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Representations of Generalized Quivers

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to my advisor

It is sometimes said of someone that one hasn't found anyone who could say a bad thing about them. Of Prof. Gothen, I can say I haven't been able to find any one person who could refrain from praising him. As his student I can only second that general apreciation. His character is apparent to anyone who has ever exchanged even a few words with him. His mathematical prowess I am far from qualified to appraise, and I can say only the following: I have scarcely been able to find anyone with such a sharp and organized mind, and I cannot imagine how I'd have done without his mathematical versatility and penchant to go straight to the essential points.

It was all but straightforward, the road that led me to this point, one that seemed oftentimes bleak. In retrospective, however, it is almost tempting to believe it was arranged by a higher order. In no small portion is that impression due to my adviser. I am left only regretting not even nearly measuring up to the student he deserved. I hope that in dedicating to him this thesis, I can partly excuse myself for that shortcoming.

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It is costumary to start by thanking one's adviser, but I could only repeat myself. I will instead start with the person who's been a reference for a longer period, my mother. Her role in my life is well beyond description, but during the specific period of my PhD I must mention her for being a role model and a beacon of strength she has been through the emotional upheaval of my father's illness and passing.

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Abstract

Generalized quivers are Lie-theoretic entities, determined by the restricted adjoint action of a group on pieces of the Lie algebra of a larger group containing it. Their relevance is in their unification of various objects with wide and important applications. For particular choices of classical groups, one can interpret a generalized quiver in terms of directed graphs, also known as quivers. This thesis systematically explains these interpretations. In thegeneral case, it studies the questions of definition and characterization of stability condition determining moduli spaces of representations of generalized quivers.

The following are the main original contributions of this thesis. We introduce a notion of duality on quivers using (anti)involutions which allows us to give a uniform treatment to many examples in the literature, and which constitutes an advancement toward understanding generalized quivers for classical groups. We explicitly understand the stability properties of representation of generalized quivers in finite dimensions, and establish an inductive formula for the Poincaré polynomial of their moduli spaces. We introduce the notion of generalized quiver bundle, and derive stability conditions for them. Finally, we study the case of generalized orthogonal quiver bundles in detail, and obtain a complete Hitchin-Kobayashi correspondence for them. On the general question of stability in gauge theory, we contribute in this thesis to a systematic and general explanation for the existence of 'extra parameters,' proving a Hitchin-Kobayashi correspondence which accounts for them.

Resumo

Aljavas generalizadas são objectos definidos em termos da teoria de groups de Lie e são determinados pela restricção da acção adjunta de um grupo reductivo sobre partes da álgebra de Lie de um grupo que o contém. A sua relevância prende-se com o facto de que providenciam uma unificação de vário objectos importantes de uso corrente. Dadas escolhas particulares de grupos clássicos, pode-se interpretar aljavas generalizadas em termos de grafos orientados, conhecidos neste contexto por aljavas. A presente tese faz um estudo sistematico destas interpretações. No caso geral, estuda a definição e caracterização das condições de estabilidade necessárias para a construção dos seus espaços d parâmetros.

As contribuições originais desta tese são as seguintes. Introduzimos uma noção de dualidade para representações de aljavas através do uso sistemático de (anti)involuções em álgebras, um passo fundamental para compreender explicitamente as aljavas generalizadas para grupos clássicos. Fazemos um estudo explícito da estabilidade de representações em dimensões finitas e estabelecemos uma fórmula indutiva para o Polinómio de Poincaré equivariante do seu espaço de paraâmetros. Introduzimos também a noção de uma aljava em fibrados generalizada e derivamos condições de estabilidade para elas. Sobre a questão geral de estabilidade em teoria de gauge, contribuimos nesta tese para uma compreensão uniforme e sistematica da existência dos chamados 'parâmetros extra', provando uma correspondência de Hitchin-Kobayashi que as explica.

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Introdution

The present thesis follows two conducting lines. The first is a study of the stability conditions involved in the construction of moduli spaces; the second is the systematic use of those stabilty conditions to study generalized quivers. Explaining the concept of stability is rather involved, and so it must be deferred to the thesis proper. We will here try to contextualize generalized quivers, emphasizing the kind of goals that their theory would aim to accomplish. Given the substantial importance and applications of quivers, which as the name indicates are the inspiration for generalized quivers, we will necessarily be very partial in our overview. This in particular means that we will orient ourselves by their interaction with gauge theory, at the expense of other areas.

On the question of stability, the main contribution of this thesis is the proof of a Hitchin-Kobayashi correspondence which provides a systematic and general explanation for the existence of 'extra parameters.' The other original contributions of this thesis refer to generalized quivers. We introduce a notion of duality on quivers using (anti)involutions which allows us to give a uniform treatment to many examples in the literature, and which constitutes an advancement toward understanding generalized quivers for classical groups. We explicitly understand the stability properties of representation of generalized quivers in finite dimensions, and establish an inductive formula for the Poincaré polynomial of their moduli spaces. We introduce the notion of generalized quiver bundle, and derive stability conditions for them. Finally, we study the case of generalized orthogonal quiver bundles in detail, and obtain a complete Hitchin-Kobayashi correspondence for them.

0.1 Generalized quivers

Originally introduced by Gabriel as representation-theoretic entitites, the representations of quivers is an extensively studied topic with deep connections to many areas in mathematics. In geometry, the construction of their moduli spaces is a non-trivial example of the intersection between Algebraic Geometry (via Geometric Invariant Theory) and Symplectic Geometry, the moduli spaces themselves are also important examples of (hyper-)Kähler manifolds furnishing finite dimensional/discrete versions of many relevant gauge-theoretical moduli spaces. A sign of their relevance is their spread across many branches of mathematics, including geometric representation theory, gauge theory, and mirror symmetry, which makes them a well established area of research for a number of years from various points of view. The definitions are as follows.

0.1.1 Definition. 1. A quiver Q is a finite directed graph, with set of vertices I, and set of arrows A. We let $t : A \to I$ and $h : A \to I$ be the tail and head functions, respectively.

2. A representation (V,φ) of Q is a realization of the diagram Q in the category of finite dimensional spaces; equivalently, a representation is an assignment of a vector space V_i for each vertex $i \in I$, and a linear map $\varphi_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ for every arrow α.

Noting that $\varphi_\alpha \in \mathrm{Hom}(V_{t(\alpha)}, V_{h(\alpha)})$, a representation of a quiver as above naturally belongs to a product of vector spaces. Hence, the category of all representations V of Q with a given total space $V_\Sigma := \bigoplus_{\alpha \in A} V_\alpha$ naturally constitutes a vector space. Further, the product of the automorphism groups of each V_i naturally acts on this vector space by the componentwise conjugation of the linear maps (in concrete terms, by simultaneous change of bases of all linear maps involved.) Therefore, it makes sense to ask for a moduli space of representations of quivers. Their construction was accomplished by using a mix of algebraic- and symplectic-geometric methods, providing perhaps the simplest non-trivial example of the coincidence of Mumford's Geometric Invariant Theory, and Marsden-Weinstein's theory of symplectic reduction, through the Kempf-Ness theorem. The original construction from this dual point of view is due to King [21]; a useful reference is also Reineke [37].

An important variation on the theme are quiver *bundles*. These represent our diagram Q in the category of holomorphic vector bundles over a given complex variety X, and yield the usual representations when X is a point. In this case, one considers the action of the gauge group on the bundles, and one can also consider the question of construction and study of a moduli of such bundles. A non-trivial associated question is the existence of preferred Hermitian bundle metrics satisfying Einstein-Hermitian type equations, which basically corresponds to asking for an infinite dimensional Kempf-Ness theorem. Indeed a so-called complete Hitchin-Kabayashi correspondence has been established by Álvarez-Consul and García-Prada [1], and characterizes the solutions of such equations. This characterization is in terms of an algebraic notion of stability used in constructing gauge theoretical moduli (in fact, one must include a slightly more general notion of quiver *sheaves* when considering base manifolds with dimension greater than one for technical reasons.)

Quiver bundles are interesting since there has been a wealth of examples that have naturally arisen in gauge theory. Higgs bundles over a Riemann surface are one especially important example, since they play a crucial role in an incredible variety of areas, including physics. A further example comes straight from the study of the moduli of Higgs bundles themselves: so-called chains arise as fixed points for a $\mathbb{C}^*$ action on the moduli space as noted by Hitchin [18], Simpson [**?**], and Gothen [**?**]. Perhaps more surprisingly, finite-dimensional representations of quivers have been found to play an important role in gauge-theoretical matters as well. We illustrate this by discussing a few examples.

1. **Local structure of the moduli space of vector bundles:** Using the GIT construction of the moduli space, together with Luna's slice theorem, Seshadri [40] proved the the local structure of the moduli of vector bundles coincides with the local structure of the quotient space of representations of the so-called local quivers. In particular, he proved that to each polystable vector bundle $\mathbb{V}$ we can associate a quiver Q such that the deformation space of the bundle and the representation space of the quiver coincide. Furthermore, this identification is such that we can also equivariantly identify the automorphism group of $\mathbb{V}$ with the symmetry group of Q, which means that the quotient space of the deformations of the vector bundle coincide with the quotient variety of its local quiver. Luna's slice theorem can then be used to yield an étale map between a neighbourhood of the origin in the latter quotient space and a neighbourhood of the

point of the moduli determined by $\mathbb{V}$. Laszlo [24] then showed that this identification is efficient in the sense that it is highly computable; in particular, he managed to compute completions of the local rings at some point corresponding to strictly polystable points. (In fact, in his thesis Bocklandt [7] has further demonstrated that quotients of quiver representations allow for explicit computions by using an analogous slice theorem of LeBruyn and Procesi [25] to characterize quivers for which the quotient space is smooth.)

2. **Functorial construction of moduli space of vector bundles:** Completing work that goes back to le Potier [26], Alvarez-Consul and King [**?**] have constructed an embedding of the category of coherent sheaves over a projective scheme into a category of representations of a particular quiver in such a way that the functor identifies stability conditions. This allows for a functorial construction of the moduli of coherent sheaves as the quotient space of representations of quivers. This construction greatly simplifies the algebraic construction of the moduli space, and could in principle increse the relevance of the algebraic machinery in its study. In fact, Hoskins [19] showed that by finetuning the particular functor used for the embedding, one can hope for the coincidence of the Harder-Narasimhan stratifications on either side.

3. **Instantons on ALE spaces:** Nakajima quiver varieties are not, strictly speaking, quiver varieties, as the quotient is not take with respect to the full symmetry group of the quiver involved, but rather by considering it in some sense as a cotangent space. Nonetheless, it is an extension of the theory that has found tremendous uses. Among them is the finite-dimensional construction of the moduli space of instantons over ALE spaces when the quiver is of affine type, and the production of a variety of hyperkähler spaces.

We have focused only on applications to gauge-theoretical moduli spaces, since this lies closest to the topics of this thesis. But representations of quivers have found many more applications, for example to the classification of semisimple Lie algebras, or to Kac-Moody Lie algebras.

Given their wealth of applications, it is only natural to wonder about possible extensions to arbitrary symmetry groups, in the hope that we may extend the applications to arbitrary principal bundles. Endowing this statement with a precise meaning is of course hard until a definite answer has been found, but we may take it provisorily as follows. We have seen above that the moduli problem of finite-dimensional representations reduces to the action of a product of general linear groups. It is natural to try to extend consideration to cases where in some vertices we consider a classical group other than the full general linear group. We will later in this thesis see how to accomplish this in general, but the first step in this direction was taken by Derksen and Weyman [12], and allows for completely orthogonal, or completely symplectic symmetries.

0.1.2 Definition. 1. A symmetric quiver (Q,σ) is a quiver Q equipped with an involution σ on the sets I and A such that $\sigma t(\alpha) = h\sigma(\alpha)$, and vice-versa.

2. An orthogonal (resp. symplectic) representation (V,C,φ) is a representation (V,φ) of Q that comes equipped with a non-degenerate symmetric (resp. anti-symmetric) quadratic form C on its total space $V_\Sigma = \bigoplus_{i\in Q_0} V_i$ which is zero on $V_i \times V_j$ if $j \neq \sigma(i)$, and such that

$$C(\varphi_\alpha v, w) + C(v, \varphi_{\sigma(\alpha)} w) = 0$$

The quadratic form C determines an orthogonal subgroup of the full symmetry group of Q: those graded automorphism that preserve C. It is *not* simply a product of orthogonal groups, since not all vertices are fixed by σ, a fact that is important for us. Nonetheless, the symmetries of orthogonal representations clearly are orthogonal in a very concrete sense. In the same way, the condition on the maps in an orthogonal representation are precisely the condition that they be alternating with repsect to C (in a precise sense, if we extend each φ_α be zero to the whole total space.)

The main motivation for the present thesis is Derksen-Weyman's theorem to the effect that the symmetric quivers above have a full Lie-theoretic incarnation, the so-called *generalized quivers*. This are representation-theoretic objects defined for arbitrary reductive Lie groups, and we propose to probe one such definition with a view to providing an extension of the application of the theory of quivers.

0.1.3 Definition. Let G be a reductive group, $\mathfrak{g}$ its Lie agebra.

1. A *generalized G-quiver $\tilde{Q}$ with dimension vector* is a pair $(R, \mathrm{Rep}(\tilde{Q}))$ where R is a reductive subgroup of G, and $\mathrm{Rep}(\tilde{Q})$ a finite-dimensional representation of R (the *representation space*.) We require the irreducible factors of the representation also to be irreducible factors of $\mathfrak{g}$ as an $\mathrm{Ad}R$-module, and the trivial representation to not occur.

2. A generalized quiver *of type Z* is a generalized quiver for which R can be realized as a centralizer in G of a closed abelian reductive subgroup.

3. A *representation of $\tilde{Q}$* is a vector $\varphi \in \mathrm{Rep}(\tilde{Q})$.

Note that Derksen-Weyman's definition always require generalized quivers to be of type Z, but the more general case should be of interest as well. It is not hard to see, as we shall do below, that generalized quivers of type Z for classical groups give objects of the type of symmetric quivers, namely quivers with extra data encoding a restriction of symmetries to smaller classical groups. In fact, many such incarnations have been defined already: supermixed quivers [46] [47], and $\mathfrak{Q}$-mixed quivers [Lopatin and Zubkov]. They all fit into the generalized quiver framework, and we shall take a systematic approach to them using algebras.

It is clear from the definitions that the problem of a quotient for representations of generalized quivers makes sense, and involves both the techniques used for classical quivers, as well as some Lie theory. One wants to construct the quotients for such representations, which come in families. One is also interested in extracting cohomological information, in the form of Poincaré polynomials. On the other hand, it makes sense to ask what would constitute a generalized quiver bundle, and pose for those the corresponding questions concerning stability and Einstein-Hermitian metrics. These are the particular questions we consider in this thesis.

0.2 Summary of contents

We now describe the contents of this thesis in more detail. This should also serve a reading guide, hoping to help the reader to pick apart the parts of interest to them. We have noted that the thesis is oriented by two guiding lines – stability conditions and generalized quivers–, both of which have finite and infinite dimensional interpretations. We will consider both aspects of the questions, and this accounts for a certain inevitable split ‘along the middle’ of the thesis as we transition from finite dimensional representations to generalized quiver bundles. The fact that both halves are different sides of the same question – and so the cohesiveness of the thesis– is clear for the case of generalized quivers. In the case of stability conditions, we must acknowledge that within the present framework it is hard to see more than an analogy connecting the different dimensional sides. We can do no more that point out that by reducing the moduli problem to a GIT quotient, the algebraic theory establishes a direct connection between stability for bundles and stability for quotients of varieties. Alas, a study of the algebraic theory of generalized quiver bundles would require another work of the same size as the present thesis. We certainly hope in the future to come to this topic in the future, but we have had to skip it altogether here.

The first chapter of the thesis is expository in nature. We start with a fairly detailed expostion of the simplest (projective) case of the theory of quotients in algebraic and symplectic theory. We motivate and introduce the definition of a Geometric Invariant Theory (GIT) quotient for projective varieties, as well as the corresponding quotients for affine ones which one obtains by considering them as quasi-projective. In particular, we explain why one typically needs to remove some closed subset from the variety in order to obtain satisfactory quotients, which takes us directly to the notion of stability. In analysing this condition we are naturally taken to the question of its computability, and so to the Hilbert-Mumford criterion. Having considered the algebraic case, we proceed then to consider quotients of differentiable manifolds, and introduce the symplectic reduction space associated with a Hamiltonian action of a compact group. We proceed the consider how this picture fits naturally into the context of Kähler geometry, obtaing a rather nice differential-geometric picture. The second half of the first chapter considers the overlap of the two formalisms we just mentioned. We first explain and prove the Kempf-Ness Theorem, which says that for projective varieties the GIT and symplectic quotients coincide. Inspired by this theorem, we then work up to the Kirwan-Ness Theorem, which shows that we may find stratifications of the original variety in both formalisms, and that these stratifications also coincide. As we explain, this is especially important in view of the fact that these stratifications are equivariantly perfect, and so can be used to extract cohomological information.

Some comments are in order about this first chapter. The first concerns the question of the groups under consideration, namely our choice of dealing with linearly reductive groups. It is well known that this condition is unreasonably restrictive in positive characteristic, in which setting one should require only geometric reductivity. We have found, however, that using linear reductivity significantly tidies up the arguments, and it is perfectly well adapted to the complex case, in which we are ultimately interested. A second question concerns the projective setting itself. In fact, our main quotient problem is that of a linear action on an affine space. The results in this first chapter do indeed extend to our case, but they require independent (even if analogous) proofs. This makes the first chapter seem superfluous, which it certainly is from a strictly mathematical point of view. Nonetheless, we feel that even the very definition of GIT quotients of affine varieties can only be understood after having studied the projective case. The reader who can do away with such motivation may skip the first chapter almost entirely, except for the general results on the stratifications. Finally, this first chapter serves a further motivational purpose: that of motivating the Hitchin-Kobayashi correspondence. Whereas the statement of these correspondences are essentailly a Hilbert-Mumford type of result, their proof is strongly analogous to that of the Kempf-Ness Theorem. Further, the full details involved in the projective case is very much the motivation for Kähler quotients, of which gauge-theoretical moduli spaces are but particular examples. Again, the reader interested in generalized quiver bundles, but acquainted with these ideas may skip the first chapter entirely.

The second chapter takes up the question of constructing and studying quotients of finite-dimensional representations of geralized quivers for general reductive groups G. We completely characterize the stability conditions for these objects, and using this characterization we are able to give explicit descriptions of so-called Jordan-Hölder objects which parametrize polystability types, as well as so-called Harder-Narasimhan objects, which parametrize instability types. These last objects in particular are important for the main result of this chapter, namely an inductive formula for the Poincaré polynomial of the quotient, where the induction is taken over the semisimple ranks of certain Levi subgroups of G. The point here is that in constructing the appropriate stratifications of the representation space, each stratum retracts nicely to a smaller variety composed of objects of the same 'Harder-Narasimhan type.' Our explicit description of these objects then identifies this retracted stratum with a representation space for an induced generalized quiver for a Levi subgroup of G. We finish this chapter by showing how our results recover well-known results about quiver representations, and using this as a blueprint to study representations of supermixed quivers. This in particular allows us to work out explicitly a family of examples of orthogonal representations, and to show how the inductive formula does indeed terminate.

The final chapter is concerned with introducing and studying generalized quiver bundles. We begin by introducting the definitions themselves, and with a concise exposition of their moduli problem. We find a first justification for our definition by showing how to recover quiver bundles from this framework, as well as a generalization of Derksen-Weyman's result for the bundle case. An important part of this chapter is to show how the existence of 'extra parameters' which arise for quiver bundles may be systematically understood from the point of view of the Hitchin-Kobayashi correspondences. This is then put to fruition in the study of stability conditions for generalized orthogonal quiver bundles, in which case we actually obtain a complete characterization of the solutions to the gauge equation (a complete Hitchin-Kobayashi correspondence for this particular case.) The chapter finishes by pointing out further examples of generalized quiver bundles which greatly illustrate the interest of this new definition we introduce.

Chapter 1

Quotients in Algebraic and Symplectic Geometry

In this chapter, we discuss the foundations of the theory of quotients in algebraic and symplectic geometry. This framework will be used throughout the thesis. The section on affine quotients is closely modelled on Mukai's book [30]. The material on projective quotients is essentially an expanded version of Thomas' notes [43], though we've also extensively used Dolgachev's book [13]. The differential geometric sections have drawn mostly from Audin [4] and Cannas da Silva [10], and the proof of the Kempf-Ness Theorem is that of Woodward [**?**]. Finally, the material on the stratifications and cohomology of quotients follow closely Kirwan's thesis [22] as well as Dolvachev and Hu's article [14].

1.1 Geometric Invariant Theory

Fix an algebraically closed field k of characteristic zero, over which all varieties will lie. Let X be a variety, and G be an algebraic group. The group acts on X if each of its elements determines an automorphism of X in a way that is compatible with multiplication. It would be natural to try to define an action as a morphism $G \to \mathrm{Aut}(X)$, but the problem is that we have then to define an appropriate structure of variety on $\mathrm{Aut}(X)$. Here is a definition that avoids this problem: *an action of G on X* is an algebraic map $\sigma : G \times X \to X$ such that if $\mu : G \times G \to G$ is the multiplication map and $\varepsilon : \mathrm{Spm}\ k \to G$ is the identity, we have $\sigma \circ (\mu \times \mathrm{id}_X) = \sigma \circ (\mathrm{id}_G \times \sigma)$, and $\sigma(\varepsilon \times \mathrm{id}_X) = \mathrm{id}_X$ (since $\mathrm{Spm}\ k$ is terminal in varieties over k, we are implicitly using that $X \times \mathrm{Spm}\ k = X$. In other words, the following diagrams commute

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times \sigma} & G \times X \\ {\scriptstyle \mu \times \mathrm{id}_X}\downarrow & & \downarrow{\scriptstyle \sigma} \\ G \times X & \xrightarrow[\sigma]{} & X \end{array} \qquad \begin{array}{ccc} \mathrm{Spm}\ k \times X & \xrightarrow{\simeq} & X \\ {\scriptstyle \varepsilon \times \mathrm{id}_X}\downarrow & & \| \\ G \times X & \xrightarrow[\sigma]{} & X \end{array}$$

We will denote the image of (g,x) by $g \cdot x$. We say a morphism $\varphi : X \to Z$ is G-invariant if $\varphi(g \cdot x) = \varphi(x)$ for all g and x. We can write this in terms of morphisms as follows. Denote by $p_X : G \times X \to X$ the canonical projection to the second factor, i.e. $p_X(g,x) = x$. Then, φ is invariant if

$\varphi \circ \sigma = \varphi \circ p_X$. In the case where $Z = \mathbb{A}^1$, we are in fact considering *invariant functions* on X, which form a subring $k[X]^G \subset k[X]$ of the the coordinate ring of X.

1.1.1 Affine quotients

Categorical quotients When we speak about a quotient, we generally mean one of two different things: an orbit space or a categorical quotient. Whereas in the category **Set** both concepts coincide, it is a very important fact that they almost never do in the geometric context. We will start by considering actions on affine varieties, where everything can be written down explicitly, and so it is easy to compare both notions. It turns out that the categorical quotient of an affine variety is not well behaved, but it still provides the basic ingredients towards an appropriate construction and understanding of quotients, following Mumford [31]. This section is, therefore, lays the ground for all that follows.

1.1.1 Definition. Suppose a group G acts on a variety X. A *categorical quotient of* X is a G-invariant morphism $\Phi : X \to Y$ that is universal, i.e., given any other invariant morphism $f : X \to Z$ to an arbitrary variety, there is a unique arrow $\bar{f} : Y \to Z$ such that $f = \bar{f}\pi$.

It is costumary to denote a categorical quotient by $X /\!\!/ G$, to distinguish it from a orbit spaces (themselves usually denoted by X/G.)

Suppose that a quotient $Y := X /\!\!/ G$ of X exists. Then, the usual pullback of global regular functions $\pi^*\mathscr{O}_Y(Y)$ must lie in the invariant ring $\mathscr{O}_X(X)^G$, since π is invariant. On the other hand, looking at an invariant f as a function $f : X \to k$, from the universal property we get a regular funtion on Y. It should be clear that these two processes are inverse to each other, and a little more effort proves that they are indeed isomorphisms $\mathscr{O}_Y(Y) \simeq \mathscr{O}_X(X)^G$ inverse to each other.

The key observation here is that in the category of rings, $R := \mathscr{O}_X(X)$ has a universal property which is dual to the property of quotients. Namely, with respect to the co-action of G on R, the image of any co-invariant map $A \to R$ must lie in R^G. For this reason, we restrict to $X = \mathrm{Spm}\, R$ affine, hoping to be able to construct the quotient out of R^G; in other words, given the ring of regular functions, we want to reconstruct Y. This is not as easy as it might seem, because this ring can be rather badly behaved. A further restriction on the group itself solves this issue.

1.1.2 Definition. An group G is *linearly reductive* if for every algebraic representation $G \to \mathrm{GL}(V)$, there is an invariant complement to every invariant subspace under the G action. In other words, if $V_1 \subset V$ is a sub-vector space of V invariant under the action of the group, then there is another invariant sub-space V_2 such that $V = V_1 \oplus V_2$.

1.1.3 Theorem (Hilbert, [30] ch.4)**.** *Let G be a linearly reductive group acting on an affine variety X. Then the ring of invariants R^G is finitely generated.*

This theorem solves our problem when we're dealing with the action of a reductive group, for then the ring of invariants is a finitely generated integral domain, and so $Y = \mathrm{Spm}\, R^G$ is an affine variety.

1.1.4 Theorem. *If G is a reductive action acting on an affine variety $X = \mathrm{Spm}\, R$, then the natural map $\Phi : X \to Y := \mathrm{Spm}\, R^G$ is a categorical quotient.*

Proof. First, the map Φ is clearly G-invariant. Let $f : X \to Z$ be a regular map into an arbitrary variety, and $Z_i = \text{Spm } A_i$ be an affine cover of Z. Since the map is invariant and X affine, then $f^{-1}(Z_i)$ is a G-invariant affine subvariety $X_i = \text{Spm } R_i$ of X. Now, since both Z_i and X_i are affine, the restriction map $f_i : X_i \to Z_i$ certainly factors through $\Phi_i : X_i \to X_i /\!/ G$, i.e., we have unique map $\bar{f}_i : X_i /\!/ G \to Z_i$ such that $f_i = \bar{f}_i \Phi_i$. If we compose with the inclusion, we get a collection of maps $X_i /\!/ G \to Z$, and so we are reduced to prove the following two things: (1) that we can build $X /\!/ G$ as a glueing of the X_i; (2) that then we can glue the $\bar{f}_i$ into a global map $\bar{f} : X /\!/ G \to Z$.

1. *We can realize $X /\!/ G$ as a glueing of the $X_i /\!/ G$:* Since for each i, X_i is G-invariant, we can assume that R_i is actually a localization of R at an invariant element. But then, clearly $R^G \subset R_i^G$, which induces a dual map $X_i /\!/ G \to X /\!/ G$. All these maps are clearly compatible, since they all come from inclusions of R^G, so we have only to prove that the $X_i /\!/ G$ cover $X /\!/ G$. But this is obvious, since the X_i cover X.

2. *The maps $\bar{f}_i$ glue into a map $\bar{f} : X /\!/ G \to Z$:* All there is to check is that if $X_i /\!/ G \cap X_j /\!/ G \neq$ as subsets of $X /\!/ G$, then $\bar{f}_i = \bar{f}_j$ on the intersection. But we first have that $\bar{f}_i \Phi_i = \bar{f}_j \Phi_j$ by definition, and also that Φ_i and Φ_j both restrict to the quotient map Φ_{ij} of $X_{ij} = X_i \cap X_j$, and this map is epimorphic. Therefore, the maps are compatible and glue.

□

What is the geometry underlying this algebraic construction? We're in fact looking at an embedding free version of the image of a universal G-invariant map $\phi : X \to \mathbb{A}^n$. Indeed, given invariant functions $f_1, ..., f_n$, the universal property of $\mathbb{A}^n$ as a product yields one such map defined as $x \mapsto (f_1(x), ..., f_n(x))$, the G-invariance of which follows from that of the f_i. By looking at all possible maps of the kind when the f_i are linearly independent, we might hope to find a variety that could serve as a quotient. The first problem is that in general we might have to increase the number of such invariant *ad aeternum*; but Hilbert's Theorem tells us that this is not the case for a linearly reductive group. What is more surprising, is that linear reductivity solves the second, more difficult problem: that the image of a regular map between varieties need not be a variety at all.

1.1.5 Proposition. *If G is linearly reductive, then for every ϕ as above where the f_i are generators, $\phi(X)$ is closed.*

Proof. Let $a = (a_1, ..., a_n) \in Y$ be a point in Y and consider the homomorphism $\pi : R \oplus ... \oplus R \to R$ defined as

$$\pi(b_1, ..., b_n) = \sum b_i(f_i - a_i)$$

This is not only a morphism of R-modules, but also of representations of G, since each $f_i - a_i$ is invariant. Further, the image of the restriction of π to invariants is the maximal ideal $\mathfrak{m}_a$ corresponding to the point a. By reductivity, π itself is not surjective, and so its image must be contained in some maximal ideal $\mathfrak{m} \subset R$. We necessarily have $\mathfrak{m} \cap R^G = \mathfrak{m}_a$, so a is the image of the point corresponding to $\mathfrak{m}$. □

With this proposition, we can take the image of one such map as the quotient of X by G. What Theorem 1.1.4 does for us is to avoid a (non-canonical) choice of generators for invariants, and thus to give us the embedding free version of this. In fact, this quotient has relatively nice properties, which we gather in a new definition

1.1.6 Definition. A G-invariant morphism $\Phi : X \to Y$ is a *good quotient* if:

1. The natural map $\mathscr{O}_Y \to \Phi_* \mathscr{O}_X^G$ is an isomorphism.
2. If Z is an G-invariant closed subset, then $\pi(Z) \subset X /\!\!/ G$ is also closed.
3. If Z_1 and Z_2 are two closed invariant sets, then $\Phi(Z_1)$ and $\Phi(Z_2)$ are also disjoint.

The natural map mentioned in the definition is nothing but the pullback $\Phi^* : k[U] \to k[\Phi^{-1}(U)]^G$ at each open U, which we considered in the beginning of this section; we just made the presence of the structure sheaves more prominent, because for a general variety this precise condition is the only one that makes sense.

We have purposely not required the map Φ to already be a quotient: it is left as an exercise to prove that a good quotient is actually a quotient, that is, it has the universal property. Whereas this definition seems rather obscure at first, it ensures that the quotient has properties one comes to expect of any reasonable quotient

1.1.7 Proposition. *Let Φ be a good quotient. Then,*

1. *Φ is surjective.*
2. *Φ is a submersion, i.e., $U \subset Y$ is open if and only if $\Phi^{-1}(U)$ is open.*
3. *The natural map $\mathscr{O}_Y \to \Phi_* \mathscr{O}_X^G$ is an isomorphism.*

Proof. We start by remarking that it is enough to prove surjectivity. Indeed, the condition on sheaves is the same, and property 2 of the definition together with surjectivity imply that Φ is a submersion. To prove this, let $U \subset Y$ be a subset with open preimage, i.e., $Z := X - \Phi^{-1}(U)$ is closed. By property 2, $\Phi(Z)$ is closed, whereas by surjectivity, $U = Y - \Phi(Z)$, which implies the result.

Suppose then that Φ is a good quotient. Condition 3 implies that Φ is dominant, and condition 2 it has closed image, implying that Φ is surjective. □

It turns out that the definition of good quotient is just slightly stronger that the conditions on this proposition, in the sense that if the fibres of the map Φ are orbits, then the proposition implies the properties of the definition. The result we're aiming at is the following.

1.1.8 Theorem. *Let X be an affine variety, and G a linearly reductive group acting on it. The categorical quotient $X /\!\!/ G$ is a good quotient.*

Almost everything we will prove about affine categorical quotients follows from this theorem. We will give independent proofs of those facts, but the reader can check that indeed that many follow formally from this.

The preimage of any open in $X /\!\!/ G$ is G-invariant, so its coordinate ring is determined by localizations at invariant elements; this implies the required isomorphism of sheaves. The other two properties are consequences of the following lemmas.

1.1.9 Lemma. *If Z is an G-invariant closed subset, then $\pi(Z) \subset X /\!\!/ G$ is also closed.*

Proof. Let Z be an invariant closed subset, and $\mathfrak{a}$ its ideal. Since Z is invariant, there is an invariant complement $R/\mathfrak{a}$ to $\mathfrak{a}$. Then, every invariant on R^G is a sum of invariants in $\mathfrak{a}^G$ and in $(R/\mathfrak{a})^G$. In particular, we have an isomorphism

$$R^G/\mathfrak{a}^G \simeq (R/\mathfrak{a})^G$$

But $R/\mathfrak{a}$ is the coordinate ring of Z, so $(R/\mathfrak{a})^G = k[Z /\!\!/ G]$. Also, $R^G/\mathfrak{a}^G$ is the coordinate ring of of the closure of $\phi(Z)$, and by Proposition 1.1.5, $Z \to Z /\!\!/ G$ is surjective. We conclude that $Z \to \overline{\phi(Z)}$ is surjective, i.e., $\phi(Z)$ is closed. □

1.1.10 Lemma. *If Z_1 and Z_2 are two invariant closed sets, then $\Phi(Z_1)$ is disjoint from $\Phi(Z_2)$.*

Proof. □

1.1.11 Remark. The proofs of our results so far have not assumed anything on the characteristic of the field, but they are also rather vacuous. The problem is the condition of linear reductivity, which is inordinately strong in positive characteristic. Instead, one can impose a weaker condition, geometric reductivity, which nonetheless is equivalent to linear reductivity in characteristic zero, but under which we can reprove all of the results above (the adptation of Hilbert's theorem to this case is due to Nagata.) We will not dwell on such points, however, since our ultimate goals concern only the complex case.

Finally, note that the reductivity of the group was instrumental to prove that the affine quotient is good, and not just to prove that the ring of invariants is finitely generated. In fact, one can find examples of non-reductive group actions for which the invariants are finitely generated, and so for which one can define the categorical quotient, which nonetheless are *not* good quotients.

Orbit spaces From now on, G will always be a linearly reductive group. We have seen how to construct a categorical quotient for the action of such groups on affine varieties. We shall now see that it doesn't correlate very well with what we mean by a quotient in the sense of an orbit space.

1.1.12 Definition. A quotient $\pi : X \to Y$ is *geometric* if the image of the map $a : G \times X \to X \times X$ sending $(g,x) \mapsto (x,gx)$ is precisely $X \times_Y X$.

Recall that the points of $X \times_Y X$ are pairs (x_1,x_2) such that $\pi(x_1) = \pi(x_2)$; this definition then just says that if the quotient is geometric, then in fact $x_2 = g \cdot x_1$ for some g.

The affine categorical quotients we just constructed are generally not geometric. There is a concrete and elementary example of this: consider the action of $\mathbb{G}_m$ (the multiplicative group of non-zero scalars in k) on $\mathbb{A}^{n+1}$ as coordinatewise multiplication. The only invariants under this action are the constant functions, and so our construction yields $\mathbb{A}^{n+1} /\!\!/ \mathbb{G}_m$ as a single point space. This is very far from an orbit space, since every ray through the origin is an orbit. But in fact, since every regular function is continuous, each invariant function must send the entire closure of an orbit to the same value. For this reason, we introduce an equivalence relation, often called S-equivalence, which identifies two different orbits if and only if their Zarisky closures intersect.

1.1.13 Lemma (Mumford, Nagata)**.** *Let G be a linearly reductive group acting on an affine variety X. Given two orbits O and O' for this action, the following are equivalent:*

1. *The two orbits are S-equivalent, i.e.,* $\overline{O} \cap \overline{O'} \neq \emptyset$.

2. *There's a sequence of orbits* $O_1 = O, O_2, ..., O_n = O'$ *with* $\overline{O_i} \cap \overline{O_{i+1}} \neq \emptyset$.

3. O *and* O' *fail to be separated by the G-invariants* R^G.

Proof. $1 \Rightarrow 2 \Rightarrow 3$ is clear. We need to prove that if two orbits have disjoint closures, then they can be distinguished by invariants. First we note that they can be distinguished by regular functions, i.e., if $\mathfrak{a}$ and $\mathfrak{a}'$ are the ideals vanishing at $\overline{O}$ and $\overline{O}'$, respectively, then if the intersection of the closures is empty, $\mathfrak{a} + \mathfrak{a}' = R$ by the Nullstellensatz.

Now, since $\overline{O}$ is G-invariant, $\mathfrak{a}$ is a subrepresentation of G. By reductivity, there is a complementary invariant subspace, and this must be contained in $\mathfrak{a}'$. Then, since the action diagonalizes, it's clear that the invariants R^G are sums of invariants of $\mathfrak{a} \cap R^G$ and $\mathfrak{a}' \cap R^G$. In particular, there are invariants $f \in \mathfrak{a} \cap R^G$ and $f' \in \mathfrak{a}' \cap R^G$ such that $f + f' = 1$, as desired. □

1.1.14 Proposition. *There's a unique closed orbit in every S-equivalence class.*

Proof. From the previous lemma, it's clear that each S-equivalence class contains at most one closed orbit, because distinct closed orbits can be distinguished by invariants. We shall prove that there is at least one in each class. In particular, the orbit O of minimum dimension in a given class must be closed. Indeed, $\overline{O}$ is G-invariant, so if O is not closed, $\overline{O} - O$ must contain an orbit that is of smaller dimension than O, which is a contradiction. □

1.1.15 Corollary. *The quotient space* $X /\!\!/ G = \mathrm{Spm}\ R^G$ *parametrizes the closed orbits of the G-action.*

For the multiplicative action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$, the zero is the unique closed orbit, and indeed the quotient space, as we saw, is a single point set. Essentially, what is happening is that the ring of invariants just isn't large enough.

There's a second, related problem that is even more concerning: though we've done things so that the regular functions on $X /\!\!/ G$ precisely correspond to G-invariant regular functions on X, the same is not true for rational functions at all.

1.1.16 Definition. A quotient Y is birational if $k(Y) = k(X)^G$.

1.1.17 Example. Consider the multiplicative action on $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$, i.e., $t \cdot (x_0, ..., x_n) \mapsto (tx_0, ..., tx_n)$. A simple calculation shows that $k[x_0, ..., x_n]^G = k$, so that $X /\!\!/ G = \{*\}$ is the one-point space, so that also $k(X /\!\!/ G) = k$. But one can also show that $k(x_0, ..., k_n)^G$ is generated by the set of all ratios of homogeneous functions of the same degree. It is then clear that the categorical quotient is not birational. It turns out, however, that it is easy to construct 'by hand' a space which is almost a quotient, and is 'birational' in the sense that its field of rational functions is precisely the invariant rational functions on X. In a loose sense, since the rational functions on a space is the union of all regular functions on its open sets, we chop the affine space into pieces at turn, so that the quotient of each piece consists precisely of the ratios we need. Then, by glue this quotients together, we find a space that has all of the necessary rational functions. It turns out, however, that it is enough to do it at the coordinate functions.

Consider the localization R_i of $k[x_1,...,x_n]$ at a coordinate function x_i, and its degree zero part

$$R_{i,0} = \left\{ \frac{f}{x_i^k} | k \geq 0, f \text{ homogeneous with } \deg f = k \right\}$$

This is a subring of $k(X_0,...,X_n)$, and so it spectrum is a variety; in fact, $R_{i,0} = R_i^G$, so that Spm $R_{i,0}$ is the affine categorical quotient of $\mathbb{A}^{n+1} - \{x_i = 0\}$. Now, given any other $j \neq i$, the ring $R_{ij,0} := R_{i,0}R_{j,0}$ corresponds to a localization of $R_{i,0}$ at x_j and vice-versa. With such data, we can make a simple, separated glueing of the two varieties, defined by the inclusions $R_{i,0} \to R_{ij,0} \leftarrow R_{j,0}$ ([30], 3.4.) The result of such glueing is is precisely the projective space $\mathbb{P}^n$, and in fact this construction is just the proj of the polynomial ring under the grading by degree. (As we mentioned above, we could have considered the degree zero part of the localization at any homogeneous polynomial, but if we added these to the glueing, we wouldn't get anything new.)

The construction in this example is not a quotient of affine space, of course, since in the course of chopping it up, we ended up leaving out the origin. In what sense, then, can $\mathbb{P}^n$ be regarded as a quotient? It is in fact very close to reconciling both notions of quotient we've defined, as the next proposition shows, if we're willing to overlook the one point we missed.

1.1.18 Proposition. *The map $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ is a birational, good geometric quotient.*

We leave the proof as an exercise (an important one, since we *will* use this result later.) This example is important for two big reasons. The first one is the realization that a nice solution for quotient problems in algebraic geometry will always have to give up on some (small) subset of points for which the orbits are just too degenerate. In fact, in this example, once we remove the origin, every orbit behaves very nicely. Second, this example will be instrumental later to define projective quotients.

1.1.19 Example. The previous example is mesleading in the sense that it might lead one to think that all we have to do to replicate the result is to remove a bunch of small closed orbits. The following example should be kept in mind to avoid such mistake. Consider $\mathbb{G}_m$ acting on $\mathbb{A}^2$ as $(x,y) \mapsto (tx, t^{-1}y)$. There are three kinds of orbits: hyperbolae that pass through a point where both x and y are non-zero; the two axes without the origin; and the origin. By looking at the action, it is immediate that the invariants for the action are generated by the polynomial $f(x,y) = xy$, which does not distinguish the orbits of the last two kinds, i.e., both axes and the origin are mapped to the same point. This agrees with out previous analysis, since the two axes without the origin are not closed, and so collapse down to the unique closed orbit in their S-equivalence class; all orbits of the first kind are closed. In this case, however, we need not extract any orbit from affine space, since $\mathbb{A}^2 /\!\!/ \mathbb{G}_m = \text{Spm } k[xy] = \mathbb{A}^1$, and $k(x,y)^{\mathbb{G}_m} = k(xy)$. In fact, if we try to remove the origin, we obtain the affine line with double origin: though as a topological space this better aproximates a quotient as an orbit space, it is a non-separated variety, which is badly behaved. Later, we shall use projective space to consistently construct rational qotients, and we'll see that such method does indeed remove the origin, but drags with it one of the axes as well.

Stability The rather different behaviour of the two actions of the multiplicative group in the examples above can be explained by a fundamental difference between them: the existence of the so-called stable points.

1.1.20 Definition. A point $x \in X$ is *simple* if it has a zero-dimensional stabilizer; a simple point is *stable* if its orbit is closed.

Note that if stable points exist, they are in fact dense:

1.1.21 Proposition. *The set X^s of all stable points and its image $\Phi(X^s)$ are open.*

Proof. Let Z be the set of points with positive dimensional stabilizer. Then, X^s is the complement of $\Phi^{-1}(\Phi(Z))$ in X. Indeed, if $\Phi(x) \in \Phi(Z)$, either $x \in Z$, and the stabilizer is not zero-dimensional, or $x \notin Z$, in which case its orbit is not closed. To see this, note that the orbit of x is not of minimal dimension in its S-equivalence class (there is a point with positive stabilizer mapping to the same point in the moduli,) and we proved that the unique closed orbit in each class is the one with minimal dimension. Therefore, $\Phi^{-1}(\Phi(Z)) \subset X - X^s$. Conversely, if a point is not stable, either its stabilizer is positive-dimensional, and so belongs to Z, or its orbit is not closed, and the closed orbit in its S-equivalence class is positive dimensional, whereas the moduli classifies these classes. Therefore, $X - X^s \subset \Phi^{-1}(\Phi(Z))$.

Now, we prove that Z is closed, and so that X^s is open by Lemma 1.1.9. Consider the map $a : G \times X \to X \times X$ defined by $(g,x) \mapsto (gx,x)$. Denoting by $\Delta \subset X \times X$ the diagonal, the subset Z is precisely the set where the fibers of the projection $a^*\Delta \to X$ are positive dimensional.

Finally, to prove the statement for $\Phi(X^s)$, we note that Z is closed and G invariant, so $\Phi(Z)$ is closed by Proposition 1.1.9. □

Because at the locus of stable points the action of G is so nice, the quotient map has important properties when restricted to X^s. One of them is that both notions of a quotient space above coincide in this case.

1.1.22 Proposition. *The quotient map $\Phi : X^s \to \Phi(X^s)$, obtained by restriction of Φ, is a geometric quotient.*

1.1.23 Lemma. *Suppose x and y do not belong to the same orbit, and one of them is stable. Then, they can be separated by an invariant function, that is, there's an invariant function attaining different values at x and at y.*

Proof. Suppose for definiteness, that x is stable. Then, since its orbit is closed, it is the orbit of minimal dimension in its S-equivalence class. Yet, since the stabilizer of x is zero-dimensional, it is also of maximal dimension. Therefore, $G \cdot x$ is the only orbit in its class, and so if $y \notin G \cdot x$, then $\overline{G \cdot y} \cap \overline{G \cdot x} = \emptyset$. By Mumford-Nagata, theorem 1.1.13, the orbits must be separated by invariants. □

Proof of Proposition 1.1.22. By the very definition of affine quotient, orbits are distinguished in the moduli if and only if they can be separated by invariants. But we saw that a stable orbit can be separated from any other orbit. □

The quotient of the locus of stable points is actually rather nice.

1.1.24 Theorem. *The restriction $\Phi : X^s \to \Phi(X^s)$ is a good geometric quotient.*

Proof. We only have left to prove that each natural map $\mathscr{O}_Y(U) \to \Phi_*\mathscr{O}_X(U)^G$ is an isomorphism. Let $U \subset \Phi(X^s)$ be an open subset; we may assume that it is defined by the non-vanishing locus of an invariant function $f \in k[X^s]$. But then, every regular function on U is of the form g/f for $g \in k[X^s]$; this is only is only invariant of g also is. □

1.1.25 Theorem. *If X^s is non-empty, then $k(X /\!\!/ G) = k(X)^G$.*

Proof. Since a variety is birationally equivalent to any of its non-empty open sets, we are done if we show the result for X^s, and so we may assume that X only contains stable points (since the restriction of the quotient map to X^s is the quotient map for X^s.)

