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Regular Transitions of Physical Measures in Nonuniformly Hyperbolic Systems

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Thesis carried out as part of the UC|UP Joint PhD Program in Mathematics
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To the memory of Moses Etubi.

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Abstract

We study the regularity of one-parameter families of *physical measures*. Firstly, we consider a d degree family of intermittent circle maps, introduced in [79], which has a unique *physical or Sinai-Ruelle-Bowen (SRB) measure* and using the cone technique of Baladi and Todd [28], show some form of weak differentiability of this measure, giving *linear response* in the process. Lifting the regularity from the base dynamics of the solenoid map with intermittency, we show that this family is *statistically stable*.

Subsequently, considering a class of multi-dimensional piecewise expanding maps we obtain, using the Keller-Liverani perturbation results [58], the Hölder continuity in the parameter, of the invariant densities and entropies of the parameterised family of the physical measures. We apply these results to a certain family of two-dimensional tent maps.

Resumo

Estudamos a regularidade de famílias a um parâmetro de medidas físicas. Primeiramente, consideramos uma família de transformações intermitentes no círculo, de grau d, introduzida em [79], que tem uma única medida *física ou Sinai-Ruelle-Bowen (SRB)* e usando a técnica do cone de Baladi e Todd [28], mostramos uma forma de diferenciabilidade fraca desta medida, obtendo *resposta linear* do processo. Fazendo o levantamento da regularidade da dinâmica de base do solenoide com intermitência, mostramos que esta família é *estatisticamente estável.*

Subsequentemente, considerando uma classe de transformações multidimensionais expansivas por pedaços, obtemos, usando os resultados de perturbação de Keller-Liverani [58], a dependência Hölder do parâmetro, as densidades invariante e as entropias da família parametrizada de medidas físicas. Aplicamos estes resultados a uma certa família bidimensional de *tent maps.*

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Chapter 1

Introduction

In the theory of Dynamical Systems, it is common to encounter examples of systems with simple governing laws that exhibit highly complex and unpredictable behaviour. Prominent examples include one-dimensional quadratic maps, two-dimensional Hénon quadratic diffeomorphisms, and the Lorenz system of quadratic differential equations in three-dimensional Euclidean space. Despite their straightforward formulation, these systems display intricate dynamical properties that have inspired significant mathematical developments over the past few decades.

An approach to the understanding of Dynamical Systems exhibiting some chaos is by studying their statistical properties. At the heart of this approach is the idea that it might be far more easier to predict the evolution of measures rather than the behaviour of single points. More precisely, the study of statistical properties of the dynamical systems concerns itself with the evolution of such measures[1].

One of the central areas of investigation in the theory of Dynamical Systems has been the study of invariant measures that describe the statistical behavior of these systems over time. Among these, *Sinai-Ruelle-Bowen (SRB) measures*, also known as *physical measures*[2], play a pivotal role. SRB measures are particularly important because they provide a way to understand the asymptotic distribution of orbits for a wide range of initial conditions, typically those that are of full measure with respect to the Lebesgue (volume) measure on the phase space. In other words, they allow us to describe the long-term statistical behavior of almost all initial points in a given region, making them essential tools for analysing chaotic systems where individual trajectories may be unpredictable, but the statistical distribution of orbits remains stable.

A key area of research in this field involves understanding the conditions under which SRB measures exist, as well as their robustness to perturbations in the system. The continuous dependence of these measures on the underlying dynamics is particularly significant, as it reflects the stability of the system's statistical properties in response to small changes. This is not only important from a theoretical standpoint but also has practical implications, as

[1]The *SRB measure* or *physical measure*. In some settings, it may mean absolutely continuous invariant measures.

[2]For subtlety on how both of them may differ, we refer the reader to [80]. In this thesis however, we shall overlook this distinction and use them interchangeably.

small perturbations often arise in real-world systems due to noise or external influences. In particular, the continuity of the *metric entropy* – a quantity that measures the complexity and unpredictability of the system – associated with SRB measures is a subject of extensive study. Metric entropy quantifies the rate of information production in the system and is directly related to the degree of chaos present.

Many results in the literature have been devoted to the study of SRB measures and their associated metric entropies in various dynamical settings, including [4, 7, 10, 18, 29–32, 36–38, 54, 60–62, 66, 70, 74]. These results range from classical systems like hyperbolic maps and diffeomorphisms to more complex systems involving non-uniformly hyperbolic dynamics or systems with piecewise smooth structures. Understanding how SRB measures and their entropies behave under perturbations provides valuable insights into the stability and predictability of chaotic systems, a topic that has been extensively explored in works such as [8, 9, 12, 14–17, 22, 47, 48, 56]. This ongoing research continues to shed light on the delicate interplay between deterministic chaos and statistical regularity, offering a deeper comprehension of the fundamental nature of chaotic dynamical systems.

