ON A SOLUTION CONCEPT TO IMPULSIVE DIFFERENTIAL SYSTEMS

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Abstract. In this article we present a solution concept for measure driven differential systems whose singular set-valued dynamics depend on the time and state variables. Models of this kind arise in a wide variety of applications ranging from space navigation to investment problems as well as ecological management. The approach underlying this solution concept provides a convenient framework to derive optimality conditions in the form of either a Maximum Principle or Hamilton-Jacobi-Bellman equations. Another class of problems related to the addressed systems concerns the state estimation. The redesign of the approach for impulsive control systems in the framework of the new solution concept is presented through the characterization of the reachable set as a level set of the value function regarded as a solution of Hamilton-Jacobi-Bellman equations.

§ 1. Introduction

We study the following measure differential inclusion problem

$$(P) \quad \text{Minimize } h(x(1))$$

$$dx(t) \in F(t, x(t))dt + \mathbf{G}(t, x(t))\mu(dt) \quad \forall t \in [0,1] \tag{1}$$

$$x(0) = x_0, \quad \mu \in \mathcal{K}$$

where

$$h : \mathbb{R}^n \mapsto \mathbb{R}, \quad F : [0,1] \times \mathbb{R}^n \hookrightarrow \mathcal{P}(\mathbb{R}^n),$$

$$\mathbf{G} : [0,1] \times \mathbb{R}^n \hookrightarrow \mathcal{P}(\mathbb{R}^{n \times q}), \quad \mathcal{K} := C^*([0,1]; K) \tag{2}$$

and K is a positive pointed convex cone in $\mathbb{R}^q$.

We consider mild assumptions on the data, i.e., we study the problem (P) under Lipschitz continuity dependence on the state variable and we do not assume the commutativity of the singular vector fields.

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Therefore, the first question that arises in Problem (P) is how to define the solution $x(\cdot)$ to (1) or to its differential inclusion interpretation:

$$x(t) = x(0) + \int_0^t f(\tau, x(\tau))d\tau + \int_{[0,t]} G(\tau, x(\cdot))\mu(d\tau) \qquad \forall t \in [0,1],$$

where f and G are suitable selections of F and $\mathbf{G}$.

The main problem in this context is to define correctly the interaction between the evolving trajectory and the impulsive integrating measure. The approach presented here enables a definition of a solution concept which ensures the well posedness of the dynamic optimization control problem. The technique to derive optimality conditions is based on the reparameterization procedure that reduces the original problem to an auxiliary conventional one. Then, we apply existing conditions to this new problem and express them in terms of the data of the original problem.

§ 2. The Concept of Proper Solution

Following [9], we introduce the solution concept

Definition 1. We will call $x(\cdot) \in BV^+([0,1]; \mathbb{R}^n)$ as a *proper solution* to the measure differential inclusion (1) iff there exist $\mathcal{L}$-integrable f and $|\mu|$-integrable g, with

$$f(t) \in F(t, x(t)) \ \ \mathcal{L} \text{ a.e. and } g(t) \in \tilde{G}(t, x(t^-); \mu(\{t\})) \ \ |\mu|\text{-a.e.,} \quad \text{s.t.:}$$

$$x(t) = x(0) + \int_0^t f(\tau)d\tau + \int_{[0,t]} g(\tau)|\mu|(d\tau) \ \ \forall t \in (0,1],$$

where $\tilde{G} : [0,1] \times \mathbb{R}^n \times K \hookrightarrow \mathbb{R}^n$ is given by

$$\tilde{G}(t,z;\alpha) := \begin{cases} \left\{ G(t,z)\dfrac{\mathrm{d}\mu}{\mathrm{d}|\mu|} \right\}, & \text{if } |\alpha| = 0 \\ \Big\{ \frac{\xi(\eta(t)) - \xi(\eta(t^-))}{|\alpha|} : \dot{\xi}(s) \in G(t, \xi(s))\dot{\gamma}(s) \ \ \bar{\eta}\text{-a.e.} & \\ \qquad \xi(\eta(t^-)) = z \text{ and } \gamma(\eta(t)) - \gamma(\eta(t^-)) = \alpha \Big\}, & \text{if } |\alpha| > 0. \end{cases} \tag{3}$$

