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Rigid Tube Model Predictive Control for Linear Sampled–data Systems

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Abstract: We consider the problem of robust model predictive control for linear sampled–data dynamical systems subject to state and control constraints and additive and bounded disturbances. We propose a rigid tube model predictive control algorithm utilizing recent and topologically compatible notions for the sampled–data forward reach sets as well as robust positively invariant sets. The proposed method inherits almost all desirable features associated with rigid tube model predictive control of discrete-time systems, and, in addition, it ensures robust constraint satisfaction and safety in a continuous-time sense.

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1. INTRODUCTION

This article deals with three prominent issues that are almost ubiquitous in the advanced control of systems with some reasonable complexity: presence of hard constraints on the input and on the state; presence of disturbances; and a sampled–data context. A sampled–data system arises every time a plant with variables evolving in continuous–time is controlled using a digital device, which is the most frequent situation. The guarantee that constraints are enforced at all times in critical sampled–data systems requires tools that characterize the inter–sample behavior of trajectories. Clearly, the constraint satisfaction at the sampling instants is not a guarantee of constraint satisfaction within the inter–sampling intervals. Moreover, the constraint satisfaction problem with a finite sampling rate is further amplified in the presence of disturbances. This is because the information of a disturbance occurring anytime can only be counteracted to at the end of the corresponding inter–sample interval.

Model predictive control (MPC) is one of rare control techniques with ability to address effectively the presence of constraints. The guarantee of robust stability and constraint satisfaction in the presence of bounded disturbances is an additional strength of MPC. Furthermore, MPC is naturally adapted to sampled–data systems: the measurement, computation of the control laws (involving optimization), and actuation cannot be performed instantaneously. Thus, while the trajectory of a physical system generally evolves in continuous–time, the control values can only be updated at discrete–time instants. Despite evident importance, it is surprising that a widely accepted framework for MPC of constrained sampled–data systems subject to bounded disturbances is unavailable.

In this paper we consider linear sampled–data systems with hard state and input constraints and subject to bounded disturbances. We ensure constraint satisfaction throughout the whole inter–sampling intervals by using recently developed robust positive invariance tools for sampled–data systems. A tube MPC algorithm with guaranteed robust stabilizing properties is proposed. The setting considered in this paper is relevant to systems with unmeasured disturbances, in which control updates occur at discrete-time instants, while state and control constraints must be satisfied at and between these time instants. Specific applications include spacecraft relative motion control (see, e.g., Di Cairano et al. (2012)) during rendezvous and docking maneuvers, in which case the trajectory must be confined to a specified set, such as the Line of Sight Cone, not only at but also in–between sampling instants. While this application provides a specific practical motivation for this work, its treatment falls beyond the scope of the present conference paper but will be reported in future publications.

Relevant previous work for the results developed here include the study of stability for sampled–data feedback systems (see e.g. Clarke et al. (1997)), and also works on robust positive invariance: (Kolmanovsky and Gilbert, 1998; Raković et al., 2005; Raković and Kouramas, 2007). Regarding MPC, although the majority of the literature has been using just discrete–time models, there are several sampled–data MPC frameworks reported: Chen and Allgöwer (1998); Fontes (2001, 2003); Magni and Scattolini (2004); Fontes et al. (2007); Worthmann et al. (2014); Nešić and Grüne (2006); Gyurkovics and Elaiw (2004). There are also sampled–data robust MPC frameworks developed, which take explicitly into account the presence of disturbances: Fontes and Magni (2003); Kogel and Findeisen (2015); Blanchini et al. (2016). Regarding tube MPC, it has witnessed developments of several generations in the discrete–time setting (see e.g. rigid, homotetic, parameterized, and elastic tube MPC discussed in Mayne et al. (2005); Raković et al. (2012b,a, 2016b)). The rigid tube

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MPC was applied to a sampled–data setting in Farina and Scattolini (2012). However, contrary to that paper, the sampled–data tube MPC developed here considers that all feedbacks (both for the local and nominal system) are sampled–data feedbacks. This, in turn, requires a different set of tools for the underlying analysis and synthesis. In particular, we use recent results on robust invariance of sampled–data systems reported in Raković et al. (2016a).

