{|\omega_{\tilde{\upsilon}}|}{|\omega_\upsilon|} = \dfrac{|\omega_{j+1}|}{|\omega_j|} \geq {\sigma_1}^{q_j}\varepsilon^{2\gamma\beta/\sigma}e^{(1-\frac{2\beta}{\sigma})\ |r_j|}$, where $q = \tilde{\upsilon} -\ (\upsilon +\ p)$. Let us consider that $|\omega_{\tilde{\upsilon}}| \leq e$, $|r| > 1$. Thus, we have from the equality

$$|\omega_{n+1}| = |\omega_1| \prod_{j=1}^{n} \frac{|\omega_{j+1}|}{|\omega_j|} \tag{2.34}$$

that

$$|\omega_1| \prod_{j=1}^{n} \frac{|\omega_{j+1}|}{|\omega_j|} = |\omega_{\tilde{\upsilon}}| \leq e$$

$${\sigma_1}^{q_0}\varepsilon^{\gamma-\beta/\sigma}e^{-(1-\beta/\sigma)|r|} \prod_{j=1}^{n} {\sigma_1}^{q_j}\varepsilon^{2\gamma\beta/\sigma}e^{(1-\frac{\beta}{\sigma})\ |r_j|} \leq |\omega_{\tilde{\upsilon}}| \leq e$$

it implies that

$$\sigma_1^{\sum_{j=0}^{n} q_j} \varepsilon^{\gamma-\beta/\sigma} \varepsilon^{2\gamma\beta/\sigma} e^{-(1-\beta/\sigma)|r|} e^{\sum_{j=1}^{n}(1-2\beta/\sigma)|r_j|} \leq |\omega_{\tilde{v}}| \leq e$$

$$e^{\sum_{j=1}^{n}(1-2\beta/\sigma)|r_j|} \leq e^{(2-\beta/\sigma)|r|} \; \sigma_1^{-\sum_{j=0}^{n} q_j} / \varepsilon^{\gamma+\beta/\sigma(2\gamma-1)}.$$

Therefore,

$$\sum_{j=1}^{n} |r_j| \leq \frac{(2-\beta/\sigma)|r| - \sum_{j=0}^{n} q_j \; \log \sigma_1}{(1-2\beta/\sigma)\; \varepsilon^{\gamma+\beta/\sigma(2\gamma-1)}},$$

$$S \leq C_4^1 |r|, \text{ where } \quad C_4^1 = \text{const}(\beta, \sigma, \gamma, \varepsilon) > 0.$$

2. $|\omega_1| \geq \sigma_1{}^{q_0} C_2 \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)|r|}$ and $\dfrac{|\omega_{\tilde{v}}|}{|\omega_v|} \geq \sigma_1{}^{q_j} C_2 \varepsilon^{2\gamma\beta/\sigma} e^{(1-\frac{2\beta}{\sigma})\,|r_j|}$, where $q = \tilde{v} - (v + p)$. The same way as in case (1), since $|\omega_{\tilde{v}}| \leq e$, $|r| > 1$ from (2.34) and from Lemma 2.1.4, it follows that

$$\sigma_1{}^{q_0} C_2 \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)|r|} \prod_{j=1}^{n} \sigma_1{}^{q_j} C_2 \varepsilon^{2\gamma\beta/\sigma} e^{(1-\frac{2\beta}{\sigma})|r_j|} \leq e$$

it implies that

$$C^2_{\varepsilon,\lambda} \; \sigma_1^{\sum_{j=0}^{n} q_j} \varepsilon^{\gamma-\beta/\sigma+2\gamma\beta/\sigma} e^{-(1-\beta/\sigma)|r|} \; e^{\sum_{j=1}^{n}(1-2\beta/\sigma)|r_j|} \leq e$$

$$C^2_{\varepsilon,\lambda} \sigma_1^{\sum_{j=0}^{n} q_j} \; \varepsilon^{\gamma+\beta/\sigma(2\gamma-1)} e^{\sum_{j=1}^{n}(1-\beta/\sigma)|r_j|} \leq e^{(2-\beta/\sigma)|r|}$$

$$C^2_{\varepsilon,\lambda} \sigma_1^{\sum_{j=0}^{n} q_j} \; \varepsilon^{\gamma+\beta/\sigma(2\gamma-1)} e^{\sum_{j=1}^{n}(1-\beta/\sigma)|r_j|} \leq e^{(2-\beta/\sigma)|r|}$$

$$e^{\sum_{j=0}^{n}|r_j|(1-2\beta/\sigma)} \leq e^{(2-\beta/\sigma)|r|} \; \sigma_1^{-\sum_{J=0}^{n} q_j} / C^2(\varepsilon, \lambda) \; \varepsilon^{\gamma+\beta/\sigma(2\gamma-1)}.$$

Therefore,

$$\sum_{j=1}^{n} |r_j| \leq \frac{(2-\beta/\sigma)|r|}{(1-2\beta/\sigma)C^2(\varepsilon,\lambda)\ \varepsilon^{\gamma+\beta/\sigma(2\gamma-1)}} \leq C_4^2\,|r|$$

and

$$S \leq C_4^2\,|r|.$$

3. $|\omega_1| \geq {\sigma_1}^{q_0}\varepsilon^{\gamma-\beta/\sigma}e^{-(1-\beta/\sigma)|r|}$ and $\dfrac{|\omega_{\tilde{v}}|}{|\omega_v|} \geq {\sigma_1}^{q_j}C_2\varepsilon^{2\gamma\beta/\sigma}e^{(1-\frac{2\beta}{\sigma})\,|r_j|}$, similarly, from (2.34) since $|\omega_{\tilde{v}}| \leq e$ we have,

$${\sigma_1}^{q_0}\varepsilon^{\gamma-\beta/\sigma}e^{-(1-\beta/\sigma)|r|}\prod_{j=1}^{n}{\sigma_1}^{q_j}C_2\varepsilon^{2\gamma\beta/\sigma}e^{(1-\frac{2\beta}{\sigma})\,|r_j|} \leq e,$$

it implies that

$$S \leq \sum_{j=1}^{n}|r_j| \leq \frac{(2-\beta/\sigma)}{C_2(1-2\beta/\sigma)\ \varepsilon^{\gamma-\beta/\sigma+2\gamma\beta/\sigma}}\ |r|\ .$$

Therefore,

$$S \leq C_4^3\,|r|.$$

4. $|\omega_1| \geq {\sigma_1}^{q_0}C_2\varepsilon^{\gamma-\beta/\sigma}e^{-(1-\beta/\sigma)|r|}$ and $\dfrac{|\omega_{\tilde{v}}|}{|\omega_v|} \geq {\sigma_1}^{q_j}\varepsilon^{2\gamma\beta/\sigma}e^{(1-\frac{2\beta}{\sigma})\,|r_j|}$.
Using the equality (2.34), since $|\omega_{\tilde{v}}| \leq e$ and $|r| > 1$, it follows that

$${\sigma_1}^{q_0}C_2\varepsilon^{\gamma-\beta/\sigma}e^{-(1-\beta/\sigma)|r|}\prod_{j=1}^{n}{\sigma_1}^{q_j}\varepsilon^{2\gamma\beta/\sigma}e^{(1-\frac{2\beta}{\sigma})\,|r_j|} \leq e,$$

it implies that,

$${\sigma_1}^{\sum_{j=0}^{n} q_j}C_2\varepsilon^{\gamma-\beta/\sigma+2\gamma\beta/\sigma}e^{\sum_{j=1}^{n}(1-2\beta/\sigma)|r_j|} \leq e^{(2-\beta/\sigma)|r|}.$$

Therefore,

$$S \leq \sum_{j=1}^{n} |r_j| \leq \frac{(2-\beta/\sigma)}{C_2(1-2\beta/\sigma)\ \varepsilon^{\gamma-\beta/\sigma+2\gamma\beta/\sigma}}\ |r| \leq C_4^4\ |r|.$$

5. $|\omega_1| \geq {\sigma_1}^{q_0} \varepsilon^{-\iota/2} \varepsilon^{\gamma} e^{-|r|}$ and $\dfrac{|\omega_{\tilde{v}}|}{|\omega_v|} \geq {\sigma_1}^{q_j} \varepsilon^{2\gamma\beta/\sigma} e^{(1-\frac{2\beta}{\sigma})\ |r_j|}$.