Here, $|\mu|$ is the total variation measure associated with μ, $\eta(\cdot)$ is a *time reparameterization* and $(\gamma(\cdot), \theta(\cdot))$ is a *h-graph completion*. We remind that the graph completion of a measure $\mu \in C^+(0,1)$ is a pair $(\theta, \gamma) : [0,1] \mapsto (\mathbb{R}^+)^{q+1}$ defined by

$$\gamma(s) := \begin{cases} M(\theta(s)), & \text{if } |\mu|(\{\theta(s)\}) = 0 \\ M(t^-) + \int_{\eta(t^-)}^{s} v(\sigma)d\sigma, & \text{otherwise} \end{cases},$$

and $\theta : [0,1] \mapsto [0,1]$ is s.t. $\theta(s) = t$ for all $s \in \bar{\eta}(t)$ where

$$v(\cdot) \in V^t := \{v : \bar{\eta}(t) \to K \,|\, \sum_{i=1}^{q} v_i(s) = 1, \ \ \int_{\bar{\eta}(t)} v(s)ds = \mu(\{t\})\},$$

$$M_i(0) := 0 \text{ and } M_i(t) := \int_{[0,t]} \mu_i(ds) \text{ for all } t > [0,1], \quad i = 1, \dots, q,$$

$$\eta(t) := t + \sum_{i=1}^{q} M_i(t), \text{ and } \bar{\eta}(t) := \begin{cases} \{\eta(t)\} & \text{if } |\mu|(\{t\}) = 0 \\ [\eta(t^-), \eta(t)] & \text{if } |\mu|(\{t\}) > 0. \end{cases}$$

The *reparameterized system* is

$$\dot{y}(s) \in F(\theta(s), y(s))\dot{\theta}(s) + G(\theta(s), y(s))\dot{\gamma}(s),$$

being $\dot{\gamma}(s)$ the variation rate of the control measure in the reparameterized time.

For a given μ and a pair of measurable selections (f, G) of (F, G), we have a set of reparametrized trajectories satisfying :

$$\mathcal{F}_{\mu,f,G} := \big\{ y(\cdot) : \dot{y}(s) = f(\theta(s), y(s))\dot{\theta}(s) + G(\theta(s), y(s))\dot{\gamma}(s), \quad \dot{\gamma}(s) \in K,\ (\dot{\theta}(s), \dot{\gamma}(s)) \in \Omega,\ [0,1] \text{ a.e.} \big\}, \tag{4}$$

where $\Omega := \{w \in \mathbb{R}^+ \times K : \sum_{i=0}^{q} w_i = 1\}$, $\gamma(0) = 0$ and $\gamma(\eta(t)) = \mu([0,t])$, $\forall t \in [0,1]$.

Definition 2. The pair (θ, γ) is a *h-graph completion* associated to (μ, f, G) if

$$(\theta, \gamma) \in \operatorname{argmin}\{h(y(1)) : y(\cdot) \in \mathcal{F}_{\mu,f,G}\}.$$

Let $\sum_h := \{(y, \theta, \gamma) : y \in \mathcal{F}_{\mu,f,G}\ \ \forall$ h-graph completion $(\theta, \gamma)\}$ be the set of all h-standard reparametrized control processes.