Let $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ be a parameterised family of maps, where $\mathcal{A}$ is the parameter space. Suppose that $\mathcal{A}$ is a neighbourhood of 0, we may view f_0 as the unperturbed map and f_α, $\alpha \neq 0$ as its perturbation. A central question that is of interest to us in this thesis is:

> *Suppose that μ_α is the unique SRB measure of a one parameter family of dynamical systems $\{f_\alpha\}_{\alpha\in\mathcal{A}}$, how does this measure vary with respect to the parameter when the dynamical system is perturbed?*

In some cases, $\alpha \mapsto \mu_\alpha$ changes continuously (called *statistical stability*)(see [5, 8, 15, 17]), in some other cases it varies Hölder continuously [77], Lipschitz continuously, or is differentiable (in such a case, we say that the system admits *linear response*) that is, it is the first order approximation of the SRB of the system with respect to the SRB of the unperturbed system and in that case a formula for this derivative is called the *linear response formula* of the system.

One way that the question of statistical stability has been posed in the literature is to ask whether the perturbed density h_α converges to the unperturbed density h_0 in the L^1-norm, this is usually known as *strong statistical stability.* However, the notion of statistical stability we shall consider in this thesis is given in terms of the weak* convergence of the measures μ_{α_n} to μ_{α_0} as $\alpha_n \to \alpha_0$, and sometimes it is referred to as *weak statistical stability.* Without making any distinction, we shall simply call it statistical stability.

The notion of linear response has been around for quite some time in statistical mechanics. A pioneering result in this area has been the work of Ruelle for Axiom A attractors [71] and in the Anosov case [55], and for system with exponential decay of correlation or at least summable decay of correlations [25, 27, 40, 46, 51, 71], and in random systems [23]. However, linear response have been reported to fail in some cases [25–27]. To circumvent the lack of linear response formula in a tent like family of maps, Bahsoun and Galatolo in [20] introduced unbounded derivatives at the turning point of such families. For cases of systems which decays rather slowly, linear response have been shown for the Pomeau-Manneville type maps, particularly, system containing an intermittent fixed point albeit using different techniques [24, 28, 59].

An idea that has been shown to yield extraordinary results over the years in the study of the statistical properties of piecewise expanding maps is that of understanding the spectral properties of the *Perron-Frobenius operator* associated with the dynamics [39, 41] and using these properties to deduce various statistical properties such as the existence of invariant probability measures [1, 4, 38, 60], decay of correlations [38], large deviation [3] etc. We should however note that it is not immediate, since the success of this approach very much depends on choice of a suitable Banach space $(\mathcal{B}, \|\cdot\|), \mathcal{B} \subset L^1$, such that the Perron-Frobenius operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ is *quasi-compact*. Suppose that $f_\alpha : X \to X$ is a nonsingular map, quasi-compactness allows us to recover good results, such as the existence of absolutely continuous invariant measures, with density lying in $\mathcal{B}$. Several function spaces have been used so far, such as bounded variation [1, 4, 38, 60], generalized bounded p-variation [21, 22, 57] and quasi-Hölder spaces [3, 73]. For the particular setting we have in mind, we shall perform our analysis in the space of functions of bounded variation. A pioneering result in dimension 1 was put forth by Lasota and Yorke [60], where they showed the existence of absolutely continuous invariant measures for piecewise expanding C^2 maps. Extensions to higher dimensions were rather not so straight forward, due to the geometric intricacies partly due to the unavailability of a precise definition of what it means for a function to have bounded variation in higher dimension. A breakthrough in this direction was achieved when a distributional definition was given in [49].

An overview of the structure of this thesis is as follows. Firstly, in Chapter 2, we introduce key concepts and fundamental results that will be useful throughout the thesis.

In Chapter 3 we study the linear response for the intermittent circle map introduced in [79]. Obtaining linear response in the strong form, that is, in some topology [20, 23, 24], is not always possible. However, linear response may be given in a weak sense [28, 59]. By this, we mean that for a fixed observable, say ψ, in a suitable class, the function

$$\begin{aligned} \mathcal{R}_\psi : [0, 1) &\to \mathbb{R} \\ \alpha &\mapsto \int \psi d\mu_\alpha, \end{aligned}$$

with μ_α the unique SRB measure of f_α is instead shown to be differentiable at 0. If the formula for this derivative exists in terms of the unperturbed terms of f_0, μ_0, ψ and the vector field $v_0 := \partial_\alpha f_\alpha|_{\alpha=0}$, then we call this the linear response formula. We show that for any $\psi \in L^q, q \geq 1$, $\mathcal{R}_\psi(\alpha)$ is differentiable on $\alpha \in [0, 1 - 1/q)$. Another interesting dynamical system to us is the solenoid map with intermittency, originally introduced in [13], where the $2x \bmod 1$ map in the base dynamics of the classical solenoid map was replaced by the intermittent circle map. The linear response result in the base map of this dynamics implies statistical stability and using the techniques from Alves and Soufi in [16] lift this regularity to the SRB measure of the solenoid map with intermittency.

In Chapter 4, we consider a multi-dimensional parameterised family of piecewise expanding maps. We apply the perturbation theory of Keller-Liverani to study the spectral properties of the perturbed operators, a seminal result developed in [58] to show Hölder continuity of the spectrum, working with more "refined" spaces and assuming among other things, quasi-compactness. As in the case of the one-dimensional dynamics, we use the notion of bounded variation in multi-dimension.

Finally, we give future research directions and possible extensions in Chapter 5.

