

BIRATIONALITY OF MODULI SPACES OF TWISTED $U(p, q)$ -HIGGS BUNDLES

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ABSTRACT. A $U(p, q)$ -Higgs bundle on a Riemann surface (twisted by a line bundle) consists of a pair of holomorphic vector bundles, together with a pair of (twisted) maps between them. Their moduli spaces depend on a real parameter α . In this paper we study wall crossing for the moduli spaces of α -polystable twisted $U(p, q)$ -Higgs bundles. Our main result is that the moduli spaces are birational for a certain range of the parameter and we deduce irreducibility results using known results on Higgs bundles. Quiver bundles and the Hitchin–Kobayashi correspondence play an essential role.

1. INTRODUCTION

Holomorphic vector bundles with extra structure on a Riemann surface X have been intensively studied over the last decades. Higgs bundles constitute an important example, not least due to the non-abelian Hodge Theorem [12, 13, 21, 31, 32], which identifies the moduli space of Higgs bundles with the character variety for representations of the fundamental group. Another important example is that of quiver bundles. A quiver Q is a directed graph and a Q -bundle on X is a collection of vector bundles, indexed by the vertices of Q , and morphisms, indexed by the arrows of Q . The natural stability condition for quiver bundles depends on real parameters and hence so do the corresponding moduli spaces. The stability condition stays the same in chambers but wall-crossing phenomena arise and can be used in the study of the moduli spaces. An early spectacular success for this approach is Thaddeus’ proof of the rank two Verlinde formula [33], using Bradlow pairs [6], which are examples of *triples*. Triples are Q -bundles for a quiver with two vertices and a single arrow connecting them. Moduli spaces of triples have been studied extensively, using wall-crossing techniques. Without being exhaustive, we mention [9], where connectedness and irreducibility results for triples were studied, and the work of Muñoz [23, 24, 25] and Muñoz–Ortega–Vázquez-Gallo [26, 27] on finer topological invariants, such as Hodge numbers. More generally, chains (introduced by Álvarez-Cónsul–García-Prada in [1]) are Q -bundles for a quiver of type A_n . Chains have also been studied using wall crossing techniques; we mention here the work of Álvarez-Consul–García-Prada–Schmitt [3], García-Prada–Heinloth–Schmitt [15] and García-Prada–Heinloth [14].

A natural question to ask is to what extent wall crossing techniques can be extended to moduli of Q -bundles for more general quivers. Our aim in this paper is to investigate the situation when Q has oriented cycles, as opposed to the case of chains. Since the number

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of effective stability parameters is one less than the number of vertices of the quiver, in order to encounter wall crossing phenomena, we are led to considering the following quiver as the simplest non-trivial case:

$$(1.1) \quad \begin{array}{c} \bullet \quad \curvearrowright \quad \bullet \\ \curvearrowleft \end{array}$$

For such quivers it becomes relevant to consider *twisted* Q -bundles, meaning that to each arrow one associates a fixed line bundle twisting the corresponding morphism.

Quiver bundles for the quiver (1.1) are closely related to Higgs bundles through the notion of G -Higgs bundles. These are the appropriate objects for extending the non-abelian Hodge Theorem to representations of the fundamental group in a *real* reductive Lie group G (see, e.g., [18, 19]). The relevant case here is that of $G = \mathrm{U}(p, q)$. Indeed, a $\mathrm{U}(p, q)$ -Higgs bundle is a twisted Q -bundle for the quiver (1.1), twisted by the canonical bundle K of X . Allowing for twisting by an arbitrary line bundle L on X , an L -twisted $\mathrm{U}(p, q)$ -Higgs bundle is a quadruple $E = (V, W, \beta, \gamma)$, where V and W are vector bundles of rank p and q , respectively, and the morphisms are $\beta: W \rightarrow V \otimes L$ and $\gamma: V \rightarrow W \otimes L$. The stability notion for Q -bundles for the quiver (1.1) depends on a real parameter α and the value which is relevant for the non-abelian Hodge Theorem is $\alpha = 0$.

We denote by $\mathcal{M}_\alpha(t)$ the moduli space of α -semistable L -twisted $\mathrm{U}(p, q)$ -Higgs bundles of type $t = (p, q, a, b) = (\mathrm{rk}(V), \mathrm{rk}(W), \mathrm{deg}(V), \mathrm{deg}(W))$ and by $\mathcal{M}_\alpha^s(t) \subset \mathcal{M}_\alpha(t)$ the subspace of α -stable L -twisted $\mathrm{U}(p, q)$ -Higgs bundles. We show that the parameter α is constrained to lie in an interval $\alpha_m \leq \alpha \leq \alpha_M$ (with $\alpha_m = -\infty$ and $\alpha_M = \infty$ if $p = q$) and the stability condition changes at a discrete set of critical values α_c for α .

Our main result is the following theorem (see Theorem 5.3 below).

Theorem A. *Fix a type $t = (p, q, a, b)$. Let α_c be a critical value. If either one of the following conditions holds:*

- (1) $a/p - b/q > -\mathrm{deg}(L)$, $q \leq p$ and $0 \leq \alpha_c^\pm < \frac{2pq}{pq - q^2 + p + q}(b/q - a/p - \mathrm{deg}(L)) + \mathrm{deg}(L)$,
- (2) $a/p - b/q < \mathrm{deg}(L)$, $p \leq q$ and $\frac{2pq}{pq - p^2 + p + q}(b/q - a/p + \mathrm{deg}(L)) - \mathrm{deg}(L) < \alpha_c^\pm \leq 0$.

Then the moduli spaces $\mathcal{M}_{\alpha_c^-}^s(t)$ and $\mathcal{M}_{\alpha_c^+}^s(t)$ are birationally equivalent.

Under suitable co-primality conditions on the topological invariants (p, q, a, b) we also have results for the full moduli spaces $\mathcal{M}_\alpha(t)$; we refer to Theorem 5.3 below for the precise result.

A systematic study of $\mathrm{U}(p, q)$ -Higgs bundles was carried out in [8], based on results for holomorphic triples from [7, 9]. In particular, it was shown that the moduli space of $\mathrm{U}(p, q)$ -Higgs bundles is irreducible (again under suitable co-primality conditions). Using these results, we deduce the following corollary to our main theorem (see Theorem 5.5 below).

Theorem B. *Let $L = K$ and fix a type $t = (p, q, a, b)$. Suppose that $(p + q, a + b) = 1$ and that $\tau = \frac{2pq}{p+q}(a/p - b/q)$ satisfies $|\tau| \leq \min\{p, q\}(2g - 2)$. Suppose that either one of the following conditions holds:*

- (1) $a/p - b/q > -(2g - 2)$, $q \leq p$ and $0 \leq \alpha < \frac{2pq}{pq - q^2 + p + q}(b/q - a/p - (2g - 2)) + 2g - 2$,
- (2) $a/p - b/q < 2g - 2$, $p \leq q$ and $\frac{2pq}{pq - p^2 + p + q}(b/q - a/p + 2g - 2) - (2g - 2) < \alpha \leq 0$.

Then the moduli space $\mathcal{M}_\alpha(t)$ is irreducible.

A related work is the recent preprint by Biquard–García-Prada–Rúbio [5], which studies G -Higgs bundles for *any* non-compact G of hermitian type. Their focus is different from ours, in that they adopt a general Lie theoretic approach and study special properties such as rigidity for maximal G -Higgs bundles, whereas wall crossing phenomena are not

studied. On the other hand it is similar in spirit in allowing for arbitrary values of the stability parameter and, indeed, our Proposition 3.3 for twisted $U(p, q)$ -Higgs bundles is a special case of the Milnor–Wood inequality for general G proved by these authors (when $L = K$). A different generalization, namely to *parabolic* $U(p, q)$ -Higgs bundles, has appeared in the work of García-Prada–Logares–Muñoz [16].

This paper is organized as follows. In Section 2 we give some definitions and basic results on quiver bundles. In Section 3 we analyze how the α -stability condition constrains the parameter range for fixed type $t = (p, q, a, b)$, prove the Milnor–Wood type inequality for α -semistable twisted $U(p, q)$ -Higgs bundles mentioned above, and study vanishing of the second hypercohomology group of the deformation complex and deduce smoothness results for the moduli space. These results provide essential input for the analysis in Section 4 of the loci where the moduli space changes when crossing a critical value. Finally, in Section 5, we put our results together and prove our main theorems.

This paper is, in part, based on the second author’s Ph.D. thesis [29].

2. DEFINITIONS AND BASIC RESULTS

In this section we recall definitions and relevant facts on quiver bundles, from [20] and [2], that will be needed in the sequel. We give the results for general Q -bundles. This generality is needed since more general Q -bundles naturally appear in the study of twisted $U(p, q)$ -Higgs bundles (see Section 4.2).

2.1. Quivers. A *quiver* Q is a directed graph specified by a set of vertices Q_0 , a set of arrows Q_1 and head and tail maps $h, t : Q_1 \rightarrow Q_0$. We shall assume that Q is finite.

2.2. Twisted quiver sheaves and bundles. Let X be a compact Riemann surface, let Q be a quiver and let $M = \{M_a\}_{a \in Q_1}$ be a collection of finite rank locally free sheaves of \mathcal{O}_X -modules.

Definition 2.1. An *M -twisted Q -sheaf* on X is a pair $E = (V, \varphi)$, where V is a collection of coherent sheaves V_i on X , for each $i \in Q_0$, and φ is a collection of morphisms $\varphi_a : V_{ta} \otimes M_a \rightarrow V_{ha}$, for each $a \in Q_1$, such that $V_i = 0$ for all but finitely many $i \in Q_0$, and $\varphi_a = 0$ for all but finitely many $a \in Q_1$.

A *holomorphic M -twisted Q -bundle* is an M -twisted Q -sheaf $E = (V, \varphi)$ such that the sheaf V_i is a holomorphic vector bundle, for each $i \in Q_0$.

We shall not distinguish vector bundles and locally free finite rank sheaves.

A morphism between twisted Q -sheaves (V, φ) and (W, ψ) on X is given by a collection of morphisms $f_i : V_i \rightarrow W_i$, for each $i \in Q_0$, such that the diagrams

$$\begin{array}{ccc} V_{ta} \otimes M_a & \xrightarrow{\varphi_a} & V_{ha} \\ \downarrow f_{ta} \otimes 1 & & \downarrow f_{ha} \\ W_{ta} \otimes M_a & \xrightarrow{\psi_a} & W_{ha} \end{array}$$

commute for every $a \in Q_1$.

In this way M -twisted Q -sheaves form an abelian category. The notions of Q -subbundles and quotient Q -bundles, as well as simple Q -bundles are defined in the obvious way. The subobjects $(0, 0)$ and E itself are called the *trivial subobjects*. The *type* of a Q -bundle $E = (V, \varphi)$ is given by

$$t(E) = (\mathrm{rk}(V_i); \mathrm{deg}(V_i))_{i \in Q_0},$$

where $\mathrm{rk}(V_i)$ and $\mathrm{deg}(V_i)$ are the rank and degree of V_i , respectively. We sometimes write $\mathrm{rk}(E) = \mathrm{rk}(\bigoplus V_i)$ and call it the *rank of E* . Note that the type is independent of φ .

2.3. Stability. Fix a tuple $\alpha = (\alpha_i) \in \mathbb{R}^{|Q_0|}$ of real numbers. For a non-zero Q -bundle $E = (V, \varphi)$, the associated α -slope is defined as

$$\mu_\alpha(E) = \frac{\sum_{i \in Q_0} (\alpha_i \operatorname{rk}(V_i) + \deg(V_i))}{\sum_{i \in Q_0} \operatorname{rk}(V_i)}.$$

Definition 2.2. A Q -bundle $E = (V, \varphi)$ is said to be α -(semi)stable if, for all non-trivial subobjects F of E , $\mu_\alpha(F) < (\leq) \mu_\alpha(E)$. An α -polystable Q -bundle is a finite direct sum of α -stable Q -bundles, all of them with the same α -slope.

A Q -bundle E is *strictly α -semistable* if and only if there is a non-trivial subobject $F \subset E$ such that $\mu_\alpha(F) = \mu_\alpha(E)$.

Remark 2.3. If we translate the parameter vector $\alpha = (\alpha_i)_{i \in Q_0}$ by a global constant $c \in \mathbb{R}$, obtaining $\alpha' = (\alpha'_i)_{i \in Q_0}$, with $\alpha'_i = \alpha_i + c$, then $\mu_{\alpha'}(E) = \mu_\alpha(E) - c$. Hence the stability condition does not change under global translations. So we may assume that $\alpha_0 = 0$.

The following is a well-known fact (see, e.g., [30, Exercise 2.5.6.6]). Consider a strictly α -semistable Q -bundle $E = (V, \varphi)$. As it is not α -stable, E admits a subobject $F \subset E$ of the same α -slope. If F is a non-zero subobject of E of minimal rank and the same α -slope, it follows that F is α -stable. Then, by induction, one obtains a flag of subobjects

$$F_0 = 0 \subset F_1 \subset \cdots \subset F_m = E$$

where $\mu_\alpha(F_i/F_{i-1}) = \mu_\alpha(E)$ for $1 \leq i \leq m$, and where the subquotients F_i/F_{i-1} are α -stable Q -bundles. This is the *Jordan-Hölder filtration* of \mathcal{E} , and it is not unique. However, the *associated graded object*

$$\operatorname{Gr}(E) := \bigoplus_{i=1}^m F_i/F_{i-1}$$

is unique up to isomorphism.

Definition 2.4. Two semi-stable Q -bundles E and E' are said to be *S -equivalent* if $\operatorname{Gr}(E) \cong \operatorname{Gr}(E')$.

Remark 2.5. It is a standard fact that each S -equivalence class contains a unique polystable representative. Moreover, if a Q -bundle E is stable, then the induced Jordan-Hölder filtration is trivial, and so the S -equivalence class of E coincides with its isomorphism class.

