STABILITY OF RESET SWITCHED SYSTEMS

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Abstract: In this paper we define a reset switched system as a switched system where, once switching occurs, the state is forced to assume (is reset to) a new value which is a linear function of the previous state. Using a different approach from the ones that have already been proposed in the literature, we show that, by carefully selecting te reset laws, it is always possible to achieve stability under arbitrary switching.

Keywords: Switched systems, stability, common quadratic Lyapunov functions, reset.

1. INTRODUCTION

A switched linear system is a special type of time varying system that can be viewed as a family of time invariant linear systems together with a switching law. The switching law determines which of the linear system within the family is active at each time instant, hence defining how the time invariant systems commute among themselves. This type of systems may appear either as a direct result of the mathematical modeling of a phenomenon or as the consequence of certain control techniques using switching schemes, see, for instance, (?) (?). In these schemes, instead of using a unique controller for a given system, a bank of controllers (multi-controller) is considered and the control procedure is made by commutation within the bank. In this context, finding conditions that guarantee that the obtained switched system is stable for every switching control law is a crucial issue, (?).

The most common approach when dealing with switched systems is not to allow jumps in the state during the switching instances. In such case, even if each individual time invariant system is stable the correspondent switched system may be unstable, (?). The stability of switched systems with continuous state trajectories has been widely investigated, see, for instance, (?), (?), (?), (?) and (?). In particular, it has been shown that the existence of a common quadratic Lyapunov function (CQLF) for a set of state-space models $\{\Sigma_p, p \in \mathcal{P}\}$ implies the stability of the overall switched system, (?).

However, in some situations it is natural and profitable to allow discontinuous state jumps during switching instants. In fact, many processes may experience abrupt state changes at certain moments of time, for instance in drug administration, (?). Also, it is possible to use a state reset in order to construct a multi-controller that stabilizes a given process, (?).

In this paper, we define reset switched systems as switched systems where the state may change according with a certain linear reset map when switching occurs. This reset map does not depend on the instant when the switching occurs itself,

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but only on the value of the switching signal before and after the switching. Reset switched systems t may be regarded as, some authors call, systems with impulse effects, (?), (?), (?). We shall show that, by carefully selecting the resets, it is always possible to stabilize the switched system. This resembles what has been done in (?) using different framework and techniques.

2. PRELIMINARIES

Let $\mathcal{P}$ be a finite index set, $\{\Sigma_p, p \in \mathcal{P}\}$ a family of time invariant linear systems and (A_p, B_p, C_p, D_p) the state model representation of Σ_p, for $p \in \mathcal{P}$. Additionally, define a switching law $\sigma\colon [0, +\infty[\to \mathcal{P}$ to be a piecewise constant function of time, *i. e.*,

$$\sigma(t) = i_k, \text{ for } t_k \leq t < t_{k+1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots$. The time instants t_k, $k \in \mathbb{N}_0$, are called switching instants. The corresponding switched system Σ_σ has the following representation

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} \tag{1}$$

where $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output and $x(t) \in \mathbb{R}^n$ is the state. At each switching instant, t_k, the state is considered to be such that

$$x(t_k) = R_{(i_{k-1},i_k)}x(t_k^-) \,, \tag{2}$$

where $x(t_k^-) := \lim\limits_{t \to t_k \;\, t<t_k} x(t)$ and $R_{(i_{k-1},i_k)}$ is an invertible square matrix, for $k \in \mathbb{N}$. The matrices $R_{(i_{k-1},i_k)}$ are called *reset matrices* or *resets.*

Notice that, as above mentioned, $R_{(i_{k-1},i_k)}$ is determined by the linear systems (within the family) active before and after the commutation; it does not depend on the switching instant.

When all reset matrices, $R_{(i_{k-1},i_k)}$, are the identity matrix, the state replacement does not exist. In this case, Σ_σ is called a *switched system without reset* or, simply, a *switched system.* In the contrary, Σ_σ is called a *reset switched system.* In the first case, it is assumed that there are no discontinuous state jumps during the switching instants, while, in the second case, discontinuous state jumps during the switching instants are allowed, and determined by the reset matrices.

Definition 1. The system Σ_σ is globally uniformly exponentially stable if there exists $\gamma, \lambda \in \mathbb{R}^+$ such that, for every $t_0 \in \mathbb{R}$ and every $x_0 \in \mathbb{R}^n$, the solution $x(t)$ of $\dot{x}(t) = A_{\sigma(t)}x(t)$, with $x(t_0) = x_0$, satisfies $||x(t)|| \leq \gamma e^{-\lambda(t-t_0)}||x_0||$, for $t \geq t_0$.