   Using the equality (2.34), since $|\omega_{\tilde{v}}| \leq e$ and $|r| > 1$, it follows that

$$ {\sigma_1}^{q_0} \varepsilon^{-\iota/2} \varepsilon^{\gamma} e^{-|r|} \prod_{j=1}^{n} {\sigma_1}^{q_j} \varepsilon^{2\gamma\beta/\sigma} e^{(1-2\beta/\sigma)|r_j|} \leq e,$$

So,

$${\sigma_1}^{\sum\limits_{j=0}^{n} q_j} e^{\sum\limits_{j=1}^{n}(1-2\beta/\sigma)|r_j|} \leq \frac{e^{2|r|}}{\varepsilon^{-\iota/2+\gamma+2\gamma\beta/\sigma}},$$

it implies,

$$S \leq \sum_{j=1}^{n} \leq \frac{2}{(1-2\beta/\sigma)\varepsilon^{-\iota/2+(1+2\beta/\sigma)\gamma}}|r|.$$

Thus,

$$S \leq \sum_{j=1}^{n} |r_j| \leq C_4^5\ |r|.$$

Finally the last case:

6. $|\omega_1| \geq {\sigma_1}^{q_0} \varepsilon^{-\iota/2} \varepsilon^{\gamma} e^{-|r|}$ and $\dfrac{|\omega_{\tilde{v}}|}{|\omega_v|} \geq {\sigma_1}^{q_j} C_2 \varepsilon^{2\gamma\beta/\sigma} e^{(1-\frac{2\beta}{\sigma})\ |r_j|}$.

   Using the equality (2.34), since $|\omega_{\tilde{v}}| \leq e$ and $|r| > 1$ and as in previous case, we are taking $\gamma, \beta, \sigma, \iota$ such that $(1+2\beta/\sigma)\gamma > \iota/2$. Furthermore, we have

$${\sigma_1}^{q_0} \varepsilon^{-\iota/2} \varepsilon^{\gamma} e^{-|r|} \prod_{J=1}^{n} {\sigma_1}^{q_j} C_2 \varepsilon^{2\gamma\beta/\sigma} e^{(1-\frac{2\beta}{\sigma})\ |r_j|}.$$

Analogously as before, we get

$$S \leq \sum_{j=1}^{n} |r_j| \leq \frac{2}{C_2(1-2\beta/\sigma)\varepsilon^{-\iota/2+(1+2\beta/\sigma)\gamma}}|r| \leq C_4^6\,|r|.$$

In this way, we believe that the fundamental cases were covered. Now, we can take the constant $C_4 = \max\{C_4^1, C_4^2, C_4^3, C_4^4, C_4^5, C_4^6\} > 0$ such that,

$$S \leq C_4|r|$$

□

2.3 Probability of essential returns with a certain depth

In the previous section we studied the depth of returns and we saw that only essential returns are important, because these returns are above to the inessential and bounded returns. Now, we proceed with the study of the second basic idea for the proof of our main result: *the chances of occurring a very deep essential return are very small.* In fact, they are less than $e^{-\tau\rho}$, where $\tau > 0$ is constant and ρ is the depth as we can see it on the Proposition 2.3.1 and Corollary 2.3.2. The main ingredient of the proof is bounded distortion.

From Lemma 2.1.3, it follows that

$$\frac{2|r|}{\beta + \log C_0} - K \leq p, \qquad \text{with } K = \frac{\beta + 2 + \log(\varepsilon^\gamma \eta_2^+)}{\beta \log C_0}.$$

Let Θ be threshold large enough such that

$$K \leq \frac{\Theta}{\beta + \log C_0} \Leftrightarrow -K \geq -\frac{\Theta}{\beta + \log C_0}.$$

Thus,

$$\frac{2|r|}{\beta+\log C_0} - K \leq p \Leftrightarrow \frac{2|r|}{\beta+\log C_0} - \frac{\Theta}{\beta+\log C_0} \leq p.$$

As $|r| \leq \Theta$, we have

$$\frac{2\Theta}{\beta+\log C_0} - \frac{\Theta}{\beta+\log C_0} \leq p.$$

Which implies that,

$$\frac{1}{\beta+\log C_0}\Theta \leq p.$$

As during the bounded period there is not any essential return situation, dividing n by the lower bound of p, we obtain a bound for d_n sequence of essential returns from 1 up to n. Therefore,

$$d_n \leq \frac{n}{\Theta/\beta+\log C_0} = \frac{\beta+\log C_0}{\Theta} n \tag{2.35}$$

For each $x \in I$, $n \in \mathbb{N}$ there is a unique element ω in the partition $\mathcal{P}_n$ such that $x \in \omega$. Let $u_n(x)$ be the *number of the essential return situation* of ω between 1 and n, $s_n(x)$ is the number of those essential *return situations which are actual essential returns* times and denote by $d_n(x)$ the number of those essential returns which have *deep essential return* with depth above the threshold $\Theta \geq \Lambda$. Moreover, the essential return situation is a *chopping time* and it would be either a return time or an escape time for every chopping time component, so $u_n(x) - s_n(x)$ is exactly number of escaping times of ω.

Given an integer d, n fixed such that $1 \leq d \leq n\frac{\beta+\log C_0}{\Theta}$, there is an integer u with $d \leq u \leq n$ and d-integers $r_1, \ldots, r_d \geq \Theta$, we define the set of events,

$$A^{u,d}_{r_1,\ldots,r_d}(n) = \left\{ x \in I : \begin{array}{l} u_n(x) = u,\ \ d_n(x) = d, \text{ and the depth of the } j\text{-} \\ \text{th deep essential return is } r_j \text{ for } j \in \{1,\ldots,d\} \end{array} \right\}$$

Now, we want to estimate the size of this set

Proposition 2.3.1. *We have*

$$\left|A^{u,d}_{r_1,\dots,r_d}(n)\right| \leq \binom{u}{d} \exp\left(-\left(1-\frac{\beta}{\sigma}\right)\sum_{j=1}^{d} r_j\right).$$

Proof. Take $n \in \mathbb{N}$ fixed and $\omega_0 \in \mathcal{P}_0$. Note that the functions u_n and d_n are constant for every $\omega \in \mathcal{P}_n$. Let $\omega \subseteq \omega_0 \cap \mathcal{P}_n$ be such that $u_n(\omega) = u$ and ω_i denote the element of the partition $\mathcal{P}_{t_i}$ containing ω, where t_i is the i-th return situation. Then, there is a sequence $0 \leq t_1 \leq t_2 \leq \dots. \leq t_u \leq_n$ of *essential return situations.* We have $\omega_0 \supseteq \omega_1 \supseteq \dots. \supseteq \omega_u = \omega$. Consider that $\omega_j = \emptyset$ whenever $i > u$.

For each $i \in \{0, \dots, n\}$ we define the set $Q_i = \bigcup_{\omega \in \mathcal{P}_n \cap \omega_0} \omega_i$ and its partition,

$$\mathcal{Q}_i = \{\omega_i : \omega \in \mathcal{P}_n \quad \text{and } \omega \subseteq \omega_0 \text{ with } u_n(\omega) = u\}.$$

Fix d integers $0 \leq m_1 \leq m_2 \leq \dots. \leq m_d \leq u$ with m_j indicating that the j-th deep essential return occurs in the m_j-th essential return situation (i.e., t_{m_j} is the j-th deep essential return time).

Now, we just consider those elements of the partition $\mathcal{P}_n$ which are subsets of ω_0 with u-times essential return situation and at its m_j-th essential return situation its j-th deep essential return time occurs with depht r_j. We set $V(0) = Q_0 = \omega_0$ and fix d integers $1 \leq m_1 \leq \dots. \leq m_d \leq u$. On the other hand, for $i \leq u$, we define $V(i)$ recursively. Suppose that $V(i-1)$ is already defined and $m_{j-1} < i < m_j$. Then, we set

$$V(i) = \bigcup_{\omega \in \mathcal{Q}_i} \omega \cap f^{-t_i}(I \setminus \Delta_\Lambda) \cap V(i-1),$$

and if $i = m_j$ we set

$$V(i) = \bigcup_{\omega \in \mathcal{Q}_i} \omega \cap f^{-t_i}(I_{r_j} \cap I_{-r_j}) \cap V(i-1)$$

Observe that for every $i \in \{1, \ldots, u\}$ we have that $\frac{|V(i)|}{|V(i-1)|} \leq 1$. Therefore, we concentrate in finding a better estimate for $\frac{|V(m_j)|}{|V(m_j-1)|}$. Consider that $\omega_{m_j} \in V(m_j) \cap \mathcal{Q}_{m_j}$ and $\omega_{m_j-1} \in V(m_j-1) \cap \mathcal{Q}_{m_j-1}$.We consider two situations depending on whether t_{u_j-1} is an *escaping situation* or an *essential return*.