2.4. The gauge theory equations. Throughout this paper, given a smooth bundle M on X , $\Omega^k(M)$ (resp. $\Omega^{i,j}(M)$) is the space of smooth M -valued k -forms (resp. (i, j) -forms) on X , ω is a fixed Kähler form on X , and $\Lambda : \Omega^{i,j}(M) \rightarrow \Omega^{i-1,j-1}(M)$ is contraction with ω . The gauge equations will also depend on a fixed collection q of Hermitian metrics q_a on M_a , for each $a \in Q_1$, which we fix once and for all. Let $\mathcal{E} = (V, \varphi)$ be a M -twisted Q -bundle on X . A Hermitian metric on \mathcal{E} is a collection H of Hermitian metrics H_i on V_i , for each $i \in Q_0$ with $V_i \neq 0$. To define the gauge equations on \mathcal{E} , we note that $\varphi_a : V_{ta} \otimes M_a \rightarrow V_{ha}$ has a smooth adjoint morphism $\varphi_a^* : V_{ha} \rightarrow V_{ta} \otimes M_a$ with respect to the Hermitian metrics $H_{ta} \otimes q_a$ on $V_{ta} \otimes M_a$ and H_{ha} on V_{ha} , for each $a \in Q_1$, so it makes sense to consider the compositions $\varphi_a \circ \varphi_a^*$ and $\varphi_a^* \circ \varphi_a$. The following definitions are found in [2]. Let α be the stability parameter.

Define τ to be collections of real numbers τ_i , for which

$$(2.1) \quad \tau_i = \mu_\alpha(\mathcal{E}) - \alpha_i, \quad i \in Q_0.$$

Then, by using Remark 2.3, α can be recovered from τ as follows

$$\alpha_i = \tau_0 - \tau_i, \quad i \in Q_0.$$

Definition 2.6. A Hermitian metric H satisfies the *quiver τ -vortex equations* if

$$(2.2) \quad \sqrt{-1}\Lambda F(V_i) + \sum_{i=ha} \varphi_a \varphi_a^* - \sum_{i=ta} \varphi_a^* \varphi_a = \tau_i \text{Id}_{V_i}$$

for each $i \in Q_0$ such that $V_i \neq 0$, where $F(V_i)$ is the curvature of the Chern connection associated to the metric H_i on the holomorphic vector bundle V_i .

The following is the *Hitchin-Kobayashi correspondence* between the twisted quiver vortex equations and the stability condition for holomorphic twisted quiver bundles, given in [2, Theorem 3.1]:

Theorem 2.7. *A holomorphic Q -bundle E is α -polystable if and only if it admits a Hermitian metric H satisfying the quiver τ -vortex equations (2.2), where α and τ are related by (2.1).*

2.5. Twisted $U(p, q)$ -Higgs bundles. An important example of twisted Q -bundles, which is our main object study in this paper, is that of twisted $U(p, q)$ -Higgs bundles on X given in the following. It is to be noted that twisted $U(p, q)$ -Higgs bundles in our study is twisted with the same line bundle for each arrow.

Definition 2.8. Let L be a line bundle on X . A *L -twisted $U(p, q)$ -Higgs bundle* is a twisted Q -bundle for the quiver



where each arrow is twisted by L , and such that $\text{rk}(V) = p$ and $\text{rk}(W) = q$. Thus a L -twisted $U(p, q)$ -Higgs bundle is a quadruple $E = (V, W, \beta, \gamma)$, where V and W are holomorphic vector bundles on X of ranks p and q respectively, and

$$\beta : W \longrightarrow V \otimes L,$$

$$\gamma : V \longrightarrow W \otimes L,$$

are holomorphic maps. The *type* of a twisted $U(p, q)$ -Higgs bundle $E = (V, W, \beta, \gamma)$ is defined by a tuple of integers $t(E) := (p, q, a, b)$ determined by ranks and degrees of V and W , respectively.

Note that K -twisted $U(p, q)$ -Higgs bundles can be seen as a special case of G -Higgs bundles ([22], see also [8, 10, 18, 19]), where G is a real form of a complex reductive Lie group and K is the canonical bundle of the Riemann surface X .

2.6. Gauge equations. For this L -twisted quiver bundle one can consider the general quiver equations as defined in (2.2).

Let $\tau = (\tau_1, \tau_2)$ be a pair of real numbers. A Hermitian metric H satisfies the *L -twisted quiver τ -vortex equations* on twisted $U(p, q)$ -Higgs bundle E if

$$(2.3) \quad \begin{aligned} \sqrt{-1}\Lambda F_{H_V} + \beta\beta^* - \gamma^*\gamma &= \tau_1 \text{Id}_V, \\ \sqrt{-1}\Lambda F_{H_W} + \gamma\gamma^* - \beta^*\beta &= \tau_2 \text{Id}_W. \end{aligned}$$

where F_{H_V} and F_{H_W} are the curvatures of the Chern connections associated to the metrics H_V and H_W , respectively.

Remark 2.9. (i) If a holomorphic twisted $U(p, q)$ -bundle E admits a Hermitian metric satisfying the τ -vortex equations, then taking traces in (2.3), summing for V and W , and integrating over X , we see that the parameters τ_1 and τ_2 are constrained by $p\tau_1 + q\tau_2 = \text{deg}(V) + \text{deg}(W)$.

- (ii) If $L = K$ the equations are conformally invariant and so depend only on the Riemann surface structure on X . In this case they are *the Hitchin equations* for the $U(p, q)$ -Higgs bundle.

2.7. Stability. Let $E = (V, W, \beta, \gamma)$ be a twisted $U(p, q)$ -Higgs bundle, and α be a real number; α is called the *stability parameter*. The definitions of the previous section specialize as follows. The α -slope of E is defined to be

$$\mu_\alpha(E) = \mu(E) + \alpha \frac{p}{p+q},$$

where $\mu(E) := \mu(V \oplus W)$. A twisted $U(p, q)$ -bundle E is α -semistable if, for every proper (non-trivial) subobject $F \subset E$,

$$\mu_\alpha(F) \leq \mu_\alpha(E).$$

Further, E is α -stable if this inequality is always strict. A twisted $U(p, q)$ -bundle is called α -polystable if it is the direct sum of α -stable twisted $U(p, q)$ -Higgs bundles of the same α -slope.

Remark 2.10. The stability can be defined using quotients as for vector bundles. Note that for any subobject $E' = (V', W')$ we obtain an induced quotient bundle $E/E' = (V/V', W/W', \bar{\beta}, \bar{\gamma})$ and E is α -(semi)stable if $\mu_\alpha(E/E') (\geq) > \mu_\alpha(E)$.

The following is a special case of the Hitchin-Kobayashi correspondence between the twisted quiver vortex equations and the stability condition for holomorphic twisted quiver bundles, stated in Proposition 2.7.

Theorem 2.11. *A solution to (2.3) exists if and only if E is α -polystable for $\alpha = \tau_2 - \tau_1$.*

2.7.1. Critical values. A twisted $U(p, q)$ -Higgs bundle E is strictly α -semistable if and only if there is a proper subobject $F = (V', W')$ such that $\mu_\alpha(F) = \mu_\alpha(E)$, i.e.,

$$\mu(V' \oplus W') + \alpha \frac{p'}{p'+q'} = \mu(V \oplus W) + \alpha \frac{p}{p+q}.$$

The case in which the terms containing α drop from the above equality and E is strictly α -semistable for all values of α , i.e.,

$$\frac{p}{p+q} = \frac{p'}{p'+q'}, \text{ and} \\ \mu(V \oplus W) = \mu(V' \oplus W')$$

is called α -independent strict semistability.

Definition 2.12. For a fixed type (p, q, a, b) a value of α is called a *critical value* if there exist integers p', q', a' and b' such that $\frac{p'}{p'+q'} \neq \frac{p}{p+q}$ and $\frac{a'+b'}{p'+q'} + \alpha \frac{p'}{p'+q'} = \frac{a+b}{p+q} + \alpha \frac{p}{p+q}$, with $0 \leq p' \leq p$, $0 \leq q' \leq q$ and $(p', q') \neq (0, 0)$. We say that α is *generic* if it is not critical.

Lemma 2.13. *In the following situations α -independent semistability cannot occur:*

- (i) [9, Lemma 2.7] *There is an integer m such that $\text{GCD}(p+q, d_1 + d_2 - mp) = 1$.*
- (ii) *If $\text{GCD}(p, q) = 1$, for $p \neq q$.*

Proof. To prove (ii), on the contrary assume that $E = (V, W, \beta, \gamma)$ is a α -semistable twisted $U(p, q)$ -Higgs bundle with a proper subobject $E' = (V', W', \beta', \gamma')$ such that

$$\mu(V' \oplus W') + \alpha \frac{p'}{p'+q'} = \mu(V \oplus W) + \alpha \frac{p}{p+q}$$

and

$$\frac{p'}{p'+q'} = \frac{p}{p+q},$$

where p' and q' are the ranks of V' and W' respectively. Since E' is proper, either $p' < p$ or $q' < q$ and then the equality $\frac{p'}{p' + q'} = \frac{p}{p + q}$ contradicts that p and q are co-prime. \square

Fix a type $t = (p, q, a, b)$. We denote the moduli space of α -polystable twisted $U(p, q)$ -Higgs bundles $E = (V, W, \beta, \gamma)$ which have the type $t(E) = (p, q, a, b)$, where $a = \deg(V)$ and $b = \deg(W)$, by

$$\mathcal{M}_\alpha(t) = \mathcal{M}_\alpha(p, q, a, b),$$

and the moduli space of α -stable twisted $U(p, q)$ -Higgs bundles by $\mathcal{M}_\alpha^s(t)$. A Geometric Invariant Theory construction of the moduli space follows from the general constructions of Schmitt [30, Theorem 2.5.6.13], thinking of twisted $U(p, q)$ -Higgs bundles in terms of quivers.

2.8. Deformation theory of twisted $U(p, q)$ -Higgs bundles.

Definition 2.14. Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. We introduce the following notation:

$$\begin{aligned} \mathcal{H}om^0 &= \text{Hom}(V', V) \oplus \text{Hom}(W', W), \\ \mathcal{H}om^1 &= \text{Hom}(V', W \otimes L) \oplus \text{Hom}(W', V \otimes L). \end{aligned}$$

With this notation we consider the complex of sheaves

$$(2.4) \quad \mathcal{H}om^\bullet(E', E) : \mathcal{H}om^0 \xrightarrow{a_0} \mathcal{H}om^1$$

defined by

$$a_0(f_1, f_2) = (\phi_a(f_1, f_2), \phi_b(f_1, f_2)), \text{ for } (f_1, f_2) \in \mathcal{H}om^0$$

where

$$\phi_a : \mathcal{H}om^0 \rightarrow \text{Hom}(V', W \otimes L) \hookrightarrow \mathcal{H}om^1 \text{ and } \phi_b : \mathcal{H}om^0 \rightarrow \text{Hom}(W', V \otimes L) \hookrightarrow \mathcal{H}om^1$$

are given by

$$\begin{aligned} \phi_a(f_1, f_2) &= (f_2 \otimes Id_L) \circ \gamma' - \gamma \circ f_1, \\ \phi_b(f_1, f_2) &= (f_1 \otimes Id_L) \circ \beta' - \beta \circ f_2. \end{aligned}$$

The complex $\mathcal{H}om^\bullet(E', E)$ is called the *Hom-complex*. This is a special case of the Hom-complex for Q -bundles defined in [20], and also for G -Higgs bundles (see, e.g., [4]). We shall write $\mathcal{E}nd^\bullet(E)$ for $\mathcal{H}om^\bullet(E, E)$.

The following proposition follows from [20, Theorem 4.1 and Theorem 5.1].

Proposition 2.15. *Let E be a L -twisted $U(p, q)$ -Higgs bundle and E' a L -twisted $U(p', q')$ -Higgs bundle. Then there are natural isomorphisms*

$$\begin{aligned} \text{Hom}(E', E) &\cong \mathbb{H}^0(\mathcal{H}om^\bullet(E', E)) \\ \text{Ext}^1(E', E) &\cong \mathbb{H}^1(\mathcal{H}om^\bullet(E', E)) \end{aligned}$$

and a long exact sequence associated to the complex $\mathcal{H}om^\bullet(E', E)$:

$$(2.5) \quad 0 \longrightarrow \mathbb{H}^0(\mathcal{H}om^\bullet(E', E)) \longrightarrow H^0(\mathcal{H}om^0) \longrightarrow H^0(\mathcal{H}om^1) \longrightarrow \mathbb{H}^1(\mathcal{H}om^\bullet(E', E)) \\ \longrightarrow H^1(\mathcal{H}om^0) \longrightarrow H^1(\mathcal{H}om^1) \longrightarrow \mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) \longrightarrow 0.$$

When $E = E'$, we have $\text{End}(E) = \text{Hom}(E, E) \cong \mathbb{H}^0(\mathcal{H}om^\bullet(E, E))$.

Definition 2.16. We denote by $\chi(E', E)$ the hypercohomology Euler characteristic for the complex $\mathcal{H}om^\bullet(E', E)$, i.e.

$$\chi(E', E) = \dim \mathbb{H}^0(\mathcal{H}om^\bullet(E', E)) - \dim \mathbb{H}^1(\mathcal{H}om^\bullet(E', E)) + \dim \mathbb{H}^2(\mathcal{H}om^\bullet(E', E)).$$

As an immediate consequence of the long exact sequence (2.5) and the Riemann-Roch formula we can obtain the following.

Proposition 2.17. *For any twisted $U(p, q)$ -Higgs bundle E and twisted $U(p', q')$ -Higgs bundle E' we have*

$$\begin{aligned}\chi(E', E) &= \chi(\mathcal{H}om^0) - \chi(\mathcal{H}om^1) \\ &= (1 - g)(\text{rk}(\mathcal{H}om^0) - \text{rk}(\mathcal{H}om^1)) + \text{deg}(\mathcal{H}om^0) - \text{deg}(\mathcal{H}om^1) \\ &= (1 - g)(p'p + q'q - p'q - q'p) + (q' - p')(\text{deg}(W) - \text{deg}(V)) + \\ &\quad (q - p)(\text{deg}(V') - \text{deg}(W')) - (pq' + p'q) \text{deg}(L)\end{aligned}$$

Recall that the type of a $U(p, q)$ -Higgs bundle $E = (V, W, \beta, \gamma)$ is $t(E) = (p, q, a, b)$, where $a = \text{deg}(V)$, $b = \text{deg}(W)$. The previous proposition shows that $\chi(E', E)$ only depends on the types $t' = t(E')$ and $t = t(E)$ of E' and E , respectively, so we may use the notation

$$\chi(t', t) := \chi(E', E).$$

Remark 2.18. Suppose that $E = E' \oplus E''$. Then it is clear that the Hom-complexes satisfy:

$$\mathcal{H}om^\bullet(E, E) = \mathcal{H}om^\bullet(E', E') \oplus \mathcal{H}om^\bullet(E'', E'') \oplus \mathcal{H}om^\bullet(E'', E') \oplus \mathcal{H}om^\bullet(E', E''),$$

and so the hypercohomology groups have an analogous direct sum decomposition.

