Hyperelliptic surfaces with $K^2 < 4\chi - 6$

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Abstract

Let S be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves of minimal genus g. We prove that if $K_S^2 < 4\chi(\mathcal{O}_S) - 6$, then g is bounded. The surface S is determined by the branch locus of the covering $S \to S/i$, where i is the hyperelliptic involution of S. For $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, we show how to determine the possibilities for this branch curve. As an application, given $g > 4$ and $K_S^2 - 3\chi(\mathcal{O}_S) < -6$, we compute the maximum value for $\chi(\mathcal{O}_S)$. This list of possibilities is sharp.
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1 Introduction

Our motivation for this work is the following. In [AK] Ashikaga and Konno consider surfaces S of general type with $K_S^2 = 3\chi(\mathcal{O}_S) - 10$. For these surfaces the canonical map is of degree 1 or 2. In the degree 2 case, the canonical image is a ruled surface, thus if S is regular, it has a pencil of hyperelliptic curves. By a result of Xiao [Xi2, Thm. 1] if $\chi(\mathcal{O}_S) \geq 47$, then S has such an hyperelliptic pencil of curves of genus ≤ 4. But for $\chi(\mathcal{O}_S) \leq 46$ this result gives no information (for $\chi(\mathcal{O}_S) = 46$ the slope formula [Xi1, Thm. 2] implies $g \leq 5 \vee g \geq 9$; we show that in this case S has an hyperelliptic pencil of minimal genus $g \leq 10$ and the cases $g = 9$, $g = 10$ do occur). Ashikaga and Konno study only the case $g \leq 4$ (there is an infinite number of possibilities). Nothing is said for the possibilities with $g \geq 5$ and $\chi(\mathcal{O}_S) \leq 46$. A similar situation occurs in [K].

In this paper we study smooth minimal surfaces S of general type which have a pencil of hyperelliptic curves (by *pencil* we mean a linear system of dimension 1). We say that S has such a pencil of *minimal genus* g if it has an hyperelliptic pencil of genus g and all hyperelliptic pencils of S are of genus $\geq g$. We are mainly interested in the case $g > 4$ and $\chi(\mathcal{O}_S)$ small (i.e. where [Xi2, Thm. 1] is not useful).

For S such that $K_S^2 < 4\chi(\mathcal{O}_S) - 6$, we give bounds for the minimal genus g (Theorem 1).

The surface S is the smooth minimal model of a double cover of an Hirzebruch surface $\mathbb{F}_e$ ramified over a curve $\overline{B}$ (which determines S). We prove that if $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then $\overline{B}$ has at most points of multiplicity 8 and we show how to determine the possibilities for $\overline{B}$ (Proposition 2).

As an application, given $g > 4$ and $K_S^2 - 3\chi(\mathcal{O}_S) < -6$, we compute the maximum value for $\chi(\mathcal{O}_S)$; this list of possibilities is sharp (Theorem 3).

The paper is organized as follows. In Section 2 we present the main results of the paper. The hyperelliptic involutions of the fibres of S induce an involution i of S, so in Section 3 we review some general facts on involutions. Since the quotient S/i is a rational surface, a smooth minimal model of S/i is not unique. We make a choice for this minimal model in Section 4 (which is due to Xiao [Xi3]) and we show some consequences of it. Section 5 contains the key result of the paper, which allow us to compute bounds for the minimal genus of the hyperelliptic fibration. This is done via a carefully analysis of the possibilities for the branch locus of the covering $S \to S/i$ considering the restrictions imposed by the choice of minimal model. Finally this is used in Section 6 to prove the main results, stated in Section 2.

Notation

We work over the complex numbers; all varieties are assumed to be projective algebraic. A (-2)-*curve* or *nodal curve* A on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $A^2 = -2$. An $(m_1, m_2, \ldots)$-point of a curve, or point of type $(m_1, m_2, \ldots)$, is a singular point of multiplicity m_1, which resolves to a point of multiplicity m_2 after one blow-up, etc. By *double cover* we mean a finite morphism of degree 2. The rest of the notation is standard in Algebraic Geometry.

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2 Main results

Theorem 1. *Let S be a minimal smooth surface of general type with a pencil of hyperelliptic curves of minimal genus g.*

If $K_S^2 < 4\chi(\mathcal{O}_S) - 6$, then g is not greater than

$$\max\left\{-1+\frac{8\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S)-K_S^2-6}, 1+\frac{8\chi(\mathcal{O}_S)-16}{4\chi(\mathcal{O}_S)-K_S^2-6}, 1+\frac{8\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S)-K_S^2-3}, \frac{3+\sqrt{1+8\chi(\mathcal{O}_S)}}{2}\right\}.$$

Let $B \subset W$ be the branch locus of a double cover $V \to W$, where V and W are smooth surfaces (thus B is also smooth). Let $\rho : W \to P$ be the projection of W onto a minimal model and denote by $\overline{B}$ the projection $\rho(B)$.

