COMPARISON OF THE PROPERTY OF STATE-RECONSTRUCTIBILITY WITH BEHAVIORAL RECONSTRUCTIBILITY FOR PERIODIC SYSTEMS

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Abstract: In this paper we view a classical periodic state space system as a behavioral system and compare the property of state-reconstructibility with behavioral reconstructibility. It turns out that, like it happens for the time-invariant case, the behavioral reconstructibility of a periodic state space system is equivalent to its complete state-reconstructibility.

Keywords: Mathematical systems theory; Linear systems; Discrete-time systems; Time-invariant systems; Time-varying systems; Difference equations; Behavior; Dynamic systems

1. INTRODUCTION

The behavioral approach to dynamical systems, introduced by Jan C. Willems in the eighties (Willems, 1989; Willems, 1991), views a system essentially as a set of admissible trajectories, known as the system *behavior*, where no distinction is made *a priori* between input and output variables. Similar to what happens for "classical" systems, such as, for instance, state space systems, several structural properties have been defined and characterized for behaviors. Of particular interest among them are the properties of observability and reconstructibility (Willems, 1989; Willems, 1991; Polderman and Willems, 1998; Valcher and Willems, 1999*b*; Valcher and Willems, 1999*a*).

If the system variable w is partitioned into two sub-variables w_1 and w_2, the fact that one of them, say, w_2, is *observable* from the other one (w_1) corresponds to the possibility of obtaining full information on w_2 from the knowledge of w_1. According to the definitions given in (Willems, 1989; Willems, 1991; Polderman and Willems, 1998), for linear time-invariant systems this amounts to say that whenever the whole trajectory w_1 is null, the same happens for the whole trajectory w_2.

On the other hand, the property of reconstructibility corresponds, roughly speaking, to the possibility of recovering some of the system variables from the other ones, but with some delay. More concretely, according to the definition given in (Valcher and Willems, 1999*a*) for linear time-invariant systems over the nonnegative discrete time-axis, w_2 is said to be reconstructible from w_1 if whenever the trajectory w_1 is null, i.e., $w_1(k) = 0,\ k \geq 0$, w_2 becomes null after some finite time δ, i.e. , $w_2(k) = 0,\ k \geq \delta$.

In (Aleixo and Rocha, 2007; Aleixo, 2008), the notion of reconstructibility was extended for linear time-invariant systems over $\mathbb{Z}$, allowing to conclude that a time-invariant state space system is completely state-reconstructible, in the classical sense, (Urbano, 1987), if and only if it is reconstructible in the behavioral sense.

The aim of this paper is to investigate whether this result has, or not, extension for the case of periodic systems.

2. BACKGROUND

2.1 Behavioral P-periodic systems

In the behavioral framework a dynamical system Σ is defined as a triple $\Sigma=(\mathbb{T},\mathbb{W},\mathfrak{B})$, with $\mathbb{T}\subseteq\mathbb{R}$ as the time set, $\mathbb{W}$ as the signal space and $\mathfrak{B}\subseteq\mathbb{W}^{\mathbb{T}}$ as the behavior. Here we focus on the discrete-time case, that is, $\mathbb{T}=\mathbb{Z}$, assuming furthermore that our signal space is $\mathbb{W}=\mathbb{R}^q$ with $q\in\mathbb{N}$.

Let the λ-shift

$$\sigma^\lambda : (\mathbb{R}^q)^{\mathbb{Z}} \to (\mathbb{R}^q)^{\mathbb{Z}},$$

be defined by $\left(\sigma^\lambda w\right)(k):=w(k+\lambda)$.

Whereas the behavior of a time-invariant system is characterized by its invariance under the time shift, that is,

$$\sigma\mathfrak{B}=\mathfrak{B},$$

periodic behaviors, with period P, are characterized by their invariance with respect to the P-shift ($P\in\mathbb{N}$), as stated in the next definition.

Definition 1. (Kuijper and Willems, 1997) A system Σ is said to be P-periodic (with $P\in\mathbb{N}$) if its behavior $\mathfrak{B}$ satisfies $\sigma^P\mathfrak{B}=\mathfrak{B}$.

