JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 124, No. 1, pp. 55–77, January 2005 (© 2005)

Necessary Conditions for Impulsive Nonlinear Optimal Control Problems without a priori Normality Assumptions[1]

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Communicated by B. Polyak

Abstract. First-order and second-order necessary conditions of optimality for an impulsive control problem that remain informative for abnormal control processes are presented and derived. One of the main features of these conditions is that no a priori normality assumptions are required. This feature follows from the fact that these conditions rely on an extremal principle which is proved for an abstract minimization problem with equality constraints, inequality constraints, and constraints given by an inclusion in a convex cone. Two simple examples illustrate the power of the main result.

Key Words. Optimal impulsive control, extremal principle, second-order optimality conditions, abnormality.

1. Introduction

Let us consider the following fixed-time optimal control problem:

$$\text{(A)}\quad \min \quad J(x_0, u, w) = L_0(a), \tag{1}$$

$$\text{s.t.}\quad dx(t) = f(t, x(t), u(t))dt + G(t, x(t))dw(t), \quad t \in [t_0, t_1], \tag{2}$$

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[1]The first author was partially supported by the Russian Foundation for Basic Research Grant 02-01-00334. The second author was partially supported by the Russian Foundation for Basic Research Grant 00-01-00869. The third author was partially supported by Fundacao para a Ciencia e Tecnologia and by INVOTAN Grant.

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0022-3239/05/0100-0055/0

$$L_1(a) \leq 0, \quad L_2(a) = 0, \tag{3}$$

$$dw \in \mathcal{K}. \tag{4}$$

Here,

$$a = (x(t_0), x(t_1)), \quad x(t_0) = x(t_0^-) = x_0, \quad x(t_1) = x_1, \quad t_0 < t_1$$

are given. The mappings

$$f : [t_0, t_1] \times R^n \times R^m \to R^n, \quad G : [t_0, t_1] \times R^n \to R^{n \times k},$$
$$L_i : R^n \times R^n \to R^{d(L_i)}, \quad i = 0, 1, 2,$$

are given, with $d(L_i)$ the dimension of the vector function L_i, $d(L_0) = 1$, and dw is a k-dimensional Borel measure associated with the function of bounded variation $w(t)$, right continuous on $(t_0, t_1]$. The cone $\mathcal{K}$ is defined by

$$\mathcal{K} = \{dw \in C^*([t_0, t_1]; R^k) : \forall \text{ continuous } \phi \text{ such that}$$
$$\phi(t) \in K^0 \, \forall t, \int_B \phi(t) dw \geq 0, \forall \text{ Borel } B \subset [t_0, t_1]\},$$

where K is a given convex, closed, pointed cone from R^k and K^0 is its dual. In another words, the measure dw satisfies

$$\int_B dw(t) \in K, \quad \text{for all Borel subsets } B.$$

The pair (u, w) is called an admissible control if $u \in L_\infty^m$ and $w \in BV^k$ is such that $dw \in \mathcal{K}$.

Let us describe our assumptions for problem A:

(H1) The functions L_0, L_1, L_2 are C^2.

(H2) The function f is twice differentiable w.r.t. x and u for almost all $t \in [t_0, t_1]$; the function f plus the first-order and second-order derivatives are measurable w.r.t. t and bounded on any bounded subset.

(H3) The matrix function $G \in C^2$.

(H4) The matrix G satisfies the Frobenius condition, i.e.,

$$G_x^i(t, x) G^j(t, x) - G_x^j(t, x) G^i(t, x) \equiv 0, \tag{5}$$

where G^i is the ith column of G.

Notice that, under (H4), the dynamic system (2) is robust w.r.t. approximations of the generalized control dw by conventional controls $v(\cdot) \in L^k_\infty([t_0, t_1]; K)$; see Refs. 1–4. If the Frobenius condition holds, then for any given admissible control (u, w) and initial condition x_0, the corresponding trajectory (whose existence is assumed) is the unique right-continuous function of bounded variation on $(t_0, t_1]$, with $x(t_0) = x_0$ such that

$$x(t) = x_0 + \int_{t_0}^{t} f(\theta, x(\theta), u(\theta))d\theta + \int_{[t_0,t]} G(\theta, x(\theta))dw_c(\theta) + \sum_{S_i \le t} (z(1; s_i, c^i) - x(s_i^-)). \tag{6}$$

Here, dw_c represents the continuous part of dw,

$$dw_a(t) := \sum c^i \delta_{S_i}$$

is the atomic part, $s_i \in [t_0, t_1]$ are the jump times of dw (times of impulses), δ_S is the Dirac measure at time $s, c^i \in K$ are the jumps of dw, and the function $z^i(\tau) = z(\tau; s_i, c_i)$ is the solution to the limiting system

$$dz^i/d\tau = G(s_i, z^i)c_i, \quad z^i(0) = x(S_i^-); \tag{7}$$

hence,

$$z^i(1) = x(s_i^+).$$

The robustness of the system (2), due to (H4), implies that the solution (6) belongs to the closure of the set of absolutely continuous solutions of equation (2) corresponding to $(u, w) \in L_\infty \times AC$.

An admissible control process is a triplet (x_0, u, w), where (u, w) is an admissible control and the corresponding state trajectory satisfies the given endpoint constraints. The problem under consideration is to minimize J over the set of admissible control processes.

By (x_0^*, u^*, w^*) and x^*, we denote respectively an admissible control process and the corresponding state trajectory investigated for a minimum of problem (A). It is assumed that this control process satisfies the following additional assumption:

$$\text{(H5)} \quad dw^*(t) = v^*(t)dt + \sum_{s \in S^*} c^s \delta_s(t), \tag{8}$$

where $\nu^*(t)=\dot{w}^*(t)$ a.e. with respect to the Lebesgue measure on $[t_0, t_1]$[5], $S^* \subset [t_0, t_1]$ is the set of jump times of $w^*(\cdot)$, assumed to be finite, and

$$c^s = [w^*(s)] := w^*(s^+) - w^*(s^-),$$

i.e., the function $w^*(\cdot)$ has no singular continuous part and has a finite number of jump times.

Moreover, since (x_0^*, u^*, w^*) is investigated for a local minimum only (in the sense of Definition 1.1 below), then without loss of generality we can assume that all endpoint inequality constraints are active at the optimal trajectory x^*, i.e.,

$$L_1(a^*)=0, \quad \text{where } a^*=(x^*(t_0), x^*(t_1)). \tag{9}$$

Dynamic optimization problems arising in a variety of application areas such as finance, mechanics, resources management, and space navigation (see Refs. 4–10, just to mention a small but representative sample of references), whose solutions might involve discontinuous trajectories, have been considered over the years, motivating a significant research effort on the impulsive control problem.

In order to not obscure the aim of this article, we selected the simplest control problem paradigm enabling us to deal with the issues relevant to first-order and second-order conditions for impulsive control problems that remain informative, even for abnormal control processes. It is not difficult to see that this result can be derived for a number of different and more complex control formulations. In particular, by standard state-variable manipulations, one can convert Bolza and Lagrange types of cost functionals into the one stated here.

The approach of this article can be used to derive these optimality conditions for problems with regular control constriants of the type $R(u,t)=0$. Under regularity assumptions (see Ref. 11), the implicit function theorem can be used to solve (for each t) this equation in u, thus converting the control problem into the one considered here.

