OPTIMALITY CONDITIONS FOR ASYMPTOTICALLY STABLE IMPULSIVE CONTROL SYSTEMS

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Abstract: We consider an infinite horizon optimal impulsive control problems for which a performance criteria is minimized by choosing asymptotically stable control strategies. We present necessary optimality conditions in the form of a maximum principle and show how they can be derived from an auxiliary conventional (nonimpulsive) optimal control problem. As far as we know, results of this kind have not been previously derived for problems with trajectories restricted to the set of stabilizing ones.

Keywords: optimal stability, impulsive control, measure driven differential inclusion, necessary conditions of optimality.

1. INTRODUCTION

This article concerns the derivation of necessary conditions of optimality for an infinity horizon impulsive control problem in which the admissible trajectories - i.e., solutions to the measure driven differential inclusion satisfying the endpoint constraints - are asymptotically stable.

The dynamics are given by a differential inclusion driven by a vector valued control measure. An infinite horizon extension of the concept of proper solution adopted in (Pereira and Silva, 2000) - which, in turn, is a development of the results in (Silva and Vinter, 1996) and (Silva and Vinter, 1997) - will be used. The fundamental issue in this concept is how to ensure a consistent definition of the state trajectory on the set of points in which it exhibits discontinuities. In particular, one key issue for a trajectory to be well defined is the existence of a path joining the jump endpoints that satisfies the singular dynamics. By this, it is meant that the derivative of the state variable with respect to a certain parameter is in the singular velocity set. This parameter takes values in an interval whose length is proportional to the variation of the control measure on the support of the considered atom.

Let (P) be the problem

$$\begin{aligned} \text{Minimize } & h(x(0), \xi) && (1)\\ \text{such that } & dx(t) \in F(t, x(t))dt + \mathbf{G}(t, x(t))\mu(dt) && (2)\\ & x(0) \in C, \quad x(t) \to \xi \in S && (3)\\ & \mu \in \mathcal{K}. && (4) \end{aligned}$$

Here, F and $\mathbf{G}$ are set-valued maps from $[0, \infty) \times \mathbb{R}^n$ to, respectively, $\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{R}^{n\times q})$, $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a given function and the sets C, and S are closed subsets of $\mathbb{R}^n$. $\mathcal{K} \subset C^*([0, \infty), \mathbb{R}^q)$ is a cone

[1] Apoio da Fundação para a Ciência e Tecnologia - convénio com a FAPESP
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of measures with range in a positive, convex, closed, pointed cone K of $\mathbb{R}^q$, i.e., for all μ-measurable $A \in [0, \infty)$, $\mu(A) \in K$. Moreover, a feasible control measure μ may have unbounded total variation but its total variation on any bounded set is finite, i.e., $\bar{\mu}(A) < \infty$ for any finite $A \subset [0, \infty)$ (here and in what follows $\bar{\mu}$ denotes the total variation measure associated with μ).

Obviously, intrinsic to the well-posedness of this problem is the existence of at least an equilibrium point. The point $\xi \in \mathbb{R}^n$ is an equilibrium of the dynamic system (2) if there exists a feasible control process $(x(\cdot), \mathrm{d}\mu)$ satisfying $\lim_{t\to\infty} x(t)=\xi$, and for which

$$0 \in \lim_{t\to\infty} \{F(t, x(t))\mathrm{d}t + \mathbf{G}(t, x(t))\mu(\mathrm{d}t)\}.$$

The new feature of this problem is that the optimization is conducted over trajectories which asymptotically stabilize the system over all the equilibrium points in the given set S. This includes the case where it is not possible to steer the state to the equilibrium point in finite time. This feature distinguishes the problem addressed in this article from the usual finite time control problem.

This optimal control problem is a natural impulsive extension of the optimal control problem addressed in (Pereira and Silva, 2006) for which a control formulation with only ordinary Borel measurable controls was considered, that is, one with the data as in (P) with $K = \{0\}$. However, in this article, the set of admissible trajectories was restricted to the set of those whose norm converged exponentially, with a prescribed minimum rate, i.e.,

$$\frac{d}{dt}(\|x(t) - \xi\|^2) \le -2\gamma\|x(t) - \xi\|^2$$

where γ is some positive number. Notice that this constraint is, in fact, of the mixed constraint type

$$(x(t) - \xi)^T f(x(t), u(t)) \le -\gamma\|x(t) - \xi\|^2,$$

which can be written in the more common form $h(y, u) \le 0$, where $y = col(x, \xi)$, with $y(0) \in C \times S$, $\dot{y} = col(f(x, u), 0)$, and $h(y, u) = (x - \xi)^T f(x, u) + \gamma\|x - \xi\|^2$. This definition of the stabilizing state trajectory constraint implies that the set of feasible trajectories is much smaller than the one considered here.

