Implementation of autonomous multidimensional behaviors

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***Abstract*— In this paper, we consider linear shift-invariant discrete n-dimensional systems over $\mathbb{Z}^n$ in the behavioral context (nD behaviors) and investigate the *regular* implementation of autonomous behaviors with different degrees of autonomy. Taking into account that stable nD behaviors have the highest degree of autonomy, we apply the previous results to characterize all stabilizable behaviors.**

I. Introduction

The possibility of obtaining (or implementing) a given control objective by means of regular interconnection is a central question in behavioral control [2], [8], [9], [10], [16].

Roughly speaking, regular implementation corresponds to the possibility of intersecting a given behavior with a suitable "non redundant" controller in order to obtain the desired controlled behavior. This is a crucial issue in the context of feedback control [6], [16].

In several applications, such as for instance pole-placement and stabilization, the desired controlled behavior is required to be autonomous, i.e., to have no free variables.

Whereas in the 1D case the property of autonomy is equivalent to the finite dimensionality of a behavior (meaning that each trajectory is generated from a finite number of initial conditions), nD autonomous behavior are generally infinite-dimensional. But even in this case the amount of information (initial conditions) necessary to generate the trajectories of an autonomous nD behavior may vary. This has led to the notion of autonomy degree proposed in [12].

In this paper we consider the problem of regular implementation of autonomous nD behaviors with different autonomous degrees, and give conditions in terms of the original (to be controlled) behavior for the solvability of this problem. The obtained results are then applied to the stabilization of nD behaviors, allowing to complete the analysis carried out in [7], [3].

The paper is organized as follows: we begin by introducing some necessary background from the field of nD discrete behaviors over $\mathbb{Z}^n$, centering around concepts such as controllability, autonomy, orthogonal module, etc. Section 3 is devoted to an exposition of the different degrees of autonomy. In Section 4 we investigate the regular implementation of autonomous behaviors. Finally in Section 5 we apply the results of Section 4 to characterize all stabilizable behaviors.

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II. Preliminaries: Discrete multidimensional behaviors over $\mathbb{Z}^n$

In order to state more precisely the questions to be considered, we introduce some preliminary notions and results. We consider nD behaviors $\mathfrak{B}$ defined over $\mathbb{Z}^n$ that can be described by a set of linear partial difference equations, i.e.,

$$\mathfrak{B} = \ker\, R(\underline{\sigma}, \underline{\sigma}^{-1}) := \{w \in \mathcal{U} \mid R(\underline{\sigma}, \underline{\sigma}^{-1})w \equiv 0\},$$

where $\mathcal{U}$ is the trajectory universe, here taken to be $(\mathbb{C}^q)^{\mathbb{Z}^n}$, $\underline{\sigma} = (\sigma_1, \dots, \sigma_n)$, $\underline{\sigma}^{-1} = (\sigma_1^{-1}, \dots, \sigma_n^{-1})$, the σ_i's are the elementary nD shift operators (defined by $\sigma_i w(\underline{k}) = w(\underline{k} + e_i)$, for $\underline{k} \in \mathbb{Z}^n$, where e_i is the ith element of the canonical basis of $\mathbb{R}^n$) and $R(\underline{s}, \underline{s}^{-1})$, $\underline{s} = (s_1, \dots, s_n)$, $\underline{s}^{-1} = (s_1^{-1}, \dots, s_n^{-1})$, is an nD Laurent-polynomial matrix known as *representation* of $\mathfrak{B}$. These behaviors are known as *kernel behaviors*, however throughout this paper we simply refer to them as *behaviors*.