Lemma 2.19. *For any extension $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ of twisted $U(p, q)$ -Higgs bundles,*

$$\chi(E, E) = \chi(E', E') + \chi(E'', E'') + \chi(E'', E') + \chi(E', E'').$$

Proof. Since the Euler characteristic is topological, we may assume that $E = E' \oplus E''$. Now the result is immediate in view of Remark 2.18. \square

Given the identification of $\mathbb{H}^0(\mathcal{H}om^\bullet(E', E))$ with $\text{Hom}(E', E)$, by Proposition 2.15, the following is the direct analogue of the corresponding result for semistable vector bundles.

Proposition 2.20. *Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. If E and E' are both α -semistable, then the following holds:*

- (1) *If $\mu_\alpha(E) < \mu_\alpha(E')$, then $\mathbb{H}^0(\mathcal{H}om^\bullet(E', E)) = 0$.*
- (2) *If $\mu_\alpha(E') = \mu_\alpha(E)$, and E' is α -stable, then*

$$\mathbb{H}^0(\mathcal{H}om^\bullet(E', E)) \cong \begin{cases} 0 & \text{if } E \not\cong E' \\ \mathbb{C} & \text{if } E \cong E'. \end{cases}$$

Definition 2.21. A twisted $U(p, q)$ -Higgs bundle $E = (V, W, \varphi = \beta + \gamma)$ is *infinitesimally simple* if $\text{End}(E) \cong \mathbb{C}$ and it is *simple* if $\text{Aut}(E) \cong \mathbb{C}^*$, where $\text{Aut}(E)$ denotes the automorphism group of E .

Since L -twisted $U(p, q)$ -Higgs bundles form an abelian category, any automorphism is also an endomorphism. Hence, if (V, W, β, γ) is infinitesimally simple then it is simple. Thus Proposition 2.20 implies the following lemma.

Lemma 2.22. *Let (V, W, β, γ) be a twisted $U(p, q)$ -Higgs bundle. If (V, W, β, γ) is α -stable then it is simple.*

Proposition 2.23. *Let $E = (V, W, \beta, \gamma)$ be an α -stable twisted $U(p, q)$ -Higgs bundle of type $t = (p, q, a, b)$.*

- (1) *The space of infinitesimal deformations of E is isomorphic to the first hypercohomology group $\mathbb{H}^1(\mathcal{E}nd^\bullet(E))$.*
- (2) *If $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$, then the moduli space $\mathcal{M}_\alpha^s(t)$ is smooth in a neighborhood of the point defined by E and*

$$\begin{aligned} \dim \mathcal{M}_\alpha^s(t) &= \dim \mathbb{H}^1(\mathcal{E}nd^\bullet(E)) \\ &= 1 - \chi(E, E) = (g-1)(q-p)^2 + 2pq \deg(L) + 1. \end{aligned}$$

Proof. Statement (1) follows from [4, Theorem 2.3]. Statement (2) is analogous to Proposition 2.14 of [11]. \square

3. CONSEQUENCES OF STABILITY AND PROPERTIES OF THE MODULI SPACE

3.1. Bounds on the topological invariants and Milnor–Wood inequality. In this section we explore the constraints imposed by stability on the topological invariants of $U(p, q)$ -Higgs bundles and on the stability parameter α .

Proposition 3.1. *Let $E = (V, W, \beta, \gamma)$ be an α -semistable twisted $U(p, q)$ -Higgs bundle. Then the following inequalities hold.*

$$(3.1) \quad \frac{2pq}{p+q}(\mu(V) - \mu(W)) \leq \operatorname{rk}(\gamma) \deg(L) + \alpha \left(\operatorname{rk}(\gamma) - \frac{2pq}{p+q} \right),$$

$$(3.2) \quad \frac{2pq}{p+q}(\mu(W) - \mu(V)) \leq \operatorname{rk}(\beta) \deg(L) + \alpha \left(\frac{2pq}{p+q} - \operatorname{rk}(\beta) \right).$$

Moreover, if $\deg(L) + \alpha > 0$ and equality holds in (3.1) then either E is strictly semistable or $p = q$ and γ is an isomorphism $\gamma: V \xrightarrow{\cong} W \otimes L$. Similarly, if $\deg(L) - \alpha > 0$ and equality holds in (3.2) then either E is strictly semistable or $p = q$ and β is an isomorphism $\beta: W \xrightarrow{\cong} V \otimes L$.

Proof. An argument similar to that given in [8, Lemma 3.24] shows that

$$2p(\mu(V) - \mu_\alpha(E)) \leq \operatorname{rk}(\gamma) \deg(L) + \alpha(\operatorname{rk}(\gamma) - 2p);$$

Similarly,

$$2q(\mu(W) - \mu_\alpha(E)) \leq \operatorname{rk}(\beta) \deg(L) - \operatorname{rk}(\beta)\alpha,$$

Using this, the result follows immediately using the following identities:

$$\begin{aligned} \mu(V) - \mu_\alpha(E) &= \frac{q}{p+q}(\mu(V) - \mu(W)) - \alpha \frac{p}{p+q}, \\ \mu(W) - \mu_\alpha(E) &= \frac{p}{p+q}(\mu(W) - \mu(V)) - \alpha \frac{p}{p+q}. \end{aligned}$$

The statement about equality for $\deg(L) - \alpha > 0$ also follows as in loc. cit. \square

By analogy with the case of $U(p, q)$ -Higgs bundles (cf. [8]) we make the following definition.

Definition 3.2. The *Toledo invariant* of a twisted $U(p, q)$ -Higgs bundle $E = (V, W, \beta, \gamma)$ is

$$\tau(E) = 2 \frac{q \deg(V) - p \deg(W)}{p+q} = \frac{2pq}{p+q}(\mu(V) - \mu(W)).$$

The following is the analogue of the Milnor–Wood inequality for $U(p, q)$ -Higgs bundles ([8, Corollary 3.27]). When $L = K$, it is a special case of a general result of Biquard–García-Prada–Rubio [5, Theorem 4.5], which is valid for G -Higgs bundles for any semisimple G of Hermitian type.

Proposition 3.3. *Let $E = (V, W, \beta, \gamma)$ be an α -semistable twisted $U(p, q)$ -Higgs bundle. Then the following inequality holds:*

$$-\operatorname{rk}(\beta) \deg(L) + \alpha \left(\operatorname{rk}(\beta) - \frac{2pq}{p+q} \right) \leq \tau(E) \leq \operatorname{rk}(\gamma) \deg(L) + \alpha \left(\operatorname{rk}(\gamma) - \frac{2pq}{p+q} \right).$$

Proof. In view of the definition of $\tau(E)$, we can write (3.1) and (3.2) as

$$(3.3) \quad \tau(E) \leq \operatorname{rk}(\gamma) \deg(L) + \alpha \left(\operatorname{rk}(\gamma) - \frac{2pq}{p+q} \right),$$

$$(3.4) \quad -\tau(E) \leq \operatorname{rk}(\beta) \deg(L) + \alpha \left(\frac{2pq}{p+q} - \operatorname{rk}(\beta) \right)$$

from which the result is immediate. \square

When equality holds in the Milnor–Wood inequality, more information on the maps β and γ can be obtained from Proposition 3.1. In this respect we have the following result.

Proposition 3.4. *Let $E = (V, W, \beta, \gamma)$ be an α -semistable twisted $U(p, q)$ -Higgs bundle.*

(1) *Assume that $\alpha > -\deg(L)$. Then*

$$\tau(E) \leq \min\{p, q\} \left(\deg(L) - \alpha \frac{|p-q|}{p+q} \right).$$

and if equality holds then $p \leq q$ and γ is an isomorphism onto its image.

(2) *Assume that $\alpha \leq -\deg(L)$. Then*

$$\tau(E) \leq -\alpha \frac{2pq}{p+q}$$

and if equality holds and $\alpha < -\deg(L)$ then $\gamma = 0$.

(3) *Assume that $\alpha < \deg(L)$. Then*

$$\tau(E) \geq \min\{p, q\} \left(-\alpha \frac{|p-q|}{p+q} - \deg(L) \right)$$

and if equality holds then $q \leq p$ and β is an isomorphism onto its image.

(4) *Assume that $\alpha \geq \deg(L)$. Then*

$$\tau(E) \geq -\alpha \frac{2pq}{p+q}$$

and if equality holds and $\alpha > \deg(L)$ then $\beta = 0$.

Proof. We rewrite (3.3) as $\tau(E) \leq \operatorname{rk}(\gamma)(\deg(L) + \alpha) - \alpha \frac{2pq}{p+q}$. Then (1) and (2) are immediate from Proposition 3.1. Similarly, (3) and (4) follow rewriting (3.4) as $\tau(E) \geq \operatorname{rk}(\beta)(\alpha - \deg(L)) - \alpha \frac{2pq}{p+q}$. \square

In the case when $|\alpha| < \deg(L)$ we can write the inequality of the preceding proposition in a more suggestive manner as follows.

Corollary 3.5. *Assume that $|\alpha| < \deg(L)$ and let E be an α -semistable twisted $U(p, q)$ -Higgs bundle. Then*

$$|\tau(E)| \leq \min\{p, q\} \left(\deg(L) - \alpha \frac{|p-q|}{p+q} \right).$$

Remark 3.6. In the cases of Proposition 3.4 when one of the Higgs fields β and γ is an isomorphism onto its image, it is natural to explore rigidity phenomena for twisted $U(p, q)$ -Hitchin pairs, along the lines of [8] (for $U(p, q)$ -Higgs bundles) and Biquard–García-Prada–Rubio [5] (for parameter dependent G -Higgs bundles when G is Hermitian of tube type). This line of inquiry will be pursued elsewhere.

3.2. Range for the stability parameter. In the following we determine a range for the stability parameter whenever $p \neq q$. We denote the minimum and the maximum value for α by α_m and α_M , respectively.

Proposition 3.7. *Assume that $p \neq q$ and let E be a α -semistable twisted $U(p, q)$ -Higgs bundle. Then $\alpha_m \leq \alpha \leq \alpha_M$, where*

$$\alpha_m = \begin{cases} -\frac{2\max\{p, q\}}{|q-p|}(\mu(V) - \mu(W)) - \frac{p+q}{|q-p|} \deg(L) & \text{if } \mu(V) - \mu(W) > -\deg(L), \\ -(\mu(V) - \mu(W)) & \text{if } \mu(V) - \mu(W) \leq -\deg(L), \end{cases}$$

and

$$\alpha_M = \begin{cases} -\frac{2\max\{p, q\}}{|q-p|}(\mu(V) - \mu(W)) + \frac{p+q}{|q-p|} \deg(L) & \text{if } \mu(V) - \mu(W) < \deg(L), \\ -(\mu(V) - \mu(W)) & \text{if } \mu(V) - \mu(W) \geq \deg(L). \end{cases}$$

Proof. First we determine α_M . Using (3.3) we get

$$\alpha \left(\frac{2pq}{p+q} - \text{rk}(\gamma) \right) \leq \text{rk}(\gamma) \deg(L) - \tau(E)$$

since $p \neq q$ therefore $\frac{2pq}{p+q} - \text{rk}(\gamma) > 0$. Hence the above inequality yields

$$\alpha \leq \frac{p+q}{2pq - (p+q) \text{rk}(\gamma)} (\text{rk}(\gamma) \deg(L) - \tau(E)).$$

In order to find an upper bound for α we maximize the right hand side of this inequality as a function of $\text{rk}(\gamma)$. Thus we study monotonicity of the function $f(r) = \frac{rd - \tau}{c - r}$, where $c = \frac{2pq}{p+q}$, $d = \deg(L)$ and $r \in [0, \min\{p, q\}]$. We obtain the following:

(a) If $\deg(L) = \mu(V) - \mu(W)$ then f is constant and

$$\alpha \leq \mu(W) - \mu(V).$$

(b) If $\deg(L) > \mu(V) - \mu(W)$ then f is increasing so

$$\alpha \leq \frac{p+q}{|q-p|} \left(\deg(L) - \frac{\tau(E)}{\min\{p, q\}} \right) = \frac{p+q}{|q-p|} \deg(L) - \frac{2\max\{p, q\}}{|q-p|} (\mu(V) - \mu(W))$$

and, if equality holds then $\text{rk}(\gamma) = \min\{p, q\}$.

(c) If $\deg(L) < \mu(V) - \mu(W)$ then f is decreasing so

$$\alpha \leq \mu(W) - \mu(V)$$

and, if equality holds then $\gamma = 0$.

Now we determine the lower bound α_m . The inequality (3.4) yields

$$\alpha \geq \frac{\text{rk}(\beta) \deg(L) + \tau(E)}{\text{rk}(\beta) - \frac{2pq}{p+q}}.$$

Similarly to the above, by studying the monotonicity of $g(r) = \frac{rd + \tau}{r - c}$, we obtain the following:

(a)' If $\mu(V) - \mu(W) = -\deg(L)$ then g is constant and

$$\alpha \geq \mu(W) - \mu(V).$$

(b)' If $\mu(V) - \mu(W) < -\deg(L)$ then g is increasing, so

$$\alpha \geq \mu(W) - \mu(V),$$

and, if equality holds then $\beta = 0$.

(c)' If $\mu(V) - \mu(W) > -\deg(L)$ then g is decreasing, so

$$\alpha \geq -\frac{p+q}{|q-p|} \left(\deg(L) + \frac{\tau(E)}{\min\{p, q\}} \right) = -\frac{p+q}{|q-p|} \deg(L) - \frac{2\max\{p, q\}}{|q-p|} (\mu(V) - \mu(W)),$$

and, if equality holds then $\text{rk}(\beta) = \min\{p, q\}$.

Note that if $\mu(V) - \mu(W) \geq 0$ then $\mu(V) - \mu(W) \geq -\deg(L)$, and if $\mu(V) - \mu(W) \leq 0$ then $\mu(V) - \mu(W) < \deg(L)$. Hence the result follows. \square

Remark 3.8. The preceding proof gives the following additional information when α equals one of the extreme values α_m and α_M :

- if $\mu(V) - \mu(W) < \deg(L)$ and $\alpha = \alpha_M$ then $\text{rk}(\gamma) = \min\{p, q\}$;
- if $\mu(V) - \mu(W) > \deg(L)$ and $\alpha = \alpha_M$ then $\gamma = 0$;
- if $\mu(V) - \mu(W) > -\deg(L)$ and $\alpha = \alpha_m$ then $\text{rk}(\beta) = \min\{p, q\}$, and
- if $\mu(V) - \mu(W) < -\deg(L)$ and $\alpha = \alpha_m$ then $\beta = 0$.

The following corollary is relevant because $\alpha = 0$ is the value of stability parameter for which the Non-abelian Hodge Theorem gives the correspondence between $U(p, q)$ -Higgs bundles and representations of the fundamental group of X .