This article contributes to the extension of the rich body of results of optimal control theory for systems with absolutely continuous trajectories to that of systems with trajectories of bounded variation, (Warga, 1962; Rishel, 1965; Rockafellar, 1976; Rockafellar, 1981; Gurman, 1985; Vinter and Pereira, 1988; Bressan and Rampazzo, 1991; Bressan and Rampazzo, 1994; Motta and Rampazzo, 1995; Kolokolnikova, 1996; Dykhta, 1996; Miller, 1996; Pereira and Silva, 2000; Pereira and Silva, 2002; Pereira and Silva, 2004; Arutyunov *et al.*, 2005*b*; Arutyunov *et al.*, 2005*a*). These developments have been strongly fueled by the motivation underlying the important classes of applications - space navigation (Marec, 1979; Dykhta and Samsonyuk, 2000), socioeconomic problems (Gurman and (eds.), 2000), resources management (Baumeister, 2001; Clark *et al.*, 1979; Dykhta and Samsonyuk, 2000), impact mechanics (Brogliato, 1996), to name just a few of the many classes of multi-phase systems, (Zavalischin and Sesekin, 1991) - for which dynamic control systems naturally exhibit discontinuities.

We would like to remark that there is an appealing engineering motivation behind this solution concept. For important classes of engineering systems, it is of interest to control a dynamical system that might operate in several viable configurations. These systems are often referred to as hybrid systems, (Aubin, 2000). Although transitions between configurations - modeled here by trajectory jumps - are non productive and their duration negligible, the way they evolve might affect the overall performance of the system. Therefore, it is relevant to incorporate the dynamics of the jump as an essential aspect of the dynamic optimization problem.

This paper is organized as follows. In the next section, the adopted solution concept is presented. This is an infinite horizon extension of the one discussed in (Pereira and Silva, 2000). Then, the necessary conditions of optimality are stated together with pertinent comments in section 3. A sketch of the proof is outlined in section 4, just before the conclusions.

2. SOLUTION TO MEASURE DIFFERENTIAL INCLUSIONS

Let $F : [0, \infty) \times \mathbb{R}^n \hookrightarrow \mathcal{P}(\mathbb{R}^n)$, and $\mathbf{G} : [0, \infty) \times \mathbb{R}^n \hookrightarrow \mathcal{P}(\mathbb{R}^{n\times q})$ be given set-valued functions and $\mu \in C^*([0, \infty); K)$, the set in the dual space of continuous functions from $[0, \infty)$ into $\mathbb{R}^q$ with values in K, a positive, convex, and pointed cone in $\mathbb{R}^q$. Then, the well established notation for differential systems can be extended to describe the class of impulsive dynamic control systems by

$$\begin{aligned} \mathrm{d}x(t) &\in F(t, x(t))\mathrm{d}t + \mathbf{G}(t, x(t))\mu(\mathrm{d}t), \\ x(0) &= x_0. \end{aligned} \tag{5}$$

However, it can be easily seen that this short notation is not clear at all. More specifically, what should be the value of $x(t)$ when t is on the support of an atom of μ? This issue is addressed by extending the solution concept introduced by several authors (e.g., (Bressan and Rampazzo, 1994; Silva and Vinter, 1997; Pereira and Silva, 2000)) to (5). We will include here some key results for the sake of the clarification of concepts.