Definition 1.1. We say that the admissible process (x_0^*, u^*, w^*) is a local minimizer of the problem (A) if $\exists \varepsilon > 0$ and, for any finite-dimensional subspace $R \subset L_\infty^m[t_0, t_1]$, $\exists \varepsilon_R > 0$ such that process (x_0^*, u^*, w^*) yields the minimum to problem (1)–(4) with the additional constraints

$$\|a - a^*\| < \varepsilon, \quad \|dw - dw^*\|_{C^*([t_0,t_1];R^k)} < \varepsilon,$$
$$\|u - u^*\|_{L_\infty^m[t_0,t_1]} < \varepsilon_R, \quad u(\cdot) \in R.$$

[5]Heretofore, $\mathcal{L}$-a.e. denotes a.e. w.r.t. the Lebesgue measure.

The defined type of local minimum is finite dimensional in u and weak in dw.

In this article, we obtain first-order and second-order necessary conditions of optimality for the problem under consideration. The main features of the results is that no a priori normality assumptions are required and that they are informative for abnormal control processes as well. Another issue concerns the fact that, in the problem considered, the function G depends also on x. The proof of these conditions is based on a nonlinear transformation of the initial problem A (Ref. 4) into another one for which G does not depend on x and first-order and second-order necessary conditions of optimality were derived in Ref. 12.

In spite of the well-developed theory of higher-order necessary conditions of optimality for conventional optimal control problems (see for example, Refs. 11, 12, 14), it is somewhat surprising that, from the vast amount of literature addressing optimal impulsive control problems (Refs. 1–3, 15–22), only a few publications are available (Refs. 23–25, 27).

We notice that, while the conditions in Refs. 23, 24 become trivial (i.e., degenerate, for abnormal problems), ours remain informative. Also, our results differ substantially from these conditions as it can be seen from the fact that these follow directly from the maximum principle in the case the optimal trajectory is absolutely continuous, i.e., with no impulses.

In Ref. 3, second-order necessary conditions of optimality of the Legendre-Jacobi-Morse type for time-optimal control are derived by using in an essential way an extremal principle and the notion of index of quasiextremality provided in Ref. 26.

However, the approach followed here differs substantially from all the ones in the references cited above as we regard this problem as a specific instance of a general abstract problem for which powerful second-order optimality conditions are derived.

This article is organized as follows. In Section 2, we introduce key definitions and state first-order and second-order necessary conditions of optimality for the dynamic optimization problem described in Section 1. Issues concerning abnormality, geometric interpretation, and computation are also discussed. In Section 3, we present the proof, which is organized in three parts: transformation of the given problem into another one for which there are first-order and second-order necessary conditions of optimality available; statement of the mentioned optimality conditions for the problem considered; and decoding of the thus obtained first-order and of second-order conditions in terms of the data of the original problem. Finally, in Section 4, two examples illustrate the application of these conditions.

2. Second-Order Necessary Conditions of Optimality

Let us state the necessary conditions of optimality for problem A. Before presenting the main result, we discuss some auxiliary concepts which are fundamental for the statement of our main result: local maximum principle, critical cone, and quadratic form.

2.1. Local Maximum Principle. Let

$$F(t,x,u,v)=f(t,x,u)+G(t,x)v$$

and let

$$\psi\in R^n,\quad \lambda=(\lambda_0,\lambda_1,\lambda_2)\in R^1\times R^{d(L_1)}\times R^{d(L_2)}.$$

Define the Pontryagin function $H=H_0+H_1$ and the endpoint Lagrangian l^λ by

$$H_0(t,x,\psi,u)=\langle\psi, f(t,x,u)\rangle,$$
$$H_1(t,x,\psi,v)=\langle\psi, G(t,x)v\rangle,$$
$$l^\lambda(a)=\lambda_0L_0(a)+\langle\lambda_1, L_1(a)\rangle+\langle\lambda_2, L_2(a)\rangle.$$

Definition 2.1. We say that a process (x_0^*, u^*, w^*) satisfies the Euler-Lagrange conditions or the local maximum principle if there exists $\lambda\neq 0$ such that

$$\lambda_0\geq 0,\quad \lambda_1\geq 0,\quad \langle\lambda_1, L_1(a^*)\rangle=0 \tag{10}$$

and the vector function ψ, solution to the adjoint system

$$-d\psi(t)=H_{0x}(t)dt+H_{x_v}(t)dw^*(t),\quad -\psi(t_1)=l^\lambda_{x_1}(a^*), \tag{11}$$

which satisfy the following conditions:

$$\psi(t_0)=l^\lambda_{x_0}(a^*), \tag{12}$$
$$H_u(t)=0,\qquad \mathcal{L}\text{-a.e.}, \tag{13}$$
$$\langle H_v(t), v\rangle\leq 0,\qquad \forall(t,v)\in[t_0,t_1]\times K, \tag{14}$$
$$\langle H_v(t), \bar\omega^*(t)\rangle=0,\qquad dw^*\text{-a.e.}, \tag{15}$$

where

$$\bar\omega^*(t)=dw^*(t)/d|w^*(t)|$$

is the Radon-Nicodym derivative of the measure dw^* with respect to its total variation measure.

Notice that the solution to the adjoint system (11) is in the same sense as the one to (6), i.e.,

$$\psi(t_1) = -l^{\lambda}_{x_1}(a^*)$$

and that

$$\psi(t) = -l^{\lambda}_{x_1}(a^*) + \int_t^{t_1} H_{0x}(\theta)d\theta + \int_t^{t_1} H_{xv}(\theta)dw^*_c(\theta) + \sum_{s_i > t}(\psi(s_i) - q(0; s_i, c^i)), \quad t \in [t_0, t_1). \tag{16}$$

Here, the functions $q^i(\tau) = q^i(\tau; s_i, c^i)$ are solutions to the adjoint limiting system

$$-dq^i/d\tau = H_{xv}(s_i, z^i(\tau), q^i(\tau))c^i, \quad q^i(1) = \psi(s_i), \tag{17}$$

with the corresponding solution $z^i(\tau)$ to the system (7) when $x(s_i^-) = x^*(s_i^-)$. The notation

$$H_{xv}(t) = \frac{\partial^2 H}{\partial v \partial x}(t)$$

refers to the evaluation of the function H_{xv} along the process examined (this notation is adopted also for other functions in similar contexts). We remark that any adjoint trajectory $\psi(t)$ and the function $H(t)$ depend on λ due to the transversality condition (12).

Denote by

$$\Lambda = \Lambda(x_0^*, u^*, w^*)$$

the set of all normalized Lagrange multipliers λ, $\|\lambda\| = 1$, satisfying the local maximum principle. It is well known that $\Lambda \neq \emptyset$ is a first-order necessary condition for a weak local minimum for problem A. However, we shall prove here that it is also necessary for the local minimum in the sense of Definition 1.1. Note that the local maximum principle holds without (H5).