A function $V(x)$ is said to be a common quadratic Lyapunov function (CQLF) for the switched system Σ_σ, defined by equations (1), (and for the corresponding set of matrices $\{A_p, p \in \mathcal{P}\}$) if it is a quadratic Lyapunov function for each of the systems $\Sigma_p = (A_p, B_p, C_p, D_p)$. Moreover, it is easy to justify that if P is a square symmetric positive definite matrix and $V(x) = x^T Px$, then the function $V(x)$ is a CQLF for the switched system Σ_σ if and only if

$$A_p^T P + PA_p < 0 \,, \text{ for all } p \in \mathcal{P} \,.$$

In the following, we use the next well-known sufficient condition for stability for switched systems without reset, (?).

Theorem 2. If there exists a CQLF for the switched system Σ_σ, then Σ_σ is stable, for every switching signal.

Notice that the reciprocal of Theorem 2 does not hold. In fact, there are stable switched systems, for every switching signal, with no CQLF, (?). On the other hand, several authors have tried to establish conditions in order to guarantee the existence of a CQLF for a switched system, (?), (?), (?) and (?).

3. STATE TRAJECTORIES ANALYSIS AND STABILITY

In this section we will study the stability of a reset switched system by relating its state trajectories with the ones of a time-varying system without resets, i.e., with no discontinuous jumps on the state. Our aim is to identify cases where that associated time-varying system is indeed a switched system without resets, according with our definition, (which is particular type of the former ones).

For the sake of simplicity and considering that our interest is focussed on the system stability properties, from now on let consider the reset switched system represented by

$$\Sigma_\sigma := \dot{x}(t) = A_{\sigma(t)}x(t), \tag{3}$$

associated to a switching signal σ, with switching instances $0 = t_0 < t_1 < \ldots < t_k < \ldots, k \in \mathbb{N}$, where

$$x(t_k) = R_{(i_{k-1},i_k)}x(t_k^-) \tag{4}$$

with $\sigma(t) = i_k$ for $t_k \leq t < t_{k+1}$ and $R_{(i_{k-1},i_k)}$, $k \in \mathbb{N}$ an invertible square matrix.

We next produce a switching dynamic without reset that will allow an analysis of the trajectories of the reset switched system Σ_σ . The trajectory of this new dynamic will be denoted by $\widetilde{x}(t)$. Here, to simplify the notation we define $c_k = (i_{k-1}, i_k)$, for $k \in \mathbb{N}$.

Lemma 3. Let Σ_σ be a reset switched system defined according to (3) and (4). The following time-varying linear system

$$\dot{\widetilde{x}}(t) = \widetilde{A}(t)\widetilde{x}(t); \tag{5}$$

where

$$\widetilde{A}(t) = A_{i_0}, \text{ for } 0 \le t < t_1$$
$$\widetilde{A}(t) = \left(\prod_{m=k}^{1} R_{c_m}\right)^{-1} A_{i_k} \prod_{m=k}^{1} R_{c_m}, \quad \text{for } t_k \le t < t_{k+1},\ k \ge 1,$$

and $\widetilde{x}(t_k) = \tilde{x}(t_k^-)$, is such that

$$x(t) = \prod_{m=k}^{0} R_{c_m} \widetilde{x}(t), \text{ for } t_k \le t < t_{k+1},\ k \ge 0.$$

By convention, $R_{c_0} = I$.

Proof.

For $0 = t_0 \le t < t_1$, we have $\dot{x}(t) = A_{i_0}x(t)$, that is equivalent to

$$\dot{\widetilde{x}}_0(t) = \widetilde{A}(t)\widetilde{x}_0(t); \widetilde{x}_0(t_0) = x(t_0), \tag{6}$$

with

$$\widetilde{x}_0(t) = x(t) \tag{7}$$
$$\widetilde{A}(t) = A_{i_0}.$$

For $t_1 \le t < t_2$,

$$\dot{x}(t) = A_{i_1}x(t),$$

with $x(t_1) = R_{c_1}x(t_1^-)$. So

$$R_{c_1}^{-1}\dot{x}(t) = R_{c_1}^{-1}A_{i_1}R_{c_1}R_{c_1}^{-1}x(t);$$
$$R_{c_1}^{-1}x(t_1) = x(t_1^-).$$

But, by (7), $x(t_1^-) = \widetilde{x}_0(t_1^-)$. Then,

$$R_{c_1}^{-1}\dot{x}(t) = R_{c_1}^{-1}A_{i_1}R_{c_1}R_{c_1}^{-1}x(t), \tag{8}$$

where $R_{c_1}^{-1}x(t_1) = \widetilde{x}_0(t_1^-)$.