Corollary 3.9. *With the notation of Proposition 3.7, the inequality $\alpha_M \geq 0$ holds if and only if $\tau(E) \leq \min\{p, q\} \deg(L)$ and the inequality $\alpha_m \leq 0$ holds if and only if $\tau(E) \geq -\min\{p, q\} \deg(L)$. Thus $0 \in [\alpha_m, \alpha_M]$ if and only if $|\tau(E)| \leq \min\{p, q\} \deg(L)$.*

Proof. Immediate from Proposition 3.7. \square

Remark 3.10. Note that the condition $|\tau(E)| \leq \min\{p, q\} \deg(L)$ is stronger than the condition $|\mu(V) - \mu(W)| \leq \deg(L)$.

3.3. Parameters forcing special properties of the Higgs fields. In this section we use a variation on the preceding arguments to find a parameter range where β and γ have special properties. Assume that the twisted $U(p, q)$ -Higgs bundle $E = (V, W, \beta, \gamma)$ has type (p, q, a, b) .

For the following proposition it is convenient to introduce the following notation. For $0 \leq i < q \leq p$, let

$$\alpha_i = \frac{2pq}{q(p-q) + (i+1)(p+q)} (\mu(W) - \mu(V) - \deg(L)) + \deg(L),$$

and for $0 \leq j < p \leq q$, let

$$\alpha'_j = \frac{2pq}{p(q-p) + (j+1)(p+q)} (\mu(W) - \mu(V) + \deg(L)) - \deg(L).$$

Proposition 3.11. *Let $E = (V, W, \beta, \gamma)$ be an α -semistable twisted $U(p, q)$ -Higgs bundle. Then we have the following:*

- (i) *Assume that $p \geq q$ and $\mu(V) - \mu(W) > -\deg(L)$. If $\alpha < \alpha_{i-1}$ then $\text{rk}(\ker(\beta)) < i$. In particular β is injective whenever*

$$\alpha < \alpha_0 = \frac{2pq}{pq - q^2 + p + q} (\mu(W) - \mu(V) - \deg(L)) + \deg(L).$$

- (ii) Assume that $p \geq q$ and $\mu(V) - \mu(W) < -\deg(L)$. If $\alpha < \alpha_{i-1}$ then $\text{rk}(\ker(\beta)) > i$. In particular β is zero whenever

$$\alpha < \alpha_{q-2} = \frac{2pq}{2pq - p - q} (\mu(W) - \mu(V) - \deg(L)) + \deg(L).$$

- (iii) Assume that $p \leq q$ and $\mu(V) - \mu(W) < \deg(L)$. If $\alpha > \alpha'_j$ then $\text{rk}(\ker(\gamma)) < j$. In particular γ is injective whenever

$$\alpha > \alpha'_0 = \frac{2pq}{pq - p^2 + p + q} (\mu(W) - \mu(V) + \deg(L)) - \deg(L).$$

- (iv) Assume that $p \leq q$ and $\mu(V) - \mu(W) > \deg(L)$. If $\alpha > \alpha'_j$ then $\text{rk}(\ker(\gamma)) > j$. In particular γ is zero whenever

$$\alpha > \alpha'_{p-2} = \frac{2pq}{2pq - p - q} (\mu(W) - \mu(V) + \deg(L)) - \deg(L).$$

Proof. We shall only prove parts (i) and (ii). One can deduce the other parts in a similar way. Suppose that $\text{rk}(\ker(\beta)) = n > 0$. The inequality (3.2) yields

$$\alpha \geq \frac{2pq}{n(p+q) + q(p-q)} (\mu(W) - \mu(V) - \deg(L)) + \deg(L) = \alpha_{n-1}.$$

Now suppose $\mu(W) - \mu(V) - \deg(L) < 0$, then α_i increases with i and so, if $n \geq i$ then $\alpha \geq \alpha_{i-1}$. Hence, if $\alpha < i - 1$ then $n < i$. In particular, if $\alpha < \alpha_0$ then β is injective, which gives part (i).

On the other hand, if $\mu(W) - \mu(V) - \deg(L) > 0$, then α_i decreases with i and so, if $n \leq i$ then $\alpha \geq \alpha_{i-1}$. Hence, if $\alpha < \alpha_{i-1}$ then $n > i$. In particular, if $\alpha < \alpha_{q-2}$ then β is zero, proving part (ii). \square

Remark 3.12. Note that the signs of α_0 and α'_0 given in the preceding proposition are related to the Toledo invariant as follows:

- $\alpha_0 > 0$ if and only if $\tau(E) < -(q-1)\deg(L)$.
- $\alpha'_0 < 0$ if and only if $\tau(E) > (p-1)\deg(L)$.

Remark 3.13. Associated to $E = (V, W, \beta, \gamma)$ there is a dual L -twisted $U(p, q)$ -Higgs bundle $E^* = (V^*, W^*, \gamma^*, \beta^*)$. Clearly there is a one-to-one correspondence between subobjects of E and quotients of E^* , and $\mu_{-\alpha}(E) = -\mu_{\alpha}(E^*)$. Therefore α -stability of E^* is equivalent to $-\alpha$ -stability of E .

Corollary 3.14. *Let $E = (V, W, \beta, \gamma)$ be an α -semistable twisted $U(p, q)$ -Higgs bundle. Then we have the following:*

- (i) If $p \geq q$ and $\mu(W) - \mu(V) > -\deg(L)$ then γ is surjective whenever

$$\alpha > \alpha_t := \frac{2pq}{pq - q^2 + p + q} (\mu(W) - \mu(V) + \deg(L)) - \deg(L).$$

- (ii) If $p \leq q$ and $\mu(W) - \mu(V) < \deg(L)$ then β is surjective whenever

$$\alpha < \alpha'_t := \frac{2pq}{pq - p^2 + p + q} (\mu(W) - \mu(V) - \deg(L)) + \deg(L).$$

Proof. Using Proposition 3.11 we can find a range for the stability parameter of E^* where β^* and γ^* are injective. Hence the result follows by using Remark 3.13 to relate the stability parameters of E and E^* . \square

The following results shows that the bounds in Proposition 3.11 are meaningful in view of the bounds for α of Proposition 3.7.

Proposition 3.15. *Let α_0 and α'_0 be given in Proposition 3.11. Then the following holds.*

- (i) Assume that $p > q$. If $\mu(V) - \mu(W) > -\deg(L)$ then $\alpha_0 > \alpha_m$, and if $\mu(V) - \mu(W) < -\deg(L)$ then $\alpha_{q-2} > \alpha_m$.
- (ii) Assume that $p < q$. If $\mu(V) - \mu(W) < \deg(L)$ then $\alpha'_0 < \alpha_M$, and if $\mu(V) - \mu(W) > \deg(L)$ then $\alpha'_{p-2} < \alpha_M$.

Proof. For (i), using $\mu(V) - \mu(W) > -\deg(L)$ we get

$$\begin{aligned} \alpha_0 - \alpha_m &= (\mu(V) - \mu(W)) \left(\frac{-2pq}{q(p-q) + p + q} + \frac{2p}{p-q} \right) \\ &\quad + \deg(L) \left(\frac{-2pq}{q(p-q) + p + q} + 1 + \frac{p+q}{p-q} \right) \\ &> \deg(L) \left(-\frac{2p}{p-q} + 1 + \frac{p+q}{p-q} \right) = 0, \end{aligned}$$

where we have used that $p > q$ makes the term which multiplies $\mu(V) - \mu(W)$ positive. Thus $\alpha_0 > \alpha_m$. Moreover, when $\mu(V) - \mu(W) < -\deg(L)$ and $p > q$, we have $\alpha_m = \alpha_{q-1} < \alpha_{q-2}$ (cf. the proof of Proposition 3.11). This finishes the proof of (i).

For (ii), using $\mu(V) - \mu(W) < \deg(L)$ we obtain the following

$$\begin{aligned} \alpha_M - \alpha'_0 &= (\mu(V) - \mu(W)) \left(\frac{-2q}{q-p} + \frac{2pq}{p(q-p) + p + q} \right) \\ &\quad + \deg(L) \left(\frac{p+q}{q-p} - \frac{2pq}{p(q-p) + p + q} + 1 \right) \\ &> \deg(L) \left(-\frac{2q}{q-p} + 1 + \frac{p+q}{q-p} \right) = 0, \end{aligned}$$

where we have used that $p < q$ makes the term which multiplies $\mu(V) - \mu(W)$ negative. Hence $\alpha'_0 < \alpha_M$. Moreover, when $\mu(V) - \mu(W) > \deg(L)$ and $p < q$, we have $\alpha_M = \alpha'_{p-1} > \alpha'_{p-2}$ (again, cf. the proof of Proposition 3.11). This finishes the proof of (ii). \square

3.4. The comparison between $U(p, q)$ -Higgs bundles and $GL(p+q, \mathbb{C})$ -Higgs bundles. Any $U(p, q)$ -Higgs bundle gives rise to a $GL(p+q, \mathbb{C})$ -Higgs bundle. In this section we compare the respective stability conditions. We shall not need these results in the remainder of the paper but for completeness we have chosen to include them, since the question is a natural one to consider.

We recall the following about $GL(n, \mathbb{C})$ -Higgs bundles. A $GL(n, \mathbb{C})$ -Higgs bundle on X is a pair (E, ϕ) , where E is a rank n holomorphic vector bundle over X and $\phi \in H^0(\text{End}(E) \otimes K)$ is a holomorphic endomorphism of E twisted by the canonical bundle K of X . More generally, replacing K by an arbitrary line bundle on X , we obtain the notion of a L -twisted $GL(n, \mathbb{C})$ -Higgs bundle on X . The $GL(n, \mathbb{C})$ -Higgs bundle (E, ϕ) is stable if the slope stability condition

$$\mu(E') < \mu(E)$$

holds for all non-zero proper ϕ -invariant subbundles E' of E . Semistability is defined by replacing the strict inequality with a weak inequality. A twisted Higgs bundle is called polystable if it is the direct sum of stable twisted Higgs bundles with the same slope.

Remark 3.16. Nitsure [28] was the first to study twisted Higgs bundles in a systematic way. For some of his results he needs to make the assumption $\deg(L) \geq 2g - 2$ (similarly, for example, to our Proposition 3.22 below). However, the comparison of stability conditions which we carry out here is valid for any L .

For any twisted $U(p, q)$ -Higgs bundle $E = (V, W, \beta, \gamma)$ we can associate a twisted $GL(p+q, \mathbb{C})$ -Higgs bundle defined by taking $\tilde{E} = V \oplus W$ and $\phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$.

The following result is reminiscent of Theorem 3.26 of [17], which is a result for $Sp(2n, \mathbb{R})$ -Higgs bundles. The corresponding result for 0-semistable $U(p, q)$ -Higgs bundles can be found in the appendix to the first preprint version of [11] and the proof given there easily adapts to the present situation. We include it here for the convenience of the reader.

Recall from Proposition 3.11 that for $p = q$,

$$(3.5) \quad \alpha_0 = p(\mu(W) - \mu(V) - \deg(L)) + \deg(L),$$

$$(3.6) \quad \alpha'_0 = p(\mu(W) - \mu(V) + \deg(L)) - \deg(L).$$

Proposition 3.17. *Let $E = (V, W, \beta, \gamma)$ be an α -semistable twisted $U(p, q)$ -Higgs bundle such that $p = q$ and let α_0 and let α'_0 be given by (3.5) and (3.6), respectively. Suppose that one of the following conditions holds:*

- (1) $\mu(V) - \mu(W) > -\deg(L)$ and $0 \leq \alpha < \alpha_0$.
- (2) $\mu(V) - \mu(W) < \deg(L)$ and $\alpha'_0 < \alpha \leq 0$.

Then the associated $GL(2p, \mathbb{C})$ -Higgs bundle \tilde{E} is semistable. Moreover α -stability of E implies stability of \tilde{E} unless there is an isomorphism $f : V \rightarrow W$ such that $\beta f = f^{-1}\gamma$. In this case (\tilde{E}, ϕ) is polystable and decomposes as

$$(\tilde{E}, \phi) = (\tilde{E}_1, \phi_1) \oplus (\tilde{E}_2, \phi_2)$$

where each summand is a stable $GL(p, \mathbb{C})$ -Higgs bundle isomorphic to $(V, \beta f)$.

Proof. Let \tilde{E}' be an invariant subbundle of \tilde{E} . By projecting onto V and W and taking the kernels and images, we get the following short exact sequences:

$$(3.7) \quad \begin{aligned} 0 &\rightarrow W'' \rightarrow \tilde{E}' \rightarrow V' \rightarrow 0, \\ 0 &\rightarrow V'' \rightarrow \tilde{E}' \rightarrow W' \rightarrow 0. \end{aligned}$$

We can then deduce that

$$(3.8) \quad \begin{aligned} \deg W'' + \deg V' &= \deg \tilde{E}' = \deg V'' + \deg W' \\ q'' + p' &= \text{rk } \tilde{E}' = p'' + q' \end{aligned}$$

where q'' , q' , p'' and p' denote the rank of W'' , W' , V'' and V' , respectively. Note that (V', W') and (V'', W'') define subobjects of E . The α -semistability conditions applied to these subobjects imply

$$(3.9) \quad \deg V' + \deg W' \leq \mu(E)(p' + q') + \frac{q' - p'}{2}\alpha$$

$$(3.10) \quad \deg V'' + \deg W'' \leq \mu(E)(p'' + q'') + \frac{q'' - p''}{2}\alpha$$

Adding these two inequalities and using (3.8), we get

$$(3.11) \quad \mu(\tilde{E}') \leq \mu(\tilde{E}) + \frac{q' - p' + q'' - p''}{2(p' + p'' + q' + q'')}\alpha = \mu(\tilde{E}) + \frac{q' - p'}{p' + p'' + q' + q''}\alpha$$

From Proposition 3.11 we obtain the injectivity of β and γ by using the hypotheses (1) and (2), respectively. Injectivity of β and γ yield $q' \leq p'$ and $q' \geq p'$, respectively. Hence, in either case $(q' - p')\alpha$ is negative. Therefore (3.11) proves that \tilde{E} is semistable.

