RECONSTRUCTIBILITY AND FORWARD-OBSERVABILITY OF BEHAVIORS OVER $\mathbb{Z}$

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Abstract: The properties of reconstructibility and forward-observability for systems over the whole time axis $\mathbb{Z}$ are introduced and characterized in terms of appropriate rank conditions. A comparison is made with the existing results in the behavioral setting as well as in the classical state space framework.

Keywords: Behavior; Difference equations; Discrete-time systems; Dynamic systems; Linear systems; Mathematical systems theory; Time-invariant systems.

1. INTRODUCTION

The behavioral approach to dynamical systems, introduced by J.C.Willems in the eighties (Willems, 1989; Willems, 1991), views a system essentially as a set of admissible trajectories, known as the system *behavior*, where no distinction is made a priori between input and output variables. Similar to what happens for "classical" systems, such as, for instance, state space systems, several structural properties have been defined and characterized for behaviors. Of particular interest among them are the properties of observability and reconstructibility (Willems, 1989; Willems, 1991; Polderman and Willems, 1998; Valcher and Willems, 1999*b*; Valcher and Willems, 1999*a*).

If the system variable $\mathbf{w}$ is partitioned into two subvariables $\mathbf{w}_1$ and $\mathbf{w}_2$, the fact that one of them, say, $\mathbf{w}_2$, is *observable* from the other one ($\mathbf{w}_1$) corresponds to the possibility of obtaining full information on $\mathbf{w}_2$ from the knowledge of $\mathbf{w}_1$. According to the definitions given in (Willems, 1989; Willems, 1991; Polderman and Willems, 1998), for linear systems this amounts to say that whenever the whole trajectory $\mathbf{w}_1$ is null, the same happens for the whole trajectory $\mathbf{w}_2$.

On the other hand, the property of reconstructibility corresponds, roughly speaking, to the possibility of recovering some of the system variables from the other ones, but with some delay. More concretely, according to the definitions given in (Valcher and Willems, 1999*a*) for linear time-invariant systems over the nonnegative discrete time-axis, $\mathbf{w}_2$ is said to be reconstructible from $\mathbf{w}_1$ if whenever the trajectory $\mathbf{w}_1$ is null, i.e., $\mathbf{w}_1(k) = 0,\ k \geq 0$, $\mathbf{w}_2$ becomes null after some finite time δ, i.e. , $\mathbf{w}_2(k) = 0,\ k \geq \delta$.

As happens for the case of state space systems, both these properties can be characterized by means of (column) rank conditions on certain matrices.

One of the aims of this paper is to extend the notion of reconstructibility for systems over $\mathbb{Z}$. This leads to a definition of observability which differs from the original observability definition given in (Willems, 1989; Willems, 1991; Polderman and Willems, 1998), but can as well be viewed as a generalization of the definition adopted for systems over the nonnegative time-axis $\mathbb{Z}_+$. In order to make a distinction, we shall refer to our observability notion as forward-observability, whereas the definition in (Willems, 1989; Willems, 1991; Polderman and Willems, 1998) will be called simply observability or Willems-observability. The characterizations of the newly defined properties are similar to the ones obtained in (Valcher and Willems, 1999*a*) for the nonnegative time-axis case.

In Section 2 we introduce some preliminary notions and facts about behaviors. Section 3 is devoted to the definition of reconstructibility and forward-observability. These properties are compared with the existing behavioral ones and characterized in terms of the matrices that are used in the corresponding behavior description. Moreover, an application to the case of (classical) state space systems is presented. Conclusions are drawn in Section 4.

2. PRELIMINARIES

In this paper, in the framework of the behavioral approach, we deal with the class of discrete-time systems with *kernel behaviors*, more concretely, systems $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ that are defined over the whole discrete time-axis $\mathbb{T} = \mathbb{Z}$, with vector-valued variable taking values in $\mathbb{W} = \mathbb{R}^q$, for some $q \in \mathbb{N}$, and whose set of admissible trajectories is a behavior $\mathfrak{B}$ given by the kernel of a matrix polynomial shift operator $\mathbf{R}\left(\sigma, \sigma^{-1}\right)$. Here, $\mathbf{R}^{\bullet \times q}$ is a Laurent-polynomial matrix of the form

$$\mathbf{R}^{-M}\xi^{-M} + \cdots + \mathbf{R}^0 + \cdots + \mathbf{R}^N\xi^N$$

with $N, M \in \mathbb{Z}_+$, and $\sigma^{\pm 1}$ denotes the backward/forward shift defined by

$$\left(\sigma^{\pm 1}\mathbf{w}\right)(k) = \mathbf{w}\left(k \pm 1\right), \quad k \in \mathbb{Z}.$$