Let f be an invariant rational function on X. It is constant on orbits, and orbits are in bijective correspondence to points in Y, so the expression $\Phi_* f(\Phi(x)) = f(x)$ is well defined. Thus, if we prove that $\Phi_* f$ is a rational function on Y, we have a map $\Phi_* : k(X)^G \to k(Y)$ that inverts the natural map $k(Y) \to k(X)^G$ as desired. But f can be seen as a regular function $X \to \mathbb{P}^1$, and the map $\Phi_* f$ is the uniquely induced map by the universal property of the quotient, showing that it is indeed a rational function. □

1.1.2 Projective quotients

The first part of this section should make clear what problems are involved in constructing quotient spaces in algebraic geometry. This part will determine a systematic method for the construction of reasonable quotients in the setting of projective varieties. All this is grounded in our construction of projective space as a birational quotient, and in fact the procedure, when suitably interpreted, greatly generalizes. As a first motivation for the search of such quotients, one can consider the following theorem on the existence of birational quotients.

1.1.26 Theorem (Rosenlicht)**.** *Assume X is irreducible. Then there is an open set U such that a good geometric quotient $U \to U/G$ exists with quasi-projective U/G. The field of rational functions on U/G is isomorphic to the subfield $k(X)^G$ of G-invariant rational functions on X.*

For a proof, the reader should check [13]. We saw how to accomplish this for the multiplicative action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1}$, and our strategy, following Mumford [31], will basically make the most out of this construction. We will see that using the affine quotient above, plus the properties of projective space, we can design a consistent method for constructing reasonable quotients of projective varieties.

Subvarieties of $\mathbb{P}^n$ We are now going to put ourselves in the following situation: let G be a linearly reductive group acting on $\mathbb{P}^n$ through a representation $\rho : G \to \mathrm{GL}(\mathbb{A}^{n+1})$, and X be an invariant subvariety, so that we have a restricted action $G \circlearrowleft X$. These are non-trivial conditions, but in fact we can always get such a situation with relatively minor changes. The point is that the quotient of affine varieties was constructed relying on the universal property of the ring of invariants, which is not helpful in other situations; but under such conditions, the action of G lifts to the affine cone $\tilde{X} \subset \mathbb{A}^{n+1}$ over X. The idea of the GIT quotient is then simple: take the affine quotient of this larger, affine variety, and then project back down to some projective space.

Suppose X is defined by a homogenous ideal $\mathfrak{a}$, i.e., $X = \text{Proj } k[x_0,...,x_n]/\mathfrak{a}$, then the cone is $\tilde{X} = \text{Spm } k[x_0,...,x_n]/\mathfrak{a}$. The quotient we know how to take is Spm $(k[x_0,...,x_n]/\mathfrak{a})^G$, which is a good categorical quotient of $\tilde{X}$. Its coordinate ring still has a grading the inclusion in $k[x_0,...,x_n]/\mathfrak{a}$, so that we can then take Proj $(k[x_0,...,x_n]/\mathfrak{a})^G$. Written invariantly, this is

$$X /\!\!/ G = \text{Proj} \bigoplus_{n\geq 0} H^0(X,\mathscr{O}(n))^G$$

That this is the same thing as what we just described follows from the results discussed above: that the graded ring $\bigoplus_{n\geq 0} H^0(X,\mathscr{O}(n))$ is isomorphic to the coordinate ring of $\tilde{X}$ with the grading by degree. Of course, we do *not* have a regular map $X \to X /\!\!/ G$, but rather a rational one. We shall see that just as in the case of projective space, this is actually rather convenient.

In fact, this construction has hidden benefits beyond extending the machinery to projective varieties. We have seen above that the major obstruction for an affine categorical quotient to be geometrically nice was the lack of 'enough' invariants in some sense, and that in fact the right number of invariants was better captured by rational functions, rather then regular ones. This construction has an implicit extension of invariants: the action of G on X factors through an action of some quotient of G by a central subgroup; the invariants of X alone are then the invariants of this quotient, which are included in, but do not exhaust the invariants of G on the cone. We will see below that if we embed an affine variety as a quasi-projective variety, then in fact the invariants of the cone is much larger than the invariants of X itself. This will allows us to construct many variations of the affine quotient, which can be interpreted as a systematic generalization of the construction of projective space.

To see this construction geometrically, recall that taking quotient of the cone by the action of G implies looking at the image of maps $\tilde{X} \to \mathbb{A}^{N+1}$ induced by a number of independent G-invariants $f_0,...,f_N$. If we try to track specific points, the procedure is then the following: we lift a point $x \in X$ to a point $\tilde{x} \in \tilde{X}$; we project it to the categorical quotient, and then to projective space. This is well defined since diferent lifts differ by the action of $\mathbb{G}_m$, and that's factored out when we take Proj , i.e., the point with *homogenous coordintes* $[f_0(x),...,f_n(x)]$ is well defined. But after projecting to $\mathbb{P}^N$, the image of a point in $\tilde{X}$ is only defined if it didn't map to zero in $\mathbb{A}^{N+1}$. In other words, we can only define a quotient map at a point $\tilde{x} \in \tilde{X}$ if there is at least one invariant $f \in R^G$ for which $f(\tilde{x}) \neq 0$. In practice, we considered the induced co-action of G on $\bigoplus H^0(X,\mathscr{O}(n))$. The quotient map $X \dashrightarrow \mathbb{P}^N$ is then defined by looking at maps $x \mapsto [s_0(x) : ... : s_N(x)]$ for independent sections $s_o,...,s_N \in \bigoplus H^0(X,\mathscr{O}(n))^G$. A point x has a lift where an invariant doesn't vanish if and only if there is some section $s \in H^0(X,\mathscr{O}(n))^G$ not vanishing at x, so the map is defined precisely at the points x for which there is an invariant section s such that $s(x) \neq 0$.

1.1.27 Definition. Let G be a linearly reductive group acting on $X \subset \mathbb{P}^n$ as above.

1. A point x is ρ-semistable if there is some invariant section $s \in H^0(X,\mathscr{O}(n))^G$ such that $s(x) \neq 0$; the set of all semistable points is denoted $X^{\rho-ss}$.

2. A point is ρ-polystable if it is semistable, and its orbit is closed in $X^{\rho-ss}$.

3. A point x is ρ-stable if it is simple (cf. Definition 1.1.20) and polystable; the set is denoted $X^{\rho-st}$.

4. A point is ρ-unstable if it is not semistable.

Keeping the representation ρ in the notation is a clumsy way of reminding us that this concept of semistability depends on the particular way that G acts on the whole cone; below we'll have a more general and systematic way of denoting this. As a first word of caution, when we extend this GT quotient to affine varieties, for which there is already a notion of stable points, we'll always have $X^{st} \subset X^{\rho-st}$, but almost never equality.

The following is a clear analogue of the results for the affine case, and should convince the reader that the GIT quotient is indeed a good definition.

1.1.28 Proposition. *Let G be a linearly reductive group acting on $X \subset \mathbb{P}^n$ as above.*

1. *The set $X^{\rho-ss}$ is open in X, and the natural map $\Phi : X^{\rho-ss} \to X /\!\!/ G$ is a good categorical quotient which parametrizes polystable orbits.*
2. *The set $X^{\rho-st}$ and its image $\Phi(X^{\rho-st})$ are open, and the restriction map $\Phi : X^{\rho-st} \to \Phi(X^{\rho-st})$ is a good geometric quotient.*
3. *If $X^{\rho-st}$ is non-empty, then $k(X /\!\!/ G) \simeq k(X)^G$.*

Parametrization of polystable orbits is important, but not as obvious as it might seem from our results in the affine case: when $\mathbb{G}_m$ acts on $\mathbb{P}^1$ by extending the multiplication of $\mathbb{G}_m$ on itself, without removing unstable points there are actually two closed orbits in the unique S-equivalence class!

Proof. We will derive everything from the nice properties of the affine categorical quotient together with those of projective space.

1. The set $\tilde{X}^{\rho-ss}$ of all points of $\tilde{X}$ lying over semistable points of X is the pre-image of an open subset of $X /\!\!/ G$, so it is open. The map Φ is the composition of two good quotients (the quotient map $\tilde{\Phi} : \tilde{X} \to X /\!\!/ G$ with the canonical projection $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$,) and is therefore a good quotient. Finally, the fact that it parametrizes closed orbits follows generally from the fact that it is a good quotient.
2. To prove openness, repeat Proposition 1.1.21, which only used properties of good quotients. The fact that the restriction is geometric follows from the fact that stable points have closed orbits.
3. This follows from the chain of isomorphisms

$$k(X /\!\!/ G) \simeq k(\tilde{X} /\!\!/ G)^{\mathbb{G}_m} \simeq (k(\tilde{X})^G)^{\mathbb{G}_m} \simeq (k(\tilde{X})^{\mathbb{G}_m})^G \simeq k(X)^G$$

   The first isomorphism follows from the fact that the canonical projection to projective space is a birational quotient, the second from the fact that the categorical quotient is birational when stable points exist, the third from the actions of G and $\mathbb{G}_m$ commuting, and finally the fourth again from $\mathbb{P}^n$ being a birational quotient.

□

We can extract more geometric information if we consider more carefully the relation between these notions with the behaviour of lifts to $\tilde{X}$. First, note that any non-constant homogeneous polynomial necessarily takes the value zero at the origin. Suppose we lift $x \in X$ to a point $\tilde{x} \in \tilde{X}$ over it, and that zero is a limit point of the orbit $G \cdot \tilde{x}$. Since invariant functions are constant on the closure of orbits, as we've seen, then we must have that every invariant function in R^G vanishes at $\tilde{x}$. But this means that every invariant section vanishes at x, and so this point is unstable. Therefore, if x is semistable, the orbit of a lift cannot have zero as a limit point. Analogously, if x is stable, the stabilizer of $\tilde{x}$ is zero-dimensional, since the same is true of the stabilizer of x; by the same reasoning, the orbit $G \cdot \tilde{x}$ must be closed (using that the stabilizer of a point is a closed subgroup.) It turns out that the converses of these statements are also true, which is the best outcome one could hope for.

1.1.29 Proposition. *A point $x \in X$ is semistable if and only if $0 \notin \overline{G \cdot \tilde{x}}$ for every lift $\tilde{x} \in \tilde{X}$ of x. It is stable if and only if $\tilde{x}$ has a zero-dimensional stabilizer, and a closed orbit in $\tilde{X}$.*

Proof. We only have half the theorem left to prove. Assume that the closure of the orbit does not contain zero. We saw in the last section (Lemma 1.1.13) that in the cone over X, there must be an invariant distinguishing the two disjoint closed invariant subsets. Since this invariant necessarily takes the value zero at the origin, it is nonzero at $G \cdot \tilde{x}$, and this is precisely the invariant section we need.

For the stable case, assume that $G \cdot \tilde{x}$ is closed, and that the stabilizer of $\tilde{x}$ is zero-dimensional. First of all, the stabilizer of x must be zero-dimensional: if we assume otherwise, there is an element g which fixes x but not $\tilde{x}$. Then, either $g^n \tilde{x}$ or $g^{-n} \tilde{x}$ tends to the origin. Now, assume that g_n is a sequence such that $g_n x$ tends to a point x_∞ outside $G \cdot x$. Since the orbit of $\tilde{x}$ is closed by assumption, if x_∞ were semistable, then there would be an invariant section separating x and x_∞. But this contradicts the continuity of the section, so the point x_∞ must be unstable, and the orbit $G \cdot x$ closed in the locus of semistable points. □

There is a parallel statement in terms of lifts to the line bundle $\mathscr{O}(-1)$ rather than to the cone $\tilde{X}$. We leave the proof to the reader.

1.1.30 Proposition. *A point $x \in X$ is semistable if and only if $\overline{G \cdot \tilde{x}}$ does not intersect the zero section for any lift $\tilde{x} \in \mathscr{O}(-1)$ of x. It is stable if and only if $\tilde{x}$ has a zero-dimensional stabilizer, and a closed orbit in $\mathscr{O}(-1)$.*

This interpretation of the (semi)stability of a point is a fundamental ingredient for the Hilbert-Mumford numerical criterion, which we shall mention later. For this reason, it is often called a *topological criterion for stability*.

General projective varieties The case of an action on a general projective variety doesn't really bring anything very much new to the table. Suppose now that X is a projective variety, and G is a linearly reductive group acting on X. Above, we were able to contruct a quotient by considering not only the action of G on X, but also on the cone over it. For a general projective variety, this notion does not make sense in absolute terms, but if we choose a polarization L, we can *a posteriori* indentify the the total space of L^{-n} for some n with the cone over the image of the embedding. Here, as in the beginning of the section, a polarization is an ample line bundle L: recall that maps to projective space correspond to line bundles over the variety. Our strategy will be prove that we can reduce the proofs to the previous case precusely using such a polarization. This will be done through a linearization of the action.

1.1.31 Definition. Let L be a line bundle over X. A *linearization of L for the action of G* is an isomorphism of line bundles $\sigma^*L \simeq p_X^*L$, where $p_X : G \times X \to X$ is the canonical projection.

To understand this definition, note that the projection of the line bundle $p : L \to X$ induces a map $\mathrm{id} \times p : G \times L \to G \times X$ which makes $p_X^*L = G \times L$ into a line bundle over $G \times X$. Then, there is a bijective correspondence between morphisms of line bundles $\bar{\sigma} : G \times L \to L$ such that the following diagram commutes

$$\begin{array}{ccc} G \times L & \xrightarrow{\bar{\sigma}} & L \\ \downarrow & & \downarrow \\ G \times X & \xrightarrow{\sigma} & X \end{array}$$

and morphisms $p_X^*L \to \sigma^*L$; the properties of σ then imply that the latter are always isomorphisms (through the usual argument that the elements of G are invertible.) In this way, we see that this definition is just a generalization of the setting for subvarieties of $\mathbb{P}^n$: the linearization of the action is essentially a linear action on the total space of L (except the basepoint might change,) and corresponds to the condition that G act on X through a representation $\rho : G \to \mathrm{GL}(\mathbb{A}^{n+1})$.

There is an obvious difference between a general projective variety and a subvariety of $\mathbb{P}^n$, namely that the absence of a distinguished polarization. This turns out to be a feature rather than a defect: different linearizations of the action will generally yield different quotients, and this variation shows up precisely in the flexibility of the embedding in projective space. It is important, therefore, to go through this case carefully.

1.1.32 Definition. There is the obvious question of when a given action is linearizable, and for which line bundles. Though later we'll consider what happens for different choices of a line bundle, we'll side-step the questions about the action completely, and always consider a polarized variety (X, L) with an already linearized action of G. The interested reader should consult [13], a very complete reference at this level of exposition; there is also, of course, the canonical reference, [31].

Given the analogy with the situation of a subvariety of $\mathbb{P}^n$, we make the following definion.

1.1.33 Remark. Let (X, L) be a polarized projective variety with a linearization $\bar{\sigma}$ of the action of G.

1. A point x is $(L, \bar{\sigma})$-semistable if there is some invariant section $s \in H^0(X, L^n)$ such that $s(x) \neq 0$; the set of all semistable points is denoted $X^{(L,\bar{\sigma})-ss}$.

2. A point is $(L, \bar{\sigma})$-polystable if it is semistable, and its orbit is closed in $X^{(L,\bar{\sigma})-ss}$.

3. A point x is $(L, \bar{\sigma})$-stable if it is polystable, and has a zero-dimensional stabilizer; the set of all these is denoted $X^{(L,\bar{\sigma})-st}$.

4. A point is $(L, \bar{\sigma})$-unstable if it is not semistable.

We will often ommit $\bar{\sigma}$ from the notation, it is fixed from the context. This seems like a naive procedure, but the next couple of results show that we are well justified in doing so.

1.1.34 Proposition. *Let (X,L) be a polarized variety together with a linearized action of G, and let $n>0$ be an integer such that L^n is very ample. Then, there is an action $G \to \mathrm{GL}_{r+1}$ on $\mathbb{P}^r$, and a choice of sections $s_0,\ldots,s_r \in H^0(X,L^n)$ such that the corresponding map $\phi : X \to \mathbb{P}^r$ is an equivariant embedding.*

Proof. We only sketch the proof. Assume, without loss of generality, that L is very ample. Then, the embedding induced by L into some projective space is given by $X \to \mathbb{P}(\Gamma(L))$. If the action of G has been linearized, then there is a naturally defined action of G on the space of sections of L, i.e., we have a representation $G \to \mathbb{GL}(\Gamma(L))$, and one can verify that this is the representation required by the theorem. □

1.1.35 Lemma. *The stability of a point is unchanged by taking a power of L, that is, if x is L-semistable, then it is also L^n-semistable for any $n>0$, and analogously for the other concepts.*

Proof. Note that we need only prove this for semistability, and the others follow. Suppose $s \in H^0(X,L^r)^G$ does not vanish at x; then, $s^n \in H^0(X,L^{rn})^G$ does not vanish either. □

The following theorem is a consequence of this lemma, replacing L by some power which is very ample. We should note that our geometric interpretation of the quotient also applies in this case, replacing throughout for the aproppriate line bundle. The reader can verifiy this as an exercise.

1.1.36 Theorem. *Let (X,L) be a polarized variety, with a linearized action of G. X^{L-ss} and X^{L-s} are open subsets of X. Let also*

$$X /\!\!/ G := \mathrm{Proj} \bigoplus_{n\geq 0} \Gamma(X,L^n)^G$$

This is a projective variety, and the natural map $\Phi : X^{L-ss} \to X /\!\!/ G$ is a quotient parametrizing polystable orbits. $\Phi(X^{L-st})$ is open, and the restriction $\Phi : X^{L-st} \to \Phi(X^{L-st})$ is a good geometric quotient. If X^{L-st} is non-empty, then $k(X /\!\!/ G) = k(X)^G$.

Strictly speaking, we end up not just with a projective variety, but with a polarized variety, the line bundle of which is defined as usual for a proj. This point becomes essential when we consider different linearizations of the action.

1.1.37 Remark. Our definition of the quotient for projective varieties in fact holds generally for any variety. In fact, the definition of linearization of the action of G makes sense regardless of whether L is ample or not, and the same goes for the definition of $X /\!\!/ G$ itself. What is particular to polarized projective varieties is the *proof* we gave. The proof for a general variety will involve imposing some affineness condition on semistability, taking the quotient for some affine cover, and then glueing back. It was the remarkable work of Mumford that showed that this general procedure indeed produces nice quotients. Since we shall have no use for these general results, we refer the interested reader to [13].

The Hilbert-Mumford numerical criterion Having defined a suitable quotient, the problem remains of finding the points of space, since the definitions are rather involved. The Hilbert-Mumford criterion is a fundamental tool that allows one to detect semistability without actually computing the semiinvariants of the action, which is usually rather hard. We will not prove this criterion here, but rather focus on using geometric insight to make the statement of the criterion plausible. In fact, recall that Proposition 1.1.30 characterizes (semi)stable points in a geometric manner. The Hilbert-Mumford Criterion essentially says that in detecting limit points of an orbit, it is enough to study one-dimensional paths.

1.1.38 Definition. A one-parameter subgroup is a morphism $\lambda : \mathbb{G}_m \to G$. If it extends to a map $\bar{\lambda} : \mathbb{A}^1 \to G$, then $\lim \lambda(t) = \bar{\lambda}(0)$.

if we fix a point $x \in X$, the action of the one-parameter subgroup just makes the point move on a one-dimensional path, it determines a regular map $\mathbb{G}_m \to X$ as $t \mapsto \lambda(t) \cdot x$. Because of the valuative criterion for properness, when X is projective, this path must have a limit $x_0 = \lim \lambda(t)\tilde{x}$ as $t \to 0$, and this point is a fixed point of the action of the one-parameter subgroup. To study the behaviour of a point x under a one-parameter subgroup λ, we can try to look at the action of that subgroup on the whole variety, and extract information from the lines over the limit points. Indeed, on these lines, the induced action of $\mathbb{G}_m$ on the line over x_0 is simply multiplication by a scalar $t^{\rho(x,\lambda)}$. Let $\tilde{x}$ be a lift of x; the Hilbert-Mumford Theorem states the following:

1.1.39 Theorem (Hilbert-Mumford)**.** *Let $X \in \mathbb{P}^n$ be a subvariety of projective space. Then, for a point $x \in X$,*

1. *x is semistable if and only if $\rho(x,\lambda) \leq 0$ for all one-parameter subgroups;*
2. *x is stable if and only if $\rho(x,\lambda) < 0$ for all one-parameter subgroups.*

Let's interpret this statement geometrically: first note that if $\rho(x,\lambda)$ is non-zero, then x_0 is unstable (either the given one-parameter subgroup or its inverse have limit in the zero section.) Then, if x is semistable, the orbit of $\tilde{x}$ must shoot up to infinity as it aproaches x_0, or else it's dragged down with it to the zero section. Therefore, $\rho(x,\lambda)$ must be less than zero. On the other hand, if $\rho(x,\lambda) = 0$, then x_0 itself is semistable, and the orbit of x has it as a limit point. So two things can happen: either $x_0 \in Gx$, and so by multiplying by an apropriate g, we can transform this into a situation where the one parameter subgroup is in the stabilizer; or, $x_0 \notin Gx$, and the orbit of x is not closed, and x is not polystable. Finally, note that the action on the line over x is trivial, then the image of the one-parameter subgroup lies in its stabilizer. Therefore, x has stabilizer of dimension at least one, so it cannot be stable.

1.1.3 Affine quotients, again

Before we proceed to study the quotients of abstract varieties, we'll apply the results above to the affine case. This is an essential exercise if we want to concretely understand the GIT quotient, and it is quite sufficient for a lot of applications: in [30], the moduli space of bundles over curves is constructed using only affine methods. It is also a particularly simple one, since, as we shall see, everything can be written quite explicitly. Despite that, it is only fully understood in the context of the theory above. The constructions here correspond to considering affine varieties as quasi-projective. Strictly speaking, it doesn't satisfy the conditions of our theorems, but these still hold; the proofs can be found in [30].

For an affine variety, $\mathscr{O}(-1)$ is always trivial; indeed, $X = \operatorname{Spm} R = \operatorname{Proj} R[z]$. The co-action of G splits into a co-action on R and one in z, that is, an action on the base space and one on the fibres of the line bundle, as $f \otimes z \mapsto (g \cdot f) \otimes \chi(g)^{-1} z$, where $\chi : G \to \mathbb{G}_m$ is a character of G (this is because $\mathscr{O}(X)^* = k^*$.) If $f \otimes z^n$ is invariant, then we have

$$f \otimes z^n = g \cdot (f \otimes z^n) = (\chi(g)^{-n} g \cdot f) \otimes z^n$$

In other words, $f \otimes z^n$ is invariant if and only if $g \cdot f = \chi(g)^n f$.

1.1.40 Definition. A function $f \in R$ is a semi-invariant of weight n with respect to a character χ if $g \cdot f = \chi(g)^n f$; we denote their ring as $R_{\chi,n}$.

The ring of G-invariants of the *line bundle* is precisely the ring of all semi-invariants *of* X of non-negative weight. (The semi-invariants of non-positive weight correspond to the invariants of the inverse linearization of the action.) Therefore, the GIT quotient is

$$X /\!\!/_{\chi} G = \operatorname{Proj} \bigoplus_{n \geq 0} R_{\chi,n}$$

Since for the trivial character $\chi(G) = 1$ we get back the affine categorical quotient the GIT quotient can be considered its generalization. In fact, the invariants of the action on X are always realized as the semi-invariants of weight zero, i.e., the degree zero part of the graded ring. By the properties of the proj construction, there is then always a canonical morphism

$$X /\!\!/_{\chi} G \to X /\!\!/ G$$

It turns out that this map can be completely characterized for any χ, a topic to which we'll come back below.

Let $F = \ker \chi$ be the subgroup of G that maps to one by χ; an action of F is naturally defined by considering the restriction of the action of G. An important observation is that the semi-invariants of G are the invariants of F, that is,

$$R^F = \bigoplus_{n \geq 0} R_{\chi,n}$$

One inclusion is obvious: if $f \in R_{\chi,n}$, and $a \in F$, then

$$f \otimes z^n = a \cdot (f \otimes z^n) = a \cdot f \otimes \chi(a)^{-n} z^n = a \cdot f \otimes z^n$$

On the other hand, the action of G on the ring of invariants of F factors through an action of $\operatorname{coker} \ker F = \mathbb{G}_m$, and this must have the form of a power function.

Semistability here just means that at a point x is χ-semistable if there is a semi-invariant that does not vanish at x. The reader is invited to consult [30] for direct proofs of the results above for this case.

1.1.41 Example. Consider the action of $\mathbb{G}_m$ on $\mathbb{A}^2$ as $(x,y) \mapsto (tx, t^{-1}y)$. The trivial line bundle is isomorphic to $\mathbb{A}^3$, and the induced action is given by $(x,y,z) \mapsto (tx, t^{-1}y, \chi(t)z)$. Now, the characters of the multiplicative group are just the power functions $\chi_i(t) = t^i$, but it's enough to consider the cases $i = -1, 0, 1$. To see why, note that $R_{\chi_i,n} = R_{\chi_1,ni}$, and that $\operatorname{Proj} \bigoplus_{n \geq 0} R_{ni} = \operatorname{Proj} \bigoplus_{n \geq 0} R_n$ for any graded ring R.

In all three cases, the hyperbolae are χ-stable, since they are stable in the general sense. The question, then, is reduced to the study of the orbits corresponding to the axes and the origin. The origin is a fixed point, so it is easily seen to be unstable as soon as the action on z is non-trivial: the z-axis will then be an orbit, which accumulates at the origin of $\mathbb{A}^3$. On the other hand, the stability of the axes will be determined by the behaviour of tz when we approach the origin: if it tends to zero, that axis is unstable; otherwise, it is polystable (because the orbit becomes closed when the origin is removed.) Therefore, we find:

- for $i = 0$, the action on z is trivial, so the origin is polystable and the axes are strictly semistable. The quotient map then collapses all three orbits to the same point, and we get the affine line. This is consistent with our general observation that the trivial character always yields the affine quotient.

- for $i = 1$, the action is non-trivial on z, so the origin is automatically thrown out. Now, if we consider a point $(0,y,z)$ lifting a point in the y-axis, we approach the origin by letting $t \to \infty$; but the tz has no limit, and so the y-axis is polystable. For a point $(x,0,z)$, we must consider the limit as $t \to 0$, and in this case, $tz \to 0$; therefore, the x-axis is unstable, and is therefore thrown out. We find then the quotient space to be the affine line.

- for $i = -1$, the situation is precisely the opposite of the previous one.

Note that following Proposition 1.1.34, we can realize these three quotients as the result of three different equivariant embedding into projective space. For example, for $i = 1$, we consider the action of $\mathbb{G}_m \to \mathrm{GL}_3$ on projective space defined by $t \mapsto \mathrm{diag}(t^2, 1, t)$. On the affine piece $z = 1$, we have

$$t \cdot [x,y,1] = [t^2x, y, t] = \frac{1}{t}[tx, t^{-1}y, 1] = [tx, t^{-1}y, 1]$$

The birational morphism $B : \mathbb{A}^3 - \{z = 0\} \dashrightarrow \mathbb{A}^3$ from the cone over this affine piece to its blow-up at the origin is defined by $B(x,y,z) = (z^{-1}x, z^{-1}y, z)$, so the pullback of this action is

$$t \cdot (a,b,c) = t \cdot B(x,y,z) = B(t^2x, y, tz) = (tzx, t^{-1}zy, tz) = (ta, t^{-1}b, tc)$$

which is precisely the linearization above.

In the preceding example, one must only be careful that though the reparametrization of the graded ring doesn't change the base variety of the quotient, it does change the line bundle over it. This means our considerations were enough to compute the base variety, but not the polarized variety.

A clearer example of this is the construction of projective space itself. Let $\mathbb{G}_m$ act on $\mathbb{A}^{n+1}$ by multiplication. We embed $\mathbb{A}^{n+1} \to \mathbb{P}^{n+1}$ as the hyperplane $x_{n+1} = 0$, and let $\mathbb{G}_m$ act by $t \mapsto \mathrm{diag}(t, \ldots, t, t^{i+1})$. Then, it's clear that a semi-invariant of weight one corresponds precisely to a polynomial on $x_0, \ldots, x_n$ of degree i. On the other hand, the line bundle associated with the quotient Proj R corresponds to the module $R[1]$, which is the shift by one of the graded ring. Therefore, since the semi-invariants of weight one are precisely the first graded piece of R, the polarized quotient is $(\mathbb{P}^n, \mathscr{O}(i))$.

1.2 A note on reductivity, complex analytic and algebraic

As we move on from an algebraic to a complex analytic context, as we'll do in the next section, a word is due about the nature of groups. When dealing with complex projective varieties as such, the groups we've dealt with were complex algebraic groups; when thinking of those varieties as holomorphic varieties, the natural choice are complex Lie (or complex analytic) groups. This in itself does not present difficulty, since every complex algebraic group has a structure of a complex analytic group. However, for the theory of quotients we needed to restric to *linerly reductive* groups; we'll clarify the precise sense in which this is extended and corresponds to complex analytic groups. The exposition here is necessarily very brief, sticking to the main points and skipping proofs, but we hope that it at least can serve as a guide to the vast literature on the subject. (Most definitely, a detailed understanding of the material here is not necessary for what follows.)

1.2.1 Complex Lie groups

Recal that a *Lie group* is a group object in the category of smooth manifolds. A non-trivial result is that each Lie group admits a unique structure of real analytic manifold, and so it is also a group object in the category of real analytic manifolds in a unique way; therefore, by a Lie group we will mean real analytic group. Analogously, a *complex Lie group* is a group object in the category of complex analytic manifolds.

Since every complex analytic structure on a topological space determines a smooth structure, there is a forgetful functor

$$\Phi : \mathbf{CLie} \to \mathbf{Lie}$$

It turns out that this functor admits a left adjoint, as follows from the following theorem

1.2.1 Theorem. *Given a Lie group G, there is a complex analytic group G^+ together with a morphism $g^+ : G \to G^+$ that is universal for the forgetful functor, that is, given a morphism $f : G \to H$ where H is complex, there is a unique* complex analytic *morphism $\bar{f} : G^+ \to H$ such that $f = \bar{f}g^+$.*

The universal morphism in the theorem is the *universal complexification of* G. For various reasons, this universal complexification does not comply with our intuition of what a complexification should be: to start with, it might not even be injective. Further, even if it is a real analytic embedding, it might not be half-dimensional as expected. We do have the following proposition, which follows from the construction of the universal complexification.

1.2.2 Theorem. *Suppose a real group G admits a faithful complex representation. Then, the complexification map is an immersion, and $\mathfrak{g}^+ = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.*

The following definitions make precise what we are looking for.

1.2.3 Definition. Let G be a real group. A *complexification* of G is a complex Lie group $G^{\mathbb{C}}$ together with an embedding $G \to G^{\mathbb{C}}$ such that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We say that a complexification is *global* if there is a anti-holomorphic involution θ on $G^{\mathbb{C}}$ such that $G = G^{\theta}$ is the subgroup of fixed points.

Note that as opposed to the universal complexification, a (global) complexification is not in general unique. Global complexification in particular are rather fickle objects, and depend greatly on whether we take the analytic or algebraic category, as example 1.2.5 below shows.

1.2.4 Example. Consider the multiplicative group of units $\mathbb{R}^*$ in $\mathbb{R}$. It is easy to see that the natural inclusion $\mathbb{R}^* \to \mathbb{C}^*$ is both the universal complexification, and a global complexification (the anti-holomorphic involution being conjugation.)

1.2.5 Example. If we denote by $\mathbb{R}_0^*$ the identity component of the previous example (in other words, the multiplicative group of positive real numbers,) then the inclusion morphism $\mathbb{R}_0^* \to \mathbb{C}^*$ is a complexification, but not a global one. We can also in fact prove that it is not the universal complexification: consider the logarithm map $\log : \mathbb{R}_0^* \to \mathbb{R} \subset \mathbb{C}$. A factorization of this map would imply that a global inverse of the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ existed, which is not true. On the other hand, the logarithm map on $\mathbb{R}_0^*$ is actually a diffeomorphism with $\mathbb{R}$, which implies that they have the same universal complexification, namely $\mathbb{C}$. Some care here is needed: one might be at this point be tempted to think of the inclusion $\mathbb{R}_0^* \to \mathbb{C}$ as the complexification, but in fact this is neither an embedding, nor even a group homomorphism! The universal complexification map is precisely the logarithm and it is also a global compexification.

1.2.6 Example. The *metaplectic group* Mp_{2n} is the double cover of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$. It is a non-trivial fact that because this group is semisimple but not linear (not a closed subgroup of $\mathrm{GL}(V)$ for some vector space V,) it does not admit a complexification, though the universal complexification exists.

Our main interest will lie in the particular case of compact real groups. These groups have a rather tame behaviour, as well as nice topological properties. One of their important properties is that they admit faithful (real) representations, so that Theorem 1.2.2 applies directly, and since an immersion of a compact space is automatically an embedding, we see that the universal complexifications of compact groups are complexifications. In fact, they are even global complexifications with very simple topology, as follows from the following theorem.

1.2.7 Theorem (Cartan Decomposition)**.** $G = K \exp i\mathfrak{k}$

It is important, however, to not simplify too much our picture of compact groups in general, as we can see in the following example.

1.2.8 Example. Let Σ be a connected and compact topological surface of genus 1; this is diffeomorphic to $S^1 \times S^1$, which endows it with a group structure. The universal complexification is $(\mathbb{C}^*)^2$, which can be seen to be a global complexification. This is an interesting examples since we know that Σ can be endowed with different complex analytic structures which make it into an elliptic curve. These are even projective algebraic groups, and yet the universal complexification is none of them.

1.2.2 Reductivity

We now come to the question of generalizing the reductivity condition. The most straightforward thing to do is to simply translate the definition for algebraic groups into one for complex Lie groups.

1.2.9 Definition. A complex Lie group G is said to be *reductive* if any of the following conditions hold:

1. *Linear reductivity:* any complex analytic representation $G \to \mathrm{GL}(V)$ is completely reducible, i.e., any invariant subspace has an invariant complement.

2. *Geometric reductivity:* for any complex analytic representation $G \to \mathrm{GL}(V)$, given a fixed point $v \in V$, there is a homogenous G-invariant polynomial F such that $F(v) \neq 0$.

3. The unipotent radical of G is trivial. Recall that the radical of G is its maximal normal and solvable subgroup; the unipotent radical is then the subgroup of unipotent elements in the radical of G.

We could have given the same definition for complex algebraic groups. As we have remarked above however (cf. Remark 1.1.11,) in the case of algebraic groups defined over fields of positive characteristic these three conditions are not equivalent.

We have actually already constructed an important class of complex reductive groups, namely the universal complexifications of compact groups. Every representation of a compact group admits an invariant metric, so that the representation is actually unitary; using the metric we can then choose complementary subspaces. In other words, we the following theorem.

1.2.10 Theorem. *The universal complexification determines a functor*

$$\Psi : \mathbf{CptLie} \to \mathbf{RedLie}$$

We have seen above that the universal complexifications of compact groups have a quite explicit description; it turns out that this is enough to characterize the essential image of this functor. We say that a compact real analytic subgroup $K \subset G$ is maximal if given a another compact subgroup $K' \subset G$ with $K \subset K'$, then $K = K'$. We then have

1.2.11 Theorem. *A reductive group G is in the essential image of Ψ if and only if it is a complexification of any of its maximal compact subgroups.*

In the following we will restrict our attention to reductive groups in the essential image of Ψ. The fundamental reason is that the actions of these groups are topologically tame enough for us to grasp their behaviour. A second motivation is the following remark.

1.2.12 Remark. Our discussion above makes clear that there is another distinguished class of complex reductive groups, namely the compact ones. These would even have the extra benefit of being natural candidates for applying Serre's GAGA to induce pojective algebraic structures. Their drawback, however, is that any compact complex group is abelian: they are all abelian varieties, i.e., quotients of complex vector spaces by integer lattices, and so the natural generalizations of elliptic curves we discussed above. Their theory is rather rich, but they fail to encompass any of the most common examples.

Surprisingly, the restriction to the essential image of Φ allows us to 'have our cake and eat it too,' as it were. The following theorem shows both that the restriction is mild enough to encompass pretty much all the important examples, and strong enough to still provide algebraic structure.

1.2.13 Theorem. *A reductive complex Lie group is in the essetial image of* Φ *if and only if it admits a complex analytic faithful representation. Further, a connected such group admits an affine algebraic structure, and such structure is unique.*

1.2.14 Corollary. *The category of connected faithfully representable reductive Lie groups and the category of connected affine reductive groups are isomorphic.*

The restriction to affine groups might seem inordinately strong. To put one's mind at ease, one can note a result of Chevalley [11] to the effect that every algebraic group is the extension of an abelian variety (cf. Remark 1.2.12) by an affine group.

1.2.15 Remark. For compact groups, there is an explicit construction of the universal complexification (the so-called *Chevalley complexification*) which makes clear that the complexification of a compact group is always a linear algebraic group, i.e. a closed subgroup of some $\mathrm{GL}(V)$.

1.3 Symplectic quotients

In this section we will be interested in defining a suitable quotient for symplectic manifolds, which are real objects, rather than complex. In the real context, it is compact groups that play the role that linear reductive groups played in the algebraic context. We will then start by discussing briefly the action of compact groups on differential manifolds, specialize to symplectic manifolds, and then prove the Marsden-Weinstein Reduction Theorem, to the effect that under certain conditions, a suitable symplectic quotient can be constructed by inducing a symplectic form on the set-theoretic quotient of a submanifold (the zero set of a moment map) by the action of the group.

It turns out that the symplectic case is deeply intertwined with the algebraic quotients we defined in the first section. We have seen in the previous section that the category of complex reductive Lie groups is equivalent to the category of compact real Lie groups, and so we can in fact complexify the situation just describe to get an action of a complex group on a complex manifold. This will be closely related to the compact case, and we can define a quotient at its expense. Then, theorem of Kempf and Ness, the centerpiece of this section, in fact states that if the complex manifold is projective, then the symplectic quotient so defined is in fact homeomorphic to the algebraic quotient, when considered as a complex analytic variety. This celebrated theorem is the cornerstone for a large interaction between algebraic and symplectic geometry, and will be used later for the study of the topology of the quotient.

To finish this section, we will describe a further interaction between both fields, the case of hyperkähler quotients. Hyperkähler manifolds are equipped with three different symplectic forms, two of which can be used to define a complex symplectic form. We can then find a hyperkähler quotient by first applying a complex symplectic reduction, defined as the categorical quotient of a certain subvariety, and then use the Kempf-Ness theorem to construct that algebraic quotient using symplectic techniques.

1.3.1 Symplectic Reduction

Actions of compact groups on manifolds Let M be a differential manifold, and K be a (real) Lie group. We can define an action of K on a manifold just as we did in the algebraic context: an action of

K on M is a smooth map $K \times M \to M$ satisfying $\sigma(\varepsilon \times \mathrm{id}_M) = \mathrm{id}_M$, and $\sigma(\mathrm{id}_G \times \sigma) = \sigma(\mu \times \mathrm{id}_M)$, where μ is the multiplication map of K, and ε the identity. In this context, the action of K determines a so-called *infinitesimal action* as follows: the exponential $\exp : \mathfrak{k} \to K$ defines a map $\sigma(\exp \times \mathrm{id}_M) : \mathfrak{k} \times M \to M$, the differential of which along $\mathfrak{k}$ is a map $d\sigma : \mathfrak{k} \times M \to TM$. This can be interpreted as a map $\mathfrak{k} \to \mathscr{A}^0(TM)$, assigning to each element $\beta \in \mathfrak{k}$ a vector field $\beta^\#$ on M; this map is in fact a morphism of Lie algebras (since it is defined as a differential.)

The problem of constructing a quotient is again an important, but non-trivial one. To start with, for an arbitrary Lie group, the topological quotient of M could perfectly well not even be Hausdorff, since the orbits are not necessarily closed. To see this, given a point $x \in M$, define the map $\alpha_x : \to M$ by $g \mapsto g \cdot x$. This map is smooth, since it is the restriction of the action map, and its image is precisely the orbit $K \cdot x$; in fact, it determines a bijection from G/G_x (where G_x to the stabilizer of x) to the orbit of x. We have the following:

1.3.1 Lemma. The map $\alpha_x : G/G_x \to M$ is an injective immersion.

Proof. It is enough to compute the differential of α_x, since α_x is K-equivariant. But by definition, $(d\alpha_x)_1 : \mathfrak{k} \to T_xM$ just assigns $\beta \mapsto \beta_x^\#$. In other words, $(d\alpha_x)_1(\beta) = 0$ if and only of $\beta_x^\#$, which happens if and only if β is in the Lie algebra $\mathfrak{k}_x$ of K_x. □

The problem is that injective immersions are not nearly strong enough to ensure a nice quotient. In fact, even if it happened that the image was a closed subspace of M (which is not guaranteed,) such an subspace could easily not be a submanifold. Restricting to compact groups, however, solves this issue, since an injective immersion of a compact space *is* an embedding.

1.3.2 Proposition. If K is a compact Lie group action on M, then the topological space M/K is Hausdorff, and the canonical projection $\Phi : M \to M/K$ is proper and closed.

Proof. Since each orbit is the image of a map $K \to M$, they are all compact. Since M is Hausdorff, every two such orbits can then be separated by open sets, and M/K is also Hausdorff. Given this, to prove properness of Φ, it is enough to prove that it is a closed and that the pre-image of each point is compact. The latter is obvious by the above. We prove that Φ is closed. Indeed, since M is regular, we can separate any disjoint pair of a closed set and a compact one. In particular, given any closed subset C and an orbit $K \cdot p$ not intersecting C, they are separable, and are so by K invariant neighbourhoods. But then the projection of these two neighbourhoods separates $\Phi(p)$ from $\Phi(C)$, and so $\Phi(p)$ cannot be a limit point of $\Phi(C)$. □

Since second countability is automatic, we only need to endow M/K with a differential structure, so that we can make sense of the quotient. The essential technical tool is a result known as *slice lemma*. A *slice at p* is a locally closed submanifold of M which contains p, and which is transversal to $K \cdot p$. We can always get such a slice by decomposing the tangent space at p. Indeed, the image of the map $(d\sigma)_x : \mathfrak{k} \to T_xM$ is the tangent space to the orbit, so that we can by choosing a normal space N_x to that image, we can locally integrate it so that we get a slice S such that $T_xS = N$.

1.3.3 Lemma (Slice Lemma)**.** *Suppose K acts freely on M at $p \in M$, and for $\varepsilon > 0$, let $S_\varepsilon = S \cap B_\varepsilon(x)$ be the intersection of a slice S at x with a ball of radius ε. Then, for sufficiently small ε, the restriction of the action $K \times S_\varepsilon \to M$ maps $K \times S_\varepsilon$ diffeomorphically onto a K-invariant neighbourhood of $K \cdot p$.*

Proof. Since the action is free, we know from Lemma 1.3.1 that the orbit map $K \times p \to M$ for any x is an embedding. If S is a slice at x, it follows that the differential of the restricted action $\sigma : K \times S \to M$ is bijective at every point. Since $K \times \{x\}$ is compact, the implicit function theorem asserts the existence of a neighbourhood U of x such that the restiction to $\sigma : K \times U \to \sigma(K \times U)$ is a diffeomorphism. Take then $B_\varepsilon(x) \subset U$. □

One should note that slice theorem is non-trivial, and it is the key lemma in the context of any quotient related to manifolds. Its importance far exceeds its use in endowing a differential structure. Typically, we deal with manifolds with extra structure which is also of local nature (in fact, generally tangential,) and to find a quotient of the same type we then only need to define the structure on the slice (which is generally much easier.) We'll do this below for symplectic structures. Furthermore, in infinite dimensions it is precisely the existence of a slice that is highly non-trivial, and cannot be asserted in general; as long as its existence is assumed, then it is relatively straightforward (with the appropriate functional analysis) to prove that the quotient exists.

We should note also that the slice lemma can be generalized to the situation when K does not act freely. When that is the case,one should note that the stabilizer G_x has a residual action on the normal space N_x, and so on S_ε for sufficiently small ε. Then, picking a smaller ε if necessary, there is a homeomorphism from $K \times_{K_x} S_\varepsilon$ onto a K-invariant neighbourhood of $K \cdot x$. Sometimes this is useful in understanding the singularities of the quotient.

Finally, with the slice lemma at hand, we can prove the following.

1.3.4 Theorem. *Suppose a compact group K acts freely on a manifold M. Then, the quotient M/G exists as a differential manifold, and the canonical projection $\Phi : M \to M/K$ is a principal K-bundle.*

Proof. The slice theorem tells us that given a point $x \in M$, there is a neighbourhood U and a submanifold $S_\varepsilon \subset U$ such that S_ε maps homeomorphically onto a neighbourhood of $\bar{x} \in M/K$. We use such homeomorphisms to define an atlas on M/K.

We now need to prove that $\Phi : M \to M/K$ is a principal K-bundle. Since the stabilizers are trivial, the fibres have a transitive and free action of K. On the other hand, the slice theorem gives us precisely a local trivialization, in this case. □

Note that the condition of K acting freely can be slightly relaxed as follows: the intersection $K_X := \bigcap K_x$ of the stabilizers of every point is a closed, normal subgroup of K. Then, if $K_x = K_X$ for all x, we can apply the previous theorem to K/K_X to conclude that a quotient exists.

An important extension is the case when K acts with finite stabilizers, i.e. if $K'_x := K_x/K_X$ is finite for every x. We have mentioned a generalization of the slice lemma for the case when the action is not free, and using it we can realize M/K locally around $\Phi(x)$ as a quotient of a vector space by a finite group. Such a space is called an *orbifold*, and despite being singular, its singularities are fairly mild. Indeed, even finding an orbifold quotient is rather uncommon.

The Marsden-Weinstein theorem Suppose now that (M, ω) is a symplectic manifold, and that K acts by symplectomorphisms, i.e., the action of a given $k \in K$ preserves the symplectic form.

1.3.5 Definition. A moment map is a smooth map $\mu : M \to \mathfrak{k}^*$ that is equivariant with respect to the co-adjoint action of K, and that for every $k \in \mathfrak{k}$ satisfies

$$\langle d\mu, k\rangle = \iota(k^\#)\omega$$

where $\langle \cdot, \cdot \rangle$ is the canonical contraction $\mathfrak{k}^* \times \mathfrak{k} \to \mathbb{R}$.

Just as we did not discuss the existence and uniqueness of linearizations, we will also forgo a discussion of such topics for moment maps. The name moment map comes from the following two examples.