According to (Kuijper and Willems, 1997), a behavior $\mathfrak{B}$ is a σ^P-invariant linear closed subspace of $(\mathbb{R}^q)^{\mathbb{Z}}$ (in the topology of point-wise convergence) if and only if it has a representation of the type

$$\left(R_t\left(\sigma,\sigma^{-1}\right)w\right)(Pk+t)=0,\ t=0,\dots,P-1,\ k\in\mathbb{Z}, \quad (1)$$

where $R_t\in\mathbb{R}^{g_t\times q}\left[\xi,\xi^{-1}\right]$ is the Laurent polynomial matrix at instant t in the indeterminate ξ. Remark that the Laurent-polynomial matrices R_t need not have the same number of rows (in fact we could even have some g_t equal to zero, meaning that the corresponding matrix R_t would be void and no restrictions were imposed at the time instants $Pk+t$). Note that (1) can also be written as

$$\left(R\left(\sigma,\sigma^{-1}\right)w\right)(Pk)=0,\quad k\in\mathbb{Z}, \quad (2)$$

where

$$R\left(\xi,\xi^{-1}\right):=\begin{bmatrix} R_0\left(\xi,\xi^{-1}\right)\\ \xi R_1\left(\xi,\xi^{-1}\right)\\ \vdots\\ \xi^{P-1}R_{P-1}\left(\xi,\xi^{-1}\right)\end{bmatrix}\in\mathbb{R}^{g\times q}\left[\xi,\xi^{-1}\right],$$

with $g:=\sum_{t=0}^{P-1}g_t$. Analogously to the time-invariant case, although with some abuse of language, we refer to (2) as a *P-periodic kernel representation (P-PKR).*

In order to study the desired property of reconstructibility, we shall consider that the system variable w is partitioned as (w_1,w_2), where w_1 is the observed variable and w_2 is the variable about which information is sought. In this case, the corresponding behavior description (2) will be written as

$$\left(R_2\left(\sigma,\sigma^{-1}\right)w_2\right)(Pk) = \left(R_1\left(\sigma,\sigma^{-1}\right)w_1\right)(Pk),\ k\in\mathbb{Z}, \quad (3)$$

where $R_i\in\mathbb{R}^{g\times q_i}\left[\xi,\xi^{-1}\right]$, $g:=\sum_{t=0}^{P-1}g_t$, $i=1,2$, i.e., are obtained by means of a suitable partition (and, if necessary, rearrangement) of the columns of R. We will denote representation (3) by (R_2,R_1).

By decomposing matrices R_2 and R_1 as, see (Aleixo *et al.*, 2006),

$$R_i\left(\xi,\xi^{-1}\right)=R_i^L\left(\xi^P,\xi^{-P}\right)\Omega_{P,q_i}(\xi),\ i=1,2,$$

we may write down relation (3) as

$$\left(R_2^L\left(\sigma^P,\sigma^{-P}\right)\Omega_{P,q_2}(\sigma)w_2\right)(Pk) = \left(R_1^L\left(\sigma^P,\sigma^{-P}\right)\Omega_{P,q_1}(\sigma)w_1\right)(Pk),k\in\mathbb{Z}. \quad (4)$$

Defining the lifted trajectories

$$(Lw_i)(k)=\begin{bmatrix} w_i(Pk)\\ \vdots\\ w_i(Pk+P-1)\end{bmatrix},\ i=1,2,$$

see (Kuijper and Willems, 1997; Aleixo *et al.*, 2005), and noting that $L\sigma^P=\sigma L$, (4) may be written as

$$\left(R_2^L\left(\sigma,\sigma^{-1}\right)(Lw_2)\right)(k) = \left(R_1^L\left(\sigma,\sigma^{-1}\right)(Lw_1)\right)(k),\ k\in\mathbb{Z}. \quad (5)$$

Thus the time-invariant behavior $L\mathfrak{B}$, defined by $\{Lw,\ w\in\mathfrak{B}\}$ and known as *lifted behavior*, is equal to the set of trajectories

$$\left\{(Lw_1,Lw_2)\in\left(\mathbb{R}^{Pq_1}\right)^{\mathbb{Z}}\times\left(\mathbb{R}^{Pq_2}\right)^{\mathbb{Z}}\mid(5)\text{ holds}\right\}.$$

In (Aleixo, 2008) several results are obtained concerning the characterization of the behavioral reconstructibility of $\mathfrak{B}$ based on the reconstructibility of $L\mathfrak{B}$. Results concerning the property of behavioral reconstructibility for the time-invariant case can be found in (Aleixo and Rocha, 2007; Aleixo, 2008).

