FEP WORKING PAPERS
FEP WORKING PAPERS

RESEARCH WORK IN PROGRESS

N. 245, JULY 2007

OPTIMAL INVESTMENT TIMING USING MARKOV JUMP PRICE PROCESSES

FERNANDO A. C. C. FONTES*

DALILA B. M. M. FONTES**

* DEPARTAMENTO DE MATEMÁTICA PARA A CIÊNCIA E TECNOLOGIA, UNIVERSIDADE DO MINHO
** LIAAD AND FACULDADE DE ECONOMIA, UNIVERSIDADE DO PORTO

U.PORTO

FEP FACULDADE DE ECONOMIA UNIVERSIDADE DO PORTO

OPTIMAL INVESTMENT TIMING USING MARKOV JUMP PRICE PROCESSES*

Fernando A. C. C. Fontes
Depart. de Matemática para a Ciência e Tecnologia,
Universidade do Minho
4800-058 Guimarães, Portugal
Tel:351-253 510 400, fax:351-253 510 401 ffontes@mct.uminho.pt

Dalila B. M. M. Fontes
Faculdade de Economia da Universidade do Porto
Rua Dr. Roberto Frias, 4200-464 Porto, Portugal
Tel:351-225 571 100, fax:351-225 505 050 Email:fontes@fep.up.pt

ABSTRACT

In this work, we address an investment problem where the investment can either be made immediately or postponed to a later time, in the hope that market conditions become more favourable. In our case, uncertainty is introduced through market price. When the investment is undertaken, a fixed sunk cost must be paid and a series of cash flows are to be received. Therefore, we are faced with an irreversible investment. Real options analysis provides an adequate framework for this type of problems by recognizing these two characteristics, uncertainty and irreversibility, explicitly. We describe algorithmic solutions for this type of problems by modelling market prices evolution by Markov jump processes.

Keywords: Irreversible investment, optimal stopping, dynamic programming, Markov jump processes.

J.E.L. Classification. C61, G31.

*Research supported by FCT Project POCTI/MAT/61842/2004

1 INTRODUCTION

The timing of market entry decisions is a central concern for business strategy. In a context of uncertainty and irreversibility the entry timing decision becomes even more important since by making a commitment we loose the option of waiting for a better opportunity.

The problem that we discuss here is deciding on the optimal timing for an irreversible investment. A firm has an investment opportunity in the exploration of a natural resource, such as oil. When the investment decision is made, the firm pays a sunk fixed cost and from then on it will start to receive the incomes of the exploration over a long period, considered infinite. This problem falls within the real options category, since at each point in time we have the option to invest or to wait for a more appropriate time and also because the investments discussed here typically involve real assets rather than financial assets (see e.g. (Dixit and Pindyck 1996 , Trigeorgis 1996)).

The income received at each instant of time, the price of the resource, is considered to evolve according to a known stochastic dynamic model. Therefore, the expected total income will then be the integral over time of the expected discounted incomes received at each instant, i.e. the expected discounted prices of the resource integrated from the time of investment to infinity.

Usually, a simple static Net Present Value analysis of this problem is made. A comparison is made between the fixed cost of the investment and the total expected income. Questions like “Will the expected total income be greater then the investment cost?”, “If yes, what is the probability of that happening?”, and “How many years will it take for the investment to be paid?”, are typically answered by such analysis.

Here, we use a real options approach to this problem focusing on the optimal time for investment. Even if the answers to the previous questions would lead us to invest, the expected evolution of the resource price could be such that the conditions in the future would be even better. In this scenario, although a decision to invest now is a profitable one, the decision to wait and invest in the future would be more valuable, as long as the opportunity to invest remains available. In the real options literature this is known as the option to delay.

The question addressed in this work is the instant of time at which the best conditions for investment occurs, i.e. the optimal investment time. To answer this, instead of comparing the present expected total income just with the sunk cost of investment, we have to compare the present expected total income with the expected total incomes of all future times and choose the maximum one. This problem, as we will see, falls within a category of problems known in the literature as “Optimal Stopping Problems” (see e.g. (Shiryayev 1978 , Peskir and Shiryayev 2006)).

We note that the solution we seek for is not a time value, but rather a policy. That is, a rule that specifies under which conditions we should invest. In fact, the results obtained state that the investment should be undertaken once the price reaches a certain threshold value. However, for a specific price value, it is possible to obtain an estimate of the waiting time till investment.

This problem can be addressed using a set of known methodologies to analyse and price American options. The most used are the ones based on the Black-Scholes model (Black and Scholes 1973) for infinite horizon problems (perpetual American options) with prices following geometric Brownian motion, as well as the discrete-time models such as the binomial and other lattice-based models (Cox, Ross, and Rubinstein 1979). In the latter set of methodologies, most literature addresses the finite horizon case, since in opposition to the Black-Scholes based methods, these methodologies are more complicated in the infinite horizon case. Notable exceptions are the recent works of Boyarchenko and Levendorskiĭ (Boyarchenko and Levendorskiĭ 2007a, Boyarchenko and Levendorskiĭ 2007b) proposing an elegant framework to address the infinite horizon case when prices follow binomial/trinomial lattices or a random walk.

Here we consider that the evolution of the price is modelled as a Markov jump process. The randomness of the process is considered to occur not in a continuous way, but rather at specific instants of time. This is particularly suitable when the main random factors that affect the price are point events, occurring at random times and with random intensity. There are several examples of price changes in natural resources caused by financial, political or environmental events. Oil prices are a typical example of a variable whose value is affected by abnormal news and their sudden variations are more adequately modelled by jump processes. A possible application for the model and algorithms here developed is the decision to make an investment in the exploration of an oil field.

In this work, we discuss 3 solution methodologies, leading to 3 different algorithms, for the optimal stopping problem modelling an irreversible investment decision when the prices follow a Markov jump process. For illustration purposes, though, we start by discussing a discrete time Markov model.