Definition 2.1. The trajectory x, with $x(0) = x_0$, is admissible for (5) if $x(t) = x_{ac}(t) + x_s(t)$ $\forall t \in [0, \infty)$, where

$$\dot{x}_{ac}(t) = f(t) + G_{ac}(t)\cdot w_{ac}(t) \text{ a.e.}$$
$$x_s(t) = \int_{[0,t]} G_c(\tau)w_c(\tau)\mathrm{d}\bar{\mu}_{sc}(\tau) + \int_{[0,t]} g_a(\tau)\mathrm{d}\bar{\mu}_{sa}(\tau). \quad (6)$$

Here, $\bar{\mu}$ is the total variation measure associated with μ, μ_{sc}, μ_{sa} and μ_{ac} are, respectively, the singular continuous, the singular atomic, and the absolutely continuous components of μ, w_{ac} is the time derivative of μ_{ac}, w_{sc} is the Radon-Nicodym derivative of μ_{sc} with respect to its total variation, $f(\cdot)$ and $G_{ac}(\cdot)$ are Lebesgue measurable selections of, respectively, $F(\cdot, x(\cdot))$ and $\mathbf{G}(\cdot, x(\cdot))$, $G_c(\cdot)$ is a $\bar{\mu}_{sc}$ measurable selection of $\mathbf{G}(\cdot, x(\cdot))$ and $g_a(\cdot)$ is a $\bar{\mu}_{sa}$ measurable selection of the multifunction

$$\tilde{G}(t, x(t^-); \mu(\{t\})) : [0,\infty)\times\Re^n\times K \hookrightarrow \mathcal{P}(\mathbb{R}^n) \quad (7)$$

specifying the reachable set of the singular dynamics at $(t, x(t^-))$ when the control measure has an atom of "weight" $\mu(\{t\})$. In order to explain the meaning of this multifunction, the following concept of graph completion of time reparameterization function is required (see (Pereira and Silva, 2000)).

Given a, possibly unbounded, time interval T and a measure μ supported on T, a new time parameterization is defined by associating with t the range of $\bar{\eta}(t)$ defined by $[\eta(t^-), \eta(t)]$ if $\bar{\mu}(\{t\}) > 0$, and by $\{\eta(t)\}$ otherwise, being

- $\eta(t) := t + \sum_{i=1}^{q} M_i(t)$, and
- $M(\cdot) = \mathrm{col}(M_1(\cdot), ..., M_q(\cdot))$, with $M_i(0) = 0$, and $M_i(t) = \int_{[0,t]} \mu_i(\mathrm{d}s)$, $\forall t > 0$.

Definition 2.2. A family of *graph completions* associated with a measure $\mu \in C^*(0,1;K)$ is the set of pairs $(\theta,\gamma) : [0,\infty) \hookrightarrow \mathbb{R}^+ \times \mathcal{P}(K)$ where $\theta : [0,\infty) \to \mathbb{R}$ is the "inverse" of $\bar{\eta}$, and the function $\gamma : \bar{\eta}(t) \to \mathbb{R}^q$ takes values

$$\gamma(s) := \begin{cases} M(\theta(s)) & \text{if } \bar{\mu}(\{t\}) = 0 \\ M(t^-) + \int_{\eta(t^-)}^{s} v(\sigma)\mathrm{d}\sigma & \text{if } \bar{\mu}(\{t\}) > 0, \end{cases}$$

where $\bar{\mu}(\mathrm{d}t) = \sum_{i=1}^{q} \mu_i(\mathrm{d}t)$ and $v(\cdot)$ is in V^t which is defined as the set of functions v from $\bar{\eta}(t)$ to $\Re^q$ satisfying $v(s) \in K$, and $\int_{\bar{\eta}(t)} v(s)\mathrm{d}s = \mu(\{t\})$.

Now, we are in position to define the set-valued function $\tilde{G}(t,z;\alpha)$ in (7). Let $|\alpha| = \sum_{i=1}^{q} \alpha^i$, $w(\cdot)$ be the Radon-Nicodym derivative of μ w.r.t. $\bar{\mu}$, $(\xi,\gamma) \in AC([0,1]; \mathbb{R}^n \times \mathbb{R}^q)$, and the pair (θ,γ) be a *graph completion* with $\dot{\theta}(s) \equiv 0$ on $\bar{\eta}(t)$.

Then, for $|\alpha| = 0$, $\tilde{G}(t,z;\alpha)$ is given by $\{G(t,z)w(t)\}$, and, for $|\alpha| > 0$, by the set of all vectors

$$\frac{\xi(\eta(t)) - \xi(\eta(t^-))}{|\alpha|},$$

where $(\xi(\cdot), \gamma(\cdot))$ satisfies

$$(\dot{\xi}(s), \dot{\gamma}(s)) = (G(t,\xi(s))v(s), v(s)), \quad \bar{\eta}(t)\text{-a.e.},$$

being $G(t,\xi(s)) \in \mathbf{G}(t,\xi(s))$, $v(s) \in V^t$ a.e. in $\bar{\eta}(t)$, with $\xi(\eta(t^-)) = z$, and $\gamma(\eta(t)) - \gamma(\eta(t^-))) = \alpha$.

