Necessary Optimality Conditions for Optimal Control Problems with Infinite Horizon *

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1 Introduction

This article concerns the derivation of necessary conditions of optimality for infinite horizon control problems whose cost functional depend on the state at the final time which is subject to constraints. Moreover, the optimization is conducted over trajectories which converge asymptotically to an equilibrium point in the given compact set C_∞. This includes the case in which it is not possible to steer the state to the equilibrium point in finite time. This feature distinguishes the problem addressed in this article from the usual finite time control problem.

The basic problem can be stated as follows:

$$(P)\ \text{Minimize} h(\xi) \tag{1}$$

$$\text{subject to} \dot{x}(t) = f(t, x(t), u(t)),\ \mathcal{L} - a.e. \tag{2}$$

$$(x(0), \xi) \in C_0 \times C_\infty,\ \xi = \lim_{t\to\infty} x(t) \tag{3}$$

$$u \in \mathcal{U} \tag{4}$$

where $h : \boldsymbol{R}^n \to \boldsymbol{R}$, $f : [0, \infty) \times \boldsymbol{R}^n \times \boldsymbol{R}^m \to \boldsymbol{R}^n$, C_0 and C_∞ are compact sets, $\mathcal{U}$ is the set of Borel measurable functions $u : [0, \infty) \to \boldsymbol{R}^m$ with $u(t) \in \Omega$ where Ω is a given compact set.

The optimization is carried out over all feasible control processes that converge asymptotically to equilibria in an infinite time horizon. A point $\xi \in \boldsymbol{R}^n$ is an

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equilibrium as $t \to \infty$ if $\exists$ a feasible control process $(x(\cdot), u)$ such that

$$\lim_{t\to\infty} x(t) = \xi, \text{ and } 0 \in \lim_{t\to\infty} \text{Int } f(t, x(t), \Omega).$$

This problem is cast in the context of nonsmooth analysis (see [1]) due to both the assumptions on its data and the approach used to derive the optimality conditions.

The key feature is the novel notion of transversality condition at infinite time - directional inclusion at infinity - which is obtained by assuming hypotheses which are significantly weaker than those usually required by competing results currently available in the literature. This condition enables the adjoint variable to "propagate" at least a partial effect of the cost function and state constraint penalization from the final time to any finite time. This result represents a compromise between the additional wealth of information provided by the transversality condition and the extent of the applicability range of the optimality conditions. An additional feature is the nondegeneracy of the conditions resulting from an endpoint controllability assumption required to prove our main result. The approach to derive these conditions essentially consist in considering a family of finite horizon optimal control problems approaching the original one and, then, in showing that the desired result is obtained as the limit of a properly extracted subsequence.

There are a number of necessary conditions of optimality that, over the years, have been obtained for infinite horizon control problems. Back in 1974, in [2], a problem with an integral cost functional was considered, having an appropriate solution concept been given and the derived optimality conditions do not exhibit any transversality conditions. In [3], it was shown that, under a certain controllability assumption, the Hamiltonian tends to zero as time goes to infinity. Inspired in stability theory, [5] provides necessary and sufficient conditions of optimality for infinite horizon control problems with a transversality condition by imposing a regularity assumption formulated in terms of Lyapunov exponents to be satisfied by the adjoint variable. In [4], a nonsmooth maximum principle encompassing final time transversality conditions was derived for nonsmooth optimal control problems with final state dependent cost functional as well as final time state constraints both with a linear structure. In [6], an infinite horizon discounted optimal control problem is considered and a maximum principle with a transversality condition is derived under assumptions on the data of the problem which imply that the adjoint variable remains bounded.

2 Necessary conditions of optimality

We start defining the concept of directional inclusion which will enable us to make precise boundary conditions involving variables which may either become unbounded-valued or persist in a certain set as time goes to infinity.

Let $y : [0, \infty) \to \boldsymbol{R}^n$ be a continuous function. Let $\boldsymbol{P}(y) := \boldsymbol{P}_L(y) \cup \text{dir}\boldsymbol{P}^\infty(y)$, also alluded to as the set of persistency points of y, where

- $\boldsymbol{P}_L(y) := \{\xi \in \boldsymbol{R}^n : \exists\ t_i \to \infty,\ \lim_{i\to\infty} y(t_i) = \xi\}$

- $\mathrm{dir}\boldsymbol{P}^{\infty}(y) := \{\xi\in\boldsymbol{R}^n : \exists t_i\to\infty, \lambda_i\searrow 0, \lim_{i\to\infty} \lambda_i y(t_i)=\xi\}$.