2.2. Critical Cone. In order to ensure a compact statement of the second-order conditions, we shall use the total derivative w.r.t. time along the solution to the following ordinary differential system:

$$\dot{x} = F(t, x, u, v), \tag{18a}$$

$$-\dot{\psi} = H_x(t, x, \psi, u, v), \tag{18b}$$

$$\dot{w} = v, \quad v(t) \in K. \tag{18c}$$

For example,

$$(\dot{H}_v)_x = (\partial/\partial x)[(d/dt)(\partial H/\partial v)]_{|t, x^*(t), u^*(t), w^*(t))}.$$

Under the Frobenius condition, this derivative does not depend on v, but in any other case, we put always $v^*(t) = \dot{w}^*(t)$; see (8). Denote by $BV^n(S^*)$ the set of n-dimensional vector functions of bounded variation whose jump times are supported on S^*. Clearly, each term in $(x^*(\cdot), \psi(\cdot), w^*(\cdot))$ is in a $BV(S^*)$ space of the corresponding dimension.

Definition 2.2. A variation $(\delta x_0, \delta u, \delta w) \in R^n \times L^m_\infty \times BV^k(S^*)$ is called critical if the corresponding state trajectory variation $\delta x \in BV^n(S^*)$ satisfies the following conditions:

$$\langle L_{ia}(a^*), \delta a\rangle + \langle L_{ix_1}(a^*), G(t_1)\delta w_1\rangle \begin{cases} \leq 0, & i = 0, 1, \\ = 0, & i = 2, \end{cases} \tag{19}$$

$$\delta a = (\delta x(t_0), \delta x(t_1)), \quad \delta w_1 = \delta w(t_1), \tag{20}$$

$$d(\delta x)/dt = F_x(t)\delta x + F_u(t)\delta u - (\dot{H}_v)^T_\psi(t)\delta w, \quad t \notin S^*, \tag{21}$$

$$d(\delta w) \in \mathcal{K} + \text{Lin } \{dw^*\}, \quad \delta w(t_0) = 0, \tag{22}$$

$$\delta x(s) = \delta q(1; s, c), \quad \forall s \in S^*. \tag{23}$$

Here,

$$G(t_1) = G(t_1, x^*(t_1)),$$

$\delta q(\tau; s, c) := \delta q^s(\tau)$ is the solution to the system

$$d(\delta q^s)/d\tau = H_{1\psi x}(s, z^s(\tau), c)\delta q^s, \tag{24a}$$

$$\delta q^{t_0}(0) = \delta x_0, \tag{24b}$$

$$\delta q^s(0) = \delta x(s^-), \quad s > t_0, \tag{24c}$$

and the function $z^s(\tau)$ is solution of (7) when $s_i = s, x(s^-) = x^*(s^-)$; recall that $c = [w^*(s)]$.

Denote by $\mathcal{K}_{cr}$ the cone of all critical variations.

2.3. Quadratic Form. For any $\lambda \in \Lambda$, define the quadratic form

$$\Omega^\lambda(\delta x_0, \delta u, \delta w) = \delta a^T l^\lambda_{aa}(a^*)\delta a + Q^\lambda_1(\delta a, \delta w_1) - \int_{t_0}^{t_1} Q^\lambda(\delta x, \delta u, \delta w)(t)dt, \tag{25}$$

where Q^λ and Q^λ_1 are the following quadratic forms:

$$Q^\lambda(\delta x, \delta u, \delta w) = \delta u^T H^\lambda_{uu}\delta u + 2\delta x^T H^\lambda_{xu}\delta u - 2\delta w^T(\dot{H}^\lambda_v)_u\delta u - \delta w^T(\ddot{H}^\lambda_v)_v\delta w - 2\delta w^T(\dot{H}^\lambda_v)_x\delta x + \delta x^T H^\lambda_{xx}\delta x, \tag{26}$$

$$Q^\lambda_1(\delta x(\cdot), \delta w_1) = 2[\delta x(t_0)^T l^\lambda_{x_0x_1}(a^*)G(t_1) - \delta x(t_1)^T H^\lambda_{xv}(t_1)]\delta w_1 + \delta w_1^T G^T(t_1)[l^\lambda_{x_1x_1}(a^*)G(t_1) - H^\lambda_{xv}(t_1)]\delta w_1 - \sum_{s\in S^*}[\delta x^T(s)\Psi^\lambda(s)\delta x(s) - \delta x^T(s^-)\Psi^\lambda(s^-)\delta x(s^-)]. \tag{27}$$

Here, dependence on top Q^λ is omitted, H^λ refers to the Pontryagin function evaluated along (t, x^*, ψ, u^*, v^*), ψ satisfied (11) for a certain λ, and $\delta x(\cdot)$ is the corresponding solution to (24) with (22), (23), $\delta x(t_0^-) = \delta x_0$, $w(t_0^-) = 0$. Only $\Psi^\lambda(t) \in BV^{n\times n}(S^*)$ remains to be defined in formula (27). For this, let $z^*(\tau; t)$ and $q^*(\tau; t)$ be solutions of

$$dz^*/d\tau = G(t, z^*)w^*(t), \qquad z^*(1; t) = x^*(t),$$
$$-dq^*/d\tau = H_{1x}(t, z^*, q^*, w^*(t)), \qquad q^*(1; t) = \psi(t),$$

and let the $n \times n$ matrix $Z(\tau; t)$ satisfy

$$-dZ/d\tau = ZH_{1\psi x}(t, z^*(\tau, t), w^*(t)), \quad Z(0; t) = I.$$

Then,

$$\Psi^\lambda(t) = -Z^T(1; t)\left(\int_0^1 Z^{-1T}(\tau; t)H_{1xx}(\tau; t)Z^{-1}(\tau; t)d\tau\right)Z(1; t), \tag{28}$$

where the expression $H_{1xx}(\tau; t)$ is a short notation for H_{1xx} evaluated along $(t, z^*(\tau; t), q^*(\tau; t), w^*(t))$. We remark that $\Psi(t_0^-) = 0$ because $w(t_0^-) = 0$.

2.4. Main Result. Let π be the orthogonal projection from R^k onto the linear subspace N defined by

$$N = K \cap (-K).$$

Obviously, $C^*([t_0, t_1], N)$ is the maximal linear subspace contained in $\mathcal{K}$. Consider the following modified variational equation:

$$d(\delta x)/dt = F_x(t)\delta x + F_u(t)\delta u - (\dot{H}_v)_\psi(t)\pi\delta w, \quad t \notin S^*, \tag{29}$$

with jump conditions (23), (24), where

$$\delta x(t_0) = \delta x_0 \in R^n, \quad \delta u \in L_\infty^m, \quad \delta w \in L_\infty^k. \tag{30}$$

Consider the quadratic form Ω_a^λ defined on $R^n \times L_\infty^m \times L_\infty^k \times R^k$ obtained from Ω^λ by formally replacing δw_1 by h. Put $L = (L_1, L_2)$ and consider the set of all tuples $(\delta x_0, \delta u, \delta w, h) \in R^n \times L_\infty^m \times L_\infty^k \times R^k$ such that the corresponding solution of (29) with (23), (24) satisfies

$$L_a(a^*)\delta a + L_{x_1}(a^*)G(t_1)\pi h = 0, \quad h \in R^k.$$

This set, denoted by $\mathcal{K}_\pi$, is obviously a linear subspace of $R^n \times L_\infty^m \times L_\infty^k \times R^k$.