Thus

$$\mathfrak{B} = \ker \mathbf{R}\left(\sigma, \sigma^{-1}\right) := \left\{\mathbf{w} \in \left(\mathbb{R}^q\right)^{\mathbb{Z}} : \mathbf{R}\left(\sigma, \sigma^{-1}\right)\mathbf{w} = 0\right\},$$

i.e., the trajectories $\mathbf{w} \in \mathfrak{B}$ are the elements of $\left(\mathbb{R}^q\right)^{\mathbb{Z}}$ (the space of $\mathbb{R}^q$-value sequences over $\mathbb{Z}$) which constitute the solution set of a linear constant coefficient matrix difference equation

$$\mathbf{R}^{-M}\mathbf{w}\left(k - M\right) + \cdots + \mathbf{R}^{-1}\mathbf{w}\left(k-1\right) + \mathbf{R}^0\mathbf{w}\left(k\right) + \mathbf{R}^1\mathbf{w}\left(k+1\right) + \cdots + \mathbf{R}^N\mathbf{w}\left(k+N\right) = 0, \forall k \in \mathbb{Z}.$$

From now on, unless otherwise specified, the term *behavior* will exclusively refer to discrete-time kernel behaviors over $\mathbb{Z}$. Note that in particular kernel behaviors are linear, time-invariant (i.e., $\sigma\left(\mathfrak{B}\right) = \mathfrak{B}$), and complete. The completeness of a behavior $\mathfrak{B}$ means that it is possible to check whether a trajectory $\mathbf{w} \in \left(\mathbb{R}^q\right)^{\mathbb{Z}}$ belongs to $\mathfrak{B}$, by checking what happens in the set $\mathfrak{I}$ of finite intervals of $\mathbb{Z}$. More concretely: $\mathfrak{B}$ is said to be complete if

$$\left(\forall I \in \mathfrak{I}, \quad \mathbf{w}\Big|_I \in \mathfrak{B}\Big|_I\right) \Leftrightarrow \mathbf{w} \in \mathfrak{B}. \qquad (1)$$

For linear time-invariant systems, the completeness of $\mathfrak{B}$ is equivalent to say that $\mathfrak{B}$ is a closed subspace of $\left(\mathbb{R}^q\right)^{\mathbb{Z}}$, in the topology of pointwise convergence.

In order to study the desired properties of reconstructibility and observability, we shall consider that the system variable $\mathbf{w}$ is partitioned as $(\mathbf{w}_1, \mathbf{w}_2)$, where $\mathbf{w}_1$ is the observed variable and $\mathbf{w}_2$ is the variable about which information is sought. In this case, the corresponding behavior description

$$\mathbf{R}\left(\sigma, \sigma^{-1}\right)\mathbf{w} = 0$$

will be written as

$$\mathbf{R}_2\left(\sigma, \sigma^{-1}\right)\mathbf{w}_2 = \mathbf{R}_1\left(\sigma, \sigma^{-1}\right)\mathbf{w}_1, \qquad (2)$$

by means of a suitable partition (and, if necessary, rearrangement) of the columns of $\mathbf{R}$.

3. RECONSTRUCTIBILITY AND FORWARD-OBSERVABILITY

We start by formalizing the proposed definitions for reconstructibility and observability. Since it is clear that we work over the discrete time-axis $\mathbb{Z}$, for simplicity we use the interval notation to represent discrete intervals and write, for instance, $[k_1, k_2]$ instead of $[k_1, k_2] \cap \mathbb{Z}$.

Definition 1. Let $\mathfrak{B} \subset \left(\mathbb{R}^q\right)^{\mathbb{Z}}$ be a behavior whose system variable $\mathbf{w}$ is partitioned as $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$. Given $\delta \geq 0$, we say that $\mathbf{w}_2$ is *δ-reconstructible* from $\mathbf{w}_1$ if

$$\left(\mathbf{w}_1\Big|_{[k_0,+\infty)} \equiv 0 \Rightarrow \mathbf{w}_2\Big|_{[k_0+\delta,+\infty)} \equiv 0\right), \ \forall k_0 \in \mathbb{Z}. \quad (3)$$

Moreover, $\mathbf{w}_2$ is said to be *reconstructible* from $\mathbf{w}_1$ if it is δ-reconstructible from $\mathbf{w}_1$ for some $\delta \geq 0$. In particular, $\mathbf{w}_2$ is said to be *forward-observable* from $\mathbf{w}_1$ if it is 0-reconstructible from $\mathbf{w}_1$, i.e., if

$$\left(\mathbf{w}_1\Big|_{[k_0,+\infty)} \equiv 0 \Rightarrow \mathbf{w}_2\Big|_{[k_0,+\infty)} \equiv 0\right), \ \forall k_0 \in \mathbb{Z}. \quad (4)$$