Instead of characterizing $\mathfrak{B}$ by means of a representation matrix R, it is also possible to characterize it by means of its *orthogonal module* $\mathrm{Mod}(\mathfrak{B})$, which consists of all the nD Laurent-polynomial rows $r(\underline{s}, \underline{s}^{-1}) \in \mathbb{C}^{1\times q}[\underline{s}, \underline{s}^{-1}]$ such that $\mathfrak{B} \subset \ker\, r(\underline{\sigma}, \underline{\sigma}^{-1})$, and can be shown to coincide with the $\mathbb{C}[\underline{s}, \underline{s}^{-1}]$-module $\mathrm{RM}(R)$ generated by the rows of R, i.e., $\mathrm{Mod}(\mathfrak{B}) = \mathrm{RM}(R(\underline{s}, \underline{s}^{-1}))$ [11].

The notions of controllability and autonomy play an important role in the sequel.

Definition 1: A behavior $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ is said to be *controllable* if for all $w_1, w_2 \in \mathfrak{B}$ there exits $\delta > 0$ such that for all subsets $U_1, U_2 \subset \mathbb{Z}^n$ with $d(U_1, U_2) > \delta$, there exists a $w \in \mathfrak{B}$ such that $w\,|_{U_1} = w_1\,|_{U_1}$ and $w\,|_{U_2} = w_2\,|_{U_2}$.

It was shown (see [13]) that this is equivalently to say that $\mathbb{C}^{1\times q}[\underline{s}, \underline{s}^{-1}]/\mathrm{Mod}(\mathfrak{B})$ is a torsion free $\mathbb{C}[\underline{s}, \underline{s}^{-1}]$-module (see [13, Cor.2 and Th.5]).

In contrast with the one dimensional case, multidimensional behaviors admit a stronger notion of controllability called rectificability. The following theorem shows several characterizations of rectifiable behaviors that have appeared in several papers.

Theorem 2: (see [8, Lemma 2.12], [1, Prop.2.1] and [15, Th. 9 and Th. 10, page 819]) Let $\mathfrak{B} = \ker R$ be a behavior. Then the following are equivalent.

1) $\mathfrak{B}$ is rectifiable,
2) there exists an invertible operator U, where U is an nD Laurent-polynomial matrix, such that $U(\mathfrak{B}) = \ker[I_l \quad 0]$, where I_l is the $l \times l$ identity matrix, for some $l \in \{1, \dots, q\}$,

3) $\mathbb{C}^q[\underline{s},\underline{s}^{-1}]/\text{Mod}(\mathfrak{B})$ is a free $\mathbb{C}[\underline{s},\underline{s}^{-1}]$-module.

On the other hand, we say that a behavior $\mathfrak{B} = \ker R(\underline{\sigma},\underline{\sigma}^{-1})$ is autonomous if it has no free variables. This is equivalent to the condition that $R(\underline{s},\underline{s}^{-1})$ has full column rank (over $\mathbb{R}[\underline{s},\underline{s}^{-1}]$) (see [14, Lemma 5]).

It was also shown in [14] that every nD behavior $\mathfrak{B}$ can be decomposed into a sum

$$\mathfrak{B} = \mathfrak{B}^c + \mathfrak{B}^a,$$

where $\mathfrak{B}^c$ is the *controllable part* of $\mathfrak{B}$ (defined as the largest controllable sub-behavior of $\mathfrak{B}$) and $\mathfrak{B}^a$ is a (non-unique) autonomous sub-behavior said to be an *autonomous part* of $\mathfrak{B}$. In general, this cannot be made a direct sum when $n > 1$.

If the controllable-autonomous decomposition happens to be a direct sum decomposition, i.e., if $\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$, we say that the autonomous part of $\mathfrak{B}^a$ is an *autonomous direct summand* of $\mathfrak{B}$.

Remark 3: It is important to remark that if $\mathfrak{B} = \ker R$ and $\mathfrak{B}^c = \ker R^c$ then the rank of R and the rank of R^c must coincide. Indeed, by [8, Cor. 2.10], $\mathfrak{B}$ and $\mathfrak{B}^c$ must have the same number of inputs (free variables) and the same number of outputs. Moreover, by [4, Th. 2.69], the number of outputs in a behavior coincides with the rank of its representation matrices.