We mention here that under mild hypotheses a robust solution $x(\cdot)$ to the measure differential inclusion (1) exists if and only if there exists an absolutely continuous solution $y(\cdot)$ to the reparameterized differential inclusion such that

$$x(t) = y(\eta(t)) \ \ \forall t \in [0,1] \text{ and } \|x\|_{TV} \le \|y\|_{TV}.$$

§ 3. Applications of the Solution Concept

We require the following assumptions on the data:

(H1) h is Lipschitz continuous with constant K_h;

(H2) F is continuous, and for each t is Lipschitz continuous with respect to (w.r.t.) x with constant K_f;

(H3) F is a nonempty, convex and compact-valued function;

(H4) There are constants K_1 and K_2, such that,

$$\forall (t,x), \ \ \forall v \in F(t,x) \ \ |v| \le K_1 + K_2|x|;$$

(H5) G is bounded, convex-valued and Lipschitz continuous w.r.t. (t,x) with constant K_G;

(H6) F and G have closed graphs;

(H7) $\forall r > 0$, $\exists\, k_0(r) \in \mathbb{R}$ s.t. $\exists$ a solution (x, μ) to (P) s.t. $\|\mu\| \le k_0(r)$, $\|x\| \le r$.

The proper solution concept is amenable to the derivation of a wide range of results such as optimality conditions, the generalized Hamilton - Jacobi - Bellman (HJB) equations, stability properties, state estimation approaches. We will mention here only few of related theorems [9, 14, 4].

Theorem 1. Let (x, μ) solve the Problem (P) under assumptions (H1)-(H7). Then, there is $p \in BV([0,1]; \mathbb{R}^n)$ s.t.:

$$\begin{array}{rcll}
(-dp(t), dx(t)) & \in & \partial H_F(t, x(t), p(t))dt + \partial H_G(t, x(t), p(t))\mu(dt) & \mu \text{ and } \mathcal{L} \text{ a.e.}, \\
-p(1) & \in & L\partial h(x(1)), & \\
0 & \ge & \sigma_K(H_G(t, x(t), p(t))) & \forall t \in [0,1], \\
0 & \le & \sigma_K(H_G(t, x(t), p(t))) & \mu \text{ a.e.};
\end{array}$$

$$\begin{array}{rcll}
(-\dot{\alpha}_t(s), \dot{\xi}_t(s)) & \in & \partial H_G(t, \xi_t(s), \alpha_t(s)) \cdot \bar{v}(s) & \bar{\eta}(t) \text{ a.e.}, \\
0 & \le & \sigma_K(H_G(t, \xi_t(s), \alpha_t(s))) & \bar{\eta}(t) \text{ a.e.}, \\
(x^*(t^-), p(t^-)) & = & (\xi_t(\eta(t^-)), \alpha_t(\eta(t^-))), & \\
(x^*(t), p(t)) & = & (\xi_t(\eta(t)), \alpha_t(\eta(t))), &
\end{array}$$

where $\bar{v}$ satisfies $\sigma_K(H_G(t, \xi_t(s), \alpha_t(s))) = H_G(t, \xi_t(s), \alpha_t(s)) \cdot \bar{v}(s)$ and $\int_{\bar{\eta}(t)} \bar{v}(s)ds = \mu(\{t\})$, $\sigma_K(k) := sup_{\varsigma \in K}\{k \cdot \varsigma\}$, $H_F(t, x^*(t), p(t)) := \sigma_{F(t,x(t))}(p(t))$ and

$$H_G(t, x^*(t), p(t)) := \begin{cases} \{h_G(t) \cdot w^*(t)\} & \text{if } \mu(\{t\}) = 0 \\ \{h_G^s(t) : s \in \bar{\eta}(t)\} & \text{otherwise} \end{cases}$$

with

$$h_G(t) \cdot w^*(t) := \operatorname{Sup}\Big\{p^T(t)G(t, x(t)) \cdot w : w = \tfrac{d\mu}{d\bar{\mu}} \in K,\ G(t, x(t)) \in \bar{G}(t, x(t))\Big\}$$

$$h_G^s(t) \cdot \bar{v}^*(s) = \operatorname{Sup}\big\{\alpha_t(s)G(t, \xi_t(s)) \cdot v(s) : v(\cdot) \in V^t,\ G(t, \xi_t(s)) \in \bar{G}(t, \xi_t(s))\big\}.$$

Let us now consider the derivation of generalized Hamilton-Jacobi-Bellman equations in the context of this solution concept, [10]. Let $X(\tau, \xi)$ be the set of feasible trajectories starting at (τ, ξ). Then,

- $R(\tau, \xi) := \{x(1) \in \mathbb{R}^n : x \in X(\tau, \xi)\}$ is the *reachable set* at time $t = 1$ when $x(\tau) = \xi$,
- $V(\tau, \xi) := \min\{h(z) : z \in R(\tau, \xi)\}$ is the *value function*.