In order to investigate the connection between behavioral and state-reconstructibility in the periodic case, we first formalize the definition of behavioral reconstructibility.

Definition 2. (Behavioral reconstructibility) Let $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}} \simeq (\mathbb{R}^{q_1} \times \mathbb{R}^{q_2})^{\mathbb{Z}}$ be a behavior whose system variable w is partitioned as $w = (w_1, w_2)$. Given $\delta \geq 0$, we say that w_2 is *δ-reconstructible* from w_1 if

$$\left\{ w_1 \big|_{[k_0,+\infty)} \equiv 0 \right\} \Rightarrow \left\{ w_2 \big|_{[k_0+\delta,+\infty)} \equiv 0 \right\}, \ \forall k_0 \in \mathbb{Z}.$$

Moreover, w_2 is said to be *reconstructible* from w_1 if it is δ-reconstructible from w_1 for some $\delta \geq 0$.

From here on, whenever in a dynamical system, w_2 is reconstructible from w_1, we simply say that $\mathfrak{B}$ is reconstructible w.r.t. w_2.

The relationship between the reconstructibility of a periodic behavior and of its lifted version is given by the following result.

Theorem 3. (Aleixo, 2008) Let $\Sigma = (\mathbb{Z}, \mathbb{R}^{q_1} \times \mathbb{R}^{q_2}, \mathfrak{B})$ be a P-periodic system whose system variable w is partitioned as $w = (w_1, w_2)$. Suppose that the system is described by (3). Then the following are equivalent:

i) $\mathfrak{B}$ is reconstructible w.r.t. w_2;

ii) $L\mathfrak{B}$ is reconstructible w.r.t. Lw_2;

iii) $\operatorname{rank} R_2^L(\lambda, \lambda^{-1}) = Pq_2, \ \forall \lambda \in \mathbb{C} \backslash \{0\}$.

2.2 Periodic state space systems

The classical state space approach to P-periodic systems takes as starting point a description of the form:

$$\begin{cases} (\sigma x)(k) = A(k)\, x(k) + B(k)\, u(k) \\ y(k) = C(k)\, x(k) + D(k)\, u(k) \end{cases} \quad k \in \mathbb{Z}, \qquad (6)$$

where the matrices $A(k) \in \mathbb{R}^{n \times n}$, $B(k) \in \mathbb{R}^{n \times m}$, $C(k) \in \mathbb{R}^{p \times n}$ and $D(k) \in \mathbb{R}^{p \times m}$ are periodic functions of k with period P, x is the state variable and u and y are the input and output, respectively. To go into further detail we refer the reader to for instance, (Urbano, 1987; Hernández and Urbano, 1987; Bittanti and Colaneri, 2000).

The property of state-reconstructibility is there defined as follows:

Definition 4. (State-reconstructibility).

i) A state $x_1 \in \mathbb{R}^n$ is called unreconstructible (at time k_1) if for all $k_0 \leq k_1$, there exists $x_0 = x(k_0) \in \mathbb{R}^n$ such that

$$y(k) = C(k)\, \phi_A(k, k_0)\, x_0 = 0, \ k \in [k_0, k_1 - 1],$$

with $x_1 = x(k_1)$;

ii) The system (6) is called completely state-reconstructible at time k_1 if the only state x_1 that is unreconstructible is the zero state, i.e., $x_1 = 0 \in \mathbb{R}^n$. If this happens for all $k_1 \in \mathbb{Z}$, (6) is simply called *completely state-reconstructible.*

Here we shall focus on complete state-reconstructibility for periodic systems.