When the controllable part $\mathfrak{B}^c$ is rectifiable it is possible to take advantage of the simplified form of the rectified behaviors in order to derive various results. In particular, it is not difficult to obtain the next proposition, that characterizes the autonomous direct summands of a behavior.

Proposition 4: Let $\mathfrak{B} = \ker R(\underline{\sigma},\underline{\sigma}^{-1}) \subset (\mathbb{R}^q)^{\mathbb{Z}^n}$ be an nD behavior with rectifiable controllable part $\mathfrak{B}^c$ and $U(\underline{\sigma},\underline{\sigma}^{-1})$ be a corresponding rectifying operator such that $U(\underline{\sigma},\underline{\sigma}^{-1})(\mathfrak{B}^c) = \ker[I_l \quad 0]$. Then the following are equivalent.

1) $\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$,
2) $\mathfrak{B}^a = \ker\left(\begin{bmatrix} P & 0 \\ X & I_{q-l} \end{bmatrix} U\right)$, with $P(\underline{s},\underline{s}^{-1})$ such that $RU^{-1} = [P \quad 0]$ and $X(\underline{s},\underline{s}^{-1})$ an arbitrary Laurent-polynomial matrix of suitable size.

Note that the behaviors $\mathfrak{B}^a$ of Proposition 4 always exist and are autonomous. Thus, this result states that every behavior $\mathfrak{B}$ with rectifiable controllable part has an autonomous part which is a direct summand of $\mathfrak{B}$; moreover, it gives a parametrization for all such summands.

III. AUTONOMOUS nD BEHAVIORS

Given an autonomous behavior, a natural question to ask is how much information is necessary in order to fully determine the system trajectories, i.e., how large is the initial condition set. This question has been analyzed in [12] by introducing the notion of autonomy degrees for behaviors and relating them to the different types of primeness of the corresponding representation matrices. Although the results presented in [12] concern behaviors over $\mathbb{N}^n$, it is possible to extend them to behaviors over $\mathbb{Z}^n$, as we shall do in the sequel using a slightly different formulation.

We first consider some simple examples.

Example 5: 1) Let $\mathfrak{B}^1 = \ker R_1 \subset \mathbb{C}^{\mathbb{Z}^3}$, where $R_1 = (s_3)$, be an autonomous 3D behavior. Then the trajectories $w(k_1,k_2,k_3) \in \mathfrak{B}^1$ can be assigned freely on a plane, parallel to the span of e_1 and e_2.
2) Let $\mathfrak{B}^2 = \ker R_2 \subset \mathbb{C}^{\mathbb{Z}^3}$, where $R_2 = \begin{pmatrix} s_3 \\ s_2 \end{pmatrix}$, be an autonomous 3D behavior. Then the trajectories $w(k_1,k_2,k_3) \in \mathfrak{B}^2$ can be assigned freely on a line, parallel to the k_1-axis.
3) Let $\mathfrak{B}^3 = \ker R_3 \subset \mathbb{C}^{\mathbb{Z}^3}$, where $R_3 = \begin{pmatrix} s_3 \\ s_2 \\ s_1 \end{pmatrix}$, be an autonomous 3D behavior. Then the trajectories $w(k_1,k_2,k_3) \in \mathfrak{B}^3$ can be assigned freely only on a point, and $\mathfrak{B}^3$ is therefore finite dimensional.

In order to formalize the notion of autonomy degree, we define a standard (ℓ-dimensional) sublattice of $\mathbb{Z}^n$ as $\mathfrak{L} := \{(i_1,\ldots,i_n) \in \mathbb{Z}^n \mid i_{j_1} = \cdots = i_{j_{n-\ell}} = 0,\ j_1,\ldots,j_{n-\ell} \in \{1,\ldots,n\}\}$. Moreover, we define the restriction of a behavior $\mathfrak{B} = \ker R$ to a standard ℓ-dimensional sublattice $\mathfrak{L}$ of $\mathbb{Z}^n$ as $\mathfrak{B}_{\mathfrak{L}} := \{w_{\mathfrak{L}} : \mathfrak{L} \mapsto \mathbb{C}^q \mid \exists w \in \mathfrak{B} \text{ such that } w\mid_{\mathfrak{L}} = w_{\mathfrak{L}}\}$. It can be shown that $\mathfrak{B}_{\mathfrak{L}}$ is also a behavior; moreover it can clearly be identified with a behavior over $\mathbb{Z}^\ell$.