Notice that, in spite of the nonuniqueness of the graph completions of the control measure, for a given pair $(x(t^-), \mu(\{t\}))$ with $\mu(\{t\})) > 0$ and a given measurable selection of $\mathbf{G}$, $x(t^+)$ is unique only when the vector fields associated with the columns of $\mathbf{G}$ are commutative.

The following proposition proved in (Silva and Vinter, 1996) for the scalar valued measures is to be used in the proof of the main result of this article. The extension to the vector valued case is straightforward and therefore omitted.

Proposition 2.3. Let (θ,Γ) be a family of *graph completions* of $\mu \in C^*([0,+\infty);K)$ as defined above and take $\gamma(\cdot)$ a Borel measurable selection of $\Gamma(\cdot)$. Then

(i) θ and γ are Lipschitz continuous, non-negative functions satisfying
$$\dot{\theta}(s) + \sum_{i=1}^{q} \dot{\gamma}_i(s) = 1 \quad \mathcal{L}\text{a.e..}$$

(ii) For all Borel measurable $\mu \in C^*([0,\infty);K)$, integrable function $G : [0,\infty) \mapsto \Re^{n\times q}$ and Borel set $T \subset [0,\infty)$, we have
$$\int_{\theta^{-1}(T)} G(\theta(s))\dot{\gamma}(s)\mathrm{d}s = \int_{T} G(\tau)\mu(\mathrm{d}\tau).$$

(iii) For all measurable function $f : [0,\infty) \mapsto \mathbb{R}^n$ and Borel set $S \subset [0,\infty)$, $\theta(S)$ is also Borel set
$$\int_{S} f(\theta(s))\dot{\theta}(s)\mathrm{d}s = \int_{\theta(S)} f(\tau)\mathrm{d}\tau.$$

The next result established in (Silva and Vinter, 1996) for the particular case of scalar valued measures concerns the properties of the adopted solution concept. It involves two main aspects: robustness and the equivalence relationship between the impulsive control problem and the associated conventional control problem obtained by reparameterization. The extension to systems with vector-valued control measures with commutative singular vector fields is straightforward as this problem can be easily reduced to the former. Furthermore, the necessary modifications in order to encompass the noncommutative case, the one considered here, are minor and are omitted.

Proposition 2.4. Consider multifunctions F and $\mathbf{G}$ with domain $[0,+\infty) \times \mathbb{R}^n$ satisfying:

(i) $F(t,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ have closed graphs and takes as compact sets in, respectively, $\mathbb{R}^n$ and $\mathbb{R}^{n\times q}$ as values,
(ii) F is Lebesgue $\times$ Borel measurable and $\mathbf{G}$ is Borel measurable, and
(iii) $F(t,x)$ and $\mathbf{G}(t,x)$ are convex valued for all (t,x).

Consider $T > 0$ and take a sequence $\{x_0^i\}$ in $\mathbb{R}^n$ and a sequence $\{\mu_i\}$ in $C^*([0,T];K)$, and elements $x_0 \in \mathbb{R}^n$ and $\mu \in C^*([0,T];K)$ such that, as $i \to \infty$, $x_0^i \to x_0$ and $\mu_i \to^* \mu$. Take also a sequence $\{x_i\}$ in $BV^+([0,T];\mathbb{R}^n)$ such that $x_i(\cdot)$ is a solution to (5) in the sense of Definition 2.1 with μ_i in place of μ. Consider the following inclusion

$$\dot{y}(s)\in F(\theta(s),y(s))\dot{\theta}(s)+\mathbf{G}(\theta(s),y(s))\dot{\gamma}(s) \quad (8)$$

almost everywhere in $[0,T]$. For each i, assume the existence of $\beta(t)\in L^1$ and $c>0$ such that $F(t,x_i(t))\subset \beta(t)B$ a.e. and $\mathbf{G}(t,x_i(t))\subset cB$ for all t.