◇

Example 2. Consider a system $\Sigma = \left(\mathbb{Z}, \mathbb{R}^2, \mathfrak{B}\right)$ with variables $(\mathbf{w}_1, \mathbf{w}_2)$, whose behavior $\mathfrak{B}$ is described by

$$\sigma\mathbf{w}_2 = \mathbf{w}_1,$$

i.e.,

$$\mathbf{w}_2\left(k\right) = \mathbf{w}_1\left(k-1\right), \quad \forall k \in \mathbb{Z}.$$

Clearly $\mathbf{w}_2$ is 1-reconstructible from $\mathbf{w}_1$, since

$$\mathbf{w}_1\left(k\right) = 0, \quad k \geq k_0$$

implies

$$\mathbf{w}_2\left(k\right) = \mathbf{w}_1\left(k-1\right) = 0, \quad k \geq k_0 + 1.$$

It is also simple to see that $\mathbf{w}_2$ is not forward-observable from $\mathbf{w}_1$. Indeed, if $\mathbf{w}_1\left(-1\right) = 1$ and $\mathbf{w}_1\left(k\right) = 0$ for $k \neq -1$, we have that $\mathbf{w}_2\left(0\right) = 1$ and $\mathbf{w}_2\left(k\right) = 0$ for $k \neq 0$. Thus

$$\mathbf{w}_1\Big|_{[0,+\infty)} = 0, \quad \text{but} \quad \mathbf{w}_2\Big|_{[0,+\infty)} \neq 0.$$

However $\mathbf{w}_1$ is forward-observable from $\mathbf{w}_2$, as the reader can easily check. ◇

Note that, due to time-invariance, the δ-reconstructibility condition (3) in Definition 1 can be replaced by

$$\mathbf{w}_1\Big|_{[0,+\infty)} \equiv 0 \Rightarrow \mathbf{w}_2\Big|_{[\delta,+\infty)} \equiv 0,$$

whereas the forward-observability condition (4) can be replaced by

$$\mathbf{w}_1\Big|_{[0,+\infty)} \equiv 0 \Rightarrow \mathbf{w}_2\Big|_{[0,+\infty)} \equiv 0.$$

This agrees with the definitions of reconstructibility and observability given in (Valcher and Willems, 1999*a*), for discrete-time systems over $\mathbb{Z}_+$, but not with the definition of observability given in (Willems, 1991), according to which $\mathbf{w}_2$ is said to be observable from $\mathbf{w}_1$ if

$$(\mathbf{w}_1(k) = 0,\ \forall k \in \mathbb{Z}) \Rightarrow (\mathbf{w}_2(k) = 0,\ \forall k \in \mathbb{Z}). \quad (5)$$

In fact, as we next prove, when applied to systems over $\mathbb{Z}$, Willems's observability condition coincides rather with our reconstructibility property.

Proposition 3. Let $\mathfrak{B} \subset (\mathbb{R}^q)^{\mathbb{Z}}$ be a behavior whose system variable $\mathbf{w}$ is partitioned as $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$. Then $\mathbf{w}_2$ is *Willems-observable* from $\mathbf{w}_1$ if and only if it is *reconstructible* from $\mathbf{w}_1$, for some $\delta > 0$.

Proof. Assume that $\mathbf{w}_2$ is not reconstructible from $\mathbf{w}_1$. Then there exists a trajectory $\mathbf{w}' = (\mathbf{w}_1', \mathbf{w}_2') \in \mathfrak{B}$ such that

$$\mathbf{w}_1'\Big|_{[0,+\infty)} = 0 \quad \text{and} \quad \forall \delta > 0,\ \mathbf{w}_2'\Big|_{[\delta,+\infty)} \neq 0.$$

Consider a sequence of trajectories $\left(\widetilde{\mathbf{w}}^i\right)_{i\in\mathbb{N}}$ constructed in the following way: let $N_1 \in \mathbb{N}$ be the smallest positive integer such that $\mathbf{w}_2'(N_1) \neq 0$. Define

$$\widetilde{\mathbf{w}}^1 := \frac{\sigma^{N_1}\mathbf{w}'}{\mathbf{w}_2'(N_1)}.$$

For each integer $i > 1$ let N_i be the smallest integer greater than N_{i-1}, such that $\mathbf{w}_2'(N_i) \neq 0$ and define

$$\widetilde{\mathbf{w}}^i := \frac{\sigma^{N_i}\mathbf{w}'}{\mathbf{w}_2'(N_i)}.$$

Note that all the trajectories $\widetilde{\mathbf{w}}^i$ in this sequence satisfy

$$\widetilde{\mathbf{w}}_1^i\Big|_{[-N_i,+\infty)} \equiv 0 \quad \text{and} \quad \widetilde{\mathbf{w}}_2^i(0) = 1.$$

Taking this into account together with the completeness of $\mathfrak{B}$, it is not difficult to conclude that $\left(\widetilde{\mathbf{w}}^i\right)_{i\in\mathbb{N}}$ converges to a trajectory $\widetilde{\mathbf{w}} \in \mathfrak{B}$ such that $\widetilde{\mathbf{w}}_1 \equiv 0$ and $\widetilde{\mathbf{w}}_2$ is nonzero (since $\widetilde{\mathbf{w}}_2(0) = 1$). This means that $\mathbf{w}_2$ is not Willems-observable from $\mathbf{w}_1$. Hence Willems-observability implies reconstructibility.