Nomenclature: The sets of nonnegative integers and real numbers are denoted by $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$, respectively. A given sampling period $T > 0$ induces sequences of sampling instances π and sampling intervals θ both w.r.t. $\mathbb{R}_{\geq 0}$ specified via:

$$\pi := \{t_k\}_{k\in\mathbb{Z}_{\geq 0}} \text{ and } \theta := \{\mathcal{T}_k\}_{k\in\mathbb{Z}_{\geq 0}}, \text{ where } \forall k \in \mathbb{Z}_{\geq 0},$$
$$t_{k+1} := t_k + T \text{ with } t_0 := 0 \text{ and } \mathcal{T}_k := [t_k, t_{k+1}).$$

For any two sets $\mathcal{X}$ and $\mathcal{Y}$ in $\mathbb{R}^n$, the Minkowski set addition is specified by $\mathcal{X} \oplus \mathcal{Y} := \{x + y \ : \ x \in \mathcal{X},\ y \in \mathcal{Y}\}$, while the Minkowski set subtraction (a.k.a. the Pontryagin or geometric set difference) is defined by $\mathcal{X} \ominus \mathcal{Y} := \{z \ : \ z \oplus \mathcal{Y} \subseteq \mathcal{X}\}$. Given a set $\mathcal{X}$ and a real matrix M of compatible dimensions the image of $\mathcal{X}$ under M is denoted by $M\mathcal{X} := \{Mx \ : \ x \in \mathcal{X}\}$. A set $\mathcal{X}$ in $\mathbb{R}^n$ is *a C–set* if it is compact, convex, and contains the origin. A set $\mathcal{X}$ in $\mathbb{R}^n$ is *a proper C–set* if it is a C–set and contains the origin in its interior. Unless stated otherwise, we work with nonempty sets, fixed sampling period $T > 0$ and fixed sequences of related sampling instants π and intervals θ.

2. SETTING AND OBJECTIVES

We consider constrained continuous–time linear, time–invariant, systems with bounded additive disturbances

$$\dot{x}(t) = Ax(t) + Bu(t) + Ew(t), \tag{2.1}$$

where, for any time $t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $w(t) \in \mathbb{R}^p$ denote, respectively, state, control and disturbance values, while $\dot{x}(t)$ denotes the value of the state derivative with respect to time. Matrices A, B and E are known exactly and are of compatible dimensions. The hard state and control constraints are expressed as

$$x(t) \in \mathbb{X} \quad \forall t \in \mathbb{R}_{\geq 0}, \text{ and} \tag{2.2}$$
$$u(t) \in \mathbb{U} \quad \text{a.e. } t \in \mathbb{R}_{\geq 0}, \tag{2.3}$$

where the state and control constraint sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$ are given and known exactly.

The disturbance values set $\mathcal{W} \subseteq \mathbb{R}^p$ is given and known exactly. The admissible disturbance functions are all disturbance functions from $\mathbb{R}_{\geq 0}$ to $\mathcal{W}$ that are piecewise constant and right–continuous in each sampling interval so that $\forall k \in \mathbb{Z}_{\geq 0},\ \forall t \in \mathcal{T}_k$

$$w(t) := w(t_k) \in \mathcal{W}. \tag{2.4}$$

We are concerned with driving towards the origin the state of the above system via sampled–data feedback control. In sampled–data feedback control (see e.g. Clarke et al. (1997)) all controls are selected constant during each inter–sample interval and the related feedbacks are sampled–data feedbacks. In this sense, the feedback control at any given time is not a function of the state at that time, rather it is a function of the state at the last sampling instant. More precisely, given any sampling period T, the related sequence of sampling instants π, and a feedback law $\kappa : \mathbb{R}^n \to \mathbb{R}^m$, we employ controls obtained via sampled–data feedback specified $\forall t \in \mathbb{R}_{\geq 0}$ by

$$u(t) := \kappa(x(\lfloor t \rfloor_\pi)), \text{ where}$$
$$\lfloor t \rfloor_\pi := \max_k \{t_k \in \pi \ : \ t_k \leq t\}. \tag{2.5}$$

In this paper, we employ a separation of the state $x(\cdot)$ into nominal and local components, $z(\cdot)$ and $s(\cdot)$, as well as a separation of control $u(\cdot)$ into nominal and local components, $v(\cdot)$ and $r(\cdot)$. Thus, for any $t \in \mathbb{R}_{\geq 0}$,

$$x(t) = z(t) + s(t) \text{ and} \tag{2.6a}$$
$$u(t) = v(t) + r(t). \tag{2.6b}$$

The nominal system is disturbance free and given by:

$$\dot{z}(t) = Az(t) + Bv(t), \tag{2.7}$$

while the local system takes into account the disturbance and is, in view of (2.1), (2.6) and (2.7), given by:

$$\dot{s}(t) = As(t) + Br(t) + Ew(t). \tag{2.8}$$

The nominal and local control components, $v(\cdot)$ and $r(\cdot)$, and related feedback controllers remain sampled–data and take the following forms, for any $k \in \mathbb{Z}_{\geq 0}$ and any $t \in \mathcal{T}_k$,

$$v(t) := v(t_k) = \kappa_z(z(t_k)) \text{ and} \tag{2.9a}$$
$$r(t) := r(t_k) = K_s s(t_k). \tag{2.9b}$$

The local control feedback control law is, thus, a linear sampled–data feedback and the related local control matrix $K_s \in \mathbb{R}^{m\times n}$ is selected offline subject to natural conditions specified in what follows. The nominal control feedback $\kappa_z(\cdot)$, or to be more precise its values, are constructed via an MPC scheme employing the nominal system (2.7) and a suitable modification of the original constraints (2.2), i.e., the state and control constraint sets $\mathbb{X}$ and $\mathbb{U}$. The main control objective is to ensure robust stability and positive invariance as well as constraint satisfaction for the sampled–data controlled uncertain continuous–time system. The remainder of this paper is dedicated to achieving these objectives under the mild and standing assumptions on the problem setting, as specified next.