Given a function $y : [0,\infty) \to \boldsymbol{R}^n$ and a set $C \subset \boldsymbol{R}^n$ we say that y satisfies the <u>weak directional inclusion in C at ∞</u> if $\boldsymbol{P}(y) \cap \mathrm{csm}\, C \neq \emptyset$. This relation can be referred to in short notation by $y \in^*_\infty C$.

2.1 Necessary Conditions of Optimality

Our necessary conditions of optimality for (P) are stated in the form of a maximum principle and they involve the <u>pseudo-Hamiltonian</u> or <u>Pontryagin function</u> which is defined as

$$H(t,x,u,p) = p^T f(t,x,u).$$

The adjoint variable $p : [0,\infty) \to \boldsymbol{R}^n$ satisfies a boundary condition at $t = \infty$. This is stated as the existence of a non empty subset of its persistency points, $\boldsymbol{P}(p)$, on the cosmic closure of the right hand set of the usual transversality condition. Moreover p is a subgradient of the value function V along the optimal trajectory, being $V(t,z) := \mathrm{Min}\{h(\xi) : \text{all admissible } (x,u) \text{ with } x(t) = z\}$. In particular, if p converges asymptotically to some point $\bar{p}$, then $\boldsymbol{P}(p) = \{\bar{p}\}$. If p approaches a limit cycle C_L at infinite time, then $\boldsymbol{P}(p) = C_L$. The pattern of realization of the limiting approach towards a given infinitely often visited set of points might not be periodic. In what follows, $N_C(c)$ denotes the normal cone to the set C at point c and by $\partial_x f(x)$ the generalized gradient of the function f, both in the sense of Clarke, [1].

Before stating the main result, we some comments are in order. We make some assumptions which reflects a sort of persistence of the velocity set at the limiting value of the state variable, $H2$ implies controllability in a neighborhood of the optimal reference trajectory as time goes to ∞. $H3$ is an initial point controllability condition with respect to the initial state constraint.

Next, we state the main result of this article.

Theorem 1. *Let (x^*,u^*) be a solution to (P). Then, there exists a multiplier (p,λ_0), with $\lambda_0 \geq 0$, satisfying:*

- $\lambda_0 + \|p\| \neq 0$ *(nontriviality).*
- $\exists p(0) \in N_{C_0}(x^*(0))$ *for which there is a solution to*

$$\begin{aligned} -\dot{p}(t) &\in \partial_x H(t,x^*(t),u^*(t),p(t)),\ \mathcal{L}\text{ -a.e.} \\ \text{s. t.} \quad -p(t) &\in \partial_x V(t,x^*(t)),\ \mathcal{L}\text{-a.e. on } [0,\infty), \\ \text{and} \quad \boldsymbol{P}(-p) &\cap csm\ (\lambda_0\partial h(\xi^*) + N_{C_\infty}(\xi^*)) \neq \emptyset \end{aligned}$$

- $u^*(t)$ *maximizes in Ω the map*

$$v \to H(t,x^*(t),v,p(t),\lambda_0),\ \mathcal{L}\text{-a.e. in } [0,\infty).$$

Remark that $\boldsymbol{P}(-p) \cap \mathrm{csm}\ (\lambda_0\partial h(\xi^*) + N_{C_\infty}(\xi^*)) \neq \emptyset$ can be interpreted as $\exists\zeta \in \lambda_0\partial h(\xi^*) + N_{C_\infty}(\xi^*)$ for which

- either $\zeta \in \boldsymbol{P}_L(-p)$, if p is bounded,
- or $\dfrac{\zeta}{|\zeta|} \in \mathrm{dir}\boldsymbol{P}^{\infty}(p)$, otherwise.

The information provided by this concept is certainly weaker than that given by the boundary condition of the adjoint variable in the finite time interval context. In general, there are many functions p that persist in an absolute or a directional sense towards a point of $\lambda_0 \partial h(\xi^*) + N_{C_\infty}(\xi^*)$ at infinite time. However, this information is still useful in delimiting the number of multipliers which satisfy the maximum condition.

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