1. First suppose that t_{m_j-1} is an essential return with depth m. Then

$$\begin{aligned} \frac{|\omega_{m_j}|}{|\omega_{m_j-1}|} &\leq \frac{|\omega_{m_j}|}{|\hat{\omega}_{m_j-1}|}, \quad \text{where } \hat{\omega}_{m_j-1} = \omega_{m_j-1} \cap f^{-t_{m_j}}(\Delta_1) \\ &\leq C \frac{|f^{t_{m_j}}(\omega_{m_j})|}{|f^{t_{m_j-1}}(\hat{\omega}_{m_j-1})|}, \quad \text{by mean value theorem and Proposition 2.1.7} \\ &\leq C \frac{2e^{-r_j}\varepsilon^{\gamma}}{|f^{t_{m_j-1}}(\hat{\omega}_{m_j-1})|}, \quad \text{by definition of } \omega_{m_j}. \end{aligned}$$

   We consider two cases,

   (a) if $\hat{\omega}_{m_j-1} = \omega_{m_j-1}$, then by parts (3b) and (3d) of Lemma 2.1.4 and knowing that t_{m_i-1} is an essential return time we have

   $$|f^{t_{m_j}}(\hat{\omega}_{m_j-1})| \geq \varepsilon^{\gamma-\beta/\sigma} e^{-(1-\beta/\sigma)r} \geq \varepsilon^{\gamma} e^{-(\beta/\sigma)r}.$$

   we recall that

   $I_r = [\varepsilon^{\gamma}e^{-r}, \ \varepsilon^{\gamma}e^{-r+1})$ for $r \geq 1$ and $I_{-r} = (-\varepsilon^{\gamma}e^{-r+1}, -\varepsilon^{\gamma}e^{-r}]$ for $-r \leq 1$. We may write $\Delta_1^0 = [\varepsilon^{\gamma}e^{-1}, \ \varepsilon^{\gamma})$, $\Delta_1^{+c} = [\varepsilon^{\gamma}e^{-1} + c, \ \varepsilon^{\gamma} + c)$ and $\Delta_1^{-c} = (-\varepsilon^{\gamma}e^{-2}, \ -\varepsilon^{\gamma}e]$, where $\Delta_1 = \Delta_1^0 \cup \Delta_1^{+c} \cup \Delta_1^{-c}$, with $\Delta_1^{\pm c} = \{\pm c\} \cup I_1$.

   (b) if $\hat{\omega}_{m_j-1} \neq \omega_{m_j-1}$, then $f^{t_{m_j}}(\hat{\omega}_{m_j-1})$ has a point outside Δ_1. Since we are assuming $\omega_{m_i} \neq 0$ and $\omega_{m_j} \subseteq \hat{\omega}_{m_j-1}$. Therefore, $f^{t_{m_j}}(\hat{\omega}_{m_j-1})$ has a point inside Δ_r and then

$$|f^{t_{m_j}}(\hat{\omega}_{m_j-1})| \geq (\varepsilon^{\gamma} + c) - (-\varepsilon^{\gamma} e^{-r+1} - c) \geq \varepsilon^{\gamma} e^{-(\beta/\sigma)r}.$$

Consequently, in both cases we have $\frac{|\omega_{m_j}|}{|\omega_{m_j-1}|} \leq 2C\frac{\varepsilon^{\gamma} e^{-r_j}}{\varepsilon^{\gamma} e^{-(\beta/\sigma)r}}$.

Note that when $m_j - 1 = m_{j-1}$, then $r = r_{u_{j-1}} \geq \Lambda$. On the other hand, if $m_j - 1 > m_{j-1}$, then t_{m_j-1} is an *essential return* with depth $r \leq \Lambda \leq r_{j-1}$. In both cases

$$\frac{|\omega_{m_j}|}{|\omega_{m_j-1}|} \leq 2C\frac{\varepsilon^{\gamma} e^{-r_j}}{\varepsilon^{\gamma} e^{-(\beta/\sigma)r_{j-1}}}. \tag{2.36}$$

2. Now suppose t_{m_i-1} is an escape situation. We have the same estimate as in (2.36). We only use Lemma 2.1.5 instead of Lemma 2.1.4 in case 1 part (a). Then, it follows that

$$\begin{aligned} |V(m_j)| &= \sum_{\omega_{m_j} \in \mathcal{Q}_{m_j} \cap V(m_j-1)} \frac{|\omega_{m_j}|}{|\omega_{m_j-1}|}|\omega_{m_j-1}| \\ &\leq 2C\varepsilon^{\gamma} e^{-r_j} \ \varepsilon^{-\gamma} \ e^{(\beta/\sigma)r_{j-1}} \sum_{\omega_{m_j} \in \mathcal{Q}_{m_j} \cap V(m_j-1)} |\omega_{m_j-1}| \\ &\leq 2Ce^{-r_j} \ e^{(\beta/\sigma)r_{j-1}}|V(m_j-1)|. \end{aligned}$$

Therefore,

$$|V(u)| = (2C)^d \exp\left(-\left(1 - \frac{\beta}{\sigma}\right)\sum_{j=1}^{d} r_j\right) e^{\frac{\beta}{\sigma}r_0}|V(0)|, \tag{2.37}$$

where $r_0 = 0$ if ω_0 be $[-a, -e\varepsilon^{\gamma}+c)$, $(-\varepsilon^{\gamma}e-c, -\varepsilon^{\gamma}e)$, $(\varepsilon^{\gamma}e, \varepsilon^{\gamma}e)$, $(\varepsilon^{\gamma}e, \varepsilon^{\gamma}e-c)$ or $(\varepsilon^{\gamma}e+c, a]$, and $r_0 = |\eta|$ if $\omega_0 = I_{\eta,K}$ for some $|\eta| \geq \Lambda$ and $k \in \{1, 2, \dots, \eta^2\}$. In addition, we have to take into account on the number of possibilities of having the occurrence of the event $V(u)$ implying the occurrence of event $A^{u,d}_{m_1,\dots,m_d}(n)$. Thus, the numbers of possible configurations with respect to different values of integers $m_1, ..., m_d$ can take is $\binom{u}{d}$. Wherefore, it follows that

$$\begin{aligned}\left|A^{u,d}_{r_1,\dots,r_d}(n)\right| &\le (2C)^d\binom{u}{d}\exp\left\{-\left(1-\frac{\beta}{\sigma}\right)\sum_{j=1}^{d} r_j\right\}\sum_{\omega_0\in\mathcal{P}_0} e^{(\beta/\sigma)r_0}|\omega_0| \\ &\le (2C)^d\binom{u}{d}\exp\left\{-\left(1-\frac{\beta}{\sigma}\right)\sum_{j=1}^{d} r_j\right\}\left(4(a-e\varepsilon^{\gamma}+c)+\sum_{\omega_0\in\mathcal{P}_0} e^{(\beta/\sigma)r_0}e^{-|r_0|}\right) \\ &\le (4a+1)(2C)^d\binom{u}{d}\exp\left\{-(1-\frac{\beta}{\sigma})\sum_{j=1}^{d} r_j\right\} \\ &\le \binom{u}{d}\exp\left\{-(1-\frac{\beta}{\sigma})\sum_{j=1}^{d} r_j\right\}.\end{aligned}$$

The last inequality comes from the fact that $d\Theta \le \sum_{j=1}^{d} r_j$ and because we can chose Θ sufficiently large ($(2C)^d$ near to 1).

□

As a consequence, we can find the probability of the event that the depth of those essential returns of its elements have reached depth m, i.e.,

$$A^{u,d}_{r,j}(n) = \left\{ x\in I : \begin{array}{c} u_n(x)=u, d_n(x)=d, \text{ and the depth of the } j\text{-th} \\ \text{deep essential return is } r \end{array} \right\} \tag{2.38}$$

for fixed $n\in\mathbb{N}$, and $\quad 1\le d\le n^{\frac{\beta+\log C_0}{\Theta}},\ \ d\le u\le n\ $ and $j\le d$.