Assumption 1. The state and control constraint sets $\mathbb{X}$, $\mathbb{U}$ and $\mathcal{W}$ are proper C–sets in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^p$, respectively.

In the above sampled–data setting, the nominal sampled–data solutions satisfy, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, for any given $z(t_0) = z$,

$$z(t_k + t) = A_d(t)z(t_k) + B_d(t)v(t_k) \text{ and}$$
$$z(t_{k+1}) = A_D z(t_k) + B_D v(t_k) \tag{2.10}$$

where, for any $t \in [0, T]$,

$$A_d(t) := e^{tA}, \quad B_d(t) := \left(\int_0^t e^{\tau A} d\tau\right) B \text{ and}$$
$$E_d(t) := \left(\int_0^t e^{\tau A} d\tau\right) E, \tag{2.11}$$

and let also,

$$A_D := A_d(T),\ B_D := B_d(T), \text{ and } E_D := E_d(T). \tag{2.12}$$

Likewise, the local sampled–data solutions satisfy, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$s(t_k + t) = (A_d(t) + B_d(t)K_s)s(t_k) + E_d(t)w(t_k) \text{ and}$$
$$s(t_{k+1}) = (A_D + B_D K_s)s(t_k) + E_D w(t_k) \tag{2.13}$$

for any given $s(t_0) = s$. Thus, the state decomposition of (2.6a), i.e. $x(t) = z(t) + s(t)$, is guaranteed for all times $t \in \mathbb{R}_{\geq 0},\ t > 0$ provided that the control decomposition of (2.6b) is utilized and that, of course, the initial state $x(0) = x$ is additively decomposed via initial nominal and local states $z(0) = z$ and $s(0) = s$ (i.e. $x = z + s$).

3. ROBUST POSITIVE INVARIANCE

In order to be able to work with compact invariant sets we assume the following

Assumption 2. There exits a sampling period T for which matrices A and B are such that the matrix pair (A_D, B_D) is strictly stabilizable. The sampling period T and a local control matrix K_s are selected in such a way that the matrix $A_D + B_D K_s$ is strictly stable.

As already pointed out, the local system takes care of the disturbances and their dynamic propagation. This is effectively done by employing an adequate notion of robust positive invariance for linear sampled–data dynamics. A detailed study of forward reach sets as well as ordinary and minimal robust positively invariant sets for linear sampled–data dynamics can be found in Raković et al. (2016a). This recent work shows that, within the context of sampled–data robust positive invariance, a demand for a subset $\mathcal{S}$ of $\mathbb{R}^n$ to satisfy dynamic conditions for robust positive invariance at sampling instants:

$$\forall x \in \mathcal{S},\ \forall w \in \mathcal{W},\ (A_D + B_D K_s)x + E_D w \in \mathcal{S},$$

or its equivalent set–theoretic reformulation

$$(A_D + B_D K_s)\mathcal{S} \oplus E_D \mathcal{W} \subseteq \mathcal{S}, \tag{3.1}$$

is natural and is, in fact, a minimal requirement to be imposed. However, the same work also demonstrates that a requirement for a subset $\mathcal{S}$ of $\mathbb{R}^n$ to satisfy dynamic conditions for robust positive invariance at the sampling instants and in the sampling intervals:

$$\forall x \in \mathcal{S}, \forall w \in \mathcal{W},\ \forall t \in [0, T],$$
$$(A_d(t) + B_d(t)K_s)x + E_d(t)w \in \mathcal{S}$$

or its equivalent set–theoretic reformulation

$$\forall t \in [0, T],\ (A_d(t) + B_d(t)K_s)\mathcal{S} \oplus E_d(t)\mathcal{W} \subseteq \mathcal{S} \tag{3.2}$$

is not natural and is, in fact, an overly conservative requirement. As elaborated on in more detail in Raković et al. (2016a), a natural notion of sampled–data robust positive invariance should guarantee discrete–time robust positive invariance and it should relax continuous–time robust positive invariance but also facilitate it if it is attainable. Clearly, it is not possible to guarantee such a flexibility with utilization of a single set $\mathcal{S}$. Instead, such a notion of robust positive invariance can be attained by employing a family of sets $\{\mathcal{S}(t)\ :\ t \in [0, T]\}$, defined over the first sampling period and used periodically in every sampling period. That facilitates a relaxed dynamic condition for robust positive invariance