Assume now that $\mathbf{w}_2$ is reconstructible from $\mathbf{w}_1$, for some $\delta > 0$. Consider a trajectory $(\mathbf{w}_1', \mathbf{w}_2') \in \mathfrak{B}$ with $\mathbf{w}_1'\Big|_{\mathbb{Z}} = 0$. In particular,

$$\mathbf{w}_1'\Big|_{[k_0,+\infty)} \equiv 0, \quad \forall k_0 \in \mathbb{Z},$$

and therefore, there exists $\delta > 0$ such that

$$\mathbf{w}_2'\Big|_{[k_0+\delta,+\infty)} \equiv 0, \quad \forall k_0 \in \mathbb{Z}.$$

This clearly implies that $\mathbf{w}_2'\Big|_{\mathbb{Z}} \equiv 0$, allowing to conclude that reconstructibility implies Willems-observability. □

3.1 Reconstructibility and forward-observability characterization

In this subsection we characterize reconstructibility and forward-observability by means of rank conditions. Given Proposition 3, reconstructibility conditions could be obtained from the characterization of Willems-observability. However, we chose to present here a direct proof, in order to give more insight.

Let $\mathbb{R}\left[\xi,\xi^{-1}\right]$ and $\mathbb{R}[\xi]$ denote respectively the rings of Laurent-polynomials and of polynomials in the indeterminate ξ. Let further $\mathbb{R}^{\ell\times m}[\xi] / \mathbb{R}^{\ell\times m}\left[\xi,\xi^{-1}\right]$ denote the set of $\ell \times m$ matrices with entries in $\mathbb{R}[\xi] / \mathbb{R}\left[\xi,\xi^{-1}\right]$.

Theorem 4. Consider the dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^{q_1+q_2}, \mathfrak{B})$ described by

$$\mathfrak{B} := \Big\{(\mathbf{w}_1,\mathbf{w}_2) \in \left(\mathbb{R}^{q_1+q_2}\right)^{\mathbb{Z}} \mid \left(\mathbf{P}\left(\sigma,\sigma^{-1}\right)\mathbf{w}_2\right)(k) = \left(\mathbf{Q}\left(\sigma,\sigma^{-1}\right)\mathbf{w}_1\right)(k),\ k \in \mathbb{Z}\Big\},$$

with $\mathbf{P}\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times q_2}\left[\xi,\xi^{-1}\right]$, $\mathbf{Q}\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times q_1}\left[\xi,\xi^{-1}\right]$. Then

(i) $\mathbf{w}_2$ is reconstructible from $\mathbf{w}_1$ if and only if

$$\operatorname{rank}\mathbf{P}\left(\lambda,\lambda^{-1}\right) = q_2, \quad \forall \lambda \in \mathbb{C}\setminus\{0\}, \quad (6)$$

or, equivalently, if and only if $\mathbf{P}\left(\xi,\xi^{-1}\right)$ is a right-prime matrix over $\mathbb{R}\left[\xi,\xi^{-1}\right]$;

(ii) $\mathbf{w}_2$ is forward-observable from $\mathbf{w}_1$ if and only if there exist $\widetilde{\mathbf{P}}(\xi) \in \mathbb{R}^{g\times q_2}[\xi]$ and $\widetilde{\mathbf{Q}}(\xi) \in \mathbb{R}^{g\times q_1}[\xi]$ s.t. $\mathfrak{B}$ is described by $\widetilde{\mathbf{P}}(\sigma)\mathbf{w}_2 = \widetilde{\mathbf{Q}}(\sigma)\mathbf{w}_1$, with

$$\operatorname{rank}\widetilde{\mathbf{P}}(\lambda) = q_2, \quad \forall \lambda \in \mathbb{C}, \quad (7)$$

i.e., with $\widetilde{\mathbf{P}}(\xi)$ right-prime over $\mathbb{R}[\xi]$.

Proof. For the equivalence between the rank and primeness conditions we refer to (Kučera, 1991).