Corollary 2.3.2. *If Θ is large enough, then*

$$\left|A^{u,d}_{r,j}(n)\right| \le \binom{u}{d} e^{-\left(1-\frac{\beta}{\sigma}\right)r}.$$

Proof. Note that

$$A_{r,j}^{u,d}(n) = \bigcup_{\substack{r_i \geq \Theta \\ i \neq j}} A_{r_1,\ldots,r_{j-1},r,r_{j+1},\ldots,r_d}^{u,d}(n).$$

It implies that

$$|A_{r,j}^{u,d}(n)| = \bigcup_{\substack{r_i \geq \Theta \\ i \neq j}} |A_{r_1,\ldots,r_{j-1},r,r_{j+1},\ldots,r_d}^{u,d}(n)|.$$

By Proposition 2.3.1,

$$\begin{aligned} \left|A_{r,j}^{u,d}(n)\right| &\leq \binom{u}{d} e^{-(1-\frac{\beta}{\sigma})r} \left(\sum_{\eta=\Theta}^{\infty} e^{-(1-\frac{\beta}{\sigma})\eta}\right)^{d-1} \\ &\leq \binom{u}{d} e^{-(1-\frac{\beta}{\sigma})r}, \end{aligned}$$

for Θ large enough such that $\sum\limits_{\eta=\Theta}^{\infty} e^{-(1-\frac{\beta}{\sigma})\eta} \leq 1$, the result is as follows

$$\left|A_{r,j}^{u,d}(n)\right| \leq \binom{u}{d} e^{-(1-\frac{\beta}{\sigma})r}$$

□

Chapter 3

The measure of the tail set and statistical stability

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> ”....Não há jogada fora da lei
> Não interessa o que diz o ditado
> Não interessa o que o Estado diz
> Não falamos outra língua....somos
> passes livres , isso é pior doque a
> prisão......somos um exército, o
> exército de um homem só....”
>
> Humberto Gessinger

Here we complete the proof of theorem A. First, we are going to check the non-uniform expansion for $\mathcal{F}$, that is (1.3) holds for Lebesgue almost all points $x \in [-a, a]$.

Moreover, there are positive constants B_0 and τ_1 such that for all $n \in \mathbb{N}$,

$$\left|\{x \in I : \mathcal{E}^{\dashv}(x) > n\}\right| \leq B_0 e^{-\tau_1 n}$$

and later the slow recurrence to the critical/singular set, that is we must see that (1.4) holds for Lebesgue almost all points $x \in [-a, a]$. Moreover, there are positive constants B_2 and τ_2 such that for all $n \in \mathbb{N}$, $\left|\{x \in I : \ \mathcal{R}^a_{\varepsilon,\delta}(x) > n\}\right| \leq B_2 \ e^{-\tau_2 n}$.

Recall that the tail set is defined as $\Gamma^a_n(\varepsilon, \delta) = \{x \in [-a, a] : \mathcal{E}^a(x) > n\} \cup \{x \in [-a, a] : \mathcal{R}^a_{\varepsilon,\delta}(x) > n\}$. In the next subsections, we are going to see that the volume of $\{\mathcal{E}^a(x) > n\}$ and $\{\mathcal{R}^a_{\varepsilon,\delta}(x) > n\}$ decays exponentially very fast (uniformly on the parameters $a \in \mathcal{A}$) with n. This completes the proof.

3.1 Non-uniform expansion

Assume that n is a fixed large integer. Take $\alpha > 0$, $I = [-a, \ a]$. We define the sets below ,

$$\mathsf{E}(n) = \mathsf{E}^-(n) \cup \mathsf{E}^0(n) \cup \mathsf{E}^+(n), \tag{3.1}$$

where

$$\begin{aligned}
\mathsf{E}^0(n) &= \left\{x \in I : \exists i \in \{1, ..., n\} : |f^i_a(x)| \leq e^{-\alpha n}\right\}, \\
\mathsf{E}^+(n) &= \{x \in I : \exists i \in \{1, ..., n\} : |f^i_a(x) - c| \leq e^{-\alpha n}\}, \\
\mathsf{E}^-(n) &= \{x \in I : \exists i \in \{1, ..., n\} : |f^i_a(x) + c| \leq e^{-\alpha n}\}.
\end{aligned}$$

For each $x \in I \setminus \mathsf{E}(n)$, suppose that $v_1, v_2, ..., v_{\gamma_n}$ are return times of x (either essential or not) up to time n. Let p_i be the associated bound period originated by return times. Set up initially putting $v_0 = 0$ whether $x \in \Delta^\varepsilon_r$ or not. Also, set $p_0 = 0$

if $x \neq \Delta_r^\varepsilon$ and as usual, see definition 2.1.1, if not.

We define $q_i = \upsilon_{i+1} - (\upsilon_i + p_i)$ for $i = 0, \dots, \gamma - 1$ and

$$q_\gamma = \begin{cases} 0 & \text{if } n < \upsilon_\gamma + p_\gamma, \\ n - (\upsilon_\gamma + p_\gamma) & \text{if } n \geq \upsilon_\gamma + p_\gamma. \end{cases}$$

Take $\tilde{c} = \sigma/2 - \beta - 2\alpha$. If $n \geq \gamma + p_\gamma$ and $\log \sigma_1 > \sigma/2 - \beta$, then by lemmas 2.1.2 and 2.1.3 we get

$$\begin{aligned} |(f^n)'(x)| &= \prod_{i=0}^{\gamma} |(f^{q_i})'(f^{\upsilon_i + p_i}(x))||(f^{p_i})'(f^{\upsilon_i}(x))| \\ &\geq e^{\log \sigma_1 \sum_{i=0}^{\gamma} q_i} \varepsilon^{2\beta\gamma/\sigma} e^{(\sigma/2-\beta) \sum_{i=0}^{\gamma} p_i} \\ &\geq \varepsilon^{2\beta\gamma/\sigma} e^{(\sigma/2-\beta) \sum_{i=0}^{\gamma} p_i + q_i} \\ &\geq c' e^{(\sigma/2-\beta-2\alpha)\ n}\ e^{2\alpha n}, \quad c' = \varepsilon^{2\beta\gamma/\sigma} \\ &\geq e^{\tilde{c}n}, \quad n \text{ big enough such that } c' e^{2\alpha n} > 1 \end{aligned} \tag{3.2}$$

On the other side, if $n \leq \upsilon_\gamma + p_\gamma$ then by the same lemmas, using condition (M_2), since $x \notin \mathsf{E}(n)$ and considering for simplicity only points x close to $+c$, the other cases are treated the similar form. It follows that

$$|(f^n)'(x)| = |f'(f^{\upsilon_\gamma}(x))||(f^{n-(\upsilon_\gamma+1)})'(f^{\upsilon_\gamma+1}(x))| \prod_{i=0}^{\gamma-1} |(f^{q_i})'(f^{\upsilon_i+p_i}(x))||(f^{p_i})'(f^{\upsilon_i}(x))| \tag{3.3}$$

$$\begin{aligned}
&\geq \frac{2\eta_2^-}{C_1} \, |f^{\upsilon_\gamma}(x) - c| \, e^{\log\sigma_1 \, (n-(\upsilon_\gamma+1))} \, \varepsilon^{2\beta\gamma/\sigma} \, e^{(\sigma/2-\beta) \sum_{i=0}^{\gamma-1} p_i+q_i} \\
&\geq \frac{2\eta_2^-}{C_1} \, e^{-\alpha n} \, e^{\log\sigma_1 \, (n-(\upsilon_\gamma+1))} \, \varepsilon^{2\beta\gamma/\sigma} \, e^{(\sigma/2-\beta) \sum_{i=0}^{\gamma-1} p_i+q_i}, \qquad x \notin \mathsf{E}(n) \\
&\geq \frac{2\eta_2^-}{C_1} \, e^{-\alpha n} \, e^{\log\sigma_1 \, (n-(\upsilon_\gamma+1))} \, \varepsilon^{2\beta\gamma/\sigma} \, e^{(\sigma/2-\beta)(n-1)} \\
&\geq \frac{2\eta_2^-}{C_1} \, \varepsilon^{2\beta\gamma/\sigma} \, e^{-\alpha n} e^{(\sigma/2-\beta)(n-1)}, \text{ since } \quad q_\gamma = 0 \\
&\geq \frac{2\eta_2^-}{C_1} \, \varepsilon^{2\beta\gamma/\sigma} \, e^{-\alpha n} e^{(\tilde{c}+2\alpha)(n-1)} \\
&\geq e^{\tilde{c}n}, \quad n \text{ large such that } \{2\eta_2^-/C_1\} \, \varepsilon^{2\beta\gamma/\sigma} \, e^{2\alpha n-\tilde{c}-2\alpha} > e^{\alpha n} \qquad (3.4)
\end{aligned}$$

Therefore, we just proved that: if $x \in I \setminus \mathsf{E}(n)$, then $|(f^n)'(x)| \geq e^{\tilde{c}n}$ for some $\tilde{c} > 0$. Now we will show that,

$$|\mathsf{E}(n)| \leq e^{-\tau_1 n}, \qquad \forall n \geq N_1 \tag{3.5}$$

for some constant $\tau_1(\alpha, \beta)$ and an integer $N_1(\Lambda, \tau_1)$.