$$\forall x \in \mathcal{S}(0),\ \forall w \in \mathcal{W},\ \forall t \in [0, T],$$
$$(A_d(t) + B_d(t)K_s)x + E_d(t)w \in \mathcal{S}(t) \text{ and } \mathcal{S}(T) \subseteq \mathcal{S}(0).$$

or its equivalent set–theoretic reformulation

$$\forall t \in [0, T],\ (A_d(t) + B_d(t)K_s)\mathcal{S}(0) \oplus E_d(t)\mathcal{W} \subseteq \mathcal{S}(t) \text{ and } \mathcal{S}(T) \subseteq \mathcal{S}(0). \tag{3.3}$$

Thus, similarly as it is done for set invariance under output feedback in Artstein and Raković (2011), in this work we employ the related generalized, and, in fact, relaxed, notion of robust positively invariant family of sets utilized in (Raković et al., 2016a).

Definition 1. A family of sets

$$\mathfrak{S} := \{\mathcal{S}(t)\ :\ t \in [0, T]\}, \tag{3.4}$$

where, for every $t \in [0, T]$, $\mathcal{S}(t)$ is a subset of $\mathbb{R}^n$, is a *robust positively invariant family of sets* for uncertain local sampled–data linear dynamics, specified via (2.4), (2.8) and (2.9b), and constraint sets $(\mathbb{X}, \mathbb{U}, \mathcal{W})$ if

$$(I)\ \forall t \in [0, T],\ (A_d(t) + B_d(t)K_s)\mathcal{S}(0) \oplus E_d(t)\mathcal{W} \subseteq \mathcal{S}(t) \text{ and } \mathcal{S}(T) \subseteq \mathcal{S}(0);$$
$$(II)\ \forall t \in [0, T],\ \mathcal{S}(t) \subseteq \mathbb{X}; \text{ and}$$
$$(III)\ \mathcal{R} := K_s\mathcal{S}(0) \subseteq \mathbb{U}. \tag{3.5}$$

Strictly speaking, the notion of robust positive invariance, as introduced in the above definition, is entirely compatible with the topological structure of the considered sampled–data setting. Clearly, if there exists a subset $\mathcal{S}$ in $\mathbb{R}^n$ that verifies dynamic relations (3.1) and (3.2), the related collection of sets $\mathfrak{S}$ satisfying the dynamic condition (I) of (3.5) can be constructed by setting, for all $t \in [0, T]$, $\mathcal{S}(t) := \mathcal{S}$. Furthermore, a suitable family of sets $\mathfrak{S}$ satisfying just the dynamic condition (I) of (3.5) can be constructed easily given a subset $\mathcal{S}$ in $\mathbb{R}^n$ that verifies only relation (3.1). To this end, it suffices to put

$$\forall t \in [0, T],\ \mathcal{S}(t) := (A_d(t) + B_d(t)K_s)\mathcal{S} \oplus E_d(t)\mathcal{W}. \tag{3.6}$$

More importantly, such a family of sets is as easy to construct and work with as usual discrete–time robust positively invariant sets, namely its members $\mathcal{S}(t),\ t \in [0, T]$ (and hence family itself) are implicitly characterized by sets $\mathcal{S}$ and $\mathcal{W}$ as specified in (3.6). These facts motivate a natural requirement in terms of the existence of a robust positively invariant family of sets as well as their deployment for the design of rigid tube model predictive control as elaborated in what follows.

Assumption 3. A set $\mathcal{S}$ satisfying (3.1) is selected in such a way that it is a proper C–set, $K_s\mathcal{S} \subseteq \text{interior}(\mathbb{U})$ and the sets $\mathcal{S}(t),\ t \in [0, T]$, specified by (3.6), satisfy $\mathcal{S}(T) \subseteq \mathcal{S}(0) = \mathcal{S}$ as well as $\mathcal{S}(t) \subseteq \text{interior}(\mathbb{X})$ for all $t \in [0, T]$.