Suppose that $\overline{B}$ has singular points $x_1, \ldots, x_n$ (possibly infinitely near). For each x_i there is an exceptional divisor E_i and a number $r_i \in 2\mathbb{N}$ such that

$$E_i^2 = -1,$$
$$K_W \equiv \rho^*(K_P) + \sum E_i,$$
$$B = \rho^*(\overline{B}) - \sum r_i E_i.$$

Notice that r_i is not the multiplicity of the singular point x_i, it is the multiplicity of the corresponding singularity in the *canonical resolution* (see [BHPV, III. 7.]). For example, in the case of a point of type $(2r-1, 2r-1)$ one has $r_1 = 2r-2$ and $r_2 = 2r$.

Since, from Theorem 1, we have a bound for the genus g, we also have a bound for the multiplicities r_i. For the case $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, we prove the result below.

Let N_j be the number of singular points x_i of $\overline{B}$ (possibly infinitely near) such that $r_i = j$. Denote by C_0 and F the negative section and a ruling of the Hirzebruch surface $\mathbb{F}_e$.

Proposition 2. *Let S be a minimal smooth surface of general type with an hyperelliptic pencil of minimal genus $(k-2)/2$. If $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then S is the smooth minimal model of a double cover $S' \to \mathbb{F}_e$ with branch curve $\overline{B} \equiv kC_0 + (ek/2 + l)F$ such that:*

a) $r_i \leq \min\{8,\ k/2+2,\ l-k/2+2\}\ \forall i$

b) $N_4 + N_6 = 15 + K_{S''}^2 - 3\chi(\mathcal{O}_S) - \frac{1}{4}(k-10)(l-10)$

c) $\chi(\mathcal{O}_S) = 1 + \frac{1}{4}(k-2)(l-2) - N_4 - 3N_6 - 6N_8$

where $S'' \longrightarrow S'$ is the canonical resolution.

Proposition 2 can be used to restrict possibilities for $\overline{B}$. We show the following:

Theorem 3. *Let S be a minimal smooth surface of general type with an hyperelliptic pencil of minimal genus $g > 4$. If $K_S^2 < 3\chi(\mathcal{O}_S) - 6$, then $\chi(\mathcal{O}_S)$ is bounded by the number given in the table below (emptiness means non-existence). All these cases do exist.*

g \ $K^2 - 3\chi$	-7	-8	-9	-10	-11	-12	-13	-14	-15	-16
5	61	56	51	46	41	36	31	26	21	16
6	49	46	43	40	37	34	27	28		22
7	42	43	43	35	35	36	28		29	22
8	44	44	45		36		37		29	
9		45		46			37			
10				46						

Remark 4. *This result gives* 3 *examples where Theorem 1 is almost sharp: in the cases* $(g, K^2 - 3\chi) = (10, -10), (9, -13), (8, -15)$ *we get from Theorem 1 that* $\chi \leq 47, 38, 30$, *respectively.*

There is at least one case where Theorem 1 is sharp: a double plane with branch locus a curve of degree 18 *with* 8 *points of multiplicity* 6. *In this case* $\chi = 5$, $K^2 = 8$ *and* $g = 5$.

3 Involutions

Let S be a smooth minimal surface of general type with a (rational) pencil of hyperelliptic curves. This hyperelliptic structure induces an involution (i.e. an automorphism of order 2) i of S. The quotient S/i is a rational surface.

Since S is minimal of general type, this involution is biregular. The fixed locus of i is the union of a smooth curve R'' (possibly empty) and of $t \geq 0$ isolated points $P_1, \ldots, P_t$. Let $p : S \to S/i$ be the projection onto the quotient. The surface S/i has nodes at the points $Q_i := p(P_i)$, $i = 1, \ldots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via p of R'' is a smooth curve B'' not containing the singular points Q_i, $i = 1, \ldots, t$. Let now $h : V \to S$ be the blow-up of S at $P_1, \ldots, P_t$ and set $R' = h^*(R'')$. The involution i induces a biregular involution $\widetilde{i}$ on V whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$. The quotient $W := V/\widetilde{i}$ is smooth and one has a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{g} & S/i \end{array}$$

where $\pi : V \to W$ is the projection onto the quotient and $g : W \to S/i$ is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,$$

are (-2)-curves and $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$.

Set $B' := g^*(B'')$. Since π is a double cover, its branch locus $B' + \sum_1^t A_i$ is *even*, i.e. there is a line bundle L on W such that

$$2L \equiv B := B' + \sum_1^t A_i.$$

4 Choice of minimal model

Part of this section may be found in [Xi3]. We use the notation introduced so far. As above, W is a rational surface.

(*). *Blowing-up, if necessary,* $\mathbb{P}^2$ *at a point, we can suppose that* $W \neq \mathbb{P}^2$.

Thus there is a birational morphism

$$\rho : W \longrightarrow \mathbb{F}_e,$$

where $\mathbb{F}_e$ is an Hirzebruch surface. Let $\overline{B} := \rho(B)$ and consider the double cover $S' \longrightarrow \mathbb{F}_e$ with branch locus $\overline{B}$. If $\overline{B}$ is singular then S' is also singular and S is isomorphic to the minimal smooth resolution of S'.