1.3.6 Example. Let ω be the canonical symplectic form on $M = T^*\mathbb{R}^3$, and let G be the group of translations of $\mathbb{R}^3$ (which is not compact, but that is irrelevant at this point.) G acts on M in a natural way, and the moment map for this action is precisely the linear momentum. Indeed, given an element $v \in \mathbb{R}^3$, its one-parameter group is $x + tv$, and so

$$\iota(v^\#)\omega = \sum \frac{d}{dt}(x_i + tv_i)dp_i = d(v \cdot p)$$

In other words, the moment map is inner product with the linear momentum.

With this example, the idea for reduction becomes very intuitive. In elementary physics, under the absence of external forces, one typically assumes that the linear momentum is zero.

1.3.7 Definition. Suppose a moment map μ for the action of K on M exists. Then, the *reduced phase space* is the quotient $\mu^{-1}(0)/K$.

Strictly speaking, the quotient we just defined is only defined topologically, and so it might very well lie outside the 'symplectic category.' There are two obstruction to this: first, the pre-image $\mu^{-1}(0)$ might not be a manifold; second, even if it is, we still have to proof that there is an induced symplectic structure on the quotient. Of course, since the group is compact, we know that the quotient is not only a manifold, but also a principal bundle when the group acts freely. To prove that there is a canonical symplectic form, the essential ingredient for the proof is again the slice theorem.

1.3.8 Proposition. *Suppose a compact group K acts freely on a symplectic manifold M. Then, M/K carries a unique symplectic form ω_r such that*

$$\Phi^*\omega_r = \omega \tag{1.1}$$

where $\Phi : M \to M/K$ is the canonical projection.

Proof. We need only to define a symplectic structure. To do this, we note that $d\pi$ is a surjection at every point, so it is enough to set for $u, v \in T_xM$

$$\omega_r(\Phi_*u, \Phi_*v) = \omega(u, v)$$

This is well defined because the kernel of π is precisely the tangent space to the orbit of x, and ω is K-invariant. Also, by its very definition, it satisfies $\Phi^*\omega_r = \omega$, and since Φ is a submersion, it is unique. To see that is is smooth and closed, we use that the restriction of $\Phi^*\omega_r$ to a slice as above is just the restriction to ω.

We have yet to prove that ω_r is non-degenerate. But indeed, the tangent space to the slice is obtained by factoring out the tangent space to the orbits. Since ω is K invariant, the latter are isotropic, so the restriction of ω defines a symplectic form. □

We now come back to the question of whether $\mu^{-1}(0)$ is a manifold.

1.3.9 Definition. A point x is *infinitesimally simple* if $\mathfrak{g}_x = 0$.

From our computation we conclude the following.

1.3.10 Lemma. *A point x is a regular point of the moment map μ if and only if it is infinitesimally simple.*

Proof. Recall that x is a regular point if the differential is surjective, so we must find its image. By Definition 1.3.5, the differential $(d\mu)_x$ is the transpose of the map $\beta \mapsto (\iota(\beta^\#)\omega)_x$, and $\mathrm{im}(d\mu)_x \simeq (\mathfrak{k}/\ker(d\mu)_x^t)^*$. Since ω is non-degenerate, the latter kernel is the set of all β such that $\beta^\# = 0$; but the set of all such is precisely $\mathfrak{k}_x$. We have proved then that x is a regular point if and only if $\mathfrak{k}/\mathfrak{k}_x = \mathfrak{k}$; this happens if and only if $\mathfrak{k}_x = 0$. □

Putting the above results together, we get the following.

1.3.11 Theorem (Marsden-Weinstein)**.** *Let K be compact, and suppose a moment map $\mu : M \to \mathfrak{k}^*$ exists for the action of K on M. If K acts freely on $\mu^{-1}(\xi)$ for some central element $\xi \in \mathfrak{k}$, then the reduced phase space $\mu^{-1}(\xi)/K$ is a symplectic manifold.*

Note that we did *not* require ξ to be a regular value. Indeed, by our computation of the differential of μ, this follows by the requirement that K act freely on the level set. That $\mu^{-1}(0)$ is a submanifold of dimension $n = \dim M - \dim G$ then follows by the inverse function theorem. With this, the theorem follows by Theorem 1.3.8.

To finish, we note that the definition of reduced space can be given for any action of a group on a manifold. What is special about compact groups is that as long as 0 is a regular value, the reduction always defines a symplectic manifold. For other groups, our computation of the differential of the moment map only ensures that if 0 is a regular value, then the stabilizers are discreet, so that in general even if $\mu^{-1}(0)$ is smooth, the reduced space will be an orbifold or worse (cf. the discussion Theorem 1.3.4.)

1.3.2 The Kempf-Ness Theorem

Complex group actions on Kähler manifolds We will now turn to quotients in complex geometry, a meeting ground between the algebraic and symplectic theory. The discussion will start with so-called *Kähler manifolds*, a special class of complex manifolds quite closely to the algebraic category in their properties, even if they might lack a proper algebraic structure. Recall that a complex manifold X is Kähler if the has a hermitian form $h \in \mathscr{A}^0((TX \otimes \overline{TX})^*)$ with antisymmetric part a closed $(1,1)$ form. In other words, if $i\omega/2$ is the antisymmetrization of h, then $\omega \in \mathscr{A}^0(\Lambda^2 TX)$ – the *Kähler form* of X– is a symplectic form on X. A discussion of such manifolds can be found in [16].

Let then X be a Kähler manifold, and G be a faithfully representable complex reductive Lie group acting on X. Assume that the action of the maximal compact K of G not only preserves the symplectic form, but that a moment map $\mu : X \to \mathfrak{k}^*$ exists. The results of the previous section allow us then to define the reduced phase space $\mu^{-1}(0)/K$; the purpose of this section is to explain how this reduced space is related to the action of G on X.

We start with a lemma, which we note does not make any assumption on stabilizers:

1.3.12 Lemma. *If $x \in \mu^{-1}(0)$, then $G \cdot x \cap \mu^{-1}(0) = K \cdot x$.*

Proof. By Lemma 1.2.7, we need to prove that given $x \in \mu^{-1}(0)$ and $\beta \in \mathfrak{k}$, if $\exp(i\beta)x \in \mu^{-1}(0)$, then $\exp(i\beta)x = x$. Define $h(t) = \langle \mu(\exp(it\beta)x), \beta\rangle$. This function vanishes at 0 and 1, so, by the mean value theorem, there must be a $t_0 \in [0,1]$ such that

$$0 = h'(t) = \langle d\mu_y(i\beta), \beta\rangle = \omega_y(i\beta_y, \beta_y)$$

where $y = \exp(it_0\beta)x$. This means that $\exp(it\beta)y = y$, so that in particular, $\exp(i\beta)x = x$. □

This lemma has the following immediate consequence:

1.3.13 Theorem. *Suppose M is compact Kähler. Then, the natural map $\mu^{-1}(0)/K \to G\mu^{-1}(0)/G$ is a homeomorphism.*

Proof. The natural map in the theorem is clearly continuous, since it is induced by the K invariance of the map $\mu^{-1}(0) \to G\mu^{-1}(0)/G$. It is surjective by definition, and Lemma 1.3.12 ensures that it is actually bijective. Since μ^{-1}/K is compact, it is enough then to prove that $G\mu^{-1}(0)/G$ is Hausdorff, as any continuous bijection from a compact to a Hausdorff is a homeomorphism.

To prove that $G\mu^{-1}(0)/G$ is Hausdorff, (KIRWAN LEMMA 7.3.) □

1.3.14 Corollary. *The symplectic reduction is a Kähler manifold if K acts freely on $\mu^{-1}(0)$.*

A word is in order about this result, because the previous three result show in a clear way why we restrict to reductive groups throughout. In fact, lemma 1.2.7 shows that these groups have simple structures, in particular, that they retract to they maximal compact subgroup K (using the paths determined by the exponential map.) Corollary 1.3.14 is merely a reflection of this: we saw that the slice lemma ensures the existence of slices transverse to the K orbits, and the structure of G shows that if we can find slices transverse to the orbits of G, they must retract to the slices transverse to the K orbit. This is essentially the content of the theorem, but we will abstain from proving this explicitly.

1.3.15 Example. We now want to use the results above to construct the projective spaces as symplectic quotients. This will give us a standard symplectic and moment maps on $\mathbb{P}^n$ which we'll need to relate the symplectic and GIT constructions.

Let V be a Hermitian vector space with metric $(\cdot,\cdot)$. The imaginary part of the metric gives a a symplectic form $\omega_V(v,w) = 2\mathrm{Im}(v,w)$, and its real part a Riemannian metric $g(v,w) = 2\mathrm{Re}(w,v)$; note that $g(v,w) = \omega_V(v,iw)$. Let the multiplicative Lie group $\mathbb{C}^*$ act on V by scalar multiplication, so that the unitary group $U(1) = S^1$ then is a compact group with a restricted action. This action of the unitary group fixes the Hermitian metric, so it also fixes the symplectic and Riemannian forms separately. It is straightforward to apply the Marsden-Weinstein theorem to this case. First of all, the moment map is given by $\mu_V(x) = i||x||$. Indeed, we have

$$\frac{d}{dt}\left(i\beta(x+ty), x+ty\right)|_{t=0} = (i\beta x, y) + (i\beta y, x) = 2\mathrm{Im}(\beta x, y) = \omega_V(\beta_x^{\#}, y)$$

The level of the moment map is $\mu_V^{-1}(0) = S^{2n+1}$, where $n+1$ is the (complex) dimension of V, so that the Marsden-Weinstein yields the usual description of $\mathbb{P}_V^n$ as a quotient of the unit sphere with antipodal points identified.

1.3.16 Remark. Some remarks are in order concerning this example:

1. Our definition of the symplectic form might have looked strange, and off by a factor of 2 from the usual definition. In general, we could multipliy ω by any real number, and when we choose a different factor for the the symplectic form, it is easy to see from the computations above that the moment map will also change by a factor. In particular, this implies that the zero-level set of the moment map will change, becoming spheres of different radii, or possibly even empty sets.
2. The remark above can alternatively be explained by an indeterminancy on the choice of the moment map. As we could have noted already in the general case, the moment map is not uniquely defined by its properties. Indeed, it is easy to see that, in our case, any addition of a constant to the moment map does not change its properties. In the general case, we can add to a given moment map any central element of the Lie algebra $\mathfrak{k}$. This is analogous to the situation in GIT, where there is a inherent indeterminancy in the choice of linearization of the action, and we'll see later that in fact they are intrinsically the same problem, i.e., different linearizations of an action on a projective variety will correspond to different choices of a hermitian metric on the cone over it. The key to understanding this fact will be the Kempf-Ness Theorem.

Since we are mostly interested in projective varieties, it will be important for us to get an explicit formula for the induced symplectic form on projective space, as well as one for the moment map with respect to the induced action of a subgroup of $\mathrm{GL}(n+1,\mathbb{C})$. At a given point $x \in \mathbb{P}_V^n$, given a lift $\tilde{x} \in S^{2n+1} \subset V$, the tangent space at x and $\tilde{x}$ identify. Using this identification, the definition of the symplectic form in the proof of the Marsden-Weinstein Theorem is simply $\omega_{\mathbb{P}^n}(v,w) = \omega_V(v,w)$. Then, it is easy to see that the moment map can be defined as $\langle \mu_{\mathbb{P}^n}(x), \beta\rangle = \langle \mu_V(\tilde{x}), \beta\rangle = (i\beta\tilde{x}, \tilde{x})$. It is straightforward to verify that this is independent of the choice of the lift. If we want to use an *arbitrary* lift $\tilde{x} \in V$, one must only first divide by the norm of the lift, i.e.,

$$\langle \mu_{\mathbb{P}^n}(x), \beta\rangle = \langle \mu_V\left(\frac{\tilde{x}}{||\tilde{x}||}\right), \beta\rangle = \frac{(i\beta\tilde{x}, \tilde{x})}{(\tilde{x}, \tilde{x})}$$

The theorem We come now to the core part of this section. Let G a smooth reductive algebraic group that acts on $\mathbb{CP}^n$ through a representation $\rho : G \to \mathrm{GL}_{n+1}$ that restricts to $K \to \mathrm{U}_n$, and let X be a smooth, invariant subvariety. Both G and X determine associated complex analytic varieties, which, because they are smooth, are also Kähler manifolds. Therefore, for this action, we can define two quotients: the GIT quotient, defined in section 1.1, and the Kähler symplectic reduction, defined just above. It turns out that these two coincide, which is the content of the Kempf-Ness theorem.

Let, as above, $||\cdot||$ be a norm induced by an inner product $(\cdot,\cdot)$ on $\mathbb{A}^{n+1}_{\mathbb{C}}$, and ω and μ be the canonically associated symplectic form and moment maps on $\mathbb{P}^n$. Since semistability is related to the adherence of orbits to zero, the main idea is to detect this using the norm. For that, we introduce the *Kempf-Ness function*:

$$\psi_{\tilde{x}}(g) = \frac{1}{2}\log||g\tilde{x}||^2$$

Since K acts unitarily, this is a function $\psi_{\tilde{x}} : G/K \to \mathbb{R}$. In fact, we can rewrite it in a much more useful way.

1.3.17 Lemma. *The Kempf-Ness function factors to a function* $\psi_{\tilde{x}} : \mathfrak{k}/\mathfrak{k}_{\tilde{x}} \to \mathbb{R}$

Proof. Since $G = K\exp(i\mathfrak{k})$ (cf. section 1.2,) and $\psi_{\tilde{x}}$ factors through G/K, we can compose with the exponential to get a function $\overline{\psi}_{\tilde{x}} : \mathfrak{k} \to \mathbb{R}$. But now, this obviously factors through a function $\mathfrak{k}/\mathfrak{k}_{\tilde{x}}$, since $\exp(i\mathfrak{k}_{\tilde{x}}$ leaves $\tilde{x}$ invariant. □

By an abuse of language, we'll denote by $\psi_{\tilde{x}}$ the factorization in the lemma, and also call it the Kempf-Ness function. The relation with the symplectic theory is explained by the next lemma.

1.3.18 Lemma. *The moment map* μ *is the gradient of* $\psi_{\tilde{x}}$.

Proof. It follows by a simple computation:

$$D_\beta \psi_{\tilde{x}}(g) = \frac{1}{2}\frac{d}{dt}\left(\log||\exp(it\beta)g\tilde{x}||^2\right)\Big|_{t=0} = \frac{(i\beta g\tilde{x}, g\tilde{x})}{(g\tilde{x}, g\tilde{x})} = -\langle\mu(gx),\beta\rangle$$

□

From this, we can immediately deduce some important convexity results.

1.3.19 Lemma.

1. *For any $x \in X$, and any non-zero lift $\tilde{x}$, $\psi_{\tilde{x}}$ is a convex function, the critical points of which are $g \in G$ such that $\mu(gx) = 0$.*
2. *The second derivative $D^2_\beta\psi(e)$ is positive if and only if $\beta \in \mathfrak{k} - \mathfrak{k}_x$, and so $\psi_{\tilde{x}}$ is a strictly convex function.*
3. *For $\beta \in \mathfrak{k}_x$, we have $\psi_{\tilde{x}}(\exp(i\beta)) = \psi_{\tilde{x}}(e) + 2\langle\mu(x),\beta\rangle$.*

Proof. To prove convexity, we compute

$$\begin{aligned} D^2_\beta\psi_{\tilde{x}}(g) &= -\frac{d}{dt}\langle\mu(\exp(it\beta)gx),\beta\rangle|_{t=0} \\ &= -\langle d\mu_{gx}(i\beta),\beta\rangle \\ &= -\omega_{gx}(i\beta,\beta) \\ &= ||\beta||^2 \geq 0 \end{aligned}$$

The rest of the statements follow from this and lemma 1.3.18, plus noting that $\mathfrak{k}/\mathfrak{k}_{\tilde{x}} \simeq \mathfrak{k}^{\perp}_{\tilde{x}}$. □

Finally, we'll prove a fundamental fact for our main theorem.

1.3.20 Lemma. *The Kempf-Ness function is proper.*

Proof. Let β_j be a sequence such that $\psi_{\tilde{x}}(\beta_j)$ is bounded; we will prove that b_j is bounded, so that it must have a convergent subsequence ($\mathfrak{k}/\mathfrak{k}_{\tilde{x}}$ is finite-dimensional.) Indeed, by strict convexity of the Kempf-Ness function, there are constants C_1 and C_2 such that

$$C_1||\beta|| + C_2 < \psi_{\tilde{x}}(\beta)$$

The result follows. □

1.3.21 Theorem (Kempf-Ness)**.** *A G-orbit is semistable if and only if its closure contains a zero of the moment map. Further, a G-orbit is polystable if and only if it contains a zero of the moment map. In particular, if $X /\!\!/ G$ denotes the GIT quotient, there is a homeomorphism $X /\!\!/ G \simeq \mu^{-1}(0)/K$, when the former is considered as a complex analytic variety.*

Proof. For the semistability, recall that there is a unique closed orbit in each S-equivalence class. Therefore, if an orbit $G \cdot x$ is semistable, its closure cannot contain the origin, and the the Kempf-Ness function is bounded from below. Find a minimizing sequence for $\psi_{\tilde{x}}$; this must converge to some critical point, which is a zero of the moment map. Conversely, if it's not semistable, then its closure contains the origin. This means that $\psi_{\tilde{x}}$ is not bounded from below, and since it is convex, it cannot have a critical point. Therefore, no zero of the moment map can be in the closure of the orbit.

Polystability is a bit more complicated. Let $G \cdot x$ be a polystable orbit. Recall that this means the the orbit $G \cdot \tilde{x}$ is closed, and thus the Kempf-Ness function is bounded below on it. If it wasn't, then the orbit must necessarily must accumulate at the origin, and it wouldn't be closed. Since the function is convex, it must attain a minimum. By lemma 1.3.19, this corresponds to a zero of the moment map.

Conversely, suppose that the orbit contains a zero of the moment map, say x; we want to prove that the orbit is closed. We need to prove that given some sequence g_j, if $g_j\tilde{x} \to \tilde{x}_\infty$ for some point $\tilde{x}_\infty$, then $\tilde{x}_\infty \in G \cdot \tilde{x}$. Since the orbit has a zero of the moment map, we proved above that $\tilde{x}_\infty \neq 0$, and so $\psi_{\tilde{x}}(g_j)$ is bounded. Writing $g_j = \exp(i\beta_j)$, then the sequence β_j is bounded in $\mathfrak{k}/\mathfrak{k}_{\tilde{x}}$. Since Kempf-Ness function is proper by Lemma 1.3.18, after passing to a subsequence we have $\beta_j \to \beta$. By continuity, it follows that if $g = \exp i\beta$, then $g\tilde{x} = \tilde{x}_\infty$.

Finally, from what we just proved, it follows that the inclusion $\iota : \mu^{-1}(0) \to X$ induces a natural map $\bar{\iota} : \mu^{-1}(0)/K \to X /\!\!/ G$. It is clearly surjective, by the above. It is also injective, by lemma 1.3.12. But then it is a bijection from a compact space to a Hausdorff one, so it is a homeomorphism. □

We'll construct a nice inverse to the map $\bar{\iota} : \mu^{-1}(0)/K \to X /\!\!/ G$ in the proof in a later section.

The linear criterion Using the Kempf-Ness Theorem, we can come up with an analytic linear criterion analogous to the Hilbert-Mumford one in the algebraic context. We still assume that $X \subset \mathbb{P}^n$. The natural thing to do in the differential context is to find flows of the action of vector fields.

1.3.22 Lemma. *Given $\beta \in \mathfrak{k}$, the trajectory from any $x \in X$ of $-\nabla\langle\mu, \beta\rangle$ is $r(t) = \exp(-it\beta)x$.*

Proof. We compute the gradient: let $x \in X$, $v \in T_xX$, and $(\cdot,\cdot)$ be the Riemannian metric on X. Then,

$$(v, \nabla\langle\mu(x), \beta\rangle) = d(\langle\mu(x), \beta\rangle)(v) = \omega_x(v, \beta_x^\#) = (v, i\beta_x^\#)$$

□

As the point moves in this trajectory, it gets closer to the stationary point $x_0 = \lim_{t\to\infty} \exp(-it\beta)x$.

From Hilbert-Mumford, we know that the stability of x is determined by the behaviour of a lift. In fact, if x_0 is not in the orbit of x, the latter is polystable if as we aproach x_0, the norm of the lift increases without bound. Analytically speaking, this corresponds to the derivative of the Kempf-Ness map being positive along the trajectory, and bounded from below. But we've computed the gradient of the Kempf-Ness function to be the moment map, so we need to consider

$$\lambda_x(\beta,t) = \langle \mu(\exp(-it\beta)x), \beta\rangle$$

and define

$$\lambda_x(\beta) = \lim_{t\to\infty} \lambda_x(\beta,t)$$

This is called the maximal weight of x in the direction β. Though we do not provide a proof, the next theorem should then be credible.

1.3.23 Theorem. *Let $X \subset \mathbb{P}^n$ be a subvariety of projective space, and G a reductive Lie group acting on it through a linear representation. Then, given a point $x \in X$,*

1. *x is stable if and only if $\lambda_x(\beta) > 0$ for all nonzero $\beta \in i\mathfrak{k}$;*
2. *x is polystable if and only if there is a $g \in G$ such that $\lambda_{gx}(\beta) \geq 0$ for all $\beta \in i\mathfrak{k}$, with strict inequality for $\beta \in i\mathfrak{k} - i\mathfrak{k}_{gx}$.*

Proof. We prove that under the conditions of the theorem, the criterion coincides with the Hilbert-Mumford. Since x_0 is fixed by the one-parameter subgroup, $\beta \in \mathfrak{k}_{x_0}$. From lemma 1.3.19, for $s \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{1}{2}\log||\exp(is\beta)\tilde{x}_0||^2 &= \psi_{\tilde{x}_0}(\exp(is\beta)) \\
&= \psi_{\tilde{x}_0}(e) + s\langle\mu(x_0),\beta\rangle \\
&= \frac{1}{2}\log e^{s\langle\mu(x_0),\beta\rangle}||\tilde{x}_0||^2 \\
&= \frac{1}{2}\log||t^{\langle\mu(x_0),\beta\rangle}\tilde{x}_0||^2
\end{aligned}$$

Proving it in the reverse is perhaps more intuitive, so we repeat the proof: the action of the one-parameter subgroup on the fixed point x_0 is just multiplication by the constant $t^{\rho(x)}$. But this means the norms are multiplied by $|t^{\rho(x)}|$. If we take logarithm of its square, we have

$$\frac{1}{2}\log||\lambda(t)\tilde{x}_0||^2 = \frac{1}{2}\log|t^{\rho(x)}|^2||\tilde{x}_0||^2 = \rho(x)\log|t| + \frac{1}{2}\log||\tilde{x}_0||^2$$

Thus, we reduced the proof to showing that the derivative of the Kempf-Ness function along the direction of β is the highest weight. By lemma 1.3.18, this derivative is $\langle\mu(x_0),\beta\rangle$. The result then follows by continuity of the moment map. □

1.3.24 Remark. A complete analytic proof of this result can be found in both [**?**] and [42]; the fundamental ideas are already in [22]. In the general case of a Kähler manifold, there isn't any simple criterion for semistability. Still, the extension of this linear criterion to general Kähler manifolds is essential for the proof of various Hitchin-Kobayashi correspondences that can be found in the literature.

1.4 Some structural remarks on reductive groups

As we proceed from the construction of quotients to a study of their topology, we'll need a more detailed understanding of the structure of reductive groups.[1] Once again, we can only graze the surface of these topics, but we're hoping that presenting them coherently here can benefit the reader by providing a guiding line while reading the literature. Good sources are [Garcia-Prada et al.] and [**?**].

1.4.1 Parabolic subgroups

In the theory of quotients, parabolic and Levi subgroups play a prominent role. We briefly review their definition, as well as some facts that we'll need later on. Good references are [Garcia-Prada et al.] [41].

The definitions

Fix a group complex reductive G with fixed maximal compact K, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebras. If $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and T a maximal torus of K, there is a choice of Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{z}\oplus\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$, where $\mathfrak{t} = \text{Lie } T$. Let Δ be a choice of simple roots for the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For any subset $A = \{\alpha_{i_1}, ..., \alpha_{i_s}\} \subset \Delta$, define

$$D_A = \{\alpha \in R \mid \alpha = \sum m_j \alpha_j,\ m_{i_t} \geq 0 \text{ for } 1 \leq t \leq s\}$$

The *parabolic subalgebra* associated to A is

$$\mathfrak{p}_A = \mathfrak{z}\oplus\mathfrak{h}\oplus\bigoplus_{\alpha\in D_A}\mathfrak{g}_\alpha$$

This subalgebra determines a subgroup P_A of G, the *standard parabolic subgroup* determined by A.

1.4.1 Definition. A *parabolic subgroup* of G is a subgroup P conjugate to some standard parabolic subgroup P_A. A *Levi subgroup* of P is a maximal reductive subgroup of P.

For the case of standard parabolic subgroups there is a "natural" choice of a Levi subgroup. Let $D^0_A \subset D_A$ be the set of roots with $m_j = 0$ for $\alpha_j \in A$. Then,

$$\mathfrak{l}_A = \mathfrak{z}\oplus\mathfrak{h}\oplus\bigoplus_{\alpha\in D^0_A}\mathfrak{g}_\alpha$$

is the standard Levi subalgebra of $\mathfrak{p}_A$, and the connected subgroup L_A determined by $\mathfrak{l}_A$ is a Levi subgroup of P_A.

[1] It might be worth revisiting our conventions from section 1.2.

We will need an important construction of parabolic subgroups. For any $\beta \in \mathfrak{k}$, let

$$\mathfrak{p}(\beta) := \{x \in \mathfrak{g} | \mathrm{Ad}(\exp it\beta)x \text{ remains bounded as } t \to 0\}$$
$$\mathfrak{l}(\beta) := \{x \in \mathfrak{g} | [\beta, x] = 0\}$$

Let $P(\beta)$ and $L(\beta)$ be the corresponding closed subgroups of G and their Levis. The following is in [Garcia-Prada et al.]:

1.4.2 Proposition. *The sugroups $P(\beta)$ and $L(\beta)$ are a parabolic, resp. Levi subgroup of G. Conversely, for any parabolic subgroup of P there is a $\beta \in \mathfrak{k}$ such that $P = P(\beta)$.*

Dominant elements

The description above is in fact a statement about parabolic subgroups and their dominant characters. Note that the characters of $\mathfrak{p}_A$ are in bijection with elements of $\mathfrak{z}^* \oplus \mathfrak{c}_A^*$.

1.4.3 Definition. An *dominant character* of $\mathfrak{p}_A$ is an element of the form $\chi = z + \sum_{\delta \in A} n_\delta \lambda_\delta$ where n_δ is a non-positive real number. It is strictly dominant if those integers are actually strictly negative. The *dominant weights* are the correspoding elements of $\mathfrak{z} \oplus \mathfrak{c}_A$ through the chosen invariant form on the latter.

We have [Garcia-Prada et al.]:

1.4.4 Proposition. *Given a dominant weight β of P, $P \subset P(\beta)$, equality holding if and only if β is strictly anti-dominant.*

Interpretation in terms of flags

The construction above is key to understanding an interpretation of parabolic subgroups in terms of flags. This interpretation will be important for us below in establishing a connection between the Lie-theoretic framework, and the theory of classical quivers.

Fix a faithful representation $\rho : K \to U(V)$ (complexifying, also $\rho : G \to GL(V)$.) This induces an isomorphism $\mathfrak{k} \simeq \mathfrak{k}^*$, which we shall use implicitly. Because ρ is a unitary representation, the image $\rho_*\beta$ of any element $\beta \in i\mathfrak{k}$ in $\mathrm{GL}(V)$ is Hermitian, so it diagonalizes with real eigenvalues $\lambda_1 < \ldots < \lambda_r$, and induces a filtration

$$0 \neq V^1 \subsetneq \ldots \subsetneq V^r = V$$

where $V^k = \bigoplus_{i \leq k} V_{\lambda_k}$ is the sum of all eigenspaces V_{λ_i} with $i \leq k$. Let $St(\beta) \subset \mathrm{GL}(V)$ be the subgroup stabilizing this flag; in other words, $g \in St(\beta)$ if and only if $g \cdot V^r \subset V^r$ for all r.

1.4.5 Proposition. *For any $\beta \in i\mathfrak{k}$, we have $\rho^{-1}(St(\beta)) = P(\beta)$.*

Using Proposition 1.4.2, it is now easy to characterize parabolic subgroups of G as stabilizers of certain flags. Levi subgroups then correspond to stabilizers of the associated graded vector space, i.e., stabilizers of the decompositions

$$V = \bigoplus V_{\lambda_i}$$

induced by an element $\beta \in i\mathfrak{k}$.

It is important, however, to keep in mind that this applies to filtrations induced by an element of $i\mathfrak{k}$, and not just any filtration of V. For $\mathrm{GL}(V)$, it is true that any flag is so induced. But for an orthogonal or symplectic group, for example, the flags induced in this way are *isotropic*: they satisfy $V^{r-k} = (V^k)^\perp$ (so that for $k \leq r/2$, the spaces are actually isotropic.)

Algebraic groups

For algebraic groups, the natural objects are one-parameter subgroups, and not elements of the Lie algebra. Given a one-parameter subgroup (OPS) λ of G, we can, in fact, define the analogue of the above parabolic and Levi subgroups by

$$P(\lambda) := \{g \in G \,\Big|\, \lim_{t\to 0} \mathrm{Ad}(\lambda(t))g \text{ exists}\}$$

$$L(\lambda) := \{g \in G \,\Big|\, \lim_{t\to 0} \mathrm{Ad}(\lambda(t))g = g\}$$

In fact, the analogue of Proposition 1.4.2 holds, that is, this construction yields all parabolic subgroups of G ([41] Proposition 8.4.5.) But in fact we can relate both constructions, since any one-parameter subgroup of G is conjugate to some other which sends the maximal compact subgroup $U(1)$ into the maximal compact K. In other words, each one-parameter subgroup is uniquely determined by an element of $\mathfrak{k}$, since

1.4.6 Proposition. $\mathrm{Hom}(U(1),K) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathfrak{k}$

One sees in this way that the two settings are completely interchangeable.

1.5 The topology of quotients

The previous sections tried to explain the procedures used in constructing quotients for actions of groups on projective varieties. This involved extracting from the original variety a small closed subset of unstable points. In this section, we show that a close analysis of this unstable locus allows one to establish a cohomological formula for the equivariant Poincaré polynomials of the remaining open set, which means that under suitable assumptions on the action, we can extract the Poincaré polynomial of the quotient itself. This whole section is a re-presentation of Kirwan's thesis [22].

1.5.1 Stratifications and cohomology

We start by setting forth the general framework which motivates the discussion below. Let X be a manifold; the idea is that by cutting up the manifold in a suitable way, one can extract cohomological information from each of the pieces. The following definition makes precise what we mean by 'cutting up.'

1.5.1 Definition. A finite collection $\{S_\beta, \beta \in B\}$ of subsets of X is a smooth stratification of X if each S_β is a locally-closed submanifold, X is the disjoint union of all the S_β, and there is a strict partial ordering $<$ on B such that

$$\overline{S}_\beta \subset \bigcup_{\beta\leq\gamma} S_\gamma$$

If X is acted upon by a group G, we say that the stratification is G-invariant if each stratum S_β is preserved by the action of G.

Recall that the Poincaré polynomial of X is given by

$$P_t(X) = \sum_i t^i \dim H^i(X)$$

If we are given a smooth stratification of X, it turns out that this polynomial is related to the polynomials of each of the stratum. In fact, using the Thom-Gysin sequence for each stratum, one can deduce the following.

1.5.2 Proposition ([22] 2.13). *If $\{S_\beta, \beta \in B\}$ is a smooth stratification of X, then there is a polynomial $R(t)$ with non-negative integer coefficients such that*

$$P_t(X) = \sum_\beta t^{d(\beta)} P_t(S_\beta) - (1+t)R(t)$$

Here, $d(\beta)$ is the codimension of $S_{[\beta]}$, which we have assumed constant for simplicity.

This type of results is known as Morse-inequalities, due to the fact that major examples are the stratifications obtained using Morse functions. If $R(t) = 0$, we say that the stratification is perfect. There is an important criterion of Atiyah-Bott for the stratification to be perfect.

1.5.3 Lemma ([22] 2.18). *Suppose that for each β, the Euler class of the normal bundle to S_β in X is not a zero-divisor. Then, the stratification is equivariantly perfect.*

Suppose now that X is acted upon by a group G. Let $EG \to BG$ be the classifying bundle for G; if G is compact, EG is simply a contractible space on which G acts freely, and $BG = EG/G$. We will be interested in the equivariant cohomology of X:

$$H^*_G(X) = H^*(EG \times_G Y)$$

The equivariant Poincaré polynomial is defined using the equivariant cohomology of X:

$$P^G_t(X) = \sum_i t^i \dim H^i_G(X)$$

We refer the reader to [3] for a thorough discussion of these concepts. The following is the result relevant for us.

1.5.4 Lemma ([22] 2.18). *Suppose $\{S_\beta, \beta \in B\}$ is a smooth G-invariant stratification of X. Then, the equivariant Morse inequalities hold: there is a polynomial $R(t)$ with non-negative integer coefficients such that*

$$P^G_t(X) = \sum_\beta t^{d(\beta)} P^G_t(S_\beta) - (1+t)R(t)$$

Here, $d(\beta)$ is the codimension od $S_{[\beta]}$, which we have assumed constant for simplicity. A sufficient condition for $R(t)$ to be zero is that for each β, the equivariant Euler class of the normal bundle to S_β in X be a non-zero divisor.

1.5.2 The moment map and the Morse stratification

Let X be a symplectic manifold, and K be a compact group actingon X with moment map μ. We will now discuss how the moment map can be used to obtain a stratification on X that is equivariantly perfect. Fix a a K-invariant product on Lie agebra of $\mathfrak{k}$. We define a function $f : X \to \mathbb{R}$ by

$$f(x) = ||\mu(x)||^2$$

The critical set The stratification will be defined using f in a way that is completely analogous to Morse theory, so we will first be interested in the critical subsets of f. The differential of f is easily computed to be

$$df = 2\langle d\mu, \mu\rangle$$

Let $\mu^* : X \to \mathfrak{k}$ be the co-moment map, the composition of μ with the fixed isomorphism $\mathfrak{k} \simeq \mathfrak{k}^*$. Using the definition of moment map, we have

1.5.5 Proposition ([22] 3.1)**.** *A point $x \in X$ is critical for f if and only if $\mu^*(x)^\dagger_x = 0$.*

Recall here that for $\beta \in \mathfrak{k}$, $\beta^\dagger$ denotes the vector field on X induced by the action of K. Our first goal is to prove the following result.

1.5.6 Lemma. *Assume X is compact, and let C be the critical set of f. Then, $\mu(C)$ is the union of a finite number of coadjoint orbits.*

Fix a maximal torus T of K, and denote by W^+ the corresponding positive Weyl chamber. Recall that every coadjoint orbit has a unique element in W^+, so that we may index the coadjoint orbits in $\mu(C)$ by a finite set B of elements of W^+. Denote $C_{[\beta]} = \mu^{-1}(K\cdot\beta) \cap C$. The lemma above implies that the critical set is the topological coproduct

$$C =_{\beta \in B} C_{[\beta]}$$

Indeed, the lemma says that it is the disjoint union of the C_β. K being compact means that all coadjoint orbits are closed, which together with finiteness of B ensures that the union is in fact a coproduct by showing that each $C_{[\beta]}$ is closed in X.

Let us prove the lemma. The trick is to reduce the matters to the action of the maximal torus T. First note that f is K-invariant, and so $x \in X$ is critical for f if and only if $k \cdot x$ is also. Since the moment map is equivariant with respect to the coadjoint action, after conjugation we may assume that $\beta := \mu^*(x) \in \mathfrak{t}_+$. Now, the restricted action of T has moment map $\mu_T = p_T\mu$, the composition of μ with the restriction $p_T : \mathfrak{k}^* \to \mathfrak{t}^*$. Also, if $\mu(x) \in \mathfrak{t}$, then x is critical for f if and only if it is critical for $f_T := ||\mu_T||^2$. We are thus reduced to showing that there only a finite number of β indexing the critical strata of f_T. The lemma now follows from the following two results.

1.5.7 Theorem ([2] Th.1). *The image of the fixed point set of T is a finite set A of points in $\mathfrak{t}^*$, and $\mu(X)$ is their convex hull.*

1.5.8 Proposition ([22] 3.12). *If an element β indexes a critical set of f, then it is the closest point to the origin of the convex hull of some subset of A.*

The idea here is to bootstrap Theorem 1.5.7, by selecting a submanifold Z_β containing x on which T acts. (Recall that x is a critical point of f with $\mu^*(x) = \beta$.) First, let $\mu_\beta = \langle \mu, \beta \rangle$. By the definition of the moment map, the differential $d\mu_\beta$ is ω-dual to the vector field $\beta^\dagger$. We define Z_β to be the union of the connected components of the critical set of μ_β where it takes the value $||\beta||^2$.

1.5.9 Proposition. *The set Z_β is a submanifold of X.*

Proof. We are going to show that in fact the whole critical set of μ_β is a submanifold, by showing that it is the fixed point set of a subtorus of T. Indeed, if we let T_β be the closure of $\exp(\mathbb{R}\beta)$ in T, T_β is a subtorus of T. The differential $d\mu_\beta$ is ω-dual to the vector field $\beta^\dagger$, by the definition of moment map. But then, x being critical for μ_β is precisely the condition that it be fixed by the action of $\exp(\mathbb{R}\beta)$, and so fixed by T_β. It is well known that the fixed point set of a torus action is a submanifold. □

Proof of Propostion 1.5.8. We will do this by steps.

1. **$\mu^*(Z_\beta)$ lies in the hyperplane defined by** $(v - \beta, \beta) = 0$**:** Indeed, by definition, if $x \in Z_\beta$, then $\mu_\beta(x) = \langle \mu(x), \beta \rangle = (\mu^*(x), \beta) = ||\beta||^2 = (\beta, \beta) \iff (\mu^*(x), \beta) - (\beta, \beta) = 0$.

2. **β is the closest point to the origin in $\mu_T(Z_\beta)$:** since $\mu_T(Z_\beta) \subset \mu(Z_\beta)$, by the previous point it is enough to prove that $\beta \in \mu_T(Z_\beta)$ (indeed, β is the closest point to the origin in the whole hyperplane.) But we started by assuming that we were given an $x \in X$ with $\mu(x) = \beta$, and which was critical for μ, and so for μ_β, which means that $x \in Z_\beta$.

3. **Z_β is T-invariant:** This follows from μ_β being a T-invariant map, so that critical points are sent to critical points.

4. **The set $\mu_T(Z_\beta)$ is the convex hull of some subset of A:** This is now just an application of Theorem 1.5.7 to the action of T on Z_β.

□

Note that in the course of the proof we also proved that

$$C_{[\beta]} = K(Z_\beta \cap \mu^{-1}(\beta))$$

We finish by noting that Proposition 1.5.8 imples also the following result, which is needed for the definition of stratification.

1.5.10 Lemma. *Set $\beta < \beta'$ if $||\beta||^2 < ||\beta'||^2$. This is a strict order on the set B above.*

The strata To define the strata, we must fix now a Riemannian metric g on X so as to make sense of the gradient ∇f of f. For each $x \in X$ there is then a path of steepest descent for f: it is the solution γ to the ODE problem given by

$$\frac{d\gamma_x}{dt}(t) = -\nabla f(\gamma_x(t))$$

with initial value $\gamma_x(0) = x$.

1.5.11 Assumptions. We will make the following two assumptions:

1. The negative gradient flow of f at any point $x \in X$ is contained in some compact neighbourhood of X;

2. The critical set C of f is a topological coproduct of a finite number of closed subsets $C_{[\beta]}$, $\beta \in B$, on each of which f takes constant value, and such that $\beta < \beta$ if $f(C_{[\beta]}) < f(C_{[\beta']})$ is a strict ordering of B.

These two conditions hold if X is compact: compact manifolds are complete, and we just proved above that the second condition also holds. However, there are important examples of non-compact manifolds X for which these conditions hold, so it pays to consider this full generality.

Because of condition 1, for any point $x \in X$ the path of steepest descent $\gamma_x(t)$ is defined for all $t > 0$; denote its limit set by

$$\omega(x) = \{y \in X | \text{every neighbourhood of } y \text{ contains points } \gamma(t) \text{ for } t \text{ arbitrarily large}\}$$

This is a non-empty closed, connected set, and its elements are all critical points of f. It follows that there is a unique $\beta \in B$ such that $\omega(x) \subset C_{[\beta]}$. Therefore, if we define

$$S_{[\beta]} = \left\{x \in X | \ \omega(x) \subset C_{[\beta]}\right\} \tag{1.2}$$

then X is the disjoint union of the finite collection of all $S_{[\beta]}$. The proof of the following result is an easy adaptation of [22] 10.7.

1.5.12 Theorem. *The collection $S_{[\beta]}$ is a stratification of X, i.e.,*

$$\overline{S}_{[\beta]} \subset \bigcup_{\gamma \geq \beta} S_{[\gamma]}$$

Proof. Suppose $x \in \overline{S}_{[\beta]}$ but that $x \in S_{[\beta']}$ for some $\beta' \neq \beta$, and let γ_x be the path of steepest descent for x. This is just the restriction to $\{x\} \times \mathbb{R}^+$ of the negative grandient flow $\gamma : X \times \mathbb{R}^+ \to X$. Since the limit set $\omega(x)$ is contained in $C_{[\beta']}$ and the negative gradient flow is continuous, for each $\delta > 0$ there is an open $x \in V \subset X$ and a $T_1 \in \mathbb{R}^+$ such that for all $y \in V$, $f(\gamma_y(T_1)) < f(C_{[\beta']}) + \delta$ (recall that f is strictly decreasing along the path of steepest descent.) In particular, since $x \in \overline{S}_{[\beta]}$, this is true of some $y \in S_{[\beta]}$.

Our aim is to show that T_1 can be so chosen that $f(\gamma_y(T_1)) \geq f(C_{[\beta]}) + \delta$, for which we will need to use assumption 1 above. Indeed, intersect $C_{[\beta']}$ with some compact neighbourhood containing γ_x; we may assume that y belongs to the compact neighbourhood. Intersect also $C_{[\beta]}$ with a compact neighbourhood for γ_y. The resulting intersections are compact, so they can be separated by two opens $U_{\beta'}$ and U_β also with compact closure. This compactness together with the fact that f is strictly decreasing along the flow mean that there is a $\delta > 0$ such that for $x' \in S_{[\beta']} \cap \partial U_{\beta'}$ we have $f(x') \geq f(C_{[\beta']}) + \delta$, and the same for β (with the same δ!) By restricting the neighbourhood V if necessary, we may assume that $\gamma_y(T_1) \in U_{\beta'}$; but since $U_{\beta'} \cap U_\beta = \varnothing$, and $\omega(y) \subset C_{[\beta]}$, this means that $\gamma_y(T_2) \in \partial U_\beta$ for some $T_2 > T_1$. Then,

$$f(C_{[\beta]}) + \delta > f(\gamma_y(T_1)) > f(\gamma_y(T_2) \geq f(C_{[\beta]}) + \delta$$

The result follows. □

This stratification is known as the Morse stratification of X. It is equivariant and smooth for a suitable (but always available) choice of Riemannian metric, so that by Theorem 1.5.2, the (equivariant) Morse inequalities hold. In fact, one can further prove that the stratification is equivariantly perfect, but proving this would takes us astray; the interested reader can check [22].

We do want to properly write the Morse formula, but this is somewhat complicated by the fact that the strata do not have constant codimension. We first must split each stratum into strata of constant codimension, which are indexed by a finite number of integers m. We have

$$S_{[\beta]} =_m S_{[\beta],m}$$

The codimension of each $S_{[\beta],m}$ is $d(\beta,m) = m - \dim K + \dim \mathrm{Stab}\beta$; this is in fact the index of the Hessian of f on the critical set $C_{[\beta],m}$. We order the pairs (β,m) lexicographically.

1.5.13 Theorem ([22] 5.4, 9)**.** *For a suitable, always existing choice of Riemannian metric, the collection $\{S_{[\beta],m}\}$ defined above is an equivariantly perfect smooth stratification of X over the rationals. Therefore, the equivariant Poincaré polynomial of X is given by*

$$P_t^K(X) = \sum_{\beta,m} t^{d(\beta,m)} P_t^K(S_{[\beta],m})$$

1.5.14 Remark.

1. In the particular case of interest for us – that of a smooth projective variety –, we will provide below an alternative proof of smoothness which can be generalized to arbitrary compact Kähler manifolds.

2. For the interested reader, the condition on the metric has to do with the tangentiality of the gradiant flow to the minimizing submanifolds for f. These minimizing submanifolds referred to in the theorem are locally closed submanifolds containing the critical subsets, and on which the function is minimized precisely along the critical sets (cf. [22] 10.)

3. It is also possible to prove that the inclusion $C_{[\beta],m} \subset S_{[\beta],m}$ is an equivalence of (K-equivariant) Čech cohomology ([22] 10.17.) This implies that we can replace the polynomials of the strata by the ones of the critical sets above. We won't have much use for that formula here, however.

1.5.3 Instability and the Hesselink stratification

Let k be an algebraically closed field of characteristic zero, and $X \subset \mathbb{P}^n$ be a subvariety of projective space- Suppose a linearly reductive group G acts on X through a representation $G \to \mathrm{GL}_{n+1}(k)$. We will build a stratification of X from a systematic study of instability of points.

Instability The Hilbert-Mumford criterion suggests that instability can be roughly measured by the speed to convergence to zero on the fixed fibre of some one-parameter subgroup. Furthermore, this convergence is polynomial, so we'd actually like to optimize this convergence to make sense of a 'maximal destabilization.' One way we could try to do this is just by minimizing the Hilbert-Mumford number. Recall that this is an integer $\rho(x,\lambda)$ defined by the condition that the action of $\lambda(\mathbb{G}_m)$ restricted to the cone-fiber over $x_0 = \lim \lambda(t)x$ is just multiplication by $t^{\rho(x,\lambda)}$. This number has the following properties

1. $\rho(gx, g\lambda g^{-1}) = \rho(x,\lambda)$ for any $g \in G$.
2. $\rho(x, g\lambda g^{-1}) = \rho(x,\lambda)$ for any $g \in P(\lambda)$.
3. $\rho(x_0,\lambda) = \rho(x,\lambda)$.