We consider $N_1(\Lambda, \tau_1)$ such that for all $n \geq N_1$ the estimates (3.2), (3.4) and (3.5) hold. Consequently, for every $n \geq N_1$ we get $|(f_a^n)'(x)| \geq e^{\tilde{c}n}$, except for a set $E(n)$ of points $x \in I$ satisfying (3.5).

We take $\Lambda = \Theta$, $1 \leq d \leq n\frac{\beta+\log C_0}{\Lambda}$, and define

$$A_r^{u,d}(n) = \left\{ x \in I : \begin{array}{l} u_n(x) = u, \ d_n(x) = d, \text{ and there is an essential} \\ \text{return with depth } r \end{array} \right\}.$$

Fixe u, d such that $d \le u \le n$, with $r \ge \Lambda$, and define

$$A_r(n) = \big\{x \in I : \exists\, t \le n \text{ such that } t \text{ is an essential return and } |f^t(x)| \in I_r\big\}, \quad (3.6)$$

Since, $A_r^{u,d}(n) = \bigcup_{j=1}^{d} A_{r,j}^{u,d}(n)$ by the Corollary 2.3.2, it follows that

$$\big|A_r^{u,d}(n)\big| \le \sum_{j=1}^{d} \Big|A_{r,j}^{u,d}(n)\Big| \le d \binom{u}{d} e^{-\left(1-\frac{\beta}{\sigma}\right)r}. \quad (3.7)$$

Also, since $A_r(n) = \bigcup_{d=1}^{n^{\frac{\beta+\log C_0}{\Lambda}}} \bigcup_{u=d}^{n} A_r^{u,d}(n)$, by inequality (3.7), we have

$$\begin{aligned}
|A_r(n)| &\le \sum_{d=1}^{n^{\frac{\beta+\log C_0}{\Lambda}}} \sum_{u=d}^{n} \big|A_r^{u,d}(n)\big| \\
&\le \sum_{d=1}^{n^{\frac{\beta+\log C_0}{\Lambda}}} \sum_{u=d}^{n} d \binom{u}{d} e^{-\left(1-\frac{\beta}{\sigma}\right)r}, \quad \text{by (3.7)} \\
&\le d\, e^{-\left(1-\frac{\beta}{\sigma}\right)r} \sum_{d=1}^{n^{\frac{\beta+\log C_0}{\Lambda}}} \sum_{u=d}^{n} \binom{u}{d} \\
&\le d\, n e^{-\left(1-\frac{\beta}{\sigma}\right)r} \sum_{d=1}^{n^{\frac{\beta+\log C_0}{\Lambda}}} \binom{n}{d} \\
&\le d\, n\, n \frac{\beta+\log C_0}{\Lambda} \binom{n}{n^{\frac{\beta+\log C_0}{\Lambda}}} e^{-\left(1-\frac{\beta}{\sigma}\right)r} \le \frac{n^3}{\Lambda} \binom{n}{n^{\frac{\beta+\log C_0}{\Lambda}}} e^{-\left(1-\frac{\beta}{\sigma}\right)r}.
\end{aligned}$$

Taking $R = n^{\frac{\beta+\log C_0}{\Lambda}}$, $C_0 > 0$ and using the Stirling Formula

$$\sqrt{2\pi n}\, n^n e^{-n} \le n! \le \sqrt{2\pi n}\, n^n e^{-n}\Big(1 + \frac{1}{4n}\Big),$$

implies that

$$\binom{n}{R} \le \text{const} \frac{n^n}{(n-R)^{n-R}(R)^R}.$$

$$\binom{n}{R} \leq const \left(\left(1+\frac{\frac{R}{n}}{1-\frac{R}{n}}\right)\left(1+\frac{1-\frac{R}{n}}{\frac{R}{n}}\right)^{\frac{\frac{R}{n}}{1-\frac{R}{n}}} \right)^{(1-\frac{R}{n})n}.$$

We can take $h(\Lambda) = (1 - \frac{\beta+\log C_0}{\Lambda}) \log \left(\left(1+\frac{\frac{\beta+\log C_0}{\Lambda}}{1-\frac{\beta+\log C_0}{\Lambda}}\right)\left(1+\frac{1-\frac{\beta+\log C_0}{\Lambda}}{\frac{\beta+\log C_0}{\Lambda}}\right)^{\frac{\frac{\beta+\log C_0}{\Lambda}}{1-\frac{\beta+\log C_0}{\Lambda}}} \right)$, and so that $h(\Lambda) \to 0$, when $\Lambda \to +\infty$, and

$$\binom{n}{R} \leq \text{const } e^{h(\Lambda)n}.$$

Recalling that, $I_r = [\varepsilon^\gamma e^{-r},\ \varepsilon^\gamma e^{-r+1})$, $r,\ \alpha > 0$, $I = [-a,\ a]$, $\Delta_r^\varepsilon = \Delta_r^{-c} \cup \Delta_r^0 \cup \Delta_r^{+c}$, from (3.1), since the depths of inessential and bound returns are less than depth of the essential returns (see 2.2.1 and 2.2.2) preceding them for n such that $\alpha n > |r|$, we have

$$\mathsf{E}(n) \subseteq \bigcup_{r=\alpha n}^{+\infty} A_r(n) \tag{3.8}$$

Now, we can write the set in (3.6) as the union $A_r(n) = A_r^-(n) \cup A_r^0(n) \cup A_r^+(n)$, where

$$A_r^-(n) = \{x \in I : \exists\, t \leq n \text{ such that } t \text{ is an essential return and } |f^t(x)| \in I_r^{-c}\};$$

$$A_r^0(n) = \{x \in I : \exists\, t \leq n \text{ such that } t \text{ is an essential return and } |f^t(x)| \in I_r\};$$

$$A_r^+(n) = \{x \in I : \exists\, t \leq n \text{ such that } t \text{ is an essential return and } |f^t(x)| \in I_r^{+c}\}.$$

Note that,

$$|\mathsf{E}^-(n)| \leq \sum_{r=\alpha n}^{+\infty} |A_r^-(n)|, \quad |\mathsf{E}^0(n)| \leq \sum_{r=\alpha n}^{+\infty} |A_r^0(n)| \text{ and } |\mathsf{E}^+(n)| \leq \sum_{r=\alpha n}^{+\infty} |A_r^+(n)|.$$

From (3.8) implies that,

$$|\mathsf{E}(n)| \leq \sum_{r=\alpha n}^{+\infty} \Big(|A_r^-(n)| + |A_r^0(n)| + |A_r^+(n)| \Big).$$

We want to proof that,

$$|\mathsf{E}(n)| \leq \sum_{r=\alpha n}^{+\infty} |A_r(n)| \leq e^{-\tau_1 n} \quad \forall\, n \geq N_1,$$

for some N_1 integer and n fixed.

Let allows us take Λ larger enough such that $h(\Lambda) \leq (1-\frac{\beta}{\sigma})\frac{\alpha}{2}$. Thus, we have for the following situations:

$$\begin{aligned}
|\mathsf{E}^0(n)| &\leq \text{const}\frac{n^3}{\Lambda} e^{h(\Lambda)n} \sum_{r=\alpha n}^{+\infty} e^{-(1-\frac{\beta}{\sigma})r} \\
&\leq \text{const}\frac{n^3}{\Lambda} e^{h(\Lambda)n} e^{-(1-\frac{\beta}{\sigma})\alpha n} \\
&\leq \text{const}\frac{n^3}{\Lambda}\, e^{\left(h(\Lambda)n-(1-\frac{\beta}{\sigma})\alpha n\right)} \\
&\leq \text{const}\frac{n^3}{\Lambda} e^{-(1-\frac{\beta}{\sigma})\frac{\alpha n}{2}} \\
&\leq \text{const}\frac{n^3}{\Lambda} e^{-2\tau_1^0 n} \\
&\leq e^{-\tau_1^0\, n}
\end{aligned}$$

where $\tau_1^0 = (1-\frac{\beta}{\sigma})\frac{\alpha}{4}$, and n is large enough such that $\text{const}\frac{n^3}{\Lambda}e^{-\tau_1^0\, n} \leq 1$.