We can define k and l such that

$$\overline{B} \equiv: kC_0 + \left(\frac{ek}{2} + l\right) F,$$

where C_0 and F are, respectively, the negative section and a ruling of $\mathbb{F}_e$ (thus $C_0^2 = -e$, $C_0F = 1$, $F^2 = 0$). Notice that $\overline{B}^2 = 2kl$ and $K_P\overline{B} = -2k - 2l$.

(*). *Among all the possibilities for the map* ρ*, we choose one satisfying, in this order:*

1) *the degree* k *of* $\overline{B}$ *over a section is minimal;*

2) *the greatest order of the singularities of* $\overline{B}$ *is minimal;*

3) *the number of singularities with greatest order is also minimal.*

Recall that a $(2r-1, 2r-1)$ singularity of $\overline{B}$ is a pair (x_j, x_k) such that x_k is infinitely near to x_j and $r_j = 2r-2$, $r_k = 2r$.

Let

$$r_m := \max\{r_i\}$$

or $r_m := 0$ if $\overline{B}$ is smooth.

By *elementary transformation* over $x_i \in \mathbb{F}_e$ we mean the blow-up of x_i followed by the blow-down of the strict transform of the ruling of $\mathbb{F}_e$ that contains x_i.

The following is a consequence of the choice (*) of the map ρ.

Proposition 5 ([Xi3])**.** *We have:*

a) *If* $k \equiv 0 \pmod 4$, *then* $r_m \leq \frac{k}{2}+2$ *and the equality holds only if* x_m *belongs to a singularity* $\left(\frac{k}{2}+1, \frac{k}{2}+1\right)$. *In this case* $l \geq k+2$ *and all the branches of the singularity are tangent to the ruling of* $\mathbb{F}_e$ *that contains it.*

b) *If* $k \equiv 2 \pmod 4$, *then* $r_m \leq \frac{k}{2}+1$ *and the equality holds only if* x_m *belongs to a singularity* $\left(\frac{k}{2}, \frac{k}{2}\right)$. *In this case* $l \geq k$.

In a similar vein:

Proposition 6. *We have that:*

a) *if* $l = k+2$ *and* $k > 8$, *there are at most two* $\left(\frac{k}{2}+1, \frac{k}{2}+1\right)$*-points.*

b) $l \geq \frac{k}{2}$ *and* $l \geq \frac{k}{2}+r_m-2$;

c) *if* $l = \frac{k}{2}+r_m-2$, *then either:*

- $e=2$, $l=k-2$, *the branch locus* $\overline{B}$ *has a* $\left(\frac{k}{2}-1, \frac{k}{2}-1\right)$*-point and all singularities are of multiplicity* $< \frac{k}{2}$, *or*
- *we can suppose* $e=1$, *the negative section* C_0 *of* $\mathbb{F}_1$ *is contained in* $\overline{B}$, $\overline{B}$ *has a point of multiplicity* r_m *contained in* C_0 *and the remaining singularities are of multiplicity* $< r_m$.

Proof:

a) This is due to Borrelli ([Bo]). Suppose that there are three singularities $(k/2+1, k/2+1)$. The rulings of $\mathbb{F}_e$ through these points are contained in $\overline{B}$ and then $\overline{B}C_0 = l - \frac{ek}{2} \geq 4$ ($\overline{B}C_0$ is even). This implies $e \leq 1$. Making, if necessary, an elementary transformation over one of these points, we can suppose that $e=1$.

Let ρ be as above and E_i, E_i', $i = 1, 2, 3$, be the exceptional divisors corresponding to three singularities $(k/2+1, k/2+1)$ of $\overline{B}$. The general element of the linear system $|\rho^*(4C_0 + 5F) - \sum_1^3 (2E_i + 2E_i')|$ is a smooth and irreducible rational curve C such that $CB < k$. This contradicts the choice (*) of the map ρ.

b) If $r_m > \frac{k}{2}$ then the result follows from Proposition 5. Suppose now $r_m \leq \frac{k}{2}$. We have $\overline{B}C_0 \geq -e$, i.e. $l - \frac{ek}{2} \geq -e$. Therefore if $e \geq 2$, then

$$l \geq k - 2 \geq \frac{k}{2} \quad \text{and} \quad l \geq k - 2 \geq \frac{k}{2} + r_m - 2.$$

When $e = 0$ we obtain immediately $l \geq k$, by the choice of the map ρ, thus $l \geq \frac{k}{2} + r_m$.