Above assumption is natural in that the existence of a proper C–set in $\mathbb{R}^n$ satisfying (3.1) is, under Assumption 1 and 2, guaranteed so that it is necessary to require to such a set is also constraint admissible. Clearly, Assumption 3 ensures that $\mathcal{S}(T) \subseteq \mathcal{S}(0) = \mathcal{S}$ and that the family of sets $\{(A_d(t)+B_d(t)K_s)\mathcal{S}\oplus E_d(t)\mathcal{W}\ :\ t \in [0, T]\}$ is a robust positively invariant family of sets as specified in Definition 1. Another relevant consequence of Assumption 3 concerns the sets $\mathcal{V}$ and $\mathcal{Z}(t)$ specified, for all $t \in [0, T]$, by

$$\mathcal{V} := \mathbb{U} \ominus \mathcal{R} \text{ with } \mathcal{R} := K_s\mathcal{S}, \text{ and}$$
$$\mathcal{Z}(t) := \mathbb{X} \ominus \mathcal{S}(t), \tag{3.7}$$

where the sets $\mathcal{S}(t),\ t \in [0, T]$ are specified by (3.6).

Proposition 1. Suppose Assumptions 1, 2 and 3 hold. The sets $\mathcal{V}$ and $\mathcal{Z}(t)$ specified by (3.7) are, respectively, proper C–sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ for all $t \in [0, T]$.

We also observe one more relevant consequence of Assumption 3. To this end, consider the exact forward reach sets of local linear sampled–data dynamics given, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, by

$$\mathcal{S}^e(t_k + t) = (A_d(t) + B_d(t)K_s)\mathcal{S}^e(t_k) \oplus E_d(t)\mathcal{W} \text{ and}$$
$$\mathcal{S}^e(t_{k+1}) = (A_D + B_D K_s)\mathcal{S}^e(t_k) \oplus E_D\mathcal{W} \tag{3.8}$$

with $\mathcal{S}^e(0)$ being arbitrary compact subset of the set $\mathcal{S}$.

Proposition 2. Suppose Assumptions 1, 2 and 3 hold. Consider the sets $\mathcal{S}(t),\ t \in [0,T]$ and $\mathcal{S}^e(t),\ t \in \mathbb{R}_{\geq 0}$ specified, respectively, by (3.6) and (3.8) with $\mathcal{S}^e(0) \subseteq \mathcal{S}(0)$. Then, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$,

$$\mathcal{S}^e(t_k + t) \subseteq \mathcal{S}(t) \text{ and } \mathcal{S}^e(t_{k+1}) \subseteq \mathcal{S}(T) \subseteq \mathcal{S}(0). \quad (3.9)$$

4. RIGID TUBES

The state and control separation (2.6) in conjunction with Proposition 2 motivates the utilization of the state tubes $\mathcal{X}(\cdot)$ specified, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, by

$$\mathcal{X}(t_k + t) := z(t_k + t) \oplus \mathcal{S}(t), \quad (4.1)$$

and control tubes $\mathcal{U}(\cdot)$ specified, for all $k \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathcal{T}_0$, by

$$\begin{aligned}\mathcal{U}(t_k + t) &:= v(t_k + t) \oplus \mathcal{R}(t)\\ &= v(t_k) \oplus K_s\mathcal{S}(0). \end{aligned} \quad (4.2)$$

The related functions $z(\cdot)$ and $v(\cdot)$ represent the central paths of the state and control tubes $\mathcal{X}(\cdot)$ and $\mathcal{U}(\cdot)$.

The state and control tubes $\mathcal{X}(\cdot)$ and $\mathcal{U}(\cdot)$ are employed globally as well as locally. The global deployment of the state and control tubes $\mathcal{X}(\cdot)$ and $\mathcal{U}(\cdot)$ exploits relations (2.9a) and (2.10) for the determination of the related central paths $z(\cdot)$ and $v(\cdot)$ over the prediction horizon $[0, NT]$ so that, for all $k \in \mathbb{Z}_{\geq 0},\ k < N$ and all $t \in \mathcal{T}_0$,

$$\begin{aligned}&z(t_k + t) = A_d(t)z(t_k) + B_d(t)v(t_k) \text{ and}\\ &z(t_{k+1}) = A_D z(t_k) + B_D v(t_k) \text{ with } z(t_0) = z. \end{aligned} \quad (4.3)$$

With this in mind, the local utilization of the state and and control tubes $\mathcal{X}(\cdot)$ and $\mathcal{U}(\cdot)$ results locally in the related central paths $z(\cdot)$ and $v(\cdot)$ over the prolongation $[NT, \infty)$ of the prediction horizon $[0, NT]$ satisfying, for all $k \in \mathbb{Z}_{\geq 0},\ k \geq N$ and all $t \in \mathcal{T}_0$. Thus, the global and local state and and control tubes $\mathcal{X}(\cdot)$ and $\mathcal{U}(\cdot)$ are induced, and entirely determined, by the central paths $z(\cdot)$ and $v(\cdot)$ and the set $\mathcal{S}$.