Definition 6: Let $\mathfrak{B} = \ker R \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ be a nonzero nD behavior. We define the *autonomous degree* of $\mathfrak{B}$, denoted by $autodeg(\mathfrak{B})$, as $n - \ell$, where ℓ is the largest value for which there exists a standard ℓ-dimensional sublattice $\mathfrak{L}$ of $\mathbb{Z}^n$ such that $\mathfrak{B}_{\mathfrak{L}}$ is not autonomous. The autonomy degree of the zero behavior is defined to be ∞. Note that the larger the autonomy degree, the smallest is the freedom to assign initial conditions. Indeed, a behavior that is not autonomous has autonomous degree equal to zero.

Example 7: Consider $\mathfrak{B}^1, \mathfrak{B}^2$ and $\mathfrak{B}^3$ in Example 5. Define $\mathfrak{L}_1 := \{(k_1,k_2,k_3) \in \mathbb{Z}^3 \mid k_3 = 0\}$. Then $\mathfrak{B}^1_{\mathfrak{L}_1} := \{w_{\mathfrak{L}_1} : \mathfrak{L}_1 \mapsto \mathbb{C} \mid \exists w \in \mathfrak{B}^1 \text{ such that } w(k_1,k_2,0) = w_{\mathfrak{L}_1}\} \approx \{w(k_1,k_2) \in (\mathbb{C})^{\mathbb{Z}^2}\}$, which is not autonomous and $\dim(\mathfrak{L}_1) = 2$. Hence $autodeg(\mathfrak{B}^1) = 1$. Equivalently, for part 2) define $\mathfrak{L}_2 := \{(k_1,k_2,k_3) \in \mathbb{Z}^3 \mid k_2 = k_3 = 0\}$. Then $\mathfrak{B}^2_{\mathfrak{L}_2} := \{w_{\mathfrak{L}_2} : \mathfrak{L}_2 \mapsto \mathbb{C} \mid \exists w \in \mathfrak{B}^2 \text{ such that } w(k_1,0,0) = w_{\mathfrak{L}_2}\}$, which is not autonomous and $\dim(\mathfrak{L}_2) = 1$. In this example it is easy to see that there does not exist a standard 2-dimensional sublattice $\overline{\mathfrak{L}}$ of $\mathbb{Z}^3$ such that $\mathfrak{B}_{\overline{\mathfrak{L}}}$ is not autonomous. Hence $autodeg(\mathfrak{B}^2) = 2$. Finally, for part 3) define $\mathfrak{L}_3 := \{(k_1,k_2,k_3) \in \mathbb{Z}^3 \mid k_1 = k_2 = k_3 = 0\}$. Then $\mathfrak{B}^3_{\mathfrak{L}_3} := \{w_{\mathfrak{L}_3} : \mathfrak{L}_3 \mapsto \mathbb{C} \mid \exists w \in \mathfrak{B}^3 \text{ such that } w(0,0,0) = w_{\mathfrak{L}_3}\}$, which is not autonomous and $\dim(\mathfrak{L}_3) = 0$. Hence $autodeg(\mathfrak{B}^3) = 3$.

It is possible to relate the autonomy degree of $\mathfrak{B} = \ker R$ with the right primeness degree of $\mathfrak{B}$, which we define as follows.