If $e = 1$ then $\overline{B}C_0 = l - \frac{k}{2} \geq 0$. Blow-down C_0. We obtain a singularity of order at most $l - \frac{k}{2} + 1$, hence the choice of the minimal model implies $r_m \leq l - \frac{k}{2} + 2$ (notice that the equality happens only if the order of the singularity is $(r_m - 1, r_m - 1)$).

c) Assume that $l = k/2 + r_m - 2$. Proposition 5 implies $r_m \leq k/2$. From $\overline{B}C_0 \geq -e$ we obtain $k/2 + r_m - 2 = l \geq \frac{ek}{2} - e$, thus either $e = 1$ or $e = 2$ and $r_m = k/2$ (notice that $e = 0$ implies $l \geq k$).

In the case $e = 1$ we can, as in the proof of b), contract the section with self-intersection (-1) to obtain a branch curve in $\mathbb{P}^2$ with at most singularities of type $(l - k/2 + 1, l - k/2 + 1)$.

Suppose now that $e = 2$ and there is a point x_i of multiplicity $k/2$. In this case $\overline{B}C_0 = -2$, hence $x_i \notin C_0$. We make an elementary transformation over x_i to obtain the case $e = 1$ also with $l = k - 2$. □

5 Bound of genus

In this section we prove the key result to establish bounds for the minimal genus of the hyperelliptic fibrations.

From [Ri] (cf. also [CM]), we get the following:

Proposition 7. *Let $S'' \to S'$ be the canonical resolution of a double cover $S' \to \mathbb{F}_e$ with branch locus $\overline{B} \equiv kC_0 + (ek/2 + l)F$. Let S be the minimal model of S'' and $t := K_S^2 - K_{S''}^2$. If S is of general type, then:*

a) $\sum(r_i-2)(k-r_i-2)=H$

b) $2l=G+\sum(r_i-2)$,

where
$H=2k^2-k\,(4\chi(\mathcal{O}_S)+t-K_S^2+8)+16\chi(\mathcal{O}_S)+2t-2K_S^2$
and
$G=-2k+4\chi(\mathcal{O}_S)+t-K_S^2+8.$

Proof: From [Ri, Propositions 2 and 3, a)] one gets:

(a) $2kl=-48+12l+12k-8\chi(\mathcal{O}_S)+4K_S^2-4t+\sum(r_i-2)(r_i-4)$

(b) $2k+2l=8+4\chi(\mathcal{O}_S)+t-K_S^2+\sum(r_i-2)$.

The result is obtained replacing (a) by (a)+$(6-k)$(b). □

The next result is a fundamental tool in the proof of Proposition 9 below.

Lemma 8. *Suppose that $k>8$. With the above notation, we have*

a) $2l\leq G+\dfrac{H}{k-r_m-2}$, *and*

b) *if r_m is obtained only from singularities of type (r_m-1,r_m-1), then*

$$2l\leq G+\frac{H}{(r_m-4)(k-r_m)+(r_m-2)(k-r_m-2)}(2r_m-6).$$

Proof: The first statement follows from Proposition 7 and Proposition 5, a).

Next we prove b). By the assumptions, if x_i does not belong to a (r_m-1,r_m-1) singularity, we have $r_i<r_m$. Let $n\geq 1$ be the number of singularities of type (r_m-1,r_m-1) and $s\geq 0$ be the number of singular points x_j of another type. As seen in Section 4, each singularity (r_m-1,r_m-1) corresponds to two infinitely near singular points x_k, x_{k+1} with $r_k=r_m-2$, $r_{k+1}=r_m$. Therefore

$$\sum_{i=1}^{2n+s}(r_i-2)=n(2r_m-6)+\sum_{j=1}^{s}(r_j-2),$$

with $r_j<r_m$. Thus from Proposition 7, b) we get

$$2l=G+n(2r_m-6)+\sum_{j=1}^{s}(r_j-2).$$

By Proposition 7, a),

$$H = n\Big((r_m-4)(k-r_m)+(r_m-2)(k-r_m-2)\Big)+\sum_{j=1}^{s}(r_j-2)(k-r_j-2),$$

hence

$$n = \frac{H-\sum_{j=1}^{s}(r_j-2)(k-r_j-2)}{(r_m-4)(k-r_m)+(r_m-2)(k-r_m-2)}$$

and then

$$2l = G+\frac{H-\sum_{j=1}^{s}(r_j-2)(k-r_j-2)}{(r_m-4)(k-r_m)+(r_m-2)(k-r_m-2)}(2r_m-6)+\sum_{j=1}^{s}(r_j-2). \quad (1)$$

Since $r_j < r_m$, $j = 1,\dots,s$,

$$(r_m-4)(k-r_m)+(r_m-2)(k-r_m-2) \le (2r_m-6)(k-r_j-2).$$

This implies

$$\sum_{j=1}^{s}(r_j-2) \le \sum_{j=1}^{s}\frac{(r_j-2)(k-r_j-2)(2r_m-6)}{(r_m-4)(k-r_m)+(r_m-2)(k-r_m-2)}$$

and the result follows from (1). □

The following proposition will allow us to give bounds for k. Notice that, since $\overline{B}$ is even and $\overline{B}C_0 = l-\frac{ek}{2}$,

$$k \equiv 0 \pmod 4 \implies l \equiv 0 \pmod 2.$$

Proposition 9. *In the conditions of Proposition 7, suppose that $k > 8$.*

If $k \equiv 0 \pmod 4$, one of the following holds:

- **a)** $r_m = k/2+2$, $l = k+2$ *and*
  $(4\chi(\mathcal{O}_S)+t-K_S^2-8)\,k \le 16\chi(\mathcal{O}_S)-16$, *with* $t \ge 2$;
- **b)** $r_m = k/2+2$, $l \ge k+4$ *and*
  $(4\chi(\mathcal{O}_S)+t-K_S^2-8)k^2-16\chi(\mathcal{O}_S)k+32\chi(\mathcal{O}_S) \le 0$, *with* $t \ge 2$;
- **c)** $r_m = k/2$, $l = k-2$ *and*
  $(4\chi(\mathcal{O}_S)+t-K_S^2-4)k^2+(-48\chi(\mathcal{O}_S)-8t+8K_S^2+32)k+$
  $160\chi(\mathcal{O}_S)+16t-16K_S^2-96 \le 0$, *with* $t \ge 1$,
  or
  $(4\chi(\mathcal{O}_S)+t-K_S^2+2)k \le 32\chi(\mathcal{O}_S)+4t-4K_S^2-8$, *with* $t \ge 1$,
  or
  $(4\chi(\mathcal{O}_S)+t-K_S^2-5)k^2+(-48\chi(\mathcal{O}_S)-8t+8K_S^2+44)k+$
  $160\chi(\mathcal{O}_S)+16t-16K_S^2-128 \le 0$, *with* $t \ge 2$;

d) $r_m = k/2$, $l = k + j$, $j \geq 0$, *and*
$(4\chi(\mathcal{O}_S) + t - K_S^2 + 8 + 2j - 2n)\, k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8n$, *with* $n \leq j + 7$, *where* n *is the number of points of multiplicity* $k/2$.

e) $r_m \leq k/2 - 2$ *and*
$k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$, *or*
$(4\chi(\mathcal{O}_S) + t - K_S^2)k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2$.

If $k \equiv 2 \pmod 4$, *one of the following holds:*

f) $r_m = k/2 + 1$ *and*
$(4\chi(\mathcal{O}_S) + t - K_S^2 - 2)k \leq 24\chi(\mathcal{O}_S) + 2t - 2K_S^2 - 20$, *with* $t \geq 1$, *or*
$(4\chi(\mathcal{O}_S) + t - K_S^2 - 8)k^2 + (-32\chi(\mathcal{O}_S) - 4t + 4K_S^2 + 48)k +$
$80\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 96 \leq 0$, *with* $t \geq 2$;

g) $r_m \leq k/2 - 1$ *and*
$k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$, *or*
$(4\chi(\mathcal{O}_S) + t - K_S^2 - 6)\, k \leq 24\chi(\mathcal{O}_S) + 2t - 2K_S^2 - 28$.

Proof: Let H, G be as defined in Proposition 7 and let

$$P_1(l, r_m, G, H, k) := (2l - G)(k - r_m - 2) - H,$$

$$P_2(l, r_m, G, H, k) := (2l-G)\big((r_m-4)(k-r_m)+(r_m-2)(k-r_m-2)\big)-H(2r_m-6).$$

From Lemma 8,

$$P_1 \leq 0 \quad \text{and} \quad P_2 \leq 0.$$

a) Let n be the number of $(k/2+1, k/2+1)$ points. From Proposition 5, b), d), $n = 1$ or 2. From Proposition 7, we have

$$\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),$$

where

$$H' = H - n\left(k/2(k/2 - 4) + (k/2 - 2)^2\right), \quad G' = G + n(k-2)$$

and $r_i \leq k/2$, $\forall i$.

The result follows from

$$P_1(k + 2, k/2, G', H', k) \leq 0.$$

b) From Proposition 5, there are at most $(k/2+1, k/2+1)$ singularities. The inequality

$$P_2(k+4, k/2+2, G, H, k) \leq 0$$

gives the result.

c) Let n be the number of points of multiplicity $k/2$ and m be the number of $(k/2-1, k/2-1)$ singularities. From Proposition 6, c), $n = 0$ or 1.

If $n = 0$, then $r_m = k/2$ implies $m \geq 1$ (thus $t \geq 1$). From

$$P_2(k-2, k/2, G, H, k) \leq 0$$

one gets the first inequality.

Suppose $n = 1$. Notice that, as shown in the proof of Proposition 6, c), the point of multiplicity $k/2$ is obtained from the blow-up of $\mathbb{P}^2$ at a point of type $(k/2-1, k/2-1)$. Hence $t \geq 1$.

Let

$$H' := H - (k/2-2)^2, \quad G' = G + k/2 - 2.$$

If $m = 0$, then

$$P_1(k-2, k/2-2, G', H', k) \leq 0$$

implies the second inequality.