Suppose now that E is α -stable. Then, by the above argument, \tilde{E} is semistable and it is stable if (3.11) is strict for all non-trivial subbundles $\tilde{E}' \subset \tilde{E}$. The equality holds in (3.11) if it holds in both (3.9) and (3.10). Since E is α -stable the only way in which a non-trivial subbundle $\tilde{E}' \subset \tilde{E}$ can yield equality in (3.11) is that

$$V' \oplus W' = V \oplus W \text{ and } V'' \oplus W''.$$

In this case from (3.7) we obtain isomorphisms $E' \rightarrow V$ and $E' \rightarrow W$. Therefore, combining these, we get an isomorphism $f : V \rightarrow W$ such that $\beta f = f^{-1}\gamma$. Hence, if there is no such isomorphism between V and W then (\tilde{E}, ϕ) is α -stable.

Now suppose that there exists such an isomorphism $f : V \rightarrow W$, define

$$\begin{aligned} (\tilde{E}_1, \phi_1) &= (\{(v, f(v)) \in \tilde{E} | v \in V\}, \phi|_{\tilde{E}_1}), \\ (\tilde{E}_2, \phi_2) &= (\{(v, -f(v)) \in \tilde{E} | v \in V\}, \phi|_{\tilde{E}_2}). \end{aligned}$$

The fact that $\beta f = f^{-1}\gamma$ implies that (E_i, ϕ_i) , $i = 1, 2$, define $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundles isomorphic to $(V, \beta f)$. We have

$$(\tilde{E}, \phi) = (\tilde{E}_1, \phi_1) \oplus (\tilde{E}_2, \phi_2),$$

with

$$\mu(\tilde{E}_1) = \mu(\tilde{E}) = \mu(\tilde{E}_2).$$

To show that each summand is a stable $\mathrm{GL}(n, \mathbb{C})$ -Higgs bundle, note that any non-trivial subbundle \tilde{E}' of \tilde{E}_i is a subbundle of \tilde{E} and hence $\mu(\tilde{E}') < \mu(\tilde{E}) = \mu(\tilde{E}_i)$. \square

Remark 3.18. We can also conclude from the proof of the above proposition that a twisted $\mathrm{U}(p, q)$ -Higgs bundle is α -semistable for $\alpha = 0$ if and only if the associated $\mathrm{GL}(p+q, \mathbb{C})$ -Higgs bundle is semistable. Equivalence also holds for stability, unless there is an isomorphism $f : V \rightarrow W$ such that $\beta f = f^{-1}\gamma$.

3.5. Vanishing of hypercohomology in degree two. In order to study smoothness of the moduli space we investigate vanishing of the second hypercohomology group of the deformation complex (cf. Proposition 2.23). This vanishing will also play an important role in the analysis of the flip loci in Section 4. We note that vanishing is not guaranteed by α -stability for $\alpha \neq 0$, in contrast to the case of triples (and chains), where vanishing is guaranteed for $\alpha > 0$.

By using the obvious symmetry of the quiver interchanging the vertices we can associate to a $\mathrm{U}(p, q)$ -Higgs bundle a $\mathrm{U}(q, p)$ -Higgs bundle. The following proposition is immediate.

Proposition 3.19. *Let $E = (V, W, \beta, \gamma)$ be a $\mathrm{U}(p, q)$ -Higgs bundle and let $\sigma(E) = (W, V, \gamma, \beta)$ be the associated $\mathrm{U}(q, p)$ -Higgs bundle. Then E is α -stable if and only if $\sigma(E)$ is $-\alpha$ -stable, and similarly for poly- and semi-stability. \square*

The next result uses this construction and Serre duality to identify the second hypercohomology of the Hom-complex with the dual of a zeroth hypercohomology group.

Lemma 3.20. *Let $E = (V, W, \beta, \gamma)$ be a L -twisted $\mathrm{U}(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $\mathrm{U}(p', q')$ -Higgs bundle. Let $E'' = \sigma(E') \otimes L^{-1}K = (W' \otimes L^{-1}K, V' \otimes L^{-1}K, \gamma \otimes 1, \beta \otimes 1)$. Then*

$$\mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) \cong \mathbb{H}^0(\mathcal{H}om^\bullet(E, E''))^*.$$

Proof. By Serre duality for hypercohomology

$$\mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) \cong \mathbb{H}^0(\mathcal{H}om^{\bullet\vee}(E', E) \otimes K)^*$$

where the dual complex twisted by K is

$$\begin{aligned} \mathcal{H}om^{\bullet \vee}(E', E) \otimes K &: (\text{Hom}(V, W' \otimes L^{-1}) \oplus \text{Hom}(W, V' \otimes L^{-1})) \otimes K \\ &\rightarrow (\text{Hom}(V, V') \oplus \text{Hom}(W, W')) \otimes K. \end{aligned}$$

One easily checks that the differentials correspond, so that

$$\mathcal{H}om^{\bullet \vee}(E', E) \otimes K \cong \mathcal{H}om^{\bullet}(E, E'').$$

This completes the proof. \square

Lemma 3.21. *Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. As above let $E'' = \sigma(E') \otimes L^{-1}K = (W' \otimes L^{-1}K, V' \otimes L^{-1}K, \gamma' \otimes 1, \beta' \otimes 1)$. Let $f \in \mathbb{H}^0(\mathcal{H}om^{\bullet}(E, E''))$ viewed as morphism $f: E \rightarrow E''$ and write $\lambda(f) = \frac{\text{rk}(f(V))}{\text{rk}(f(V)) + \text{rk}(f(W))}$. Then, if $f \neq 0$, the inequality*

$$(3.12) \quad \alpha(2\lambda(f) - 1) + 2g - 2 - \deg(L) \geq 0$$

holds. Moreover, if E and E'' are α -stable, then strict inequality holds unless $f: E \xrightarrow{\cong} E''$ is an isomorphism.

Proof. Write $N = \ker(f) \subset E$ and $I = \text{im}(f) \subset E''$. Then α -semistability of E implies that $\mu_{\alpha}(N) \leq \mu_{\alpha}(E)$, which is equivalent to

$$(3.13) \quad \mu_{\alpha}(I) \geq \mu_{\alpha}(E);$$

note that this also holds if $N = 0$, since then $I \cong E$. Moreover, by Proposition 3.19, E'' is $-\alpha$ -semistable and so $\mu_{-\alpha}(I) \leq \mu_{-\alpha}(E'')$. This, using that $\mu_{-\alpha}(I) = \mu_{\alpha}(I) - 2\alpha\lambda(f)$ and $\mu_{-\alpha}(E'') = \mu_{\alpha}(E) - \alpha + (2g - 2 - \deg(L))$, is equivalent to

$$(3.14) \quad \mu_{\alpha}(I) \leq \mu_{\alpha}(E) + 2\alpha\lambda(f) - \alpha + 2g - 2 - \deg(L).$$

Combining (3.13) and (3.14) gives the result. The statement about strict inequality is easy. \square

The following is our first main result on vanishing of \mathbb{H}^2 . It should be compared with [9, Proposition 3.6]. The reason why extra conditions are required for the vanishing is essentially that the ‘‘total Higgs field’’ $\beta + \gamma \in H^0(\text{End}(V \oplus W) \otimes L)$ is not nilpotent, contrary to the case of triples.

Proposition 3.22. *Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. Assume that E and E' are α -semistable with $\mu_{\alpha}(E) = \mu_{\alpha}(E')$. Let $E'' = \sigma(E') \otimes L^{-1}K$. Assume that one of the following hypotheses hold:*

- (A) $\deg(L) > 2g - 2$;
- (B) $\deg(L) = 2g - 2$, both E and E' are α -stable and there is no isomorphism $f: E \xrightarrow{\cong} E''$.

Then $\mathbb{H}^2(\mathcal{H}om^{\bullet}(E', E)) = 0$ if one of the following additional conditions holds:

- (1) $\alpha = 0$;
- (2) $\alpha > 0$ and either β' is injective or β is surjective;
- (3) $\alpha < 0$ and either γ' is injective or γ is surjective.

Proof. Suppose first that $\alpha = 0$. Then either of the conditions (A) and (B) guarantee that strict inequality holds in (3.12). Hence Lemmas 3.20 and 3.21 imply the stated vanishing of \mathbb{H}^2 .

Now suppose that $\beta': W' \rightarrow V' \otimes L$ is injective. If $f: E \rightarrow E''$ is non-zero then, since f is a morphism of twisted $U(p, q)$ -Higgs bundles, we have $\text{rk}(f(W)) \geq \text{rk}(f(V))$.

Hence $\lambda(f) = \frac{\text{rk}(f(V))}{\text{rk}(f(V)) + \text{rk}(f(W))}$ satisfies $\lambda(f) \leq 1/2$. If additionally $\alpha > 0$, it follows that $\alpha(2\lambda(f) - 1) \leq 0$ which contradicts Lemma 3.21 under either of the conditions (A) and (B). Therefore there are no non-zero morphisms $f: E \rightarrow E''$ and so Lemma 3.20 implies vanishing of $\mathbb{H}^2(\mathcal{H}om^\bullet(E', E))$.

We have deduced vanishing of \mathbb{H}^2 under the conditions $\alpha > 0$ and β' injective. The remaining conditions in (2) and (3) for vanishing of \mathbb{H}^2 can now be deduced by using symmetry arguments as follows.

Suppose first that $\alpha < 0$ and γ' is injective. Then, using Proposition 3.19, $\sigma(E)$ is an $-\alpha$ -semistable $U(p, q)$ -Higgs bundle and similarly for $\sigma(E')$. Moreover, the β -map (which is $\sigma(\gamma')$) of $\sigma(E')$ is injective. Observe that

$$\mathcal{H}om^\bullet(\sigma(E'), \sigma(E)) \cong \mathcal{H}om^\bullet(E', E).$$

Hence, noting that $-\alpha > 0$, the conclusion follows from the previous case applied to the pair $(\sigma(E'), \sigma(E))$.

Next suppose that $\alpha < 0$ and γ is surjective. Then the dual $U(p, q)$ -Higgs bundle E^* is $-\alpha$ -semistable, and similarly for E'^* . Moreover, the β -map (which is γ^*) of E^* is injective. Observe that

$$\mathcal{H}om^\bullet(E^*, E'^*) \cong \mathcal{H}om^\bullet(E', E).$$

Hence again the conclusion follows from the previous case, applied to the pair (E^*, E'^*) .

The final case, $\alpha > 0$ and β surjective, follows in a similar way, combining the two previous constructions. \square

In the case when $q = 1$ we can improve on Proposition 3.22, as follows.

Proposition 3.23. *Let E be an α -semistable L -twisted $U(p, 1)$ -Higgs bundle with $p \geq 2$. Assume that $\deg(L) > 2g - 2$. Then $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$ for all α in the range*

$$p(\mu(V) - \mu(W)) - (p+1)(\deg(L) - 2g + 2) < \alpha < p(\mu(V) - \mu(W)) + (p+1)(\deg(L) - 2g + 2).$$

Proof. Assume first that $\alpha \geq 0$. Note that an isomorphism as in (B) of the hypothesis of Proposition 3.22 cannot exist when $p \neq q$. Hence the proposition immediate gives the result if $\alpha = 0$. Moreover, if $\beta \neq 0$, then it is injective, and hence $\mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) = 0$ by (2) of the proposition. We may thus assume that $\beta = 0$ and consider the L -twisted triple $E_T: \gamma: V \rightarrow W \otimes L$. We have that

$$\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = \mathbb{H}^2(\mathcal{E}nd^\bullet(E_T)) \oplus H^1(\text{Hom}(W, V) \otimes L),$$

where $\mathcal{E}nd^\bullet(E_T)$ is the deformation complex of the triple. The vanishing of $\mathbb{H}^2(\mathcal{E}nd^\bullet(E_T))$ for an α -semistable triple when $\alpha > 0$ is well known¹ (cf. [9]). Hence it remains to show that $H^1(\text{Hom}(W, V) \otimes L) = 0$ which, by Serre duality, is equivalent to the vanishing

$$H^0(\text{Hom}(V, W) \otimes L^{-1}K) = 0.$$

So assume we have a non-zero $f: V \rightarrow W \otimes L^{-1}K$. Then f induces a non-zero map of line bundles $f: V/\ker(f) \rightarrow W \otimes L^{-1}K$ and hence

$$(3.15) \quad \deg(W) - \deg(L) + 2g - 2 \geq \deg(V) - \deg(\ker(f)).$$

On the other hand, since $\beta = 0$ we can consider the subobject $(\ker(f), W, 0, \gamma)$ of E and hence, by α -semistability,

$$(3.16) \quad \begin{aligned} & \mu_\alpha(\ker(f) \oplus W) \leq \mu_\alpha(V \oplus W) \\ \iff & (p+1) \deg(\ker(f)) + \deg(W) \leq p \deg(V) + \alpha, \end{aligned}$$

¹Note that the stability parameter for the corresponding untwisted triple as considered in [9] is $\alpha + \deg(L)$.

where we have used that $\text{rk}(\ker(f)) = p - 1$ and $\text{rk}(W) = 1$. Now combining (3.15) and (3.16) we obtain

$$\alpha \geq p(\mu(V) - \mu(W)) + (p + 1)(\deg(L) - 2g + 2).$$

This establishes the vanishing of \mathbb{H}^2 for α in the range

$$0 \leq \alpha < p(\mu(V) - \mu(W)) + (p + 1)(\deg(L) - 2g + 2).$$

On the other hand, if $\alpha \leq 0$, applying the preceding result to the dual twisted $U(p, q)$ -Higgs bundle $(V^*, W^*, \gamma^*, \beta^*)$ gives vanishing of \mathbb{H}^2 for α in the range

$$0 \geq \alpha > p(\mu(V) - \mu(W)) - (p + 1)(\deg(L) - 2g + 2).$$

This finishes the proof. \square

In general the preceding proposition does not guarantee vanishing of \mathbb{H}^2 for all values of the parameter α . But for some values of the topological invariants, the upper bound of the preceding proposition is actually larger than the maximal value for the parameter α . More precisely, we have the following result.

Proposition 3.24. *Let E be an α -semistable L -twisted $U(p, 1)$ -Higgs bundle with $p \geq 2$. Assume that $\deg(L) > 2g - 2$. We have the following:*

- (1) *If $p(\mu(V) - \mu(W)) > 2g - 2 - (p - 2)(\deg(L) - (2g - 2))$ then $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$ for all $\alpha \geq 0$*
- (2) *If $p(\mu(V) - \mu(W)) < -2g + 2 + (p - 2)(\deg(L) - (2g - 2))$ then $\mathbb{H}^2(\mathcal{E}nd^\bullet(E)) = 0$ for all $\alpha \leq 0$*

Proof. The upper and lower bound for α given in Proposition 3.7 is, in this case

$$\begin{aligned} \alpha_M &= -\frac{2p}{p-1}(\mu(V) - \mu(W)) + \frac{p+1}{p-1}\deg(L), \\ \alpha_m &= -\frac{2p}{p-1}(\mu(V) - \mu(W)) - \frac{p+1}{p-1}\deg(L). \end{aligned}$$

It is simple to check that the inequalities of the statements are equivalent to α_M being less than the upper bound and α_m being bigger than the lower bound for α of Proposition 3.23. \square

The following trivial observation is sometimes useful.