Define the linear operator $\mathcal{A}: \mathcal{K}_\pi \to R^{d(L)}$ by the formula

$$\mathcal{A}(\delta x(0), \delta u, \delta w, h) = L_{x_0}(a^*)\delta x_0 + L_{x_1}(a^*)\delta x_1 + L_{x_1}(a^*)G(t_1)\pi h,$$

where δx is the corresponding solution to (29), (30), (23), (24). Put

$$d = \operatorname{codim}(Im\mathcal{A})$$

and denote by $\Lambda_a(x^*, u^*, w^*)$, Λ_a for short, the set of vectors $\lambda \in \Lambda(x^*, u^*, w^*)$ such that the index[6] of the form Ω_a^λ on the subspace $\mathcal{K}_\pi$ is not greater then d.

Theorem 2.1. Necessary Conditions of Optimality. Let the control process (x^*, u^*, w^*) be a local optimal to the problem A. Then, $\Lambda_a \neq \emptyset$ and, for any $(\delta x_0, \delta u, \delta w) \in \mathcal{K}_{cr}$, we have

$$\max_{\lambda \in \Lambda_a} \Omega^\lambda(\delta x_0, \delta u, \delta w) \geq 0. \tag{31}$$

The proof of this theorem is presented in Section 3.

[6]The index of a quadratic form q on a given subspace V is the dimension of a subspace of V of maximum dimension where the quadratic form is negative definite.

Notice that, by definition, the cone $\Lambda_a \subseteq \Lambda$. (therefore, Theorem 2.1) is stronger than the well-known conditions for which cone Λ_a in (31) is replaced by Λ; see Refs. 24, 25.

Remark 2.1. It can be shown easily that d is equal to the dimension of the kernel of the block matrix operator

$$\begin{bmatrix} A \\ B \\ G(t_1)\pi \end{bmatrix} : R^{d(L)} \to R^{n+k+d(L)},$$

where

$$A = L_{x_0}(a^*) + \Phi(t_1)^T L_{x_1}(a^*),$$

$$B = L_{x_1}(a^*)^T \Phi(t_1) \int_{t_0}^{t_1} \Phi^{-1}(t)\Gamma(t) \times \Gamma(t)^T \Phi^{-1}(t)^T dt \Phi(t_1)^T L_{x_1}(a^*),$$

where Φ is a fundamental solution to the system (29); i.e., Φ is the solution to the system

$$(d/dt)\Phi(t) = F_x(t)\Phi(t), \quad \mathcal{L}\text{-a.e.}, \quad \Phi(t_0) = I, \tag{32}$$

$\Gamma(t)$ is the $n \times (m+k)$ block matrix defined by

$$[F_u(t) | -(\dot{H}_v)_\psi(t)\pi],$$

and A^T denotes the transpose of A.

Remark 2.2. The second-order necessary conditions of optimality are also significant for the abnormal case (see Refs. 11, 12, 27). For problem A, the abnormality of the admissible control process $(\delta x_0, u(\cdot), w(\cdot))$ implies that the convex hull of $\Lambda(\delta x_0, u(\cdot), w(\cdot))$ contains 0. Notice that, for the abnormal case, the second-order conditions in which Λ is used instead of Λ_a in formula (31) become trivial (i.e., uninformative).

Remark 2.3. The last term in the expression of Q_1^λ features the left limits $\delta x(s^-)^T, \Psi^\lambda(s^-), s \in S^*$. We note that it is not necessary to extract limits from the left in order to compute these terms. For example, to compute $\Psi^\lambda(t^-)$ at some fixed point $t \in S^*, t > t_0$, with the formula (28), it suffices to solve first the system of differential equations

$$\begin{aligned} dz^*/d\tau &= G(t, z^*)(w^*(t) - c^t), & z^*(1;t) &= x^*(t), \\ -dq^*/d\tau &= H_{1x}(t, z^*, q^*, (w^*(t) - c^t)), & q^*(1;t) &= \psi(t). \end{aligned}$$

Then, $Z(\tau, t^-)$ is obtained by integrating the differential equation

$$-dZ/d\tau = ZH_{1\psi x}(t, z^*(\tau, t), (w^*(t) - c^t)), \quad Z(0; t) = I.$$

Clearly, only two Cauchy problems need to be solved for each atom.

3. Proof of Theorem 3.1

For simplicity, we organize the proof in four steps.

Step 1. Transformation of Problem A into Problem B. Denote by $\xi(t, y, w)$ the solution to the following system of partial differential equations which, under the Frobenius condition, is completely integrable,

$$\partial\xi/\partial w = G(t, \xi), \quad \xi|_{w=0} = y. \tag{33}$$

Since we are interested in only a local minimum, we may assume without any loss of generality that G is bounded. Hence, the global solution ξ is defined on the whole space $[t_0, t_1] \times R^n$. Let

$$\eta(t, x, w) = \xi(t, x, -w)$$

and let us apply the following nonlinear transformation (Refs. 4, 17, 18, 23) to problem A:

$$\Theta : (x(\cdot), u(\cdot), w(\cdot)) \;\to\; (y(\cdot), u(\cdot), w(\cdot), v(d\cdot)), \tag{34a}$$

$$y(t) = \eta(t, x(t), w(t)), \quad v(dt) = dw(t). \tag{34b}$$

Now, we consider the following problem:

$$\begin{aligned}
\text{(B)} \quad \min \quad & I(y_0, u, v) := \tilde{L}_0(b), \\
\text{s.t.} \quad & \tilde{L}_1(b) \le 0, \quad \tilde{L}_2(b) = 0, \\
& \dot{y} = g(t, y, u, w), \\
& dw(t) = v(dt), \quad w(t_0) = 0, \quad v \in \mathcal{K},
\end{aligned}$$

where

$$\begin{aligned}
& b := (y_0, y_1, w_1), \\
& \tilde{L}_i(b) = L_i(y_0, \xi(t_1, y_1, w_1)), \\
& (y_0, y_1, w_1) = (y(t_0), y(t_1), w(t_1)), \\
& g(t, y, u, w) = (\eta_t(t, x, w) + \eta_x(t, x, w) f(t, x, u))|_{x=\xi(t,y,w)}.
\end{aligned}$$

Now, the pair (y, w) is the state variable and the pair $(u(\cdot), v(d\cdot)) \in L^m_\infty \times (C^k)^*$ is the control variable. The minimization in problem B is carried out over all the control processes $(y_0, u(\cdot), v(dt))$ satisfying its constraints. Notice that the component $y(\cdot)$ of the trajectory $(y(\cdot), w(\cdot))$ is absolutely continuous, but the component $w(\cdot)$ is a function of bounded variation. The inverse transformation Θ^{-1} is defined by the equality

$$x(t) = \xi(t, y(t), w(t)).$$

From this property and Definition 1.1, the proposition below follows.

Proposition 3.1. The process $(x_0^*, u^*(\cdot), w^*(\cdot))$ is a local optimum for problem A if and only if the $(y_0^*, u^*(\cdot), v^*(dt)) = (x_0^*, u^*(\cdot), dw^*(\cdot))$ is a local optimum process for problem B.