Then, there exist:

a) a sequence of processes (y_i,θ_i,γ_i), solution to (8) with $y_i(0)=x_0^i$,
b) (y,θ,γ), solution to (8), with $y(0)=x_0$, and
c) a solution x to (5),

such that $x_i(t) = y_i(\eta_i(t))$, and $x(t) = y(\eta(t))$, $\forall t \in (0,T]$. Furthermore, along a subsequence, we have $\mathrm{d}x_i \to^* \mathrm{d}x$ and $x_i(t)\to x(t)$ for all $t \in \big([0,T]\setminus \mathcal{M}_\mu\big) \cup \{0,T\}$ (where $\mathcal{M}_\mu$ denotes the atoms of μ) and $y_i \to y$ strongly in $C([0,T];\mathbb{R}^n)$.

Here, the notation $\mathrm{d}x_i\to^*\mathrm{d}x$ means the weak* convergence of measures $\mathrm{d}x_i$ to $\mathrm{d}x$.

3. THE NECESSARY CONDITIONS OF OPTIMALITY

In this section, we state necessary conditions of optimality that are proved under the following set of assumptions.

H1 h is Lipschitz continuous in its arguments.
H2 F is measurably Hausdorff Lipschitz with constant $K_F(\cdot) \in L^1$.
H3 $\mathbf{G}$ is continuous in t and Hausdorff Lipschitz with constant $K_{\mathbf{G}}$.
H4 F and $\mathbf{G}$ are convex and compact valued set-valued maps with closed graphs.
H5 C and S are compact sets in $\mathbb{R}^n$ and $K\subset\mathbb{R}^q$ is a positive, pointed, convex cone.

These are by no means the weakest assumptions under which necessary conditions of optimality for impulsive control problems can be proved. At this stage, we do not worry about non-degeneracy. However, it is not difficult to show that if a certain endpoint controllability assumption holds, then the stated conditions are always informative.

Theorem 3.1. Let (x^*,μ^*) be a solution to problem (P) whose data satisfies the hypotheses stated above. Then, there exist a nonnegative number λ and a function of bounded variation p satisfying $\lambda + \|p\| \neq 0$ and

$$\begin{aligned}
(-\mathrm{d}p(t),\mathrm{d}x^*(t)) &\in \partial[H_F^*(t)+H_{\mathbf{G}}^*(t)w_{ac}^*(t)]\mathrm{d}t \\
&\quad + \partial H_{\mathbf{G}}^*(t)\mathrm{d}\mu_{sc}^*(t) \\
&\quad +\overline{\partial H_{\mathbf{G}}}(t,x^*(t^-),p(t^-);\mu_{sa}^*(\{t\}))\mathrm{d}\bar{\mu}_{sa}^*(t), \\
&\quad \forall t\in[0,\infty),\\
(p(0),-\zeta) &\in\lambda\partial g(x^*(0),\xi^*)+N_{C\times S}(x^*(0),\xi^*),\\
&\text{where } (\xi^*,\zeta)=\lim_{t\to\infty}(x^*(t),p(t))\\
0 &\geq \sigma_K(H_{\mathbf{G}}^*(t)) \qquad \forall t\in[0,\infty),\\
0 &= \sigma_K(H_{\mathbf{G}}^*(t)) = H_{\mathbf{G}}^*(t)v_{sc}^*(t), \\
&\quad \mu_{sc}^* \text{ a.e.},
\end{aligned}$$

$\forall t\in \mathrm{Supp}(\mu_{sa}^*)$, $\exists\ (\xi_t^*,\zeta_t,v_t^*):[0,\infty)\to\mathbb{R}^n\times\mathbb{R}^n\times K_1$ ($K_1:=K\cap B_1(0)$) satisfying:

$$\begin{aligned}
(-\dot{\zeta}_t(s),\dot{\xi}_t^*(s)) &\in \partial H_{\mathbf{G}}(t,\xi_t^*(s),\zeta_t(s))v_t^*(s)\ \bar{\eta}\text{-a.e.}\\
0 &= \sigma_K(H_{\mathbf{G}}(t,\xi_t^*(s),\zeta_t(s)) \quad \bar{\eta}\text{-a.e.}\\
&= H_{\mathbf{G}}(t,\xi_t^*(s),\zeta_t(s))v_t^*(s)\ \bar{\eta}\text{-a.e.}\\
(\xi_t^*,\zeta_t)(\eta(t^-)) &= (x^*(t^-),p(t^-)),\\
(\xi_t^*,\zeta_t)(\eta(t)) &= (x^*(t),p(t)),
\end{aligned}$$

with $v_t^*\in V^t$ such that $\displaystyle\int_{\bar{\eta}(t)} v_t^*(s)\mathrm{d}s = \mu_{sa}^*(\{t\})$.