Taking

$$\widetilde{x}_1(t) = R_{c_1}^{-1}x(t), \tag{9}$$

and considering (8), we obtain

$$\dot{\widetilde{x}}_1(t) = \widetilde{A}(t)\widetilde{x}_1(t)$$
$$\widetilde{x}_1(t_1) = \widetilde{x}_0(t_1^-),$$

for $t_1 \le t < t_2$ and $\widetilde{A}(t) = R_{c_1}^{-1}A_{i_1}R_{c_1}$.

For $t_2 \le t < t_3$,

$$\dot{x}(t) = A_{i_2}x(t) \tag{10}$$
$$x(t_2) = R_{c_2}x(t_2^-). \tag{11}$$

Hence, by (9), we have $R_{c_1}\widetilde{x}_1(t_2^-) = x(t_2^-)$ and consequently

$$x(t_2) = R_{c_2}R_{c_1}\widetilde{x}_1(t_2^-)\ .$$

Therefore,

$$R_{c_1}^{-1}R_{c_2}^{-1}\dot{x}(t) = R_{c_1}^{-1}R_{c_2}^{-1}A_{i_2}R_{c_2}R_{c_1}R_{c_1}^{-1}R_{c_2}^{-1}x(t), \tag{12}$$

with $R_{c_1}^{-1}R_{c_2}^{-1}x(t_2) = \widetilde{x}_1(t_2^-)$.

Taking

$$\widetilde{x}_2(t) = R_{c_1}^{-1}R_{c_2}^{-1}x(t), \tag{13}$$

(12) is equivalent to

$$\dot{\widetilde{x}}_2(t) = \widetilde{A}(t)\widetilde{x}_2(t) \tag{14}$$
$$\widetilde{x}_2(t_2) = \widetilde{x}_1(t_2^-), \tag{15}$$

for $t_2 \le t < t_3$ and $\widetilde{A}(t) = R_{c_1}^{-1}R_{c_2}^{-1}A_{i_2}R_{c_2}R_{c_1}$.

Following the previous process, we obtain, for $t_k \le t < t_{k+1}$,

$$\dot{\widetilde{x}}_k(t) = \widetilde{A}(t)\widetilde{x}_k(t)\,, \tag{16}$$

where

$$\widetilde{x}_k(t_k) = \tilde{x}_{k-1}(t_k^-),$$
$$\widetilde{A}(t) = \left(\prod_{m=1}^{k} R_{c_m}^{-1}\right) A_{i_k} \prod_{m=k}^{1} R_{c_m}$$
$$\widetilde{x}_k(t) = \left(\prod_{m=1}^{k} R_{c_m}^{-1}\right) x(t), \tag{17}$$

with $k \in \mathbb{N}$.

Considering $\widetilde{x}(t) = \widetilde{x}_k(t)$, for $t_k \le t < t_{k+1}, k \in \mathbb{N}_0$, the equations (16) can be written as

$$\dot{\widetilde{x}}(t) = \widetilde{A}(t)\widetilde{x}(t), \tag{18}$$

where

$$\widetilde{A}(t) = A_{i_0}, \text{ for } 0 \le t < t_1$$
$$\widetilde{A}(t) = \left(\prod_{m=k}^{1} R_{c_m}\right)^{-1} A_{i_k} \prod_{m=k}^{1} R_{c_m}, \quad \text{for } t_k \le t < t_{k+1},\ k \ge 1,$$

and $\widetilde{x}(t_k) = \tilde{x}(t_k^-)$. From (17), it follows that

$$x(t) = \prod_{m=k}^{0} R_{c_m} \widetilde{x}(t), \text{ for } t_k \le t < t_{k+1},\ k \ge 0.$$

□

In order to avoid long mathematical expressions, we will use the following notation:

$$\overline{R}_{\sigma,0} = I, \text{ for } 0 \le t < t_1$$
$$\overline{R}_{\sigma,k} = \prod_{m=k}^{1} R_{c_m}, \quad \text{for } t_k \le t < t_{k+1}, k \ge 1.$$