Theorem 2. Under assumptions (H1)-(H7) the value function V is locally Lipschitz continuous.

Theorem 3. Let the value function V be locally Lipschitz continuous on $[0,1] \times \mathbb{R}^n$. Then,

$$\max_{(w_0, w) \in K_1, f \in F(t,x), g \in G(t,x)} \{DV((t,x); -(w_0, f w_0 + g w))\} = 0, \quad \forall (t,x) \in (0,1) \times \mathbb{R}^n,$$

$$\max_{r\in K, g\in G(t,x)} \{DV_t(x; -gr)\} \leq 0, \quad \forall (t,x) \in [0,1] \times \mathbb{R}^n, \tag{5}$$

$$\max \{ \max_{r\in K, g\in G(1,x)} \{DV_1(x; -gr)\}, \; V(1,x) - h(x)\} = 0, \quad \forall x \in \mathbb{R}^n.$$

Here $Df(x;v)$ is the lower Dini derivative of a l.s.c. function f w.r.t. x in the direction v, and $V_t(x) = V(t,x)$.

The approach presented above may also be used to solve the state estimation problem for impulsive dynamical system. The estimation problem in the deterministic setting is studied under uncertainty conditions with set-membership description of uncertain variables considered to be unknown but bounded with known bounds, [1, 4, 5, 6]. Such problems arise from mathematical models of dynamical and physical systems for which we have either an incomplete description or a loose mode of time dependence of their generalized coordinates. The techniques to describe the trajectory tubes for impulsive differential inclusion are studied by using dynamic programming results articulated with the described concept of proper solution.

One of the main points of interest in control theory under uncertainty, [6], concerns the study of the set of all solutions $x[t] = x(t, t_0, x_0)$ to (1) with unknown but bounded initial state x_0:

$$x_0 \in X_0, \tag{6}$$

where X_0 is a compact subset of $\mathbb{R}^n$. The "guaranteed" estimation problem consists in describing the set

$$X[\cdot] = \cup\{x[\cdot] = x(\cdot, t_0, x_0)\}$$

of solutions (1) and (6), being the t - cross section of this set, $X[t]$, the reachable set (information set) at time t of the system from X_0 at time t_0.

The information set can be treated as a *level set* of a generalized solution $V(t,x)$ to the HJB equation (5), where $V(t,x)$ is a value function given by

$$V(t,x) = \inf_{x[\cdot]} \{\phi(t_0, x[t_0]) | \; x[\cdot] = x(\cdot, t_0, x_0), \; x[\cdot] \text{ solves (1) s. t. } x[t] = x\}, \tag{7}$$

being ϕ an appropriately chosen function (e.g., $\phi(t_0, x) = d^2(x, X_0)$ with X_0 as in (6) where $d(x, M)$ is the *distance function* from x to $M \subset \mathbb{R}^n$). Then, according to Theorem 2-3, techniques of proper solutions can be used in finding $V(t,x)$ and, thus, in constructing trajectory tubes and their cross sections as level sets of V [4].

§ 4. Concluding Remarks

Time reparametrization techniques that are at the root of our approach were used in [11]. However, as in [7], the original time becomes an additional state variable component in the derivation of optimality conditions. This fact forces restrictive assumptions on the data. Our methods can be regarded as a substantial refinement of the ones developed for the control context in [13] which uses the concept of "robust solution" to measure driven differential inclusions provided by [12].

Vector valued control measures have been also addressed by [2], [3], [15].

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