In this form, we have obtained $|\mathsf{E}^0(n)| \leq \sum_{r=\alpha n}^{+\infty} |A_r^0(n)|$. The same way we can get ,

$$|\mathsf{E}^-(n)| \leq \sum_{r=\alpha n}^{+\infty} |A_r^-(n)| \quad \text{and} \quad |\mathsf{E}^+(n)| \leq \bigcup \sum_{r=\alpha n}^{+\infty} |A_r^+(n)|.$$

After the three cases above and considering Λ big enough such that,

$$h(\Lambda) \leq (1 - \frac{\beta}{\sigma})\frac{\alpha}{2}.$$

Then, we obtain

$$|\mathsf{E}(n)| \leq e^{-\tau_1 n},$$

where n is large enough such that

$$\text{const}\frac{n^3}{\Lambda}e^{-\tau_1 \, n} \leq 1,$$

for a constant $\tau_1(\alpha, \beta, \sigma) > 0$, for all $\quad n \geq N_1(\Lambda, \ \tau_1)$.

Therefore, for n sufficiently large, $n > N_1$, we have $(f_a^n)'(x) \geq e^{\tilde{c}n}$, where $\tilde{c}$ is a constant, for every $x \in I \setminus \mathsf{E}(n)$. Now, we are going to exclude the points which does not verify the NUE-condition (1.3). We take $\mathsf{E} = \bigcap_{k \geq N_1} \bigcup_{n \geq k} \mathsf{E}(n)$. By the other side, since for every $k \geq N_1$,

$$\sum_{n \geq k} |\mathsf{E}(n)| \leq \text{const } e^{-\tau_1 k}, \tag{3.9}$$

by the Borel-Cantelli Lemma we have $|\mathsf{E}| = 0$. Therefore, on the full Lebesgue measure set $I \setminus \mathsf{E}$, we obtain (1.3) holds. Note that $\{x \in I : \mathcal{E}(x) > k\} \subseteq \bigcup_{n \geq k} \mathsf{E}(n)$, where $\mathcal{E}(x)$ was defined as (1.5). Thus for $k \geq N_1$ by (3.9) we get,

$$\Big|\{x \in I : \mathcal{E}(x) > k\}\Big| \leq \text{const } e^{-\tau_1 k}.$$

Therefore, there exists $B_0 = B_0(N_1, \tau_1)$ such that for all $n \in \mathbb{N}$

$$\Big|\{x \in I : \mathcal{E}(x) > n\}\Big| \leq B_0 e^{-\tau_1 n}.$$

It is enough to show item (i_1), which implies that the volume of *tail set* (1.6) decays to 0 as n goes to ∞.

Now, in the next section our focus is to show that

$$\left|\{x \in I : \mathcal{R}(x) > n\}\right| \leq B_0 e^{-\tau_2 n}.$$

3.2 Slow recurrence to the critical/singular set

We define for every $x \in [-a, a]$, and $n \in \mathbb{N}$,

$$T_n(x) = T_n^-(x) + T_n^0(x) + T_n^+(x), \tag{3.10}$$

where

$$\begin{aligned} T_n^0(x) &= \frac{1}{n}\sum_{j=0}^{n-1} -\log d_\delta(f_a^j(x), 0) \\ T_n^+(x) &= \frac{1}{n}\sum_{j=0}^{n-1} -\log d_\delta(f_a^j(x), +c) \\ T_n^-(x) &= \frac{1}{n}\sum_{j=0}^{n-1} -\log d_\delta(f_a^j(x), -c). \end{aligned}$$

We may write $T_n = T_n^- + T_n^0 + T_n^+$, where $\delta = e^{\Theta}$ is the same of condition 3.2. We consider the only points of the orbit of x that contribute to the sum in 3.10 to be *deep returns* with depth above the threshold $(\Theta \geq \Lambda)$, Λ is a big number such that the sum $\sum_{j=1}^{d} r_j \geq \Theta \geq \Lambda$. In this section, we are going to obtain the upper bound for each $T_n^*(x)$ (where, $* = 0, \pm c$) and as a consequence we get the upper bound for the sum of inessential and bound returns depths which occurs between two consecutive essential returns. Using the previous Lemmas 2.2.3 and 2.2.4 we can see that if t is an essential return time with depth $|r|$ then the sum of its depth and depth of all inessential returns and bound returns before the next essential return is less than $(C_3 + C_3\, C_4)|r|$. We define the sequence $F_n(x) = \sum_{j=1}^{d} r_j$ such that d be the number of essential returns of x with depth above Θ up to time n and r_j's are their respective depths, then it follows that

$$T_n(x) = T_n(x)^- + T_n(x)^0 + T_n(x)^+ \leq \frac{C_5}{n} F_n(x), \text{ with } C_5 = C_3 + C_3\, C_4, \quad (3.11)$$

the constants C_3 and C_4 are from Lemmas 2.2.3 and 2.2.4, each T_n^* is less than $\frac{C_5}{n} F_n(x)$. Now, we define for all $n \in \mathbb{N}$, the union below as

$$\mathsf{G}(n) = \mathsf{G}^-(n) \cup \mathsf{G}^0(n) \cup \mathsf{G}^+(n), \quad (3.12)$$

where

$$\begin{aligned} \mathsf{G}^-(n) =& \{x \in I : T_n^-(x) > \varepsilon\} \\ \mathsf{G}^0(n) =& \{x \in I : T_n^0(x) > \varepsilon\} \\ \mathsf{G}^+(n) =& \{x \in I : T_n^+(x) > \varepsilon\}. \end{aligned}$$

We define for all $n \in \mathbb{N}$,

$$\mathsf{G}(n) = \{x \in I = [-a, a] : T_n(x) > \varepsilon\}. \quad (3.13)$$

From (3.12), we may conclude that

$$|\mathsf{G}(n)| \leq |\mathsf{G}^-(n)| + |\mathsf{G}^0(n)| + |\mathsf{G}^+(n)|.$$

Also, from (3.11) and (3.13) we get

$$|\mathsf{G}(n)| \leq \Big|\{x \in I:\ F_n(x) \geq \frac{n\,\varepsilon}{C_5}\}\Big|.$$

To complete the proof of our main theoremA we should show that

$$|\mathsf{G}(n)| \leq \Big|\{x \in I:\ F_n(x) \geq \frac{n\,\varepsilon}{C_5}\}\Big| \leq e^{-\tau_2\, n}, \tag{3.14}$$

where the constant τ_2 will be get in the next subsections. We will do it using a large deviation argument for which we start by estimating the moment generating function of F_n. In what follows $\mathbb{E}$ denotes expectation with respect to *Lebesgue measure.*

Lemma 3.2.1. *Let $0 < t \leq (1-\beta/\sigma)\frac{1}{3}$. For Θ large enough, there exits $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have*

$$\mathbb{E}\left(e^{tF_n}\right) \leq const\ \frac{n^2}{\Theta} e^{h(\Theta)n},$$

where $\mathbb{E}$ is the mathematical expectation. Moreover $h(\Theta) \to 0$ when $\Theta \to +\infty$.