Definition 8: Let $R \in \mathbb{C}^{p\times q}[\underline{s},\underline{s}^{-1}]$ be a matrix with entries in $\mathbb{C}[\underline{s},\underline{s}^{-1}]$ and $p \geq q$. Let $m_1,\ldots,m_s \in \mathbb{C}[\underline{s},\underline{s}^{-1}]$ be the $q\times q$ order minors of R. The ideal generated by these minors is denoted by $I(R) =< m_1,\ldots,m_s >$ and let $Z(I(R))$ denote the set of all points in $\mathbb{C}^n$ at which every element of $I(R)$ vanishes. We define $primdeg(R) := n - dimZ(I(R))$ to be the *right primeness degree of* R.

Example 9: Consider the matrices R_1, R_2 and R_3 in Example 5. Then $I(R_1) =< s_3 >$, $\dim Z(I(R_1)) = 2$ and therefore $primdeg(R_1) = 1$, whereas $I(R_2) =< s_2, s_3 >$, $\dim Z(I(R_2)) = 1$ and therefore $primdeg(R_2) = 2$. Finally we have that $I(R_3) =<s_1, s_2, s_3 >$, $\dim Z(I(R_3)) = 0$ and therefore $primdeg(R_3) = 3$.

Note that here the primeness of the representation matrices coincide with the autonomy degrees of the associated behaviors. In fact this also holds in the general case, as stated in the following theorem, whose proof we omit.

Theorem 10: Let $\mathfrak{B} = \ker R$ be a behavior. Then the autonomy degree of $\mathfrak{B}$ is equal to the right primeness degree of R, i.e., $autodeg(\mathfrak{B}) = primdeg(R)$.

IV. Regular implementation of autonomous behaviors

Given two behaviors $\mathfrak{B}^1$ and $\mathfrak{B}^2$ their *interconnection* is defined as the intersection $\mathfrak{B}^1 \cap \mathfrak{B}^2$. This interconnection is said to be *regular* if

$$\mathrm{Mod}(\mathfrak{B}^1) \cap \mathrm{Mod}(\mathfrak{B}^2) = \{0\}.$$

The following result can be found in, for instance, [8, Lemma 3, pag 115].

Lemma 11: Given the two behaviors $\mathfrak{B}^1 = \ker R^1$ and $\mathfrak{B}^2 = \ker R^2$. The following are equivalent.

1) $\mathfrak{B}^1 \cap \mathfrak{B}^2$ is a regular interconnection,
2) $\mathfrak{B}^1 + \mathfrak{B}^2 = (\mathbb{C}^q)^{\mathbb{Z}^n}$,
3) rank R^1 + rank R^2 = rank $\begin{pmatrix} R^1 \\ R^2 \end{pmatrix}$.

Regular interconnections correspond to a lack of overlapping between the laws of the interconnected behaviors and play an important role in behavioral control, [10], [16], [6], [2].

A sub-behavior $\mathfrak{B}^d \subset \mathfrak{B}$ is said to be regularly implementable from $\mathfrak{B}$ if there exists a controller behavior $\mathcal{C}$ such that $\mathfrak{B} \cap \mathcal{C} = \mathfrak{B}^d$ and this interconnection is regular. In this case we denote $\mathfrak{B} \cap_{reg} \mathcal{C} = \mathfrak{B}^d$.

A relevant question (for instance in the framework of pole-placement) is the regular implementation of autonomous behaviors. The following proposition is a direct consequence of the results in [8], and states that every regularly implementable autonomous sub-behavior of $\mathfrak{B}$ is an autonomous part of $\mathfrak{B}$.

Proposition 12: Let $\mathfrak{B}^d \subset \mathfrak{B}$ be two behaviors, with $\mathfrak{B}^d$ autonomous. If $\mathfrak{B}^d$ is regularly implementable from $\mathfrak{B}$, then

$$\mathfrak{B} = \mathfrak{B}^c + \mathfrak{B}^d.$$

This result can be intuitively explained by the fact that an autonomous part of a behavior may be somehow considered as obstructions to the (regular) control of that behavior, as happens for instance with the non-controllable modes in the context of pole-placement for classical state-space systems.