2 PRELIMINARIES

Consider a Markov jump process $\{X_t; t \in \mathbb{R}^+\}$ with transition function P_t, generator A, and state-space E (countable). Let α be a nonnegative number. The *α-potential of the function g* (bounded, non-negative, and defined on E) for the process X is the expected value of the total discounted return of the rewards g, given by

$$U^\alpha g(i) = E_i\left[\int_0^\infty e^{-\alpha t} g\left(X_t\right) dt\right],$$

and the matrix U^α can be computed as

$$U^\alpha(i,j) = \int_0^\infty e^{-\alpha t} P_t(i,j) dt, \quad \forall i,j \in E.$$

Proposition 2.1 *Let g be a bounded function and $\alpha > 0$ then the vector $u = U^{\alpha} \cdot \underline{g}$ is the unique solution to the system of linear equations*

$$(\alpha I - A)\, u = g.$$

Proposition 2.2 *For any stopping time T*

$$U^{\alpha} g(i) = E_i \left[\int_0^T e^{-\alpha t} g\left(X_T\right) dt \right] + E_i \left[e^{-\alpha T} U^{\alpha} g\left(X_T\right) \right].$$

(See (Çinlar 1975) for the proof of these propositions.)

Let f be a finite-valued function defined on E, the state-space of a Markov process X_t with semigroup P_t, and let $r \geq 0$. Function f is said to be an *r-excessive function* for the process X if for all $x \in E$ and for all $t \geq 0$

$$f(x) \geq e^{-rt} P_t f(x),$$

or equivalently

$$r f(x) - A f(x) \geq 0.$$

A function which is 0-excessive is simply called excessive.

Theorem 2.3 *The value function V is the minimal r-excessive function that majorises g.*

(See (Shiryayev 1978) chapter 3, for a proof.)

Hence for a finite space E, V can be computed by linear programming as

$$\begin{array}{l} \min_V \sum_{j \in E} V(j) \\ \text{s.t.} \quad (A - rI)V(x) \leq 0 \\ \qquad -V(x) \leq -h(x) \\ \text{for all } x \in E, \end{array}$$

where the second set of constraints is active (satisfied as equality) when x belongs to the stopping set; otherwise the first set of constraints is active.

Hence, these constraints can also be written as the following Variational Inequality

$$\max \{AV(x) - rV(x),\ h(x) - V(x)\} = 0, \ \ \forall x \in E.$$

3 DISCRETE TIME MODEL

The problem here is to decide the best period of time to invest in the extraction of a resource given its actual price and a stochastic model of the price evolution. For simplicity of exposition, it is assumed that the investment can be done immediately, once decided, and the corresponding income starts on the next time period.

The prices are considered to evolve according to

$$p_{k+1} = (1 + w_k)p_k, \tag{1}$$

where w_k values are uncorrelated, belonging to a finite ordered set of values Ω={$\Omega_{\mathrm{x}}, ..., \Omega_{\mathrm{N}}$} distributed according to $F(\Omega_i)$=Prob($w_k \leq \Omega_i$) with corresponding mean $Ew_k = m$ and density $f(\Omega_i)$=Prob($w_k = \Omega_i$).

Our goal is to determine the investment timing that maximises the net expected discounted profit

$$\max_k \left\{ E \left[\sum_{i=k+1}^{\infty} (1+r)^{-i} p_i) - (1+r)^{-k} I \,|p_0, ..., p_k \right] \right\},$$

where I is the fixed cost of investment and r the interest rate.

Alternatively, we can write a dynamic programming recursion (see e.g (Bertsekas, D.P. 1976)) to decide at each time period whether to invest now or wait at least one more period.

$$\max J_0(p_0)$$

where

$$J_k(p_k) = \max \left\{ E \left[\sum_{i=k+1}^{\infty} (1+r)^{-i} p_i) - (1+r)^{-k} I \right], E(J_{k+1}(p_{k+1})) \right\},$$

(invest) (wait)

or using the value function at current prices

$$V_K(p_k) = (1+r)^k J_k(p_k),$$

$$V_k(p_k) = \max\left\{ \underset{\text{(invest)}}{E\left[\sum_{i=1}^{\infty}(1+r)^{-i}p_i) - I\right]}, \underset{\text{(wait)}}{(1+r)^{-1}E(V_{k+1}(p_{k+1}))} \right\}.$$

For the infinite time horizon case, the case in which we are interested in, $V_k(p) = V(p)$ for all k, and so the value function V satisfies

$$V(p_k) = \max\left\{ \underset{\text{(invest)}}{E_{p_k}\left[\sum_{i=1}^{\infty}(1+r)^{-i}p_i) - I\right]}, \underset{\text{(wait)}}{(1+r)^{-1}E_{p_k}(V(p_{k+1}))} \right\},$$

which falls within the Optimal Stopping Problems category.

3.1 Solution method

Define $h(p)$ as the net expected return of investing now, i.e. the sum of the discounted incomes from now till infinity net of the investment cost

$$\begin{aligned} h(p_k) =& E\left(\sum_{i=1}^{\infty}(1+r)^{-i}p_{i+k}\right) - I \\ =& \sum_{i=1}^{\infty}\left(\frac{1+m}{1+r}\right)^{-i} p_k - I \\ =& \frac{1+m}{r-m}p_k - I, \qquad \text{if } r > m \end{aligned}$$

and $\upsilon(p)$ as the net expected return if we wait at least one unit of time

$$\upsilon(p_k) = (1+r)^{-1}E(V_{k+1}(p_{k+1})).$$

At time k, the decision is to invest if $h(p_k) > \upsilon(p_k)$, to wait if $h(p_k) < \upsilon(p_k)$, and either decision is optimal if $h(p_k) = \upsilon(p_k)$. If in this last case we choose to invest, the decision rule is to invest if and only if $h(p_k) \geq \upsilon(p_k)$, i.e.

$$\text{iff } \frac{1+m}{r-m}p_k - I \geq \upsilon(p_k),$$

$$\text{iff } p_k \geq \frac{r-m}{1+m}(\upsilon(p_k)+I),$$

$$\text{iff } p_k \geq p^* \text{ where } p^* \text{ satisfies } p^* = \frac{r-m}{1+m}(\upsilon(p^*)+I).$$

A graphical interpretation of the decision rule is given in Fig. 1.

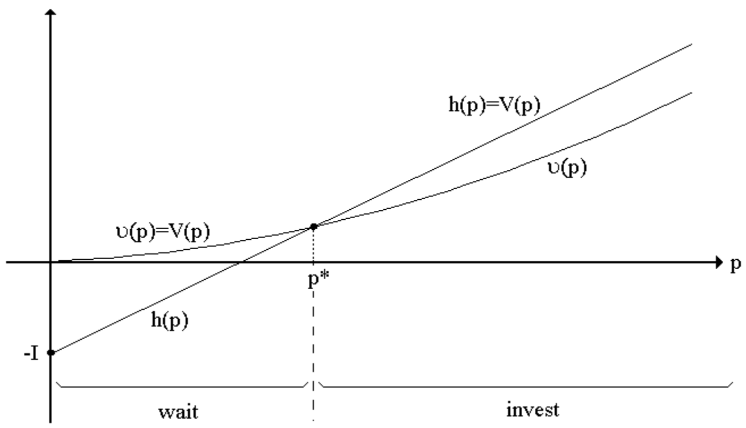

Figure 1: Optimal stopping graphical interpretation.