Chapter 2

Preliminaries on ergodic theory

This chapter introduces key concepts from ergodic theory and dynamical systems, along with several results that are essential for the proofs of certain theorems presented later in this thesis.

2.1 Invariant measures

Definition 2.1.1 (Non-singular maps)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space, a measurable map $f : X \to X$ is said to be non-singular with respect to μ if for any $A \in \mathcal{A}$,*

$$\mu(A) = 0 \iff \mu(f^{-1}(A)) = 0.$$

If $\mu(A) = \mu(f^{-1}(A))$ for all $A \in \mathcal{A}$, then μ is said to be invariant under f or measure preserving with respect to μ.

From this definition, we note that any measure preserving transformation is non-singular.

Definition 2.1.2 (Push-forward)**.** *Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $f : X \to X$ a function. The pushforward of $\mathcal{A}$ is the σ-algebra*

$$f_*\mathcal{A} := \{A \subset X | f^{-1}(A) \in \mathcal{A}\}.$$

The pushforward of μ is the function $f_*\mu : f_*\mathcal{A} \to [0, \infty]$ defined by

$$f_*\mu(A) := \mu(f^{-1}(A)), \quad \text{for } A \in f_*\mathcal{A}. \tag{2.1}$$

Definition 2.1.3 (Ergodicity)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space, a measurable map $f : X \to X$ is said to be* ergodic *if for every measurable set A satisfying $\mu(A \Delta f^{-1}(A)) = 0$, we have that $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.*

However, in practice, a useful characterization of ergodicity is given by the following proposition.

Proposition 2.1.1. *Let $(X, \mathcal{A}, \mu)$ be a measure space, a measurable map $f : X \to X$ is said to be ergodic if and only if for every $A \in \mathcal{A}$ satisfying $f^{-1}(A) = A$, then $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.*

Definition 2.1.4 (Absolute continuity). *A measure μ is said to be absolutely continuous with respect to the measure ν (written as $\mu \ll \nu$) if $\nu(A) = 0 \Rightarrow \mu(A) = 0$. If both μ and ν are absolutely continuous with respect to each other, then we say that they are equivalent.*

Theorem 2.1.2 (Radon-Nikodym). *Let $(X, \mathcal{A})$ be a measurable space, with two σ-finite measures μ and ν, such that $\mu \ll \nu$. Then there is a function $h \in L^1(\nu)$, such that for every $A \in \mathcal{A}$*

$$\mu(A) = \int_A h(x) d\nu.$$

The function h is sometimes written as $h = \dfrac{d\mu}{d\nu}$ and it called the *density* of μ with respect to ν or the *Radon-Nikodym derivative* of μ with respect to ν.

One of the most important classes of measures that captures the chaotic nature of dynamical systems is the *SRB* (or physical) measures. Let $f : M \to M$ be a measurable map on some metric space M and m the Lebesgue measure. We now define terminologies related to this measure.

Definition 2.1.5 (Weak* convergence). *A sequence of probability measures $(\mu_n)_n \in \mathbb{P}(M)$ converges in the weak* topology to $\mu \in \mathbb{P}(M)$ if for all $\varphi \in C_b(M)$,*

$$\int \varphi d\mu_n \to \int \varphi d\mu.$$

Where $C_b(M)$ is the space of bounded continuous real valued function on M and $\mathbb{P}(M)$ is the space of probability measures on the Borel sets of M.

Definition 2.1.6 (Basin of attraction). *Let μ be a Borel probability measure on M and $x \in M$, its initial states. The set*

$$B(\mu) = \left\{ x \in M : \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) \to \int_M \varphi d\mu \text{ for any } \varphi \in C^0(M) \right\} \tag{2.2}$$

is the basin of attraction *of μ.*

If $m(B(\mu)) > 0$ then μ is a *physical measure*. A physical measure is necessarily f-invariant for every Borel set $A \subset M$.

Definition 2.1.7 (Decay of Correlation). *The correlation function of observables $\varphi, \psi \in C^0(M)$ with respect to an f-invariant probability measure μ is defined as*

$$Cor_\mu(\varphi, \psi \circ f^n) = \left| \int \varphi(\psi \circ f^n)\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu \right|. \tag{2.3}$$

For sufficiently regular observables, we may obtain the $Cor_\mu(\varphi, \psi \circ f^n) \xrightarrow{n \to \infty} 0$, in that case, we say that the correlation function decays.

2.2 Entropy

In this section, we recall some definitions about entropy, particularly, the metric entropy. For a more detailed exposition we refer the reader to [78].