(i) Assume that (6) holds, i.e., that $\mathbf{P}\left(\xi,\xi^{-1}\right)$ is right-prime (over $\mathbb{R}\left[\xi,\xi^{-1}\right]$). Then, there exists a matrix $\mathbf{U}\left(\xi,\xi^{-1}\right) \in \mathbb{R}^{g\times g}\left[\xi,\xi^{-1}\right]$, which is unimodular over $\mathbb{R}\left[\xi,\xi^{-1}\right]$, such that (Polderman and Willems, 1998),

$$\mathbf{U}\left(\xi,\xi^{-1}\right)\mathbf{P}\left(\xi,\xi^{-1}\right) = \begin{bmatrix}\mathbf{I}_{q_2}\\ 0\end{bmatrix}.$$

Thus (leaving out σ and σ^{-1} in the notation, for simplicity),

$$\mathbf{P}\mathbf{w}_2 = \mathbf{Q}\mathbf{w}_1 \Leftrightarrow \mathbf{U}\mathbf{P}\mathbf{w}_2 = \mathbf{U}\mathbf{Q}\mathbf{w}_1$$
$$\Leftrightarrow \begin{bmatrix} \mathbf{I}_{q_2} \\ 0 \end{bmatrix} \mathbf{w}_2 = \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix} \mathbf{w}_1$$
$$\Leftrightarrow \mathbf{Q}_2\mathbf{w}_1 = 0 \text{ and } \mathbf{w}_2 = \mathbf{Q}_1\mathbf{w}_1,$$

with $\mathbf{UQ}$ conformably partitioned as

$$\mathbf{U}\left(\xi,\xi^{-1}\right)\mathbf{Q}\left(\xi,\xi^{-1}\right) = \begin{bmatrix} \mathbf{Q}_1\left(\xi,\xi^{-1}\right) \\ \mathbf{Q}_2\left(\xi,\xi^{-1}\right) \end{bmatrix}.$$

Let

$$\mathbf{Q}_1\left(\xi,\xi^{-1}\right) = \mathbf{Q}_1^{-M}\xi^{-M} + \cdots + \mathbf{Q}_1^{0} + \cdots + \mathbf{Q}_1^{N}\xi^{N},$$

with $N, M \in \mathbb{Z}_+$. Applying σ^M to both sides of the equality $\mathbf{w}_2 = \mathbf{Q}_1\mathbf{w}_1$, we obtain

$$\left(\sigma^M \mathbf{w}_2\right)(k) = \left(\widetilde{\mathbf{Q}}_1(\sigma)\,\mathbf{w}_1\right)(k),\ k \in \mathbb{Z},$$

allowing us to conclude that

$$\mathbf{w}_1\Big|_{[k_0,+\infty)} = 0 \Rightarrow \mathbf{w}_2\Big|_{[k_0+M,+\infty)} = 0,$$

i.e., $\mathbf{w}_2$ is M-reconstructible, and hence reconstructible, from $\mathbf{w}_1$.

Suppose now that (6) does not hold. Then, $\mathbf{P}\left(\xi,\xi^{-1}\right)$ is not right-prime (over $\mathbb{R}\left[\xi,\xi^{-1}\right]$), implying that there exists a trajectory $\mathbf{w}_2^* \in \ker \mathbf{P}\left(\sigma,\sigma^{-1}\right)$, which is nonzero (Polderman and Willems, 1998). This trajectory is such that $\mathbf{w}^* = \left(\mathbf{w}_1^* \equiv 0, \mathbf{w}_2^*\right) \in \mathfrak{B}$. If $\mathbf{w}_2$ were reconstructible from $\mathbf{w}_1$, this would imply that

$$\mathbf{w}_2^*\Big|_{[k^*,+\infty)} \equiv 0, \quad \forall k^* \in \mathbb{Z},$$

and, consequently, $\mathbf{w}_2^*$ would be null in the whole time-axis $\mathbb{Z}$, which is a contradiction. Therefore, if the rank condition (6) does not hold, $\mathbf{w}_2$ is not reconstructible from $\mathbf{w}_1$, or, in other words, the reconstructibility of $\mathbf{w}_2$ from $\mathbf{w}_1$ implies that (6) holds.

(ii) Suppose now that there exist $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ such that $\mathfrak{B}$ is described by

$$\left(\widetilde{\mathbf{P}}(\sigma)\,\mathbf{w}_2\right)(k) = \left(\widetilde{\mathbf{Q}}(\sigma)\,\mathbf{w}_1\right)(k),\ k \in \mathbb{Z},$$

with $\widetilde{\mathbf{P}}(\xi)$ satisfying (7) and hence right-prime over $\mathbb{R}[\xi]$. Then, there exists an unimodular matrix (over $\mathbb{R}[\xi]$) $\mathbf{U}(\xi)$ such that (Polderman and Willems, 1998),

$$\mathbf{U}(\xi)\,\widetilde{\mathbf{P}}(\xi) = \begin{bmatrix} \mathbf{I}_{q_2} \\ 0 \end{bmatrix}.$$