Step 2. Necessary Conditions of Optimality for Problem B. In Ref. 27, first-order and second-order necessary conditions of optimality for a local minimizer (in the sense of Def. 1.1) of problem B are obtained. Let us write them down now.

Define the Pontryagin function $\tilde{H}$ and the endpoint Lagrangian $\tilde{l}^{\mu}$ for problem B,

$$\tilde{H}(t, y, w, p, p_w, u, v) = \langle p, g(y, w, u, t)\rangle + \langle p_w, v\rangle,$$
$$\tilde{l}^{\mu}(b) = \mu_0 \tilde{L}_0(b) + \langle \mu_1, \tilde{L}_1(b)\rangle + \langle \mu_2, \tilde{L}_2(b)\rangle.$$

Here,

$$\mu = (\mu_0, \mu_1, \mu_2) \in R^1 \times R^{d(L_1)} \times R^{d(L_2)},$$

p and p_w are respectively η-dimensional and k-dimensional vectors. Due to Theorem 2.2 and its Remark in Ref. 27, the process (y_0^*, u^*, v^*) satisfies the Euler-Lagrange conditions and the second-order necessary conditions of optimality. The first-order conditions imply the existence of $\mu \neq 0$ and vector functions (p, p_w) such that

$$\mu_0 \geq 0, \qquad \mu_1 \geq 0, \tag{35}$$
$$-\dot{p}(t) = \tilde{H}_y(t), \ \ \mathcal{L}\text{-a.e.}, \tag{36}$$
$$-\dot{p}_w(t) = \tilde{H}_w(t), \ \ \mathcal{L}\text{-a.e.}, \tag{37}$$
$$(p(t_0), -p(t_1)) = (\tilde{l}^{\mu}_{y_0}(b^*), \tilde{l}^{\mu}_{y_1}(b^*)), \tag{38}$$
$$-p_w(t_1) = \tilde{l}^{\mu}_{w_1}(b^*), \tag{39}$$
$$\tilde{H}_u(t) = 0, \ \ \mathcal{L}\text{-a.e.}, \tag{40}$$
$$\langle \tilde{H}_w(t), v\rangle = \langle p_w(t), v\rangle \leq 0, \ \ \forall v \in K, \forall t \in [t_0, t_1], \tag{41}$$
$$\langle \tilde{H}_w(t), \bar{\omega}^*(t)\rangle = \langle p_w(t), \bar{\omega}^*(t)\rangle = 0, \ \ v^*\text{-a.e.} \tag{42}$$

Here,

$$b^* = (y^*(t_0), y^*(t_1), w^*(t_1))$$

and

$$\bar{\omega}^*(t) = \frac{dv^*}{d|v^*|}(t)$$

is the Radon-Nicodym derivative of the measure v with respect to its total variation measure. Notice that, from the convention explained earlier, we have that $\tilde{L}_1(b^*) = 0$.

Denote by M the set of all normalized vectors μ that satisfy Euler-Lagrange conditions for the process (y_0^*, w^*, u^*, v^*) and consider the following variational system:

$$d(\delta_y(t))/dt = g_y(t)\delta y(t) + g_u(t)\delta u(t) + g_w(t)\delta w(t), \tag{43a}$$

$$d(\delta w)(t) = \delta v(dt), \tag{43b}$$

$$\delta w(t_0) = 0, \;\; \delta u \in L_\infty^m, \;\; \delta v \in \mathcal{K} + \text{Lin}\{v^*\}, \tag{43c}$$

with boundary conditions

$$\tilde{L}_{2b}(b^*)\delta b = 0, \;\; \tilde{L}_{1b}(b^*)\delta b \le 0, \tag{44}$$

where

$$\delta b = (\delta y(t_0), \delta y(t_1), \delta w(t_1)).$$

These conditions, together with the inequality

$$\langle \tilde{L}_{0b}(b^*), \delta b \rangle \le 0, \tag{45}$$

describe the critical cone $\tilde{\mathcal{K}}_{cr}$ at the point (y_0^*, u^*, v^*) for problem B.

For any $\mu \in M$, define the quadratic form

$$\tilde{\Omega}^\mu(\delta y_0, \delta u, \delta v) = -\int_{t_0}^{t_1} \frac{\partial^2 \tilde{H}}{\partial (y, w, u)^2}(t)[(\delta y(t), \delta w(t), \delta u(t))]^2 dt + \delta b^T \tilde{l}_{bb}^\mu(b^*)\delta b. \tag{46}$$

Here, $(\delta y, \delta w)$ is the solution to the system (43) corresponding to δu and $\delta v \in \mathcal{K} + \mathrm{Lin}\{v^*\}$ with initial condition $\delta y(t_0) = \delta y_0$.

Let

$$\mathcal{N} = C^*([t_0, t_1], N)$$

and consider the variational system (43) with dv restricted to $\mathcal{N}$. Due to the definition of the operator π, the differential equation (43a) can be written as

$$d(\delta y(t))/dt = g_y(t)\delta y(t) + g_u(t)\delta u(t) + g_w(t)\pi\delta w(t), \tag{47}$$

where δw is an arbitrary function from $BV^k(S^*)$ such that $\delta w(t_0) = 0$. The equality (47) generates a new linear operator D by the formula

$$D(\delta y_0, \delta u, \delta w) = \tilde{L}_b(b^*)\delta b, \tag{48}$$

where $\tilde{L} = (\tilde{L}_1, \tilde{L}_2)$ and δy is the solution to (47) with initial condition $\delta y(t_0) = \delta y_0$.

Let

$$\tilde{d} = \mathrm{codim}(\mathrm{Im} D).$$

Consider the set $\ker D$, i.e., the set of all triples

$$(\delta y_0, \delta u, \delta w) \in R^n \times L_\infty^m \times BV^k(S^*)$$

such that the corresponding solution to (47) satisfies

$$\tilde{L}_b(b^*)\delta b = \tilde{L}_{y_0}(b^*)\delta y_0 + \tilde{L}_{y_1}(b^*)\delta y(t_1) + \tilde{L}_{w_1}(b^*)\delta w(t_1) = 0. \tag{49}$$

Consider the set of all vectors $\mu \in M$ such that the index of the quadratic form $\tilde{\Omega}^\mu$ restricted to the set $\ker D$ is not greater than $\tilde{d}$. Denote such a set by M_a. From Theorem 2.2 and its Remark in Ref. 27, it can be seen readily that $M_a \neq \emptyset$ and that, $\forall (\delta y_0, \delta u, \delta w) \in \tilde{\mathcal{K}}_{cr}$,

$$\max_{\mu \in M_a} \tilde{\Omega}^\mu(\delta y_0, \delta u, \delta w) \geq 0. \tag{50}$$

We shall need the following extension of (50). Consider the space $E = R^n \times L_\infty^m \times L_\infty^k \times R^k$ consisting of elements $e = (\delta y_0, \delta u, \delta w, h)$. Define the quadratic form $\tilde{\Omega}_a^\mu$ on E by formally replacing $\delta w(t_1)$ by $h \in R^k$ in $\tilde{\Omega}^\mu$ and define the linear operator $\tilde{\mathcal{A}}: E \to R^{d(L)}$ by

$$\tilde{\mathcal{A}}(e) := \tilde{L}y_0(b^*)\delta y_0 + \tilde{L}_{y_1}(b^*)\delta y(t_1) + \tilde{L}_{w_1}(b^*)h, \tag{51}$$

where $\delta y(\cdot)$ is the corresponding solution to (47). Put

$$\tilde{\mathcal{K}}_\pi = \mathrm{Ker}\tilde{\mathcal{A}}.$$

Proposition 3.2. We have that $\tilde{d} = \mathrm{codim}\ (\mathrm{Im}\tilde{\mathcal{A}})$ and also $\mu \in M_a$ is equivalent to the fact that the index of the form $\tilde{\Omega}_a^\mu$ considered on the linear subspace $\tilde{\mathcal{K}}_\pi$ is not greater than $\tilde{d}$.