The reason why optimizing ρ wouldn't work is a simple one: given a one-parameter subgroup λ, $\rho(x,\lambda^n) = n\rho(x,\lambda)$. This does suggest that we normalize one-parameter subgroups in some way. Let $||\cdot||$ be a fixed, G-invariant norm on the space $\chi_*(G)$ of one-parameter subgroups of G. This always exists since, having fixed a maximal torus of G, choosing such a norm is equivalent to choosing a norm on $\chi_*(T)$ invariant under the action of the Weyl group, which is finite. (It is clear from this, however, that such a choice is in general far from unique.) Alternatively, if we recall our conventions about one-parameter subgroups, such a norm is also equivalent to a choice of a K invariant norm on $\mathfrak{k}$. We will assume that this norm is *integral*, that is, $||\lambda|| \in \mathbb{Z}$, or equivalently, $||\alpha|| \in \mathbb{Z}$ for all integral weights of $\mathfrak{k}$ (we can always use a multiple of the Killing form on $\mathfrak{k}$.) With this, we now make a couple of definitions.

1.5.15 Definition. For any $x \in X$, let

$$M_G(x) := \sup\left\{ m_G(x,\lambda) := \frac{\rho(x,\lambda)}{||\lambda||}, \lambda \in \chi_*(G)\right\}$$

Further, let $\Lambda_G(x)$ be the set of indivisible one-parameter subgroups λ such that $m(x,\lambda) = M_G(x)$.

Indivisible here means that λ is not a positive power of another one-parameter subgroup; alternatively, since $\lambda \in \chi_*(T)$ for some maximal torus $T \subset G$, and this is a lattice, indivisibility means λ is minimal in the lattice. Our first goal is to prove the following:

1.5.16 Proposition. *$M_G(x)$ is a finite number for all x, and $\Lambda_G(x)$ is non-empty.*

The proof of this proposition involves fixing a maximal torus of G and giving a geometric interpretation of $m_G(x,\lambda)$. The ideas are in fact important, since they are related to the coincidence of the index sets for the Morse and Hesselink stratification.

Proof. Let $x \in X$ be a point, and λ a one-parameter subgroup. There is a maximal torus $T \subset G$ such that λ is a one-parameter group of T. The action of T is through the representation $\rho : T \to GL_{n+1}(k)$, so consider the splitting into isotypic components

$$\mathbb{C}^{n+1} = \bigoplus_{\chi \in \chi(T)} V_\chi$$

where $V_\chi = \{v \in \mathbb{C}^{n+1} |\ g \cdot v = \chi(g)v\}$. For any $x \in X$, let $v \in \mathbb{C}^{n+1}$ be any lift, and denote

$$\mathrm{st}(x) := \{\chi \in \chi(T) | phantom.v_\chi \neq 0\}$$

where v_χ is the projection of v to V_χ; denote further $\overline{\mathrm{st}(x)}$ the convex hull of $\mathrm{st}(x)$. We have that

$$\rho(x, \lambda) = \min_{\chi \in \mathrm{st}(x)} \langle \chi, \lambda \rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard pairing of characters of T with its one-parameter subgroups induced by the chosen invariant form. We conclude from this that $m_G(x, \lambda)$ is the signed distance from the origin to the projection of $\overline{\mathrm{st}(x)}$ to the ray spanned by the vector λ. Therefore, $M_G(x)$ is a finite number, and it is also easy to see that for T there is always an element in $\Lambda_T(x) \subset \Lambda_G(x)$. □

The definitions above were first introduced by Kempf. We need an additional result also due to him.

1.5.17 Theorem ([20]). *Let x be an unstable point of X. Then,*

1. *There exists a parabolic $P(x)$ such that $P(x) = P(\lambda)$ for all $\lambda \in \Lambda_G(x)$.*
2. *The elements of $\Lambda_G(x)$ are all conjugate under $P(x)$.*
3. *If T is a maximal torus of G contained in $P(x)$, then $\Lambda_T(x)$ is a single orbit under the action of the Weyl group.*

With this, we see that we can sort instability types by their speed $d := M_G(x)$, and their conjugation class $[\lambda]$ of one-parameter subgroups under the action of G. These pairs will in fact be the index of the stratification; a careful analysis along the lines of proof that $M_G(x)$ is a finite number can also show the following.

1.5.18 Proposition. *The set of all $M_G(x)$ for $x \in X$ is finite. Further, there are only finitely many orbits $[\lambda]$ such that $\Lambda_G(x) \subset [\lambda]$ for some $x \in X$.*

The stratification We now define the stratification. For $d > 0$ we let

$$S_{d,[\lambda]} := \{x \in X |\ M_G(x) = d,\ \Lambda_G(x) \subset [\lambda]\} \tag{1.3}$$

We define also $S_0 := X^{ss}$. Because of Theorem 1.5.17, these are mutually disjoint sets. We have

1.5.19 Theorem (Hesselink). *X is the disjoint union of the sets $S_{d,[\lambda]}$.*

This is the Hesselink stratification of X. Fix $\lambda \in [\lambda]$. We can also define

$$S_{d,\lambda} := \{x \in X |\, M_G(x) = d,\ \lambda \in \Lambda_G(x)\}$$

Hesselink called these subsets blades.

1.5.20 Lemma ([22] 13.5)**.** *For $d > 0$, there is a bijective morphism $G \times_{P(\lambda)} S_{d,\lambda} \to S_{d,[\lambda]}$, which is an isomorphism if X is smooth.*

We will actually provide a different proof from Kirwan's, since it applies to more than just the projective case.

Proof. It is easy to see that there is continuous bijection $\overline{\sigma} : G \times_{P(\beta)} S_\beta \to S_{[\beta]}$ as follows: to start with, the restriction of the action $\sigma : G \times S_\beta \to S_{[\beta]}$ is a surjection since $S_{[\beta]} = GS_\beta$. Now, this map factors through $G \times_{P(\beta)} S_\beta$ because if $(g', y') = (gp^{-1}, py)$ (which makes sense since P_β stabilizes S_β,) then clearly $g'y' = gy$. This factorization is the desired continuous bijection $\overline{\sigma}$, and it is surjective because σ is so. It is injective, note that $g'y' = gy \iff y = g^{-1}g'y'$ implies that $(g'(g^{-1}g')^{-}1, g^{-1}g'y') = (g, y)$.

The problem now is to show that this is an isomorphism of varieties; it is enough to show that $\overline{\sigma}$ is a homeomorphism of the underlying topological spaces, and then to show that the infinitesimal maps on the Zarisky tangent spaces are all injective.

We will start by showing that $\overline{\sigma}$ is a homeomorphism. We will need to go about this in a somewhat roundabout way. First, $G \times_P Y$ is a fibre bundle over G/P in a natural way, and consider its product with $\overline{\sigma}$. This gives a monomorphism $\iota : G \times_{P_\beta} S_\beta \to G/P \times X$, and realizes $\overline{\sigma}$ as the second projection $G/P \times X \to X$. In particular, since G/P_β is proper, it is now enough to show that the image of ι is locally closed. To see that this is the case, let $Y := \overline{S_\beta}$, and consider the product of the projection of the first coordinate $G \times Y \to G/P_\beta$ with the restriction of the action $\sigma : G \times Y \to X$. The image of this map is closed, and S_β is easily seen to be open in it, as desired.

Finally, we consider the infinitesimal properties, and it is enough to consider only the distinguished point of G/P_β (the others follow by translation.) If we let then $m = (P_\beta, y)$ for $y \in S_\beta$, an element of $T_m(G/P_\beta \times S_\beta)$ is of the form $(a + \mathfrak{p}_\beta, \xi)$ where $a + \mathfrak{p}_\beta \in \mathfrak{g}/\mathfrak{p}_\beta$ and $\xi \in T_y X$ such that $a_y^\dagger + \xi \in T_y S_\beta$. Now, this is in the kernel of the second projection if and only if $a_y^\dagger \in T_y S_\beta$, which implies that $a \in \mathfrak{p}_\beta$, or in other words that $(a + \mathfrak{p}_\beta, \xi)$ is the zero element as desired. □

The retraction We now want to define the 'critical' Hesselink strata. The point here is that there is a natural action of $\mathbb{G}_m$ on $S_{d,\lambda}$ through the one parameter subgroup λ. We need some results of Bialynicki-Birula. In fact, if $x_0 = \lim \lambda(t)x$, then by the properties of the Hilbert-Mumford pairing, if x belongs to $S_{d,\lambda}$, then so does x_0. Let $Z_{d,\lambda}$ be the set of all such limit points. We can also define Z_λ in the following way: $\mathbb{G}_m$ acts on X through λ, and there is a retration $p_\lambda : X \to X^\lambda$, where X^λ is the fixed point set. We define $Z_\lambda = p_\lambda(S_\lambda)$, and the restriction of this retraction defines a morphism

$$p_{d,\lambda} : S_{d,\lambda} \to Z_{d,\lambda}$$

1.5.21 Proposition. *For any d and λ, we have $Z_{d,\lambda} = \{x \in S_{d,\lambda} |\, \lambda(\mathbb{G}_m) \subset G_x\}$. Further, we have $S_{d,\lambda} = p_\lambda^{-1}(Z_{d,\lambda})$.*

We will now realize $Z_{d,\lambda}$ as the locus of semistable points of some subvariety $X_{d,\lambda} \subset X$ for the action of the Levi subgroup $L(\lambda)$. It will follow that $Z_{d,\lambda}$ is a locally closed subvariety. In fact, when X is smooth, this will show that $Z_{d,\lambda}$ is an open subset of a smooth variety; by the results of Bialynicki-Birula, S_λ has then the structure of a vector bundle over Z_λ, and using Lemma 1.5.20 it follows that the Hesselink strata of a smooth variety are smooth.

Recall that X^λ is the fixed point set of the action of λ on X. Let also

$$V = \bigoplus_{i\in\mathbb{Z}} V_i$$

where $V_i = \{v \in V |\ \lambda(t)v = t^i v\}$, be the the decomposition of V into isotypic components. For a fixed d, we obtain a closed imersion $\mathbb{P}(V_d) \subset \mathbb{P}(V)$. We define

$$X_{d,\lambda} := X^\lambda \cap \mathbb{P}(V_d)$$

By definition, the elements of $L(\lambda)$ centralized λ, so that they preserve the isotypic components of V. It follows that there is an induced action of $L(\lambda)$ on $X_{d,\lambda}$. The following is a result of Ness.

1.5.22 Proposition ([33] 9.4)**.** $Z_{d,\lambda} = X^{ss}_{d,\lambda}$

1.5.4 The Kirwan-Ness Theorem

Let now $X \subset \mathbb{P}^n$ be smooth. We have defined two stratifications on X, and the Kempf-Ness Theorem says precisely that S_0 is the same in both cases. It turns out that this is true of the whole stratification. To define the relationship, note that if β is a rational point in the chosen positive Weyl chamber, then for some n, $n\beta$ is a primitive integral point, and so it defines a one parameter subgroup λ_β. The assignment

$$\beta \mapsto \sigma(\beta) := (||\beta||, [\lambda_\beta])$$

Is then a map from the index set of the Morse stratification to the index set of the Hesselink stratification.

1.5.23 Theorem (Kirwan [22], Ness [33])**.** *The Morse and Hesselink stratifications coincide:* $S_{[\beta]} = S_{\sigma(\beta)}$.

1.5.24 Remark. It is possible (and indeed necessary to prove the theorem) to provide Morse-theoretic interpretations of the sets $S_{d,\lambda}$ and $Z_{d,\lambda}$. In fact, the set $X_{d,\lambda}$ above coincides with the set Z_β involved in the construction of the Morse stratification.

We now use Theorem 1.5.13, Lemma 1.5.20 and Bialynicki-Birula to conclude that we may write

$$P_t^G(X^{ss}) = P_t^G(X) + \sum_{\beta\neq 0,m} t^{d(\beta,m)} P_t^{L(\beta)}(Z_{d,\lambda_\beta,m})$$

Here we have used the fact that G and K are homotopically equivalent, so that the stratification is G-equivariantly parfect because it is K-equivariantly perfect. The interest in this formula is twofold: first, if the action of G on X^{ss} is free, then $P_t^G(X^{ss}) = P_t(X /\!\!/ G)$. Second, $Z_{d,\lambda}$ are themselves semistable loci for actions on lower dimensional varieties, so that this formula is inductive on dimension; in some examples, they even have natural interpretations which make them natural to compute.

Chapter 2

Generalized Quivers

In the last chapter, we studied how one can construct quotients in algebraic geometry, including their differential-geometric incarnations, and how to extract cohomological information about them. This chapter is in a certain sense an extended example, except that we will be working with affine spaces, rather than projective varieties. For that reason, we have introduced some remarks in the appropriate places, including references to the sources where the proof for our case can be found. The examples we study, those of generalized quivers, are not only interesting in themselves, but stand in relation to important topics in various mathematical areas, as we have remarked in the introduction.

2.1 Generalized Quivers

Our main problem is the following: classify representantions of generalized quivers up to isomorphism; in other words, construct and characterize the quotient $\mathrm{Rep}(\tilde{Q}) /\!\!/ R$.

2.1.1 Definition. Let G be a reductive group, $\mathfrak{g}$ its Lie agebra.

1. A *generalized G-quiver $\tilde{Q}$ with dimension vector* is a pair $(R, \mathrm{Rep}(\tilde{Q}))$ where R is a closed reductive subgroup of G, and $\mathrm{Rep}(\tilde{Q})$ a finite-dimensional representation of R (the *representation space*.) We require the irreducible factors of the representation also to be irreducible factors of $\mathfrak{g}$ as an AdR-module, and the trivial representation to not occur.

2. A generalized quiver *of type Z* is a generalized quiver for which R can be realized as a centralizer in G of some closed abelian reductive subgroup.

3. A *representation of $\tilde{Q}$* is a vector $\varphi \in \mathrm{Rep}(\tilde{Q})$.

This definition is essentially due to Derksen-Weyman [12], though they require generalized quivers always to be of type Z. (This has the important consequence that R is then a Levi subgroup.) Note that the definition above applies equally well to real or to complex Lie groups, as well as to reductive algebraic groups over some field k (in fact, Derksen-Weyman's original setting.) However, apart from an incidental appearance of unitary groups, in this thesis we will restrict to *affine, linearly reductive complex groups*; recall that this implies not only that the groups are smooth, but actually also linear. From the analytic point of view, this implies that the associated complex analytic variety is then a linearly reductive complex Lie group which admits a faithful representation, or equivalently, it is the complexification of any of its maximal compacts. Conversely, any connected Lie group of this kind determines a unique algebraic group satisfying our conditions.

2.2 The case of classical groups

In this section we show how the definition of generalized quivers for some classical groups give interesting geometric objects parametrized by directed graphs. For the general linear case, these are the well-known quiver representations. For the case of linear groups defined by bilinear forms, we get quivers which are 'enriched' in some sense, and we need to impose a compatibility of representations with that extra structure.

2.2.1 Classical quivers

We start by exposing the tight relations between generalized quivers, which are Lie theoretic entities, and the classical theory of quivers, which come from 'graphical interpretations.' In fact, plain quivers, with which we start, were the very motivation for generalized quivers.

2.2.1 Definition. Let **Vec** be the category of finite dimensional complex vector spaces.

1. A *quiver* Q is a finite directed graph, with set of vertices I, and set of arrows A. We let $t : A \to I$ and $h : A \to I$ be the tail and head functions, respectively.
2. A *representation* (V, φ) of Q is a realization of the diagram Q in **Vec**; equivalently, a representation is an assignment of a vector space V_i for each vertex $i \in I$, and a linear map $\varphi_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ for every arrow α.

Given a representation of a quiver, let $n_i = \dim V_i$; we call the vector $\mathbf{n} = (n_i) \in \mathbb{N}_0^I$ the *dimension vector* of the representation. It is clear that two representations of Q can only be isomorphic if they have the same dimension vector. Therefore, we always consider this vector as given and fixed. Then, a representation of Q with a prescribed dimension vector is precisely a choice of an element in

$$\mathrm{Rep}(Q, \mathbf{n}) = \bigoplus_{\alpha \in A} \mathrm{Hom}(V_{t(\alpha)}, V_{h(\alpha)})$$

On this space, we have an action of the product group $G(\mathbf{n}) = \prod \mathrm{GL}(V_i)$ acting by the apropriate conjugation, namely, an element $g = (g_i) \in G(\mathbf{n})$ acts as $g \cdot \varphi = (g_{h(\alpha)} \varphi_\alpha g_{t(\alpha)}^{-1})$. The classical theory of quiver representations is precisely the construction of a suitable quotient for this action.

Consider now the direct sum $V = \bigoplus V_i$; this is called the *total space* of the representation. It is clear that $\mathrm{Hom}(V_i, V_j)$ can be considered as a subspace of $\mathrm{End}(V)$, by extending every element by zero, so that in fact given an arrow φ_α in the representation, $\varphi_\alpha \in \mathrm{End}(V)$. In the same way, any automorphism $g_i \in \mathrm{GL}(V_i)$ can be seen as an element in $G(\mathbf{n})$, extending it by the identity; in fact, the whole group fits in, that is, $G(\mathbf{n}) \subset \mathrm{GL}(V)$. If we let $H = \{\prod \lambda_i \mathrm{id}_i | \lambda_i \in \mathbb{C}^*\}$, we can characterize $R = G(\mathbf{n})$ precisely as the centralizer of H in $\mathrm{GL}(V)$. Further, under the adjoint action of R, the Lie algebra $\mathfrak{g}$ of G decomposes precisely as $\mathfrak{g} = \bigoplus_{i,j} \mathrm{Hom}(V_i, V_j)$. It is now clear that $(H, G(\mathbf{n}), \mathrm{Rep}(Q, \mathbf{n}))$ determines a generalized $\mathrm{GL}(V)$-quiver.

2.2.2 Theorem. *There is a bijective correspondence between generalized* $\mathrm{GL}(V)$*-quivers* $\tilde{Q}$ *of type* Z *and classical quivers* Q *together with dimension vectors* $\mathbf{n}$ *in such a way that* $\mathrm{Rep}(\tilde{Q}) = \mathrm{Rep}(Q, \mathbf{n})$.

2.2.2 Symmetric quivers

We want to consider a natural setting for the concept of orthogonal and symplectic symmetries of representations, the symmetric quivers. Derksen-Weyman [12] established that in fact symmetric quivers completely characterize generalized quivers of type Z for the orthogonal and symplectic group. Since Derksen-Weyman's result in [12] will be instrumental later on, we carefully review it now. In fact, for us it will be important to understand explicitly some isomorphisms that Derksen-Weyman take as implicit, so we will go through their proof carefully – but we want to note that the proof is entirely theirs. Let us first recall the definition of symmetric quiver.

2.2.3 Definition.

1. A *symmetric quiver* (Q, σ) is a quiver Q equipped with an involution σ on the sets of vertices and arrows such that $\sigma t(\alpha) = h\sigma(\alpha)$, $\sigma h(\alpha) = t\sigma(\alpha)$, and that if $t(\alpha) = \sigma h(\alpha)$, then $\alpha = \sigma(\alpha)$.

2. An *orthogonal, resp. symplectic, representation* $(V, C, varphi)$ is a representation (V, φ) of Q that comes with a non-degenerate symmetric, resp. anti-symmetric, quadratic form C on its total space $V_\Sigma = \bigoplus_{i \in Q_0} V_i$ which is zero on $V_i \times V_j$ if $j \neq \sigma(i)$, and such that

$$C(\varphi_\alpha v, w) + C(v, \varphi_{\sigma(\alpha)} w) = 0 \tag{2.1}$$

Note that a dimension vector for an orthogonal representation must have $n_i = n_{\sigma(i)}$; we say that such a dimension vector is 'compatible.'

Let (Q, σ) be a symmetric quiver, and $\mathbf{n}$ a compatible dimension vector with $n = \sum n_i$. Let I_1 be the set of vertices fixed by σ, and I_2 a choice of a unique representative from each orbit of order two under σ. Write $\mathbb{C}^n$ as

$$\mathbb{C}^n = \bigoplus_{i \in I_1} \mathbb{C}^{n_i} \oplus \bigoplus_{i \in I_2} (\mathbb{C}^{n_i} \oplus (\mathbb{C}^{n_i})^*)$$

This splitting determines a quadratic form C by requiring that it be the standard quadratic form on $\mathbb{C}^{n_i}$ for $i \in I_1$, and the standard pairing on $\mathbb{C}^{n_i} \oplus (\mathbb{C}^{n_i})^*$ for $i \in I_2$. This quadratic form determines not only a representative of $\mathrm{O}(n, \mathbb{C})$, but by its definition also a group of graded orthogonal automorphisms:

$$\mathrm{O}(\mathbf{n}) = \left(\prod_{i \in I_1} \mathrm{O}(n_i, \mathbb{C}) \right) \times \left(\prod_{i \in I_2} \mathrm{GL}(n_i, \mathbb{C}) \right)$$

Note that in $\mathrm{O}(n, \mathbb{C})$, the group $\mathrm{O}(\mathbf{n})$ is the centralizer of its center. Also, $\mathrm{O}(\mathbf{n})$ is a subgroup of the group $G(\mathbf{n})$ of symmetries of plain representations of Q.

Denote by $\mathrm{Rep}(Q, C, \mathbf{n})$ the space of orthogonal representations of Q with dimension vector $\mathbf{n}$ and quadratic form C. This is a subspace of $\mathrm{Rep}(Q, \mathbf{n})$ in a natural way. Indeed, we define an anti-involution by sending $(\varphi_\alpha)_{\alpha \in A} \in \mathrm{Rep}(Q, \mathbf{n})$ to $\sigma(\varphi)$ with $\sigma(\varphi)_\alpha = -\varphi^t_{\sigma(\alpha)}$. The space $\mathrm{Rep}_0(Q, C, \mathbf{n})$ is precisely the set of fixed points. With this we see that the group $\mathrm{O}(\mathbf{n})$ acts naturally on $\mathrm{Rep}(Q, C, \mathbf{n})$ as the group of symmetries of orthogonal representations.

We may motivate Derksen-Weyman's Theorem now by noting that the irreducible summands of $\mathrm{Rep}(Q,C,\mathbf{n})$ are isomorphic to summands of the Lie algebra $\mathfrak{o}(n,\mathbb{C})$ of $\mathrm{O}(n,\mathbb{C})$ under the adjoint action of $\mathrm{O}(\mathbf{n})$. In other words, $(\mathrm{O}(\mathbf{n}),\mathrm{Rep}(Q,C,\mathbf{n}))$ is a well defined generalized quiver of type Z for the group $\mathrm{O}(n,\mathbb{C})$. The following theorem is a converse.

2.2.4 Theorem (Derksen-Weyman [12] 2.3)**.** *Let $G=\mathrm{O}(n,\mathbb{C})$ (resp. $\mathrm{Sp}(n,\mathbb{C})$.) Then, to every generalized G-quiver $\tilde{Q}=(R,\mathrm{Rep}(\tilde{Q}))$ of type Z we can associate a symmetric quiver Q, a dimension vector* $\mathbf{n}$*, and a quadratic form C such that R is isomorphic to* $\mathrm{O}(\mathbf{n})$*, and there is an equivariant isomorphism*

$$\mathrm{Rep}(\tilde{Q})\simeq\mathrm{Rep}(Q,C,\mathbf{n})$$

Conversely, every symmetric quiver with dimension vector determines a generalized $\mathrm{O}(n,\mathbb{C})$*-generalized quiver.*

Proof. Following our convention, we prove only the orthogonal case, the symplectic one being no different. Consider the standard representation of G on $W=\mathbb{C}^n$, with $C(\cdot,\cdot)$ the induced symmetric non-degenerate quadratic form. If we denote by H an abelian group for which $R=Z_G(H)$, W decomposes under the action of H into the direct sum

$$W=\bigoplus W_{\chi_i}$$

where W_{χ_i} is the isotypic component of the character χ_i of H (we consider only those characters for which this component is non-empty, so the sum is in fact finite.) The presence of the quadratic form imposes restrictions on this decomposition. In particular, if $v\in W_\chi$ and $w\in W_\mu$, then for all $h\in H$,

$$C(v,w)=C(h\cdot v,h\cdot w)=\chi(h)\mu(h)C(v,w)$$

Therefore, the restriction of the quadratic form to $W_\chi\times W_\mu$ must be zero if $\chi\mu$ is not trivial. Since the form is non-degenerate, it also follows that for any χ in the decomposition, χ^{-1} must also appear, and the restriction to $W_\chi\times W_{\chi^{-1}}$ is non-degenerate. We relabel the character in the decomposition so that $\mu_1,\ldots,\mu_l$ are the characters with $\mu_i^2=1$, and $\chi_1,\ldots,\chi_r$ is the maximal collection of characters that are not their own inverses or of each other. If we denote $V_i=W_{\chi_i}$, and $W_i=W_{\mu_i}$, we see that the quadratic form establishes isomorphisms $V_i^*=W_{\chi_1^{-1}}$ and $W_i^*=W_i$. The decomposition of W is then of the form

$$W=\left(\bigoplus_{i=1}^{i=r}V_i\right)\oplus\left(\bigoplus_{i=1}^{i=r}V_i^*\right)\oplus\left(\bigoplus_{i=1}^{i=l}W_i\right)$$

The centralizer R of H is precisely the group of all orthogonal endomorphisms of W preserving such a decomposition, i.e.,

$$R=\left(\prod_{i=1}^{r}R_i\right)\times\left(\prod_{i=1}^{l}\mathrm{O}(W_i)\right)$$

To describe R_i, note that we just proved that the quadratic form restricts to a symmetric quadratic form on $V_i\times V_i^*$. Then, $R_i\subset\mathrm{O}(V_i\times V_i^*)$ is the subgroup respecting the decomposition: its elements are in fact determined by the transformation at V_i, since the transformation at V_i^* must be dual to it. Hence, we have an isomorphism

$$R = \left(\prod_{i=1}^{r} \mathrm{GL}(V_i)\right) \times \left(\prod_{i=1}^{l} \mathrm{O}(W_i)\right) \tag{2.2}$$

Now, for a vector space V, the adjoint representation of $O(V)$ can be identified with $\Lambda^2(V)$. In the particular case of $V_i \times V_i^*$, the adjoint representation $\Lambda^2(V_i \times V_i^*)$ of $\mathrm{O}(V_i \times V_i^*)$, under the action of R_i splits into irreducible summands

$$\Lambda^2(V_i \times V_i^*) = \Lambda^2(V_i) \oplus E_i \oplus \Lambda^2(V_i^*)$$

Here, $\Lambda^2(V_i)$ is to be seen as the subspace of $\mathrm{Hom}(V_i^*, V_i) \subset \mathfrak{gl}(V_i \times V_i^*)$ which is alternating with respect to the quadratic form C, i.e., condition (2.1) is satisfied for all $\varphi \in \Lambda^2(V_i)$; analogously $\Lambda^2(V_i^*) \subset \mathrm{Hom}(V_i, V_i^*)$. The summand $E_i \subset \mathrm{End}(V_i) \oplus \mathrm{End}(V_i^*)$ is the subspace satisfying the same alternating condition, which amounts to a pair (φ, ψ) such that $\psi = -\varphi^t$. Again, we have an isomorphism $E_i = \mathrm{End}(V_i)$ induced from the isomorphism for R_i in (2.2) (indeed $E_i = \mathrm{Lie} R_i$, which also shows that this piece is in fact irreducible.) This splitting can easily be checked by writing matrices in two-by-two blocks, and putting C in a standard form.

When we extend the analysis to other summands of W, we see the same kind of coupling we found for E_i, where the irreducible pieces are subspaces of sums of Hom spaces. We conclude that under the action of R, the adjoint representation of $\mathrm{O}(n, \mathbb{C})$ splits into factors of the form

$$\Lambda^2(V_i),\ \Lambda^2(V_i^*),\ E_i,\ \Lambda^2(W_i),$$
$$V_{ij} = \mathrm{Hom}(V_i, V_j),\ V_{i\bar{j}} = \mathrm{Hom}(V_i, V_j^*),\ V_{\bar{i}j} = \mathrm{Hom}(V_i^*, V_j),\ W_{ij} = \mathrm{Hom}(W_i, W_j)$$
$$VW_{ij} = \mathrm{Hom}(V_i, W_j),\ VW_{\bar{i}j} = \mathrm{Hom}(V_i^*, W_j)$$

In the second and third line, i and j are to be taken as different. The equalities are isomorphism that we get just as for E_i. For example, $V_{ij} \subset \mathrm{Hom}(V_i, V_j) \oplus \mathrm{Hom}(V_i^*, V_j^*)$ is the subspace of (φ, ψ) such that $\psi = -\varphi^t$.

Finally, we construct the quiver. We draw one vertex for each summand in the decomposition of W. We will label the ones corresponding to V_i as q_i, the ones corresponding to V_i^* as q_i^*, and to W_i as p_i. For the arrows we must look into the representation $\mathrm{Rep}(\tilde{Q})$ that comes with the generalized quiver. Write $\mathrm{Rep}(\tilde{Q}, V) = \bigoplus Z_\alpha$ with Z_α irreducible. Since we assume that the trivial representation does not occur in the representation, we may assume that Z_α is not trivial, so that it must be isomorphic to one of the pieces above. To draw the arrows, we go through these pieces one by one as follows

If $Z_\alpha = \Lambda^2(V_i)$	draw an arrow $g_\alpha = g_\alpha^* : q_i^* \to q_i$
If $Z_\alpha = \Lambda^2(V_i^*)$	draw an arrow $g_\alpha = g_\alpha^* : q_i \to q_i^*$
If $Z_\alpha = E_i$	draw arrows $g_\alpha = q_i \to q_i$ and $g_\alpha^* : q_i^* \to q_i^*$
If $Z_\alpha = \Lambda^2(W_i)$	draw an arrow $g_\alpha = g_\alpha^* : p_i \to p_i$
If $Z_\alpha = V_{ij}$	draw arrows $g_\alpha : q_i \to q_j$ and $g_\alpha^* : q_j^* \to q_i^*$
If $Z_\alpha = V_{\bar{i}j}$	draw arrows $g_\alpha : q_i^* \to q_j$ and $g_\alpha^* : q_j^* \to q_i$
If $Z_\alpha = V_{i\bar{j}}$	draw arrows $g_\alpha : q_i \to q_j^*$ and $g_\alpha^* : q_j \to q_i^*$
If $Z_\alpha = W_{ij}$	draw arrows $g_\alpha : p_i \to p_j$ and $g_\alpha^* : p_j \to p_i$
If $Z_\alpha = VW_{ij}$	draw arrows $g_\alpha : q_i \to p_j$ and $g_\alpha^* : p_j \to q_i^*$
If $Z_\alpha = VW_{\bar{i}j}$	draw arrows $g_\alpha : q_i^* \to p_j$ and $g_\alpha^* : p_j \to q_i$

We now have to define the involution. This is easy with the notation above: switch starred and unstarred elements of the same kind and label: $\sigma(q_i) = q_i^*$, $\sigma(p_i) = p_i$, and $\sigma(g_\alpha) = g_\alpha^*$. It is an easy exercise to verify that with this involution we have a symmetric quiver.

We have left to show that there is a bijection between representations. But this is obvious from our description of the irreducible summands Z_α. A representation of the generalized quiver is a vector $v_\alpha \in Z_\alpha$, which is a space of morphisms, and so it is in fact a representation for the arrows. We have seen that v_α is either a morphism in $\Lambda^2(V_i)$, $\Lambda^2(V_i)^*$, $\Lambda^2(W_i)$, in which case the involution we defined fixes the arrow; or it is an element in the other pieces, and it is in fact a pair of morphisms satisfying the condition for an orthogonal representation. □

2.2.3 Quivers with duality

Let K be a field; though almost everything in this paper can be extended to any characteristic, we will assume char $K \neq 2$ for simplicity. If A is a finite-dimensional simple K-algebra, a theorem of Wedderburn says that there is a division algebra D with K in the center (D central if A is central) for which $A \cong M_n(D)$. For K algebraically closed (which we'll also assume,) the Brauer group is trivial, and so we can always take $D = K$. In these circumstances, a corollary to Wedderburn's theorem is the following: to each semisimple K-algebra A we can associate a graded K-vector space $V = \bigoplus V_i$ (finite-dimensional if A is so) such that A identifies with the graded endomorphisms of V, i.e., $A \cong \prod \mathrm{End}(V_i)$. This splitting can be seen intrinsically by writing a decomposition $1 = \sum e_i$ into central irreducible idempotents of A.

Let now M be a semisimple A-bimodule, and write $M_{ij} = e_i M e_j$. Then, M_{ij} is an A_i-A_j-bimodule, and in fact we have an identification $M_{ij} \cong \mathrm{Hom}(V_i, V_j) \otimes K^{n_{ij}}$, and clearly $M = \bigoplus M_{ij}$. We construct a quiver in the following way: draw one vertex for each index in the grading of V, and draw n_{ij} arrows from i to j. Then, the bimodule M identifies with $\mathrm{Rep}(Q,V)$, the space of representations of Q with total space V. In other words this construction establishes a correspondence between pairs (S,M) of algebra and bimodule, on one hand, and quivers Q together with total spaces V, on the other.

If now V comes with an orthogonal, symplectic or hermitian form which respects the granding (e.g., but not exclusively, if it restricts to a form on each graded piece,) then its algebra of graded endomorphisms, as well as $\mathrm{Rep}(Q,V)$ come with an anti-involution, namely, taking the adjoint with respect to that form. In this section, we define a very general duality on quiver representations precisely through anti-involutions of this kind. The study of the relation between anti-involutions on central simple algebras and quadratic/hermitian forms goes back to Brauer, Noether and Albert, and is by now well established in great generality. In this sense, the starting point of this section can be seen as an extension of this theory into semi-simple algebras through geometric methods.

It was originally Weil's idea that all classical groups could actually be obtained through such a systematic use of anti-involutions. In a strict sense, one needs also a notion of a determinant map, which can be extended to quiver representations in an appropriately relaxed way. The objects we thus obtain have shown up already in the literature in the work of Lopatin-Zubkov [Lopatin and Zubkov] as $\mathfrak{Q}$-mixed quivers. In fact, Bocklandt in [7] already presents a restricted subset of those (namely, supermixed quivers,) through the use of linear involutions. However, here we organize this point of view through a systematic use of the general theory of anti-involutions, which allows us to consider as well the anti-linear case. We further complement it by a study of involutions, which allows us to define 'real forms' of representations, in some sense. Again, we do this through systematically, and so we actually encounter a wider set of objects.

Involutions on K-algebras

We will start by discussing the case of involutions on K-algebras. We know of no source where this material is set down, so we give a few details about the proofs; the results, however, are certainly no new.

Let A be a K-algebra. By an involution on A we will always mean a ring automorphism $\overline{(-)} : A \to A$ of order at most two. Note that such an anti-involution restricts to an automorphism of the center of order at most two, so it necessarily also restricts to an automorphism χ of K. This determines the linearity properties of the anti-involution as a map of algebras: if χ is trivial, then the involution is linear. If χ is non-trivial, we denote the fixed field as k, and the involution is then linear only with respect to k. This realizes K as a separable quadratic field extension, and by Galois theory, we also pinned down the *only* k-automorphism of K. If k is fixed in the context, as we will eventually do, we say the involution is anti-linear, the involution χ being implicit.

Since K is algebraically closed, for A a simple K-algebra, Wedderburn's theorem establishes an isomorphism $A \cong \mathrm{End}(V)$ for some K vector space V, which we will take as fixed. Note in particular that from this isomorphism we conclude that every simple K algebra is central, i.e., its center is precisely K. This allows us in particular to use the Skolem-Noether Theorem, which says that any algebra automorphism of A is inner, i.e., it is conjugation by some invertible element.

Our aim is to characterize involutions on A through extra structure on V. This will come in the form either of a splitting $V = V_1 \oplus V_2$ over K, or a k-structure (again, k is a fixed quadratic subfield of K,) by which we mean a choice of k vector space V_1, and a presentation $V = V_1 \otimes_k K$.

Note first that each of these structures induces an involution on $\mathrm{End}(V)$, by way of an involution $\overline{(-)}$ on V itself. In the case of a K-splitting $V = V_1 \oplus V_2$, we define an involution on V by fixing elements of V_1 and multiplying elements of V_2 by -1 (here is the only point in this section where we make essential use of char $K \neq 2$.) In the case of a k-structure, the involution is induced by the k-automorphism χ of K through the universal property of the tensor product. Either way, the involution on $\mathrm{End}(V)$ is then defined by

$$\overline{f}(v) = \overline{f\overline{v}}$$

2.2.5 Theorem. *Let V be a K vector space. Any linear involution on* $\mathrm{End}(V)$ *is induced by a splitting of V; analogously, any χ-linear involution is induced by a k-structure, for some quadratic subfield k of K.*

Proof. It is enough to consider the case of $V = K^n$. If we consider $v \in V$ as a column vector and send it to the matrix $(0,...,0,v)$, we identify V with the ideal of A consisting of the matrices which are zero except for the last column. This shows that the involution on A induces an involution on V itself. We just saw that this involution on V induces another involution on A, and a simple computation shows that it must be the original one. □

Another case of interest is when we have *both* a linear and an anti-linear involution which commute. In view of this, it is natural to consider the case when A is not simple, but rather a product $A = A_1 \times A_2$ of two simple algebras. Fix k, and suppose such an A has an antilinear involution which does not restrict to each of the factors (if it does, then everything reduces to the results above.)

2.2.6 Proposition. *There is an isomorphism $(A, \overline{(-)}) \cong (A_1 \times \overline{A_1}, \tau)$, where τ is the factor interchange.*

Proof. The splitting $A_1 \times A_2$ determines an orthogonal decomposition of the identity into indecomposable central idempotents: $1 = e_1 + e_2$. The involution does not fix these idempotents, since it does not restrict to each factor by assumption. This implies that it interchanges them. But A_1 and A_2 are simple, so the involution determines an isomorphism $A_1 \to A_2$ of $A_1 \to \overline{A}_2$ according to the involution being either linear or antilinear. □

The interpretation in terms of vector spaces is now simple: since $A_1 \cong \mathrm{End}(V_1)$ and $A_2 \cong \mathrm{End}(V_2)$ for some V_1 and V_2, the lemma gives an isomorphism $V_2 \cong \overline{V_1}$ which induces an isomorphism $A_2 \cong \overline{A_1}$, and an embedding of A into $\mathrm{End}(V_1 \oplus \overline{V_1})$. In other words, this involution is a k-structure on the sum of the two spaces, and not on each one separately.

2.2.7 Remark. Our consideration of the case $A = A_1 \times A_2$ is not completely arbitrary, and neither is it only justifiable by our ultimate purpose. Let A be a K-algebra with an involution. We say that A is simple as an algebra-with-involution if the only ideals left invariant by the involution are the zero ideal and the whole of A; it is central if the only endomorphisms commuting with the involution are the multiplication by the fixed field k of the involution (so, if it is linear, $k = K$.) One can prove that if A is a central simple algebra with involution, then the center of A is a quadratic étale extension of k. This means that either the center is a quadratic extension of k, i.e., it is precisely K, and A itself is a central simple K-algebra; or the center is a product $K \times K$, and since K is algebraically closed, A itself decomposes as a product of central simple algebras. These comments apply equally well to anti-involutions below.

Anti-involutions on K-algebras

An *anti-involution* is a ring anti-automorphism $(-)^* : A \to A$ of order two, i.e., it satisfies $(xy)^* = y^*x^*$ (rather than the opposite,) and $x^{**} = x$. Again, as in the case of involutions, an anti-involution necessarily also restricts to an automorphism χ of K. In the anti-involution case, it is costumary to classify as anti-involutions of the *first kind* those for which this automorphism is trivial, and of the *second kind* those where it is not. For anti-involutions of the second kind, χ is non-trivial, and we denote the fixed field as k. The anti-involution is then linear only with respect to k. This realizes K as a separable quadratic field extension, and by Galois theory, we also pinned down the *only* k-automorphism of K. If k is fixed in the context, as we will eventually do, we say the anti-involution is anti-linear, the automorphism χ being implicit.

Alternatively, anti-involutions are in bijection with isomorphisms of A with its opposite ring A^{op}. Suppose now that A is a K-algebra. Then, anti-involutions of the first kind correspond to K-linear isomorphisms with its opposite algebra; those of the second kind are isomorphisms twisted by the involution on K, i.e., $\varphi(\lambda x) = \chi(\lambda)\varphi(x)$ (so that it is still k-linear.)

Fix an isomorphism $A \cong \mathrm{End}(V)$ given by Wedderburn's theorem. It turns out that there is a relation between anti-involutions on A and hermitian forms on V.

2.2.8 Definition. A χ*-hermitian form* on V is a map $h : V \times V \to K$ which satisfies:

1. $h(x+y,z) = h(x,z)+h(y,z)$ and $h(x,y+z) = h(x,y)+h(x,z)$ for all $x,y,z,\in V$.
2. $h(\lambda x,y) = h(x,\chi(\lambda)y) = \lambda h(x,y)$ for all $x,y \in V$, and $\lambda \in K$.
3. $h(x,y) = \pm\chi(h(y,x))$

Such a form is *non-degenerate* if $h(x,x) \neq 0$ for all x.

This notion, of course, is more general than what is usually meant by 'hermitian,' which corresponds to the case where χ is the k-automorphism of K. But if χ is the identity, a χ-hermitian form is simply a bilinear form; as usual, we will say it is orthogonal if it is non-degenerate and symmetric (+ sign in (3),) and symplectic if it is non-degenerate and skew-symmetric. Whenever k is fixed, since χ is unique, we'll designate a form with the linearity properties in the definition (i.e., (1) and (2)) as *sesqui-linear*.

Given an χ-hermitian form, we can define an anti-involution on $\mathrm{End}(V)$ by sending a endomorphism φ to its adjoint, i.e., the map φ^* such that

$$h(\varphi(v),w) = h(v,\varphi^*(w))$$

The following theorem is a converse of this, and will be used intensively in the sequel. The proof is essentially the same as the case of involutions, cf. [38].

2.2.9 Theorem. *Let V be a K vector space. Any anti-involution on* $\mathrm{End}(V)$ *is induced by some χ-hermitian form on V, and this form is unique up to multiplication by $\lambda \in K^\times$.*

2.2.10 Lemma. *Let $A = A_1 \times A_2$ be a product of simple algebras with an anti-involution which does not restrict to the individual factors. There is an isomorphism $(A,*) \cong (A_1 \times A_1^{op},\tau)$, where τ is the factor interchange.*

A remark analogous to Remark 2.2.7 applies to this case as well.

To interpret this result, note that the essential step was the isomorphism $A_1^{op} \cong A_2$. Such an isomorphism translates into χ-linear pairings between the two vector spaces.

2.2.11 Definition. Fix an involution χ of K. A χ-linear pairing between two K vector spaces V_1 and V_2 is a map $h : V_1 \times V_2 \to K$ satisfying

1. $h(x+y,z) = h(x,z) + h(y,z)$ and $h(x,y+z) = h(x,y) + h(x,z)$ for all $x,y,z, \in V$.
2. $h(\lambda x,y) = h(x,\chi(\lambda)y) = \lambda h(x,y)$ for all $x,y \in V$, and $\lambda \in K$.

It is non-degenerate if for every element of each vector space there is one in the other which pairs to a non-zero scalar.

Again, as above, if χ is trivial, this is just a bilinear pairing. Also, when a quadratic subfield k of K is fixed, χ is left implicit and we speak of a sesqui-linear pairing. One easily sees that a χ pairing induces an isomorphism $\mathrm{End}(V_2) \cong \mathrm{End}(V_1)^{op}$. The converse of this statement follows from the case of simple algebras.

2.2.12 Proposition. *Any isomorphism* $\mathrm{End}(V_2) \cong \mathrm{End}(V_1)^{op}$ *is induced by some* χ*-linear pairing on* $V = V_1 \oplus V_2$ *which respects the splitting.*

Our final result can now be stated as follows, and is a consequence of Lemma 2.2.10.

2.2.13 Theorem. *Any anti-involution on* $\mathrm{End}(V_1) \times \mathrm{End}(V_2)$ *is induced by some* χ*-linear pairing, unique up to multiplication by a scalar in the fixed field.*

Dualized and k-quivers

In the following, we will always consider the quadratic subfield k of K as fixed (and so, also the k-automorphism χ of K.) The results of last section extend easily to a semisimple algebra by iteration. When we add the additional information of a bimodule, quivers naturally appear into the picture. We will see that the (anti)involutions have a nice interpretation on the representations of these quivers.

2.2.14 Remark. It is a good time perhaps to remark that the constructions here can be made in much greater generality. In particular, the results on anti-involutions can be extended to the setting where k is fixed, and rather than a quadratic extension K, we take a central simple algebra with anti-involution E over k. This includes two paradigmatic cases, when E is k itself, or when it is a quaternionic algebra over k (and, of course, K itself as a central simple algebra-with-antiinvolution of the second kind;) it includes many other examples also. The idea is that once we fix E, we can define hermitian forms over E on left E-modules, and then characterize anti-involutions on their endomorphism spaces over E. We would get dualized quivers just as in the K case, and the representations would need the obvious adaptations. For the basic material, see [23].

Definition of k-quivers. Suppose we are given a semisimple algebra A together with an involution; we will call such an involution a (total) k-structure. When discussing involutions, non-trivial linear ones will not be of great interest since they simply correspond to splittings of a vector space. Because of this, *we always assume that the involution is completely anti-linear*. (We could require only that the involution be trivial on any factor on which it was linear, so that we would get a 'partial real structure'; nothing terribly interesting comes out of it.)

Suppose additionally that we have an A-bimodule M with an involution such that $\overline{amb} = \overline{a} \cdot \overline{m} \cdot \overline{b}$, and define

$$k(M) = \{m \in M \,|\overline{m} = m\}$$

A *k-representation* of (A,M) is defined to be an element of $k(M)$. To understand these elements, we make the following definition.

2.2.15 Definition. Let Q be a quiver with set of vertices I, and set of arrows A.

1. A *quadratic quiver* is a pair (Q,σ) where Q is a quiver, and σ is an involution on the sets of vertices I and arrows A (separately) which commutes with the head and tail maps, and such that if $t(\alpha) = \sigma h(\alpha)$, then $\alpha = \sigma(\alpha)$.
2. If k is a quadratic subfield of K, a *representation with k-symmetry* is a representation of the quiver Q together with a K-antilinear involution $\overline{(-)} : V \to V$ such that $\overline{V_i} = V_{\sigma(i)}$, and $\overline{\varphi_\alpha} = \varphi_{\sigma(\alpha)}$ (where here the involution is induced by that of V.)