Thus

$$\widetilde{\mathbf{P}}\mathbf{w}_2 = \widetilde{\mathbf{Q}}\mathbf{w}_1 \Leftrightarrow \mathbf{U}\widetilde{\mathbf{P}}\mathbf{w}_2 = \mathbf{U}\widetilde{\mathbf{Q}}\mathbf{w}_1$$
$$\Leftrightarrow \begin{bmatrix} \mathbf{I}_{q_2} \\ 0 \end{bmatrix} \mathbf{w}_2 = \begin{bmatrix} \widetilde{\mathbf{Q}}_1 \\ \widetilde{\mathbf{Q}}_2 \end{bmatrix} \mathbf{w}_1$$
$$\Leftrightarrow \widetilde{\mathbf{Q}}_2\mathbf{w}_1 = 0 \text{ and } \mathbf{w}_2 = \widetilde{\mathbf{Q}}_1\mathbf{w}_1,$$

with $\mathbf{U}\widetilde{\mathbf{Q}}$ conformably partitioned as

$$\mathbf{U}(\xi)\,\widetilde{\mathbf{Q}}(\xi) = \begin{bmatrix} \widetilde{\mathbf{Q}}_1(\xi) \\ \widetilde{\mathbf{Q}}_2(\xi) \end{bmatrix}.$$

Thus, if $\mathbf{w}_1(k) = 0$ for $k \in [k_0,+\infty)$, then

$$\left(\widetilde{\mathbf{Q}}_1(\sigma)\,\mathbf{w}_1\right)(k) = 0, \text{ for } k \in [k_0,+\infty)$$

and hence

$$\mathbf{w}_2(k) = 0, \quad \text{for } k \in [k_0,+\infty),$$

which allows to conclude that $\mathbf{w}_2$ is forward-observable from $\mathbf{w}_1$.

Assume now that $\mathbf{w}_2$ is forward-observable from $\mathbf{w}_1$ and let

$$\left(\widehat{\mathbf{P}}(\sigma)\,\mathbf{w}_2\right)(k) = \left(\widehat{\mathbf{Q}}(\sigma)\,\mathbf{w}_1\right)(k),\ k \in \mathbb{Z},$$

be a representation of $\mathfrak{B}$. Consider a trajectory $(\mathbf{w}_1,\mathbf{w}_2) \in \mathfrak{B}$ such that $\mathbf{w}_1 \equiv 0$. Then, by the forward-observability of $\mathfrak{B}$, this implies that

$$\forall k_0 \in \mathbb{Z},\ \mathbf{w}_2(k) = 0,\ k \geq k_0,$$

or, in other words, $\mathbf{w}_2 \equiv 0$. This means that $\ker \widehat{\mathbf{P}}(\sigma) = \{0\}$, which is equivalent to say that

$$\operatorname{rank} \widehat{\mathbf{P}}(\lambda) = \text{const}, \quad \forall \lambda \in \mathbb{C}\setminus\{0\},$$

i.e., $\widehat{\mathbf{P}}(\xi)$ is right-prime over $\mathbb{R}\left[\xi,\xi^{-1}\right]$. Note that this also follows immediately from the previous item and from noticing that forward-observability implies reconstructibility. Let now $\mathbf{U}(\xi)$ and $\mathbf{V}(\xi)$ be unimodular matrices (over $\mathbb{R}[\xi]$) that bring $\widehat{\mathbf{P}}$ into its Smith form, i.e.,

$$\mathbf{U}\widehat{\mathbf{P}}\mathbf{V} = \left[\begin{array}{ccc} \xi^{\ell_1} & & \\ & \ddots & \\ & & \xi^{\ell_{q_2}} \\ \hline & 0 & \end{array}\right] =: \left[\begin{array}{c} \mathbf{\Xi} \\ \hline 0 \end{array}\right].$$

Then,

$$\widehat{\mathbf{P}}\mathbf{w}_2 = \widehat{\mathbf{Q}}\mathbf{w}_1 \Leftrightarrow \mathbf{U}\widehat{\mathbf{P}}\mathbf{w}_2 = \mathbf{U}\widehat{\mathbf{Q}}\mathbf{w}_1$$
$$\Leftrightarrow \left[\begin{array}{c} \mathbf{\Xi} \\ \hline 0 \end{array}\right] \mathbf{V}^{-1}\mathbf{w}_2 = \begin{bmatrix} \widehat{\mathbf{Q}}_1 \\ \widehat{\mathbf{Q}}_2 \end{bmatrix} \mathbf{w}_1,$$

which is equivalent to

$$\widehat{\mathbf{Q}}_2\mathbf{w}_1 = 0 \quad \text{and} \quad \mathbf{w}_2 = \mathbf{V}\mathbf{\Xi}^{-1}\widehat{\mathbf{Q}}_1\mathbf{w}_1,$$

with $\mathbf{U}\widehat{\mathbf{Q}}$ conformably partitioned as

$$\mathbf{U}(\xi)\,\widehat{\mathbf{Q}}(\xi) = \begin{bmatrix} \widehat{\mathbf{Q}}_1(\xi) \\ \widehat{\mathbf{Q}}_2(\xi) \end{bmatrix}.$$