Note that for p greater than p^* (inside the stopping region), $h(p)$ is greater than $\upsilon(p)$ because we are in fact loosing an opportunity if we do not make the correct decision even if it is only for one instant of time. The value function V coincides with $\upsilon(p)$ for p less than or equal to p^* and coincides with $h(p)$ for p greater than or equal to p^*.

To achieve the solution it remains only to compute $\upsilon(p)$.

3.1.1 Computation of υ(p)

We know that $\upsilon(p_k)$ satisfies

$$\upsilon(0) = 0,$$
$$\upsilon(p*) = h(p*),$$
$$\upsilon(p) = (1+r)^{-1}\mathrm{E}\{V[(1+w)p]\} = (1+r)^{-1}\ \mathrm{E}\{\max\{h[(1+w)p], \upsilon[(1+w)p]\}\}.$$

Let us consider separately the cases when (a)$(1+w)p > p*$, and when (b) $(1+w)p \leq p$*.

a) Case $(1+w)p > p* \Leftrightarrow w > \frac{p*}{p} - 1$

Let $N1$ be such that $\Omega_{N1} = \min\{\Omega_i \in \Omega : \Omega_i > \frac{p*}{p} - 1\}$, i.e. $N1$ is the index of the first term in Ω that satisfies the condition of being in this case.

$$v^a(p) = (1+r)^{-1} E\{h[(1+w)p] | w > \Omega_{N1}\} =$$

$$= (1+r)^{-1} \left(\frac{1+m}{r-m} E[(1+w)p | w > \Omega_{N1}] - I \right) =$$

$$= (1+r)^{-1} \left(\frac{1+m}{r-m} \frac{\sum_{i=N1}^{N} (1+\Omega_i) f(\Omega_i)}{\sum_{i=N1}^{N} f(\Omega_i)} p - I \right).$$

And the probability of being in this case is

$$\sum_{i=N1}^{N} f(\Omega_i) = 1 - \text{Prob}\left(w \leq \frac{p*}{p} - 1 \right) = 1 - F\left(\frac{p*}{p} - 1 \right).$$

b) Case $(1+w)p \leq p* \Leftrightarrow w \leq \frac{p*}{p} - 1$

Let $N2$ be such that $\Omega_{N2} = \max\left\{\Omega_i \in \Omega : \Omega_i \leq \frac{p*}{p} - 1\right\}$. i.e. $N2$ is the index of the last term in Ω that satisfies the condition of being in this case.

$$v^b(p) = (1+r)^{-1} E\{v[(1+w)p] | w \leq \Omega_{N1}\} =$$

$$= (1+r)^{-1} \frac{\sum_{i=1}^{N2} \nu\left[(1+\Omega_i)p\right] f(\Omega_i)}{\sum_{i=1}^{N2} f(\Omega_i)}.$$

And the probability of being in this case is

$$\sum_{i=1}^{N2} f(\Omega_i) = \text{Prob}\left(w \leq \frac{p*}{p} - 1\right) = F\left(\frac{p*}{p} - 1\right).$$

Finally, using the Bayes rule, $\upsilon(p)$ is given by

$$(1+r)\upsilon(p) = \left[1 - F\left(\frac{p*}{p} - 1\right)\right]\upsilon^a(p) + F\left(\frac{p*}{p} - 1\right)\upsilon^b(p),$$

$$(1+r)\upsilon(p) = \frac{I+m}{r-m}\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)p - I\left[1 - F\left(\frac{p*}{p} - 1\right)\right] + \sum_{i=1}^{N2}\upsilon\left[(1+\Omega_i)p\right]f(\Omega_i).$$

3.1.2 Computing an estimate for the Optimal "waiting" time

Recall that the original problem was

$$\max_k \left\{E\left[\sum_{i=k+1}^{\infty}(1+r)^{-i}p_i) - (1+r)^{-k}I|p_0, ..., p_k\right]\right\}.$$

In the case that at the present time k, the decision obtained is to wait, we may wish to estimate (given the present data $p_0, ..., p_k$) for how long we should wait before investing. The answer is given by

$$\tau = \min\{t \geq 0 : E[h(p_{k+t})|p_0, ..., p_k] \geq E[\upsilon(p_{k+t})|p_k]\} =$$

$$= \min\{t \geq 0 : E_{p_k}[h(p_{k+t})] \geq \quad E_{p_k}[\upsilon(p_{k+t})]\},$$

which, as we have seen, is equivalent to

$$\tau = \min\{t \geq 0 : E[p_{k+t} \quad |p_0, ..., p_k] \geq p^*\}.$$

As $E(p_{k+t}|p_0, ..., p_k) = E(p_{k+t}|p_k)$=(1+$m)^t p_k$, we get

$$\tau = \min\{t \geq 0 : (1+m)^t p_k \geq p^*\},$$

and so $\tau = \min\left\{t \geq 0 : t \geq \dfrac{\ln(p^*/p_k)}{\ln(1+m)}\right\}$.

Hence for each value of p_k, the corresponding stopping time can be directly determined as $\tau = \dfrac{\ln(p^*/p_k)}{\ln(1+m)}$.

3.1.3 Special Case

In the special case where the prices are monotonically increasing (i.e. Prob(w >0)=1) we have that $F\left(\frac{p*}{p} - 1\right) = 0$ for $p = p$*, and so the expression for $\upsilon(p)$ simplifies to

$$\upsilon(p^*) = (1+r)^{-1}\left\{\frac{1+m}{r-m}\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)p^* - I\right\}.$$

Since $\upsilon(p^*) = h(p^*)$ and $\sum\limits_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)$=1+$m$, we obtain

$$(1+r)^{-1}\left[\frac{(1+m)^2}{r-m}p^* - I\right] = \frac{1+m}{r-m}p^* - I.$$

Finally, after some algebra we get

$$p^* = \frac{rI}{1+m}. \tag{2}$$

So p^*can be determined explicitly in closed form for this special case. In the general case, this equation can be used as an educated initial guess for p^* in an iterative procedure.