We want to emphasize the distinction with symmetric quivers. Here, the involution on the quiver commutes with the head and tail maps, rather than interchanging them. If we think of a quiver as a 'category,' the involution in both cases can be thought of as a functor; in the symmetric quiver case it is contravariant, and in the quadratic quiver case it is covariant.

We will always fix a total space V and the antilinear involution on it. We will denote by $\mathrm{Rep}_k(Q,V)$ the space of representations with k-symmetry. The following theorem is a complete characterization of elements of $k(M)$.

2.2.16 Theorem. *There is a bijective correspondence between pairs (A,M) with a k-structure and quadratic quivers with dimension vectors in such a way that $k(M)$ identifies with the space $\mathrm{Rep}_k(Q,V)$ of representations with k-symmetry.*

Proof. A splitting of $A = \prod A_i$ into simple factors is equivalent to a decomposition $1 = \sum e_i$ of the identity into primitive central idempotents (just recall the isomorphism $A_i \cong \mathrm{End}(V_i)$ for each factor.) Because the idempotents are primitive and central, the involution necessarily restricts to an involution on the set $\{e_1,...,e_n\}$; we will denote by σ the corresponding involution on the index set I. Now, if i is fixed by σ, then the involution restricts to the factor A_i; otherwise, it restricts to the product $A_i \times A_{\sigma(i)}$. We now bootstrap Theorems 2.2.5 and 2.2.6 to the corresponding summands of $V = \bigoplus V_i$ to find a k-structure for V. To finish, we can see that that $M_{ij} = e_i M e_j$ is isomoprhic to a number of copies of $\mathrm{Hom}(V_i,V_j)$. This determines a quiver Q: the vertices are the indices i, and the arrows are the irreducible bimodules in M_{ij}. The k structure on V induces an involution on M; a straightforward computation shows that it is the original one as desired. □

Now, we can naturally associate to (A,M) another quiver for which representations 'forget' the larger field K. Indeed, if we have a vector space $V = V_1 \otimes_k K$ where V_1 is a k-vector space, an endomorphism of V is fixed by the induced anti-involution precisely if it restricts to a k-endomorphism of V_1. Then, we define a quiver $\tilde{Q}$ whose vertices and morphisms are the orbits of the involution σ on the symmetric quiver we associated to (A,M) above. To show that this defines a quiver, we need to define head and tail maps. But in fact, σ commutes with the head and tail maps, so we can just define the head and tail of a given orbit as the head and tail of any arrow in it. Note that $\tilde{Q}$ is a plain quiver, i.e., we do not define any extra structure on it. Our goal is the following theorem.

2.2.17 Theorem. *Let V_1 be the k-form of V (i.e., the space fixed by the involution on V.) Then,* $\mathrm{Rep}_k(Q,V) = \mathrm{Rep}(\tilde{Q},V_1)$.

Proof. We consider the pairs (A_0,M_0) corresponding to $\mathrm{Rep}(\tilde{Q},V_1)$, and (A,M) correspoding to $\mathrm{Rep}(Q,V)$. The map we defined above just realizes (A_0,M_0) as the fixed point set of the involutions on (A,M). □

Dualized quivers. Recall that throughout we have fixed a quadratic subfield k of K. We start with a definition.

2.2.18 Definition. Let $V = \bigoplus V_i$ be a graded K vector space, with index set I. A *dualization* of V is a tuple (σ,ε,s,C) where: σ is an involution on I; $s : I \to \{0.1\}$ and $\varepsilon : I \to \{\pm 1\}$ are maps such that $s\sigma = s$ and $\varepsilon\sigma = \varepsilon$; and $C : V \times V \to K$ is an additive function which is linear on the first variable, zero on $V_i \times V_j$ if $j \neq \sigma(i)$, and for $w \in V_i$

1. $C(zv,w) = C(v,\chi^{s_i}(z)w) = C(v,w)$
2. $C(v,w) = \varepsilon_i \chi^{s_i}(C(v,w))$

Note that because of the other assumptions, the last condition is consistent. Essentially, this structure means the following:

1. if i is fixed by σ, then the dualizing structure restricts to S_i, and it is induced by some non-degenerate χ-hermitian form
2. if i is not fixed by σ, $n_i = n_{\sigma(i)}$, the dualizing structure restricts to $S_i \times S_{\sigma(i)}$, and is induced by a non-degenerate χ-linear pairing which comes from a non-degenerate χ-hermitian form on the sum of the two spaces.

From this description, the following theorem is now to be expected.

2.2.19 Theorem. *Let A be a semisimple K algebra and $V = \bigoplus V_i$ the corresponding graded vector space. Then, any anti-involution on A as a k-algebra is induced by some dualizing structure on V.*

Proof. To start with, the splitting $A = \prod A_i$ into simple factors yields a decomposition $1 = \sum e_i$ into primitive central idempotents, and the anti-involution on A necessarily restricts to an involution on the set $\{e_1,...,e_n\}$; we will denote by σ the corresponding involution on the index set.

The rest of the proof is essentially an induction. If i is fixed by σ, then the anti-involution restricts to A_i, and we use Theorem 2.2.9; otherwise, it restricts to an anti-involution on $A_i \times A_{\sigma(i)}$, and we use Theorem 2.2.13. □

Suppose now that we're given an A-bimodule M with an anti-involution $*$ which satisfies $(amb)^* = b^*m^*a^*$ for all $m \in M$ and $a,b \in A$. We will call the pair (A,M) with involutions a *dualmod* (DUalized ALgebra and biMODule.) A *dualized representation* of such a dualmod is a skew-element of M, i.e., an element of

$$D(M) := \{m \in M \,|\, m^* = -m\}$$

The following definitions should now be well motivated.

2.2.20 Definition. Let Q be a quiver with set of vertices I, and set of arrows A.

1. A *dualizing structure* on Q is a a triple (σ, s, ε) where: σ is an involution on the sets of vertices I and arrows A (separately) such that $\sigma t(\alpha) = h\sigma(\alpha)$, and vice-versa, and that if $t(\alpha) = \sigma h(\alpha)$, then $\alpha = \sigma(\alpha)$; $s : I \cup A \to \{0,1\}$ a map satisfying $s = s\sigma$; and $\varepsilon : I \cup A \to \{1,-1\}$ is a map such that $\varepsilon = \varepsilon\sigma$. A quiver with dualizing structure will be called a *dualized quiver*.

2. A *dualized representation* is a representation of Q where the total space comes with a dualization $(\sigma, \varepsilon, s, C)$ on V where σ, ε and λ are the same as those of Q, and satisfying

$$C(\phi_\alpha v, w) + C(v, \phi_{\sigma(\alpha)} w) = 0$$

In the literature, the case when s is identically one is referred to as that of *supermixed quivers*. To emphasize the distinction, we'll refer to the underlying quiver Q as a 'plain quiver.' These definitions are tailor-made for the following theorem.

2.2.21 Theorem. *There is a bijective correspondence between dualmods and dualized quivers with dimension vectors in such a way that $D(M)$ identifies equivariantly with the space of dualized representations.*

The proof is just as in the case of k-structures, noting that M already determines a plain quiver, and then showing that the anti-involution induces a dualizing structure for Q.

Dualized quivers with k-structures.

Suppose that on top of our dualized quiver, we impose a k-structure which (as an involution) commutes with the anti-involutions defining the dualizing structure. The involution will pick out real forms of the total space and of the representation space, and the dualizing structure will then restrict to a k-linear anti-involution on those. In other words, recalling the k-form $\tilde{Q}$ of Q we defined above (cf. Theorem 2.2.17,) this yields the following result.

2.2.22 Theorem. *Let $(Q, \tilde{\sigma})$ be a quiver with quadratic structure, and $\tilde{Q}$ the associated quiver. Then, a dualizing structure on Q which commutes with the quadratic structure induces a supermixed structure on $\tilde{Q}$ in such a way that dualized representations of Q with k-structure identify with supermixed representations of $\tilde{Q}$.*

2.3 Quotients of affine spaces

We want to apply the machinery of the first chapter to obtain cohomological information of the moduli space of representations of generalized quivers. Stricty speaking, there we considered only the projective case, but for linear actions on affine spaces there is indeed an extension of the theory that we can use. We will briefly discuss it, pointing out the points where something essentially new appears. We have tried to make this section self-contained, so that only minimal reference is made to the previous chapter. This means that we give a brief overview of the construction of quotients in this setting, even though it already fits into the framework of that chapter. This also serves to set the notation.

The setting throughout the section is as follows. Let V be a hermitian vector space, a finite dimensional complex vector space with a fixed hermitian form $(\cdot,\cdot)$. The anti-symmetrization of the hermitian form is a (real) symplectic form on V:

$$\omega(v,w) = 2\mathrm{Im}(v,w)$$

We will further denote the coordinate ring of V by $R := \mathbb{C}[V]$.

Let G be a connected, linearly reductive linear algebraic group over the complex numbers, with a fixed maximal compact K. Suppose G acts on V through a regular representation $\rho : G \to \mathrm{GL}(V)$ which restricts to a unitary representation $K \to \mathrm{U}(V)$; we allow ρ to have a kernel Δ.

2.3.1 Geometric Invariant Theory

The quotient

Recall from Chpater 1 that the algebraic construction of a quotient involves a choice of a character $\chi : G \to \mathbb{G}_m$ of G. In general we must always have $\chi([G,G]) = 1$, and we will also require that $\chi(\Delta) = 1$.[1] If we denote $F := \ker\chi$, and $R_{\chi,n}$ the set of elements in R such that $g \cdot r = \chi(g)^n r$ (the *semi-invariants* of weight n), we have

$$R^F = \bigoplus R_{\chi,n} \tag{2.3}$$

where R^F is the ring of F-invariants. We then define the *GIT quotient* to be

$$V /\!\!/_{\chi} G := \mathrm{Proj}\ R_{\chi,n}$$

We can make sense of this definition as follows. With respect to the action of a reductive group F, the ring of invariants R^F is final for the subrings made up of invariant elements of R. By a theorem of Hilbert, it is also an affine ring, which means that Spec R^F is an affine variety with the following universal property: every F-invariant map $V \to Y$ factors through the natural map $\pi : V \to \mathrm{Spec}\ R^F$ induced by the inclusion $R^F \to R$. In other words, Spec R^F is a categorical quotient. In the sense that Proj factors out the $\mathbb{G}_m$ action, we may then see $V /\!\!/_{\chi} G$ in fact as a quotient by G. An important fact to notice, however, is that precisely because this definition involves a projective quotient, there is only a rational map $V \dashrightarrow V /\!\!/_{\chi} G$, which is not generally regular. This means that in a strict sense, we are finding a quotient only for a (Zarisky) open set. In fact, the map is defined only in the open locus of *semistable points*, where the relevant definitions are as follows.

2.3.1 Definition. A point $x \in X$ is

1. χ*-semistable* if there is some semi-invariant which does not vanish at x.

2. χ*-polystable* if its orbit is closed in the set $X^{\chi-ss}$ of semistable points.

[1] We could omit this condition, but it is necessary to ensure the existence of semistable points as we shall see below (cf. also [21].)

3. χ-*stable* if it is polystable and it is simple, i.e., its stabilizer is precisely Δ.

The complement of X^{ss} (i.e. the locus where the map is undefined) is called the *null cone* of X. Note that polystable points are necessarily semistable, and in fact, the GIT quotient parametrizes closed orbits in $X^{\chi-ss}$. Thus, this quotient is in fact an orbit space for polystable points. On stable points as defined, the quotient has nice geometric properties, in particular it is geometric (it actually parametrizes orbits) and smooth. We want to remark that in the literature, it is often only required that the quotient of stable points be geometric, not smoooth; the corresponding condition on stabilizers is just that it contain Δ with finite index.

2.3.2 Remark. We could also have taken the categorical quotient for G, and in fact it is retrieved for the trivial character. This would have the advantage that *every* point would be semistable. However, such a quotient is usually very restrictive, as can already be seen in the simplest examples. The choice of the character goes a great way to remedy this. One interesting thing to note is that, because invariants separate closed orbits, every stable point for the trivial character is stable for *any* character.

Geometrically speaking, the quotient we just defined amounts to the quotient of V as a quasi-projective variety. With the character χ we can make G act on the affine ring $R[z]$ by letting $g \cdot z = \chi(g)^{-1}z$. Again by the theorem of Hilbert, $R[z]^G$ is also an affine ring, and it inherits a grading from $R[z]$ according to the powers of z. A simple observation is the following: an element $r \otimes z^n$ is G-invariant if and only if $r \in R_{\chi,n}$. This implies in particular that $(R[z]^G)_n = R_{\chi,n}$, which means that

$$V /\!\!/_{\chi} G = \operatorname{Proj} R[z]^G$$

Now, $R[z] = R \otimes_k k[z]$ is the coordinate ring of the trivial line bundle L over V, and the co-action of G on $R[z]$ through the character χ corresponds to an action of G on L^{-1} by the formula

$$g \cdot (v,z) = (g \cdot v, \chi(g)^{-1}z)$$

Now, L is the pullback of the anticanonical line bundle $\mathscr{O}(1)$ for any embedding $X \to \mathbb{P}^n$ into some projective space, and so $L^{-1} = \mathscr{O}_V(-1)$ is the blow-up of the corresponding affine cone over V at the origin. Further, $\bigoplus_{n\geq 0} \mathscr{O}(n)$ is the coordinate ring of that cone, so that Proj $R[z]^G$ essentially corresponds to taking the quotient of that affine cone and then projecting down to some projective space. This geometric interpretation explains the so-called *topological criterion*.

2.3.3 Lemma. *Let $v \in V$ be any point, and $\hat{v}$ be an arbitrary lift of v to the total space of L^{-1}. Then,*

1. *The point v is χ-semistable if and only if the closure of the orbit $G \cdot \hat{v}$ is disjoint from the zero section;*

2. *The point v is χ-stable if and only if $G \cdot \hat{v}$ is closed, and its stabilizer is precisely Δ.*

The Hilbert-Mumford criterion

The topological criterion for a GIT quotient shows that we need to consider the existence of certain limit points in the zero section. The Hilbert-Mumford criterion essentially states that we can do that by checking one-dimensional paths generated by elements of G. We will here explain this criterion in the affine case.

Let λ be a one-parameter subgroup (OPS) of G. If $\lambda(t)\cdot x$ does not converge to any point, then clearly for any lift $\hat{x}_0$, the orbit $\lambda\cdot\hat{x}_0$ is disjoint from the zero section. Suppose then that the limit $x_0 = \lim_{t\to 0}\lambda(t)\cdot x$ exists. The point x_0 is necessarily a fixed point of the action of λ (i.e., the action of $\mathbb{C}^*$ through λ,) so that on L^{-1} this action restricts to the fibre over x_0. This action on that fibre is just mulitplication by $\chi(\lambda(t))^{-1} = t^{-\langle\chi,\lambda\rangle}$. In other words, for any lift $\hat{x}_0$ of x_0 we have

$$\lambda(t)\cdot\hat{x}_0 = t^{-\langle\chi,\lambda\rangle}\hat{x}_0$$

Clearly, if $\langle\chi,\lambda\rangle$ is strictly negative, then as $t\to 0$ the orbit $\lambda\cdot\hat{x}_0$ adheres to the zero of the fibre. This is also true for any lift $\hat{x}$ of x, and so the topological criterion implies that x is unstable. The content of the following theorem is that unstability can always be checked in this way.

2.3.4 Theorem ([21] 2.5)**.** *Let $v\in V$. Then,*

1. *The point v is χ-semistable if and only if $\langle\chi,\lambda\rangle\geq 0$ for all one-parameter subgroups λ for which $\lim\lambda(t)\cdot v$ exists.*
2. *The point v is χ-stable if and only if it is semistable, and the only λ for which $\lim\lambda(t)\cdot v$ exists and $\langle\chi,\lambda\rangle = 0$ are in Δ.*

This criterion explains the requirement that $\chi(\Delta) = 1$, for otherwise some one-parameter subgroup of Δ would destabilize every point. It is hard to overstate the importance of this criterion. In many important examples (classical quivers included,) an explicit calculation of this criterion leads to slope-conditions that are very explicit and descriptive.

The number $\langle\chi,\lambda\rangle$ is the *Hilbert-Mumford pairing*. It is worth noting that, in contrast with the projective case, this pairing is independent of the point x. On the other hand, one considers not all one-parameter subgroups, but only those in the set

$$\chi_*(G,v) := \{\lambda\in\chi_*(G) |\ \lim_{t\to 0}\lambda(t)\cdot v \text{ exists}\}$$

We want to use this pairing to classify the instability of points as in the projective case. This relied on important properties of the pairing in that case, the analogues of which are in the following proposition.

2.3.5 Proposition. *Let $v\in V$ and $\lambda\in\chi_*(G,v)$ be arbitrary, and denote $v_0 = \lim_{t\to 0}\lambda(t)\cdot v$. Then,*

1. *$\chi_*(G,g\cdot v) = g\chi_*(G,v)g^{-1}$ for any $g\in G$.*
2. *$g\lambda g^{-1}\in\chi_*(G,v)$ for any $g\in P(\lambda)$.*
3. *$\langle\chi,g\lambda g^{-1}\rangle = \langle\chi,\lambda\rangle$ for any $g\in G$.*

Let $||\cdot||$ be a fixed, G-invariant norm on the space $\chi_*(G)$ of one-parameter subgroups of G. This always exists since, having fixed a maximal torus of G, choosing such a norm is equivalent to choosing a norm on $\chi_*(T)$ invariant under the action of the Weyl group, which is finite. (It is clear from this, however, that such a choice is in general far from unique.) Alternatively, such a norm is also equivalent to a choice of a K invariant norm on $\mathfrak{k}$. We will assume that this norm is *integral*, that is, for all $\lambda \in \chi_*(G)$ we have $||\lambda|| \in \mathbb{Z}$, or equivalently that $||\alpha|| \in \mathbb{Z}$ for all integral weights of $\mathfrak{k}$ (we can always use a multiple of the Killing form on $\mathfrak{k}$.)

2.3.6 Definition. For any $v \in V$, let

$$M_G^\chi(v) = \inf\left\{ m_\chi(\lambda) := \frac{\langle \chi, \lambda \rangle}{||\lambda||}, \lambda \in \chi_*(G,v) \right\} \tag{2.4}$$

Further, let $\Lambda_G(v) \subset \chi_*(G,v)$ be the set of indivisible λ with $m_\chi(\lambda) = M_G^\chi(v)$.

Indivisible here means that λ is not a positive power of another one-parameter subgroup; alternatively, since $\lambda \in \chi_*(T)$ for some maximal torus $T \subset G$, and $\chi_*(T)$ is a lattice, indivisibility means λ is minimal in the lattice. Our first goal is to prove the following:

2.3.7 Proposition. *$M_G^\chi(v)$ is a finite number for all $v \in V$, and $\Lambda_G(v)$ is non-empty.*

Proof. There is a maximal torus T of G for which λ is also a one-parameter subgroup. Through the chosen invariant form, χ determines an element of $\mathfrak{t}$, which we also denote by χ. The number $m_\chi(\lambda)$ is clearly the component of λ along χ. Since the norms of indivisible λ are bounded, so is is the collection of $m_\chi(\lambda)$, so that $M_G^\chi(v)$ is finite. It is also easy to see from this that for the action of T there is certainly a minimum, so that $\Lambda_G(x)$ is non-empty. □

The Hesselink stratification

We now define the Hesselink stratification for V. Note that in the affine case, the instability of a point is completely characterized by the set $\Lambda_G(v)$, since $m_\chi(\lambda)$ is independent of the point v for all λ. The following result of Kempf shows that this set is contained in some adjoint orbit, just as in the projective case.

2.3.8 Lemma (Kempf). *Let $v \in V$ be unstable. Then, $\Lambda_c(v)$ is non-empty, and there is a (unique) parabolic $P(v)$ such that $P(v) = P(\lambda)$ for every $\lambda \in \Lambda_c(v)$. Furthermore, for $\lambda, \lambda' \in \Lambda_c(v)$, we have $\mathrm{Ad}(g)\lambda = \lambda'$ if and only if $g \in P(v)$.*

With this we now define B to be the set of adjoint orbits $[\lambda]$ of one-parameter subgroups for which there is a v such that $\Lambda_G(v) \subset [\lambda]$. This will be the index set for the stratification. We define an (strict) ordering on B by setting $[\lambda] < [\lambda']$ if $m_\chi([\lambda]) < m_\chi([\lambda'])$.

2.3.9 Definition. Let $[\lambda]$ be a conjugacy class of one parameter subgroups of G. The *Hesselink stratum indexed by* $[\lambda]$ is the set

$$S_{[\lambda]} := \{ v \in V^{us} \mid \Lambda_c(v) \subset [\lambda] \}$$

For each $\lambda' \in [\lambda]$, the *blade defined by* λ' is

$$S_{\lambda'} := \{ v \in V^{us} \bigm| \lambda' \in \Lambda_c(v) \} \subset S_{[\lambda]}$$

2.3.10 Theorem (Hoskins [19] 2.16). *The collection $S_{d,[\lambda]}$ is a stratification of X, i.e., X is the disjoint union of the sets, and the ordering on B is such that*

$$\overline{S}_{[\lambda]} \subset \bigcup_{[\lambda]\leq[\lambda']} S_{[\lambda']}$$

Our proofs in the projective case apply to show the following results. First,

2.3.11 Lemma. *For any λ, $S_{[\lambda]} \simeq G \times_{P(\lambda)} S_\lambda$.*

Also, the action of λ defines a $\mathbb{C}^*$-action on V which stabilizes S_λ. In fact, if $p_\lambda : V \to V^\lambda$ is the retration onto the fixed point set, we define

$$Z_\lambda := p_\lambda(S_\lambda)$$

We have that $S_\lambda = p_\lambda^{-1}(Z_\lambda)$. We also conclude that

$$H_G^* = H_{P(\lambda)}^*(S_\lambda) = H_{L(\lambda)}^*(Z_\lambda) \tag{2.5}$$

Just as in the projective case, we can give an intepretation of Z_λ as a semistable locus for the action of λ on a certain subvariety of V.

2.3.2 Symplectic reduction

We may consider V as a Kähler manifold (with constant Kähler form,) and the action of G satisfies the requirements for the construction of a Kähler quotient. A moment map for the action of K is a map $\mu : V \to \mathfrak{k}^*$ which is equivariant with the respect to the coadjoint action of K on $\mathfrak{k}^*$, and which satisfies the condition

$$\langle d\mu, \beta\rangle = \iota(\eta^\dagger)\omega$$

Here, $\langle\cdot,\cdot\rangle$ is the canonical contraction on $\mathfrak{k}^* \times \mathfrak{k}$, $\beta^\dagger$ is the vector field induced by the infinitesimal action of $\beta \in \mathfrak{k}$, and $\iota(\cdot)$ is the contraction with the vector field. Under our assumptions, there is a natural choice of moment map determined by the expression

$$\langle \mu(x), \beta\rangle = (i\beta x, x) \tag{2.6}$$

for $\beta \in \mathfrak{k}$. (Recall that β identifies with an endomorphism of V by the representation.)

The *Marsden-Weinstein (or symplectic) reduction* is the quotient

$$V /\!\!/_\mu G := \mu^{-1}(0)/K \tag{2.7}$$

where on the right we mean the actual orbit space. If K acts with finite stabilizers on $\mu^{-1}(0)$, then $G\mu^{-1}(0)$ is actually an open set. On points where K acts freely, the reduction inherits a Kähler structure. Our work in the previous chapter shows that this notation is abusive, but not innocently so. To first approximation, we may justify the presence of the group G by remarking that the natural map $\mu^{-1}(0)/K \to G\mu^{-1}(0)/G$ is a homeomorphism. The following definitions are also motivated by our discussion in that chapter.

2.3.12 Definition. Let Δ the the intersection of the stabilizers of all points of X. A point $x \in X$ is

1. μ-*semistable* if $\overline{G \cdot x} \cap \mu^{-1}(0) \neq \varnothing$.

2. μ-*polystable* if $G \cdot x \cap \mu^{-1}(0) \neq \varnothing$.

3. μ-*stable* if it is polystable and its stabilizer is precisely Δ.

The set of μ-semistable points is open, usually rather large, and the symplectic reduction parametrizes its closed orbits.

The Morse stratification

Fix a K-invariant product on $\mathfrak{k}$, and recall that the real part of the hermitian product defines a Riemannian metric g on V (in fact, just a positive definite quadratic form on V itself.) Define

$$f(v) = ||\mu(v)||^2$$

This defines a smooth function on V, and its critical points are determined by the equation $i\mu^*(v) \cdot v = 0$; indeed,

$$(df)_v = 2(d\mu(v), \mu(v)) = 2\langle d\mu(v), \mu^*(v)\rangle = 2\omega(i\mu^*(v) \cdot v, v)$$

We may also consider paths of steepest descent from any point $v \in V$, namely the solutions γ to the ODE problem given by

$$\frac{d\gamma_v}{dt}(t) = -\nabla f(\gamma_v(t))$$

with initial value $\gamma_v(0) = v$.

Recall that in defining the Morse stratification in the last chapter, we made two assumptions:

1. The negative gradient flow of f at any point $x \in X$ is contained in some compact neighbourhood of X;

2. The critical set C of f is a topological coproduct of a finite number of closed subsets $C_{[\beta]}$, $\beta \in B$, on each of which f takes constant value, and such that $\beta < \beta$ if $f(C_{[\beta]}) < f(C_{[\beta']})$ is a strict ordering of B.

Compactness is a sufficient condition for these two assumptions. But V is an affine space, so that the conditions need to be proven for this case. The proof of the condition on the flows is due to Harada-Wilkin [17] Lemma 3.3. The assumption on the indices was proven by Hoskins [19] section 3.3.

We will not reproduce the proofs here, but we want to give the description of the indices, from which finiteness will follow. Whereas for a compact symplectic manifold, the image of the moment map is a convex polytope, the imagine of the moment map for an affine space is a polyhedral cone. Indeed, it is the cone generated by the weights of the action of the maximal torus of K on the space V, shifted by the vector χ^* determined by the character χ by the chosen invariant pairing. The indices of the stratification are then the closest point to the origin of the cone generated by some subset of weights. It follows that there are only finitely many indices β, and they are all rational in the sense that for some integer n, $n\beta$ exponentiates to a one-parameter subgroup.

It follows from finiteness of the indexing set B that if C is a connected component of the critical set of f, then $\mu^*(C)$ must lie in a single adjoint orbit of $\mathfrak{k}$ (in fact, map onto it, since the moment map is equivariant with respect to the adjoint action.) To see this, we have just to note that K is compact, so that the adjoint orbits of $\mathfrak{k}$ are all closed. With this in mind, given an element $\beta \in \mathfrak{k}$, we let $C_{[\beta]}$ be the set of critical points v of f with $\mu^*(v)$ conjugate to β by K, equipped with the subspace topology. If we fix a positive Weyl chamber W^+, we conclude that the topological coproduct

$$C =_{\beta \in W^+} C_{[\beta]}$$

is actually isomorphic to the critical set of f. Also, if for any point $v \in V$ we denote the path of steepest descent by $\gamma_v(t)$, assumption 1 guarantees that $\gamma_v(t)$ converges to a unique point v_∞ which is critical for f ([17] Lemmas 3.6, 3.7.) It then makes sense to define

2.3.13 Definition. For any $\beta \in B$, let

$$S_{[\beta]} := \left\{ x \in X \mid x_\infty \in C_{[\beta]} \right\}$$

It follows that V is the disjoint union of all $S_{[\beta]}$. The following now follows straightforwardly from our work in Chapter 1. Recall that we denote by $S_{[\beta],m}$ the component of the Morse stratum with codimension m; this corresponds to the component of the critical set $C_{[\beta]}$ where the Hessian of f has index m.

2.3.14 Theorem. *For a suitable, always existing choice of Riemannian metric, the collection $\{S_{[\beta],m}\}$ defined above is an equivariantly perfect smooth stratification of X over the rationals. Therefore, the equivariant Poincaré polynomial of X is given by*

$$P_t^K(X) = \sum_{\beta,m} t^{d(\beta,m)} P_t^K(S_{[\beta],m})$$

2.3.3 The Kempf-Ness and Kirwan-Ness Theorems

We now establish the relation between the algebraic and symplectic constructions above. Since the various linearizations in principle determine different quotients, it doesn't make sense to compare them to the fixed moment map we defined above for the Marsden-Weinstein reduction. However, the derivative $d\chi$ of the character determines an element in $\mathfrak{k}^*$ which is central, and we define the shifted moment map

$$\mu^\chi = \mu + d\chi$$

The following is the affine version of the Kempf-Ness Theorem **??**, originally proved by King.

2.3.15 Theorem ([21] Thm. 6.1)**.** *A point $x \in V$ is χ-(semi,poly)stable if and only if it is μ^χ-(semi,poly)stable. Consequently, $V /\!\!/_\chi G$ and $V /\!\!/_{\mu^\chi} G$ are homeomorphic.*

Note that we could have alternatively have seen the characters as determining symplectic reductions at different level sets of the same fixed moment map.

This coincidence extends from the quotients to the stratifications. Suppose β is an index for a Morse stratum. There is an integer n such that $n\beta$ is an integral point of $\mathfrak{k}$, and so defines a one-parameter subgroup λ_β. This turns out to index a Hesselink stratum. In fact, Hoskins established the following analogue of the Kirwan-Ness Theorem.

2.3.16 Theorem ([19] Thm. 4.12)**.** *The Morse stratum $S_{[\beta]}$ and the Hesselink stratum $S_{[\lambda_\beta]}$ coincide.*

We now bootstrap Theorem 2.3.14 with (**??**) to establish the formula

$$P_t^G(V^{ss}) = P_t(BG) - \sum_{\beta \neq 0, m} t^{d(\beta,m)} P_t^{L(\beta)}(Z_{\beta,m}) \tag{2.8}$$

2.4 The moduli space of representations of generalized quivers

2.4.1 Linearizations and moment maps for generalized quivers

Let $\tilde{Q} = (R, \mathrm{Rep}(\tilde{Q}))$ be a generalized G-quiver, and fix a character $\chi : G \to \mathbb{G}_m$ as well as maximal compacts $K_R \subset K$ of R and G, respectively. Given an element $\beta \in \mathfrak{k}_R$, we can define two Levi subgroups

$$L_R(\beta) = \{g \in R \mid \exp(it\beta) g \exp(-it\beta) = g\}$$
$$L(\beta) = \{g \in G \mid \exp(it\beta) g \exp(-it\beta) = g\}$$

of R and G respectively. We trivially have $L_R(\beta) \subset L(\beta)$, and so $L_R(\beta)$ has a restricted adjoint action on $\mathfrak{l}(\beta)$. This action coincides with the action on $\mathfrak{g}$, where $\mathfrak{l}(\beta)$ is an invariant subspace. Given a decomposition of $\mathrm{Rep}(\tilde{Q}) = \bigoplus Z_\alpha$ as an Ad R-module, it then makes sense to consider the intersection $Z_\alpha(\beta) = Z_\alpha \cap \mathfrak{l}(\beta)$ of *modules*, since Z_α is isomorphic to a unique irreducible piece of the module $\mathfrak{g}$, and define $\mathrm{Rep}(\tilde{Q}_\beta) := \bigoplus Z_\alpha(\beta)$. We then have:

2.4.1 Lemma. *As defined above, $\tilde{Q}_\beta = (L_R(\beta), \mathrm{Rep}(\tilde{Q}_\beta))$ is a generalized $L(\beta)$-quiver. If $\tilde{Q}$ is a quiver of type Z with $R = Z_G(H)$, then $\tilde{Q}_\beta$ is of type Z and $L_R(\beta) = Z_{L(\beta)}(H)$.*

Note that this new generalized quiver is independent of the particular β we pick to realize $L = L(\beta)$, and we could just have started with an arbitrary Levi $L \subset G$ such that $L_R = L \cap R$ is a Levi of R; we will often speak of $\tilde{Q}_L$ when we do not wish to emphasize β. We can give an interpretation of this result by fixing a faithful representation $K \to U(V)$. We have seen that β determines a grading of V, and that elements of the Levi subgroups above are precisely those that stabilized the splitting. On the other hand, under the identification of $\mathfrak{g}$ as endomorphisms of V, the elements of $Z_\alpha(\beta)$ are precisely those of Z_α which also split as graded endomorphisms of V. We can then make sense of subrepresentations of the original representation, so that the representations of $\tilde{Q}_\beta$ are precisely the splittings of representations of $\tilde{Q}$ according to the action of β. It is useful to keep this interpretation in mind as we discuss stability, and when discussing classical quivers we will be able to see this splitting very explicitly (also in that setting the abelian group's role in the story will become apparent.)

Many of our results will relate stability properties of representations of $\tilde{Q}$ with those of $\tilde{Q}_L$, so we will need to define a suitable linearization for $\tilde{Q}_L$ starting from the character χ. Now, χ is of course a character of L itself, but it is not suitable for the following simple reason: if $L = L(\beta) \neq G$, whereas

$\exp(\beta)$ is in the kernel of the representation of L_R on $\mathfrak{l}$, it is not in the kernel of the representation of R on $\mathfrak{g}$. It is therefore perfectly possible that $\chi(\exp(\beta)) \neq 1$, which by, the Hilbert-Mumford criterion, makes the stability condition for $\tilde{Q}_L$ empty, as we've remarked above. In this setting, we need to correct the choice of character for χ_L on $\mathrm{Rep}(\tilde{Q}_L)$ by projecting out the new elements in the kernel of the representation. For this, we choose an L_R invariant inner product $(\cdot,\cdot)$ on $\mathfrak{l}_R$; since $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{z}(\mathfrak{l}_R)$ is invariant, we use the inner product to choose a complementary space, and denote $p_{\mathfrak{z}(L)}$ the projection onto that complement; we then define χ_L as the character determined by $(\chi_L)_* = \chi_* \circ p_{\mathfrak{z}(L)}$. For the symplectic point of view, if μ_s determines the standard moment map given by the hermitian metric, we have see that the moment map corresponding to χ is $\mu = \mu_s - \chi_*$; we conclude then that the moment map adapted to χ_L is $\mu_L = \mu - \chi_* \circ p_{\mathfrak{z}(L)}$.

2.4.2 Stability of generalized quivers

We will begin by characterizing the convergence for one-parameter subgroups. Let $\tilde{Q} = (R, \mathrm{Rep}(\tilde{Q}))$ be a generalized G-quiver, and fix a character χ of G.

2.4.2 Lemma. *Let $\varphi \in \mathrm{Rep}(\tilde{Q})$ be a representation of $\tilde{Q}$, and let λ be a one-parameter subgroup of R.*

1. *The limit $\lim \lambda(t) \cdot \varphi$ exists if and only if $\varphi_\alpha \in \mathfrak{p}_G(\lambda)$ for all α.*
2. *If it exists, $\varphi_0 := \lim \lambda(t) \cdot \varphi \in \mathrm{Rep}(\tilde{Q}_\lambda) \subset \mathrm{Rep}(\tilde{Q})$.*
3. *If φ_0 is semistable as a representation of $\tilde{Q}$, then φ is semistable.*
4. *Suppose $\langle \chi, \lambda' \rangle = 0$ for all λ' such that $P_R(\lambda') = P_R(\lambda)$, and that φ_0 exists. Then, φ is semistable if and only if φ_0 is a semistable representation of $\tilde{Q}_\lambda$.*
5. *Under the conditions of the last point, if φ is semistable, then φ_0 is a semistable representation of $\tilde{Q}$.*

Proof. (1) is obvious from the definition of $\mathfrak{p}(\lambda)$. To prove (2), it is enough to prove that $\varphi_{0,\alpha} \in \mathfrak{l}_G(\lambda)$. We have

$$\mathrm{Ad}(\lambda(t))\varphi_0 = \mathrm{Ad}(\lambda(t)) \lim_{u\to 0} \mathrm{Ad}(\lambda(u))\varphi = \lim_{u\to 0} \mathrm{Ad}(\lambda(ut))\varphi = \varphi_0$$

Point (3) follows from the fact that the set of unstable points is closed.

Now, point (4) from the fact that all such λ' generate the center of $L(\lambda)$, and so the condition in the theorem ensures that the character χ_L on $\mathrm{Rep}(\tilde{Q}_\lambda)$ is precisely χ. Point (5) follows immediately from this, since the coincidence of the characters guarantees that semistable points of $\mathrm{Rep}(\tilde{Q}_\lambda)$ are sent to semistable representations of $\tilde{Q}$. ☐

One must be careful in interpreting this lemma. Let φ be a representations, and suppose $\lim \lambda(t) \cdot \varphi$ exists. Given point (1) above, and since a parabolic is determined by its strictly dominant elements,[2] one might be tempted to conclude that $\lim \lambda'(t) \cdot \varphi$ exists for any dominant λ' of $P_R(\lambda)$. However, this does not quite follow from (1), because we need λ' do be dominant for $P_G(\lambda)$, *not* $P_R(\lambda)$! There is in

[2] Recall here that dominant elements are dual of dominant characters in $\mathfrak{z} \oplus \mathfrak{c}$.

fact a difference in the components along the center of $P_R(\lambda)$: a dominant for this latter group has an arbitrary component along the center, whereas dominants for $P_G(\lambda)$ do not (they are only arbitrary along the smaller center of $P_G(\lambda)$ itself.) We will use the term G,R-*dominant* to refer to dominants of both $P_R(\lambda)$ and $P_G(\lambda)$; or equivalently, for dominants of $P_G(\lambda)$ which happen to belong to $P_R(\lambda)$.

What is clear is that the stability condition is not really a matter of the one-parameter subgroups, but rather on the parabolics themselves. This is made clear in the next proposition, which resembles stability conditions in gauge theory.

2.4.3 Proposition. *Let $\varphi \in \mathrm{Rep}(\tilde{Q})$ be a representation, and $\mathscr{P}(\varphi)$ be the set of parabolics P of G such that $\varphi \in \mathfrak{p}$, and $P = P(\lambda)$ for some OPS λ of R. Then, φ is*

1. *semistable if and only if for every G,R-dominant weight β of $P \in \mathscr{P}(\varphi)$ we have $\chi_*(\beta) \geq 0$.*
2. *stable if and only if for every G,R-dominant weight β of $P \in \mathscr{P}(\varphi)$ we have $\chi_*(\beta) > 0$.*

Proof. The only thing we need to prove is that it is enough to check that the Hilbert-Mumford pairing can be computed with the derivative of the character. If β is integral, then this is the following computation:

$$\chi(\lambda_\beta(e^s)) = \chi(\exp(s\beta)) = e^{s\chi_*(\beta)} = t^{\chi_*(\beta)}$$

Otherwise, there is always a positive integer n such that $n\beta$ is an integral point, and we have $\langle \chi, \lambda_{n\beta} \rangle = \chi_*(n\beta) = n\chi_*(\beta)$, so that the sign does not change. □

2.4.3 Jordan-Hölder objects

2.4.4 Definition. A pair of parabolic subgroups $(P_R \subset R, P \subset G)$, is *admissible* if $P \cap R = P_R$ and $\langle \chi, \lambda' \rangle = 0$ for all OPS λ' *of the group* R such that $P(\lambda') = P$.

We will need the existence of admissible parabolics below.

2.4.5 Lemma. *If φ is strictly semistable, then $\varphi \in P$ for some admissible pair (P_R, P). Furthermore, there is a minimal such admissible P.*

Proof. If φ is strictly semistable, then there is a one-parameter subgroup λ_β with $\langle \chi, \lambda_\beta \rangle = 0$ for which the limit exists. The restriction that λ_β be strictly dominant for both $P(\lambda_\beta)$ and $P_R(\lambda_\beta)$ is precisely that there is a decomposition

$$\beta = z_\beta + \sum_j \beta_j \alpha_j^G + \sum_i \beta_i \alpha_i$$

where $z_\beta \in \mathfrak{z}(G)$, $z_\beta + \sum_j \alpha_j^G \in \mathfrak{z}(R)$ with α_j^G corresponding to positive combinations of simple weights corresponding to $P_G(\lambda)$, and the α_i are the simple weights corresponding to $P_R(\lambda)$. From the fact that β is strictly dominant for both $P_R(\lambda)$ and $P_G(\lambda)$ it follows that $\beta_j < 0$ and $\beta_i < 0$. We also assumed that χ is trivial on the center of G to ensure the existence of semistable points, so that $\chi_*(z_\beta) = 0$; it follows that if $\chi_*(\beta) = 0$, then $\chi_*(\alpha_j^G) = \chi_*(\alpha_i) = 0$. But every dominant of $P_R(\beta)$ can be expressed in terms of the same α_j^G and α_i, so $P_R(\lambda)$ is admissible.

To prove that there is a minimal admissible, it is enough to remark the following: if P_1 and P_2 are admissible, and defined by sets of simple roots A_1 and A_2, respectively, then $A_1 \cup A_2$ defines an admissible parabolic smaller than both. This is enough since this reduces the semisimple rank, which is finite to start with. □

We will also need the following result:

2.4.6 Lemma. *Suppose (P_R, P) is admissible for φ, and let $P_R = L_R U_R$ and $P = LU$ be the Levi decompositions with $L_R = L \cap R$. If $(P'_R \subset L_R, P' \subset L)$ is admissible for $\varphi_0 = \lim \lambda(t) \cdot \varphi$, then $(P'_R U_R, P'U)$ is admissible for φ.*

Proof. This follows from the fact that for every dominant β of $P_1 U$ that is some integer n such that $n\beta = \beta_1 + \beta'$ where β_1 is a dominant of P_1 and β' is a dominant of P (cf. [35] 3.5.9.) □

Recall that two points of V are *S-equivalent* if their orbit closures intersect, or, alternatively, both closures share a (necessarily unique) closed orbit. Jordan-Hölder objects select a representative in the closed orbit of each S-equivalence class, and can now be constructed along standard lines by an inductive process.

The next result determines the existence of Jordan-Hölder objects for generalized quivers.

2.4.7 Proposition. *Let $\varphi \in \mathrm{Rep}(\tilde{Q})$ be a semistable representation. Then, there is a parabolic subgroups $P_R \subset R$ and $P \subset G$, $P_R = P \cap R$ with Lie algebra $\mathfrak{p}$ such that $\varphi_\alpha \in \mathfrak{p}$, and if $p : P \to L$ is the projection onto a Levi subgroup, $\varphi_{JH} := p_*(\varphi)$ is a stable representation of $\tilde{Q}_L$. Furthermore, under the inclusion as a representation of $\tilde{Q}$, φ_{JH} is polystable and S-equivalent to φ.*

We should remark here that generally speaking, *closed* orbits on the boundary of $R \cdot \varphi$ can always be reached by some one-parameter subgroup. However, the statement in the theorem is stronger insofar as it determines another quiver setting for which the Jordan-Hölder object is stable.

Proof. If φ is stable, nothing needs to be proven. Otherwise, take a minimal admissible parabolic (P_R^{min}, P^{min}) for φ. Let L_R and L be Levis of P_R and P, respectively, with $L_R = L \cap R$, and let $p_{min} : P^{min} \to L^{min}$ be the projection. We claim that the L_R^{min} representation $\varphi_{JH} := p_{min}(\varphi)$ is stable; it is certainly semistable by Lemma 2.4.2. On the other hand, if we assume it is not stable, it admits a pair of parabolic $(P_R \subset L_R^{min}, P \subset L^{min})$. But by Lemma 2.4.6 we have seen that then we can from (P_R, P) construct an admissible pair $(P'_R \subset P_R^{min}, P' \subsetneq P^{min})$, which is a contradiction. As a G-representations, φ_{JH} is certainly S-equivalent to φ, since this projection is the limit of the flow by λ_β for some dominant β of P_R^{min}, and so the closures of the two orbits intersect. Finally, we must prove that again as a G-representation it is polystable, i.e., that the orbit $R \cdot \varphi_0$ is closed. But in fact, this orbit is the image of $L_R(\lambda) \cdot \varphi_0$ under the action of $R/P_R(\varphi)$; the latter is proper and the former is a closed subset of $\mathrm{Rep}(\tilde{Q}_{min})$, so the image is also closed since it is the action of a proper group. □

2.4.8 Remark. Note that we can reach a minimal admissible parabolic by successively considering maximal admissible parabolics, and so arrive at an inductive process which more closely resembles the usual construction of Jordan-Hölder objects. To make this precise, assuming that φ is strictly semistable, choose a maximal admissible parabolic P_1; from Lemma 2.4.2, we conclude that $\varphi_1 := p_1(\varphi)$ is semistable. If it is stable, we are done; otherwise, choose a parabolic P_2 in L_1 that is maximally admissible for φ_1 and repeat. Since the semisimple rank keeps decreasing and also generalized quivers determind by tori are automatically stable, the process must stop at a finite number of steps. That this is the same as above follows again by the construction above for each $P_i \subset L_{i-1}$ of an parabolic $P_i' \subset G$ that is admissible for φ. We conclude that the process stops precisely when P_i' is a minimal admissible.

2.4.9 Corollary. *Two representations φ and φ' are S-equivalent if and only if there is an $r \in R$ such that $\varphi_{JH} = r \cdot \varphi'_{JH}$.*

2.4.4 The local structure of the quotient

We will now investigate the local structure of the quotient, starting with the deformation theory of generalized quivers.

2.4.10 Lemma. *Let φ be a polystable representation. The deformation space N_φ of φ is a representation space of a G generalized quiver $\tilde{Q}_\varphi$ with symmetry group $R_\varphi := \mathrm{Stab}(\varphi)$*

Proof. Since our variety is an affine space, this reduces to the following sequence of vector spaces:

$$0 \longrightarrow \mathrm{ad}(\mathfrak{r})\varphi \longrightarrow \mathrm{Rep}(\tilde{Q}) \longrightarrow N_\varphi \longrightarrow 0$$

Picking a hermitian metric, we can now find a splitting of $\mathrm{Rep}(\tilde{Q})$ which is also a splitting as an $\mathrm{ad}(\mathfrak{r})$-module, which allows us to identify N_φ as a subspace of $\mathrm{Rep}(\tilde{Q})$. On the other hand, the action of $R_\varphi := \mathrm{Stab}(\varphi)$ respects this splitting, so that N_φ is a sum of $\mathrm{Ad}(\mathrm{Stab}(\varphi))$-submodules of $\mathfrak{g}$. □

A immediate application follows by Luna's results [29] III.1.