Definition 2.2.1. *Let $(X, \mathcal{A}, m)$ be a measure space, a family $\mathcal{P}$ of subsets of X is an m* mod 0 *partition of X if there exists a measurable set $X_0 \subset X$ for which $m(X \setminus X_0) = 0$ such that the elements in the family $\{\omega \cap X_0 : \omega \in \mathcal{P}\}$ are pairwise disjoint and satisfy $X_0 = \bigcup_{\omega \in \mathcal{P}} \omega$. In a situation where X_0 can be equal to X, $\mathcal{P}$ is referred to as the partition of X.*

Two families $\mathcal{P}$ and $\mathcal{Q}$ of subsets of X are said to be m mod 0 *equal* if there exists a measurable set $X_0 \subset X$ for which $m(X \setminus X_0) = 0$ such that the families $\{\omega \cap X_0 : \omega \in \mathcal{P}\}$ and $\{\omega \cap X_0 : \omega \in \mathcal{Q}\}$ coincide. The elements of the partitions are called *atoms*. Given a mod 0 partition $\mathcal{P}$, $F : X \to X$ the dynamics on X, set for $n \geq 0$,

$$F^{-n}(\mathcal{P}) = \{F^{-n}(\omega) : \omega \in \mathcal{P}\}. \tag{2.4}$$

Now, for all $n \geq 1$, we have

$$\mathcal{P}_n = \bigvee_{i=0}^{n-1} F^{-i}\mathcal{P} = \{\omega_0 \cap F^{-1}(\omega_1) \cap \cdots \cap F^{-n+1}(\omega_{n-1}) : \omega_0, \ldots, \omega_{n-1} \in \mathcal{P}\}$$

and

$$\bigvee_{n=0}^{\infty} F^{-n}\mathcal{P} = \{\omega_0 \cap F^{-1}(\omega_1) \cap \cdots : \omega_n \in \mathcal{P} \text{ for all } n \geq 0\}.$$

The increasing sequence $(\mathcal{P}_n)_n$ of countable mod 0 partitions is a *basis* of Ξ_0 if $(\mathcal{P}_n)_n$ generates $\mathcal{A}$ (mod 0) and $\bigvee_{n=0}^{\infty} \mathcal{P}_n$ is a partition into single points (mod 0).

The entropy of a finite partition $\mathcal{P}$ is defined as

$$H_\mu(\mathcal{P}) := -\sum_{i=0}^{n-1} \mu(\omega_i) \log(\mu(\omega_i)).$$

We make the following definition of entropy in a "static" sense as follows

$$h(\mu) = H_\mu(\bigvee_{i=1}^{n} \mathcal{P}_i).$$

Assume that the transformation in (2.4) is measure preserving, we have the following definition.

Definition 2.2.2. *For $\mathcal{P}$ a finite partition of $(X, \mathcal{A}, \mu, F)$, then the entropy of F with respect to the $\mathcal{P}$ is*

$$h_\mu(F, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\mathcal{P}_n)$$

Finally, the entropy of F with respect to μ is given by

$$h_\mu(F) = \sup_{\mathcal{P}} h_\mu(F, \mathcal{P})$$

where the supremum is taken over all partitions with finite entropy.

Theorem 2.2.1 (Kolmogorov-Sinai). *Let $\mathcal{P}_1 \prec \cdots \prec \mathcal{P}_n \prec \cdots$ be a non-decreasing sequence of partitions with finite entropy such that $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ generates the σ-algebra of measurable sets, up to measure zero. Then,*

$$h_\mu(F) = \lim_n h_\mu(F, \mathcal{P}_n).$$

Theorem 2.2.2 (Shannon-McMillan-Breiman). *Given any partition $\mathcal{P}$, with $H_\mu(\mathcal{P}) < \infty$, the limit*

$$h_\mu(F, \mathcal{P}, x) = \lim_n -\frac{1}{n} \log \mu(\mathcal{P}^n(x)) \quad \textit{exists at } \mu\textit{-almost every point.} \tag{2.5}$$

The function $x \mapsto h_\mu(F, \mathcal{P}, x)$ is μ-integrable, and the limit in (2.5) also holds in $L^1(\mu)$. Moreover,

$$\int h_\mu(F, \mathcal{P}, x)\, d\mu(x) = h_\mu(F, \mathcal{P})$$

If (F, μ) is ergodic then $h_\mu(F, \mathcal{P}, x) = h_\mu(F, \mathcal{P})$ at μ-almost every point.

Theorem 2.2.3 (Rohlin's formula). *Let $F : X \to X$ be a locally invertible transformation and μ be an F-measure preserving probability measure. If the the partition $\mathcal{P}$ is a generator of $\mathcal{A}\,(\mathrm{mod}\ 0)$ with finite entropy and every $\omega_i \in \mathcal{P}$ is an invertibility domain of F, then $h_\mu(F) = \int \log J_\mu F\, d\mu$.*

2.3 Perron-Frobenius operator

One of the main actors in this thesis will be the Perron-Frobenius operator. Here, we briefly give its definition and state some of its properties.

Definition 2.3.1. *Let $(X, \mathcal{A}, \mu, f)$ be a measure preserving dynamical system. Then the Koopman operator with respect to f is the linear operator $U_f : L^\infty(\mu) \to L^\infty(\mu)$, defined as*

$$U_f(\psi) = \psi \circ f \quad \forall\, \psi \in L^\infty(\mu). \tag{2.6}$$

Definition 2.3.2 (Perron-Frobenius operator). *The Perron-Frobenius operator $\mathcal{L} : L^1 \to L^1$ of the function $f : X \to X$ is the dual of the Koopman operator, defined as*

$$\int_X \mathcal{L}\varphi \cdot \psi\, d\mu = \int_X \varphi \cdot U_f \psi\, d\mu = \int_X \varphi \cdot \psi \circ f\, d\mu, \quad \psi \in L^\infty, \varphi \in L^1. \tag{2.7}$$

It is even more interesting to note that more properties of the dynamical system f like existence of absolutely continuous invariant measures, mixing, ergodicity are translated to spectral properties of $\mathcal{L}$. Hence, using spectral theory, we can gain a lot of information about the dynamical system f. Another useful representation of the Perron-Frobenius operator is given as follows.