This proposition follows from the fact that BV^k is dense in L_∞^k with respect to the L_2^k metric and its proof is based on standard Lebesgue integration arguments.

Step 3. Decoding of the Local Maximum Principle. From the definition of ξ and η, it follows that η satisfies the partial differential system

$$\eta_w(t,x,w) + \eta_x(t,x,w)G(t,x) = 0, \tag{52}$$

with boundary condition $\eta(t,x,0) = x$. Moreover, the following equalities hold:

$$\eta(t,\xi(t,y,w),w) = y, \quad \xi(t,\eta(t,x,w),w) = x. \tag{53}$$

We shall use these relations and their consequences obtained by differentiation.

Let us consider the following proposition.

Proposition 3.3. The function $\sigma(t,y,p,w) = \eta_x^T(t,\xi(t,y,w),w)p$ is a solution to the following completely integrable system:

$$\sigma_w = -H_{xv}(t,\xi(t,y,w),\sigma,u,v), \quad \sigma|_{w=0} = p. \tag{54}$$

Proof. Obviously, σ satisfies the initial condition. Denote the left-hand side of (52) by $\mathcal{F}(t,x,w)$. By differentiating σ with respect to w, and using $\mathcal{F}_x \equiv 0$, we obtain

$$\begin{aligned}\sigma_w(t,y,p,w) &= \eta_{xx}^T(t,\xi(t,y,w),w)G^T(t,\xi(t,y,w))p \\ &\quad + \eta_{wx}(t,\xi(t,y,w),w)p \\ &= \mathcal{F}_x(t,\xi(t,y,w),w)p \\ &\quad - G_x^T(t,\xi(t,y,w))\eta_x^T(t,\xi(t,y,w),w)p \\ &= -H_{xv}(t,\xi(t,y,w),\sigma(t,y,p,w)).\end{aligned}$$

This proves Proposition 3.3. □

Proposition 3.4. Let

$$s(t,x,\psi,w)=\xi_y^T(t,\eta_x(t,x,w),w)\psi.$$

Then, the function

$$p(t)=s(t,x^*(t),\psi(t),w^*(t)) \tag{55}$$

is a solution to (36) if and only if the function

$$\psi(t)=\sigma(t,y^*(t),p(t),w^*(t)) \tag{56}$$

is solution to the system (11).

Proof. Any solution $(x(\cdot),\psi(\cdot))$ to the systems (2), (11) can be approximated in the weak star topology of the space of functions of bounded variation by solutions to the system (18). Hence, it is sufficient to prove that the relations (55) and (56) hold for any absolutely continuous trajectory of the system (18). For this, it suffices to prove that the formula

$$(d/dt)\sigma(t,y,p,w)=-H_x(t,\xi(t,y,w),\sigma(t,y,w),u,v) \tag{57}$$

holds, with

$$(\dot y,\dot p,\dot w)=(g,-\tilde H_y,v).$$

We have

$$\begin{aligned}(d/dt)\sigma(t,y,p,w)=&-\eta_x^T(t,\xi(t,y,w),w)\tilde H_y(t,y,p,u,w)\\ &+(d/dt)\eta_x^T(t,\xi(t,y,w),w)p.\end{aligned} \tag{58}$$

Note that the function $\tilde H$ may be represented in the form

$$\begin{aligned}\tilde H(t,y,p,u,w)=&H_0(t,\xi(t,y,w),\sigma(t,y,p,w),u)\\ &+\langle p,\eta_t(t,\xi(t,y,w),w)\rangle.\end{aligned}$$

By differentiating w.r.t. y, we obtain

$$\begin{aligned}\tilde H_y(t,y,p,u,w)=&\xi_y^T(t,y,w)[H_{0x}(t,\xi(t,y,w),\sigma(t,y,p,w),u)\\ &+[\eta_x^T(t,\xi(t,y,w),w)p]_x f(t,\xi(t,y,w),u)\\ &+\eta_{tx}(t,\xi(t,y,w),w)p].\end{aligned}$$

By substituting this equality in (58), we obtain

$$\begin{aligned}(d/dt)\sigma(t,y,p,w) = &-H_{0x}(t,\xi(t,y,w),\sigma(t,y,p,w),u)\\ &-[\eta_x^T(t,\xi(t,y,w),w)p]_x f(t,\xi(t,y,w),u)\\ &-\eta_{tx}^T(t,\xi(t,y,w),w)p\\ &+(d/dt)\eta_x^T(t,\xi(t,y,w),w)p. \qquad (59)\end{aligned}$$

By considering the identity $\mathcal{F}_x(t,x,w)\equiv 0$ and using it in the last term of (59), we obtain

$$\begin{aligned}&(d/dt)[\eta_x^T(t,\xi(t,y,w),w)p]_x\\ &=[\eta_x^T(t,\xi(t,y,w),w)p]_t+[\eta_x^T(t,\xi(t,y,w),w)p]_x f(t,\xi(t,y,w),u)\\ &\quad -H_{xv}(t,\xi(t,y,w),\zeta(t,y,p,w),u,v)v.\end{aligned}$$

By substituting in (59), we obtain (57). This proves Proposition 3.4. □

In a similar way (see the details in Refs. 4, 18, 23), we can obtain the important equality

$$-\dot{p}_w=\tilde{H}_w(t,y,p,u,w)=-(d/dt)H_v(t,\xi(t,y,w),\sigma(t,y,p,w)). \qquad (60)$$

Let us consider the transversality conditions for the adjoint systems. We have

$$\begin{aligned}&p(t_0)=\tilde{L}_{y_0}(b^*)=\xi_y^T(t_0,y^*(t_0),0)L_{x_0}(a^*)=L_{x_0}(a^*),\\ &-p(t_1)=\tilde{L}_{y_1}(b^*)=\xi_y^T(t_1,y^*(t_1),w^*)L_{x_1}(a^*),\\ &-p_w(t_1)=\tilde{L}_{w_1}(b^*)=G^T(t_1,x^*(t_1))L_{x_1}(a^*)=H_v(t_1,x_1^*,\psi(t_1)).\end{aligned}$$

From these relations and Proposition 3.4, it follows that the set of adjoint trajectories $p(\cdot)$ and $\psi(\cdot)$ can be obtained from each other by coordinates transformation. Moreover, bearing in mind the equality (60), we obtain

$$p_w(t)=-H_v(t), \quad \forall t\in[t_0,t_1],$$

since under (H4) the function $t\to H_v(t)\in AC$. Due to this and the obvious equality $\tilde{H}_u(t)=H_u(t)$, we conclude that the sets Λ and M are distinguished only by the notation. From now on, we shall use the common notation λ and Λ instead of μ and M. This conclusion is stated as follows.