Proposition 3.25. *Let E and E' be L -twisted $U(p, q)$ -Higgs bundles such that $\mathbb{H}^2(\mathcal{E}nd^\bullet(E \oplus E')) = 0$. Then*

$$\mathbb{H}^2(\mathcal{H}om^\bullet(E', E)) = \mathbb{H}^2(\mathcal{H}om^\bullet(E, E')) = 0.$$

Proof. Immediate in view of Remark 2.18. \square

We can summarize our main results on vanishing of \mathbb{H}^2 as follows.

Lemma 3.26. *Fix a type $t = (p, q, a, b)$ and let E be an α -semistable L -twisted $U(p, q)$ -Higgs bundle of type t with $\deg(L) \geq 2g - 2$. If $\deg(L) = 2g - 2$ assume moreover that E is α -stable. If either one of the following conditions holds:*

- (1) $q = 1$, $p \geq 2$ and $p(a/p - b/q) - \deg(L)(p + 1) < \alpha < p(a/p - b/q) + \deg(L)(p + 1)$,
- (2) $a/p - b/q > -\deg(L)$ and $0 \leq \alpha < \frac{2pq}{\min\{p, q\}|p-q|+p+q}(b/q - a/p - \deg(L)) + \deg(L)$,
- (3) $a/p - b/q < \deg(L)$ and $\frac{2pq}{\min\{p, q\}|p-q|+p+q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha \leq 0$.

Then $\mathbb{H}^2(\mathcal{E}nd^\bullet(E))$ vanishes.

Proof. For part (1), use Proposition 3.23. The other parts follow from Proposition 3.11, Corollary 3.14, and Proposition 3.22. \square

3.6. Moduli space of twisted $U(p, q)$ -Higgs bundles. Finally, we are in a position to make statements about smoothness of the moduli space. Recall that we denote the moduli space of α -polystable twisted $U(p, q)$ -Higgs bundles with type $t = (p, q, a, b)$ by

$$\mathcal{M}_\alpha(t) = \mathcal{M}_\alpha(p, q, a, b),$$

and the moduli space of α -stable twisted $U(p, q)$ -Higgs bundle by $\mathcal{M}_\alpha^s(t) \subset \mathcal{M}_\alpha(t)$.

Proposition 3.27. *Fix a type $t = (p, q, a, b)$. If either one of the following conditions holds:*

- (1) $q = 1, p \geq 2$ and $p(a/p - b/q) - \deg(L)(p+1) < \alpha < p(a/p - b/q) + \deg(L)(p+1)$,
- (2) $a/p - b/q > -\deg(L)$ and $0 \leq \alpha < \frac{2pq}{\min\{p, q\}|p-q|+p+q}(b/q - a/p - \deg(L)) + \deg(L)$,
- (3) $a/p - b/q < \deg(L)$ and $\frac{2pq}{\min\{p, q\}|p-q|+p+q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha \leq 0$.

Then the moduli space $\mathcal{M}_\alpha^s(t)$ is smooth.

Proof. Combine Lemma 3.26 and Proposition 2.23. □

4. CROSSING CRITICAL VALUES

4.1. Flip loci. In this section we study the variation with α of the moduli spaces $\mathcal{M}_\alpha^s(t)$ for fixed type $t = (p, q, a, b)$. We are using a method similar to the one for chains given in [3], which in turn is based on [9].

Let α_c be a critical value. We adopt the following notation:

$$\alpha_c^+ = \alpha_c + \epsilon, \quad \alpha_c^- = \alpha_c - \epsilon,$$

where $\epsilon > 0$ is small enough so that α_c is the only critical value in the interval (α_c^-, α_c^+) . We begin with a set theoretic description of the differences between two spaces $\mathcal{M}_{\alpha_c^+}$ and $\mathcal{M}_{\alpha_c^-}$.

Definition 4.1. We define *flip loci* $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{M}_{\alpha_c^\pm}^s$ by the condition that the points in $\mathcal{S}_{\alpha_c^+}$ represent twisted $U(p, q)$ -Higgs bundles which are α_c^+ -stable but α_c^- -unstable, and analogously for $\mathcal{S}_{\alpha_c^-}$.

A twisted $U(p, q)$ -Higgs bundle $E \in \mathcal{S}_{\alpha_c^\pm}$ is strictly α_c -semistable and so we can use the Jordan-Hölder filtrations of E in order to estimate the codimension of $\mathcal{S}_{\alpha_c^\pm}$ in $\mathcal{M}_{\alpha_c^\pm}^s$.

The following is an analogue for twisted $U(p, q)$ -Higgs bundles of [3, Proposition 4.3], which is a result for chains.

Proposition 4.2. *Fix a type $t = (p, q, a, b)$. Let α_c be a critical value and let \mathcal{S} be a family of α_c -semistable twisted $U(p, q)$ -Higgs bundles E of type t , all of them pairwise non-isomorphic, and whose Jordan-Hölder filtrations has an associated graded of the form $\text{Gr}(E) = \bigoplus_{i=1}^m Q_i$, with Q_i twisted $U(p, q)$ -Higgs bundle of type t_i . If either one of the following conditions holds:*

- (1) $q = 1, p \geq 2$ and $p(a/p - b/q) - \deg(L)(p+1) < \alpha_c < p(a/p - b/q) + \deg(L)(p+1)$,
- (2) $a/p - b/q > -\deg(L)$ and $0 \leq \alpha_c < \frac{2pq}{\min\{p, q\}|p-q|+p+q}(b/q - a/p - \deg(L)) + \deg(L)$,
- (3) $a/p - b/q < \deg(L)$ and $\frac{2pq}{\min\{p, q\}|p-q|+p+q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha_c \leq 0$.

Then

$$(4.1) \quad \dim \mathcal{S} \leq - \sum_{i \leq j} \chi(t_j, t_i) - \frac{m(m-3)}{2}.$$

Proof. Once appropriate vanishing of \mathbb{H}^2 is ensured, the proof is similar to the proof of [3, Proposition 4.3]; we indicate the idea for $m = 2$. In view of the definition of \mathcal{S} , there exists an injective canonical map

$$i : \mathcal{S} \rightarrow \mathcal{M}_{\alpha_c}^s(t_1) \times \mathcal{M}_{\alpha_c}^s(t_2)$$

with $i^{-1}(Q_1, Q_2) \cong \mathbb{P}(\text{Ext}^1(Q_2, Q_1))$, where $\mathbb{P}(\text{Ext}^1(Q_2, Q_1))$ parametrizes equivalence classes of extensions

$$0 \rightarrow Q_1 \rightarrow E \rightarrow Q_2 \rightarrow 0.$$

Notice that Q_1 and Q_2 satisfy the hypothesis of Proposition 3.22 (or, in case $q = 1$, Proposition 3.23; cf. Proposition 3.25) and therefore, cf. Proposition 2.17, $\dim(\mathbb{P} \text{Ext}^1(Q_2, Q_1))$ is constant as Q_1 and Q_2 vary in their corresponding moduli spaces. Hence, we obtain

$$\dim \mathcal{S} \leq \dim \mathcal{M}_{\alpha_c}^s(t_1) + \dim \mathcal{M}_{\alpha_c}^s(t_2) + \dim \mathbb{P}(\text{Ext}^1(Q_2, Q_1)).$$

The general case follows by induction on m as in loc. cit. \square

In order to show that the flip loci $\mathcal{S}_{\alpha_c^\pm}$ has positive codimension we need to bound the values of $\chi(t_i, t_j)$ in (4.1). This is what we do next.

4.2. Bound for χ . Here we consider a Q -bundle associated to the complex $\mathcal{H}om^\bullet(E', E)$ and construct a solution to the vortex equations on this Q -bundle from solutions on E' and E . The quiver Q is the following:



The construction generalizes the one of [9] Lemma 4.2.

4.2.1. The Q -bundle associated to $\mathcal{H}om^\bullet(E', E)$. Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. Let us consider the following twisted Q -bundle \tilde{E} (the morphisms are twisted by L for each arrow):

$$(4.2) \quad \begin{array}{ccccccc} & & \phi_d & & \phi_c & & \\ & \curvearrowright & & \curvearrowleft & & \curvearrowright & \\ \text{Hom}(W', V) & & \text{Hom}(V', V) \oplus \text{Hom}(W', W) & & \text{Hom}(V', W) & & \\ & \curvearrowleft & & \curvearrowright & & \curvearrowleft & \\ & & \phi_b & & \phi_a & & \end{array}$$

where

$$\begin{aligned} \phi_a(f_1, f_2) &= (f_2 \otimes 1_L) \circ \gamma' - \gamma \circ f_1, \\ \phi_b(f_1, f_2) &= (f_1 \otimes 1_L) \circ \beta' - \beta \circ f_2, \\ \phi_c(g) &= (\beta \circ g, (g \otimes 1_L) \circ \beta'), \\ \phi_d(h) &= ((h \otimes 1_L) \circ \gamma', \gamma \circ h). \end{aligned}$$

We will write briefly as \tilde{E}

$$\begin{array}{ccccc} & \phi_d & & \phi_c & \\ & \xrightarrow{L} & & \xleftarrow{L} & \\ \mathcal{H}om^{12} & & \mathcal{H}om^0 & & \mathcal{H}om^{11} \\ & \xleftarrow{L} & & \xrightarrow{L} & \\ & \phi_b & & \phi_a & \end{array}.$$

Note that $\mathcal{H}om^1 = \mathcal{H}om^{11} \oplus \mathcal{H}om^{12}$ and $a_0 = (\phi_a, \phi_b)$, where $a_0 : \mathcal{H}om^0 \rightarrow \mathcal{H}om^1$ is the Hom-complex (2.4).

In this section, by using Proposition 2.11, we prove that if E' and E are α -polystable then \tilde{E} is α -polystable for a suitable choice of α .

Lemma 4.3. *Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. Suppose, moreover, we have solutions to the (τ_1, τ_2) -vortex equations on E and the (τ'_1, τ'_2) -vortex equations on E' such that $\tau_1 - \tau'_1 = \tau_2 - \tau'_2$. Then the induced Hermitian metric on the Q -bundle \tilde{E} satisfies the vortex equations*

$$\begin{aligned}\sqrt{-1}\Lambda F(\mathcal{H}om^{12}) + \phi_b\phi_b^* - \phi_d^*\phi_d &= \tilde{\tau}_2\text{Id}_{\mathcal{H}om^{12}}, \\ \sqrt{-1}\Lambda F(\mathcal{H}om^0) + \phi_c\phi_c^* + \phi_d\phi_d^* - \phi_a^*\phi_a - \phi_b^*\phi_b &= \tilde{\tau}_1\text{Id}_{\mathcal{H}om^0}, \\ \sqrt{-1}\Lambda F(\mathcal{H}om^{11}) + \phi_a\phi_a^* - \phi_c^*\phi_c &= \tilde{\tau}_0\text{Id}_{\mathcal{H}om^{11}}.\end{aligned}$$

For $\tau = (\tilde{\tau}_0, \tilde{\tau}_1, \tilde{\tau}_2)$ given by

$$\begin{aligned}\tilde{\tau}_0 &= \tau_2 - \tau'_1, \\ \tilde{\tau}_1 &= \tau_1 - \tau'_1 = \tau_2 - \tau'_2, \\ \tilde{\tau}_2 &= \tau_1 - \tau'_2.\end{aligned}$$

Proof. The vortex equations for E and E' are

$$\begin{aligned}\sqrt{-1}\Lambda F(V) + \beta\beta^* - \gamma^*\gamma &= \tau_1\text{Id}_V, \\ \sqrt{-1}\Lambda F(W) + \gamma\gamma^* - \beta^*\beta &= \tau_2\text{Id}_W, \\ \sqrt{-1}\Lambda F(V') + \beta'\beta'^* - \gamma'^*\gamma' &= \tau'_1\text{Id}_{V'}, \\ \sqrt{-1}\Lambda F(W') + \gamma'\gamma'^* - \beta'^*\beta' &= \tau'_2\text{Id}_{W'}.\end{aligned}$$

We have

$$F(\mathcal{H}om^0)(\psi, \eta) = (F(V) \circ \psi - \psi \circ F(V'), F(W) \circ \eta - \eta \circ F(W')).$$

Now we calculate ϕ_a^* and ϕ_b^* : for $(f_1, f_2) \in \mathcal{H}om^0$, $g \in \mathcal{H}om^{11}$ and $h \in \mathcal{H}om^{12}$ we have,

$$\begin{aligned}\langle \phi_a^*(g), (f_1, f_2) \rangle_{\mathcal{H}om^0} &= \langle g, \phi_a((f_1, f_2)) \rangle_{\mathcal{H}om^{11}} \\ &= \langle g, (f_2 \otimes 1_L) \circ \gamma' - \gamma \circ f_1 \rangle_{\mathcal{H}om^{11}} \\ &= \langle g, (f_2 \otimes 1_L) \circ \gamma' \rangle_{C_{11}} - \langle g, \gamma \circ f_1 \rangle_{\mathcal{H}om^{11}} \\ &= \langle (g \circ \gamma'^*) \otimes 1_{L^*}, f_2 \rangle_{\text{Hom}(W', W)} + \langle -\gamma^* \circ g, f_1 \rangle_{\text{Hom}(V', V)} \\ &= \langle (-\gamma^* \circ g, (g \circ \gamma'^*) \otimes 1_{L^*}), (f_1, f_2) \rangle_{\mathcal{H}om^0}\end{aligned}$$

and

$$\begin{aligned}\langle \phi_b^*(h), (f_1, f_2) \rangle_{\mathcal{H}om^0} &= \langle h, \phi_b((f_1, f_2)) \rangle_{\mathcal{H}om^{12}} \\ &= \langle h, (f_1 \otimes 1_L) \circ \beta' - \beta \circ f_2 \rangle_{\mathcal{H}om^{12}} \\ &= \langle (h \circ \beta'^*) \otimes 1_{L^*}, f_1 \rangle_{\text{Hom}(V', V)} - \langle \beta^* \circ h, f_2 \rangle_{\text{Hom}(W', W)} \\ &= \langle ((h \circ \beta'^*) \otimes 1_{L^*}, -\beta^* \circ h), (f_1, f_2) \rangle_{\mathcal{H}om^0}\end{aligned}$$