Here,

- η and $\bar{\eta}$ are as defined in the previous section;
- $\mu^*(\mathrm{d}t) = w_{ac}^*(t)\mathrm{d}t + \mu_{sc}^*(\mathrm{d}t) + \mu_{sa}^*(\mathrm{d}t)$ is the usual canonical decomposition of the measure μ^*, whose continuous part is denoted by $\mathrm{d}\mu_c^*$, and the Radon-Nycodim derivative of $\mathrm{d}\mu_{sc}^*$ w.r.t. its total variation measure, $\bar{\mu}_{sc}^*$, by v_{sc}^*;
- $H_F^*(t)$ and $H_{\mathbf{G}}^*(t)$ are short representations of the Hamiltonian functions $H_F(t,x^*(t),p(t))$ and $H_{\mathbf{G}}(t,x^*(t),p(t))$, respectively, which are defined by:

$$H_F^*(t) = \max\{\langle p(t),v\rangle : v\in F(t,x^*(t))\}$$

$$H_{\mathbf{G}}^*(t) = \begin{cases} \{h_{\mathbf{G}}(t)\} & \text{if } t\in\mathrm{Supp}(\mu_c^*)\\ \{h_{\mathbf{G}}^t(s) : s\in\bar{\eta}(t)\} & \text{if } t\in\mathrm{Supp}(\mu_{sa}^*)\end{cases}$$

where

$$\langle h_{\mathbf{G}}(t),v_c^*(t)\rangle = \max_{w\in K,G\in\mathbf{G}}\{\langle p(t),G(t,x^*(t))w\rangle\},$$

and

$$\langle h_{\mathbf{G}}^t(s),v_t^*(s)\rangle = \max_{w\in K_1,G\in\mathbf{G}}\{\langle\zeta_t(s),G(t,\xi_t^*(s))w\rangle.$$

Here, $v_c^*(t)$ denotes $w_{ac}^*(t)$ and $v_{sc}^*(t)$ on the supports of the, respectively, absolutely continuous and singular continuous components of the optimal control measure μ^*;

- The generalized gradients of the Hamiltonian functions are taken with respect to the pair (x^*, p) or their graph completions at time t, (ξ_t^*, ζ_t^*);
- $\sigma_K(\cdot)$ denotes the usual support function to the set K;
- Supp(μ) denotes the support of the measure μ, i.e., the smallest set of points $S \subset [0, \infty)$ for which $\mu([0, \infty) \setminus S) = 0$.

Obviously, the solution to the measure driven Hamiltonian inclusion appearing in these conditions is interpreted in the same sense as that presented and discussed in the previous section.

By examining these conditions, it seems at first that they only provide a characterization of the support of the optimal control measure. However, these conditions, together with all the constraints of the problem suffice to determine the optimal control process.

4. SKETCH OF THE PROOF

Let (x^*, μ^*) be an optimal control process for problem (P).

Let us construct the sequences $\{a_i\}$ and $\{m_i(\cdot)\}$, respectively in $\mathbb{R}^n$ and in $L^1_{loc}([0, \infty))$, satisfying:

$$a_i \in C, \text{ and } a_i \to x^*(0)$$

$$m_i(t) \in K \text{ and } m_i(t)\mathrm{d}t \rightharpoonup^* \mathrm{d}\mu^*(t).$$

We recall that the later means that, $\forall \phi \in C([0, \infty))$,

$$\int_0^\infty \phi(t) m_i(t)\mathrm{d}t \to \int_{[0,\infty)} \phi(t)\mathrm{d}\mu^*(t).$$

It can be shown that there exists a trajectory x_i solution to

$$\dot{x}_i(t) \in F(t, x_i(t)) + \mathbf{G}(t, x_i(t)) m_i(t)$$

with $x_i(0) = a_i$, and satisfying $\dot{x}_i(t)\mathrm{d}t \rightharpoonup^* \mathrm{d}x^*(t)$ and, also, $x_i(t) \to x^*(t)$, a.a. $t \in [0, \infty)$, as $i \to \infty$, see (Silva and Vinter, 1996; Silva and Vinter, 1997).