Proposition 3.5. $\Lambda \neq \emptyset \Leftrightarrow M \neq \emptyset$. That is, the conditions of the local maximum principle in problems A and B are equivalent.

Step 4. Decoding of the Second-Order Conditions. We shall prove that the cone $\tilde{\mathcal{K}}_{cr}$ and the form $\tilde{\Omega}^{\lambda}$ are transformed into $\mathcal{K}_{cr}$ and Ω^{λ} respectively by the simple linear mapping of the y-variations

$$\Theta : \delta x(t) = \xi_y(t)\delta y(t). \tag{61}$$

Since this transformation is invertible [det $\xi_y(t) \neq 0$ on $[t_0, t_1]$ due to (33)], then in order to obtain the second-order necessary conditions of optimality for problem A from those for problem B, we need to show the following proposition.

Proposition 3.6. The following equalities hold:

$$\Theta' \circ \tilde{\mathcal{K}}_{cr} = \mathcal{K}_{cr}, \quad \Theta' \circ \tilde{\Omega}^{\lambda} = \Omega^{\lambda}, \quad \forall \lambda \in \Lambda. \tag{62}$$

Proof. The linear relations (44), (45) specifying the critical cone can be transformed easily into (19). Let us prove that, for $t \notin S^*$, the following formulas hold:

$$g_y(t) = (\eta_x(t) f_x(t) + \dot{\eta}_x(t))\xi_y(t), \tag{63}$$
$$g_w(t) = \eta_x(t)(f_x(t)G(t) - \dot{G}(t)) = -\eta_x(t)(\dot{H}_v)_\psi(t), \tag{64}$$
$$g_u(t) = \eta_x(t) f_u(t). \tag{65}$$

In fact, by using the equality $\mathcal{F}_x \equiv 0$, we obtain

$$\begin{aligned} g_y(t) &= [\eta_{xx}(t) f(t) + f_x(t)\eta_x(t) + \eta_{tx}(t)]\xi_y(t) \\ &= \{[F_x(t)\eta_x(t) + (d/dt)\eta_x(t)] - \mathcal{F}_x(t)v^*(t)\}\xi_y(t). \end{aligned}$$

Analogously, by using the additional equality $\mathcal{F}_t \equiv 0$, we obtain (64) and (65).

From (63)–(65) and (61), we see that (43a) is rewritten in the form (21). For any $s \in S_d(w^*)$, let

$$\begin{aligned} z(\tau; s) &= \xi(s, y^*(s), w^*(s^-) + \tau[w^*(s)]), && \tau \in [0, 1], \\ -q(\tau, s) &= \xi_y(s, y^*(s), w^*(s^-) + \tau[w^*(s)])\delta y(s), && \tau \in [0, 1]. \end{aligned}$$

It is easy to check that these functions satisfy equations (7), (24) and describe the jump conditions of the variation δx in (23) and (24). The proof of the second equality in (64) is analogous to the corresponding one in Ref. 23; therefore, it is omitted. □

From Propositions 3.2 and 3.6, it follows that condition (31) is equivalent to condition (50) for problem (B); hence, (31) is obtained. Theorem 3.1 is proved. □

4. Examples

Example 4.1. Take $n \geq 5, k = n-1, x = \mathrm{col}(x_1, \dots, x_n) \in R^n$, and let Q be a symmetric $k \times k$ matrix such that the index of each of the matrices Q and $-Q$ is not less than 2. The case $k = 4$ and $Q = \mathrm{diag}(1, 1, -1, -1)$ is a good example. Consider the problem

$$
\begin{aligned}
\min \quad & J = \langle \zeta, (x_1(1), \dots, x_k(1)) \rangle, \\
\text{s.t.} \quad & dx_i = f_i(x,t)dt + dw_i, \quad i = \overline{1,k}, \quad w = \mathrm{col}(w_1, \dots, w_k), \\
& dx_n = f_n(x,t)dt + \langle Q\mathrm{col}(x_1, \dots, x_k), dw \rangle, \\
& t \in [0,1], \quad x(0) = 0, \quad x_n(1) = 0, \quad K = R^k,
\end{aligned}
$$

where $\zeta \in R^k$ is a given nonzero vector; for $i = l, \dots, n$, let the f_i be arbitrarily given smooth functions such that

$$f_i(0,t) \equiv 0, \quad f_{ix}(0,t) \equiv 0, \quad f_{nxx}(0,t) \equiv 0.$$

Because of the symmetry of Q, it can be shown easily that the Frobenius condition holds. We investigate the admissible control process (0,0,0) and prove that it is not a locally optimal control process.

Fix any $\lambda \in \Lambda$. From (14), for $\psi(\cdot) = \psi^\lambda(\cdot) = (\psi_1(\cdot), \dots \psi_n(\cdot))$, we obtain $\psi_i(t) \equiv 0, i = l, \dots, k$, and from (11), we have $\psi_n(t) \equiv \psi_{n,0} = \mathrm{const}$. Hence, by using (11), (12), and $\zeta \neq 0$, we obtain

$$\Lambda = \{\lambda : \lambda_0 = 0, \lambda_{2,i} = 0, i = \overline{1, n-1}, \lambda_{2,n} = -\lambda_{2,n+1}\};$$

consequently, Λ consists of only two vectors,

$$\bar{\bar{\lambda}} = -\bar{\lambda}, \ \ \bar{\lambda} = (1/\sqrt{2})(0, \dots, 0, 1, -1) \quad \text{and} \quad \psi_{n,0} = \pm 1/\sqrt{2}.$$

It can be shown easily that

$$d = 1, \ \ \Omega_a^\lambda(\delta w) = \psi_{n,0} \int_0^1 \langle Q\mathrm{col}(\delta x_1, \dots, \delta x_k), \delta w \rangle dt.$$

Hence,

$$\Omega_a^\lambda(\delta w) = (1/2)\psi_{n,0} \langle Q\mathrm{col}(\delta x_1(1), \dots, \delta x_k(1)), \mathrm{col}(\delta x_1(1), \dots, \delta x_k(1)) \rangle.$$

This implies that, for any $\psi_{n,0}=\pm 1$, the index of the function Ω_a^λ is not less than 2. So, $\Lambda_a=\emptyset$; consequently, the process (0,0,0) is not optimal. Also notice that this process is abnormal, that

$$\max_{\lambda\in\Lambda}\Omega^\lambda(\delta w)\geq 0,\quad \forall\delta w,$$

because of $\bar{\lambda}, \bar{\bar{\lambda}}\in\Lambda$, and that the last inequality is not useful.