The first constraint is to ensure validity of the state and control separation (2.6), as expressed by:

$$x \in z \oplus \mathcal{S}(0)$$

Since the nominal state z is an internal construction, it is not expected to be directly available for measurement. Therefore, assuming the full state x is measurable, we select z from a set dependent on x

$$z \in \mathcal{Z}_0(x) := \{z \in \mathbb{R}^n : x \in z \oplus \mathcal{S}(0)\}. \quad (4.4)$$

The robust constraint satisfaction reduces to the requirement that the state and control tubes $\mathcal{X}(\cdot)$ and $\mathcal{U}(\cdot)$ are admissible w.r.t. the state and control constraints.

Thus, the global admissibility of the state tube $\mathcal{X}(\cdot)$ requires that, for all $k \in \mathbb{Z}_{\geq 0},\ k < N$ and all $t \in \mathcal{T}_0$,

$$\mathcal{X}(t_k + t) \subseteq \mathbb{X}$$

or equivalently that, for all $k \in \mathbb{Z}_{\geq 0},\ k < N$ and all $t \in \mathcal{T}_0$,

$$z(t_k + t) \in \mathcal{Z}(t), \quad (4.5)$$

where $\mathcal{Z}(t)$ is given by (3.7). Likewise, the global admissibility of the control tube $\mathcal{U}(\cdot)$ requires that, for all $k \in \mathbb{Z}_{\geq 0},\ k < N$ and all $t \in \mathcal{T}_0$,

$$\mathcal{U}(t_k + t) \subseteq \mathbb{U}$$

or equivalently that, for all $k \in \mathbb{Z}_{\geq 0},\ k < N$ and all $t \in \mathcal{T}_0$,

$$v(t_k) \in \mathcal{V}. \quad (4.6)$$

where $\mathcal{V}$ is given by (3.7).

The state and control admissibility of the local state and control tubes $\mathcal{X}(\cdot)$ and $\mathcal{U}(\cdot)$ is, as it is customary, ensured by invoking a suitable terminal constraint in the open-loop optimal control problems, which, in this paper, will be expressed in terms of the nominal terminal constraint set $\mathcal{Z}_f(0)$ as

$$z(t_N) \in \mathcal{Z}_f(0), \quad (4.7)$$

and where the set $\mathcal{Z}_f(0)$ will satisfy natural positive invariant conditions as specified next.

In analogy with Definition 1, we employ the following notion of sampled–data positive invariance.

Definition 2. A family of sets

$$\mathfrak{Z}_f := \{\mathcal{Z}_f(t) \ : \ t \in [0,T]\}, \quad (4.8)$$

where, for every $t \in [0,T]$, $\mathcal{Z}_f(t)$ is a subset of $\mathbb{R}^n$, is a positively invariant family of sets for nominal local sampled–data linear dynamics, specified via (2.7) and (2.9a) with $\kappa_z(z(t_k)) = K_f z(t_k)$ (applied to all times), and time–varying constraint sets $(\mathcal{Z}(t), \mathcal{V}),\ t \in [0,T]$ if

$$\begin{aligned}&(I)\ \forall t \in [0,T],\ (A_d(t) + B_d(t)K_f)\mathcal{Z}_f(0) \subseteq \mathcal{Z}_f(t)\\ &\qquad \text{and } \ \mathcal{Z}_f(T) \subseteq \mathcal{Z}_f(0);\\ &(II)\ \forall t \in [0,T],\ \mathcal{Z}_f(t) \subseteq \mathcal{Z}(t); \text{ and}\\ &(III)\ K_f\mathcal{Z}_f(0) \subseteq \mathcal{V}. \end{aligned} \quad (4.9)$$

Similarly as before, a suitable family of sets $\mathfrak{Z}_f$ satisfying the dynamic condition (I) of (4.9) can be constructed easily given a subset $\mathcal{Z}_f$ in $\mathbb{R}^n$ that verifies only discrete–time positive invariance relation:

$$(A_D + B_D K_f)\mathcal{Z}_f \subseteq \mathcal{Z}_f.$$

This can be achieved by letting

$$\forall t \in [0,T],\ \mathcal{Z}_f(t) := (A_d(t) + B_d(t)K_f)\mathcal{Z}_f. \quad (4.10)$$

Once again, such a family of sets is as easy to construct and work with as usual discrete–time positively invariant sets, namely its members $\mathcal{Z}_f(t),\ t \in [0,T]$ (and hence family itself) are implicitly characterized by the set $\mathcal{Z}_f$ as specified in (4.10). These facts motivate a natural sampled–data positive invariance terminal conditions.

Assumption 4. A nominal control matrix K_f and a proper C–set $\mathcal{Z}_f$ in $\mathbb{R}^n$ are selected in such a way that: (i) the matrix $A_D + B_D K_f$ is strictly stable; (ii) $K_f\mathcal{Z}_f \subseteq \mathcal{V}$ and (iii) the sets $\mathcal{Z}_f(t)$ defined for $t \in [0,T]$ by (4.10) satisfy $\mathcal{Z}_f(T) \subseteq \mathcal{Z}_f(0) = \mathcal{Z}_f$ as well as $\mathcal{Z}_f(t) \subseteq \mathcal{Z}(t)$ for all $t \in [0,T]$.