3.2 Algorithm

An algorithm to compute the solution to this problem is the following:

1. Iteration index K=1

2. Set initial guess for $p*^1$ (e.g. by eq. (2))

3. Initialise $\upsilon(p)$ as straight lines

For $p =$0...$p\star^1$

$$\upsilon(p) = \frac{h(p^{*1})}{p^{*1}}p,$$

For $p = p\star^1$...P_{max}

$$\upsilon(p) = h(p).$$

4. Update estimate of $\upsilon(p)$

For p=0..P_{max}

$$\upsilon(p) = (1+r)^{-1}\left\{\frac{1+m}{r-m}\sum_{i=N1}^{N}(1+\Omega_i)f(\Omega_i)p - I\left[1 - F\left(\frac{p*}{p} - I\right)\right] + \right.$$
$$\left. + \sum_{i=1}^{N2}\upsilon\left[(1+\Omega_i)p\right]f(\Omega_i)\right\}.$$

5. Stopping condition

If $\max_p |\upsilon(p) - \upsilon_{OLD}(p)| < \varepsilon$ then STOP.

6. Update estimate of $p*$

$$p*^{k+1} = \min\{p : \upsilon(p) = h(p)\}.$$

7. k=k+1; GOTO 4.

3.3 Alternative Solution Methods

3.3.1 Reformulating as a finite state Markov chain model

As the evolution of the prices considered before is a process with independent increments, it satisfies the Markov property. In order to get a finite state-space (the previous case had not a finite state-space) we first apply logarithms to both sides of equation (1) getting

$$\log P_{k+1} = \log P_k + \log\left(1 + w_k\right).$$

Define

$$X_k = \log P_k,$$

we have now a countable state-space for the process $\{X_k,\ k \in \mathbb{N}\}$. By setting upper and lower bounds for the state-space, $X_{\max}$ and $X_{\min}$, which we define as $x_1 = X_{\min}$ and $x_n = X_{\max}$, we get a finite state-space $X = \{x_1, x_2, ..., x_n\}$. Naturally, by clipping the state-space in this way, we get a different process, but in a real problem application, the majority of the possible prices will certainly be within a bounded interval with high probability and so the difference will not be meaningful.

This being case, the new gain function $h(x)$ is defined as

$$h(x) = E\left[\sum_{i=1}^{\infty} (1+r)^{-i} e^{X_i}\right] - I,$$

which is the $(1 + r)$-potential of the exponential function minus the investment cost I.

Defining

$$\begin{aligned} \underline{h} &= \left[h\left(x_1\right), ..., h\left(x_n\right)\right]^T, \\ \underline{f} &= \left[\exp\left(x_1\right), ..., \exp\left(x_n\right)\right]^T, \\ \underline{I} &= I * \left[1, 1, ..., 1\right]^T, \end{aligned}$$

and I_n as the identity matrix of dimension n.

By Proposition 2.1

$$\underline{h} = \left(I_n - (1+r)Q\right)\underline{f} - \underline{I}.$$

Thus, the problem

$$V\left(X_0\right) = \max_k E\left[(1+r)^{-k} h\left(X_k\right)\right],$$

can be solved for this finite state-space case, simply by calculating V for all $x \in X$, such that V satisfies

$$V(x) = \max\left\{h(x), TV(x)\right\},$$

where

$$TV\left(X_k\right) = (1+r)^{-1} EV\left(X_{k+1}\right) = (1+r)^{-1} \sum_{x_j \in X} Q\left(X_k, x_j\right) V\left(x_j\right).$$

Alternative Solution Method 1 The function V can be computed iteratively as

$$\begin{cases} V_0(x) = h(x) \\ V_{m+1}(x) = \max\{h(x),\, TV_m(x)\}, \end{cases}$$

and $V_m \to V$ as $m \to \infty$.

Alternative Solution Method 2 Knowing that the value function V is the minimum $(1+r)$-excessive function that majorises h, it can be computed by linear programming as

$$\begin{array}{ll} \min & \sum_{x \in X} V(x) \\ s.t. & V(x) \geq h(x) \\ & V(x) \geq (1+r)^{-1} TV(x) \\ & V(x) \geq 0. \end{array}$$

These alternative solution methods are adapted and implemented for the case of the Markov jump processes, as developed in the next section.

4 Markov Jump Model

In this section, we consider the price to be modelled as a Markov jump process. Such process is able to capture price jumps that are induced by the occurrence of rare events typically related to the arrival of new information. This information might be of different nature: technological (innovations)(Greenwood, Hercowitz , and Krussell 1997 , Galor Tsiddon 1997 , Grenadier and Weiss 1997), competition (new product or competitor entry)(Ghemawat Kennedy 1999 , Bresnahan and Greenstein 1999), political (wars, expropriation, change in legislation) (Clark 1997 , Wagner 1997), natural conditions (natural hazards, hurricanes)or even various types of rare events simultaneously (Martzoukosand Trigeorgis 2002).

Consider the resource price p_t to evolve according to

$$p_t = e^{X_t},$$

where X_t is a piecewise constant Markov process, taking values in a finite ordered set $X = \{x_1, x_2, ..., x_n\}$, having a jump rate λ and with the post-jump location defined by the transition probabilities $Q(i,j) = \text{Prob}\,(X_T = x_j | X_T = x_i)$ in which $Q(i,i) = 0$ for i=0...N.

For a constant jump rate λ the transition function is given by

$$P_t = \sum_{n=0}^{\infty} Q^n \frac{e^{-\lambda t}(\lambda t)^n}{n!}\ ,\ t \geq 0\,.$$

If the jump rate λ is dependent on the state, the transition function is given by

$$P_t(i,j) = e^{-\lambda(i)t}\delta_{ij} + \int_0^t \lambda(i)e^{-\lambda(i)s} \sum_{k\in E} Q(i,k)P_{t-s}(k,j)ds,$$

where $\delta_{ij} = \begin{cases} 0 \text{ if } i \neq j, \\ 1 \text{ if } i = j. \end{cases}$

The generator A for this process is given by

$$A(i,j) = \begin{cases} -\lambda(i) & \text{if } \ i = j, \\ \lambda(i)Q(i,j) & \text{if } i \neq j, \end{cases}$$

or in matrix notation $A = \Lambda(Q - I)$ where $\Lambda = \text{diag}(\lambda(1), \lambda(1), \ldots, \lambda(N))$.