Notice that, in the previous proof we have associated to the original reset switched system a time-varying system with a linear and piecewise constant dynamic. That dynamic is determine by the following set of stable matrices

$$\widetilde{\mathcal{A}} = \left\{ A_{i_0}, \overline{R}_{\sigma,1}^{-1} A_{i_1} \overline{R}_{\sigma,1}, \overline{R}_{\sigma,2}^{-1} A_{i_2} \overline{R}_{\sigma,2}, \dots \right\}.$$

Although the number of distinct reset matrices is finite, the set $\widetilde{\mathcal{A}}$ may be infinite. In this case, the time-varying system (18) does not fit into our definition of switched system. Nevertheless, if $\widetilde{\mathcal{A}}$ is finite, then (18) can be considered a switched system without reset, associated to the corresponding finite family of time-invariant systems and a switching signal $\widetilde{\sigma}$, that supervises the switching between those systems. This switching signal, $\widetilde{\sigma}$, will have switching instances in the same set of switching instances of σ.

Example 1. Let $\Sigma_\sigma := \dot{x}(t) = A_{\sigma(t)} x(t)$ be a reset switched system such that, for each switching signal σ, with switching instances $0 = t_0 < t_1 < \dots < t_k < \dots, k \in \mathbb{N}$, taking values in $\mathcal{P} = \{1, 2\}$, the reset of the state is given by $x(t_k) = R_{(i_{k-1}, i_k)} x(t_k^-)$, where

$$R_{(i_{k-1}, i_k)} = \begin{cases} R, & \text{if } i_k = 1 \\ R^{-1}, & \text{if } i_k = 2 \end{cases}.$$

Let us consider that $\sigma(t) = 1$, for $0 = t_0 \le t < t_1$. Then, $\overline{R}_{\sigma,0} = I$, $\overline{R}_{\sigma,1} = R_{c_1} = R^{-1}$, $\overline{R}_{\sigma,2} = R_{c_2} R_{c_1} = R R^{-1} = I, \dots$.

Thus,

$$\overline{R}_{\sigma,k} = \begin{cases} I, & \text{if } k \text{ is even} \\ R^{-1}, & \text{if } k \text{ is odd} \end{cases}.$$

Since $\left\{ \overline{R}_{\sigma,k} : k \in \mathbb{N}_0 \right\} = \left\{ I, R^{-1} \right\}$ is finite, the time-varying system obtained from the first system is a switched system, without reset, with a finite switching bank, given by

$$\begin{aligned} \widetilde{\mathcal{A}} &= \left\{ A_{i_0}, \overline{R}_{\sigma,1}^{-1} A_{i_1} \overline{R}_{\sigma,1}, \overline{R}_{\sigma,2}^{-1} A_{i_2} \overline{R}_{\sigma,2}, \dots \right\} \\ &= \left\{ A_1, R A_2 R^{-1} \right\} \end{aligned}$$

and associated to the same switching signal, σ.

The next result assures that, for a switching signal σ and under certain conditions, if the time-varying system without reset (18) is stable, then the reset switched system Σ_σ is stable too.

Theorem 4. Let Σ_σ be a reset switched system, defined in (3) and (4), for which $\left\{ \left\| \overline{R}_{\sigma,k} \right\| : k \in \mathbb{N} \right\}$ is upper bounded. If the time-varying system (5) is stable, then Σ_σ is stable for the switching signal σ.

Proof. Let $\Sigma_{\mathcal{R}_\sigma} := \dot{x}(t) = A_{\sigma(t)} x(t)$ a reset switched system associated to a switching signal σ, with switching instances $0 = t_0 < t_1 < \dots < t_k < \dots, k \in \mathbb{N}$, and

$$x(t_k) = R_{c_k} x(t_k^-), k \in \mathbb{N},$$

for certain invertible matrices R_{c_k}.

By Lemma 3 , we conclude that

$$\widetilde{x}_k(t) = \widetilde{x}(t) \;\text{ e }\; \sigma(t_k) = \sigma(t), \text{ for any } t > 0,$$

where $\widetilde{x}(t)$ is the trajectory of the system (5), i.e., $\dot{\widetilde{x}}(t) = \widetilde{A}(t) \widetilde{x}(t)$. Then, for any $t > 0$, there exists $k \in \mathbb{N}$ such that

$$\widetilde{x}(t) = \overline{R}_{\sigma,k}^{-1} x(t).$$

Therefore for each $t > 0$,

$$\|x(t)\| \le \left\| \overline{R}_{\sigma,k} \right\| \|\widetilde{x}(t)\|, \text{ for some } k \in \mathbb{N}.$$

But $\left\{ \left\| \overline{R}_{\sigma,k} \right\| : k \in \mathbb{N} \right\}$ is an upper bounded set, then there exists $L > 0$ such that $\left\| \overline{R}_{\sigma,k} \right\| < L$. Consequently, $\|x(t)\| \le L \|\widetilde{x}(t)\|$ and the reset switched system Σ_σ is stable. □

Remark 5. Notice that, additionally, if the set $\left\{ \left\| \overline{R}_{\sigma,k} \right\|^{-1} : k \in \mathbb{N} \right\}$ is upper bounded, then the sufficient condition of the theorem is also necessary. For example, this happens when the set of matrices $\overline{R}_{\sigma,k}, k \in \mathbb{N}$ is finite.