2.4.11 Theorem. *There is an étale map from a neighbourhood of the origin in $\mathrm{Rep}(\tilde{Q}_\varphi) /\!\!/ R_\varphi$ to a neighbourhood of φ in the quotient $\mathrm{Rep}(\tilde{Q}) /\!\!/ R$.*

Since we're working with complex varieties, recall that this result in particular implies that there is a biholomorphism between neighbourhoods of the points in question in the classical topology.

2.4.12 Remark. Given our characterization of Jordan-Hölder objects above, one might be tempted to try to characterize the Luna strata in terms of certain Levi subgroups (especially since something of the sort can be accomplished for the Hesselink strata, as we'll see below.) However, a more careful analysis easily shows that this is not something we can expect to be possible, as the Luna stratification depends on stabilizers of representations, and a characterization of stabilizers will in general involve reductive subgroups that are smaller than Levis.

2.4.5 Instability and the Hesselink stratification for generalized quivers

We start by proving the following.

2.4.13 Theorem. *Let φ be a representation, and $(P_R(\lambda), P(\lambda))$ be a pair of parabolics with $\varphi \in \mathfrak{p}$, and such that*

1. *If $p : \mathfrak{p} \to \mathfrak{l}$ is the projection onto a Levi, $p(\varphi)$ is a semistable representation of $\tilde{Q}_L$; and*
2. *For every G, R-dominant element $\beta \in \mathfrak{p}$, we have $\chi_*(\beta) > 0$.*

Then, φ is unstable, and $P_R(\lambda) = P(\varphi)$. Conversely, if φ is unstable, the pair $P(\varphi), P(\lambda)$, λ any of the most destabilizing OPS for φ, satisfies the properties above.

Note that here $P(\varphi)$ is the canonical parabolic subgroup associated to φ by the most destabilizing class of OPS; the group $P_G(\varphi)$.

Proof. Let φ be an unstable representation, λ_β a most destabilizing OPS (so that $P(\varphi) = P(\beta)$,) and suppose that $p(\varphi)$ is unstable. In particular, there is an OPS $\lambda_{\beta'}$ of L which destabilizes $p(\varphi)$ and we have

$$\chi_*(\beta+\beta') = \chi_*(\beta) + (\chi_L)_*(\beta') + m_\chi(\beta)(\beta,\beta')$$

From this, we can deduce that if β is taken in the same Cartan subalgebra as β' (which we can always do,) then $\langle \chi, \lambda_\beta\lambda_{\beta'}\rangle \geq \langle \chi, \lambda_\beta\rangle$, which is a contradiction. Therefore, $p(\varphi)$ is semistable. On the other hand, if $\beta = \sum \beta_i\alpha_i$ is the decomposition of β into the simple weights of $P_G(\lambda)$, we have that $\beta_i < 0$ for all i because β is strictly dominant. Property (2) will follow if we prove that λ_β being most destabilizing implies that $\chi_*(\alpha_i) < 0$ (recall that the simple weights are *anti*dominant, not dominant!) But indeed, suppose $\chi_*(\alpha_j) \geq 0$ for some j, and let $\beta' = \sum_{i\neq j}\beta_i\alpha_i$. We certainly have $m_\chi(\beta') \geq m_\chi(\beta)$, so that we will obtain a contradiction if we prove that $\lim \lambda_{\beta'}(t)\cdot\varphi$ exists. To prove this it is enough to remark that $P_G(\beta) \subset P_G(\beta')$.

Conversely, suppose P satisfies (1) and (2). The first property immediately implies that φ is unstable, since any strictly G,R-dominant character yields a destabilizing one-parameter subgroup. Suppose $P_R(\lambda) \neq P(\varphi)$, and let λ_β be a most destabilizing OPS for φ in a maximal torus contained in $P_R(\lambda)$. We then have $\langle \chi_L, \lambda_\beta\rangle > 0$, so that $p(\varphi)$ is not semistable, a contradiction. □

We will in particular need the following easy consequence.

2.4.14 Corollary. *Let φ and φ' be two unstable representations of $\tilde{Q}$. Then, $P(\varphi) = P(\varphi')$ if and only if $\varphi' \in p^{-1}(\mathrm{Rep}(\tilde{Q}_L)^{ss})$ (and vice versa.)*

Another interesting corollary of the proof is the following:

2.4.15 Corollary. *$(P_G(\varphi), P(\varphi))$ is the maximal pair of parabolics with property (1) in the theorem.*

Proof. In the course of the proof we proved the following: if a parabolic satisfies (2) and is not the maximal destabilizing, then it does not satisfy (1). In other words, if P is a parabolic determined by a set A of simple roots which satisfies (1), then necessarily there is $\alpha \in A$ such that $\chi_*(\alpha) \geq 0$, unless $P = P(\varphi)$. Let P' be the parabolic determined by $A - \{\alpha\}$; by definition, $P \subset P'$, and P' satisfies (1). To see this, note that $(\chi_{L'})_* = (\chi_L)_* - m_\chi(\alpha)\alpha^*$ so that they differ only along the dominants colinear with α, and for those we have $(\chi_{L'})_*(\beta) \geq (\chi_L)_*(\beta)$ because both $m_\chi(\alpha)$ and $-(\alpha,\beta)$ are positive. □

The stratification

we can now explicitly characterize the Hesselink strata. In particular, we have the following result.

2.4.16 Corollary. *Let $\beta \in \mathfrak{t}$, and $S_{[\beta]}$ the corresponding Hesselink stratum in the space of representations. Then, $H^*_G(S_{[\beta]}) = H^*_{L(\beta)}(\mathrm{Rep}(\tilde{Q}_{L(\beta)})^{ss})$*

Proof. We have $H^*_G(S_{[\lambda]}) = H^*_{P(\lambda)}(S_\lambda)$. On the other hand, on S_λ we have the action of $\mathbb{G}_m$ through the OPS λ, so that by a theorem of Bialinicky-Birula [6], there is a retraction onto the fixed point set. But it follows from 2.4.13 that this fixed point set is precisely $\mathrm{Rep}^{ss}(\tilde{Q}_{L(\beta)})$, so that we have an isomorphism $H^*_{P(\lambda)}(S_\lambda) = H^*_{L(\beta)}(\mathrm{Rep}(\tilde{Q}_{L(\beta)})^{ss})$. □

In other words, the cohomology of each stratum can be computed in terms of the quotient of representations for a Levi of G, which has lower semisimple rank. We shall see below that in conjunction with Morse theory, this in fact yields a suitable inductive formula for equivariant cohomology.

2.4.6 Morse Theory and the inductive cohomological formula

Let G be a complex reductive Lie group, K_G a maximal compact of G, and $\tilde{Q} = (R, \mathrm{Rep}(\tilde{Q}))$ be a generalized G-quiver; pick a maximal compact K_R of R such that $K_R \subset K_G$. By definition, we have a decomposition $\mathrm{Rep} = \bigoplus \mathfrak{g}_\alpha$ as an R-module, where $\mathfrak{g}_\alpha \subset \mathfrak{g}$ is a complex subspace. Now, given any complex reductive group G, there is a choice of Hermitian metric on its Lie algebra $\mathfrak{g}$ such that K acts unitarily, and this metric restricts to each $\mathfrak{g}_\alpha$. This implies that $\mathrm{Rep}(\tilde{Q})$ is a Hermitian space. Further, since $K_R \subset K_G$, the group K_R acts unitarily on each $\mathfrak{g}_\alpha$, and consequently also on $\mathrm{Rep}(\tilde{Q})$. For the case of a classical group, all of these are induced by a choice of a Hermitian metric for the standard representation.

We can now, therefore, apply the methods we just reviewed above to conclude that there is a naturally defined moment map $\mathrm{Rep}(\tilde{Q}) \to \mathfrak{k}_R^*$ for the action of K_R. Again, denote by f the square of the moment map, i.e., $f(\varphi) = ||\mu(\varphi)||^2$. Let $\beta \in i\mathfrak{k}_R$, and recall the Levi subgroups

$$L_R(\beta) = \{g \in R |\, \exp(i\beta t) g \exp(-i\beta t) = g\}$$
$$L_G(\beta) = \{g \in G |\, \exp(i\beta t) g \exp(-i\beta t) = g\}$$

Using these, we can characterize the critical points of the square of the moment map. Recall for the next proposition that we have systematically defined moment maps for each generalized quiver determined by a Levi. In the case when β determined critical components, we can actually simplify that moment map, like we did for the Hesselink stata. In fact, it is easy to see that the moment map for β indexing a critical stratum is just

$$\mu_\beta = \mu - \beta^*$$

where β^* is the dual of β through the fixed invariant inner product.

2.4.17 Proposition. *Let $\varphi \in \mathrm{Rep}(\tilde{Q})$, and $\beta = \mu(\varphi)$. Then, φ is a critical point of f if and only if φ defines a zero of the moment map as a generalized $L_G(\beta)$-quiver representation of $\tilde{Q}_\beta$.*

Proof. We have seen above that the critical points $\varphi = (\varphi_\alpha)$ are determined by

$$i\beta \cdot \varphi_\alpha = \mathrm{ad}(i\beta)\varphi_\alpha = 0$$

for each α. Since $i\beta \in \mathfrak{k}_R \subset \mathfrak{k}_G$, we immediately conclude that if φ is a critical point of f, $\varphi_\alpha \in \mathfrak{l}_G(i\beta)$ for all α. That this is a zero of the moment map then follows by definition. Conversely, if φ defines a zero of the moment map as a $\tilde{Q}_\beta$-representation, then $\mu^*(\varphi) = \beta$, and since $\varphi \in \mathfrak{l}(\beta)$, we necessarily have $i\mu^*(\varphi) \cdot \varphi = 0$. □

We will abstain from further descriptions of the strata in Morse theoretic terms, since we know it coincides with the algebraic one above, and it is certainly more natural in that language. We make only one further comment: whereas the Hesselink and Morse strata coincide, this is not true at all of the corresponding 'critical strata.' We can see this very clearly from this proposition: the 'critical Hesselink stratum' corresponding to a β (i.e., S_β) is the set of semistable representations of the appropriate $L_G(\beta)$-quiver; the Morse critical stratum, on the other hand, is only the set of polystable such representations.

We want to apply the general results at the beginning of this section to the case of generalized quivers. For that, we first need to compute codimensions for the Morse strata, and it turns out that in this case the codimension is constant for each entire stratum. To see this, recall first Propostion **??**, which implies that

$$\dim S_{[\beta]} = \dim(R/P_R(\beta)) + \dim S_\beta$$

Now, we have identified S_β as $p^{-1}(\mathrm{Rep}(\tilde{Q}_\beta)^{ss})$, where $p : \mathfrak{p}(\beta) \to \mathfrak{l}(\beta)$ is the projection onto the Levi. But this is an open set in $\mathrm{Rep}(\tilde{Q}) \cap \mathfrak{p}(\beta)$, so that the dimension is constant. Further, we identify $\mathfrak{k}/\mathfrak{p}(\beta) \simeq \mathfrak{u}(-\beta)$ and $R/P(\beta) \simeq U(-\beta)$, where $U(-\beta)$ and $\mathfrak{u}(-\beta)$ are respectively the unipotent radical of $P(\beta)$ and its Lie algebra (i.e., the nilpotent radical of $\mathfrak{p}(-\beta)$.) We then find

$$\begin{aligned}\text{codim } S_{[\beta]} &= \dim \mathrm{Rep}(\tilde{Q}) - \dim(\mathrm{Rep}(\tilde{Q}) \cap \mathfrak{p}(\beta)) + \dim(R/P_R(\beta)) \\ &= \dim(\mathrm{Rep}(\tilde{Q}) \cap \mathfrak{u}(-\beta) - \dim \mathfrak{u}(-\beta)\end{aligned}$$

We have therefore shown the following.

2.4.18 Theorem. *Let H be the set of $\beta \in W^+$ indexing the Morse stratification. Then,*

$$P_t^G(\mathrm{Rep}(\tilde{Q})^{ss}) = P_t(BG) - \sum_{\beta \in H} t^{2d(\beta)} P_t^{L(\beta)}(\mathrm{Rep}(\tilde{Q}_\beta)^{ss})$$

where $d(\beta) = \dim(\mathrm{Rep}(\tilde{Q}) \cap \mathfrak{u}(-\beta)) - \dim \mathfrak{u}(-\beta)$.

We finish by noting that following Harada-Wilkin, one can now use the flow of the Morse map to obtain local coordinates at any point. We will not, however, do this at present.

2.4.7 Classical Quivers

This section is an extended example. The results of this section are all established and well known, and we refer to [21] and [37] for more details. We will deduce the results, however, from our general set up for generalized quivers, hoping to exemplify in a familiar setting the meaning of the results we just obtained. The new ingredient here is the flag interpretation of parabolic subgroups.

Stability of classical quivers

We will start by showing how the slope-stability criterion arises naturally from our results. We need to start by fixing a total space $V = \bigoplus V_i$, and we denote $n_i := \dim V_i$ (the vector $\mathbf{n} = (n_i)$ is the *dimension* *vector*.) The group of symmetries determined by such a total space is $\mathrm{GL}(\mathbf{i}) := \prod \mathrm{GL}(V_i)$; a character of this group is of the form

$$\chi(g_i) = \prod (\det g_i)^{\theta_i}$$

for some collection of integers θ_i. This collection can be used to define a 'θ-functional', the value of which at a arbitrary representation $M = (W_i, \varphi)$ is

$$\theta(M) = \sum \theta_i \dim W_i$$

In choosing such character to define semistability, the condition that the character be trivial on the kernel of the representation means in this case that $\sum \theta_i n_i = 0$, or in other words that $\theta(M) = 0$ for every representation with the chosen dimension vector $\mathbf{n}$.

A *subrepresentation* of the representation (V, φ) is a representation determined by a subspace $W \subset V$ that is φ-stable in the sense that $\varphi(W) \subset W$, i.e., the pair (W, φ). The following is the result we are working toward., and is due to King [21].

2.4.19 Proposition. *The representation M is χ-semistable if and only if for any non-trivial subrepresentation M' we have $\theta(M') \leq 0$; it is stable is strict inequality always applies.*

Proof. Given a one-parameter subgroup λ, we know that the limit exists if and only if $\varphi \in \mathfrak{p}_G(\lambda)$. Concretely this means that λ induces a flag $0 \neq V^1 \subset \ldots \subset V^l = V$ of V as a *graded* vector space, determined by its eigenvalues. The condition on φ is then that φ restrict to each V^j, so that the pairs $M_j := (V_j, \varphi)$ are well-defined representations. A computation shows then that

$$\langle \chi, \lambda \rangle = \sum \theta(M_j)$$

This shows half of the theorem using Hilbert-Mumford. The other half comes from just considering the one-term flag for each subrepresentation. □

Suppose we are given an arbitrary collection of integers θ_i, which also induces a linear functional θ on $\mathbb{Z}^I$; let also dim define the functional $(n_i) \mapsto \sum n_i$. Then, the functional $\theta' = (\dim V)\theta + \theta(\mathbf{n}) \dim$ clearly is integral, and satisfies $\theta'(\mathbf{n}) = 0$. In other words, it obeys the condition on characters for the existence of semistable points. The condition on the proposition is now that for a subrepresentation M',

$$\theta'(M') = (\dim V)\theta(M') + \theta(\mathbf{n}) \dim M' \leq 0$$

If we define the *slope* of a representation $M = (V, \varphi)$ with dimension vector $\mathbf{n}$ as

$$s(M) = \frac{\theta(\mathbf{n})}{\dim \mathbf{n}}$$

we get an immediate corollary.

2.4.20 Corollary. *A representation is semistable if and only if $s(M') \leq s(M)$ for every non-trivial subrepresentation $M' \subsetneq M$; it is stable if strict inequality always applies.*

In general, distinct collections θ_i do not necessarily yield different semistability conditions in this way. In fact, semistability is invariant under multiplication of θ by integers, and sums of integer times the dim-functional.

A consequence of this result is the characterization of polystable objects in terms of so-called *Jordan-Hölder filtrations* of semistable representations. First note that given a subrepresentation M' of M as above, one can define the quotient representation $M'' := M/M'$ by taking the quotient of the total spaces, and noting that since φ restricts to M', it also factors through to the quotient. A Jordan-Hölder filtration to M is then a filtration

$$0 \neq M_1 \subsetneq \ldots \subsetneq M_n = M$$

such that the successive quotients M_i/M_{i-1} are stable. This filtration can be inductively constructed as follows: we may assume that the representation is strictly semistable, and we pick a minimal dimensional subrepresentation M' such that $s(M') = s(M)$, and set $M_1 = M'$; this subrepresentation is necessarily stable. If M/M_1 is not itself stable, we repeat the process.

The Jordan-Hölder filtration is not unique, but its associated graded object is so up to isomorphism. Instead of proving this directly, we will show that this in nothing else but an intepretation of Proposition 2.4.7 in terms of flags.

2.4.21 Proposition. *The graded Jordan-Hölder objects coincide with the polystable representatives in Proposition 2.4.7.*

Proof. The result follows from a careful comparison of the procedure just described with the inductive proof of that proposition. In particular, we need to understand the inductive step in terms of subrepresentations. We note that the maximal parabolics are those fixing a minimal flag, i.e., those with only one non-trivial step $0 \neq M' \subset M$. From the computation above of the Hilbert-Mumford pairing for a filtration, we see that such parabolic is admissible if and only if $s(M') = 0$. This is precisely the inductive step in constructing a Jordan-Hölder filtration. □

An analogous study can be made for the instability type of the representation, in terms of the *Harder-Narasimhan filtration*. This is the *unique* filtration

$$0 \neq M_1 \subsetneq \ldots \subsetneq M_n = M \tag{2.9}$$

such that the successive quotients $N_i := M_i/M_{i-1}$ are semistable, and $s(N_1) > s(N_2) > \ldots > s(N_n)$. We call the vector $(s(N_1), \ldots, s(N_n))$ the *Harder-Narasimhan type* of the representation.

2.4.22 Proposition. *Each most destabilizing conjugacy class of OPS determines a unique Harder-Narasimahn type for which the Hesselink stratum $S_{[\beta]}$ of the class is precisely the set of all representations of that type. Further, each blade S_β is determined by further specifying a specific filtration of the total space (the other possible ones are conjugate.) Finally, the retraction Z_β of S_β by Bialinicky-Birula is precisely the set of graded objects for such types with fixed filtration.*

If we use the coincidence of the Morse and Hesselink indices for the strata, a proof follows from Proposition 3.10 in [17], and it was carried out in [19] and [45]. We will obtain an alternative proof by showing that the conditions on the filtration imply the conditions on the parabolic in Theorem 2.4.13.

Proof. Let P be the parabolic subgroup corresponding to the filtration (2.9). The fact that the representation factors through that filtration is equivalent to the fact that $\varphi \in \mathfrak{p}$, and the condition on the semistability of the successive quotients is equivalent to the semistability of the projection $p(\varphi)$ to Levi subalgebra $\mathfrak{l}$. We have then to interpret the condition on the slopes. For some strictly dominant OPS to have the Hilbert-Mumford pairing with the character to be negative, is is necessary that some simple weight also have, so we may assume the one-parameter subgroup is defined by such a simple weight. Now, the simple weight will induce a subfiltration of (2.9), i.e., for some step M_j in that filtration, the filtration of the simple weight α is

$$0 \neq M_j \subsetneq M$$

Futher, the pairing $\langle \chi, -\alpha \rangle = \theta(M)\dim(M_j) - \dim(M)\theta(M_j)$. It follows that $s(N_j) \leq s(M)$, which contradicts the properties of the Harder-Narasimhan filtration. □

We finish the algebraic discussion by remarking that the slice theorem for the case of classical quivers was worked out by LeBruyn-Procesi [25], and it is extremely explicit and computational.

Morse Theory

The results that follow are all due to Harada-Wilkin [17]. We highly recommend that paper not only for the details for these results, but also for a more details on our approach here. We will first compute the level set of the moment map following King [21], using formula (2.6). Fix a total space $V = \bigoplus V_i$; we need to introduce a hermitian metric on $\mathrm{Rep}(Q,V)$, which can be easily done by picking a hermitian metric separately on each V_i, and then defining on each $\mathrm{Hom}(V_i,V_j)$ the metric $(\varphi,\psi) = \mathrm{tr}(\varphi\psi^*)$. This automatically determines a maximal compact of $\mathrm{GL}(\underline{i})$, and the infinitesimal of its Lie algebra is $\beta \cdot \varphi = \beta_{h(\alpha)}\varphi_\alpha - \varphi_\alpha\beta_{t(\alpha)}$. It is now a simple computation to show

$$(\beta\cdot\varphi,\varphi) = \sum_\alpha \mathrm{tr}\left(\beta_{h(\alpha)}\varphi_\alpha - \varphi_\alpha\beta_{t(\alpha)}\right) = \sum_i \mathrm{tr}\left(\beta_i\left(\sum_{h(\alpha)=i}\varphi_\alpha\varphi_\alpha^* - \sum_{t(\alpha)=i}\varphi_\alpha^*\varphi_\alpha\right)\right)$$

The last expression is clearly the standard pairing with β, and so actually gives a formula for the moment map after identification of $\mathfrak{u}(\mathbf{i})$ with its dual. To obtain compatibility with the algebraic side, we know we have to shift this moment map by the derivative of the character. Since this character is determined by the choice of integers θ_i, we actually obtain the equation

$$\sum_{h(\alpha)=i}\varphi_\alpha\varphi_\alpha^* - \sum_{t(\alpha)=i}\varphi_\alpha^*\varphi_\alpha = \theta_i I_i$$

where I_i is the identity on V_i.

Given a quiver representation A, the *Harder-Narasimhan-Jordan-Hölder filtration* of A is the double filtration obtained by first finding the Harder-Narasimhan filtration of A, and then combining it with the Jordan-Hölder filtrations of each factor (which are by definition semistable;) we then speak of the HNJH object associated to A to refeer to the graded object of this double filtration. Given our use of the flags associated to parabolic subgroups to interpret the poly- and instability of representations, the following shouldn't be too surprising.

2.4.23 Proposition ([17] Theorem 5.3)**.** *Let A be a quiver representation. Then, its limit point A_∞ under the square of the moment map is isomorphic to its HNJH-graded object A^{HNJH}.*

The proof of this within our framework is a straightforward application of Proposition 2.4.17 together with Kempf-Ness. The inductive formula in Corollary 2.4.18 is easily seen to correspond to formula (7.10) in [17] (and, as mentioned in that paper, also to Reineke's formula in [36] when all semistable points are stable.)

2.4.8 Supermixed quivers

We now work out explicitly the stability of supermixed quivers using the flag interpretation of parabolics and their dominant elements, just as in the case of classical quivers. The results in this section are, however, original (previous work on supermixed focused only on trivial stability conditions.)

Stability of supermixed representations

Since there is, up to isomorphism, a unique finite dimensional vector space in each dimension, and since over the complex numbers the (anti)symmetry uniquely determines the quadratic form, we may as well, in discussing supermixed quivers, fix a total space V and the quadratic form C. Denote by $\mathrm{Rep}(Q,V,C)$ the space of supermixed representations; since every such representation is in particular a representation of Q, there is a 'forgetful map'

$$f : \mathrm{Rep}(Q,V,C) \to \mathrm{Rep}(Q,V)$$

which is clearly injective. Indeed, we can indetify the first as a subspace of the second explicitly as follows: the quadratic form C induces an involution $* : \mathrm{Rep}(Q,V) \to \mathrm{Rep}(Q,V)$, namely transposition; the first space is then the -1 eigenspace of this involution. The symmetry group of a supermixed quiver can be found in the same way: there is also an adjoint map defined on $\mathrm{GL}(\mathbf{n})$, and the symmetry group is the group $\mathrm{O}(\mathbf{n})$ of elements such that $g^*g = gg^* = 1$ (we're here abusing notation, since the group in general is not a subgroup of the orthogonal group, but this avoids introducing new notation.) This group is isomorphic to a product

$$\mathrm{O}(\mathbf{n}) \simeq \prod_{\substack{i=\sigma(i)\\ \varepsilon_i=1}} \mathrm{O}(V_i) \times \prod_{\substack{i=\sigma(i)\\ \varepsilon_i=-1}} \mathrm{Sp}(V_i) \times \prod_{\substack{[i]\\ i\neq\sigma(i)}} \mathrm{GL}(V_i)$$

where in the last product, we mean to take on factor for each orbit of σ, and not for each i. Denote the set of indices in the first product by O, the second by S, and a fixed set of representatives for the orbits indexing the third product by G.

The map f naturally induces a semistability condition on $\mathrm{Rep}(Q,V,C)$ by restriction of a character χ to $\mathrm{O}(\mathbf{n})$. For such concordance of stability conditions, the map f naturally descends to a map between the quotients of reprsentations. It is a result of Zubkov [46] that for the trivial character on both, this natural map is actually a closed embedding.

However, a look at the above isomorphism of groups shows that these induced characters only give a small subset of possibilities. Instead, take integers θ_i for $i \in O\cup S\cup G$ with $\theta_i = 0,1$ for $i \in O$, and $\theta_i = 0$ for $i \in S$. Such vector of integers parametrizes the complete set of characters of $\mathrm{O}(\mathbf{n})$. Since we also want to apply the symplectic machinery, we will always consider $\theta_i = 0$ for $i \in O \cup S$; the condition on the kernel of the representation implies $\sum \theta_i n_i = 0$.

We will now study the resulting stability properties in a way that is analogous to the case of classical quivers. The first thing to be done is to deduce a slope condition for stability. This can be done exactly like in the classical case: take a one-parameter subgroup λ of $\mathrm{O}(\mathbf{n})$, and consider the associated filtration of V. We define the theta functional just as above, except we take only one summand for each i not fixed by σ, i.e.,

$$\theta(M) = \sum_{i \in G} \theta_i n_i$$

since $\theta_i = 0$ for $O \cup S$. King's computation straightforwardly extends to show that $\langle \chi, \lambda \rangle = \sum \theta(M_l)$, where M_l are the steps in the filtration induced by λ. What we need now is to characterize the subrepresentations determined by parabolics of $\mathrm{O}(\mathbf{n})$. Given a total space V, denote

$$V' = \bigoplus_{i \in O \cup S} V_i \qquad V'' = \bigoplus_{i \in G} (V_i \oplus V_{\sigma(i)})$$

Then, the parabolic of $\mathrm{O}(\mathbf{n})$ induces a filtration $0 \subset V_1 \subset \ldots . V_l \subset \ldots \subset V$ which is induced by filtrations on each of the vertices. In particular, it is a concatenation of filtrations on V' and V'', and we have

- The corresponding flag of V' is isotropic;
- The filtration on each $V_i \oplus V_{\sigma(i)}$ is a 'transposition,' in the sense that the filtration on V_i is arbitrary, and the filtration on $V_{\sigma(i)}$ is dual filtration naturally induced by C.

Given a subrepresentation $M' = (W, \varphi) \subset M$ such that W satisfies this conditions with respect to V, C, we will say it is an *isotropic subrepresentation*, though again we are here appropriating terminology that is specific to the exclusively orthogonal or symplectic case. The result is then

2.4.24 Proposition. *The representation M is θ-semistable if and only if for any non-trivial, isotropic subrepresentation $M' \subsetneq M$ we have $\theta(M') \leq 0$; it is stable if strict inequality always applies.*

2.4.25 Remark. Since the θ-functional only depends on half of the non-fixed vertices, it might be tempting to think that only those determine the stability of a representation. For example, one might want to extract the subquiver determined by those vertices and consider the induced representations of that new quiver by truncation. One should keep in mind, however, that whether a given subrepresentation of this new quiver is a subrepresentation of the old one is controlled also by the orthogonal and symplectic vertices, and so in fact they are always in the background conditioning the representations.

Just as for classical quivers, one can – and should – relax the condition on the numbers θ_i. For a representation M with dimension vector $\mathbf{n}$, define then

$$\dim'(M) = \sum_{i \in G} n_i$$

Since we want to keep $\theta_i = 0$ for $i \in O \cup S$, we can only add multiples of dim$'$, and not multiples of dim. Therefore, the slope of the representation is defined as

$$s(M) = \frac{\theta(M)}{\dim'(M)}$$

Repeating the argument for classical quivers for a collection of θ_i, $i \in G$, arbitrary, we get

2.4.26 Corollary. *A representation M is semistable if and only if $s(M') \leq s(M)$ for all non-trivial subrepresentations $M' \subsetneq M$; it is stable if strict inequality always applies.*

Using this slope condition, we can now formally define Jordan-Hölder objects and Harder-Narasimhan filtrations. To construct the first, suppose the representation M is strictly semistable, and choose a minimally dimensional, non-trivial isotropic subrepresentation $M_1 \subset M$; this subrepresentation determines a maximal parabolic stabilizing the flag

$$0 \neq M_1 \subsetneq M_1^{\perp} \subsetneq M$$

(The last inclusion is strict since the quadratic form is non-degenerate.) The graded representation W_l associated with this filtration is naturally a representation for some Levi subgroup L_1. We form the quotient $M_1^{\perp}/M_1$, which is a well-defined supermixed representation, and repeat the process, finding a chain of representations corresponding to a chain $L_1 \supset L_2 \supset ...$ of successively smaller Levis. This process must stop at some step l, for at some point M_l is necessarily stable or of minimal rank. The associated representation W_l must be stable for as a representation associated with L_l: for otherwise a destabilizing one-parameter subgroup would imply some M_j is not stable. We conclude then

2.4.27 Proposition. *The graded representation W_l obtained by the inductive process above is precisely the Jordan-Hölder object for M.*

We want now to characterize also the Hesselink strata in terms of filtrations; in other words, we want to find the Harder-Narasimhan object for a given representation. Assume M is an unstable representation, and let M_1 an isotropic subrepresentation of maximal slope, and maximal dimension with that property. Again, this fits into a flag

$$0 \neq M_1 \subsetneq M_1^{\perp} \subsetneq M$$

corresponding to some parabolic subgroup P_1. The associated graded object (i.e., the object corresponding to the projection to the Levi subalgebra) is

$$M_{\mathrm{gr},1} = (M_1 \oplus M_1^*) \oplus M_1^{\perp}/M_1$$

where recall that using the quadratic form we get an isomorphism $M_1^* = M/M_1^{\perp}$. This is a splitting as an orthogonal representation, since both $M_1 \oplus M_1^*$ and $M_1^{\perp}/M_1$ are orthogonal representations. The condition on M_1 ensures that $M_1 \oplus M_1^*$ is actually semistable. If $M_1^{\perp}/M_1$ is not, then we repeat the procedure. The result is a filtration

$$0 \neq M_1 \subsetneq ...M_l \subsetneq M_l^{\perp} \subsetneq ... \subsetneq M_1^{\perp} \subsetneq M$$

where we have $\mu(M_1) > ... > \mu(M_l)$, and also that $M_k/M_{k-1} \oplus M_{k-1}^{\perp}/M_k^{\perp} k = 1,...,l-1$ and $M_l^{\perp}/M_l$ are semistable orthogonal representations. In analogy with the plain case, we will refer to this filtration as the Harder-Narasimhan filtration of M. Arguing as in Proposition 2.4.22 we can prove the following proposition.

2.4.28 Proposition. *Each most destabilizing conjugacy class of OPS determines a unique Harder-Narasimhan type for which the Hesselink stratum $S_{[\beta]}$ of the class is precisely the set of all representations of that type. Further, each blade S_β is determined by further specifying a specific filtration of the total space (the other possible ones are conjugate.) Finally, the retraction Z_β of S_β by Bialinicky-Birula is precisely the set of graded objects for such types with fixed filtration.*

An example

We will now apply the inductive formula we deduced above to particular examples of orthogonal representations of the symmetric quiver

$$Q: \qquad 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} \overset{\overset{\gamma}{\circlearrowleft}}{3} \xrightarrow{\sigma(\beta)} \sigma(2) \xrightarrow{\sigma(\alpha)} \sigma(1)$$

Here, vertex 3 and arrow γ are fixed by the involution; we will fix the dimension vector $d = (1,1,n)$. The choice of a stability condition is the choice of two integers θ_1 and θ_2. Stability of representations depends principally on the relative value of these parameters.

- $\theta_1 = \theta_2$: This is the case of the trivial character, and the inexistence of instability renders our formula quite trivial. However, generators for the coordinate ring of the moduli of representations have been computed by Zubkov [46] [47] and Serman [39].

- $\theta_1 < \theta_2$: Since $d_1 = d_2 = 1$, a subrepresentation $M' \subset M$ is destabilizing if and only if it has dimension vector $d' = (0,1,n')$, and this is only possible if $\alpha = 0$. The most destabilizing E is then determined by the maximally isotropic $E_3 \subset V_3$ with $\gamma(E_3) \subset E_3$. It is then a semistable plain representation of the quiver

  $$Q': \qquad 2 \xleftarrow{\beta} \overset{\overset{\gamma}{\circlearrowleft}}{3}$$

  The induced stability condition is trivial: its subrepresentation have slope either zero or $\mu(E') = \theta_2 > 0$. Denote $n_1 = \dim E_3$, and $n_2 = n - 2n_1$; then, in the Harder-Narasimhan splitting $M = E \oplus E^* \oplus D$, $n_2 = \dim D_3$. Further, D is an orthogonal representation of Q' above, satsifying two extra conditions: first, the map $\beta_D : D_3 \to D_2$ is non-zero if and only if $\beta(E_3) = 0$; second, the map γ_D cannot fix any isotropic subspace (by definition of E_3,) and this is equivalent to the representation D actually being orthogonally stable for the trivial character. In other words, if we let

  $$Q'': \qquad \overset{\overset{\gamma}{\circlearrowleft}}{3}$$

then each critical Hesselink stratum Z_β will be either of the form

$$Z_1(n_1,n_2) := \mathrm{Rep}(Q',1,n_1) \oplus \mathrm{Rep}_0^{\mathrm{st}}(Q'',n_2)$$

or

$$Z_2(n_1,n_2) := \mathrm{Rep}(Q'',n_1) \oplus \mathrm{Rep}_0^{\mathrm{st}}(Q',1,n_2)$$

These spaces are in fact the same, but their different notation denotes also a different action of the Levi. The corresponding Levi is just $L(n_1,n_2) = \mathbb{C}^* \times \mathrm{GL}(n_1) \times \mathrm{O}(n_2)$. The codimension of $S_{[\beta]}$ is

$$d(n_1,n_2) = \frac{n^2+n+2}{2} - n_1^2 - n_1 - \frac{n_2(n_2-1)}{2} - 1$$

Conversely, every such combination for two integers n_1 and n_2 with $2n_1+n_2 = n$ give a Hesselink stratum. To apply our inductive formula, we must first choose a unique representative in the conjugacy class $S_{[\beta]}$. This corresponds to the choice of a unique isotropic $E_3 \subset V_3$ up to conjugation, and these are indexed precisely by the dimension of E_3. Therefore, in our inductive formula we will have precisely one summand for each combination (n_1,n_2), i.e.,

$$P_t^{\mathrm{O}(1,1,n)}(\mathrm{Rep}_0(Q,1,1,n)) = P_t(B\mathrm{O}(1,1,n)) + \sum_{\substack{n_1 \\ 2n_1=n}} t^{2d(n_1,n-2n_1)} P_t^{L(n_1,n_2)}(Z_1(n_1,n_2)) + \\ + \sum_{\substack{n_1 \\ 2n_1=n}} t^{2d(n_1,n-2n_1)} P_t^{L(n_1,n_2)}(Z_2(n_1,n_2))$$

Finally, we note that since the stability conditions on Z_1 and Z_2 are trivial, only the cycle part of the quiver contributes to its equivariant cohomology. In other words, if we define

$$Z(n_1) := \mathrm{Rep}(Q'',n_1) \oplus \mathrm{Rep}_0^{\mathrm{st}}(Q'',n-2n_1)$$
$$L(n_1) := \mathrm{GL}(n_1) \times \mathrm{O}(n-2n_1)$$

our formula reduces to

$$P_t^{\mathrm{O}(1,1,n)}(\mathrm{Rep}_0(Q,1,1,n)) = P_t(B\mathrm{O}(1,1,n)) + 2\sum_{\substack{n_1 \\ 2n_1=n}} t^{2d(n_1,n-2n_1)} P_t^{L(n_1)}(Z(n_1))$$

We have therefore reduced the induction to the computation of the equivariant cohomology of previously known cases. In fact, note that these cases are all of the adjoint representation proper.

- $\theta_1 > \theta_2$: Here a destabilizing representation must have dimension vector $d' = (1,0,n_1)$, which implies that the restriction of α and β are both zero. Therefore, the most destabilizing representation is just the choice of a maximal isotropic E_3 fixed by γ, and so this is parametrized by representations of Q'' with dimension vector n_1. The corresponding D in the Harder-Narasimhan splitting is again a stable representation of Q' with dimension vector $(1,n_2)$. The critical Hesselink stratum is then

$$Z(n_1,n_2) := \mathrm{Rep}(Q'',n_1) \oplus \mathrm{Rep}_0^{\mathrm{st}}(Q',1,n_2)$$

The induced stability conditions are again trivial. We can proceed as above to reduce in this way the inductive formula to known cases of the one-loop quiver.

Chapter 3

Generalized Quiver Bundles

In this chapter, we introduce a definition of generalized quiver bundle, and discuss its classification problem. In particular, we translate the classification problem from a holomorphic into a gauge-theoretical setting. We then show how the case of classical groups can be parametrized by classical quiver bundles with extra data, in parallel with the finite-dimensional case. Finally, we give an intepretation of the classification problem in terms of Kähler geomtry, and extract stability conditions generalized quivers bundles.

3.1 Generalized quiver bundles

3.1.1 The definitions

Let X be a compact Kähler manifold, and G a complex reductive Lie group. Fix a generalized G-quiver $\tilde{Q} = (R, \mathrm{Rep}(\tilde{Q}))$. Recall that the action of R on $\mathrm{Rep}(\tilde{Q})$ is identified with the restricted adjoint action on R on $\mathfrak{g}$, the Lie algebra of G.

3.1.1 Definition. A *$\tilde{Q}$-bundle* is a pair (E, φ) where E is a holomorphic principal R-bundle over X, and $\varphi \in \Omega^0(E \times_{\mathrm{Ad}} \mathrm{Rep}(\tilde{Q}))$ is a holomorphic section.

We will speak of *generalized quiver bundles* to refer generically to $\tilde{Q}$-bundles for some $\tilde{Q}$.

We may also introduce a twisted version of this definition as follows. Let $\mathrm{Rep}(\tilde{Q}) = \bigoplus_{\alpha \in A} Z_\alpha$ be the decomposition into irreducible pieces (of course, the Z_α might be isomorphic for different α.) Given $\mathbf{n} \in \mathbb{N}^A$, denote

$$\mathrm{Rep}(\tilde{Q}, \mathbf{n}) = \bigoplus_{\alpha \in A} Z_\alpha \otimes \mathbb{C}^{n_\alpha}$$

Note that there is a natural action of $R(\mathbf{n}) := R \times \prod_{\alpha \in A} \mathrm{GL}(n_\alpha, \mathbb{C})$ on $\mathrm{Rep}(\tilde{Q}, \mathbf{n})$, which by an abuse of notation we will still denote by Ad.

3.1.2 Definition. A *twisting for $\tilde{Q}$-bundles* is a choice of a vector $\mathbf{n} \in \mathbb{N}^A$ together with a holomorphic principal $\prod \mathrm{GL}(n_\alpha, \mathbb{C})$-bundle F over X. A *twisted $\tilde{Q}$-bundle* is a pair (E, φ), where E is a holomorphic principal R-bundle over X, and $\varphi \in \Omega^0((E \times F) \times_{\mathrm{Ad}} \mathrm{Rep}(\tilde{Q}, \mathbf{n}))$ is a holomorphic section.

We must remark here that $E \times F$ denotes the product of E and F in the category of principal bundles over X, and not in the category of complex manifolds. Indeed, the manifold underlying $E \times F$

(the image through the forgetful functor from principal bundles to complex manifolds) is the fibred product $E \times_X F$.

3.1.2 The classification problem

The problem

Fix a generalized G-quiver $\tilde{Q}$, and let $\mathbb{E}$ be a fixed smooth principal R-bundle over X. A generalized quiver bundle can be seen as a choice of a holomorphic structure on $\mathbb{E}$ together with a section of

$$\mathbb{R}\mathrm{ep}(\tilde{Q}) = \mathbb{E} \times_{\mathrm{Ad}} \mathrm{Rep}(\tilde{Q})$$

which is holomorphic with respect to the chosen holomorphic structure. This holomorphic condition can be expressed in terms of an operator

$$\overline{\partial} : \mathbb{R}\mathrm{ep}(\tilde{Q}) \to \mathbb{R}\mathrm{ep}(\tilde{Q})$$

Namely, each holmorphic structure induces such an operator, and the holmorphic sections are precisely elements in the kernel of $\overline{\partial}$. There is an induced action of the automorphism group $\mathscr{G}^{\mathbb{C}} = \mathscr{A}^0(\mathbb{E} \times_{\mathrm{Ad}} G)$ of $\mathbb{E}$ on $\mathbb{R}\mathrm{ep}(\tilde{Q})$, as well as on such operators, which corresponds to isomorphisms of holmorphic structures. This action preserves the holomorphicity condition, and we may think as determining isomorphisms of generalized quiver bundles.

The general task of the study of generalized quiver bundles is the following:

3.1.3 Classification Problem. Classify $\tilde{Q}$-bundles with underlying smooth bundle $\mathbb{E}$ up to isomorphism.

The underlying bundle of a $\tilde{Q}$-bundle (E, φ) is the underlying smooth bundle of E. Since the smooth classification of principal bundles is a topological one, the restriction on the underlying bundle is not as stringent as it might seem. Indeed, the topological classification is discrete.

We will translate this classification problem into one of constructing a gauge-theoretical quotient in a standard way. We will need to fix several data. Fix a maximal compact subgroup K of R, as well as an invariant inner product on $\mathfrak{k}$. Also, fix a hermitian metric on $\mathrm{Rep}(\tilde{Q})$ that is K-invariant (i.e., so that the action of K is unitary,) and a reduction $E_K \hookrightarrow \mathbb{E}$ to K.

Recall [**?**] that there is a bijective correspondence between holomorphic structures on $\mathbb{E}$ and (smooth) connections on E_K with curvature of type $(1,1)$. Under this correspondence, isomorphisms of holomorphic bundles induce an action of the complex gauge group $\mathscr{G}^{\mathbb{C}} := \mathscr{A}^0(\mathbb{E} \times_{\mathrm{Ad}} G)$ on the space $\mathscr{A}$ of connections on E_K.[1] Suppose A is one such connection, and denote by E_A the corresponding holomorphic principal R-bundle, i.e., the bundle $\mathbb{E}$ with the holomorphic structure determined by A. We have seen above that this holomorphic structure determines an operator

$$\overline{\partial}_A : \mathbb{R}\mathrm{ep}(\tilde{Q}) \to \mathbb{R}\mathrm{ep}(\tilde{Q})$$

[1] Note, however, that this induced action is *not* the restriction of the usual action of $\mathscr{G}^{\mathbb{C}}$ on connections on $\mathbb{E}$, since this does not preserve the metric requirement.

the kernel of which is precisely the space of holomorphic sections with respect to the holmorphic structure fixed by A. In other words, $\Omega^0(E_A \times_{\mathrm{Ad}} \mathrm{Rep}(\tilde{Q})) = \ker \overline{\partial}_A$. Importantly, the holomorphicity condition is preserved by the action of the gauge group: if $\overline{\partial}_A \varphi = 0$, then also $\overline{\partial}_{g\cdot A}(g\cdot\varphi) = 0$ for any transformation $g \in \mathscr{G}^{\mathbb{C}}$.

We conclude from this discussion that our classification problem above is equivalent to

3.1.4 Classification Problem. Classify pairs (A,φ) where A is a connection on E_K with curvature of type $(1,1)$, and $\varphi \in \mathscr{A}^0(\mathbb{R}\mathrm{ep}(\tilde{Q}))$ is a smooth section satisfying $\overline{\partial}_A\varphi = 0$, under the action of the complex gauge group $\mathscr{G}^{\mathbb{C}}$.

There is an analogous formulation for the twisted case, by replacing $\mathrm{Rep}(\tilde{Q})$ by $\mathrm{Rep}(\tilde{Q},\mathbf{n})$ throughout, and fixing additional data for the twisting group. We will denote $\mathfrak{Rep}(\tilde{Q}) := \mathscr{A}^0(\mathbb{R}\mathrm{ep}(\tilde{Q}))$, the *representation space*.

The gauge equations

The above is a typical gauge-theoretic moduli problem, to which we can apply standard symplectic machinery. As it is an affine space modelled on $\mathscr{A}^1(\mathrm{ad}\, E_K) = \mathscr{A}^0(T^*X \otimes \mathrm{ad}\, E_K)$, the space of connections $\mathscr{A}$ on E_K is a Kähler space in a natural way. Its tangent space at any point can be canonically identified with that space of sections, so that if I is the almost complex structure on X, then $-I^* \otimes 1$ is an integrable almost complex structure on $\mathscr{A}$. We can define a compatible symplectic structure by

$$\omega_{\mathscr{A}}(\alpha,\beta) = \int_X \Lambda\langle\alpha\wedge\beta\rangle\omega_X \tag{3.1}$$

Here, Λ is the Kähler endomorphism, i.e. the adjoint of wedging with the Kähler form ω_X on X; and $\langle\cdot\wedge\cdot\rangle$ is the combination of the usual wedge product on the T^*X with the fixed K-invariant pairing on $\mathfrak{k}$. Let $\mathscr{A}^{1,1}$ be the space of connections with curvature of typ $(1,1)$. On its smoooth locus, it inherits a Kähler structure by restructing $\omega_{\mathscr{A}}$.

On the other hand, the fixed hermitian metric on $\mathrm{Rep}(\tilde{Q})$ determines a well defined Kähler structure with Kähler form $\omega = 2\mathrm{Im}(\cdot,\cdot)$. This induces a Kähler structure on $\mathfrak{Rep}(\tilde{Q})$. Indeed, if $\pi_R : \mathbb{R}\mathrm{ep}(\tilde{Q}) \to X$ is the structural morphism, then $T_\varphi\mathfrak{Rep}(\tilde{Q}) = \mathscr{A}^0(\varphi^*(\ker d\pi))$. But $(\ker d\pi)_x$ is the space of vertical vectors, which naturally identifies with the tangent space to $\mathrm{Rep}(\tilde{Q})$, and so with $\mathrm{Rep}(\tilde{Q})$ itself. Using this identification, for $\alpha,\beta \in T_\varphi\mathfrak{Rep}(\tilde{Q})$, we define $(I\alpha)_x = i\alpha_x$, and

$$\omega_{\mathfrak{R}} := \int_X \omega_R(\alpha,\beta)\omega_X$$

Summing up, the data fixed above not only establishes a correspondence of classification problems of generalized quiver bundles and pairs (A,φ), but also fixes a Kähler structure on the configuration space $\mathscr{A}^{1,1} \times \mathfrak{Rep}(\tilde{Q})$, the Kähler form being simply the sum $\omega = \omega_{\mathscr{A}} + \omega_{\mathfrak{R}}$ (omitting pullbacks for notational simplicity.) Denote the unitary gauge group by $\mathscr{G} := \mathscr{A}^0(E_K \times_{\mathrm{Ad}} K)$.