Hence,

$$\begin{aligned}\phi_a^*(g) &= (-\gamma^* \circ g, (g \circ \gamma'^*) \otimes 1_{L^*}), \\ \phi_b^*(h) &= ((h \circ \beta'^*) \otimes 1_{L^*}, -\beta^* \circ h).\end{aligned}$$

By a similar calculation as above, we have

$$\begin{aligned}\phi_c^*(f_1, f_2) &= (f_2 \circ \beta'^*) \otimes 1_{L^*} - \beta^* \circ f_1, \\ \phi_d^*(f_1, f_2) &= (f_1 \circ \gamma'^*) \otimes 1_{L^*} - \gamma^* \circ f_2.\end{aligned}$$

Let $g \in \mathcal{H}om^{11}$ and $h \in \mathcal{H}om^{12}$, then we have:

$$\begin{aligned}\phi_c^* \phi_c(g) &= \phi_c^*(\beta \circ g, (g \otimes 1_L) \circ \beta') \\ &= \beta^* \beta \circ g + g \circ \beta' \beta'^*.\end{aligned}$$

$$\begin{aligned}\phi_d^* \phi_d(h) &= \phi_d^*((h \otimes 1_L) \circ \gamma', \gamma \circ h) \\ &= h \circ \gamma' \gamma'^* - \gamma^* \gamma \circ h.\end{aligned}$$

and

$$\begin{aligned}\phi_b \phi_b^*(h) &= \phi_b(h \circ \beta'^* \otimes 1_{L^*}, \beta^* \circ h) \\ &= h \circ \beta'^* \beta' - \beta \beta^*.\end{aligned}$$

$$\begin{aligned}\phi_a \phi_a^*(g) &= \phi_a(g \circ \gamma'^* \otimes 1_{L^*}, -\gamma^* \circ g) \\ &= g \circ \gamma'^* \gamma' + \gamma \gamma^* \circ g.\end{aligned}$$

Thus,

$$\begin{aligned}\phi_b \phi_b^* - \phi_d^* \phi_d(h) &= h \circ \beta'^* \beta' - \beta \beta^* \circ h - h \circ \gamma' \gamma'^* + \gamma^* \gamma \circ h \\ \phi_a \phi_a^* - \phi_c^* \phi_c(g) &= g \circ \gamma'^* \gamma' + \gamma \gamma^* \circ g - \beta^* \beta \circ g - g \circ \beta' \beta'^*\end{aligned}$$

Hence for $g \in \mathcal{H}om^{11}$ and $h \in \mathcal{H}om^{12}$ we have,

$$\begin{aligned}(\sqrt{-1}\Lambda F(\mathcal{H}om^{11}) + \phi_a \phi_a^* - \phi_c^* \phi_c)(g) &= \sqrt{-1}\Lambda(F(W) \circ g - g \circ F(V')) + \phi_a \phi_a^* - \phi_c^* \phi_c(g) \\ &= (\sqrt{-1}\Lambda F(W) + \gamma \gamma^* - \beta^* \beta) \circ g + g \circ (-\sqrt{-1}\Lambda F(V') + \gamma'^* \gamma' - \beta' \beta'^*) \\ &= \tau_2 \text{Id}_W \circ g - g \circ \tau_1' \text{Id}_{V'} \\ &= (\tau_2 - \tau_1')g\end{aligned}$$

$$\begin{aligned}(\sqrt{-1}\Lambda F(\mathcal{H}om^{12}) + \phi_b \phi_b^* - \phi_d^* \phi_d)(h) &= \sqrt{-1}\Lambda(\otimes F(V) \circ h - h \circ F(W')) + \phi_b \phi_b^* - \phi_d^* \phi_d(h) \\ &= (\sqrt{-1}\Lambda F(V) + \gamma^* \gamma - \beta \beta^*) \circ h + h \circ (-\sqrt{-1}\Lambda F(W') + \beta'^* \beta' - \gamma' \gamma'^*) \\ &= \tau_1 \text{Id}_V \circ h - h \circ \tau_2' \text{Id}_{W'} \\ &= (\tau_1 - \tau_2')h.\end{aligned}$$

Similarly for $(f_1, f_2) \in \mathcal{H}om^0$ we have,

$$\begin{aligned}\phi_c \phi_c^*(f_1, f_2) &= \phi_c((f_2 \circ \beta'^*) \otimes 1_{L^*} - \beta^* \circ f_1) \\ &= (\beta \beta^* \circ f_1 - \beta \circ (f_2 \circ \beta'^*) \otimes 1_{L^*}), f_2 \circ \beta'^* \beta' - (\beta^* \circ f_1 \otimes 1_L) \otimes \beta'\end{aligned}$$

$$\begin{aligned}\phi_d \phi_d^*(f_1, f_2) &= \phi_d((f_1 \circ \gamma'^*) \otimes 1_{L^*} - \gamma^* \circ f_2) \\ &= (f_1 \circ \gamma'^* \gamma' - \gamma^* \circ f_2 \otimes 1_L \circ \gamma', \gamma \circ (f_1 \circ \gamma'^* \otimes 1_{L^*}) - \gamma \gamma^* \circ f_2)\end{aligned}$$

and

$$\begin{aligned}\phi_a^* \phi_a(f_1, f_2) &= \phi_a^*(f_2 \otimes 1_L \circ \gamma' - \gamma \circ f_1) \\ &= (-\gamma^* \circ f_2 \otimes 1_L \circ \gamma' + \gamma^* \gamma \circ f_1, (f_2 \circ \gamma' \gamma'^* - \gamma \circ f_1 \circ \gamma'^* \otimes 1_{L^*}))\end{aligned}$$

$$\begin{aligned}\phi_b^* \phi_b(f_1, f_2) &= \phi_b^*(f_1 \otimes 1_L \circ \beta' - \beta \circ f_2) \\ &= (f_1 \circ \beta' \beta'^* - \beta \circ f_2 \circ \beta'^* \otimes 1_{L^*}, \beta^* \circ f_1 \otimes 1_L \circ \beta' - \beta^* \beta \circ f_2)\end{aligned}$$

So,

$$\begin{aligned} & (\phi_c \phi_c^* + \phi_d \phi_d^* - \phi_a^* \phi_a - \phi_b^* \phi_b)(f_1, f_2) \\ &= (\beta \beta^* \circ f_1 + f_1 \circ \gamma'^* \gamma' - \gamma^* \gamma \circ f_1 - f_1 \circ \beta' \beta'^*, f_2 \circ \beta'^* \beta' - \gamma \gamma^* \circ f_2 - f_2 \circ \gamma' \gamma'^* + \beta^* \beta \circ f_2) \end{aligned}$$

Hence we have,

$$\begin{aligned} & (\sqrt{-1} \Lambda F(\mathcal{H}om^0) + \phi_c \phi_c^* + \phi_d \phi_d^* - \phi_a^* \phi_a - \phi_b^* \phi_b)(f_1, f_2) \\ &= \left(\sqrt{-1} \Lambda(F(V) \circ f_1 - f_1 \circ F(V')), \sqrt{-1} \Lambda(F(W) \circ f_2 - f_2 \circ F(W')) \right) + \\ & \quad \left(\beta \beta^* \circ f_1 + f_1 \circ \gamma'^* \gamma' - \gamma^* \gamma \circ f_1 - f_1 \circ \beta' \beta'^*, f_2 \circ \beta'^* \beta' - \gamma \gamma^* \circ f_2 - f_2 \circ \gamma' \gamma'^* + \beta^* \beta \circ f_2 \right) \\ &= \left((\sqrt{-1} \Lambda F(V) + \beta \beta^* - \gamma^* \gamma) \circ f_1 + f_1 \circ (-\sqrt{-1} \Lambda F(V') + \gamma'^* \gamma' - \beta' \beta'^*), \right. \\ & \quad \left. (\sqrt{-1} \Lambda F(W) + \gamma \gamma^* - \beta^* \beta) \circ f_2 + f_2 \circ (-\sqrt{-1} \Lambda F(W') + \beta'^* \beta' - \gamma' \gamma'^*) \right) \\ &= \left((\tau_1 - \tau'_1) f_1, (\tau_2 - \tau'_2) f_2 \right). \end{aligned}$$

The proof is completed, since by assumption $\tau_1 - \tau'_1 = \tau_2 - \tau'_2$. \square

Theorem 4.4. *Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. Then the Q -bundle \tilde{E} is α -polystable for $\alpha = (\alpha, 2\alpha)$.*

Proof. Since E and E' are α -polystable, from Theorem 2.11 follows that they support solutions to the (τ_1, τ_2) - and (τ'_1, τ'_2) -vortex equations where $\alpha = \tau_2 - \tau_1 = \tau'_2 - \tau'_1$. Using Lemma 4.3 it follows that the Q -bundle \tilde{E} admits a Hermitian metric such that vortex equations are satisfied for $\tau = (\tau_2 - \tau'_1, \tau_2 - \tau'_2, \tau_1 - \tau'_2)$. Now from Theorem 2.7 we get that \tilde{E} is α -polystable for

$$\begin{aligned} \alpha_1 &= \tau_2 - \tau'_1 - \tau_2 + \tau'_2 = \alpha, \\ \alpha_2 &= \tau_2 - \tau'_1 - \tau_1 + \tau'_2 = 2\alpha. \end{aligned}$$

\square

4.2.2. *Bound for $\chi(E', E)$.* We are using the method in [9] and we start with some lemmas needed to estimate $\chi(E', E)$.

Lemma 4.5. *Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. Let $\mathcal{H}om^\bullet(E', E)$ be the deformation complex of E and E' , as in (2.14). Then the following inequalities hold.*

$$(4.3) \quad \deg(\ker(a_0) \leq \text{rk}(\ker(a_0))(\mu_\alpha(E') - \mu_\alpha(E))$$

$$(4.4) \quad \deg(\text{im}(a_0) \leq (\text{rk}(\mathcal{H}om^1) - \text{rk}(\text{im}(a_0)))(\mu_\alpha(E) - \mu_\alpha(E') - \deg(L)) - \alpha(\text{rk}(\mathcal{H}om^1) - \text{rk}(\text{im}(a_0)) - 2 \text{rk}(\text{coker}(\phi_b))) + \deg(\mathcal{H}om^1).$$

Proof. Assume that $\text{rk}(\ker(a_0)) > 0$ as if it is zero then (4.3) is obvious. It follows from Proposition 4.4 that the Q -bundle \tilde{E} is $\alpha = (\alpha, 2\alpha)$ -polystable. We can define a subobject of \tilde{E} by

$$\mathcal{K} : \begin{array}{ccccc} & \curvearrowright & & \curvearrowright & \\ & 0 & \rightarrow & \ker(a_0) & \rightarrow & 0. \\ & \curvearrowleft & & \curvearrowleft & \end{array}$$

It follows from the α -polystability that

$$\mu_\alpha(\mathcal{K}) = \mu(\ker(a_0)) + \alpha \leq \mu_\alpha(\tilde{E}) = \mu_\alpha(E') - \mu_\alpha(E) + \alpha.$$

Thus we have

$$\mu(\ker(a_0)) \leq \mu_\alpha(E') - \mu_\alpha(E),$$

which is equivalent to (4.3). The second inequality is obvious when $\text{rk}(\text{im}(a_0)) = \text{rk}(\mathcal{H}om^1)$. We thus assume $\text{rk}(\text{im}(a_0)) < \text{rk}(\mathcal{H}om^1)$. We define a quotient of the bundle \tilde{E} by

$$\mathcal{Q} : \text{coker}(\phi_b) \otimes L^{-1} \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} 0 \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} \text{coker}(\phi_a) \otimes L^{-1}$$

(we take the saturation if cokernels are not torsion free). By the α -polystability of \tilde{E} we have

$$(4.5) \quad \mu_\alpha(\mathcal{Q}) = \mu(\mathcal{Q}) + 2\alpha \frac{\text{rk}(\text{coker}(\phi_b))}{\text{rk}(\text{coker}(\phi_a)) + \text{rk}(\text{coker}(\phi_b))} \geq \mu_\alpha(\tilde{E}) = \mu_\alpha(E') - \mu_\alpha(E) + \alpha.$$

Note that $\mu(\mathcal{Q}) = \mu(\text{coker}(a_0)) - \text{deg}(L)$. This and (4.5), together with the fact that

$$\mu(\text{coker}(a_0)) \leq \frac{\text{deg}(\mathcal{H}om^1) - \text{deg}(\text{im}(a_0))}{\text{rk}(\mathcal{H}om^1) - \text{rk}(\text{im}(a_0))},$$

lead us to (4.4). □

Lemma 4.6. *Let $E = (V, W, \beta, \gamma)$ be a L -twisted $U(p, q)$ -Higgs bundle and $E' = (V', W', \beta', \gamma')$ a L -twisted $U(p', q')$ -Higgs bundle. Assume that $p - q$ and $p' - q'$ have the same sign, and suppose that the following conditions hold:*

- $-\text{deg}(L) \leq \alpha \leq \text{deg}(L)$ and $\text{deg}(L) \geq 2g - 2$,
- E and E' are α -polystable with $\mu_\alpha(E) = \mu_\alpha(E')$,
- the map a_0 is not an isomorphism.

Then

$$\chi(E', E) \leq 1 - g,$$

if the map a_0 is not generically an isomorphism, otherwise $\chi(E', E) < 0$.