Our problem is to decide the time instant in which to invest that maximises the net expected discounted profit

$$\rho(x) = \sup_{\tau \geq 0} E_x \left[e^{-r\tau} h(X_\tau) \right],$$

where h is the expected income at the time of investment.

4.1 Solution Method

In order to determine the optimal stopping policy, we first compute the gain function $h,$ the net expected income at time of investment. Define

$$f(X_t) = \exp(X_t)$$

and $V(x)$, the r-potential of f, as

$$V(x) = U^r f(x) = E_x \left[\int_0^\infty e^{-rt} f(X_t)dt \right].$$

The vector $\underline{V}$=[$V(x_1)$, $V(x_2)$,..., $V(x_N)$]^T can be computed as the unique solution of the system of equations

$$(rI_N - A)\underline{V} = \underline{f}$$

where $\underline{f}$=[$f(x_1)$, $f(x_2)$,..., $f(x_N)$]^T and I_N is the identity matrix. Hence, the vector $\underline{h}$=[$h(x_1)$, $h(x_2)$,..., $h(x_N)$]^T is given by

$$h = V - I,$$

where $\underline{I}$=I*[1,1,...,1]^T.

To compute the value function we develop, for the Markov jump processes case, the three methods referred to previously corresponding to the following3 different algorithms:

1. By using dynamic programming arguments, we can compute an approximation of the value function when the inter-decision time δ is a "small" value.

2. By using known properties of the value function we can reformulate the problem as a linear programming one.

3. By using dynamic programming arguments, the value function can be computed by analysing an increasing sequence of allowed jumps.

4.1.1 Solution Method 1

This method follows closely the first method described in the previous section and it is based on determining an approximation of the function ρ.

Defining

$$\nu_\delta(x_t) = e^{-r\delta}\rho(x_{t+\delta}), \tag{3}$$

as the net expected return if we wait a small time δ, we have the obvious result

$$\lim_{\delta \downarrow 0} \nu_\delta(x) \to \rho(x).$$

This means that, for very small δ, ν_δ gives us a good estimate of ρ, and so an "almost optimal" solution can be achieved by applying the following rule

Invest

iff $h(X_t) \geq \nu_\delta(X_t)$,

iff $X_t \geq X^*_\delta$,

where X^*_δ is given by

$$X^*_\delta = \inf\{x \in X : h(x) \geq \nu_\delta(x)\};$$

Otherwise, wait for a time τ_δ, given by

$$\tau_\delta = \inf\{t \geq 0 : h(X_t) \geq \nu_\delta(X_t)\}.$$

Now, we develop ν_δ, knowing that it is given by

$$\nu_\delta(X_t) = e^{-r\delta}\rho(X_t) = e^{-r\delta}E\max\{h(X_{t+\delta}), \nu_\delta(X_{t+\delta})\}.$$

Similar to what we have done for the discrete time case, let us consider separately the two cases when a) $X_{t+\delta} \geq X_\delta^*$ and when b) $X_{t+\delta} < X_\delta^*$.

a) Case $X_{t+\delta} \geq X_\delta^*$

We start by defining P_a, the probability of being in this case as

$$P_a = Prob(X_{t+\delta} \geq X_\delta^* \quad |X_t = x_i) = \sum_{j=1}^{N} p(i,j,\delta) I_{x_j \geq X^*}.$$

Since a Poisson process for small intervals [t,t+δ[can be given by

$$\text{Prob}(N_\delta = m) = \begin{cases} \lambda\delta + o(\delta) \text{ if } m = 1 \\ o(\delta) \text{ if } m > 1 \\ 1 - \lambda\delta + o(\delta) \text{ if } m = 0\,, \end{cases} \tag{4}$$

P_a can be rewritten as

$$P_a = \lambda\delta \sum_{j=1}^{N} q_{ij} I_{x_j \geq X^*} + (1{-}\lambda\delta) I_{x_j \geq X^*}. \tag{5}$$

And so

$$\nu_\delta^a(X_t) = e^{-r\delta}E\max\{h(X_{t+\delta})|X_{t+\delta} \geq X_\delta^*\} =$$

$$= e^{-r\delta}\frac{\lambda\delta \sum_{j=1}^{N} h(x_j) q_{ij} I_{x_j \geq X^*} + h(x_i)(1{-}\lambda\delta) I_{x_i \geq X^*}}{P_a}.$$

b) Case $X_{t+\delta} < X_\delta^*$

Defining P_b the probability of being in this case as,

P_b=Prob($X_{t+\delta} < X^*_\delta \quad | X_t = x_i$)=

$$= \lambda\delta \sum_{j=1}^{N} q_{ij} I_{x_j<X^*} + (1-\lambda\delta) I_{x_j<X^*}, \tag{6}$$

we have

$$\nu^b_\delta(X_t) = e^{-r\delta} E \max \{\nu(X_{t+\delta}) | X_{t+\delta} < X^*_\delta\} =$$

$$= e^{-r\delta} \frac{\lambda\delta \sum_{j=1}^{N} \nu(x_j) q_{ij} I_{x_j<X^*} + \nu(x_i)(1-\lambda\delta) I_{x_i<X^*}}{P_b}$$

Finally $\nu_\delta(x)$ is then given by

$$\nu_\delta(x) = \mathrm{g}\nu^a_\delta(X_t)P_a + \nu^b_\delta(X_t)P_b =$$
$$= e^{-r\delta}\left\{\lambda\delta \sum_{j=1}^{N} q_{ij}[h(x_j)I_{x_j\geq X^*} + \nu(x_j)I_{x_j<X^*}] + (1-\lambda\delta)[h(x_i)I_{x_i\geq X^*} + \nu(x_i)I_{x_i<X^*}]\right\}. \tag{7}$$

Having a recursive equation for $\nu_\delta(x)$ we only need a fixed point to start with.

Choosing this point to be the largest value in X, say x_N, we get

$$\nu_\delta(x_N) = h(x_N).$$

If this is not true for this point, then it is also not true for any other, and then we would have the trivial solution “never to invest”.