3.1.5 Theorem. *The action of the complex gauge group $\mathscr{G}^{\mathbb{C}}$ on $\mathscr{A}^{1,1} \times \mathfrak{Rep}(\tilde{Q})$ is holomorphic, and preserves the subspace defined by the condition $\overline{\partial}_A\varphi = 0$. Further, the action of the compact gauge* *group $\mathscr{G}$ is hamiltonian, with moment map*

$$\mu(A,\varphi) := \Lambda F_A + \mu_R(\varphi)$$

Here F_A is the curvature of A, and μ_R is the moment map for the action of K induced by the fixed hermitian metric.

The theory of Kähler quotients determines now a whole set of quotients for this action, parametrized by central elements $c \in \mathfrak{k}$. Recalling that each such element determines a constant section of $\mathscr{A}^0(\mathrm{ad}E_K)$, these are just

$$\mathscr{M}(c) := \{(A,\varphi) |\ \overline{\partial}_A\varphi = 0, \mu(A,\varphi) = c\}/\mathscr{G}$$

The equation on the moment map is in fact a system of partial differential equations known as the *gauge equations*. Because we are working with an infinite-dimensional setting, the construction of this moduli space is rather technically involved. We wil not be using the moduli spaces themselves, so we will not have much to say about their construction; the interested reader may consult [**?**] [**?**]. Instead, we will focus on the stability conditions derived from the condition on the moment map.

Additional parameters

There is a variation on the construction above that will be interesting for us, since it allows for additional parameters on the moduli spaces. This in principle makes a difference for the stability conditions, as we shall see below for particular cases; indeed, the additional parameter of the type we introduced here first appeared for quiver bundles in [1].

Suppose that the Lie algebra of K splits as a direct sum $\mathfrak{k} = \bigoplus_{i\in I}\mathfrak{k}_i$. Because the adjoint action of K is inner, there is an induced splitting

$$\mathrm{ad}E_K = \bigoplus_{i\in I} E_K(\mathfrak{k}_i)$$

where we denote $E_K(\mathfrak{k}_i) := E_K \times_{\mathrm{Ad}} \mathfrak{k}_i$. Accordingly, if we pick an origin for $\mathscr{A}$ and use the commutativity of the tensor product with colimits, there is a splitting of the space of connections as $\mathscr{A} = \prod_{i\in I}\mathscr{A}_i$; but one sees easily that this splitting is independent of the choice of identification of $\mathscr{A}$ with $\mathrm{ad}E_K$. If we assume the invariant form on $\mathfrak{k}$ respects the splitting, then formula (3.1) now makes sense in each $\mathscr{A}_i$ separately, defining symplectic forms ω_i with $\omega_{\mathscr{A}} = \sum\omega_i$. However, we may weigh the sum using positive numbers a_i, $i \in I$, defining

$$\omega_{\mathscr{A},a} = \sum_{i\in I} a_i\omega_i$$

The resulting system of equations defining the moduli spaces are then indexed by the vector $a = (a_i)$ and a central element $c \in \mathfrak{k}$, and given by

$$\begin{aligned} a_i\Lambda F_i + \mu_R(\varphi)_i &= c \\ \overline{\partial}_A\varphi &= 0 \end{aligned}$$

An important case of this setting is when the group R itself splits as a product of Lie groups. This in fact will be the case in the examples below, and the one for which we will prove a Hitchin-Kobayashi correspondence.

3.2 The classical groups

In analogy with the finite dimensional case, which we treated in section 2.2, in this section we characterize generalized quiver bundles for classical groups in terms of quiver bundles with extra data.

3.2.1 Quiver bundles

We begin with the standard definitions.

3.2.1 Definition. Let X be a compact Kähler manifold, and Q be a quiver with vertex set I, arrow set A and head and tail map $h,t : A \to I$, respectively.

1. A *Q-bundle* is a representation of Q in the category of holomorphic vector bundles over X: a vector bundle V_i for each vertex $i \in I$, and a linear bundle morphism $\varphi_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ for every $\alpha \in A$.
2. A *twisting* for Q is a choice of a holomorphic vector bundle M_α over X for each arrow in Q.
3. A *twisted Q-bundle* is a choice of a holomorphic vector bundle V_i for each $i \in I$, and a holomorphic morphism $\varphi_\alpha : V_{t(\alpha)} \otimes M_\alpha \to V_{h(\alpha)}$ for each $\alpha \in A$.

Suppose (V_i, φ) is a representation, and let $\mathbf{m} = (m_i) \in \mathbb{N}_0^I$ be the *rank vector of the representation*: $m_i = \mathrm{rk} V_i$. The frame bundle E_i of V_i is a principal $\mathrm{GL}(m_i, \mathbb{C})$-bundle, and $V_i = E_i^{\mathbb{C}} \times_\rho \mathbb{C}^{m_i}$, where ρ is the defining representation of $\mathrm{GL}(m_i, \mathbb{C})$. The product of principal bundles $E = \prod E_i$, which is a principal bundle over X with structure group $\mathrm{GL}(\mathbf{m}) := \prod \mathrm{GL}(m_i, \mathbb{C})$.[2] If we denote $V := \bigoplus V_i$, we have $V = E \times_\rho \mathbb{C}^m$, where $m = \sum m_i$. We refer indifferently to $E^{\mathbb{C}}$, V and $\mathbb{V}$ as the *total space of the representation*.

Analogously, if $\mathbf{n} = (n_\alpha) \in \mathbb{N}^A$ with $n_\alpha = \mathrm{rk} M_\alpha$ is the *dimension vector of the twisting*, the frame bundle F_α of M_α is a principal $\mathrm{GL}(n_\alpha, \mathbb{C})$-bundle, and $M_\alpha = F_\alpha \times_\rho \mathbb{C}^{n_\alpha}$. The *total twisting space* is then either of $F = \prod F_\alpha$ or $M = \bigoplus M_\alpha$, noting that $M = F \times_\rho \mathbb{C}^n$, where $n = \sum n_\alpha$.

Conversely, any principal $\mathrm{GL}(\mathbf{m})$-bundle E splits $E = \prod E_i$ into factors E_i of principal $\mathrm{GL}(m_i, \mathbb{C})$-bundles. In this way, it is clear that the total spaces are indifferently determined by the vector bundle data of the representation, or the corresponding principal bundle.

Having fixed total spaces $V = \bigoplus V_i$ and $M = \bigoplus M_\alpha$, the representation is determined by the vector $\varphi \in \mathfrak{Rep}(Q, V, M)$, where

$$\mathfrak{Rep}(Q, V, M) := \bigoplus \mathrm{Hom}(V_{t(\alpha)} \otimes M_\alpha, V_{h(\alpha)})$$

is the *representation space*. Now, $\mathrm{Hom}(V_{t(\alpha)} \otimes M_\alpha, V_{h(\alpha)}) = \Omega^0 \left(\mathbb{H}\mathrm{om}(V_{t(\alpha)} \otimes M_\alpha, V_{h(\alpha)}) \right)$ is just the space of global section of the bundle of twisted homomorphisms (note that the $\mathbb{H}$om-bundle has a

[2] Here we really mean the product of the bundles as principal bundles, not of the underlying manifolds. In fact, in terms of the underlying manifolds, the product bundle is a fibred product.

holomorphic structure fixed by those of the vector bundles in its arguments.) This latter bundle can be written as the bundle associated to $E_{t(\alpha)} \times E_{h(\alpha)} \times F_\alpha$ by conjugation on $\mathrm{Hom}(\mathbb{C}^{m_{t(\alpha)}} \otimes \mathbb{C}^{n_\alpha}, \mathbb{C}^{m_{h(\alpha)}})$ of the group $\mathrm{GL}(m_{t(\alpha)}, \mathbb{C}) \times \mathrm{GL}(m_{h(\alpha)}, \mathbb{C}) \times \mathrm{GL}(n_\alpha, \mathbb{C})$. In other words, if we let $\mathrm{Rep}(Q, \mathbf{m}, \mathbf{n})$ be the finite-dimensional space of twisted Q-representations, there is an induced representation ρ of $\mathrm{GL}(\mathbf{m})$ defined as

$$\rho\left((g_i)_i \times (h_\alpha)_\alpha\right)(\varphi_\alpha)_\alpha = (g_{h(\alpha)} \circ \varphi_\alpha \circ (g_{t(\alpha)} \times h_\alpha)^{-1})_\alpha$$

We may now write

$$\mathfrak{Rep}(Q, V, M) = \Omega^0\left((E \times F) \times_\rho \mathrm{Rep}(Q, \mathbf{m}, \mathbf{n})\right)$$

Recall from the finite-dimensional theory that there is a bijective correspondence between generalized quivers $\tilde{Q}$ of type Z for $G = \mathrm{GL}(n, \mathbb{C})$, and quivers Q with dimension vector $\mathbf{n}$ such that $n = \sum n_i$ in such a way that the representation spaces are equivariantly isomorphic. The outcome of the discussion above is the following extension to the bundle case.

3.2.2 Proposition. *Let Q be a quiver with dimension vector* $\mathbf{n}$*, and let $\tilde{Q}$ the corresponding generalized quiver for* $\mathrm{GL}(n, \mathbb{C})$*. The map sending a twisting M of Q to the twisting F of $\tilde{Q}$ is a bijections. Further, the assignment $(V = \bigoplus V_i, \varphi) \mapsto (E = \prod E_i, \varphi)$ extends to an equivalence between the categories of twisted Q-bundles and twisted $\tilde{Q}$-bundles.*

The parameters

The case of quiver bundles is one in which we can introduce additional parameters for the moduli spaces, as explained above. They were first introduced in this case by Álvarez-Consul and García-Prada in [1]. We know the structure group of these representations to be a product

$$\mathrm{GL}(\mathbf{m}) = \prod_{i \in I} \mathrm{GL}(m_i, \mathbb{C})$$

Accordingly, in addition to the central parameter $c \in \mathfrak{k}$, we may introduce positive real numbers p_i, $i \in I$, as parameters. On the other hand, the central parameter itself may be written as $(c_i I_i)_{i \in I}$ where $c_i \in \mathbb{R}$ and I_i is the identity on V_i (recall X is assumed compact.)

Consider now the section $\varphi \in \mathfrak{Rep}(Q, \mathbf{m}, n)$. The moment map on this space is fibrewise the moment map of $\mathrm{Rep}(Q, \mathbf{n})$, which recall may be chosen to be

$$-i\mu_i(\varphi) = \sum_{h(\alpha)=i} \varphi_\alpha \varphi_\alpha^* - \sum_{t(\alpha)=i} \varphi_\alpha^* \varphi_\alpha =: [\varphi, \varphi^*]_i$$

where $i = \sqrt{-1}$, a duplication which we hope will not cause confusion (the letter i for the vertices is always used as an index.) Therefore, the components of the gauge equations on a pair (A, φ) may be written

$$ia_i \Lambda F_i + [\varphi, \varphi^*]_i = c_i I_i$$
$$\overline{\partial}_A \varphi = 0$$

Again, F_i is the component of the curvature of A along V_i.

3.2.2 Symmetric quiver bundles

We now want to consider the case of symmetric quivers. We start with a preliminary observation. Fix a symmetric quiver (Q,σ), and a twisting M for Q. Given a morphism $\varphi_\alpha : V_{t(\alpha)} \otimes M_\alpha \to V_{h(\alpha)}$, the transpose is a map

$$\varphi_\alpha^t : V_{h(\alpha)}^* = V_{\sigma(h(\alpha))} \to V_{t(\alpha)}^* \otimes M_\alpha^* = V_{\sigma(t(\alpha))} \otimes M_\alpha^*$$

or, equivalently, $\varphi_\alpha^t \in \mathrm{Hom}(V_{t\sigma(\alpha)} \otimes M_\alpha, V_{h\sigma(\alpha)})$. For this to be comparable to $\varphi_{\sigma(\alpha)}$, then, we must necessarily have $M_\alpha = M_{\sigma(\alpha)}$. In light of this, we make the following definition:

3.2.3 Definition. Let (Q,σ) be a symmetric quiver.

1. A *symmetric twisting* for (Q,σ) is a twisting of Q which satsfies $M_\alpha = M_{\sigma(\alpha)}$ for all $\alpha \in A$.
2. Let M be a symmetric twisting. An *M-twisted orthogonal (resp. symplectic) (Q,σ)-bundle* is an M-twisted Q-bundle (V,φ) together with a non-degenerate quadratic form $g \in \mathscr{A}^0(S^2V^*)$ (resp., $g \in \mathscr{A}^0(\Lambda^2 V^*)$) which restricts to zero on $V_i \otimes V_j$ if $j \neq \sigma(i)$, and such that

$$g(\varphi_\alpha(v \otimes m), w) + g(v, \varphi_{\sigma(\alpha)}(w \otimes m)) = 0$$

   for all $x \in X$, $v \in \mathbb{V}_{t(\alpha),x}$, $w \in \mathbb{V}_{h(\alpha),x}$, and $m \in \mathbb{M}_{\alpha,x}$.

We will often omit σ from the notation, as we will assume it fixed. Below we will consider the orthogonal case only, but the proofs have been written in a way that allows for the symplectic case after obvious small changes.

3.2.4 Remark. Let (V,φ) be an orthogonal representation of Q. Fix, for a moment, an arrow α, let $i = t(\alpha)$ and $j = h(\alpha)$, and assume for simplicity that $j = \sigma(i)$. From the condition on the twisting we have that $\varphi_\alpha \in \mathrm{Hom}(V_i,V_j) \otimes M_\alpha$, and $\varphi_{\sigma(\alpha)} \in \mathrm{Hom}(V_j,V_i) \otimes M_\alpha$. The condition on the form just requires that g restrict to an orthogonal form on $V_{ij} := V_i \times V_j$, which picks a fibrewise orthogonal group $\mathrm{O}_{ij} := \mathrm{O}(V_{ij})$. On this product, φ_α and $\varphi_{\sigma(\alpha)}$ determine fibrewise an element $\overline{\varphi}_\alpha : \mathrm{End}(V_{ij}) \otimes M_\alpha$, and the requirement on the morphisms is essentially that $\overline{\varphi}_\alpha \in \mathfrak{o}_{ij} \otimes M_\alpha$. This provides a motivation for our general definition of twisting for generalized quivers.

We want now to generalize Derksen-Weyman's theorem in chapter 2 to our bundle case. Let (V,φ) be an orthogonal representation with dimension vector $\mathbf{n}$. We have written V as a bundle associated to a principal $\mathrm{GL}(\mathbf{n})$-bundle E. In general, an orthogonal form determines a reduction of the frame bundle of V to an orthogonal group, but the conditions in the definiton above further ensure that this commutes with the grading of V, and so with the reduction to E. Therefore, the orthogonal form determines a reduction $E^0 \hookrightarrow E$ to the orthogonal group

$$\mathrm{O}(\mathbf{n}) = \left(\prod_{i=1}^r R_i\right) \times \left(\prod_{i=1}^l \mathrm{O}(W_i)\right)$$

Recall from the proof of Theorem that the W_i here denote fixed vertices, and R_i is isomorphic to the orthogonal group of the space $V_i \oplus V_{\sigma(i)}^*$ with the standard pairing. Using this reduction, we may rewrite a representation as (E^0,φ), where $\varphi \in \Omega^0((E^0 \times F) \times_\rho \mathrm{Rep}_0(Q,\sigma,\mathbf{n}))$. We know the fibre of this last associated bundle to correspond to the representation space of generalized quiver.

3.2.5 Lemma. *Let $\tilde{Q} = (R, \mathrm{Rep}(\tilde{Q}))$ be a $\mathrm{O}(V)$-generalized quiver, and Q be the corresponding symmetric quiver. Then, there is an equivariant bijection between twisted $\tilde{Q}$-bundles and twisted orthogonal bundle representations of Q.*

Proof. We have an $\mathrm{O}(\mathbf{n})$-equivariant morphism $f : \mathrm{Rep}(\tilde{Q}, M) \to \mathrm{Rep}^o(Q, M, V)$ for some choice of total space V, by Theorem 3.2.2. Also, the group R is precisely $\mathrm{O}(\mathbf{n})$ above, so that the structure group of the principal bundles on both sides coincide. But then, the product map $\mathrm{id} \times f : (E \times F) \times \mathrm{Rep}(\tilde{Q}, M) \to (E \times F) \times \mathrm{Rep}(Q, M, V)$ descends to a morphism of the fibered products. □

Relation with plain Q-bundles

The definition of symmetric quivers as quivers with extra data allows us to establish a direct relation with the case of plain quiver bundles. Let (Q, σ) be a symmetric quiver, $\mathbf{n}$ be a symmetric dimension vector, and C a quadratic form on $\bigoplus \mathbb{C}^{n_i}$. The involution σ induces an involution on $\mathrm{Rep}(Q, \mathbf{n})$ by assigning to every $\varphi = (\varphi_\alpha)$ the vector $\sigma(\varphi)$ with $\sigma(\varphi)_\alpha = -\varphi^t_{\sigma(\alpha)}$, the transpose taken with respect to C. The space of orthogonal representations $\mathrm{Rep}(Q, C, \mathbf{n})$ is then the fixed point set of this involution. This involution commutes with the action of the group $\mathrm{O}(\mathbf{n})$, and so induces a linear map

$$\mathfrak{Rep}_0(Q, \mathbf{n}, g) \hookrightarrow \mathfrak{Rep}(Q, \mathbf{n})$$

where $\mathfrak{Rep}_0(Q, \mathbf{n}, g) := (E^0 \times F) \times_\rho \mathrm{Rep}_0(Q, C, \mathbf{n})$ is the representation space of orthogonal Q-bundles. If we also use extension of the structure goup of E^0 to $\mathrm{GL}(\mathbf{n})$, we can define a 'forgetful map' which sends an orthogonal representation (E^0, φ) to a plain one (E, φ).

In gauge theoretic terms, the extension of the structure group corresponds to the following map: we may choose a maximal compact K of $\mathrm{GL}(\mathbf{n})$ such that $K_0 = K \cap \mathrm{O}(\mathbf{n})$ is a maximal compact of $\mathrm{O}(\mathbf{n})$. Fix a reduction $E_{K_0} \hookrightarrow E^0$, and by extension of groups also $E_K \hookrightarrow E$. This induces a map

$$\mathscr{A}_0^{1,1} \hookrightarrow \mathscr{A}^{1,1}$$

by noting that $\mathrm{ad}E_{K_0} \hookrightarrow \mathrm{ad}E_K$. In fact, since $\mathfrak{k}_0$ is the fixed point set $\mathfrak{k}$ under an involution, the inclusion above also realizes $\mathscr{A}_0^{1,1}$ as the fixed point set of an involution. We arrive at

3.2.6 Lemma. *The space of twisted orthogonal Q-bundles can be identified as the closed linear subspace of the twisted plain representations which is invariant under the action of the gauge group $\mathscr{G}_0 = \mathscr{A}^0(\mathbb{E}^0)$ determined by $\mathrm{O}(\mathbf{n})$.*

We now want to establish gauge equations for orthogonal representations. Since the orthogonal group in question splits as a product, it is a case where we can incorporate parameters into this picture. This is simple enough: given the explicit splitting

$$\mathrm{O}(\mathbf{n}) = \left(\prod_{i=1}^{r} R_i\right) \times \left(\prod_{i=1}^{l} \mathrm{O}(W_i)\right)$$

we choose r positive paremeters $a_1, ..., a_r$, and l positive parameters $b_1, ..., b_l$. These are the parameters we take to weigh the symplectic forms ω_i as above.

Now, as a complex subspace of the space of plain representations, $\mathscr{A}_0^{1,1} \times \mathfrak{Rep}_0(Q))$ is naturally a symplectic space as well. The parameters for the plain representation are a collection of positive real numbers p_i, $i \in I$.

3.2.7 Theorem. *Choose parameters such that $p_i + p_{\sigma(i)} = a_i$, if the vertex is not fixed by the involution, and $p_i = b_i$, otherwise. Then, the bijection in Lemma 3.2.6 is a symplectomorphism between $\mathscr{A}^{1,1} \times \mathfrak{Rep}(\tilde{Q},M)$ and $\mathscr{A}_0^{1,1} \times \mathfrak{Rep}_0(Q,M,V)$.*

Proof. By construction, the map is obviously a diffeomorphism. Hence, we just have to prove that it preserves the symplectic form. Since the isomorphism of groups identifies the symplectic forms, it comes down to checking that the parameter was apropriately chosen. Since the isomorphism only changes the factors corresponding to pairs switched by the involution, we only need to check that parameters match in that case. The dual connection is defined by $A^* = -A^t$; then, on the plain representations, we consider the pair of connections $A \oplus (-A^t)$, and on the generalized quiver, simply A. We have

$$\omega(A \oplus A^*, B \oplus B^*) = p_i \omega(A,B) + p_{\sigma(i)} \omega(-A^t, -B^t) = (p_i + p_{\sigma_i}) \omega(A,B)$$

where, by an abuse of notation, ω is in each case the appropriate symplectic form. Since we want the last one to coincide with $a_i \omega$, which is the form on the generalized quiver, we must have $p_i + p_{\sigma(i)} = a_i$. □

3.3 Hitchin-Kobayashi correspondences

3.3.1 The correspondence

Our version is essentially of the same kind as various general Hitchin-Kobayashi correspondences in the literature: [8], [5], [32], [9], [28] (with increasing generality.) However, none of these cover our case because of the presence of the parameters in the moment map for the connections part. There is also [1], which covers precisely the case of quiver bundles, but only for a very particular choice of gauge group. Still, with the exception of [28], the proof of our correspondence is essentially the same as all these articles. In fact, our proof will be very cursory, since it is a straightforward adaptation of previous proofs.

Let X be a compact Kähler manifold, K a compact Lie group splitting as finite product $K = \prod K_i$, and E a (smooth) principal K bundle. Given a Kähler manifold F with a Hamiltonian left K action σ, we can form the fibration $E(F) = E \times_\sigma F$. We will consider the gauge equations on sections of this Kähler fibration. The reader may notice the absence of the holomorphicity condition for sections of this fibration, but as remarked in the introduction of [32], this condition does not play a role in proving the Hitchin-Kobayashi correspondence. (It does play a role, of course, in the construction of the moduli spaces, which we're not considering here.)

The gauge equations

We are interested in the space $\mathscr{A}^{1,1}$ of K-connections on E, and the space $\mathscr{S}$ of holomorphic global sections of $E(F)$, both properly endowed with symplectic structures.

On the space of sections, we can induce a moment map by fibrewise extension of the symplectic form on F, i.e.,

$$\omega(\varphi_1,\varphi_2) = \int_X \omega_F(\varphi_1(x),\varphi_2(x))$$

where ω_F is the symplectic form on F.

Recall from above that since the group K splits as a product, its Lie algebra as a direct sum $\mathfrak{k} = \bigoplus_{i\in I}\mathfrak{k}_i$. Because the adjoint action of K is inner, there is an induced splitting

$$\mathrm{ad}E_K = \bigoplus_{i\in I} E_K(\mathfrak{k}_i)$$

where we denote $E_K(\mathfrak{k}_i) := E_K \times_{\mathrm{Ad}} \mathfrak{k}_i$. Accordingly, if we pick an origin for $\mathscr{A}$ and use the commutativity of the tensor product with colimits, there is a splitting of the space of connections as $\mathscr{A} = \prod_{i\in I}\mathscr{A}_i$; but one sees easily that this splitting is independent of the choice of identification of $\mathscr{A}$ with $\mathrm{ad}E_K$. If we assume the invariant form on $\mathfrak{k}$ respects the splitting, then formula (3.1) now makes sense in each $\mathscr{A}_i$ separately, defining symplectic forms ω_i with $\omega_{\mathscr{A}} = \sum\omega_i$. However, we may weigh the sum using positive numbers a_i, $i\in I$, defining

$$\omega_{\mathscr{A},a} = \sum_{i\in I} a_i\omega_i = (a_i\Lambda F_i)_i$$

Assume that a moment map for the action of K on F exists. It is easy to see that it extends fibrewise to a moment map μ on $\mathscr{S}$. The gauge equations just define level sets of the resulting moment map on the product $\mathscr{A}^{1,1}\times\mathscr{S}$. Given a collection of central elements $c_i\in\mathfrak{k}_i$, where $\mathfrak{k}_i$ is the Lie algebra of K_i, then the gauge equations are

$$a_i\Lambda F_i + \mu_i(\varphi) = c_i$$

In this equation, as always, we are using a hidden parameter, the implicit choice of an equivariant isomorphism $\mathfrak{k}\simeq\mathfrak{k}^*$ of the Lie algebra with it dual.

Parabolic subgroups

Parabolic subgroups play an important role in the abstract Hitchin-Kobayashi correspondence, so let us just recall some basic facts. The references for this material are [32] and [Garcia-Prada et al.]. Let G be a connected complex reductive Lie group, K a maximal compact, $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebras. As above, suppose a faithful representation $\rho : K\to \mathrm{U}(V)$ is given, along with an induced, implicit isomorphism $\mathfrak{k}\simeq\mathfrak{k}^*$.

If $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and T a maximal torus of K, there is a choice of Cartan subalgebra $\mathfrak{h}$ such that $\mathfrak{z}\oplus\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$, where $\mathfrak{t} = \mathrm{Lie}\,T$. Let Δ be a set of roots, and Δ' be a choice of simple roots for the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. For any subset $A = \{\alpha_{i_1},\dots,\alpha_{i_s}\}\subset\Delta'$, define

$$D_A = \{\alpha\in R \mid \alpha = \sum m_j\alpha_j,\ m_{i_t}\geq 0 \text{ for } 1\leq t\leq s\}$$

The *parabolic subalgebra* associated to A is $\mathfrak{p} = \mathfrak{z}\oplus\mathfrak{h}\oplus\bigoplus_{\alpha\in D_A}\mathfrak{g}_\alpha$; the subgroup P of G determined by this subalgebra is the *parabolic subgroup* determined by A. A *dominant (resp. antidominant)* *character* of P is a positive (resp. negative) combination of the fundamental weights $\lambda_{i_1},...,\lambda_{i_s}$ plus an element of the dual of $i(\mathfrak{z}\cap\mathfrak{k})$.

Due to our choice of Cartan subalgebra, and using our implicit isomorphism $\mathfrak{k}\simeq\mathfrak{k}^*$, an anti-dominant character χ of P, may be identified with an element of $i\mathfrak{k}$; we'll still denote by χ. We have that $\rho(\chi)$ is hermitian (since $\chi\in i\mathfrak{k}$,) it has real eigenvalues $\lambda_1<...<\lambda_j<...<\lambda_r$, and it diagonalizes. In other words, χ induces a filtration

$$0\neq V^1\subsetneq...\subsetneq V^r=V$$

where $V^k=\bigoplus_{i\leq k}V_{\lambda_k}$ is the sum of all eigenspaces V_{λ_i} with $i\leq k$. The following theorem is from [32], section 2:

3.3.1 Theorem. *Let $\chi\in i\mathfrak{k}$. Then, the pre-image by ρ of the stabilizer of the induced flag is a parabolic group $P(\chi)$, and χ is the dual of an antidominant character of $P(\chi)$. Further, given an arbitrary parabolic subgroup P with Lie algebra $\mathfrak{p}$, there is a choice of a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ contained in $\mathfrak{p}$ such that $\chi\in\mathfrak{h}$ and is antidominant with respect to P if and only if P stabilizes the flag induced by χ.*

It is important, however, to keep in mind that this applies to filtrations induced by an element of $i\mathfrak{k}$, and not just any filtration of V. For $\mathrm{GL}(V)$, it is true that any flag is induced by such an element. For an orthogonal or symplectic group, however, the flags induced in this way are *isotropic* flags, i.e., flags where $V^{r-k}=(V^k)^\perp$ (so that for $k\leq r/2$, the spaces are actually isotropic.)

Again following Mundet in [32], if we additionally are given a holomorphic reduction $\pi:X\to E(G/P)$ to P, the pair (π,χ) determines an element $g_{\pi,\chi}\in\Omega^0(E\times_{\mathrm{Ad}}i\mathfrak{k})$ which is fibrewise the dual of χ. Considered as an endomorphism of $\mathbb{V}:=E\times_\rho V$ (through the representation ρ,) it has almost-constant eigenvalues $\lambda_1<...<\lambda_k<...<\lambda_r$, and so induces a filtration

$$0\subset\mathbb{V}^1\subset...\subset\mathbb{V}^k\subset...\subset\mathbb{V}^r=\mathbb{V}$$

which is defined outside of a codimension-two submanifold. Here, $\mathbb{V}^{\lambda_k}=\bigoplus_{i\leq j}\mathbb{V}(\lambda_i)$ is the sum of all eigenbundles with $\lambda_i\leq\lambda_k$. Again, from [32], we have

3.3.2 Theorem. *If the reduction is holomorphic, for any antidominant character χ, the induced filtration is holomorphic. Conversely, an element $g\in\Omega^0(E\times_{\mathrm{Ad}}i\mathfrak{k})$ with constant eigenvalues determines a holomorphic reduction π and an antidominant character χ such that $g=g_{\pi,\chi}$.*

Stability

Suppose a faithful representation $K\to U(V)$ is given. Given a pair (σ,χ) of a holomorphic reduction $\pi:X\to E(G/P)$ to a parabolic subgroup P, and an anti-dominant character χ of P, there is a codimension-two submanifold over which the character χ induces a holomorphic filtration of the associated fibre bundle $\mathbb{V}=E(V)$:

$$0\subset\mathbb{V}^1\subset...\subset\mathbb{V}^j\subset...\subset\mathbb{V}^r=\mathbb{V}$$

where $\lambda_1 < ... < \lambda_j < ... < \lambda_r$ are the eigenvalues of $\rho(\chi)$, and $\mathbb{V}^{\lambda_j} = \bigoplus_{i\leq j}\mathbb{V}(\lambda_i)$ is the sum of all eigenbundles with $\lambda_i \leq \lambda_j$. Given such a pair, we define

$$\deg(\pi,\chi) = \lambda_r \deg(\mathbb{V}^r) + \sum_{k=1}^{r-1}(\lambda_k - \lambda_{k+1})\deg(\mathbb{V}^k)$$

where $\deg(\mathbb{V})$ is the degree of the vector bundle.

Now, just as we have taken a weighted sum for the symplectic forms, we should now do the same for the degree. A reduction π induces a reduction π_i on each of the vertices, and an anti-dominant character obviously splits $\chi = \oplus\chi_i$, like the Lie algebra. But then, the maximal weight changes accordingly, and in fact we should consider instead

$$\deg_a(\pi,\chi) = \sum a_i \deg(\pi_i,\chi_i)$$

Here, the positive numbers a_i are the parameters for the moment map, as above. We'll call this the 'a-degree.'

Consider now the action of K on F, and for any $x \in F$ and $k \in \mathfrak{k}$ let

$$\lambda_t(x,k) = \langle \mu(\exp(itk)x), k\rangle$$

where $\langle\cdot,\cdot\rangle$ is the canonical pairing of $\mathfrak{k}$ with its dual. Then, the *maximal weight* of the action of k on x is

$$\lambda(x,k) = \lim_{t\to\infty}\lambda_t(x,k)$$

This number plays the role of the Hilbert-Mumford criterion in the Kähler setting.

Finally, given a section $\varphi \in \Omega^0(E(F))$ of the associated bundle with fibre F, and a central element $c \in \mathfrak{k}$, the total c-degree (σ,χ) is defined as

$$T^{c_i}_{\varphi}(\pi,\chi) = \deg_a(\pi,\chi) + \int_X \lambda(\varphi(x), -ig_{\pi,\chi}(x)) + \langle i\chi, c\rangle \mathrm{Vol}(X)$$

Here, $g_{\pi,\chi} \in \Omega^0(E\times_{\mathrm{Ad}} i\mathfrak{k})$ is the fibrewise dual of χ. The total degree is allowed to be ∞. Henceforward, as before, we will always assume that the volume of X is normalized to one.

Finally, we define stablity:

3.3.3 Definition. Let $c_i \in \mathfrak{k}_i$ be central elements. A pair $(A,\varphi) \in \mathscr{A}^{1,1}\times\mathscr{S}$ is c_i-stable if for any submanifold $X_0 \subset X$ of complex codimension 2, for any parabolic subgroup P of G, for any holomorphic reduction $\pi \in \Gamma(X_0, E(G/P))$ defined on X_0, and for any antidominant character χ of P we have

$$T^{c_i}_{\varphi}(\pi,\chi) > 0$$

The case of quiver bundles

We want to explicitly understand the stability of quiver bundles, since it is an important and intuitive case. Since for this case the Kähler fibre is actually a vector space, $g_{\pi,\chi}(x)$ can be seen as an endomorphism of $\mathrm{Rep}(Q,V)$, and it makes sense to speak of its eigenvalues. Let $\mathscr{F}^-(\chi) \subset E(F)$ be the subset of vectors where $g_{\pi,\chi}$ acts negatively, i.e., vectors in the direct sum of all the negative eigenspaces of $g_{\pi,\chi}$. Since $g_{\pi,\chi}$ has constant eigenvalues, and π is holomorphic, $\mathscr{F}^-(\chi)$ is a holomorphic subbundle. Then, one computes that

$$\lambda(\varphi(x), g_{\pi,\chi}(x)) = \begin{cases} 0 & \text{if} \quad \varphi(x) \in \mathscr{F}^-(\chi) \\ \infty & \text{if} \quad \varphi(x) \notin \mathscr{F}^-(\chi) \end{cases} \tag{3.2}$$

This result can be intuitively understood if we just look at the action of a group of matrices on a vector space. Then, a one-parameter subroup corresponds to repeated application of the same automorphism to a vector. The action on the subgroup on an eigenspace is then t^λ, where λ is an eigenvalue of the generator. As long as all λ are negative, this tends to zero; a positive one forces a divergence to infinity.

On the other hand, $c = \oplus i\tau_i \mathrm{id}_i$, and $\chi = \oplus \chi_i$ for $\chi_i \in \mathfrak{gl}_i$, so

$$\langle i\chi, c\rangle = -\sum \tau_i \langle \chi_i, \mathrm{id}_i\rangle = -\sum \tau_i \mathrm{tr}\chi_i$$

Recall that the volume of X has been normalized to one. What we've seen so far justifies the following definition:

3.3.4 Definition. A pair $(A, \varphi) \in \mathscr{A}^{1,1} \times \Omega^0(\mathfrak{Rep})$ is (a_i, τ_i)-*stable* if for any (π, χ) with $\varphi(X) \subset \mathscr{F}^-(\chi)$ we have

$$\sum (a_i \deg(\pi, \chi_i) - \tau_i \mathrm{tr}\chi_i) > 0$$

Given a subrepresentation $\mathbb{V}' \subset \mathbb{V}$, consider the one-term flag $0 \subset \mathbb{V}' \subset \mathbb{V}$. Since we are dealing with a general linear group, the subgroup P fixing this flag is a parabolic subgroup, and there is an anti-dominant character χ of P inducing that flag. For a particular choice of χ, we have $\deg(\sigma_i, \chi_i) = \deg(\mathbb{V}_i) - \deg(\mathbb{V}'_i)$, and $\mathrm{tr}\chi_i = \mathrm{rk}\mathbb{V}_i - \mathrm{rk}\mathbb{V}'_i$, and we get the more familiar definition in terms of slopes. In this case, then, the total degree can be interpreted as the maximal weight for the moment maps we constructed. In other words, the slope condition for subrepresentations is a necessary condition for stability. In general, however, parabolic subgroups stabilize more complicated flags, and although our definition looks complicated, it generalizes to other reductive groups. Another important point is that we can restrict only to subrepresentation when working over a Riemann surface, for only over a surface can we keep inside the category of vector bundles over X. From this point of view, what is surprising is that our definition even works. The essential ingredient is the remarkable theorem of Uhlenbeck and Yau [44], which assures us that the subsheaves in the filtration define vector subbundles over a submanifold of codimension at least two. This is ultimately reliant on a Hartog-like extension theorem, and the interested reader can look up Popovici's nicely geometric proof of this fact in [34].

Essentially, we cannot always interpret the total degree as a maximal weight because we are working in the wrong category: if we recall that every coherent subsheaf defined outside of a codimension two submanifold has a unique extension as a coherent subsheaf, we see that for a general χ, we get a filtration of $\mathbb{V}$ by coherent subsheaves, and not just subbundles (in fact, we are reversing the whole story, since what one can prove is that the filtration is already by coherent subsheaves.) We have the following:

3.3.5 Lemma. *In the conditions of the definition of stability above, the morphisms restrict to each element in the filtration, i.e., if* $\varphi \in \mathscr{F}^-(X)$*, then* $\varphi(\mathbb{V}_{k,t(\alpha)} \otimes M_\alpha) \subset \mathbb{V}_{k,h(\alpha)}$ *for all k in the filtration.*

Proof. Given any morhpism φ_α in the representation; we can consider it as an element of $\mathrm{Rep}(Q,V)$ by taking all the other components to be zero. Since $\varphi(X) \in \mathscr{F}^-(\chi)$, we may assume that φ_α is an eigenvector of χ with eigenvalue $\lambda < 0$, i.e., $[\chi, \alpha] = \lambda\alpha$.

Now, the character χ also acts on the total space V, so let $x \in V_{t(\alpha)}$ be an eigenvector with eigenvalue λ_x. We have

$$\chi(\alpha(x)) = \alpha(\chi(x)) + \lambda\alpha(x) = (\lambda_x + \lambda)\alpha(x)$$

Since $\lambda < 0$, $\lambda_x + \lambda < \lambda_x$, as desired. The general case follows from here. □

The following definition, to be found in [1], should be well-motivated by the previous observations:

3.3.6 Definition. A quiver sheaf representation $(\mathscr{E}, \varphi)$ is a collection of coherent sheaves $\mathscr{E}_i$, one for each vertex of the quiver, together with a collection of sheaf morphisms $\varphi_\alpha : \mathscr{E}_{t(\alpha)} \otimes \mathscr{M}_\alpha \to \mathscr{E}_{h(\alpha)}$, one for each arrow in the quiver (the $\mathscr{M}_\alpha$ are the twisting sheaves.)

Recall that the degree of a coherent sheaf is defined as

$$\deg(\mathscr{E}) = \frac{1}{(n-1)!} \frac{2\pi}{\mathrm{Vol} X} \langle c_1(\mathscr{E}) \cup [\omega^{n-1}], [X] \rangle$$

where $c_1(\mathscr{E})$ is the first Chern class of $\mathscr{E}$, $[\omega^{n-1}]$ is the class of the Kähler form of X, and $[X]$ its fundamental class. Given a quiver sheaf, for σ and τ be collections of real numbers σ_i, τ_i with $\sigma_i > 0$, we define the degree and slope of the representation respectively to be

$$\deg_{a,\tau}(\mathscr{E}) = \sum (a_i \deg(\mathscr{E}_i) - \tau_i \mathrm{rk}(\mathscr{E}_i)) \qquad \mu_{a,\tau}(\mathscr{E}) = \frac{\deg_{a,\tau}(\mathscr{E})}{\sum a_i \mathrm{rk}(\mathscr{E}_i)}$$

Note that our situation fits into this framework since we can always take the sheaves of sections of our vector bundles. This inclusion in fact respects the stability of the representation in the following sense:

3.3.7 Proposition. *A representation* (E, φ) *is stable (as a bundle representation) if and only if* $\mu_{a,\tau}(\mathscr{F}) < \mu_{a,\tau}(E)$ *for every proper sub-sheaf representation* $0 \neq \mathscr{F} \subset E$*.*

Proof. We begin by a preliminary observation. Recall that giving a pair (π, χ) is equivalent to giving a filtration

$$0 \subset \mathbb{V}^1 \subset \ldots \subset \mathbb{V}^k \subset \ldots \subset \mathbb{V}^r = \mathbb{V}$$

by *coherent subsheaves* (not subbundles,) together with a vector $(\lambda_1, \ldots \lambda_r)$ such that $\lambda_k < \lambda_{k+1}$. On the other hand, by Lemma 3.3.5, such a pair is admissible for the stability condition if and only if φ restricts to each element in the filtration.

Start with an arbitrary such filtration with φ restricting to each step, and given a vector $(\lambda_1, ...\lambda_r)$ with $\lambda_k < \lambda_{k+1}$, define

$$\deg_i(\lambda) = \lambda_r \deg(\mathbb{V}_i^r) + \sum_{k=1}^{r-1} (\lambda_k - \lambda_{k+1}) \deg(\mathbb{V}_i^k)$$

and

$$\deg(\lambda) = \sum_i \left(a_i \deg_i(\lambda) - \tau_i \sum_k \lambda_k \mathrm{rk}(\mathbb{V}_i^k) \right)$$

To check stability, we must then check that this degree is positive for each vector λ for any such filtration.

The strategy is to enlarge the space of vectors λ to include vector for which only $\lambda_k \leq \lambda_{k+1}$ is satisfied. Then, in this enlarged space we find a basis for which this expression of degree is simplified, and then prove that it is enough to prove it for such a basis.

The elements of the basis are just

$$L_k = \sum_{j=k}^{r} e_j$$

where e_j is the jth element in the canonical basis; in other words, L_k is zero up to k, and 1 from then on. Easily,

$$\deg(L_k) = \sum_i a_i (\deg \mathbb{V}_i + \deg \mathbb{V}_i^k) - \tau_i (\mathrm{rk}\mathbb{V}_i - \mathrm{rk}\mathbb{V}_i^k)$$

The slope condition is precisely $\deg(L_k) > 0$. We want to show that if this is true for all L_k, the the representation is stable. But in fact this follows from the geometry of all allowed λ: they form a polyhedral cone, the edges of which are spanned by the L_k. It is clear that if the linear functional is positive along the edges, then it is positive along the whole cone. □

The statement

We need a technical, but important definition:

3.3.8 Definition. A pair $(A, \varphi) \in \mathscr{A}^{1,1} \times \mathscr{S}$ is infinitesimally simple if no semisimple element in $\mathrm{Lie}(\mathscr{G}_G)$ stabilizes (A, φ).

The theorem is as follows:

3.3.9 Theorem. *Let $(A, \varphi) \in \mathscr{A}^{1,1} \times \mathscr{S}$ be an infinitesimally simple pair. Then (A, φ) is stable if and only if there is a gauge transformation $g \in \mathscr{G}_G$ such that $(B, \psi) = g \cdot (A, \varphi)$ solves the gauge equations*

$$a_i \Lambda F_i + \mu_i(\varphi) = c_i \tag{3.3}$$

Furthermore, if two different $g, g' \in \mathscr{G}_G$ yield a solution, then there exists a $k \in \mathscr{G}_K$ such that $g' = kg$.

The proof of this correspondence takes up the rest of this section. The general strategy is standard, and in terms of symplectic geometry can be described as the sequence of steps: stability ⇒ properness of the integral ⇒ zero of the moment map ⇒ stability.

Preliminaries

The integral of the moment map

The central construction in the proof is that of the integral of the moment map; this is a rather general construction, and it is the (infinite-dimensional) Kähler analogue of Kempf-Ness map for smooth projective varieties. For proofs we refer the reader to [32].

Let H be a Lie group, and suppose there is a complexification G for which the inclusion $H \hookrightarrow G$ induces a surjection $\pi_1(H) \twoheadrightarrow \pi_1(G)$. Note that we are not assuming finite dimensionality, since in our case, the groups are the inifinite dimensional gauge groups.) Let M be a Kähler manifold on which H acts respecting the structure, and for which a moment map $\mu : M \to \mathfrak{h}^*$ exists. For a fixed point $p \in M$, we define a 1-form σ^p on L by the formula

$$\sigma^p_g(v) = \langle \mu(g \cdot p), -i\pi(v) \rangle$$

where $g \in G$, $v \in T_g G$, and $\pi : \mathfrak{h} \oplus i\mathfrak{h} \to i\mathfrak{h}$ is the projection onto the second factor. Then, σ is exact (it is here that the surjection $\pi_1(H) \twoheadrightarrow \pi_1(G)$ is needed,) and we denote by $\Psi_p : G \to \mathbb{R}$ the unique function such that $d\Psi_p = \sigma^p$, and $\Psi_p(1) = 1$. It turns out that the Ψ_p fit together into a smooth function $\Psi : M \times G \to \mathbb{R}$, which we call the integral of the moment map.

The properties of this map are described in the following proposition.

3.3.10 Proposition. *Let $p \in M$ be any point, and $s \in \mathfrak{h}$.*

1. $\Psi(p, \exp(is)) = \int_0^1 \langle \mu(g \cdot p), s \rangle dt = \int_0^1 \lambda_t(p, s) dt$.
2. $\partial_t \Psi(p, \exp(its))|_{t=0} = \langle \mu(p), s \rangle = \lambda_0$.
3. $\partial_t^2 \Psi(p, \exp(its))|_{t=t_0} \geq 0$ *for any* $t_0 \in \mathbb{R}$, *with equality if and only if* $\mathscr{X}_s(\exp(it_o s) \cdot p) = 0$, *where* $\mathscr{X}_s$ *is the vector field generated by s.*
4. $\Psi(p, \exp(its) \cdot p) \geq (t - t_0)\lambda_t(p, s) + C_s(p, t_0)$ *for any* $t_0 \in \mathbb{R}$, *where* C_s *is a continuous function in all variables.*
5. $\Psi(p, g) + \Psi(g \cdot p, h) = \Psi(p, hg)$ *for any* $g, h \in G$.
6. $\Psi(h \cdot p, g) = \Psi(p, h^{-1}gh$ *and* $\Psi(p, hg) = \Psi(p, g)$ *for any* $h \in H$, *and* $g \in G$.
7. $\Psi(x, 1) = 0$.