Thus, $\mathfrak{B}$ is also described by

$$\left(\widetilde{\mathbf{P}}(\sigma)\,\mathbf{w}_2\right)(k) = \left(\widetilde{\mathbf{Q}}(\sigma)\,\mathbf{w}_1\right)(k),$$

with $\widetilde{\mathbf{P}},\ \widetilde{\mathbf{Q}}$ defined by

$$\widetilde{\mathbf{P}}(\xi) := \begin{bmatrix} \mathbf{I}_{q_2} \\ 0 \end{bmatrix}$$

$$\widetilde{\mathbf{Q}}(\xi) = \begin{bmatrix} \widetilde{\mathbf{Q}}_1(\xi) \\ \widetilde{\mathbf{Q}}_2(\xi) \end{bmatrix} := \begin{bmatrix} \mathbf{V}(\xi)\,\mathbf{\Xi}^{-1}(\xi)\,\widehat{\mathbf{Q}}_1(\xi) \\ \widehat{\mathbf{Q}}_2(\xi) \end{bmatrix}.$$

Now, using the forward-observability property, it is clear that the matrix $\widetilde{\mathbf{Q}}_1 := \mathbf{V}\mathbf{\Xi}^{-1}\widehat{\mathbf{Q}}_1$ cannot have terms in ξ^{-1}. Moreover $\widetilde{\mathbf{P}}(\xi)$ has constant column rank over $\mathbb{C}$. This shows that there exists a representation of $\mathfrak{B}$ with the desired properties.

□

The observability condition (5) for behaviors over $\mathbb{Z}$, described by

$$\mathbf{P}\left(\sigma,\sigma^{-1}\right)\mathbf{w}_2=\mathbf{Q}\left(\sigma,\sigma^{-1}\right)\mathbf{w}_1,$$

is equivalent to the rank condition

$$\operatorname{rank}\mathbf{P}\left(\lambda,\lambda^{-1}\right)=q_2,\quad \forall\lambda\in\mathbb{C}\backslash\{0\},$$

see (Willems, 1991, Theorem VI.2). This coincides with the condition (6) in Theorem 4(i), thus leading to the same conclusion as Proposition 3, i.e., Willems-observability is equivalent to our notion of reconstructibility, rather than to forward-observability.

Note that the definition of observability, given in (Valcher and Willems, 1999*a*), for systems over $\mathbb{Z}_+$ can be regarded as an adaptation of Willems's definition (5), since it means that if $\mathbf{w}_1$ is the null trajectory (i.e., is zero over the time-axis $\mathbb{Z}_+$), then the same happens for $\mathbf{w}_2$. However, that notion can also be seen as an adaptation of our definition of forward-observability.

The situation is summarized in the following table

	$\mathbb{T}$	
Property	$\mathbb{Z}$	$\mathbb{Z}_+$
Forward-observability	(C1)	(C1)
Willems-observability	(C2)	(C1)
Reconstructibility	(C3)⇔(C2)	(C3)

where the conditions (C1), (C2) and (C3) are as follows:

(C1) $\mathbf{w}_1\Big|_{\mathbb{Z}_+}=0\Rightarrow\mathbf{w}_2\Big|_{\mathbb{Z}_+}=0;$

(C2) $\mathbf{w}_1\Big|_{\mathbb{Z}}=0\Rightarrow\mathbf{w}_2\Big|_{\mathbb{Z}}=0;$

(C3) $\exists\delta>0$ s.t. $\mathbf{w}_1\Big|_{\mathbb{Z}_+}=0\Rightarrow\mathbf{w}_2\Big|_{[\delta,+\infty)}=0.$

3.2 Behavioral reconstructibility and observability of state space systems

Consider a behavior $\mathfrak{B}$ consisting of the set of $(\mathbf{x},\mathbf{u},\mathbf{y})$-trajectories of an n-dimensional linear and time-invariant state space model, with m inputs and p outputs

$$\begin{cases}(\sigma\mathbf{x})(k)=\mathbf{A}\mathbf{x}(k)+\mathbf{B}\mathbf{u}(k)\\ \mathbf{y}(k)=\mathbf{C}\mathbf{x}(k)+\mathbf{D}\mathbf{u}(k)\end{cases}\quad k\in\mathbb{Z}. \tag{8}$$

Assuming, as usual, that the input-output variables $(\mathbf{u},\mathbf{y})$ can be measured and the state $\mathbf{x}$ is not available, it is convenient for analysis purposes to rewrite the previous equations as

$$\begin{bmatrix}\sigma\mathbf{I}_n-\mathbf{A}\\ \mathbf{C}\end{bmatrix}\mathbf{x}=\begin{bmatrix}\mathbf{B}&\mathbf{0}\\ -\mathbf{D}&\mathbf{I}_p\end{bmatrix}\begin{bmatrix}\mathbf{u}\\ \mathbf{y}\end{bmatrix}.$$

Clearly, by Theorem 4, $\mathbf{x}$ is reconstructible from $(\mathbf{u},\mathbf{y})$ if and only if

$$\operatorname{rank}\begin{bmatrix}\lambda\mathbf{I}_n-\mathbf{A}\\ \mathbf{C}\end{bmatrix}=n,\quad\forall\lambda\in\mathbb{C}\backslash\{0\}.$$

This coincides with the well-known (re)constructibility rank condition for state space systems over $\mathbb{Z}_+$, (Kučera, 1991), and amounts to say that if $\lambda\in\mathbb{C}$ is an unobservable mode of $(\mathbf{C},\mathbf{A})$, then $\lambda=0$.