Proof. By the estimates (4.3) and (4.4), we obtain

$$\begin{aligned} \text{deg}(\ker(a_0)) + \text{deg}(\text{im}(a_0)) &\leq (\mu_\alpha(E') - \mu_\alpha(E)) \left(\text{rk}(\ker(a_0)) + \text{rk}(\text{im}(a_0)) - \text{rk}(\mathcal{H}om^1) \right) \\ &\quad - \alpha \left(\text{rk}(\text{coker}(\phi_a)) - \text{rk}(\text{coker}(\phi_b)) \right) \\ &\quad - \text{deg}(L) \left(\text{rk}(\mathcal{H}om^1) - \text{rk}(\text{im}(a_0)) \right) + \text{deg}(\mathcal{H}om^1). \end{aligned}$$

As $\mu_\alpha(E) = \mu_\alpha(E')$ we deduce

$$\begin{aligned} &\text{deg}(\mathcal{H}om^0) - \text{deg}(\mathcal{H}om^1) \\ &\leq -\alpha \left(\text{rk}(\text{coker}(\phi_a)) - \text{rk}(\text{coker}(\phi_b)) \right) - \text{deg}(L) \left(\text{rk}(\text{coker}(\phi_a)) + \text{rk}(\text{coker}(\phi_b)) \right) \end{aligned}$$

and so

$$(4.6) \quad \text{deg}(\mathcal{H}om^0) - \text{deg}(\mathcal{H}om^1) \leq \begin{cases} -\text{deg}(L) \text{rk}(\text{coker}(\phi_b)) & \text{if } -\text{deg}(L) \leq \alpha \leq 0 \\ -\text{deg}(L) \text{rk}(\text{coker}(\phi_a)) & \text{if } 0 \leq \alpha \leq \text{deg}(L). \end{cases}$$

On the other hand we have

$$\chi(E', E) = (1 - g) \left(\text{rk}(\mathcal{H}om^0) - \text{rk}(\mathcal{H}om^1) \right) + \text{deg}(\mathcal{H}om^0) - \text{deg}(\mathcal{H}om^1).$$

Combining (4.6) with the above equality, we get

$$\chi(E', E) \leq \begin{cases} (1-g)(\operatorname{rk}(\mathcal{H}om^0) - \operatorname{rk}(\mathcal{H}om^1) + 2 \operatorname{rk} \operatorname{coker}(\phi_b)) & \text{if } -\deg(L) \leq \alpha \leq 0 \\ (1-g)(\operatorname{rk}(\mathcal{H}om^0) - \operatorname{rk}(\mathcal{H}om^1) + 2 \operatorname{rk} \operatorname{coker}(\phi_a)) & \text{if } 0 \leq \alpha \leq \deg(L). \end{cases}$$

From hypothesis we have $\operatorname{rk}(\mathcal{H}om^0) \geq \operatorname{rk}(\mathcal{H}om^1)$. If a_0 is not generically an isomorphism then either cases of the above inequality implies $\chi(E', E) \leq (1-g)$. Otherwise,

$$\chi(E', E) = \deg(\mathcal{H}om^0) - \deg(\mathcal{H}om^1) < 0$$

since equality happens only if a_0 is an isomorphism. \square

5. BIRATIONALITY OF MODULI SPACES

Let α_c , α_c^+ and α_c^- be defined as in Section 4.1, where $\epsilon > 0$ is small enough so that α_c is the only critical value in the interval (α_c^-, α_c^+) . Fix a type $t = (p, q, a, b)$.

Proposition 5.1. *Let α_c be a critical value for twisted $U(p, q)$ -Higgs bundles of type $t = (a, b, p, q)$. If either one of the following conditions holds:*

- (1) $a/p - b/q > -\deg(L)$, $q \leq p$ and $0 \leq \alpha_c^\pm < \frac{2pq}{pq - q^2 + p + q}(b/q - a/p - \deg(L)) + \deg(L)$,
- (2) $a/p - b/q < \deg(L)$, $p \leq q$ and $\frac{2pq}{pq - p^2 + p + q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha_c^\pm \leq 0$.

Then the codimension of the flip loci $\mathcal{S}_{\alpha_c^\pm} \subset \mathcal{M}_{\alpha_c^\pm}^s(t)$ is strictly positive.

Proof. From Propositions 3.27 and 2.23, $\mathcal{M}_{\alpha_c^\pm}^s$ is smooth of dimension $1 - \chi(t, t)$. Hence, using that by Lemma 2.19 $\chi(t, t) = \sum_{1 \leq i, j \leq m} \chi(t_i, t_j)$, we have

$$\begin{aligned} \operatorname{codim} \mathcal{S}_{\alpha_c^\pm} &= \dim \mathcal{M}_{\alpha_c^\pm}^s(t) - \dim \mathcal{S}_{\alpha_c^\pm} \\ &= 1 - \chi(t, t) - \dim \mathcal{S}_{\alpha_c^\pm} \\ &= 1 - \sum_{i, j} \chi(t_i, t_j) - \dim \mathcal{S}_{\alpha_c^\pm}, \end{aligned}$$

where t_i, t_j and m occur in $\operatorname{Gr}(E) = \bigoplus_{i=1}^m Q_i$ coming from a α_c -Jordan-Hölder filtration of E . Now using the inequality (4.1) we get that the codimension of the strictly semistable locus is at least

$$\begin{aligned} &\min\left\{1 - \sum_{i, j} \chi(t_i, t_j) + \sum_{i \leq j} \chi(t_j, t_i) + \frac{m(m-3)}{2}\right\} \\ &= \min\left\{-\sum_{j < i} \chi(t_j, t_i) + \frac{m(m-3) + 2}{2}\right\}, \end{aligned}$$

where the minimum is taken over all t_i and m . Now we show that Q_i and Q_j satisfy the hypotheses of Lemma 4.6. Using Proposition 3.11, the hypotheses (1) and (2) imply that β and γ are injective, respectively. Therefore in both cases $p_j - q_j$ and $p_i - q_i$ have the same sign, for all i, j . Note that there are some i and j such that the map a_0 of the Hom-complex $\mathcal{H}om^\bullet(Q_j, Q_i)$ is not an isomorphism, since otherwise $\mathcal{E}nd^\bullet(E)$ will be an isomorphism which is not possible. This is because for $p \neq q$ we have $\operatorname{rk}(\mathcal{E}nd^0) > \operatorname{rk}(\mathcal{E}nd^1)$ which implies that the map a_0 can not be an isomorphism, and for $p = q$ it can be an isomorphism only if β and γ both are isomorphisms but this is not possible since these maps are twisted with a degree positive line bundle.

Hence we have that $-\chi(t_j, t_i) > 0$ and therefore

$$\operatorname{codim} \mathcal{S}_{\alpha_c^\pm} > \min\left\{\frac{m(m-3) + 2}{2}\right\}.$$

Clearly, the minimum is attained when $m = 2$ giving the result. \square

Remark 5.2. For $q = 1$, one might have hoped to obtain a stronger result in Proposition 5.1, based on (1) of Proposition 4.2. The problem is that we also need to satisfy the hypotheses of Lemma 4.6 and this requires injectivity of β or γ .

From Proposition 5.1 we immediately obtain the following.

Theorem 5.3. *Fix a type $t = (p, q, a, b)$. Let α_c be a critical value. Suppose that either one of the following conditions holds:*

- (1) $a/p - b/q > -\deg(L)$, $q \leq p$ and $0 \leq \alpha_c^\pm < \frac{2pq}{pq - q^2 + p + q}(b/q - a/p - \deg(L)) + \deg(L)$,
- (2) $a/p - b/q < \deg(L)$, $p \leq q$ and $\frac{2pq}{pq - p^2 + p + q}(b/q - a/p + \deg(L)) - \deg(L) < \alpha_c^\pm \leq 0$.

Then the moduli spaces $\mathcal{M}_{\alpha_c^-}^s(t)$ and $\mathcal{M}_{\alpha_c^+}^s(t)$ are birationally equivalent. In particular, if either of the conditions of Lemma 2.13 holds then the moduli spaces $\mathcal{M}_{\alpha_c^-}(t)$ and $\mathcal{M}_{\alpha_c^+}(t)$ are birationally equivalent.

Remark 5.4. In view of Remark 3.12, non-emptiness of the intervals for α_c^\pm in the preceding theorem bounds the Toledo invariant. Thus the ranges for the Toledo invariant $\tau = \frac{2pq}{p+q}(a/p - b/q)$ for which the statement of the theorem is meaningful are:

- (1) $-\frac{2pq}{p+q} \deg(L) < \tau < -(q-1) \deg(L)$;
- (2) $(p-1) \deg(L) < \tau < \frac{2pq}{p+q} \deg(L)$.

Note that in case (1) we have $q \leq p$ and hence $-\frac{2pq}{p+q} \deg(L) \leq -q \deg(L)$, while in case (2) we have $p \leq q$ and hence $p \deg(L) \leq \frac{2pq}{p+q} \deg(L)$.

Finally we have the following corollary.

Theorem 5.5. *Let $L = K$ and fix a type $t = (p, q, a, b)$. Suppose that $(p+q, a+b) = 1$ and that $\tau = \frac{2pq}{p+q}(a/p - b/q)$ satisfies $|\tau| \leq \min\{p, q\}(2g-2)$. Suppose that either one of the following conditions holds:*

- (1) $a/p - b/q > -(2g-2)$, $q \leq p$ and $0 \leq \alpha < \frac{2pq}{pq - q^2 + p + q}(b/q - a/p - (2g-2)) + 2g-2$,
- (2) $a/p - b/q < 2g-2$, $p \leq q$ and $\frac{2pq}{pq - p^2 + p + q}(b/q - a/p + 2g-2) - (2g-2) < \alpha \leq 0$.

Then the moduli space $\mathcal{M}_\alpha(t)$ is irreducible.

Proof. Recall that the value of the parameter for which the non-abelian Hodge Theorem applies is $\alpha = 0$. Thus, using [8, Theorem 6.5], the moduli space $\mathcal{M}_0(t)$ is irreducible and non-empty (both the co-primality condition and the bound on the Toledo invariant are needed for this). Hence the result follows from Theorem 5.3. \square

Remark 5.6. Note that unless $p = q$, the conditions on $a/b - b/q$ in the preceding theorem are guaranteed by the hypothesis $|\tau| \leq \min\{p, q\}(2g-2)$ (cf. Remark 5.4).

Remark 5.7. In the non-coprime case it is known from [8] that the closure of the stable locus in $\mathcal{M}_0(t)$ is connected (however, irreducibility is still an open question). Thus, in the non-coprime case, the closure of the stable locus of $\mathcal{M}_\alpha(t)$ is connected under the remaining hypotheses of the preceding theorem.

REFERENCES

- [1] L. Álvarez-Cónsul and O. García-Prada, *Dimensional reduction, $SL(2, \mathbb{C})$ -equivariant bundles and stable holomorphic chains*, Int. J. Math. **12** (2001), 159–201.
- [2] L. Álvarez Cónsul and O. García-Prada, *Hitchin-Kobayashi Correspondence, Quivers, and Vortices*, Commun. Math. Phys. **238** (2003), 1–33.

- [3] L. Álvarez Cónsul, O. García-Prada and A. H. W. Schmitt, *On the geometry of moduli spaces of holomorphic chains over compact Riemann surfaces*, Internat. Math. Res. Papers, Art ID 73597 (2006) 1–82.
- [4] I. Biswas and S. Ramanan, *An infinitesimal study of the moduli of Hitchin pairs*, J. London Math. Soc. (2) **49** (1994), 219–231.
- [5] O. Biquard, O. Garcia-Prada, and R. Rubio, *Higgs bundles, Toledo invariant and the Cayley correspondence*, arXiv:1511.07751v2 [math.DG] (2015), <http://arxiv.org/abs/1511.07751>.
- [6] S. B. Bradlow, *Special metrics and stability for holomorphic bundles with global sections*, J. Differential Geom. **33** (1991), 169–213.
- [7] S.B. Bradlow and O. García-Prada, *Stable triples, equivariant bundles and dimensional reduction*, Math. Ann. **304** (1996), 225–252.
- [8] S. B. Bradlow, O. García-Prada, and P. B. Gothen, *Surface group representations and $U(p, q)$ -Higgs bundles*, J. Differential Geom. **64** (2003), 111–170.
- [9] ———, *Moduli spaces of holomorphic triples over compact Riemann surfaces*, Math. Ann. **328** (2004), 299–351.
- [10] ———, *Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces*, Geometriae Dedicata **122** (2006), 185–213.
- [11] ———, *Higgs bundles for the non-compact dual of the special orthogonal group*, Geometriae Dedicata **175** (2015), 1–48.
- [12] K. Corlette, *Flat G -bundles with canonical metrics*, J. Differential Geom. **28** (1988), 361–382.
- [13] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3) **55** (1987), 127–131.
- [14] O. García-Prada and J. Heinloth, *The y -genus of the moduli space of PGL_n -Higgs bundles on a curve (for degree coprime to n)*, Duke Math. J. **162** (2013), 2731–2749.
- [15] O. García-Prada, J. Heinloth, and A. Schmitt, *On the motives of moduli of chains and Higgs bundles*, J. Eur. Math. Soc. **16** (2014), 2617–2668.
- [16] O. García-Prada, M. Logares, and V. Muñoz, *Moduli spaces of parabolic $U(p, q)$ -Higgs bundles*, Q. J. Math. **60** (2009), 183–233.
- [17] O. García-Prada, P. B. Gothen, and I. Mundet i Riera, *Higgs bundles and surface group representations in the real symplectic group*, Journal of Topology **6** (2013), 64–118.
- [18] O. García-Prada, P. B. Gothen, and I. Mundet i Riera, *The Hitchin-Kobayashi correspondence, Higgs pairs and surface group representations*, arXiv:0909.4487v3 [math.AG] (2012).
- [19] P. B. Gothen, *Representations of surface groups and Higgs bundles*, Moduli spaces, London Math. Soc. Lecture Note Ser., vol. 411, Cambridge Univ. Press, Cambridge, 2014, pp. 151–178.
- [20] P.B. Gothen and A.D. King, *Homological algebra of twisted quiver bundles*, J. London Math. Soc. **71** (2005), 85–99.
- [21] N. J. Hitchin, *The self-duality equations on a Riemann surface*, Proc. LMS **55**, **3** (1987), 59–126.
- [22] N. J. Hitchin, *Lie groups and Teichmüller space*, Topology **31** (1992), 449–473.
- [23] V. Muñoz, *Hodge polynomials of the moduli spaces of rank 3 pairs*, Geom. Dedicata **136** (2008), 17–46.
- [24] ———, *Torelli theorem for the moduli spaces of pairs*, Math. Proc. Cambridge Philos. Soc. **146** (2009), 675–693.
- [25] ———, *Hodge structures of the moduli spaces of pairs*, Internat. J. Math. **21** (2010), 1505–1529.
- [26] V. Muñoz, D. Ortega, and M. Vázquez-Gallo, *Hodge polynomials of the moduli spaces of pairs*, Internat. J. Math. **18** (2007), 695–721.
- [27] ———, *Hodge polynomials of the moduli spaces of triples of rank (2, 2)*, Q. J. Math. **60** (2009), 235–272.
- [28] N. Nitsure, *Moduli of semistable pairs on a curve*, Proc. London Math. Soc. (3) **62** (1991), 275–300.
- [29] A. Nozad, *Hitchin pairs for the indefinite unitary group*, Ph.D. thesis, Faculty of Sciences, University of Porto, 2016.
- [30] A. H. W. Schmitt, *Geometric invariant theory and decorated principal bundles*, Zürich Lectures in Advanced Mathematics, European Mathematical Society, 2008.
- [31] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), 867–918.
- [32] ———, *Higgs bundles and local systems*, Inst. Hautes Études Sci. Publ. Math. **75** (1992), 5–95.
- [33] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. **117** (1994), 317–353.

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