Clearly, Assumption 4 ensures that the family of sets $\{(A_d(t)+B_d(t)K_f)\mathcal{Z}_f \ : \ t \in [0,T]\}$ is a positively invariant family of sets as specified in Definition 2. A constructive consequence of our design is the following.

Proposition 3. Suppose Assumptions 1, 2, 3 and 4 hold. Consider the sets $\mathcal{S}(t)$ and $\mathcal{Z}_f(t)$ specified, for all $t \in [0,T]$, by (3.6) and (4.10), respectively. Then,

$$\forall t \in [0,T],\ \mathcal{S}(t) \oplus \mathcal{Z}_f(t) \subseteq \mathbb{X}, \quad (4.11)$$

and

$$K_s\mathcal{S}(0) \oplus K_f\mathcal{Z}_f(0) \subseteq \mathbb{U}. \quad (4.12)$$

5. TUBE OPTIMAL CONTROL

The state and control tubes $\mathcal{X}(\cdot)$ and $\mathcal{U}(\cdot)$ are induced from the central paths $z(\cdot)$ and $v(\cdot)$. Therefore, we may consider the following open-loop optimal control problem on the central path, with horizon NT and depending on a parameter x.

Minimize

$$\int_0^{t_N} L\left(z(t), v(t)\right) dt + G(z(t_N)) \tag{5.1}$$

subject to

$$\dot{z}(t) = Az(t) + Bv(t) \quad a.e.\ t \in [0, t_N], \tag{5.2}$$
$$z(0) \in \mathcal{Z}_0(x), \tag{5.3}$$
$$z(t_N) \in \mathcal{Z}_f, \tag{5.4}$$
$$z(t) \in \mathcal{Z}(t) \qquad \forall t \in [0, t_N], \tag{5.5}$$
$$v(t) \in \mathcal{V} \qquad a.e.\ t \in [0, t_N], \tag{5.6}$$

We note that the initial state is not fixed but constrained to the initial set $\mathcal{Z}_0$, dependent on the parameter x, as defined in (4.4). Aligned with the sampled–data setting described, the controls are piecewise constant in each sampling interval. Also, in view of (4.3), $z(\cdot)$ and $v(\cdot)$ are for all times between 0 and t_N entirely determined by the sequences $\mathbf{z}_N := \{z_i\}_{i=0}^{N}$ and $\mathbf{v}_{N-1} := \{v_i\}_{i=0}^{N-1}$, with $z_i := z(t_i)$ and $v_i := v(t_i)$. Hence, the minimization can be considered over the decision variable $\mathbf{d}_N = (\mathbf{z}_N, \mathbf{v}_{N-1})$ [2]. Defining the stage cost during one period to be

$$\ell(z, v) := \int_0^T L(A_d(t)z + B_d(t)v, v)dt$$

we can write the optimal control problem as

$\mathcal{P}_N(x)$: Minimize$_{\mathbf{d}_N}$

$$\sum_{i=0}^{N-1} \ell\left(z_i, v_i\right) + G(z_N) \tag{5.7a}$$

subject to

$$z_{i+1} = A_D z_i + B_D v_i \qquad i \in \mathcal{I} \tag{5.7b}$$
$$z_0 \in \mathcal{Z}_0(x), \tag{5.7c}$$
$$z_N \in \mathcal{Z}_f, \tag{5.7d}$$
$$A_d(t)z_i + B_d(t)v_i \in \mathcal{Z}(t) \quad \forall t \in [0, T], i \in \mathcal{I} \tag{5.7e}$$
$$v_i \in \mathcal{V} \qquad i \in \mathcal{I}, \tag{5.7f}$$

where $\mathcal{I} := \{0, 1, \ldots, N-1\}$.

We should note that, despite the simplifications to convert most equations to discrete–time, constraint (5.7e) still has to be verified for all instants of time in each sampling interval, making the optimization problem a semi–infinite one. A computational scheme to guarantee the satisfaction of this constraint is reported elsewhere.