A more surprising result is the fact that the possibility of implementing autonomous sub-behaviors of $\mathfrak{B}$ by regular interconnection may also impose conditions in the controllable part of $\mathfrak{B}$, depending on the autonomy degree of such sub-behaviors.

Theorem 13: Let $\mathfrak{B}$ be a behavior. If $\mathfrak{B}^d \subset \mathfrak{B}$ is regularly implementable from $\mathfrak{B}$ and has autonomy degree larger than 1 then $\mathfrak{B}^c$ (the controllable part of $\mathfrak{B}$) is rectifiable.

Proof: In order to prove the result we will make use of the duality between $\mathfrak{B}$ and $\mathrm{Mod}(\mathfrak{B})$. It turns out that $\mathfrak{B} \cap_{reg} \mathcal{C} = \mathfrak{B}^d$ if and only if $\mathrm{Mod}(\mathfrak{B}) \oplus \mathrm{Mod}(\mathcal{C}) = \mathrm{Mod}(\mathfrak{B}^d)$, see for instance [16, pag.1074]. The assumption that $\mathfrak{B}^d$ has autonomy degree ≥ 2 amounts to say that the height of the annihilator of $\mathbb{C}[\underline{s},\underline{s}^{-1}]^q/(\mathrm{Mod}(\mathfrak{B}) \oplus \mathrm{Mod}(\mathcal{C}))$ is ≥ 2, see [12, Lemma 4.7, page 54]. Equivalently, the annihilator of $\mathbb{C}[\underline{s},\underline{s}^{-1}]^q/(\mathrm{Mod}(\mathfrak{B}) \oplus \mathrm{Mod}(\mathcal{C}))$ contains at least two coprime elements, see [12, Lemma 3.6.].

Further, the interconnection $\mathfrak{B} \cap \mathcal{C}$ is regular if and only if $\mathfrak{B}^c \cap \mathcal{C}^c$ is regular, where $\mathfrak{B}^c$ and $\mathcal{C}^c$ denote the corresponding controllable parts, see [3, Lemma 12]. Obviously $\mathfrak{B}^c \cap \mathcal{C}^c \subset \mathfrak{B} \cap \mathcal{C}$ and therefore $autodeg(\mathfrak{B}^c \cap \mathcal{C}^c) \geq autodeg(\mathfrak{B} \cap \mathcal{C})$.

Thus we have, by assumption, that $\mathbb{C}[\underline{s},\underline{s}^{-1}]^q/(\mathrm{Mod}(\mathfrak{B}^c) \oplus \mathrm{Mod}(\mathcal{C}^c))$ contains at least two coprime elements, say d_1, d_2. Note that $\mathbb{C}[\underline{s},\underline{s}^{-1}]^q/\mathrm{Mod}(\mathfrak{B}^c)$ and $\mathbb{C}[\underline{s},\underline{s}^{-1}]^q/\mathrm{Mod}(\mathcal{C}^c)$ are torsion free since $\mathfrak{B}^c$ and $\mathcal{C}^c$ are controllable.

Using Theorem 2 we prove that $\mathfrak{B}^c$ is rectifiable by showing that $\mathbb{C}[\underline{s},\underline{s}^{-1}]^q/\mathrm{Mod}(\mathfrak{B}^c)$ is free.

Consider an element $\xi \in \mathbb{C}[\underline{s},\underline{s}^{-1}]^q$. There are coprime elements d_1, d_2 with $d_1\xi = a_1 + b_1,\ d_2\xi = a_2 + b_2$ with $a_1, a_2 \in \mathrm{Mod}(\mathfrak{B}^c),\ b_1, b_2 \in \mathrm{Mod}(\mathcal{C}^c)$. The element $\tau_1 = \frac{a_1}{d_1} = \frac{a_2}{d_2} \in Qt(\mathbb{C}[\underline{s},\underline{s}^{-1}]) \otimes_{\mathbb{C}[\underline{s},\underline{s}^{-1}]} \mathbb{C}[\underline{s},\underline{s}^{-1}]^q$ has the property $d_1\tau_1,\ d_2\tau_1 \in \mathbb{C}[\underline{s},\underline{s}^{-1}]^q$. Since d_1, d_2 are coprime, this implies that $\tau_1 \in \mathbb{C}[\underline{s},\underline{s}^{-1}]^q$. Since $\mathbb{C}[\underline{s},\underline{s}^{-1}]^q/\mathrm{Mod}(\mathfrak{B}^c)$ has no torsion, one obtains $\tau_1 \in \mathrm{Mod}(\mathfrak{B}^c)$.