Let $f : X \to X$ be a non-singular piecewise C^1 transformation, and suppose that there exists countably many domain of smoothness $\mathcal{P} = \{\omega_i\}_{i=1}^{\infty}$ of X, such that for each i, the restriction of f to each domain ω_i is one to one with inverse branches given as f_i^{-1} and has a non-zero determinant in the interior of ω_i. Then for $\varphi \in L^1$ we define the Perron-Frobenius operator $\mathcal{L} : L^1(X) \longrightarrow L^1(X)$ as

$$\mathcal{L}\varphi = \sum_{\{i:\omega_i\in\mathcal{P}\}} \frac{\varphi \circ f_i^{-1}}{|J_f \circ f_i^{-1}|} \chi_{f(\omega_i)}, \tag{2.8}$$

where J_f is the Jacobian function defined as $J_f = |\det(Df)|$. It is well known that the following properties hold for each $\mathcal{L}$,

(C1) for all φ, ψ for which the integrals make sense, we have

$$\int_X \psi\, \mathcal{L}\varphi\, dm = \int \varphi\, \psi \circ f\, dm;$$

(C2) $|\mathcal{L}\varphi| \leq \mathcal{L}(|\varphi|)$ and $\|\mathcal{L}\varphi\|_1 \leq \|\varphi\|_1$, for all $\varphi \in L^1(X)$;

(C3) $\varphi \in L^1(X)$ is the density of an absolutely continuous f-invariant measure if and only if $\varphi \geq 0$ and $\mathcal{L}\varphi = \varphi$.

Remark 2.3.1. *For any $\varphi \in L^1$, f a piecewise monotonic and expanding interval map,* (2.8) *can be written in a more compact form as*

$$\mathcal{L}\varphi(x) = \sum_{y\in f^{-1}(x)} \frac{\varphi(y)}{|f'(y)|}. \tag{2.9}$$

The Perron-Frobenius operator enjoys other good properties such as

Proposition 2.3.2 (Linearity)**.** *$\mathcal{L} : L^1 \to L^1$ is a linear operator.*

Proposition 2.3.3 (Postivity)**.** *Suppose that $\varphi \in L^1$ and $\varphi \geq 0$. Then $\mathcal{L}\varphi \geq 0$.*

Proposition 2.3.4 (Preservation of integrals)**.**

$$\int_X \mathcal{L}\varphi\, dm = \int_X \varphi\, dm$$

Proof.

$$\int_X \mathcal{L}\varphi\, dm = \int_X \mathcal{L}\varphi\, \mathbb{1}_X\, dm \overset{(C1)}{=} \int_X \varphi\, \mathbb{1}_X \circ f\, dm = \int_X \varphi\, \mathbb{1}_{f^{-1}(X)}\, dm = \int_X \varphi\, dm$$

□

For an excellent exposition and proof of the above propositions, we refer the reader to the work of Góra and Boyarsky [39].

2.4 Bounded variation in higher dimension

We adopt the definition presented in [49] . Given $f \in L^1(\mathbb{R}^d)$ with compact support, we define the *variation* of f as

$$V(f) = \sup\left\{\int_{\mathbb{R}^d} f \operatorname{div}(g)\, dm \,:\, g \in C_0^1(\mathbb{R}^d, \mathbb{R}^d) \text{ and } \|g\| \leq 1\right\},$$

where $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$ is the set of C^1 functions from $\mathbb{R}^d$ to $\mathbb{R}^d$ with compact support, $\operatorname{div}(g)$ is the divergence of g and $\| \quad \|$ is the sup norm in $C_0^1(\mathbb{R}^d, \mathbb{R}^d)$. Integration by parts gives that if f is a C^1 function with compact support, then

$$V(f) = \int_{\mathbb{R}^d} \|Df\| \, dm. \tag{2.10}$$

We shall use the following properties of bounded variation functions whose proofs may be found in [49], respectively in Remark 2.14, Theorem 1.17 and Theorem 1.28.

(B1) If $f \in BV(\mathbb{R}^d)$ is zero outside a compact domain K whose boundary is Lipschitz continuous, $f|_K$ is continuous and $f|_{\operatorname{int}(K)}$ is C^1, then

$$V(f) = \int_{\operatorname{int}(K)} \|Df\| \, dm + \int_{\partial K} |f| \, d\bar{m},$$

where $\bar{m}$ denotes the $(d-1)$-dimensional measure on ∂K.