Now, we consider an increasing sequence of positive numbers $\{T_i\}$, with $T_i \uparrow \infty$, such that $|x_i(T_i) - x^*(T_i)| \to 0$ as $i \to \infty$, and construct the auxiliary problem

$$\begin{aligned} (P_i)\ \text{Min. } & \Phi_i(a, x, m) \\ \text{s. t. } & \dot{x}(t) \in F(t, x(t)) + \mathbf{G}(t, x(t)) m(t) \\ & m(t) \in K^i := K \cap r_i B_1(0) \\ & \qquad\qquad \text{a.e. in } [0, T_i] \\ & x(0) = a \in C \end{aligned}$$

where

$$\begin{aligned} r_i^l &:= i + \max_{j \le i} \|m_j(\cdot)\|_{L^q_\infty([0,T_i))}, \\ \phi_i(a, x, m) &:= g(a, x(T_i)) + \|x(T_i)) - x_i(T_i)\|^2 + \\ & \int_0^{T_i} \|x(t) - x^*(t)\|^2 \mathrm{d}t + M d_S(x(T_i)). \end{aligned}$$

Here, M is a certain positive constant and $d_S(a)$ denotes the distance of the point a to the set S.

Denote by w the treble (a, x, m), and let W_i be the set of all w's feasible for (P_i) with $x(0) = a$ endowed with the norm $\| \cdot \|_{W_i}$ defined by

$$\|w\|_{W_i} := |a| + \int_0^{T_i} |x(t)|\mathrm{d}t + \int_0^{T_i} |m(t)|\mathrm{d}t.$$

It can be shown that $(W_i, \| \cdot \|_{W_i})$ is a complete metric space and that there exists a sequence of positive numbers $\{\varepsilon_i\}$ satisfying $\varepsilon_i \to 0$ satisfying

$$\Phi_i(w_i) \le \inf_{w \in W_i} \{\Phi_i(w)\} + \varepsilon_i^2.$$

where $w_i = (a_i, x_i, m_i)$ is as defined above.

The requirements of Ekeland's variational principle are met and, therefore, we conclude that there exists a sequence $w_i^* = (a_i^*, x_i^*, m_i^*) \in W_i$ which is a solution over W_i to a new auxiliary optimal control problem $(\bar{P}_i)$ whose cost functional is $\phi_i^l(w) + \varepsilon_i \|w - w_i^*\|_{W_i}$. Moreover, w_i^* satisfies $\|w_i - \bar{w}_i^*\| \le \varepsilon_i$ and it can be easily shown that $a_i^* = x_i^*(0) \to x^*(0)$, and that, on $[0, T_i]$, $m_i^*(t)\mathrm{d}t \rightharpoonup^* \mathrm{d}\mu^*(t)$ and $x_i^* \to x^*$ a.e..

Now, by applying the necessary conditions of optimality, along the lines of the ones in (Pereira and Silva, 2000), to $(\bar{P}_i)$ with w_i^* as a reference, we obtain a multiplier (λ_i, p_i) satisfying certain Hamiltonian inclusion, boundary and a maximum conditions.

By taking the limits of conveniently extracted subsequences we obtain the conditions stated in the main result of this article. We draw the attention to the fact that the reparameterization technique described in section 2 is used in order to characterize the state trajectory along the arc joining the jump endpoints. Moreover, we note that, relatively to the result in (Pereira and Silva, 2000), additional technical difficulties arise here due to the fact that now the final time T_i is not constant for all terms of the sequence of auxiliary problems but becomes unbounded.

5. CONCLUSIONS

In this article, necessary conditions of optimality in the form of the Hamiltonian inclusions were given for an infinite horizon control problem whose state trajectories are required to be asymptotically stable and the equilibrium point is constrained to a given closed set. Various comments relating the obtained result are included.

A critical role is played by the solution concept which, together with the notion of equilibrium, is introduced for such a class of problems. A sketch of the proof is outlined.

Acknowledgements

The research of the first author is supported by Fundação da Ciência e Tecnologia research projects project and the research of the second author is jointly supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) of Brasil and Fundação de Amparo a Pesquisa do Estado de São Paulo (FAPESP).

6. REFERENCES

Arutyunov, A., D. Karamzin and F. Lobo Pereira (2005*a*). A nondegenerate maximum principle for the impulse control problem with state constraints. *SIAM Journal of Control and Optimization* **43**, 1812–1843.