Proof. Let us calculate the mathematical expectation,

$$\begin{aligned}
\mathbb{E}\left(e^{tF_n}\right) = \mathbb{E}\left(e^{t\sum_{j=1}^{d} r_j}\right) &:= \sum_{u,d,(r_1,\ldots,r_d)} e^{t\sum_{j=1}^{d} r_j}\left|A^{u,d}_{r_1,\ldots,r_d}(n)\right| \\
&\leq \sum_{u,d,(r_1,\ldots,r_d)} e^{t\sum_{j=1}^{d} r_j}\binom{u}{d} e^{-\left(1-\frac{\beta}{\sigma}\right)r}, \text{ by Proposition 2.3.1} \\
&\leq \sum_{u,d,(r_1,\ldots,r_d)} e^{t\sum_{j=1}^{d} r_j}\binom{u}{d}\ e^{-3t\ \sum_{j=1}^{d} r_j} \\
&\leq \sum_{u,d,Q}\binom{u}{d} e^{-2tQ}\zeta(d,Q),
\end{aligned}$$

where $\zeta(d,Q)$ is the number of integer solutions of the equation $x_1+\ldots+x_d = Q$ with

$x_j \geq \Theta$ for all j and

$$\zeta(d,Q) \leq \#\{\text{solutions of } x_1 + \ldots + x_d = Q,\ x_j \in \mathbb{N}_0\} = \binom{Q+d-1}{d-1}.$$

By Stirling Formula we have

$$\begin{aligned}\zeta(d,Q) \leq \binom{Q+d-1}{d-1} &\leq \text{const}\frac{(Q+d-1)^{Q+d-1}}{Q^Q(d-1)^{d-1}} \\ &\leq \left(\text{const}^{\frac{1}{Q}}\left(1+\frac{d-1}{Q}\right)\left(1+\frac{Q}{d-1}\right)^{\frac{d-1}{Q}}\right)^Q. \qquad (3.15)\end{aligned}$$

Since $d\Theta \leq Q$, each factor in (3.15) can be made arbitrarily close to 1 by taking Θ large enough. Therefore,

$$\binom{Q+d-1}{d-1} \leq e^{tQ}, \quad \text{as in [LV00, Lemma 4.4]}$$

and

$$\mathbb{E}\left(e^{tF_n}\right) \leq \sum_{u,d,Q}\binom{u}{d}e^{tQ}e^{-2tQ} \leq \sum_{u,d,Q}\binom{u}{d}e^{-tQ} \leq \sum_{u,d}\binom{u}{d}.$$

Now,

$$\begin{aligned}\sum_{u,d}\binom{u}{d} &\leq \sum_{d=1}^{n\frac{\beta+\log C_0}{\Theta}}\sum_{u=d}^{n}\binom{u}{d} \\ &\leq n\sum_{d=1}^{n\frac{\beta+\log C_0}{\Theta}}\binom{n}{d} \\ &\leq n\sum_{d=1}^{n\frac{\beta+\log C_0}{\Theta}}\binom{n}{n\frac{\beta+\log C_0}{\Theta}} \\ &\leq n^2\,\frac{\beta+\log C_0}{\Theta}\binom{n}{n\frac{\beta+\log C_0}{\Theta}}.\end{aligned}$$

By the Stirling Formula, we have

$$\mathbb{E}\left(e^{tF_n}\right) \leq \text{const}\ \frac{n^2}{\Theta} e^{h(\Theta)n}$$

where $h(\Theta) \to 0$ when $\Theta \to +\infty$. □

In addition, if we take $t = (1-\beta/\sigma)\frac{1}{3}$, and Θ big enough such that $2\tau_2 = \frac{t\varepsilon}{C_5} - h(\Theta) > 0$. By Chebyshev's, it inequality follows that

$$\begin{aligned}
\left|\{x \in I :\ F_n(x) \geq \frac{n\ \varepsilon}{C_5}\}\right| &\leq e^{-t\frac{n\varepsilon}{C_5}}\ \mathbb{E}\left(e^{tF_n}\right) \\
&\leq e^{-t\frac{n\varepsilon}{C_5}}\ \text{const}\frac{n^2}{\Theta} e^{h(\Theta)n} \\
&\leq \text{const}\frac{n^2}{\Theta} e^{-\left(\frac{t\varepsilon}{C_5}-h(\Theta)\right)n} \\
&\leq \text{const}\frac{n^2}{\Theta} e^{-2\tau_2 n} \\
&\leq e^{-\tau_2 n},
\end{aligned}$$

for n big enough, as $n \geq N_2$ such that $\text{const}\frac{n^2}{\Theta}e^{-\tau_2 n} < 1$. Consequently, $|G(n)| \leq \text{const}\ e^{-\tau_2 n}$, which implies that $\sum_{n\geq k} |G(n)| \leq \text{const}\ e^{-\tau_2 k}$. Applying Borel Cantelli's lemma, we may get $|G| = 0$, where $G = \bigcap_{k\geq 1} \bigcup_{n\geq k} \mathsf{G}(n)$ and finally conclude that (SR) holds on the full Lebesgue measure set $I \setminus G$. Note that, for every $k \geq N_2$, $\{x \in I : \mathcal{R}(x) > k\}$, then there exists $n \geq k$ such that $\{x \in I : T_n(x) > \varepsilon\}$. Also, $\{x \in I : \mathcal{R}(x) > k\} \subseteq \bigcup_{n\geq k} G(n)$. Thus,

$$|\{x \in I : \mathcal{R}(x) > k\}| \leq \sum_{n\geq k} |G(n)|\,,$$

it follows that

$$|\{x \in I : \mathcal{R}(x) > k\}| \leq \text{const}\ e^{-\tau_2 k}.$$

Therefore, there is $B_2 = B_2(N_2, \tau_2) > 0$ such that for all $n \in \mathbb{N}$,

$$\left|\{x \in I : \mathcal{R}(x) > n\}\right| \leq B_2 \, e^{-\tau_2 n},$$

it proves the second item (i_2). As in the end of the Section 3.1, it implies that the volume of *tail set* (1.6) decays to 0 as n goes to ∞. It completes the proof of theorem A. □

In order to complete the proof of our main result we are going to prove the Theorem 1.2.1. We remark that in [ALV09], it has been proved that there a finite number of acip's for Luzzatto-Viana family. Here, we obtain the *uniqueness* of these measures.

Consider parameters $a \in \mathcal{A} \subset [c, c+\varepsilon]$, with ε small. We observe that for $a = c$, given $\delta > 0$, $\exists\ N(\delta)$ s.t.

$$\forall I \subset [-a, a] \text{ with } |I| \geq \delta,\ f_c^{N(\delta)}(I) = [-a, a].$$

By a continuity argument, given a $\mathcal{A}$ and $\delta > 0$, $\exists\ N(\delta)$ such that

$$\forall I \subset [-a, a] \text{ with } |I| \geq \delta,\ f_a^{N(\delta)}(I) = [-a, a]. \tag{3.16}$$

On the other hand, from the Main Theorem A, Leb almost everywhere $x \in [-a, a]$ has *NUE* and *SR*. Therefore, from [Alves-Bonatti-Viana 2000] [ABV00], Leb almost everywhere $x \in [-a, a]$ has arbitrarily small neighborhoods growing to large scale δ_1 (hyperbolic times). Choosing $N(\delta_1)$ for which (3.16) holds, we deduce that f_a is transitive (actually, topologically mixing). □

3.3 Uniformity on the choice of the constants

We can say that all the constants used throughout this text do not depend on the parameter value $a \in \mathcal{A}$. It was highlighted in the remark 1.2.5, but there are other constants for analyzing. We have chosen each one depending on the other, so that it has interdependencies. Let us give a brief note to explain about the interdependencies between the constants involved.

We begin by considering the constants λ and ρ which will determine the space of parameters $\mathcal{A}$. So, take $0 < \lambda < 1/2$ and $\rho = 2^{\lambda}$. In addition, we fixe $\varepsilon > 0$ and $c > 0$ such that $0 < 2\sigma < \sigma_0 < \log\sqrt{2}$. Also, we consider $\gamma > 1,\ \iota > 0,\ \delta > 0$ such that $1 < \gamma + \delta + \iota < 1/2\lambda$. Considering the definition (2.1.1), we take $\beta > 0$ and $\alpha > 0$ small enough such that $0 < \alpha < \beta$, for instance, a good choice can be $\beta = 2\alpha$ (as in [Fr05, Section 9]).

In the section 2.1, we consider $r > 1,\ |r| > r_s$ for $\delta > 0, \varepsilon > 0$ fixed. Also, for simplicity we choose $\sigma_1 = e^{\sigma_0}$. Notice that, all the constants so far are depending on the previous parameters.

After that fixed the previous constants, we choose in the the Lemmas 2.1.3 and 2.1.4, $C_1 = C_1(\alpha, \beta) > 0$ and $C_2 = C(\varepsilon, \lambda)$, even in the section 2.1, we consider $C_0 > 0$ fixed and $A > 1$ depending of the constants α and β (as in [LV00, Section 3.3]).

In a choice over the constants so that do not depends on the parameter value a in $\mathcal{A}$, the next section all the constants depends only the previous ones $\alpha, \beta, \sigma, \gamma, C_0, \varepsilon$.