Algorithm

1. Determine $h(x)$

2. Set initial guess for X^*

3. `Initialise` $\nu_\delta(x)$ `as` $\nu_\delta(x) = h(x)$

4. `Update estimate of` $\nu_\delta(x)$ `for` $i = 1, ..., N$

$$\begin{cases} \nu_\delta\left(X_N\right) = h\left(X_N\right) \\ \nu_\delta\left(x_i\right) = ...(\texttt{by eq. (7)}) \end{cases}$$

5. `Stopping condition`

   `If` $\max_i |\nu(x_i) - \nu_{old}(x_i) < \varepsilon|$ `then stop`

6. `Update estimate of` X^*

$$X^* = \min_i \left\{x_i : \nu(x_i) \leq h(x_i)\right\}$$

7. `Goto 4.`

4.1.2 Solution Method 2

The previous method can be supported by the known result that the value function is the minimal r-excessive function that majorises h; where the r-excessive function is defined as a function $f$satisfying:

$$f(x) \geq e^{-rt} P_t f(x)$$

or

$$rf(x) - Af(x) \geq 0.$$

We no longer have to consider any value δ for a discretization of t, because we are already using an infinitesimal operator - the generator A. It follows the result of the *Variational Inequalities*

$$\max\left\{rV(x) - AV(x),\, h(x) - V(x)\right\} = 0, \;\; \forall x \; \in E,$$

which can be formulated as a linear programming problem

$$\begin{aligned} &\min \sum_{j \in E} V(j) \\ &\text{s.t.} \quad AV(x) - rV(x) \leq 0 \\ &\qquad\; V(x) - h(x) \geq 0 \\ &\text{for all } x \in E, \end{aligned}$$

An implementation of this method in MATLAB was carried out, where a MATLAB built in function for solving the linear programming problem was used.

4.1.3 Solution Method 3

Another particularity of this problem is that since r >0 and X_t is constant between jumps, the optimal solution is always be to invest immediately after a jump.

This happens since if for a certain value X_t the decision is to invest, we should do so as soon as possible, as we would only be loosing money by waiting.

This can be justified with an example.

Consider $X_t = x$ constant, then

$$h(x) = \int_0^{\infty} e^{-rt+x} dt - I = \frac{e^x}{r} - I,$$

and so

$$\rho(x) = \operatorname*{Sup}_{\tau} \left\{ e^{-r\tau} \left(\frac{e^x}{r} - I \right) \right\},$$

gives, obviously, τ=0 if r >0.

Therefore, instead of considering a small time instant δ as in the previous method, we can consider T_n, the time of next jump.

Define the best value obtainable considering up to n jumps as

$$J_n(x_i) = \operatorname*{Sup}_{\tau \le T_n} E_i \left[e^{-r\tau} h\left(X_\tau\right) \right].$$

It can also be given, as we have seen, by

$$J_n(x_i) = \max \Big\{ \underset{\text{(keep strategy)}}{J_{n-1}(x_i)}, \quad \underset{\text{(invest at the n-th jump)}}{E_i \left[e^{-rT_n} h\left(X_{T_n}\right) \right]} \Big\}.$$

Since

$$E_i \left[e^{-rT_1} h\left(X_{T_1}\right) \right] = \int\limits_0^{\infty} \lambda e^{-\lambda t} dt \cdot \sum_{j=1}^{N} Q(i,j) h(x_j) = \frac{\lambda}{\lambda + r} \left(Q\underline{h}\right)_i,$$

$J_n(x_i)$ can be computed for i=1...N and for n=1,2,... as

$$\begin{cases} J_0(x_i) = h(x_i) \\ J_n(x_i) = \max \left\{ J_{n-1}(x_i), \left(\frac{\lambda}{\lambda+r}\right)^n \left(Q^n \underline{h}\right)_i \right\} \end{cases}.$$

Moreover, as λ is a bounded function

$$\lim_{n \to \infty} T_n = \infty,$$

and so

$$J_n(x) \underset{n \to \infty}{\longrightarrow} \operatorname*{Sup}_{\tau < \infty} E_x \left[e^{-r\varepsilon} h\left(X_\tau\right) \right] = \rho(x),$$

which is our aim to determine.

Algorithm

1. `Determine` $\underline{h}$ `(as in the previous methods)`

2. n`=0, MinIter=10`

   `FOR` i`=1...`N

$$J_0(x_i) = h(x_i)$$

3. `REPEAT`

   δ_{max}`=0`

   `FOR` i`=1...`N

$$J_{n+1}(x_i) = \max\left\{J_n(x_i), \left(\frac{\lambda}{\lambda + r}\right)^{n+1} \left(Q^{n+1}\underline{h}\right)_i\right\}$$

$$-\delta = |J_{n+1}(x_i) - J_n(x_i)|$$

$$\delta_{max} = \max\{\delta, \delta_{max}\}$$

   $n = n$`+1`

   `UNTIL` $\delta_{max} < \varepsilon \ \wedge \ n \geq$ `MinIter`

4.2 Example and comparison of the algorithms

To illustrate the algorithms consider the following example:

Interest rate: r=1,

Fixed cost of inv.: I=10,

$X_{Max} = \ln(200)$,

$X_{Min} = \ln(2)$,

$X = \{X_{Min}, ..., X_{Max}\}$ with N=20 points equally spaced,

Jump rate λ=10.

For the 1st Algorithm (corresponding to solution method 1):

Error ε=1e-3.

δ=1e-4.

For the 3rd Algorithm:

Error ε=1e-8,

Results of the 1st algorithm

```
Xstar=X10=1.817

Iteration:1 Error:1.059 Xstar=X13=2.504

Iteration:10 Error:0.0428 Xstar=X15=3.102

Iteration:20 Error:0.02279 Xstar=X15=3.102

Iteration:30 Error:0.01295 Xstar=X15=3.102

Final Result:

Iteration:37 Error:0.0008942 Xstar=X15=3.102
```

This algorithm converges very fast if $\delta << \varepsilon$. Moreover, reducing δs the difference between ν_δ and h for $X >= X^*$ is reduced. However, if we reduce δ too much, we start having numerical problems (because the term that is multiplied by δ vanishes).