(B2) Given $f \in BV(\mathbb{R}^d)$, there is a sequence $(f_n)_n$ of C^∞ maps such that

$$\lim_{n\to\infty} \int |f - f_n| \, dm = 0 \quad \text{and} \quad \lim_{n\to\infty} \int \|Df_n\| \, dm = V(f).$$

(B3) There is some constant $C > 0$ such that, for any $f \in BV(\mathbb{R}^d)$,

$$\left(\int |f|^p \, dm \right)^{1/p} \leq C \, V(f), \quad \text{with} \quad p = \frac{d}{d-1}. \tag{2.11}$$

This last property is known as *Sobolev Inequality*. Notice that $p = d/(d-1)$ is the conjugate of $d \geq 1$, meaning that

$$\frac{1}{p} + \frac{1}{d} = 1. \tag{2.12}$$

Suppose that $\Omega \subset \mathbb{R}^d$, the space of *bounded variation* functions in Ω is given by

$$BV(\Omega) = \left\{ f \in L^1(\Omega) : V(f) < +\infty \right\}.$$

Property (B3) gives in particular $BV(\Omega) \subset L^p(\Omega)$, for some $p > 1$. Set for each $f \in BV(\Omega)$

$$\|f\|_{BV} = \|f\|_1 + V(f).$$

It is well known that this defines a norm, and $BV(\Omega)$ endowed with this norm becomes a Banach space; see e.g. [49, Remark 1.12].

Proposition 2.4.1 (Lasota-Yorke type inequality)**.** *There are constants $\lambda \in (0,1)$ and $K > 0$ such that for every $f \in BV(\mathbb{R}^d)$*

$$V(\mathcal{L}f) \leq \lambda V(f) + K \int |f| \, dm.$$

Theorem 2.4.2. *[53] Let $\mathcal{L} : L^1(\Omega) \to L^1(\Omega)$ and let it satisfy the following properties*

(1) $\mathcal{L} \geq 0$, $\int_\Omega \mathcal{L}f\, dm = \int_\Omega f\, dm$, for $f \in L^1(\Omega)$ which implies that

$$\|\mathcal{L}\|_1 = 1;$$

(2) there exist constants $0 < \lambda < 1$, $M > 0$ such that

$$\|\mathcal{L}f\|_{BV} \leq \lambda \|f\|_{BV} + M\|f\|_1, \quad \text{and } f \in BV(\Omega);$$

(3) the image of any bounded subset of $BV(\Omega)$ under $\mathcal{L}$ is relatively compact in $L^1(\Omega)$.

Then $\mathcal{L}$ is a quasi-compact operator on $(BV(\Omega), \|\cdot\|_{BV})$. Thus, $\mathcal{L}$ has finitely many eigenvalues $\{\alpha_1, \dots, \alpha_k\}$ of modulus 1. The corresponding eigenspace E_i are finite dimensional subspaces of $BV(\Omega)$. Furthermore, $\mathcal{L}$ admits the following decomposition

$$\mathcal{L} = \sum_{i=1}^{k} \alpha_i \Pi_i + T,$$

where $\Pi_i : BV(\Omega) \to BV(\Omega)$ are linear projections with finite dimensional range onto the E_i's and

$$T : BV(\Omega) \to BV(\Omega),$$

is a continuous linear operator. For $1 \leq i, j \leq k$ we have

$$\int \phi_i \psi_j \, dm = \delta_{ij},$$

where $\phi_i \in BV(\Omega)$ and $\psi_j \in L^\infty(\Omega)$.

2.5 (Weak) Gibbs-Markov maps

Consider $f : M \to M$ and a measure m on M. Let $\Xi_0 \subseteq M$ be a Borel set on a σ-algebra $\mathcal{F}$ of Ξ_0 such that $m(\Xi_0) < \infty$. We say that $F : \Xi_0 \to \Xi_0$ is an *induced transformation* if there exists a countable $m \mod 0$ partition $\mathcal{P}$ of Ξ_0 into pairwise disjoint subsets and a return function $R : \mathcal{P} \to \mathbb{N}$ such that

$$F|_\omega = f^{R(\omega)}|_\omega, \quad \forall \omega \in \mathcal{P}.$$

The function R is the *return time* associated with the induced map. Furthermore, if we define

$$R(x) = \inf\{n \geq 1 : f^n(x) \in \Xi_0\},$$

then we say it is the *first return time*. We say that F is a *weak Gibbs-Markov* map if conditions (G_1)-(G_5) below are satisfied.

(G_1) *Markov*: F maps each $\omega \in \mathcal{P}$ bijectively to a $\mod 0$ union of elements of $\mathcal{P}$.

(G_2) *Separability*: the sequence $(\bigvee_{i=0}^{n-1} F^{-i}\mathcal{P})_n$ is a basis of Ξ_0.

It follows from (G_2) that the *separation time*

$$s(x,y) = \min\{n \geq 0 : F^n(x) \text{ and } F^n(y) \text{ lie in distinct elements of } \mathcal{P}\}$$

is well defined and finite for distinct points x, y in a full m measure subset of Ξ_0. For definiteness, set the separation time equal to zero for all other points.

(G_3) *Nonsingular*: F has a strictly positive Jacobian J_F i.e $J_F : \Xi_0 \to (0, \infty)$ a measurable function such that for every measurable set $A \subset \omega \in \mathcal{P}$,

$$m(F(A)) = \int_A J_F \, dm.$$

(G_4) *Gibbs*: there are $C > 0$ and $0 < \beta < 1$ such that, for all $x, y \in \omega \in \mathcal{P}$

$$\log \frac{J_F(x)}{J_F(y)} \leq C\beta^{s(F(x),F(y))}.$$

(G_5) *Long branches*: there is $\delta_0 > 0$ such that $m(F(\omega)) \geq \delta_0$, for all $\omega \in \mathcal{P}$.