Finally, we fixe Λ large enough. Thus, the last sections on this text the constants $\tilde{c}, c', \tau_1, N_1, R, B_0, \tau_2, C_5, \Theta, B_2$ are depending only the previous one. Therefore, we can say that all constants used in this text depend of the one, two or more constants between $\lambda, \alpha, \beta, \gamma, \sigma, \varepsilon, \Lambda$ and ι, which were chosen uniformly on $\mathcal{A}$. So, we may say that the family in question $\mathcal{F}$ is *uniform* as referred in [Al04].

Uniformity		
Main references	Constant	Dependencies
	λ	
	δ	
	γ	
	α	
	σ	
	σ_0	
	ι	
Definition 2.1.1	β	
	ε	
	ρ	λ
Section 2.1	r_s	$\gamma, \delta, \iota, \varepsilon$
Section 2.1.2	σ_1	σ_0
Lemma 2.1.3	C_1	β, α
Section 2.1	C_0	fixed
Lemma 2.1.4	C_2	ε, λ
Lemma 2.1.6	$\tilde{C}$	fixed
Lemma 2.1.7	B	α, β
Lemma 2.2.3	C_3	$\alpha, \gamma, \varepsilon, \sigma, C_0$
Lemma 2.2.4	C_4	$C_2, \beta, \gamma, \sigma$
Section 2.3 and Lemma 3.2.1	Θ	
Section 3.1	$\tilde{c}$	α, β, σ
Section 3.1	c'	$\varepsilon, \beta, \gamma, \sigma$
(3.5)	τ_1	α, β
Section 3.1	N_1	Λ, τ_1
Section 3.1	R	β, C_0, Λ
Section 3.1	B_0	N_1, τ_1
Section 3.1	τ_2	$t, \varepsilon, C_5, \Theta$
(3.11)	C_5	C_3, C_4
Section 3.2	B_2	N_2, τ_2

Table 3.1: Uniformity of the constants

Final remarks and future study

As we have been showing in this thesis statistical stability is very recent theory and it has been used to get this type of stability for some classes of transformations. In the case of one-dimensional maps, statistical stability was proved by Freitas for the quadratic family on the Benedicks-Carleson parameters, Alves and Soufi proved for certain one-dimensional families associated to the Lorenz attractors for Rovella parameters and in this current thesis we had proved that the family of maps with critical and singular points on the Luzzatto and Viana parameters satisfies the conditions to get the statistical stability. The main difficulty was working with the presence of two critical points in the domain of the map. The present work gives a contribution to the theory, extending the number of applications where Alves theory for non-uniformly expanding maps can be applied.

With respect to future studies, we intend to extend the set of parameters $\mathcal{A}$ used in this thesis and show that the family $\mathcal{F}$ loses stability, that is the family is *statistically unstable* to parameter values within this extended set. In this direction, we will use an approach similar to that developed by Thunberg for the quadratic family, see [Th01], and a recent result of Alves and Khan for Rovella maps, see [K18].

Stochastic stability has been shown for quadratic family by Benedicks and Young in [BY92], in a weak* topology and by Baladi and Viana in [BV96], in the strong sense (L^1 convergence of invariant densities). Also, stochastic stability was proved by Metzger for Rovella family in [Me00]. In addition, we intend to obtain *strong stochastic stability* for Luzzatto-Viana maps following the result of Alves and Vilarinho in [AV13].

We also propose to study the possibility of application for Luzzatto-Viana maps of the theory developed by Freitas, Freitas and Magalhães, in [FFM17], and particularly, to understand the possible effects of the singularities and of the criticalities on the multiplicity distribution limiting point processes that are typically compound Poisson.

List of symbols and notations

$r \in \mathbb{N}$	depth
0	singular point
$\pm c$	critical points
$I_r = [\varepsilon^\gamma e^{-r}, \varepsilon^\gamma e^{-(r-1)}), \quad I_r^0 = I_r \ ; \quad (r \geq \Lambda \geq 1)$	partitions of Δ_r^0
$I_{-r} = -I_r, \quad (r \leq -\Lambda),$ with Λ is a big number	
$I_r^+ = I_r^{0^+} = [e^{-(r+1)}, e^{-(r-1)})$	
$I_r^+ = I_{r-1} \cup I_r \cup I_{r+1}$	host interval
$I_r = \bigcup_{r,i}^{r^2} I_{r,i}$ where $\quad i = 1, 2, ..., r^2$	
$I_{r,i}^+ = I_{r-1,i} \cup I_{r,i} \cup I_{r+1,i}$	
$I_r^{\pm c} = I_r^0 \pm c$	
$I_r^{\pm c} = [\varepsilon^\gamma e^{-r} \pm c, \quad \varepsilon^\gamma e^{-(r-1)} \pm c), \quad (r \geq \Lambda)$	
$I_r^{\pm c} = -I_{-r}^{\pm c}$	
$I_+ = ([-a, a] \backslash [-\varepsilon^\gamma e^{-r+1} + c, \ \varepsilon^\gamma e^{-r+1} + c] \cup [-\varepsilon^\gamma e, \ \varepsilon^\gamma e]) \cap \ \mathbb{R}^+$	
$I_- = ([-a, a] \backslash [-\varepsilon^\gamma e^{-r+1} - c, \ \varepsilon^\gamma e^{-r+1} - c] \cup [-\varepsilon^\gamma e, \ \varepsilon^\gamma e]) \cap \ \mathbb{R}^-$	

$I_r^{\pm c^+} = I_{r-1}^{\pm c} \cup I_r^{\pm c} \cup I_{r+1}^{\pm c}$	
$I_r^{\pm c} = \bigcup_{r,i}^{r^2} I_{r,i}^{\pm c}\ , \quad i = 1, ..., r^2$	
$I_{r,i}^{\pm c^+} = I_{r-1,i}^{\pm c} \cup I_{r,i}^{\pm c} \cup I_{r+1,i}^{\pm c}$	
$\Delta_r^0 = (-\varepsilon^\gamma\ e^{-r+1}, \varepsilon^\gamma\ e^{-r+1})$	neighbourhood of 0
$\Delta_r^{\pm c} = (-\varepsilon^\gamma\ e^{-r+1} \pm c, \varepsilon^\gamma\ e^{-r+1} \pm c)$	neighbourhood of $\pm c$
$\Delta_r^\varepsilon = \Delta_r^0 \cup \Delta_r^{\pm c}$	
$\Delta_r^{\pm c} = \{\pm c\} \bigcup_{\lvert r\rvert \geq 1} I_r^{\pm c}$	
$\Delta_{r_s}^0 = \{0\}\ \cup \bigcup_{\lvert r\rvert \geq r_{s+1}} I_r^0$	smaller neighbourhood of 0
Λ, Θ	large numbers such that $r \geq \Theta \geq \Lambda$
Leb	the Lebesgue measure
$x_{\sqrt{2}}$	a unique point on $(0, c)$
	such that $f'(x_{\sqrt{2}}) = \sqrt{2}$
$I = [-a, a]$	
$[c + \rho\varepsilon,\ c + \varepsilon]$	dynamical space
$\mathcal{A}$	Luzzatto-Viana parameters set
$\mathcal{F} = \{f_a\}_{a \in \mathcal{A}}$	uniform family of real maps
ψ	smooth map denined on $\mathbb{R}$
$\Phi_c = f_a^n(\pm c)$	critical orbit
$\Phi_0 = f_a^n(0)$	singular orbit
$d_\delta = (x, *), \quad * = 0,\ \pm c$	δ-truncated distance
$\mathcal{E}^a(x)$	expansion function time (ETF)

$\mathcal{R}^{a}_{\varepsilon,\delta}(x)$	recurrence function time (RTF)
$\Gamma^{n}_{a}(x)$	tail set at the time $n \in \mathbb{N}$
μ_a	measure probability
$\mathfrak{D}(\omega_j) = \min\{\lvert\omega_j\rvert, \quad \lvert\omega_j \pm c\rvert\}$	dynamic distance for partitions
$\mathbb{E}$	mathematical expectation
$\zeta(d, Q)$	number of solution of
	equation $x_1 + \cdots + x_d = Q$
F_n	sequence of return depths
p	binding period
$\{\mathcal{P}_n\}_{n\in\mathbb{N}}$	sequence of partitions
ω	element of partition
$R_n(\omega) = \{\upsilon_1, \ldots, \upsilon_{\gamma(n)}\}$	set of the return times
	of $\omega \in \mathcal{P}_n$ up to n
$Q_n(\omega) = \{(r_1, k_1), \ldots, (r_{\gamma(n)}, k_{\gamma(n)})\}$	set of records the indices
u	essential return
d	deep essential return

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