The same argument shows that $\tau_2 = \frac{b_1}{d_1} = \frac{b_2}{d_2}$ belongs to $\mathrm{Mod}(\mathcal{C}^c)$. Hence $\xi = \tau_1 + \tau_2 \in \mathrm{Mod}(\mathfrak{B}^c) \oplus \mathrm{Mod}(\mathcal{C}^c)$ and $\mathbb{C}[\underline{s},\underline{s}^{-1}]^q = \mathrm{Mod}(\mathfrak{B}^c) \oplus \mathrm{Mod}(\mathcal{C}^c)$. Then $\mathrm{Mod}(\mathfrak{B}^c)$ is a projective module and therefore free. This concludes the proof. ■

This result generalizes the one obtained in [3] for the 2D case. However, the proof given here is completely different from the one in [3], which is not adaptable to the nD case.

It is not difficult to conclude that if $\widetilde{\mathfrak{B}^a}$ is a sub-behavior of $\mathfrak{B}$ with autonomous degree not less than 2, that is regularly implementable from $\mathfrak{B}$ then $\widetilde{\mathfrak{B}^a}$ is described as

$$\widetilde{\mathfrak{B}^a} = \ker \begin{pmatrix} P & 0 \\ C_1 & C_2 \end{pmatrix} U,$$

where U is a rectifying operator such that $U(\mathfrak{B}) = \ker(P \quad 0)$, C_2 has full column rank and rank $(C_1\ C_2) =$ rank C_2. The fact that $autodeg(\widetilde{\mathfrak{B}^a}) \geq 2$ also implies that $autodeg(\ker P) \geq 2$. As a consequence, by Proposition 4, all the autonomous direct summands of $\mathfrak{B}$ must have autonomous degree larger than 2. Taking into account that such direct summands are regularly implementable from $\mathfrak{B}$, this allows to conclude the following.

Proposition 14: Let $\mathfrak{B}$ be a behavior. Then there exists a sub-behavior of $\mathfrak{B}$ with autonomous degree larger than 2 that is regularly implementable from $\mathfrak{B}$ if and only if

$$\mathfrak{B} = \mathfrak{B}^c \oplus \mathfrak{B}^a$$

with $\mathfrak{B}^c$ rectifiable and $autodeg(\mathfrak{B}^a) \geq 2$.

V. Stabilizability

In this section we apply the results obtained in the previous section to the context of stabilization and characterized all stabilizable behaviors.

A discrete 1D behavior $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}}$ is said to be *stable* if all its trajectories tend to the origin as time goes to infinity. In the nD case, we shall define stability with respect to a specified stability region, as in [7] by adapting the ideas in [5] to the discrete case. For this purpose we identify a *direction* in $\mathbb{Z}^n$ with an element $\underline{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ whose components are coprime integers, and define a *stability cone* in $\mathbb{Z}^n$ as the set of all positive integer linear combinations of n linearly independent directions.

By a *half-line* associated with a direction $\underline{d} \in \mathbb{Z}^n$ we mean the set of all points of the form $\alpha\underline{d}$ where α is a nonnegative integer; clearly, the half-lines in a stability cone S are the ones associated with the directions $\underline{d} \in S$.

Given a stability cone $S \subset \mathbb{Z}^n$, a trajectory $w \in (\mathbb{C}^q)^{\mathbb{Z}^n}$ is said to be *S-stable* if it tends to zero along every half line in S. A behavior $\mathfrak{B}$ is *S-stable* if all its trajectories are *S-stable*.