Example 4.2. Consider the following optimal control problem with parameters α_1,α_2.

$$\begin{aligned}
\min\quad & x_3(1),\\
\text{s.t.}\quad & dx_1=x_2dt, && x_1(0)=0,\\
& dx_2=dw, && x_2(0)=x_{20}<0,\\
& dx_3=(\alpha_1x_1+\alpha_2x_2)dw, && x_3(0)=0,\\
& dw\geq 0.
\end{aligned}$$

The control function

$$w^*(t)=-x_{20},\quad \forall t\in(0,1],$$

with $w^*(0)=0$ satisfies the maximum principle for any parameter values. The corresponding trajectories and adjoint variables are, respectively,

$$(x_1^*(t),x_2^*(t))=(0,0), \forall t\in(0,1],$$

with

$$(x_1^*(0),x_2^*(0))=(0,x_{20}),$$

and

$$(\psi_1(t),\psi_2(t)=(0,0),\quad \forall t\in(0,1],$$

with

$$(\psi_1(0),\psi_2(0))=x_{20}(\alpha_1,\alpha_2)\quad\text{and}\quad \psi_3\equiv -1.$$

The critical cone for the control is described by the conditions

$$\begin{aligned}
&\delta\dot{x}_1=\delta w,\quad \delta x_1(0)=\delta x_1(0^+)=0,\\
&\delta\dot{x}_2=0,\qquad \delta x_2(0)=0,\\
&d(\delta w)\in C^*([0,1],R^+)+\gamma\delta_0,\quad \gamma\in R.
\end{aligned}$$

Since $H_{1\psi x} \equiv 0, \delta x_1(\cdot)$, and $\delta x_2(\cdot)$ are continuous; hence, $\delta x_2 \equiv 0$. The fact that $H_{1xx} \equiv 0$ implies that $\psi \equiv 0$. Therefore, the form Ω is given by

$$\Omega(\delta w) = 2\alpha_1 \delta x_1(1)\delta w_1 + \alpha_2 \delta w_1^2 - 2\alpha_1 \int_0^1 \delta w(\delta w + \delta x_1)dt.$$

The necessary conditions of Theorem 2.3 amount to the inequality $\Omega \geq 0$ on k_{cr}; consequently, the function $\delta w^* \equiv 0$ has to minimize Ω on K_{cr}. From the maximum principle, we have that $\alpha_1 \leq 0$. Notice that it suffices to consider a needle-shaped variation δw concentrated at a left semineighborhood of point $t = 1$.

If $\alpha_1 < 0$ and $\alpha_2 > 0$, then the control w^* is globally optimal.

References

1. Miller, B., *The Optimality Conditions in a Problem of Control of a System That Can Be Described with a Differential Equation with Measure*, Avtomatika i Telemekhanika, Vol. 6, pp. 752– 761, 1982.
2. Zavalischin, S., and Sesekin, A., *Impulsive Processes: Models and Applications*, Nauka, Moscow, Russia, 1991.
3. Sarychev, A., *Optimization of Generalized Controls in a Nonlinear Time-Optimal Problem*, Differential Equations, Vol. 27, pp. 539–550, 1991.
4. Dykhta, V., and Samsonyuk, O. N., *Optimal Impulse Control with Applications*, Nauka, Moscow, Russia, 2000.
5. Marec, J., *Optimal Space Trajectories*, Elsevier, Amesterdam, Holland, 1979.
6. Lawden, D., *Optimal Trajectories for Space Navigation*, Butterworth, London, England, 1663.
7. Brogliato, B., *Nonsmooth Impact Mechanics: Models, Dynamics, and Control*, Lecture Notes in Control and Information Sciences, Springer Verlag, New York, NY, Vol. 220, 1996.
8. Clark, C., Clarke, F., and Munro, G., *The Optimal Exploitation of Renewable Stocks*, Econometrica, Vol. 47, pp. 25–47, 1979.
9. Krotov, V., Bukreev, V., and Gurman, V., *New Variational Methods in Flight Dynamics*, Mashinostroenie, Moscow, Russia, 1969.
10. Baumeister, J., *On Optimal Control of a Fishery*, Proceedings of NOLCOS'01, 5th IFAC Symposium on Nonlinear Control Systems, St. Petersburg, Russia, 2001.
11. Arutyunov, A., *Second-Order Necessary Conditions in Optimal Control Problems*, Doklady Mathematics, Vol. 61, pp. 158–161, 2000.
12. Arutyunov, A., *Optimality Conditions: Abnormal and Degenerate Problems*, Kluwer Academic Publishers, Dordretch, Holland, 2000.
13. Krotov, V., *Global Methods in Optimal Control*, Marcel Dekker, New York, NY, 1996.
14. Ledzewicz, U., and Schaettler, H., *Higher- Order Conditions for Optimality*, SIAM Journal on Control and Optimization, Vol. 37, pp. 33–53, 1998.

15. Bressan, A., and Rampazzo, F., *Impulsive Control Systems with Commutative Vector Fields*, Journal of Optimization Theory and Applications, Vol. 71, pp. 67–83, 1991.
16. Miller, B., and Rubinovitch, E., *Impulsive Control in Continuous and Discrete-Continuous Systems*, Kluwer Academic Publishers, Amsterdam, Holland, 2002.
17. Dykhta, V., *Conditions of Local Minimum for Singular Modes in the System with Linear Control*, Automation and Remote Control, Vol. 12, pp. 5–10, 1981.
18. Dykhta, V., *Necessary Optimality Conditions for Impulsive Processes under Constraints on the Image of the Control Measure*, Izvestiya Vyssikh Uchebnykh Zavedenyi Matematika, Vol. 12, pp. 1–9, 1996.
19. Kolokolnikova, G., *A Variational Maximum Principle for Discontinuous Trajectories of Unbounded Asymptotically Linear Control Systems*, Journal of Differential Equations, Vol. 33, pp. 1633–1640, 1997.
20. Vinter, R., and Pereira, F., *A Maximum Principle for Optimal Processes with Discontinuous Trajectories*, SIAM Journal on Control and Optimization, Vol. 26, pp. 205–229, 1988.
21. Pereira, F., and Silva, G., *Necessary Conditions of Optimality for Vector-Valued Impulsive Control Problems*, Systems and Control Letters, Vol. 40, pp. 205–215, 2000.
22. Silva, G., and Vinter, R., *Necessary Conditions for Optimal Impulsive Control Problems*, SIAM Journal on Control and Optimization, Vol. 35, pp. 1829–1846, 1997.
23. Dykhta, V., *Variational Maximum Principle and Quadratic Optimality Conditions for Impulsive and Singular Processes*, Siberian Mathematical Journal, Vol. 35, pp. 70–82, 1994.
24. Dykhta, V., *Second-Order Necessary Optimality Conditions for Impulsive Control Problems and Multiprocesses*, Singular Solutions and Perturbations in Control Systems, Edited by V. Gurman, B. Miller, M. Dmitriev, Pergamon, Elsevier Science, Kidlington, Oxford, UK, pp. 97–101, 1997.
25. Sumsonuk, O., *Quadratic Optimality Conditions for Optimal Impulsive Control Problems*, Proceedings of the 12th Baikal International Conference on Optimization Methods and Their Applications, Irkutsk, Baikal, Russia, Vol. 2, pp. 144–149, 2001.
26. Agrachev, A., and Gamkrelidze, R., *Index of Extremality and Quasiextremality*, Russian Mathematics Doklady, Vol. 284, pp. 11–14, 1985.
27. Arutyunov, A., Jacimovic, V., and Pereira, F., *Second-Order Necessary Conditions for Optimal Impulsive Control Problems*, Journal of Dynamical and Control Systems, Vol. 9, pp. 131–153, 2003.