Since, as will be seen in the sequel, the characterization of this property is based on results for time-invariant systems, we quickly review some relevant facts about the state-reconstructibility of such systems. For this purpose, let (A, B, C, D) be a time-invariant state space system. Then,

Theorem 5. The following conditions are equivalent:

i) (A, B, C, D) is completely state-reconstructible;

ii) $\operatorname{rank} \begin{bmatrix} \lambda I_n - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C} \backslash \{0\}$;

iii) $\ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \subset \ker A^n$.

In (Urbano, 1987) and (Hernández and Urbano, 1987) an invariant dynamical decomposition associated with the P-periodic state system description (6) is introduced allowing an one-to-one correspondence between a P-periodic state space system and P time-invariant state space systems.

Definition 6. (Urbano, 1987) Let Σ_s be the P-periodic state space system described by (6). The P time-invariant systems $\Sigma_t, \ t = 0, \ldots, P-1$, are defined as

$$\begin{cases} (\sigma x_t)(k) = A_t x_t(k) + B_t u_t(k) \\ y_t(k) = C_t x_t(k) + D_t u_t(k) \end{cases} \quad k \in \mathbb{Z},$$

where

$$A_t := \phi_A(t+P, t) \qquad (7)$$

$$C_t := \left[(C(t))^T \ (C(t+1)\, \phi_A(t+1, t))^T \ \cdots \ (C(t+P-1)\, \phi_A(t+P-1, t))^T \right]^T \qquad (8)$$

and

$$\phi_A(k, k_0) := A(k-1)\, A(k-2) \cdots A(k_0)$$

$$\phi_A(k_0, k_0) := I_n,$$

is the well known state transition matrix for (6).

In (Urbano, 1987) several results are obtained concerning the characterization of the state-reconstructibility of Σ_s based on the state-reconstructibility of each Σ_t and known results for the time-invariant case. In particular, the following theorem is relevant for our purposes.

Theorem 7. (Urbano, 1987) The P-periodic state space system Σ_s is completely state-reconstructible if and only if all the P time-invariant systems Σ_t are completely state-reconstructible.

3. BEHAVIORAL RECONSTRUCTIBILITY OF PERIODIC STATE SPACE SYSTEMS

In this section we view a periodic state space system as a periodic behavioral system, study its reconstructibility in behavioral terms and relate this property to the classical property of state-reconstructibility.

Note that the state space description (6) can be regarded as a particular case of (1). Indeed, letting $w := \left[u^T\ y^T\right]^T$ and $v := x$, and due to the periodicity of matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$, the state space description (6) can be written as

$$\left(R_t\left(\sigma,\sigma^{-1}\right)w\right)(Pk+t) = \left(M_t\left(\sigma,\sigma^{-1}\right)v\right)(Pk+t),\ t=0,\ldots,P\text{–}1,\ k\in\mathbb{Z},$$

with

$$R_t\left(\xi,\xi^{-1}\right) = \begin{bmatrix} B(t) & 0 \\ -D(t) & I_p \end{bmatrix}$$

and

$$M_t\left(\xi,\xi^{-1}\right) = \begin{bmatrix} \xi I_n - A(t) \\ C(t) \end{bmatrix},$$

or still

$$\left(R\left(\sigma,\sigma^{-1}\right)w\right)(Pk) = \left(M\left(\sigma,\sigma^{-1}\right)v\right)(Pk),\ k\in\mathbb{Z}, \quad (9)$$

with $R\left(\xi,\xi^{-1}\right)$ and $M\left(\xi,\xi^{-1}\right)$ given by

$$\left[\begin{array}{cc} B(0) & 0 \\ -D(0) & I_p \\ \hline \xi B(1) & 0 \\ -\xi D(1) & \xi I_p \\ \hline \vdots & \vdots \\ \hline \xi^{P-1}B(P-1) & 0 \\ -\xi^{P-1}D(P-1) & \xi^{P-1}I_p \end{array}\right]$$

and

$$\left[\begin{array}{c} \xi I_n - A(0) \\ C(0) \\ \hline \xi\left(\xi I_n - A(1)\right) \\ \xi C(1) \\ \hline \vdots \\ \hline \xi^{P-1}\left(\xi I_n - A(P-1)\right) \\ \xi^{P-1}C(P-1) \end{array}\right],$$

respectively.