If in addition to the above conditions, F satisfies (G_5') below, then F is called a *Gibbs-Markov map*.

(G_5') *Full branches*: F maps each $\omega \in \mathcal{P}$ bijectively to $\Xi_0 (\text{mod } 0)$.

Chapter 3

Linear response for intermittent circle maps

3.1 Introduction

The linear response of the *Liverani-Saussol-Vaienti (LSV) map*: $f(x) : [0,1] \to [0,1]$, given by

$$f(x) = \begin{cases} x(1 + 2^\alpha x^\alpha), & 0 \le x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} \le x \le 1. \end{cases}$$

has been tackled using several methods such as the coupling argument technique [59], inducing technique [24] and cone technique [28]. For $\alpha \in (0,1)$, these maps are mixing with polynomial decay of correlations [52, 65, 79], summable when $\alpha < 1/2$. In this chapter, we study the linear response of a parameterised family of maps originally introduced by Young in [79].

An approach that has proven highly effective in analyzing positive operators is the cone technique developed by Birkhoff [33]. This technique has found applications in tackling several problems, such as, but not limited to decay of correlations [64], polynomial loss of memory [2], in [35] among several other things to show differentiability of some equilibrium measures, linear response for solenoidal attractors, a skew product with a uniformly expanding maps as the base dynamics [19, 34], in [65] to show some ergodic theoretic property of the LSV map. The method used in [65] has been shown to be useful in various scenarios, particularly in showing the existence of invariant measures for maps with critical point [43]. Leppänen in [63] using the approach of Baladi and Todd [28], showed linear response for the class of maps introduced in [43] which is a specific case of the broader class of maps introduced in [42]. We however remark that for uniformly expanding maps on a circle, a linear response formula was proved by Baladi in [26]. The proof for the linear response formula in this chapter follows the approach in [28]. Since at the heart of the mechanism is the summability of the decay of correlation, we use a better decay estimate shown in [50] which allowed the linear response result to hold for $\alpha \in [0,1)$. The result for the linear response of the intermittent circle map is particularly useful for us since we lifted the regularity to show the statistical stability of the intermittent solenoid map.

3.2 The intermittent circle map

Let $f(x)$ and $g(x)$ be real valued functions, we write $f(x) \lesssim g(x)$ (resp. $f(x) \approx g(x)$) to mean that there exists a uniform constant $C \geq 1$, such that $f(x) \leq Cg(x)$ (resp. $C^{-1}\ \ g(x) \ \leq\ f(x) \ \leq C\, g(x)$) for all x. Observe that, $f(x) \approx g(x)$ means that both $f(x) \lesssim g(x)$ and $g(x) \lesssim f(x)$ holds. The same notation is applicable to sequences, a_n, b_n, for all n. In what follows, for functions depending on both x and the parameter α, we write the partial derivative with respect to the parameter as ∂_α and the derivative with respect to x as $(\cdot)'$.

Let $f_\alpha : S^1 \to S^1$, $S^1 = \mathbb{R}/\mathbb{Z}$, be a degree $d \geq 2$ circle map with $\alpha > 0$ satisfying:

(s1) $f_\alpha(0) = 0$ and $f'_\alpha(0) = 1$;

(s2) $f'_\alpha > 1$ on $S^1 \setminus \{0\}$;

(s3) f_α is C^2 on $S^1 \setminus \{0\}$ and $xf''_\alpha(x) \approx |x|^\alpha$, for x close to 0.

By (s1) and (s3),

$$f'_\alpha(x) - 1 \approx |x|^\alpha, \tag{3.1}$$

integrating the above,

$$f_\alpha(x) \approx x + \operatorname{sgn}(x)|x|^{\alpha+1}, \tag{3.2}$$

where sgn is the signum function, defined as

$$\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ 1, & \text{if } x > 0; \end{cases}$$

with $\operatorname{sgn}(x) = \dfrac{x}{|x|} = \dfrac{|x|}{x}$, for $x \neq 0$. A map satisfying the condition (s1) is said to have an *indifferent (or neutral) fixed point.*

3.2.1 Asymptotic behaviour near the fixed point

Restricting ourselves to $f_\alpha|_{[0,\varepsilon_0]}$ (resp. $f_\alpha|_{[\varepsilon'_0,0]}$), where $(0, \varepsilon_0]$ (resp. $[\varepsilon'_0, 0))$ is an interval where the conditions (s1)-(s3) holds. When we say a degree d map, we mean that there exists an m mod 0 partition of S^1, $\{I_1, \ldots, I_d\}$ into open intervals, such that $f_\alpha|_{I_i} : I_i \to S^1 \setminus \{0\}$ is a diffeomorphism, for each $1 \leq i \leq d$. Let $0 \sim 1$ in such a way that 0 is the infimum of I_1 and the supremum of I_d. The intervals I_1 and I_d are further partitioned into countably many subintervals J_n and J'_n respectively as follows; define the sequences $(z_n)_n$ and $(z'_n)_n$ as

$$f_\alpha(z_{n+1}) = z_n \text{ and } f_\alpha(z'_{n+1}) = z'_n, \quad n \geq 0, z_0 \in (0, \varepsilon_0], z'_0 \in [\varepsilon'_0, 0).$$