Taking into account that all the observable modes can be shifted to zero, and minding the results in (Antsaklis and Michel, 2006), the following proposition is obtained.

Proposition 5. Consider the behavior $\mathfrak{B}$ (over $\mathbb{Z}$) described by the state space equations (8). Then, the following conditions are equivalent:

(1) $\mathbf{x}$ is reconstructible from $(\mathbf{u},\mathbf{y})$;

(2) If $\lambda\in\mathbb{C}$ is an unobservable mode of $(\mathbf{C},\mathbf{A})$, then $\lambda=0$;

(3) There exists $\mathbf{L}\in\mathbb{R}^{n\times p}$ such that $\mathbf{A}+\mathbf{L}\mathbf{C}$ is nilpotent;

(4) There exists a deadbeat observer.

◇

Contrary to what happens with reconstructibility, the characterization of forward-observability for state space systems over $\mathbb{Z}$ does not coincide with the observability condition for systems over $\mathbb{Z}_+$, (Kučera, 1991),

$$\operatorname{rank}\begin{bmatrix}\lambda\mathbf{I}_n-\mathbf{A}\\ \mathbf{C}\end{bmatrix}=n,\quad\forall\lambda\in\mathbb{C}.$$

This is illustrated in the following example.

Example 6. Consider the a state space system with no inputs, state $\mathbf{x}=[x_1,x_2]^T$ and output y, described by

$$\begin{cases}(\sigma\mathbf{x})(k)=\mathbf{A}\mathbf{x}(k)\\ y(k)=\mathbf{C}\mathbf{x}(k)\end{cases}\quad k\in\mathbb{Z},$$

with

$$\mathbf{A}=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad\mathbf{C}=\begin{bmatrix}0&1\end{bmatrix}.$$

It turns out that the state $\mathbf{x}$ is forward-observable from the output y, since the system trajectories satisfy $x_1=0$ and $x_2=y$. However

$$\begin{bmatrix}\lambda\mathbf{I}_2-\mathbf{A}\\ \mathbf{C}\end{bmatrix}$$

has a rank drop for $\lambda=0$. Nevertheless, the description

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \sigma - 1 \\ 0 & 1 \end{bmatrix}}_{\widetilde{\mathbf{P}}(\sigma)} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} y$$

is such that

$$\operatorname{rank} \widetilde{\mathbf{P}}(\lambda) = 2, \quad \forall \lambda \in \mathbb{C},$$

satisfying thus the condition of Theorem 4(ii). ◇

Thus, forward-observability cannot be directly characterized in terms of the unobservable modes of $(\mathbf{C}, \mathbf{A})$. However, the following result can be proved.

Proposition 7. Consider the behavior $\mathfrak{B}$ (over $\mathbb{Z}$) described by the state space equations (8). Then $\mathbf{x}$ is forward-observable from $(\mathbf{u}, \mathbf{y})$ if and only if there exists a suitable change of variable $\bar{\mathbf{x}}(k) = \mathbf{S}\mathbf{x}(k)$, where $\mathbf{S}$ is an invertible $n \times n$ matrix, such that

$$\begin{cases} \sigma \bar{\mathbf{x}}_1 = \mathbf{A}_1 \bar{\mathbf{x}}_1 + \mathbf{B}_1 \mathbf{u} \\ \bar{\mathbf{x}}_2 = 0 \\ \mathbf{y} = \mathbf{C}_1 \bar{\mathbf{x}}_1, \end{cases} \qquad (\bar{\mathbf{x}}_1, \bar{\mathbf{x}}_2) = \bar{\mathbf{x}}$$

with $(\mathbf{C}_1, \mathbf{A}_1)$ observable. ◇

4. CONCLUSIONS

In this paper, we have introduced and characterized the properties of reconstructibility and forward-observability for systems over $\mathbb{Z}$. A comparison was made with the existing results in the behavioral setting as well as in the classical state space framework. It turned out that our reconstructibility property is equivalent to Willems-observability. Moreover, for the case of state space systems (over $\mathbb{Z}$), the characterization of reconstructibility coincides with the well-known reconstructibility condition for systems over $\mathbb{Z}_+$. However the characterization of forward-observability is different from the observability condition for state space systems over $\mathbb{Z}_+$.

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