Arutyunov, A., V. Dykhta and F. Lobo Pereira (2005*b*). Necessary conditions for impulsive nonlinear optimal control problems without a priori normality assumptions. *Journal of Optimization Theory and Applications* **124**, 55–77.

Aubin, Jean-Pierre (2000). Impulse differential equations and hybrid systems: A viability approach. Lecture Notes, University of California at Berkeley.

Baumeister, J. (2001). On optimal control of a fishery. In: *NOLCOS'01 - 5th IFAC Symposium on Nonlinear Control Systems*.

Bressan, A. and F. Rampazzo (1991). Impulsive control systems with commutative vector fields. *J. of Optimization Theory and Applications* **71(1)**, 67–83.

Bressan, A. and F. Rampazzo (1994). Impulsive control systems without commutativity assumptions. *J. of Optimization Theory and Applications* **81**(3), 435–457.

Brogliato, B. (1996). *Nonsmooth Impact Mechanics: Models, Dynamics and Control*. Lect. Notes in Control and Inform. Sci., 220. Springer-Verlag.

Clark, C., F. Clarke and G. Munro (1979). The optimal exploitation of renewable stocks. *Econometrica* **47**, 25–47.

Dykhta, V.A. (1996). Optimality conditions for impulsive processes and it's applications. *Procs of 13th World Congress of the IFAC* pp. 345–350.

Dykhta, V.A. and O.N. Samsonyuk (2000). *Optimal impulse control with applications*. Nauka. Moscow.

Gurman, V.I. (1985). *The extension principle in control problems*. Nauka. Moscow.

Gurman, V.I. and E.V. Ryumina (eds.) (2000). *Modeling the socio-ecology-economic system of a region*. Nauka, (in Russian). Moscow.

Kolokolnikova, G.A. (1996). Discontinuous trajectories optimality in the nonlinear optimal control problems. *Procs of 13th World Congress of the IFAC* pp. 353–357.

Marec, J. P. (1979). *Optimal Space Trajectories*. Elsevier.

Miller, B.M. (1996). The generalized solutions of nonlinear optimization problems with impulse control. *SIAM J. Control and Optimization*.

Motta, M. and F. Rampazzo (1995). Space-time trajectories of nonlinear systems driven by ordinary and impulse controls. *Differential and Integral Equations*.

Pereira, F. L. and G. N. Silva (2000). Necessary conditions of optimality for vector-valued impulsive control problems. *Systems & Control Letters* **40**, 205–215.

Pereira, F. L. and G. N. Silva (2002). Stability for impulsive control systems. *Dynamical Systems: an International Journal* **17**(4), 421–434.

Pereira, F. L. and G. N. Silva (2004). Lyapounov stability of measure driven differential inclusions. *Differential Equations* **40**(8), 1122–1130.

Pereira, F. L. and G. N. Silva (2006). Optimality conditions for asymptotically stable control processes. *Int. J. of Tomography & Statistics* **6**(2), 127–132.

Rishel, R. (1965). An extended pontryagin principle for control systems whose control laws contain measures. *SIAM J. Control* **3**, 191–205.

Rockafellar, R. (1976). Dual problems for arcs of bounded variation. In: *Calculus of Variations and Control Theory*. pp. 155–192. D.L. Russell ed., Academic Press, New York.

Rockafellar, R. (1981). Optimality conditions for convex control problems with nonnegative states and the possibility of jumps. In: *Game Theory and Mathematical Economics*. pp. 339–349. O. Moeschlin, D. Pallaschke eds., North Holland, Amsterdam.

Silva, G. N. and R. B. Vinter (1996). Measure differential inclusions. *J. of Mathematical Analysis and Applications* **202**, 727–746.

Silva, G. N. and R. B. Vinter (1997). Necessary conditions for optimal impulsive control problems. *SIAM J. Control and Optimization* **35**, 1829–1846.

Vinter, R.B. and F.M. Pereira (1988). A maximum principle for optimal processes with discontinuous trajectories. *SIAM J. Control and Optimization* **26**, 205–229.

Warga, J. (1962). Relaxed variational problems. *J. Math. Anal. and Applic.*

Zavalischin, S. and A. Sesekin (1991). *Impulsive processes: models and applications*. Nauka (Science). Moscow.