4. STABILITY OBTAINED BY RESET

In this section, we prove that by an adequate choice of reset matrices it is possible to be left just with a finite set of matrices $\overline{R}_{\sigma,k}, k \in \mathbb{N}_0$ and that the correspondent switched system (without resets) is stable. Hence, using Theorem 4, we conclude that the given system with resets, Σ_σ, is also stable for the switching signal σ. This is done for every possible switching signal.

We start by showing that given a family of stable matrices and an arbitrary positive definite symmetric matrix, P, there exists a set of matrices similar to the given ones that share P as CQLF.

Lemma 6. Let $\{A_p, p \in \mathcal{P}\} \subset \mathbb{R}^{n \times n}$ a set of stable matrices and P a positive definite symmetric matrix $n \times n$. Then, there exists a set of invertible matrices $\{W_p, p \in \mathcal{P}\}$ such that $\overline{A}_p = W_p A_p W_p^{-1}$, for $p \in \mathcal{P}$ and

$$\overline{A}_p^{\,T} P + P \overline{A}_p < 0, \; p \in \mathcal{P}.$$

Proof. Let us suppose $P = P^T > 0$ and $\{A_p, p \in \mathcal{P}\}$ is set of stable matrices. Then, there exists a invertible matrix M such that $P = M^T M$. On the other hand, there exists $P_p = P_p^T > 0$ such that

$$A_p^T P_p + P_p A_p < 0, \; p \in \mathcal{P},$$

because A_p, $p \in \mathcal{P}$, is a stable matrix. But, for each $p \in \mathcal{P}$, $P_p = M_p^T M_p$ for some invertible matrix M_p. Thus,

$$A_p^T M_p^T M_p + M_p^T M_p A_p < 0,\ p \in \mathcal{P}.$$

Multiplying the last inequality on the left by M_p^{-T} and on the right by M_p^{-1}, we obtain

$$M_p^{-T} A_p^T M_p^T + M_p A_p M_p^{-1} < 0,\ p \in \mathcal{P}.$$

Consequently,

$$M^T \left(M_p^{-T} A_p^T M_p^T + M_p A_p M_p^{-1}\right) M < 0,\ p \in \mathcal{P}.$$

Since $I = MM^{-1}$, the last inequality can be written as, for $p \in \mathcal{P}$,

$$M^T M_p^{-T} A_p^T M_p^T I^T M + M^T I M_p A_p M_p^{-1} M < 0\,.$$

Taking $W_p := M^{-1} M_p$ and $\overline{A}_p := W_p A_p W_p^{-1}$, we obtain

$$\overline{A}_p^{\,T} P + P\overline{A}_p < 0,\ p \in \mathcal{P}.$$

□

Based on the previous lemma, the next result shows a way to adequately select the reset matrices in order to guarantee the stability of the switched system $\dot{x}(t) = A_{\sigma(t)} x(t)$, for all switching signal σ. Note that, a similar result was obtained in (**?**) using another perspective.

Theorem 7. Let Σ_σ be a reset switched system, defined in (3) and (4). Consider a set of invertible matrices $\{S_p, p \in \mathcal{P}\}$ such that the set $\left\{S_p A_p S_p^{-1}, p \in \mathcal{P}\right\}$ has a CQLF.

If

$$R_{(i_{k-1},i_k)} = S_{i_k}^{-1} S_{i_{k-1}}, \tag{19}$$

where $\sigma(t) = i_k$ for $t_k \le t < t_{k+1}$, $k \in \mathbb{N}$, then Σ_σ is stable.

Proof. Let $\sigma : [0, +\infty[\to \mathcal{P}$ a switching signal with switching instances $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots, \in \mathbb{N}$.