Results of the 2nd algorithm

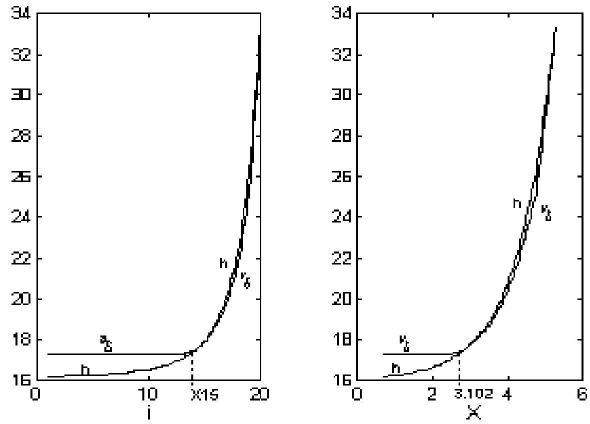

Figure 2: 1st algorithm results.

```
Xstar=X15=3.102
```

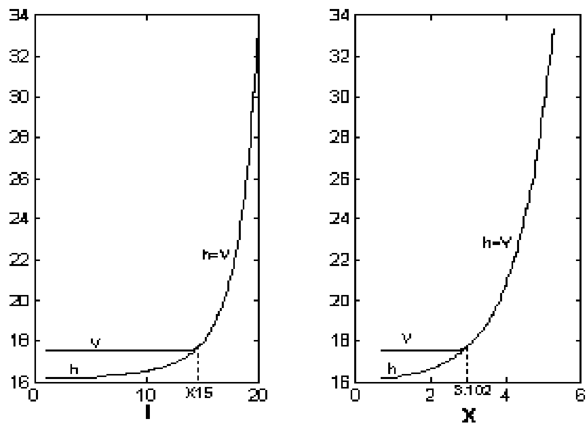

Figure 3: 2nd algorithm results.

This algorithm was the fastest to run for this example, but we have used a MATLAB built-in function to determine the optimal solution of the linear programming problem. Although linear programming optimisation methods are usually very efficient (the Simplex algorithm, for example, usually converges in few iterations), this efficiency is not guaranteed for every instance of the problem.

Results of the 3rd algorithm

```
Iteration:0 Error:0.7273
```

```
Iteration:10 Error:0.002785

Iteration:20 Error:0.00002414

Iteration:30 Error:2.093e-007

Iteration:37 Error:7.539e-009

Xstar=X15=3.102
```

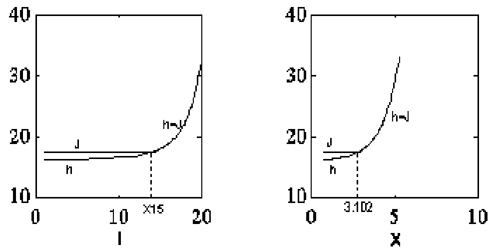

Figure 4: 3rd algorithm results.

As expected we have found the same solution as with the previous methods. Conceptually this method is, probably, the most simple, resulting in a more clear implementation. Therefore, it can be more easily expanded to cover more general Markov processes or additional problem features. In fact, we are developing this solution method for Markov processes that in addition to random jumps allow a deterministic drift – known as piecewise deterministic Markov processes (Davis 1993).

References

Bertsekas, D.P. (1976) *Dynamic Programming and Stochastic Control*, Academic Press, New York.

Black, F. and Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* 27, 637–654.

Boyarchenko, S. and Levendorskiĭ S. (2007) Optimal stopping made easy. *Journal of Mathematical Economics* 43, 201–217.

Boyarchenko, S. and Levendorskiĭ S. (2007) Practical guide to real options in discrete time. *International Economic Review* 48, 311–341.

Bresnahan, T. F. and Greenstein S. (1999) *Technological competition and the structure of the computer industry.* Journal of Industrial Economics 47, 1-40.

Campbell, J. A. (2002) Real Options analysis of the timing of IS investment decisions, *Information and Management* 39, 337-344.

Çinlar, E. (1975) *Introduction to Stochastic Processes,* Prentice Hall, Englewood Cliffs, New Jersey.

Clark E. (1997) *Valuing political risk*, Journal of International Money and Finance 16, 477-490.

Costa, O.L.V. and Davis, M.H.A. (1988) Aproximations for optimal stopping of a piecewise-deterministic process. *Math. Control Signal Systems*, **1**, 123-146

Cox, J., Ross, S., and Rubinstein, M. (1979) Option Pricing: a Simplified Approach. *Journal of Financial Economics* 12, 229–263

Davis, M.H.A. (1993) *Markov Models and Optimization,* Chapman & Hall, London

Dixit, A. K. and Pindyck R. S. (1996) *Investment under Uncertainty*. Princeton University Press.

Fontes, F.A.C.C. (1994) *Optimal Stopping Algorithms for Irreversible Investment Decisions*, MSc Thesis, Imperial College London (Supervisor: M.H.A. Davis).

Galor O. and Tsiddon D. (1997) *Technoligical progress, mobility, and economic growth* American Economic Review 87, 363-382.

Ghemawat P. and Kennedy R. E. (1999) *Competitive shocks and industrial structure: The case of polish manufacturing* International Journal of Industrial Organization 17, 847-867.

Greenwood J. , Hercowitz Z., and Krussell P. (1997) *Long-run implications of investment-specific technological change* American Economic Review 87, 342-362.

Grenadier S. R. and Weiss A. M. (1997) *Investment in technological innovations: An option pricing approach* Journal of Financial Economics .

Martzoukos, S. H. and Trigeorgis L. (2002) *Real (investment) options with multiple sources of rare events*, European Journal of Operational Research 136, 696-706.

Miller, K.D. and Folta T.B. (2002) Option Value and Entry Timing, *Strategic Management Journal* 23, 655- 665.

Peskir G. and Shiryayev, A. N. (2006) *Optimal Stopping and Free-Boundary Problems*, Birkhauser, Basel.

Shiryayev, A. N. (1978) *Optimal Stopping Rules*, Springer Verlag, Berlin.

Trigeorgis, L (1996) *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. MIT Press

Wagner D. (1997) *The resurgence of political risk insurance* Risk Management 44, 56-58.