If $m > 0$, then

$$P_2(k-2, k/2, G', H', k) \leq 0$$

gives the third inequality. In this case $t \geq 2$.

d) Let $j := l - k$ and let n be the number of points x_i (possibly infinitely near) such that $r_i = k/2$. From Proposition 7, we have

$$\sum (r_i - 2)(k - r_i - 2) = H' \quad \text{and} \quad 2l = G' + \sum (r_i - 2),$$

where

$$H' = H - n(k/2-2)^2, \quad G' = G + n(k/2-2)$$

and $r_i \leq k/2 - 2$, $\forall i$.

The inequality

$$P_1(k+j, k/2-2, G', H', k) \leq 0$$

gives

$$\left(4\chi(\mathcal{O}_S) + t - K_S^2 + 8 + 2j - 2n\right) k \leq 32\chi(\mathcal{O}_S) + 4t - 4K_S^2 - 8n.$$

It only remains to show that $n \leq j+7$.

One can verify, using the double cover formulas (see e.g. [BHPV]), that $n \geq j+8$ implies $\chi(\mathcal{O}_S) < 1$, except for $n=8$, $l=k$ and $n=10$, $k=12$, $l=14$. We *claim* that in these cases $K_S^2 \leq 0$. This is impossible because S is of general type.

Proof of the claim:
From the double cover formulas one gets that $\chi(\mathcal{O}_S) \leq 2$ and there is at least a (-2)-curve A contained in B, otherwise $K_S^2 \leq 0$. One has

$$B \equiv -\frac{k}{2}K_W + (l-k)\widetilde{F} + \sum \left(\frac{k}{2} - r_i\right) E_i,$$

where $\widetilde{F}$ is the total transform of F and each E_i is an exceptional divisor with self-intersection -1. Since $AB = -2$, $AK_W = 0$, $l \geq k$ and $r_i \leq k/2 \ \forall i$, we have $AE_i < 0$ for some i such that $r_i < k/2$. The only possibility is the existence of a $(3,3)$-point in $\overline{B}$ and $\chi(\mathcal{O}_S) = 1$. But the imposition of such a singularity in the branch locus decreases the self-intersection of the canonical divisor by 1.

e) From Proposition 6, b), $l \geq k/2 + r_m - 2$. Let

$$f(r_m) := P_1(k/2 + r_m - 2, r_m, G, H, k).$$

We have

$$f(r_m) = -2r_m^2 + br_m + c \leq 0,$$

where

$$b = 4\chi(\mathcal{O}_S) + t - K_S^2 - k + 8$$

and

$$c = k^2 - 10k - 8\chi(\mathcal{O}_S) + 24.$$

Suppose that $c = f(0) > 0$ (i.e. $k > 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$). Then $f(r_m)$ has exactly one positive root x. One has

$$4x - b = \sqrt{b^2 + 8c}$$

and $k/2 - 2 \geq r_m \geq x$ implies that

$$(4(k/2-2) - b)^2 \geq b^2 + 8c.$$

This inequality gives the result.

f) Let n be the number of points of type $(k/2, k/2)$.

If $n = 1$, we proceed as in a).

If $n > 1$, the inequality is given by

$$P_2(k, k/2+1, G, H, k) \leq 0.$$

g) It is analogous to the proof of e). □

6 Proof of main results

Proof of Theorem 1:
Consider the parabola given by $f(x) = ax^2 + bx + c$, with $a > 0$. If $f(k) \leq 0$, $f(z) \geq 0$ and $z \geq -b/2a$ (the first coordinate of the vertex), then $k \leq z$.

This fact and Proposition 9 imply that, if $K_S^2 < 4\chi(\mathcal{O}_S) - 6$, one of the following holds:

a) $k \leq \dfrac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6}$ **b)** $k \leq \dfrac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) + t - K_S^2 - 8}$, $t \geq 2$

c) $k \leq 4 + \dfrac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) + t - K_S^2 - 4}$, $t \geq 1$

c') $k \leq 4 + \dfrac{16\chi(\mathcal{O}_S) - 4}{4\chi(\mathcal{O}_S) + t - K_S^2 - 5}$, $t \geq 2$ **d)** $k \leq 4 + \dfrac{16\chi(\mathcal{O}_S) - 32}{4\chi(\mathcal{O}_S) - K_S^2 - 6}$

e) $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$ **e')** $k \leq 4 + \dfrac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2}$

f) $k \leq 2 + \dfrac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 1}$ **f')** $k \leq 2 + \dfrac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) + t - K_S^2 - 8}$, $t \geq 2$

g) $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$ **g')** $k \leq 2 + \dfrac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2}$

We want to show that k is not greater than

$$\max\left\{\tfrac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S)-K_S^2-6}, 4 + \tfrac{16\chi(\mathcal{O}_S)-32}{4\chi(\mathcal{O}_S)-K_S^2-6}, 4 + \tfrac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S)-K_S^2-3}, 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}\right\}.$$