It turns out that stable behaviors on $(\mathbb{C}^q)^{\mathbb{Z}^n}$ must be finite dimensional.

Lemma 15: ([7, Lemma 2]) Every nD behavior $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ which is stable with respect to some stability cone S is a finite dimensional linear subspace of the trajectory universe, $(\mathbb{C}^q)^{\mathbb{Z}^n}$, i.e., $autodeg(\mathfrak{B}) = n$.

As for stabilization, our definition of S-stabilizability is similar to the one proposed in [5], but has the extra requirement of regularity.

Definition 16: Given a stability cone $S \subset \mathbb{Z}^n$, we say that a behavior $\mathfrak{B} \subset (\mathbb{C}^q)^{\mathbb{Z}^n}$ is *S-stabilizable* if there exists an S-stable sub-behavior $\mathfrak{B}^s \subset \mathfrak{B}$ that is implementable from $\mathfrak{B}$ by regular interconnection.

The following theorem provides a characterization of all stabilizable behaviors.

Theorem 17: Let $\mathfrak{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1}) \subset (\mathbb{R}^q)^{\mathbb{Z}^n}$ be a behavior and $S \subset \mathbb{Z}^n$ be a stability cone. Then the following statements are equivalent.

1) $\mathfrak{B}$ is S-stabilizable,
2) $\mathfrak{B}^c$ is rectifiable and if U is a rectifiable operator such that $RU = [P \quad 0]$ then $\ker P(\underline{\sigma}, \underline{\sigma}^{-1})$ is S-stable,
3) $\mathfrak{B}^c$ is rectifiable and every autonomous direct summand of $\mathfrak{B}$ is stable.

Proof: 1 $\Rightarrow$ 2: Assume that $\mathfrak{B}$ is S-stabilizable. Then, by Lemma 15 and Theorem 13, $\mathfrak{B}^c$ is rectifiable. If $\mathfrak{B} = \ker R = \ker PR^c$ with R^c such that $\mathfrak{B}^c = \ker R^c$ and U is a rectifying operator for $\mathfrak{B}^c$ then $PR^c = P(I \quad 0)U$, $U(\mathfrak{B}) = \ker(P \quad 0)$ and $U(\mathfrak{B}^c) = \ker(I \quad 0)$.

If $\mathcal{K} = \ker(K_1\ K_2)U$ is a controller behavior such that its interconnection with $\mathfrak{B}$ is regular and yields an autonomous behavior then, by Lemma 11,

$$\text{rank} \begin{pmatrix} P & 0 \\ K_1 & K_2 \end{pmatrix} = \text{rank } (P \quad 0) + \text{rank } (K_1\ K_2).$$

On the other hand, P must have full column rank (by Remark 3) as well as K_2 (otherwise $U(\mathfrak{B} \cap \mathcal{K}) = \ker \begin{pmatrix} P & 0 \\ K_1 & K_2 \end{pmatrix}$ would not be full column rank) and therefore we have that

$$\text{rank} \begin{pmatrix} P & 0 \\ K_1 & K_2 \end{pmatrix} = \text{rank } P + \text{rank } K_2.$$

Thus rank $K_2 =$ rank $(K_1\ K_2)$.

In particular this implies that if $w_1 \in \ker P$ then there exists a trajectory w_2 such that $(w_1, w_2) \in \mathfrak{B} \cap \mathcal{K}$.

In this way, we conclude that if $\mathfrak{B}$ is S-stabilizable then P must be stable, i.e., 1 $\Rightarrow$ 2.

2 $\Rightarrow$ 1: Take $\mathcal{K}$ such that $U(\mathcal{K}) = \ker(0 \quad I)$. Thus $\mathfrak{B} \cap_{reg} \mathcal{K}$ is stable since P is stable.

2 $\Leftrightarrow$ 3: Easy from the characterization, obtained in Proposition 4, of all autonomous direct summands of $\mathfrak{B}$. □

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