Recent FEP Working Papers

Nº 244	Rui Henrique Alves and Óscar Afonso, *"Fiscal Federalism in the European Union: How Far Are We?"*, July 2007
Nº 243	Dalila B. M. M. Fontes, *"Computational results for Constrained Minimum Spanning Trees in Flow Networks"*, June 2007
Nº 242	Álvaro Aguiar and Inês Drumond, *"Business Cycle and Bank Capital: Monetary Policy Transmission under the Basel Accords"*, June 2007
Nº 241	Sandra T. Silva, Jorge M. S. Valente and Aurora A. C. Teixeira, *"An evolutionary model of industry dynamics and firms' institutional behavior with job search, bargaining and matching"*, April 2007
Nº 240	António Miguel Martins and Ana Paula Serra, *"Market Impact of International Sporting and Cultural Events"*, April 2007
Nº 239	Patrícia Teixeira Lopes and Lúcia Lima Rodrigues, *"Accounting for financial instruments: A comparison of European companies' practices with IAS 32 and IAS 39"*, March 2007
Nº 238	Jorge M. S. Valente, *"An exact approach for single machine scheduling with quadratic earliness and tardiness penalties"*, February 2007
Nº 237	Álvaro Aguiar and Ana Paula Ribeiro, *"Monetary Policy and the Political Support for a Labor Market Reform"*, February 2007
Nº 236	Jorge M. S. Valente and Rui A. F. S. Alves, *"Heuristics for the single machine scheduling problem with quadratic earliness and tardiness penalties"*, February 2007
Nº 235	Manuela Magalhães and Ana Paula Africano, *"A Panel Analysis of the FDI Impact on International Trade"*, January 2007
Nº 234	Jorge M. S. Valente, *"Heuristics for the single machine scheduling problem with early and quadratic tardy penalties"*, December 2006
Nº 233	Pedro Cosme Vieira and Aurora A. C. Teixeira, *"Are Finance, Management, and Marketing Autonomous Fields of Scientific Research? An Analysis Based on Journal Citations"*, December 2006
Nº 232	Ester Gomes da Silva and Aurora A. C. Teixeira, *"Surveying structural change: seminal contributions and a bibliometric account"*, November 2006
Nº 231	Carlos Alves and Cristina Barbot, *"Do low cost carriers have different corporate governance models?"*, November 2006
Nº 230	Ana Paula Delgado and Isabel Maria Godinho, *"Long term evolution of the size distribution of Portuguese cities"*, September 2006
Nº 229	Sandra Tavares Silva and Aurora A. C. Teixeira, *"On the divergence of evolutionary research paths in the past fifty years: a comprehensive bibliometric account"*, September 2006
Nº 228	Argentino Pessoa, *"Public-Private Sector Partnerships in Developing Countries: Prospects and Drawbacks"*, September 2006
Nº 227	Sandra Tavares Silva and Aurora A. C. Teixeira, *"An evolutionary model of firms' institutional behavior focusing on labor decisions"*, August 2006
Nº 226	Aurora A. C. Teixeira and Natércia Fortuna, *"Human capital, trade and long-run productivity. Testing the technological absorption hypothesis for the Portuguese economy, 1960-2001"*, August 2006
Nº 225	Catarina Monteiro and Aurora A. C. Teixeira, *"Local sustainable mobility management. Are Portuguese municipalities aware?"*, August 2006
Nº 224	Filipe J. Sousa and Luís M. de Castro, *"Of the significance of business relationships"*, July 2006
Nº 223	Pedro Cosme da Costa Vieira, *"Nuclear high-radioactive residues: a new economic solution based on the emergence of a global competitive market"*, July 2006
Nº 222	Paulo Santos, Aurora A. C. Teixeira and Ana Oliveira-Brochado, *"The 'de-territorialisation of closeness' - a typology of international successful R&D*

	projects involving cultural and geographic proximity", July 2006
Nº 221	Manuel M. F. Martins, "Dilemas macroeconómicos e política monetária: o caso da Zona Euro", July 2006
Nº 220	Ana Oliveira-Brochado and F. Vitorino Martins, "Examining the segment retention problem for the "Group Satellite" case", July 2006
Nº 219	Óscar Afonso Rui and Henrique Alves, "To Deficit or Not to Deficit": Should European Fiscal Rules Differ Among Countries?", July 2006
Nº 218	Rui Henrique Alves and Óscar Afonso, "The "New" Stability and Growth Pact: More Flexible, Less Stupid?", July 2006
Nº 217	J Maciej Cieślukowski and Rui Henrique Alves, "Financial Autonomy of the European Union after Enlargement", July 2006
Nº 216	Joao Correia-da-Silva and Carlos Hervés-Beloso, "Prudent Expectations Equilibrium in Economies with Uncertain Delivery", June 2006
Nº 215	Maria Rosário Moreira and Rui Alves, "How far from Just-in-time are Portuguese firms? A survey of its progress and perception", June 2006
Nº 214	Maria Fátima Rocha and Aurora A.C. Teixeira, "A cross-country evaluation of cheating in academia: is it related to 'real world' business ethics?", June 2006
Nº 213	Maria Rosário Moreira and Rui Alves, "Does Order Negotiation Improve The Job-Shop Workload Control?", June 2006
Nº 212	Pedro Cosme da Costa Vieira and Aurora A. C. Teixeira, "Human capital and corruption: a microeconomic model of the bribes market with democratic contestability", May 2006
Nº 211	Ana Teresa Tavares and Aurora A. C. Teixeira, "Is human capital a significant determinant of Portugal's FDI attractiveness?", May 2006
Nº 210	Maria Rosário Moreira and Rui Alves, A new input-output control order release mechanism: how workload control improves manufacturing operations in a job shop, April 2006
Nº 209	Patrícia Teixeira Lopes and Lúcia Lima Rodrigues, Accounting for Financial Instruments: An Analysis of the Determinants of Disclosure in the Portuguese Stock Exchange, April 2006
Nº 208	Pedro Cosme Costa Vieira, Energia nuclear: Uma solução para Portugal?, April 2006
Nº 207	Aurora A.C. Teixeira and Joana Costa, What type of firm forges closer innovation linkages with Portuguese Universities?, March 2006
Nº 206	Joao Correia-da-Silva and Carlos Hervés-Beloso, Rational Expectations Equilibrium in Economies with Uncertain Delivery, March 2006
Nº 205	Luís Delfim Santos and José Varejão, Employment, Pay and Discrimination in the Tourism Industry, February 2006
Nº 204	Carlos F. Alves and Victor Mendes, Mutual fund flows' performance reaction: does convexity apply to small markets?, February 2006

Editor: Sandra Silva (sandras@fep.up.pt)
Download available at:
http://www.fep.up.pt/investigacao/workingpapers/workingpapers.htm
also in http://ideas.repec.org/PaperSeries.html

www.fep.up.pt

Faculdade de Economia da Universidade do Porto
Rua Dr. Roberto Frias, 4200-464 Porto | Tel. 225 571 100
Tel. 225571100 | www.fep.up.pt