The result follows easily. Just notice that

$$4\chi(\mathcal{O}_S) - K_S^2 - 6 \leq 8 \Longrightarrow 2 + \frac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6} \leq \frac{16\chi(\mathcal{O}_S)}{4\chi(\mathcal{O}_S) - K_S^2 - 6}$$

and

$$4\chi(\mathcal{O}_S) - K_S^2 - 6 \geq 8 \Longrightarrow 2 + \frac{16\chi(\mathcal{O}_S) - 16}{4\chi(\mathcal{O}_S) - K_S^2 - 6} \leq 4 + \frac{16\chi(\mathcal{O}_S) - 32}{4\chi(\mathcal{O}_S) - K_S^2 - 6}.$$

□

Proof of Proposition 2:
Let (α), (β) be the equations of Proposition 7, a), b), respectively. One has that $[(\alpha) + (k-10)(\beta)]/8$ is equivalent to

$$\frac{1}{8}\sum(r_i - 2)(8 - r_i) = 15 + K_S^2 - t - 3\chi(\mathcal{O}_S) - \frac{1}{4}(k-10)(l-10) \qquad (2)$$

and (β)+(2) is equivalent to

$$\chi(\mathcal{O}_S) = 1 + \frac{1}{4}(k-2)(l-2) - \frac{1}{8}\sum r_i(r_i - 2). \qquad (3)$$

Now it suffices to show that $r_m \leq 8$.

Suppose that $K_S^2 < 3\chi(\mathcal{O}_S) - 6$.

From [Xi2, Theorem 1] one gets that if $\chi(\mathcal{O}_S) \geq 54$, then S has a pencil of hyperelliptic curves of genus ≤ 6. In this case $k \leq 14$, thus $r_m \leq k/2 + 2$ implies $r_m \leq 8$.

From the proof of Theorem 1 we obtain that if $\chi(\mathcal{O}_S) \leq 31$, then one of the possibilities below occur. In all cases $r_m \leq 8$.

a) b) $k < 16$, $r_m < 8$;

c) c') d) $k \leq 18$, $r_m = k/2 \leq 8$;

e) $k \leq 20$, $r_m \leq k/2 - 2 \leq 8$; **e')** $k \leq 16$, $r_m \leq k/2 - 2 \leq 6$;

f) $k \leq 14$, $r_m = k/2 + 1 \leq 8$; **f')** $k \leq 16$, $r_m = k/2 + 1 \leq 8$;

g) $k \leq 18$, $r_m \leq k/2 - 1 \leq 8$; **g')** $k \leq 14$, $r_m \leq k/2 - 1 \leq 6$.

Suppose now that $32 \leq \chi(\mathcal{O}_S) \leq 53$. From Theorem 1 we get that $k \leq 18$ or $k \leq 5 + \sqrt{1 + 8\chi(\mathcal{O}_S)}$. In this last case $r_m \leq k/2 - 1$ (see Proposition 9 e), g)). Thus we have $r_m \leq 18/2 + 2$ or $r_m \leq 24/2 - 1$. Since r_m is even, $r_m \leq 10$.

Let N_j be the number of points x_i such that $r_i = j$. We have

$$\sum(r_i - 2) \geq 8N_{10} + 6N_8$$

and, from (2),

$$8N_{10} \geq (k-10)(l-10) - 32.$$

Using Proposition 7, b) and the assumption $\chi(\mathcal{O}_S) \geq 32$, we obtain

$$2l + 2k \geq 15 + (k-10)(l-10) + 6N_8.$$

This is equivalent to

$$(k-12)(l-12) \leq 29 - 6N_8. \tag{4}$$

Suppose $r_m = 10$. Then Propositions 5 and 6 give two possibilities:

· $k = 16$, $l \geq k+2 = 18$, there is a singularity of type $(9,9)$ ($N_8 \geq 1$);

· $k \geq 18$, $l \geq k/2 + r_m - 2 \geq 17$.

Both cases contradict (4). We conclude that $r_m \leq 8$. □

Proof of Theorem 3:
First we claim that if A is a (-2)-curve contained in B, the image $\overline{A}$ of A in $\mathbb{F}_e$ does not intersect a negligible singularity of $\overline{B}$, unless $\overline{A}$ is the negative section of $\mathbb{F}_1$ and the only singularity of $\overline{B}$ is a double point in C_0 (this corresponds to a smooth branch curve in $\mathbb{P}^2$). In fact otherwise there is a (-1)-curve E such that $AE = 1$ or 2. If $AE = 1$, then $A + E$ can be contracted to a smooth point of the branch curve $\overline{B} \subset \mathbb{F}_e$. This is impossible because the canonical resolution blows-up only singular points of $\overline{B}$. Suppose $AE = 2$. The inverse image of A is a (-1)-curve which contracts to a smooth point of S. The inverse image of E is then contracted to a curve $\widehat{E}$ with arithmetic genus 1 and $\widehat{E}^2 = 2$. We obtain from the adjunction formula that $K_S\widehat{E} = -2$, which is a contradiction because S is of general type.