Let us consider $R_{c_k} = S_{i_k}^{-1} S_{i_{k-1}}$, where $c_k := (i_{k-1}, i_k)$ and $\sigma(t) = i_k$ for $t_k \le t < t_{k+1}$. Then, considering $\overline{R}_{\sigma,k} = \prod_{m=k}^{1} R_{c_m}$ and $\overline{R}_{\sigma,0} = I$, we obtain $\overline{R}_{\sigma,1} = R_{c_1} = S_{i_1}^{-1} S_{i_0}$, $\overline{R}_{\sigma,2} = R_{c_2} R_{c_1} = S_{i_2}^{-1} S_{i_0}, \ldots$. Thus,

$$\overline{R}_{\sigma,k} = S_{i_k}^{-1} S_{i_0},\ k \in \mathbb{N}_0$$

and $\left\{\overline{R}_{\sigma,k} : k \in \mathbb{N}_0\right\} = \{S_p S_{i_0}; p \in \mathcal{P}\}$ is a finite set. So, by Theorem 4, it is sufficient to prove that the following system

$$\dot{\widetilde{x}}(t) = \widetilde{A}(t)\widetilde{x}(t), \quad \widetilde{A}(t) \in \widetilde{\mathcal{A}} \tag{20}$$

$$\widetilde{\mathcal{A}} = \left\{A_{i_0}, \overline{R}_{\sigma,1}^{-1} A_{i_1} \overline{R}_{\sigma,1}, \overline{R}_{\sigma,2}^{-1} A_{i_2} \overline{R}_{\sigma,2}, \ldots\right\} \tag{21}$$

is stable. But,

$$\widetilde{\mathcal{A}} = \left\{S_{i_0}^{-1} \overline{A}_p S_{i_0} : p \in \mathcal{P}\right\},$$

where $\overline{A}_p = S_p A_p S_p^{-1}, p \in \mathcal{P}$. Hence the time-varying system (20) is a switched system with switching signal σ, and it is stable because the matrices $S_{i_0}^{-1} \overline{A}_p S_{i_0}, p \in \mathcal{P}$ have a CQLF. Notice that the matrices $\overline{A}_p, p \in \mathcal{P}$ have a CQLF. Then, by Theorem4, we conclude that the system Σ_σ is stable for σ. □

Remark 8. In the previous theorem, the resets applied in the switching instances are always from the same type, independently of the considered switching signal, and so the reset switched system Σ_σ is stable for any switching signal.

Remark 9. In particular, if $\overline{A}_p = S_p A_p S_p^{-1}, p \in \mathcal{P}$ are all upper triangular matrices, then they have a CQLF and the switched system is stable since the resets are chosen as in last theorem.

Finally, to illustrate the previous result, we present a example that shows a way to pick the reset matrices in order to ensure stability of a switched systems which, without reset, is unstable.

Example 2. Consider $\Sigma_\sigma := \dot{x}(t) = A_{\sigma(t)} x(t)$, a switched system, with switching signal $\sigma : [0, +\infty[\to \{1, 2\}$. Assume that

$$A_1 = \begin{pmatrix} -0.05 & 2 \\ -1 & -0.05 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} -0.05 & 1 \\ -2 & -0.05 \end{pmatrix}.$$

Note that the time-invariant systems $\dot{x}(t) = A_1 x(t)$ and $\dot{x}(t) = A_2 x(t)$ are stable, but the switched system Σ_σ is unstable, (**?**).

If we have invertible matrices S_1 and S_2, such that $S_1 A_1 S_1^{-1}$ and $S_2 A_2 S_2^{-1}$ have a CQLF, for instance the identity matrix, then choosing the resets as

$$R_{(1,2)} = S_2^{-1} S_1 \text{ and } R_{(2,1)} = S_1^{-1} S_2$$

we obtain stability of the reset switched system Σ_σ, for all switching signal σ. In fact, the pair (S_1, S_2) is not unique. We may take, for instance,

$$S_1 = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 2\sqrt{5} \end{pmatrix} \text{ and } S_2 = \begin{pmatrix} 2\sqrt{5} & 0 \\ 0 & \sqrt{10} \end{pmatrix}.$$

In this case

$$S_1 A_1 S_1^{-1} = S_2 A_2 S_2^{-1} = \begin{pmatrix} -\dfrac{1}{20} & \sqrt{2} \\ -\sqrt{2} & -\dfrac{1}{20} \end{pmatrix}.$$

and the resets to be used are

$$R_{(1,2)} = \begin{pmatrix} \dfrac{1}{\sqrt{2}} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \text{ and } R_{(2,1)} = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \dfrac{1}{\sqrt{2}} \end{pmatrix}.$$