Together with the convexity proven in the previous proposition, the next lemma is the fundamental property in the proof:

3.3.11 Lemma. *An element $g \in G$ is a critical poit of Ψ_p if and only if $\mu(g \cdot p) = 0$.*

Equivalence of C^0 and L^1 norms As usual, we will need to complete spaces of smooth maps by Sobolev norms, to get spaces that are flexible enough. In general, we want twice-differentiability, and the L^p norm needs to satisfy a bound coming from the Sobolev multiplication theorem. In particular, if $n = \dim X$, we must choose $p > 2n$. The proof of the correspondence involves a properness argument on the integral of the moment map, and to prove such properness we have to fiddle with norms. In particular, we will require an equivalence between C^0 and L^1 estimates. Choose $B<0$; we will need to restrict to the subset

$$\mathscr{M}^p_{2,B} = \left\{ s \in L^p_2(E \times_{\mathrm{Ad}} \mathfrak{k}) \quad | \quad ||\mu^c(\exp(s)(A,\varphi))||^p_{L^p} \leq B \right\}$$

3.3.12 Lemma. *There are two constants $C_1, C_2 > 0$ (which implicitly depend on B and on the parameters a_i) such that for all $s \in \mathscr{M}^p_{2,B}$ one has* $\sup|s| \leq C_1||s||_{L^1} + C_2$.

For a proof this lemma, check [1] section 3.5, which does not use anything specific to the general linear group.

3.3.13 Definition. The integral of the moment map Ψ^c satisfies the C^0 *main estimate* if there are constants $C_1, C_2 > 0$ such that

$$\sup|s| \leq C_1 \Psi^c(\exp(s)) + C_2$$

If the same condition is verified with $\sup|s|$ replaced with the L^1 norm, then we say Ψ^c satisfies the L^1 *main estimate*.

The following is an easy corollary of Lemma 3.3.12

3.3.14 Corollary. *In $\mathscr{M}^p_{2,B}$, the integral of the moment map satisfies the C^0 main estimate if and only if it satisfies the L^1 main estimate.*

As we mentioned, the point here is that the proof of the correspondence demands the properness of the integral of the moment map in the weak topology of the infinite dimensional Lie algebra involved. The main estimate is only a requirement that implies properness, but it more easily serves as an intermediary step in the proof.

3.3.15 Lemma. *If Ψ^c satisfies the main estimate, then Ψ^c is proper in the weak topology of $L^p_2(E \times_{\mathrm{Ad}} \mathfrak{k})$.*

Proof. This lemma is proven by contradiction, and is precisely the same as [8] section 3.14, or [1] section 3.30. □

Minima in $\mathscr{M}^p_{2,B}$ Our restriction to $\mathscr{M}^p_{2,B}$ only makes sense if we can prove that the minima in this subset are in fact minima in the whole of $\mathscr{M}^p_2$.

3.3.16 Lemma. *Suppose (A,φ) is an infinitesimally simple pair, and that s minimizes the integral in $\mathscr{M}^p_2$. Then, there is a gauge transformation g such that if $g\cdot(A,\varphi) = (B,\theta)$, then $\mu^c(B,\theta) = 0$.*

Proof. Define the operator $L : L^p_2(E \times_{\mathrm{Ad}} i\mathfrak{k}) \to L^p(E \times_{\mathrm{Ad}} i\mathfrak{k})$ as

$$L(u) = i\frac{d}{dt}\,\mu^c(\exp(tu)(B,\theta))|_{t=0} = i\langle d\mu^c, u\rangle(B,\theta) = i\sum a_i \langle d\mu_i, u_i\rangle + i\langle d\mu_S, u\rangle$$

Each $\langle d\mu_i, u_i\rangle$ is a Fredholm operator with index zero (indeed, up to a compact operator, it is $\partial^*_B\partial_B$, cf. [8].) But, up to a compact operator, L is a linear combination of these, so it is itself a Fredholm operator of index zero. We prove that it is also injective, implying that it is surjective. In fact, if $L(u) = 0$,

$$0 = \langle iL(u), -iu\rangle = ||\mathscr{X}_{-iu}(B,\theta)||^2$$

which implies that $-iu$ leaves (B,θ) fixed, and simplicity of (A,φ) now implies that $u=0$.

Knowing that L is surjective, we conclude that there must be an u such that $L(u) = -i\mu^c(B,\theta)$. A standard argument originally due to Simpson then shows that $\mu^c(B,\theta) = 0$, cf. [8] or [1]. $\square$

Stability implies main estimate

We start with a lemma.

3.3.17 Lemma. *If the integral of the moment map does not satisfy the main estimate, then there is an element $u_\infty \in L^2_p(E\times_{\mathrm{Ad}}\mathfrak{k})$ such that $\lambda((A,\varphi),-iu_\infty)\le 0$.*

Proof. Let C_j be a sequence of positive constants diverging to infinity. We start by finding a sequence (s_j) in $L^2_p(E\times_{\mathrm{Ad}}\mathfrak{k})$ such that $||s_j||_{L^1}\to\infty$ and $||s_j||_{L^1}\ge C_j\Psi(\exp s_j)$ (cf. [1] Lemma 3.43.) With such a sequence in hand, we set $l_j = ||s_j||_{L^1}$, and $u_j = s_j/l_j$, so that $||u_j||_{L_1} = 1$ and $\sup|u_j|\le C$. We can assume that $\lim_j \lambda^t_{a_i,\tau_i}((A,\varphi),-iu_j)$ exists.

Using the convexity of the integral of the moment map, and the fact that X is compact, we have

$$\frac{l_j-t}{l_j}\lambda^t_{a_i,\tau_i}(A,-iu_j)+\frac{1}{l_j}\int_0^t \lambda^l_{a_i,\tau_i}(A,-iu_j)dl\le C \tag{3.4}$$

for some constant C.

For a principal bundle E and a connection A on E, we have (cf. [32]):

$$\lambda_t(A,s)=\int_X\langle\Lambda F_A,s\rangle+\int_0^t||\exp(ils)\bar\partial(s)\exp(ils)||dl$$

where λ_t is the finite-time maximal weight for the Atiyah-Bott moment map. It easily follows, then, that in our case we have

$$\lambda^t_{a_i,\tau_i}=\sum a_i\left(\int_X\langle\Lambda F_{A_i},s\rangle+\int_0^t||\exp(ils)\bar\partial(s_i)\exp(ils)||dl\right)$$

Using this and (3.4) (recall that the curvature is bounded,) we can prove that $\sum a_i||\bar\partial((u_j)_i)||_{L^2}$ is bounded, and so $u_j\in L^2_1$. After passing to a subsequence, $u_j\to u_\infty$ weakly in L^2_1, since the u_j belong o the unit ball. As the embedding $L^2_1\hookrightarrow L^2$ is compact, the convergence is also strong in L^2, and $u_\infty\ne 0$ because of the uniform bound on the C^0 norm of the u_j. To see that $\lambda^t_{a_i,\tau_i}((A,\varphi),-iu\infty)\le 0$, see [1]. $\square$

Using methods due to Uhlenbeck-Yau [44] (cf.[8] and [34],) we can prove the the element u_∞ in the lemma has almost everywhere constant eigenvalues, and that it defines a filtration of V by holomorphic subbundles in the complement of a complex codimension 2 submanifold. But then (cf. [32],) this defines a reduction π of the structure group to a parabolic subgroup P, and an antidominant character χ of P with $\deg_{a_i,\tau_i}(\pi,\chi)=\lambda((A,\varphi),-iu_\infty)\le 0$, contradicting stability.

Main estimate implies solution

Here we need Lemma 3.3.15, and the proof is essentially due to Bradlow [8]. Since Ψ^c is proper in the weak topology, if $\Psi^c(\exp(s_j)$ is bounded, then $||s_j||_{L^2_p}$ is also bounded. But then, we take a minimizing sequence (s_j) of Ψ^c, and by properness it converges weakly to some s_∞ where Ψ^c attains a minimum. But we have seen that minima to Ψ^c correspond precisely to zeros of the moment map, so we only need to check smoothness, which follows from elliptic regularity.

Solution implies stability

Nothing really new happens here, since it mostly uses general properties of the integral of the moment map. It involves, however, computing some technical inequalities on the norms of the Lie algebra, so for details we refer to [32].

First of all, supposing that the orbit of an infinitesimally simple pair (A,φ) has a zero of the moment map, say $h\cdot(A,\varphi)$, one proves that $h\cdot(A,\varphi)$ is also an infinitesimally simple pair, and one with positive maximal weight. Indeed, a semisimple element contradicting stability of $h\cdot(A,\varphi)$ could not leave it fixed, since this would contradict the simplicity of (A,φ) itself. By the explicit computation of the gradient of the moment map, we arrive then at a contradiciton. Now, using suitable inequalities, one proves that

$$t\sup|g_{\pi,\chi}|\leq C_1\Psi_{(A,\varphi)}(\exp(g_{\pi,\chi})+C_2$$

It is standard from here to prove that the original pair is linearly stable, cf. section 6.3 of [32].

Uniqueness of solution

The statement on uniqueness follows on general grounds from the convexity of the integral of the moment map.

Polystability and complete Hitchin-Kobayashi correspondences

We make a detour here to discuss polystability of quiver bundles, as an example of what one means by a 'complete' Hitchin-Kobayashi correspondence. None of our remarks here extend to general fibrations, but it will paly a role below. Also for this reason, our remarks necessarily consist of generalities grazing only the surface of this topic. The main point is that we can still get an explicit description of representations satisfying the gauge equations even if we do nott restrict attention to infinitesimally simple pairs. Such a description is essential for the study of the moduli space of representations, since the moduli of stable pairs is generally not compact.

As it happens, the category of plain quiver sheaves associated with a given quiver forms an abelian category. As is well known (e.g., in the case of vector bundles,) the existence of the abelian structure significantly simplifies the description of polystability. First of all, the study of polystability implies reductions to Levi subgroups of the parabolic groups in question. We have seen above that the parabolic groups correspond naturally to certain filtrations of the representation; the reduction to a Levi subgroup, in the context of an abelian category, corresponds to taking the associated graded object. The second fundamental fact is the existence of a *Jordan-Hölder filtration*, which implies that every representation can lead to such a graded object. This latter filtration is a filtration of the form

$$0=F_0\subset F_1\subset\ldots\subset F_k=F$$

where each consecutive quotient F_i/F_{i-1} is stable. For a polystable (more generally, semistable) object, one can show that any two such filtrations have the same length, and yield the same graded object (though in general the filtrations themselves are *not* isomorphic.) A very elucidating example is the finite dimensional case, cf. [21]. The outcome of such considerations is that a representation is polystable if it is a direct sum of stable representations.

To state the correspondence as it appears in the literature, recall the definitions of (σ,τ) degree and slope. The definition and theorem that follow come straight from [1].

3.3.18 Definition. Let $(\mathscr{E},\varphi)$ be a quiver sheaf. Then, $\mathscr{E}$ is stable (resp. semistable) if for all proper quivers subsheaves $\mathscr{F}$ we have $\mu_{\sigma,\tau}(\mathscr{F}) < \mu_{\sigma,\tau}(\mathscr{E})$ (resp., $\mu_{\sigma,\tau}(\mathscr{F}) \leq \mu_{\sigma,\tau}(\mathscr{E})$); $\mathscr{E}$ is polystable if it is a direct sum of polystable quivers sheaves, all of them with the same slope.

3.3.19 Theorem. *Let $(\mathscr{E},\varphi)$ be a holomorphic twisted quiver bundle with $\deg_{\sigma,\tau}(\mathscr{E})=0$. Then, $\mathscr{E}$ is (σ,τ)-polystable if and only if it admits a hermitian metric satisfying the (σ,τ)-gauge equations. This hermitian metric is unique up to multiplication by a constant for each summand in the decomposition as a direct sum of stable subrepresentations.*

This theorem is easily proven using our theorem above after one shows that any stable pair is infinitesimally simple (in fact simple.) We will not prove this here, however, since it is not relevant to our discussion below.

3.3.2 Orthogonal quiver bundles

Stability of orthogonal representations

We now characterize the stability condition for orthogonal representations of symmetric quivers. Let $(R,\mathrm{Rep}(\tilde{Q}))$ be an $\mathrm{O}(m,\mathbb{C})$-generalized quiver. Recall the isomorphism

$$R = \mathrm{O}(\mathbf{m}) := \left(\prod_{i=1}^{r} R_i\right) \times \left(\prod_{i=1}^{l} \mathrm{O}(W_i)\right)$$

A central element $c \in \mathfrak{r}$ can be written

$$c = \bigoplus_{i=1}^{r} (\tau_i \mathrm{id}_i \oplus (-\tau_i)\mathrm{id}_{\sigma(i)}) \oplus \bigoplus_{i=1}^{l} \sigma_i \mathrm{id}_i$$

where $\tau_i \in \mathbb{R}$ and $\sigma_i = \pm 1$. Also, χ has values in $\mathfrak{r}$, each χ_i is traceless. Further, if i corresponds to a vertex that is not fixed, $\chi_i = (\psi_i, -\psi_i^t)$. Thus,

$$\langle c,\chi\rangle = \sum_{i=1}^{r} (\langle \tau_i,\psi_i\rangle + \langle -\tau_i, -\psi_i^t\rangle) + \sum_{i=1}^{l} \langle \sigma_i,\chi_i\rangle = \sum_{i=1}^{r} 2\tau_i \mathrm{tr}\psi_i$$

3.3.20 Definition. A representation $(A,\varphi) \in \mathscr{A}^{1,1} \times \Omega^0(\mathfrak{Rep}(\tilde{Q},M))$ is stable if for any reduction π to a parabolic subgroup P, and an anti-dominant character χ of P such that $\varphi(X) \subset \mathscr{F}^-(\chi)$, we have

$$\sum_{i=1}^{r} (a_i \deg(\pi,\chi_i) - 2\tau_i \mathrm{tr}\chi_i) + \sum_{i=1}^{l} b_i \deg(\pi,\chi_i) > 0$$

We want to simplify this stability condition. In analogy to the general linear case, there is a concrete interpretation of parabolic subgroups of an orthogonal group in terms of special flags. A filtration

$$0 = V^0 \subsetneq V^1 \subsetneq \ldots \subsetneq V^r = V$$

of a quadratic vector space V is said to be isotropic if $V^{r-k} = (V^k)^\perp$ for every $0 \leq k \leq r$. The subgroup of $\mathrm{O}(V)$ fixing such a flag is parabolic, and conversely, every parabolic subgroup is the stabilizer of such a flag.

In the case of a vector bundle $\mathbb{V}$ with fibre V,[3] above we easily see that the filtration induced by an antidominant character of a parabolic subgroup is also isotropic. However, this filtration is an isotropic filtration by vector bundles only outside a codimension two submanifold. Since every coherent sheaf defined outside a codimension two submanifold has a unique coherent extension to the whole manifold, what we actually get is an isotropic filtration of $\mathbb{V}$ by subsheaves.

Let us consider the case of orthogonal representations. Our structure group is

$$\mathrm{O}(\mathbf{m}) = \left(\prod_{i=1}^{r} R_i\right) \times \left(\prod_{i=1}^{l} \mathrm{O}(W_i)\right)$$

where R_i is also an orthogonal group of the product $V_i \oplus V_i^*$. This means that when we look at the splitting $\chi = \oplus\chi_i$, each component is an antidominant character of a parabolic subgroup of an orthogonal group. We have two cases:

- When $i = \sigma(i)$ is fixed by the involution, i.e., the component χ_i corresponds to a factor of the form $\mathrm{O}(W_i)$: then, just as above, χ_i (and therefore χ) induces an isotropic filtration of $\mathbb{W}_i$.
- When $i \neq \sigma(i)$ is not fixed by the involution: then, $\chi_i = (\psi_i, -\psi_i^t)$ for some $\psi_i \in \mathfrak{gl}(V_i)$. Note that $\mathbb{V}_i$ and $\mathbb{V}_i^*$ are themselves isotropic subbundles. The isotropic filtration of the direct sum might select subspaces from either of them, e.g., if ψ_i has both a positive and a negative eigenvalues (since the eigenvalues of $-\psi^t$ are the symmetric of the eigenvalues of ψ, the first half of the filtration will include an isotropic subbundle containing both subspaces from $\mathbb{V}_i$ and $\mathbb{V}_i^*$.) However, at each step in the filtration, each subbundle can be split into the two vertices.

Using the same methods as for plain quiver bundles above, we can prove the following:

3.3.21 Theorem. *The morphism φ restricts to each subsheaf in the representation.*

This proposition means that each element in the filtration in the definition of stability is actually a sheaf subrepresentation; we call such a subrepresentation an *isotropic* quiver subsheaf. Recall that every orthogonal bundle is isomorphic to its dual, and hence has degree zero. We make the following definition.

3.3.22 Definition. An orthogonal representation $(E^{\mathbb{C}}, \varphi)$ is *slope stable* if for every isotropic reflexive subsheaf representation $(\mathscr{F}, \varphi)$ we have

$$\deg^0_{a,\tau} = \sum_{i=1}^{2r} (a_i \deg(\mathscr{F}_i) - \tau_i \mathrm{rk}\mathscr{F}_i) + \sum_{i=1}^{l} b_i \deg \mathscr{F}_i < 0$$

[3] We are here departing from the notation in the previous sections. The font V here denotes the fibre, so $\mathbb{V}$ denotes the holomorphic vector bundle, not the underlying smooth bundle.

Note that the isotropic subsheaf representations are not, by definition, orthogonal representations (since the quadratic form is certainly degenerate.) From now on, when we speak of a subsheaf representation, we implicitly assume it to be 'reflexive'.

3.3.23 Proposition. *An orthogonal representation is stable if and only if it is slope stable.*

Proof. Let $\mathbb{W} \subset \mathbb{V}$ be a subsheaf of an orthonal bundle. From the short exact sequence

$$0 \to \mathbb{W}^{\perp} \to \mathbb{V}^* \to \mathbb{W}^* \to 0$$

we find that $\deg \mathbb{W}^{\perp} = \deg \mathbb{W}$. Then, given a filtration indued by an antidominant character χ, as in the definition of stability, we have for each $k \leq \lfloor r/2 \rfloor$

$$(\lambda_{r-k-1} - \lambda_{r-k}) \deg(\mathbb{V}^k)^{\perp} = (\lambda_k - \lambda_{k+1}) \deg \mathbb{V}^k$$

Therefore,

$$\deg(\pi, \chi) = \sum_{k=1}^{r} (\lambda_k - \lambda_{k+1}) \deg \mathbb{V}^k = 2 \sum_{k=1}^{\lfloor r/2 \rfloor} (\lambda_k - \lambda_{k+1}) \deg \mathbb{V}^k$$

where now each subsheaf in the sum is isotropic (recall that orthogonal bundles have degree zero.) When the vertex is fixed, nothing else needs to be said. When $\mathbb{V} = \mathbb{V}_i \oplus \mathbb{V}_{\sigma(i)}$ is the sum of two exchanged vertices, we have a splitting $\mathbb{V}^k = \mathbb{V}_i^k \oplus \mathbb{V}_{\sigma(i)}^k$, and so,

$$\deg(\pi, \chi_i) = 2 \sum_{k=1}^{\lfloor r \rfloor} (\lambda_k - \lambda_{k+1}) \left(\deg \mathbb{V}_i^k + \deg \mathbb{V}_{\sigma(i)}^k \right)$$

Finally,

$$\begin{aligned}
&\sum_{i=1}^{r'} (a_i \deg(\pi, \chi_i) - 2\tau_i \mathrm{tr} \chi_i) + \sum_{i=1}^{l} b_i \deg(\pi, \chi_i) \\
&= \sum_{i=1}^{r'} \left(2a_i \sum_{k=1}^{\lfloor r/2 \rfloor} (\lambda_k - \lambda_{k+1}) \left(\deg \mathbb{V}_i^k + \deg \mathbb{V}_{\sigma(i)}^k \right) - 2\tau_i \mathrm{tr} \chi_i \right) + \sum_{i=1}^{l} 2b_i \sum_{k=1}^{\lfloor r/2 \rfloor} (\lambda_k - \lambda_{k+1}) \deg \mathbb{W}_i^k \\
&= 2 \left(\sum_{k=1}^{\lfloor r/2 \rfloor} (\lambda_k - \lambda_{k+1}) \left(\sum_{k=1}^{2r'} (a_i \deg(\mathbb{V}_i) - \tau_i \mathrm{rk} \mathbb{V}_i) + \sum_{i=1}^{l} b_i \deg \mathbb{W}_i \right) \right)
\end{aligned}$$

Here we denoted r' the number of orbits of interchanged vertices, so not to be confused with the number of steps in the filtration. Note that by definition, $\lambda_k < \lambda_{k+1}$, and, again, that the terms involve only isotropic subsheaves. Therefore, if the representation is slope-stable, it is stable.

Conversely, given an isotropic sheaf subrepresentation, we apply the stability condition to the two term flag involving that sheaf. □

Inspired by the analogy with the plain case, we make the following definition:

3.3.24 Definition. Let (V, φ) be an orthogonal quiver bundle. Then it is *semistable* if for every isotropic subsheaf representation $\mathscr{F}$, we have $\deg^0_{a,\tau}(\mathscr{F}) \leq 0$.

Given an orthogonal representation, we then have two concepts of (semi) stability for it: as an orthogonal representation of a symmetric quiver (Q,σ) (which we defined above), or as a plain representation of the underlying quiver Q. In fact, these are closely connected.

3.3.25 Theorem. *Let (Q,σ) be a symmetric quiver, and $(\mathbb{V},\varphi)$ be an orthogonal bundle representation. Then,*

1. *$(\mathbb{V},\varphi)$ is semistable as plain representation if and only if it is semistable as an orthogonal representation.*
2. *$(\mathbb{V},\varphi)$ is orthogonally stable if and only if it is an orthogonal sum of mutually non-isomorphic sheaf subrepresentations, each of which is stable as a plain sheaf representation.*

For the proof note, that $\mu_{a,\tau}(\mathbb{V})=0$, since the vector bundle is orthogonal, and so it is isomorphic to its dual (a fact we used before.)

Proof. 1. One direction is obvious. For the other, suppose $\mathbb{V}$ is orthogonally semistable, let $\mathscr{F}$ be an arbitrary (sheaf) subrepresentation, and denote $\mathscr{E}:=\mathscr{F}\cap\mathscr{F}^{\perp}$. Then, $\mathscr{E}$ defines an isotropic subrepresentation, and we have the exact sequence

$$0\to\mathscr{E}\to\mathscr{F}\oplus\mathscr{F}^{\perp}\to\mathscr{M}\to 0$$

We have the isomorphism $\mathscr{M}=\mathscr{E}^{\perp}$ (as sheaves), and since $\mathscr{E}$ is isotropic and φ alternating, $(\mathscr{M},\varphi)$ is a subrepresentation. Indeed, φ_α restricts to a map φ_α^1 of $\mathscr{E}$, and we can write

$$\varphi_\alpha=\begin{pmatrix}\varphi_\alpha^1 & \beta\\ 0 & \varphi_\alpha^2\end{pmatrix}$$

Now, the condition on the maps states that $\varphi_{\sigma(\alpha)}=-\varphi_\alpha^t$. Since $\varphi_{\sigma(\alpha)}$ also restricts to a map in $\mathscr{E}$, this implies that $\beta=0$.

Therefore, $\deg^0_{a,\tau}(\mathscr{F}\oplus\mathscr{F}^{\perp})=\deg^0_{a,\tau}(\mathscr{E})+\deg^0_{a,\tau}(\mathscr{E}^{\perp})$. Now, since the quadratic form C is non-degenerate, it gives an isomorphism $E\simeq E^*$, while we have a short exact sequence $0\to E^{\perp}\to V\to E^*\to 0$. Hence, $\deg^0_{a,\tau}(F)=\deg^0_{a,\tau}(E)\leq 0$ by the orthogonal semistability of V.

2. Suppose $(\mathbb{V},\varphi)$ is orthogonally stable, but not stable as a plain representation, and let $(\mathscr{F},\varphi)$ be a destabilizing subrepresentation. Using the notation of the previous point, since $\mathbb{V}$ is orthogonally stable, $\mathscr{E}$ is trivial, which means that we have an orthogonal decomposition $\mathbb{V}=\mathscr{F}\oplus\mathscr{F}^{\perp}$, which is also a decomposition of orthogonal representations, since φ is alternating and the quadratic form non-degenerate. Actually, each of the representations is orthogonally stable as well, because $\mathbb{V}$ is so (though not necessarily stable as plain representations.) By induction on the rank, we decompose $\mathbb{V}=\perp_i\mathscr{F}_i$ where the $\mathscr{F}_i$ are stable as plain representations. If we have $\mathscr{F}_1\simeq\mathscr{F}_2$, then the embedding $x\mapsto(x,ix)$ gives an isotropic subrepresentation contradicting the stability of $\mathbb{V}$. Conversely, if no two summands are isomorphic, any subrepresentation of maximal degree would be a sum of some of the $\mathscr{F}_i$, and cannot be isotropic (again, the $\mathscr{F}_i$ cannot be isotropic since $\mathbb{V}$ is non-degenerate.)

□

The following is an easy corollary of the decomposition in the theorem

3.3.26 Corollary. *Let $(\mathbb{V},\varphi)$ be an orthogonally stable representation. Then it is stable as a plain representation if and only if it is orthogonally simple (i.e., its only automorphisms are $\pm I$).*

This has the following easy consequence:

3.3.27 Corollary. *If an orthogonal representation $\mathbb{V},\varphi)$ is stable as a plain representation, it is trivial on any vertex that is not fixed by the involution, i.e., $\mathbb{V}_i = 0$ if $i \neq \sigma(i)$.*

Polystability

The gauge equations for the orthogonal case are just the projection of the equations for the plain case onto the Lie algebra of the orthogonal group. Thus, if an orthogonal representation already solves the gauge equation for the orthogonal case, it also solves them for the plain case, and so it is polystable as a plain representation. This means, in particular, that we have a splitting

$$(\mathbb{V},\varphi) = \bigoplus(\mathscr{F}_i,\varphi)$$

into stable (plain sheaf) subrepresentations. Now, a given summand $(\mathscr{F}_i,\varphi)$ might very well be an orthogonal representation (i.e., the quadratic form might be non-degenerate on its total space), in which case it is orthogonally stable as well, a case which we described above. Otherwise, if the representation is not orthogonal, it must necessarily intersect its orthogonal complement in $(\mathbb{V},\varphi)$, and since stable plain representations are simple, it must be isotropic. Since $(\mathbb{V},\varphi)$ is itself orthogonal, there is a $j \neq i$ such that $(\mathscr{F}_j,\varphi) \simeq (\mathscr{F}_i,\varphi)^*$, and the quadratic form restricts to the standard orthogonal pairing. In other words, $(\mathscr{F}_i,\varphi)\oplus(\mathscr{F}_j,\varphi) = (\mathscr{F}_i,\varphi)\oplus(\mathscr{F}_i^*,\varphi)$ is stable orthogonal representation (but not, of course, stable as a plain representation.) We arrive at the following result:

3.3.28 Lemma. *Let $(\mathbb{V},\varphi)$ be an orthogonal representation which solves the Hitchin-Kobayashi correspondence. Then, we have a decomposition*

$$(\mathbb{V},\varphi) = \bigoplus(\mathscr{F}_i,\varphi)^{f_i} \oplus \bigoplus((\mathscr{E}_i,\varphi)\oplus(\mathscr{E}_i^*,\varphi))^{e_i} \oplus \bigoplus((\mathscr{S}_i,\varphi)\oplus(\mathscr{S}_i^*,\varphi))^{s_i} \tag{3.5}$$

where f_i, e_i, and s_i are positive integers, $(\mathscr{F}_i,\varphi)$ are stable orthogonal subrepresentations, $(\mathscr{S}_i,\varphi)$ and $(\mathscr{E}_i,\varphi)$ are stable plain representations respectively isomorphic and not isomorphic to their dual. Further, a given factor is not isomorphic to any other factor in the sum, and the sums $(\mathscr{E}_i,\varphi)\oplus(\mathscr{E}_i^*,\varphi)$ and $(\mathscr{S}_i,\varphi)\oplus(\mathscr{S}_i^*,\varphi)$ endowed with the standard orthogonal pairing.

3.3.29 Remark. We can make the following change in the previous composition. Let $(\mathscr{S}_i,\varphi)$ be a summand that is isomorphic to its dual. Then, if we choose an $\mathbb{C}$-linear isomorphism $\psi : \mathscr{S}_i^* \simeq \mathscr{S}_i$, it induces an isomorphism $(\mathscr{S}_i,\varphi)\oplus(\mathscr{S}_i^*,\varphi) \simeq (\mathscr{S}_i\otimes\mathbb{C}^2,\varphi)$ defined by $(f,g)\mapsto f\otimes e_1 + \psi(g)\otimes e_2$ (here we are implicitly using that $\mathscr{S}_i$ is locally free outside codimension two.) Recall that such a $\mathbb{C}$ linear isomorphism is equivalent to a pairing on $\mathscr{S}_i$, and in fact the pairing must be skew-symmetric (or else, each factor would be orthogonal itself, and would fit as an $\mathscr{F}_i$.) We will also assume that this isomorphism respects the vertices and the involution on them, in the sense of a representation of a symmetric quiver. Then, the orthogonal pairing on $\mathscr{S}_i \oplus \mathscr{S}_i^*$ coming from $\mathbb{V}$ naturally induces a skew-symmetric pairing on $\mathbb{C}^2$. On the other hand, the maps φ_α naturally induce maps $\overline{\varphi}_\alpha$ on $\mathscr{F} \otimes \mathbb{C}$, determined by the conditions $f \otimes e_i \mapsto \varphi_\alpha(f) \otimes e_1$, and $f \otimes e_2 \mapsto \psi \varphi_{\sigma(\alpha)} \psi^{-1}(g) \otimes e_2$. This map is now alternating with respect to the symplectic form on $\mathscr{F}$; in other words, the previous decomposition can be written as

$$(\mathbb{V}, \varphi) = \bigoplus (\mathscr{F}_i, \varphi)^{f_i} \oplus \bigoplus ((\mathscr{E}_i, \varphi) \oplus (\mathscr{E}_i^*, \varphi))^{e_i} \oplus \bigoplus (\mathscr{S}_i, \varphi)^{s_i}$$

where now the $\mathscr{S}_i$ are symplectic representations of the symmetric quiver. In this way, both orthogonal and symplectic representations of a symmetric quiver are necessary, and in this way we are naturally 'thrown' into the concept of supermixed quivers.

We have in fact fully characterized polystability.

3.3.30 Theorem. *Let $(\mathbb{V}, \varphi)$ be an orthogonal representation. Then, there is a hermitian metric solving the gauge equations if and only if it has a decomposition as in Lemma 3.3.28.*

Proof. If a representation has a decomposition as in Lemma 3.3.28, then it is a sum of stable representations, so it solves the plain gauge equations. Since it is already an orthogonal representation, it solves the orthogonal equations. □

We are now only missing one piece in a complete Hitchin-Kobayashi correspondence: to relate polystability in the sense of solving the gauge equations with polystability in the sense of satisfying a linear criterion like that of stability. The missing step is to characterize the Jordan-Hölder filtration associated with a semistable object, as shown in [Garcia-Prada et al.]. Strictly speaking, our case does not fit into that framework, since their correspondence needs to be tweaked in the same sense that the results in [32] are tweaked in section 3.3.1; doing this, however, should be straightforward. More meaningfully, there isn't, at present, any complete correspondence for base manifolds of higher dimension, at least within the point of view we have taken (i.e., the correspondence as a linear symplectic criterion; see, however, [28].)

3.3.3 Further Examples

The goal of the last section was to show that generalized quivers actually yield down-to-earth objects in concrete cases, despite giving an easier setting for moduli problems. We did this by carefully studying the orthogonal and symplectic case. In this section, we quickly mention a few further examples to reinforce the point.

Supermixed quiver bundles

Recall the definition of supermixed quivers from last chapter. There is a straightforward definition of supermixed quiver bundles that derives from our definition of generalized quiver bundles. This is done as for symmetric quiver bundles.

The work in the finite dimensional case is already enough for the next correspondence.

3.3.31 Theorem. *Let $\tilde{Q}$ be a generalized supermixed quiver, and Q be the corresponding supermixed quiver. Then, there is an equivariant bijection between twisted $\tilde{Q}$-bundles and supermixed twisted bundle representations of Q.*

The proof is exectly like the one for symmetric quivers. It is also straightforward to generalize the other results, and we assemble them here for reference.

3.3.32 Theorem. *Let $\tilde{Q}$ be a generalized supermixed quiver.*

1. *The space of bundle representations of $\tilde{Q}$ embeds as a subspace of the representation space of the underlying quiver Q. Further, for a particular choice of stability parameters, this embedding is symplectic (i.e., the equivariant bijection in the previous theorem is a symplectomorphism.)*
2. *A representation is stable if and only if it is slope semistable, where slope stability is also defined in terms of isotropic subsheaves.*
3. *A representation is semistable as a supermixed representation if and only if it is semistable as a plain representation.*
4. *If the representation $(\mathbb{V},\varphi)$ solves the gauge equations, it has a decomposition*

$$(\mathbb{V},\varphi)=\bigoplus(\mathscr{F},\varphi)\oplus\bigoplus((\mathscr{E}_i,\varphi)\oplus(\mathscr{E}_i^*,\varphi))\oplus\bigoplus((\mathscr{S}_i,\varphi)\oplus(\mathscr{S}_i^*,\varphi))$$

where $(\mathscr{F},\varphi)$ are orthogonal representations, $(\mathscr{S}_i,\varphi)$ a stable plain representations isomorphic to their duals such that $(\mathscr{S}_i,\varphi)\oplus(\mathscr{S}_i^*,\varphi)$ are symplectic representations, $(\mathscr{E}_i,\varphi)$ stable plain representations not isomorphic to their dual, and $(\mathscr{E}_i,\varphi)\oplus(\mathscr{E}_i^*,\varphi)$ is an orthogonal representation with the standard orthogonal pairing of a space with its dual.

$\mathfrak{Q}$-mixed quivers

Generalizations of symmetric and supermixed quivers have already been studied in the finite dimensional case, from an algebraic point of view. In [**?**], Lopatin and Zubkov introduce a unifying notion of $\mathfrak{Q}$-mixed quivers. Essentially, these quivers include more general symmetries by explicitly constructing geometric instances of generalized quivers for very particular choices of the reductive abelian group H.

We first fix some notation. We'll denote by J the standard sympletic form in $\mathbb{C}^{2n}$, that is,

$$J=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Keeping this in mind, we define:

- $S^+(n):=\{A\in \mathrm{GL}(n)|A^t=A\}$
- $S^-(n):=\{A\in \mathrm{GL}(n)|A^t=-A\}$
- $L^+(n):=\{A\in \mathrm{GL}(n)|AJ\in S^+\}$

- $L^-(n) := \{A \in \mathrm{GL}(n) | AJ \in S^-\}$

These are the Lie algebras of the orthogonal and symplectic groups and their complements.

3.3.33 Definition. A *mixed quiver setting* is a quintuple $\mathfrak{Q} = (Q, \mathbf{n}, \mathbf{g}, \mathbf{h}, \sigma)$ where Q is a quiver, $\mathbf{n}$ is a dimension vector for Q, $\mathbf{g} = (g_i)$ is a symbol sequence indexed by the vertices of Q with $g_i \in \{\mathrm{GL}, \mathrm{O}, \mathrm{SO}, \mathrm{Sp}, \mathrm{SL}\}$, $\mathbf{h} = (h_\alpha)$ are symbols indexed by the arrows of Q with $h_\alpha \in \{M, S^+, S^-, L^+, L^-\}$, and σ is an involution on the sets of vertices and arrows. These are subject to the conditions:

1. if $g_i = \mathrm{Sp}$, then n_i is even;
2. if $h_\alpha \neq M$, then $n_{t(\alpha)} = n_{h(\alpha)}$;
3. if α is a loop and $h_\alpha = S^+$ or S^-, then $g_{t(\alpha)} = \mathrm{O}$ or SO;
4. if α is a loop and $h_\alpha = L^+$ or L^-, then $g_{t(\alpha)} = \mathrm{Sp}$.
5. $n_{\sigma(i)} = n_i$;
6. if $g_i = \mathrm{O}, \mathrm{SO}, \mathrm{Sp}$, then $\sigma i = i$;
7. if α is not a loop and $h_\alpha \neq M$, then $\sigma(t(\alpha)) = h(\alpha)$ and $h_\alpha = S^+$ or S^-.

The following definition, though not quite the definition in [**?**], is its geometric interpretation.

3.3.34 Definition. Let $\mathfrak{Q}$ be a mixed quiver setting. A $\mathfrak{Q}$-*mixed representation* is a representation (V, φ) of Q, where $V_i = \mathbb{C}^{n_i}$, with the additional data:

1. $V_{\sigma i} = V_i^*$;
2. if $g_i = \mathrm{O}$ or SO, then V_i comes with the standard orthogonal form, and if $g_i = \mathrm{Sp}$, the V_i comes with the standard symplectic form;
3. if $i \neq \sigma(i)$, Then $V_i \oplus V_{\sigma(i)}$ comes with the standard orthogonal pairing;
4. if $g_i = \mathrm{SL}$ or SO, then V_i comes with a volume form;
5. if $h_\alpha = M$, S^+, S^-, L^+, or L^-, then with respect to the previous conditions, $\varphi_\alpha \in M(n_\alpha)$, $S^+(n_\alpha)$, $S^-(n_\alpha)$, $L^+(n_\alpha)$, or $L^-(n_\alpha)$, respectively.

The reduction to the generalized quiver setting is straightforward if we note the following: a $\mathfrak{Q}$-mixed representation is identified by an element of $H(\mathbf{n}, \mathbf{h}) = \bigoplus H_\alpha$, where $H_\alpha = M(n_\alpha)$, $S^+(n_\alpha)$, $S^-(n_\alpha)$, $L^+(n_\alpha)$, or $L^-(n_\alpha)$ according to whether $h_\alpha = M$, S^+, S^-, L^+, or L^-, respectively; and the symmetry group for such a representation is $G(\mathbf{n}, \mathbf{g}) = \prod G_i$, where $G_i = \mathrm{GL}(n_i)$, $\mathrm{SL}(n_i)$, $\mathrm{O}(n_i)$, $\mathrm{SO}(n_i)$, or $\mathrm{Sp}(n_i)$ according to whether $g_i = \mathrm{GL}$, SL, O, SO, or Sp, respectively.

Note that these representations include both plain representations, as well as orthgonal and symplectic representations of symmetric quivers. Signed quivers correspond to the case where $g_i = \mathrm{GL}, \mathrm{O}, \mathrm{Sp}$ for all i; mixed quivers correspond to $g_i = \mathrm{GL}$ and $h_\alpha = M$ for all vertices i and arrows α. Note also that the full generality of the inclusion of $\mathrm{SO}(n)$ and $\mathrm{SL}(n)$ is not used, from the point of view of generalized quivers, in the sense that we do not allow for a general choice of one of their abelian subgroups H (see the next example.) This is due mostly to the fact that Lopatin-Zubkov are interested in their role in yielding *semi-invariants* of representations of $\mathrm{O}(n)$ and $\mathrm{GL}(n)$, respectively.

Higgs bundles over Riemann surfaces

A Higgs bundle over a Riemann surface is the case of a single vertex and a single morphism. These provide the simplest examples of the theory, and indeed our observations about twistings apply directly to this case. In all instances, the twisting bundle is $\mathbb{M} = \mathbb{K}$, the canonical bundle of X.

Let first $G = \mathrm{GL}(n,\mathbb{C})$, and $H = Z(G) = \{\lambda \mathrm{id} | \lambda \in \mathbb{C}^*\}$ be the torus of constant-diagonal matrices. The centralizer of H in G is then all of G. Under the action of G, the Lie algebra $\mathfrak{g}$ is an irreducible module, and so, necessarily $\mathrm{Rep}(Q,V) = \mathfrak{g}^{\oplus n}$; for the Higgs bundle, we take $n = 1$. Since our representation space only has one irreducible component, a representation is especially simple: the vertex symmetry group and the twisting group act separately on the morphism and twisting vector spaces. Then, we must choose a principal $\mathrm{GL}(n,\mathbb{C})$-bundle $E^{\mathbb{C}}$, and the space of generalized quiver bundles for such a choice of H is

$$\mathfrak{Rep}(\tilde{Q},\mathfrak{g}) = \mathrm{Ad}(E) \otimes \mathbb{K}$$

Specializing to unitary/hermitian case, we need a $\mathrm{U}(n)$-bundle E, and a section $\varphi \in (\mathrm{Ad}(E) \otimes \mathbb{K})$. We almost trivially recover the classical case: under the standard representation $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}^n$, $\mathfrak{g}$ identifies with $\mathrm{End}(\mathbb{C}^n)$, and $\mathrm{Ad}(E)$ identifies with $\mathrm{End}(\mathbb{V})$ for some vector bundle $\mathbb{V}$. Then,

$$\mathfrak{Rep}(\tilde{Q},\mathfrak{g}) = \mathrm{End}(\mathbb{V}) \otimes \mathbb{K} = \mathrm{Hom}(\mathbb{V}, \mathbb{V} \otimes \mathbb{K})$$

i.e., a representation of the arrow is just a Higgs field $\varphi : \mathbb{V} \to \mathbb{V} \otimes \mathbb{K}$.

The stability for this special case was originally studied by Hitchin in the rank 2 in [18]. As we have mentioned, it is precisely on Riemann surfaces that the stability condition simplifies to a slope condition without extending the theory to coherent sheaves. In fact, defining $\mu = \deg(\mathbb{V})/\mathrm{rk}(\mathbb{V})$ for any vector bundle $\mathbb{V}$, Hitchin found that the correct semistability condition is that $\mu(\mathbb{E}) \le \mu(\mathbb{V})$ for every proper subbundle $0 \neq \mathbb{E} \subset \mathbb{V}$ that is φ invariant in the sense that $\varphi(\mathbb{E}) \subset \mathbb{E} \otimes \mathbb{K}$, stability corresponding to a strict inequality. General polystability can then be described as a splitting into a sum of non-isomorphic stable bundles of the same slope.

The case for closed linear groups $G \subset \mathrm{GL}(n,\mathbb{C})$ follows readily in an analogous manner. We take $H = Z(G) = Z(\mathrm{GL}(n,\mathbb{C})) \cap G$, finding that the centralizer of H will again be the whole of G, and obviously $\mathfrak{g}$ is irreducible as an $\mathrm{Ad}G$-module. We find that under the standard representation, the morphism representation is a section $\varphi \in (E \times_{\mathrm{Ad}} \mathfrak{g}) \otimes \mathbb{K}$. These have been amply studied.

We note that when the base manifold is of higher dimension (i.e., X is not a curve,) Simpson [**?**] established the stability conditions for Higgs bundles. However, on higher dimensional base manifolds one imposes an integrability condition on the Higgs field. Such integrability condition is required for the non-abelian Hodge theorem to hold, but it is unnatural from the point of view of quiver bundles.

G-Higgs bundles

Let G be a real reductive Lie group, and K a maximal compact, with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, respectively. *G-Higgs bundles* are generalizations of the previous example that have been the object of intense study which encompass the real case; we hope to explore the case of real Lie groups in a later stage. Note that $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is an Ad K-module (this is the isotropy representation.) A G-Higgs bundle is a $K^{\mathbb{C}}$-principal bundle E together with an element $\varphi \in E(\mathfrak{m}^{\mathbb{C}}.)$ Suppose now that G has a complexification. Then, $\mathfrak{m}^{\mathbb{C}}$ sits naturally inside $\mathfrak{g}^{\mathbb{C}}$ as an Ad $K^{\mathbb{C}}$-module. In this setting, this clearly defines a generalized G-quiver. (It is *not* of type Z.)

Note that in the definition of generalized quiver, we could prefectly well have take G to be a real reductive group, and thus included a general G-Higgs bundle (i.e., one for which the complexification of G does not necessarily exist.) In fact, in that situation, both $\mathfrak{g}$ and G come with a (global) Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and $G = K\exp(\mathfrak{m})$. Note that the first decomposition is *not* a decomposition of Lie algebras, but a decomposition of ad$\mathfrak{h}$- (or AdH-) modules. Also, $\mathfrak{m}^{\mathbb{C}}$, as a Ad$K$-module decomposes into two pieces isomorphic to $\mathfrak{m} \subset \mathfrak{g}$. Therefore, it makes sense to define the following generalized G-quiver bundle: $(Z(K), K, \mathfrak{m}^{\mathbb{C}})$.

'Traceless quivers'

Symmetric quivers are atypical in the sense that they give a full characterization of orthogonal generalized quivers. It is much harder to characterize generalized quivers for a more general group (that is, a result describing geometrically the generalized quiver for any choice of reductive abelian subgroup.) As we have mentioned, this full generality is not even used for the case $\mathrm{SL}(n)$, and we want to use these to exemplify the complexity of the situation.

Let H be an abelian reductive subgroup of $\mathrm{SL}(V)$. As in the orthogonal case, we can decompose V in isotypic components $V = \bigoplus V_\mu$ where μ is a character of H. Noting that we can find a basis of V with respect to this splitting, and denoting by ω the volume form of V determined by the choice of a special linear group, we have

$$\omega(ge_1, ..., ge_n) = (\prod_i \mu_i^{n_i})\omega(e_1, ..., e_n) \tag{3.6}$$

where the μ_i are the distinct characters in the representation, and n_i the respective multiplicity. These characters must then satisfy $\prod \mu_i^{n_i} = 1$. This condition is much weaker than the condition for the orthogonal group, and accounts for the difficulty of completely characterizing this case.

One possibility is that for every character μ_i, we have $\mu_i^{n_i} = 1$. In that case, it is easy to geometrically interpret the generalized quiver: draw one vertex for every character in the representation; let $\mathbf{n} = (n_i)$ be the dimension vector (where n_i are the multiplicities.) The Lie algebra $\mathfrak{sl}(V)$ splits into summands $\mathrm{Hom}(V_i, V_j)$ if $i \neq j$, and $\mathrm{End}_0(V_i)$ (traceless endomorphisms.) Thus, the representation space in the definition of generalized quiver can be interpreted in terms of arrows between the vertices just drawn.

It is easy to see that the previous situation must not hold. Inside $\mathrm{SL}(2)$, take H to be the group of all matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

Then, H is its own centralizer, so that $R = H$, and under the action of H, $\mathfrak{sl}_n$ splits as above. We can therefore still interpret the generalized quiver geometrically as a quiver with two vertices, but the group of symmetries now pairs the two vertices together (the bundles are dual line bundles.)

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