Recall that $t := K_S^2 - K_{S''}^2$. The following holds:

(1) $l \geq k/2$
(Because $\overline{B}C_0 = l - ek/2 \geq -e$ and $\overline{B}C_0$ is even.);

(2) $l = k/2 \Longleftrightarrow (t = 2 \wedge N_4 = N_6 = N_8 = 0)$
(In this case $e = 1$ and $\overline{B}C_0 = 0$.);

(3) $l = k/2 + 2 \Longrightarrow (N_6 = N_8 = 0 \wedge t \geq N_4 \wedge (t = N_4 \vee N_4 > 1))$;
(If $N_4 \neq 0$, this corresponds to a branch curve in $\mathbb{P}^2$ with N_4 points of type $(3,3)$ (see Proposition 6, c)).);

(4) $l = k - 2 \wedge t = 0 \Longrightarrow k/2$ even;
(As in (1), $l \geq ek/2 - e$, hence $e \leq 2$. We have $e = 1$ because $t = 0$, thus l even implies $k/2$ even.);

(5) $l < k-2 \Longrightarrow l - k/2$ even;
(As in (1), $l \geq ek/2 - e$, thus $e = 1$ and then $l - k/2 = \overline{B}C_0$ is even.)

(6) $t = 1 \wedge N_4 = N_6 = N_8 = 0 \Longrightarrow l = k-2$.
(If there are only negligible singularities, $t = 1$ is only possible if the negative section of $\mathbb{F}_2$ is an isolated component of the branch locus.)

For given values of $K_S^2 - 3\chi(\mathcal{O}_S)$ and k, we want to choose the solution of the equation given in Proposition 2, b) which maximizes the value of $\chi(\mathcal{O}_S)$, given by the equation in Proposition 2, c). We can assume $N_6 = N_8 = 0$.

It suffices to compute the numerical possibilities for Proposition 2, b), c) which satisfy conditions $(1), \ldots, (6)$. We note the following:
since $k \geq 12$, [Xi2, Thm. 1] implies $\chi(\mathcal{O}_S) \leq 69$, then Theorem 1 gives $k \leq 28$;
$l \geq k/2$, $k \geq 12$ and (2) imply $-7 \geq K_S^2 - 3\chi(\mathcal{O}_S) \geq -18 + t + N_4$, thus $K_S^2 - 3\chi(\mathcal{O}_S) \geq -18$, $t \leq 11$ and $N_4 \leq 11$.

A simple algorithm is available at
`http://home.utad.pt/~crito/magma_code.html`

The existence is easy to verify. All cases can be constructed as double covers of $\mathbb{P}^2$, $\mathbb{F}_0$, $\mathbb{F}_1$ or $\mathbb{F}_2$. The table below contains information about l or the degree of the branch curve in $\mathbb{P}^2$ and about the singularities of the branch curve, if any.

g \ $K^2 - 3\chi$	-7	-8	-9	-10	-11
5	$\mathbb{F}_0$, $l = 26$	$\mathbb{F}_0$, $l = 24$	$\mathbb{F}_0$, $l = 22$	$\mathbb{F}_1$, $l = 20$	$\mathbb{F}_0$, $l = 18$
6	$\mathbb{F}_0$, $l = 18$	$\mathbb{F}_1$, $l = 17$	$\mathbb{F}_0$, $l = 16$	$\mathbb{F}_1$, $l = 15$	$\mathbb{F}_0$, $l = 14$
7	$\mathbb{F}_1$, $l = 14$, $(3,3)$	$\mathbb{F}_2$, $l = 14$	$\mathbb{F}_1$, $l = 14$	$\mathbb{F}_1$, $l = 12$, $(3,3)$	$\mathbb{F}_1$, $l = 12$, (4)
8	$\mathbb{F}_1$, $l = 13$, $(3,3)$	$\mathbb{F}_1$, $l = 13$, (4)	$\mathbb{F}_1$, $l = 13$		$\mathbb{P}^2$, 20, $(3,3)$
9		$\mathbb{P}^2$, 22, $(3,3)$		$\mathbb{F}_1$, $l = 12$	
10				$\mathbb{P}^2$, 22	

g \ $K^2 - 3\chi$	-12	-13	-14	-15	-16
5	$\mathbb{F}_0$, $l = 16$	$\mathbb{F}_0$, $l = 14$	$\mathbb{F}_0$, $l = 12$	$\mathbb{F}_1$, $l = 10$	$\mathbb{F}_1$, $l = 8$
6	$\mathbb{F}_1$, $l = 13$	$\mathbb{F}_1$, $l = 11$, (4)	$\mathbb{F}_1$, $l = 11$		$\mathbb{F}_1$, $l = 9$
7	$\mathbb{F}_1$, $l = 12$	$\mathbb{P}^2$, 18, $(3,3)$		$\mathbb{F}_1$, $l = 10$	$\mathbb{P}^2$, 16
8		$\mathbb{F}_1$, $l = 11$		$\mathbb{P}^2$, 18	
9		$\mathbb{P}^2$, 20			
10					

□

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