Consequently, if $\mathfrak{B}$ is the behavior formed by the (w,v)-trajectories that satisfy (9) (i.e., if $\mathfrak{B}$ is the behavior of the P-periodic state space system (9)), the corresponding lifted behavior $L\mathfrak{B}$ is described by:

$$\left(R^L\left(\sigma,\sigma^{-1}\right)(Lw)\right)(k) = \left(M^L\left(\sigma,\sigma^{-1}\right)(Lv)\right)(k),\ k\in\mathbb{Z},$$

where $M^L\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{(n+p)P\times nP}\left[\xi,\xi^{-1}\right]$ is equal to

$$\left[\begin{array}{c|c|c|c} -A(0) & I_n & \cdots & 0 \\ C(0) & 0 & \cdots & 0 \\ 0 & -A(1) & \cdots & 0 \\ 0 & C(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \xi I_n & 0 & \cdots & -A(P-1) \\ 0 & 0 & \cdots & C(P-1) \end{array}\right].$$

Taking Theorem 3 into account we conclude that $\mathfrak{B}$ is reconstructible w.r.t. x if and only if

$$\operatorname{rank} M^L\left(\lambda,\lambda^{-1}\right) = nP,\ \forall\lambda\in\mathbb{C}\backslash\{0\}.$$

For the sake of simplicity, we now consider that $P=2$, but our reasonings also apply to the general case. We then have

$$M^L\left(\xi,\xi^{-1}\right) = \left[\begin{array}{c|c} -A(0) & I_n \\ C(0) & 0 \\ \xi I_n & -A(1) \\ 0 & C(1) \end{array}\right].$$

By performing the block-column operation $C_1 \leftarrow C_1 + C_2 A(0)$, where C_j is the j^{th} block-column of M^L, we obtain the following matrix

$$\left[\begin{array}{c|c} 0 & I_n \\ \hline C(0) & 0 \\ \xi I_n - A(1)A(0) & -A(1) \\ C(1)A(0) & C(1) \end{array}\right]. \quad (10)$$

Clearly the rank of the original M^L matrix coincides with the rank of matrix (10) and, therefore,

$$\forall\lambda\in\mathbb{C},\quad \operatorname{rank} M^{L}\left(\lambda,\lambda^{-1}\right)=n$$

$$+\operatorname{rank}\begin{bmatrix}\lambda I_n - A(1)A(0)\\ C(0)\\ C(1)A(0)\end{bmatrix}=n$$

$$+\operatorname{rank}\begin{bmatrix}\lambda I_n - A_0\\ C_0\end{bmatrix},$$

with $A_0,\ C_0$ as in (7), (8), respectively, that is,

$$A_0 = A(1)A(0)$$

$$C_0 = \begin{bmatrix} C(0)\\ C(1)A(0)\end{bmatrix}.$$

Therefore $\mathfrak{B}$ is behaviorally reconstructible w.r.t. x if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda I_n - A_0\\ C_0\end{bmatrix}=n,\qquad \forall\lambda\in\mathbb{C}\backslash\{0\}.$$

Suppose now that for some $\lambda^*\in\mathbb{C}\backslash\{0\}$,

$$\operatorname{rank}\begin{bmatrix}\lambda^* I_n - A_0\\ C_0\end{bmatrix}<n.$$

This means that there exists $0\neq v^*\in\mathbb{R}^{n\times 1}$ such that

$$\begin{bmatrix}\lambda^* I_n - A_0\\ C_0\end{bmatrix}v^*=0,$$

i.e.,

$$\begin{bmatrix}\lambda^* I_n - A(1)A(0)\\ C(0)\\ C(1)A(0)\end{bmatrix}v^*=0.$$

This is equivalent to

$$\left(\lambda^* I_n - A(1)A(0)\right)v^*=0;\ ^{1} \tag{11}$$

$$C(0)v^*=0; \tag{12}$$

$$C(1)A(0)v^*=0. \tag{13}$$

Consequently, the product

$$\begin{bmatrix}\lambda^* I_n - A_1\\ C_1\end{bmatrix}A(0)v^*,$$

where

$$A_1 = A(0)A(1)$$

$$C_1 = \begin{bmatrix} C(1)\\ C(0)A(1)\end{bmatrix},$$

is given by:

$$\begin{bmatrix}\lambda^* I_n - A(0)A(1)\\ C(1)\\ C(0)A(1)\end{bmatrix}A(0)v^*$$

$$=\begin{bmatrix}A(0)\lambda^* v^* - A(0)A(1)A(0)v^*\\ \underbrace{C(1)A(0)v^*}_{=0,\text{ by (13)}}\\ C(0)\underbrace{A(1)A(0)v^*}_{=\lambda^* v^*,\text{ by (11)}}\end{bmatrix}$$

$$=\begin{bmatrix}A(0)\underbrace{\left(\lambda^* v^* - A(1)A(0)v^*\right)}_{=0,\text{ by (11)}}\\ 0\\ \lambda^*\ \underbrace{C(0)v^*}_{=0,\text{ by (12)}}\end{bmatrix}=0.$$

Since $A(0)v^*\neq 0$ (otherwise v^* would be an eigenvector of $A(0)$ associated to the eigenvalue zero, which is not the case since we have assumed that $\lambda^*\neq 0$), we conclude that also

$$\operatorname{rank}\begin{bmatrix}\lambda^* I_n - A_1\\ C_1\end{bmatrix}<n.$$

Taking into account that this procedure can be reversed, this yields that

$$\left\{\operatorname{rank}\begin{bmatrix}\lambda I_n - A_0\\ C_0\end{bmatrix}=n,\ \forall\lambda\in\mathbb{C}\backslash\{0\}\right\}$$

$$\Leftrightarrow\left\{\operatorname{rank}\begin{bmatrix}\lambda I_n - A_1\\ C_1\end{bmatrix}=n,\ \forall\lambda\in\mathbb{C}\backslash\{0\}\right\}.$$

Noting that this reasoning can be easily extended to the general P-periodic case, we obtain the next result.

Theorem 8. Let Σ be a P-periodic state space system, described as in (6), and let $\Sigma_t = (A_t, B_t, C_t, D_t)$ be the P time-invariant systems obtained by the invariant dynamical decomposition, described in Definition 6. Then the following conditions are equivalent:

i) The behavior $\mathfrak{B}$ of Σ is behaviorally reconstructible with respect to x;

ii) $\operatorname{rank}\begin{bmatrix}\lambda I_n - A_t\\ C_t\end{bmatrix}=n,\quad \forall\lambda\in\mathbb{C}\backslash\{0\}$, for at least one t in $\{0,\ldots,P-1\}$;

iii) $\operatorname{rank}\begin{bmatrix}\lambda I_n - A_t\\ C_t\end{bmatrix}=n,\quad \forall\lambda\in\mathbb{C}\backslash\{0\}$, for all t in $\{0,\ldots,P-1\}$.

Combining Theorems 5, 7 and 8, we immediately conclude that:

[1] v^* is an eigenvector of $A(1)A(0)$ associated to the eigenvalue λ^*.

Theorem 9. The behavior $\mathfrak{B}$ of a P-periodic state space system Σ is (behaviorally) reconstructible with respect to x if and only if Σ is completely state-reconstructible.

Note that, by Theorems 9 and 8, one may conclude that the complete state-reconstructibility of a periodic state space system Σ_s is equivalent to the state-reconstructibility of *at least one* of the P time-invariant systems Σ_t, thus obtaining an alternative characterization to the one given by Theorem 7.

4. CONCLUSION

In this paper we considered behavioral periodic systems and studied the property of behavioral reconstructibility, comparing it with the property of state-reconstructibility defined in the context of classical periodic state space systems. It turns out that the two properties are equivalent, as happens with their dual properties of behavioral controllability and state-space controllability. Our results also give a new insight into the property of state-reconstructibility.

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