The set of possible initial parameters, a.k.a. as the tube controllability set, is given for each horizon N by:

$$\mathbb{X}_N := \{x \in \mathbb{R}^n \ : \ \mathcal{P}_N(x) \text{ is feasible}\}. \tag{5.8}$$

6. RIGID TUBE MODEL PREDICTIVE CONTROL

In this section, we describe MPC and its robust stabilizing properties. We start by detailing the sampled–data Rigid Tube MPC algorithm

SD-RMPC algorithm: The sampled–data robust MPC algorithm, starting at $k = 0$, proceeds as follows:

(1) Measure state of the plant x_k;
(2) Solve the optimal control problem $\mathcal{P}_N(x_k)$ (5.7a)-(5.7f) to determine the control sequence $\bar{v} : \bar{v}_0, \ldots, \bar{v}_{N-1}$ as well as the corresponding trajectory $\bar{z} : \bar{z}_0, \ldots, \bar{z}_N$.
(3) Apply to the plant the control $u_z(t_k) = \bar{v}_0 + K_s(x_k - \bar{z}_0)$ in the interval $t \in [t_k, t_k + T)$, disregarding the remaining control values $\bar{v}_i, i > k$;
(4) Repeat this procedure for the next sampling time instant $t_k + T$.

We aim to design an MPC scheme that guarantees robust stability. We start by defining the notion of stability in our setting and then establish a sufficient condition for stability on the design parameters of the MPC.

The following set distance functions are used. Given a non-empty, compact set $\mathcal{S} \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we define

$$d(x, \mathcal{S}) := \min\{|x - y| : y \in \mathcal{S}\},$$

and given a family of non-empty, compact sets $\mathfrak{S} := \{\mathcal{S}(t) \ : \ t \in [0, T]\}$

$$d(x, \mathfrak{S}) := \min\{|x - y| : y \in \mathcal{S}(t), t \in [0, T]\}.$$

Definition 3. (Exponential recurring convergence to a set). We say that a trajectory $t \mapsto x(t)$ of a sampled–data system with sampling period T *recurrently converges exponentially to the family of sets* $\mathfrak{S}$ *with basin of attraction* $\mathbb{X}_0$ if $\forall x(0) \in \mathbb{X}_0 \quad \exists \alpha, \beta, \Gamma > 0$ such that $\forall t > \Gamma \quad \exists \tau \in [0, T]$

$$d(x(t), \mathfrak{S}) \le \alpha d(x(0), \mathfrak{S}) \mathrm{e}^{-\beta t},$$
$$d(x(t + \tau), \mathcal{S}(0)) \le \alpha d(x(0), \mathcal{S}(0)) \mathrm{e}^{-\beta t}.$$

where $\mathfrak{S} := \{\mathcal{S}(t) \ : \ t \in [0, T]\}$ is the family of sets for which $\mathcal{S}(0) = \mathcal{S}$ is determined by the family $\mathfrak{S}$ of sets and relation (3.6).

SC Sufficient condition for stability

The design parameters: horizon N, cost functions ℓ and G, and terminal constraint set $\mathcal{Z}_f$, satisfy

(1) The terminal set $\mathcal{Z}_f$ is closed and contains the origin. The function ℓ is positive definite, $x \mapsto \ell(x, u)$ is unbounded for all u, and F is positive semi-definite.
(2) The horizon N is such that set of initial states $\mathbb{X}_0$ is contained in $\mathbb{X}_N$.
(3) For all $z \in \mathcal{Z}_f$ there exists a linear feedback matrix K_f such that

$$G((A_D + B_D K_f)z) - G(z) \le -\ell(z, K_f z), \tag{SCa}$$
$$(A_D + B_D K_f)\mathcal{Z}_f \subseteq \mathcal{Z}_f, \tag{SCb}$$

and

$$K_f Z_f \subseteq \mathcal{V}$$
$$(A_d(t) + B_d(t)K_f)\mathcal{Z}_f \subseteq \mathcal{Z}(t), \quad t \in [0, T]. \tag{SCc}$$

Theorem 1. Let Assumptions 1, 2 and 3 hold. If the choice of design parameters satisfies the stability condition SC and the initial state x_0 is in $\mathbb{X}_0$, then

(1) all the optimal control problems involved in the SD-RMPC algorithm are feasible;
(2) the trajectory generated by the SD-RMPC algorithm recurrently converges exponentially to the set $\mathcal{S}$.

The proof of this theorem follows arguments described in Fontes (2001, 1999) (for sampled–data systems) and in Mayne et al. (2000, 2005) (for discrete–time systems). Details will be reported elsewhere.

[2] In fact, the solution is completely determined by selecting only the variables $(z_0, \mathbf{v}_{N-1})$. However, the full sequence of states $\mathbf{z}_N$ is included to be closer to the way the resulting optimization problem is implemented.

7. CONCLUSION

Model predictive control is a widely researched and practically used control system design methodology, with the vast majority of the literature and implementation using discrete–time models. However, many physical plants with digital controllers are in fact sampled–data systems and its inter–sampling behavior should be studied when hard, critical constraints are present. This article shows that for linear sampled–data systems subject to bounded disturbances, robust stabilization can be carried out using to a large extent well-known discrete–time tools, but complemented with new results on robust positive invariance and stability of sampled–data systems.

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