Set for each $n \geq 1$

$$J_n = (z_n, z_{n-1}) \quad \text{ and } \quad J'_n = (z'_{n-1}, z'_n).$$

The dynamics is related to the subintervals as follows

$$f_\alpha(J_{n+1}) = J_n \quad \text{and} \quad f_\alpha(J'_{n+1}) = J'_n.$$

The local analysis in the interval containing the intermittent point is given by $z_n \approx n^{-1/\alpha}$ [6, 79], i.e there exists a uniform constant $C \geq 1$ such that

$$\frac{1}{C} n^{-1/\alpha} \leq z_n \leq C n^{-1/\alpha}. \tag{3.3}$$

3.2.2 Bounds on the invariant density

Here, we state a result that enables us to establish some bounds on the invariant density h_α of f_α. Let $\{I_i : i = 1, \dots, d\}$ be the partition of S^1 into disjoint sub-intervals. We denote by x_i the unique fixed point of each partition I_i, $1 \leq i \leq d$. Next, we define the following functions

$$G_i(x) = \begin{cases} (x - x_i) \cdot (f_\alpha(x) - x)^{-1}, & \text{if } x \in I_i, x \neq x_i, \\ 1, & \text{if } x \in S^1 \setminus I_i; \end{cases}$$

$$F_i(x) = \begin{cases} (x - x_i) \cdot (x - g_{\alpha,i}(x))^{-1}, & \text{if } x \in I_i, x \neq x_i, \\ 1, & \text{if } x \in S^1 \setminus I_i. \end{cases} \tag{3.4}$$

Where $g_{\alpha,i} : S^1 \to I_i$ is the inverse branch of $f_{\alpha,i}$, $i \in \{1, d\}$.

It follows from [6, Lemma 3.65] that f_α has an induced transformation with a unique ergodic absolutely continuous invariant measure, whose density is bounded from above and below by positive constants. We now state a result due to Thaler [76] adapted particularly to our setting.

Theorem 3.2.1. *[76, Theorem 1] Let f_α be the map satisfying (s1)-(s3), such that the induced transformation possesses an invariant density bounded from above and below by positive constants. Then, there exists positive constants c_1, c_2 such that the invariant density h_α satisfies*

$$c_1\, G_j(x) \leq h_\alpha(x) \leq c_2\, F_j(x), \quad x \in S^1 \setminus \{0\}.$$

Where j is the partition with $x = 0$.

By the above theorem, there exists positive constants c_1, c_2 such that the invariant density of f_α is bounded as follows

$$c_1 |x|^{-\alpha} \leq h_\alpha(x) \leq c_2 |x|^{-\alpha}, \quad x \in S^1 \setminus \{0\}, c_2 \geq c_1 > 0, \tag{3.5}$$

the singularity point of h_α being at $x = 0$. Indeed, from equation (3.2), for x close to 0

$$\frac{|f_\alpha(x)|}{|x|} \approx 1 + |x|^\alpha. \tag{3.6}$$

Observe that,

$$|x|^{\alpha+1} = |f_\alpha(x)|^{\alpha+1}\left(\frac{|f_\alpha(x)|}{|x|}\right)^{-(\alpha+1)},$$

which together with (3.6) yields

$$|x|^{\alpha+1} \approx |f_\alpha(x)|^{\alpha+1}\left(1+|x|^\alpha\right)^{-(\alpha+1)}.$$

Again, from equation (3.2),

$$\begin{aligned} f_\alpha(x) &\approx x + |f_\alpha(x)|^{\alpha+1}\operatorname{sgn}(x)\left(1+|x|^\alpha\right)^{-(\alpha+1)} \\ x &\approx f_\alpha(x) - \operatorname{sgn}(f_\alpha(x))\,|f_\alpha(x)|^{\alpha+1}\left(\operatorname{sgn}(x)\right)^2\left(1+|x|^\alpha\right)^{-(\alpha+1)}, \end{aligned}$$

which leads us to conclude that

$$g_{\alpha,i}(x) \approx x - \operatorname{sgn}(x)|x|^{1+\alpha}\cdot p(x), \quad i\in\{1,d\}, \tag{3.7}$$

where $p(x) = (1+|g_{\alpha,i}(x)|^\alpha)^{-(\alpha+1)}$. From equations (3.2), (3.4) and (3.7), we get

$$\begin{aligned} G_j(x) &\approx \frac{x}{\operatorname{sgn}(x)|x|^{\alpha+1}} = \frac{1}{|x|^\alpha}, \\ F_j(x) &\approx \frac{x}{\operatorname{sgn}(x)|x|^{\alpha+1}\cdot p(x)} = \frac{1}{p(x)|x|^\alpha}, \end{aligned}$$

which verifies equation (3.5).

3.3 The mechanism

Using the approach of Baladi and Todd [28], we explain the mechanism that will be used in showing the linear response for the family of maps introduced in the previous section.

Firstly, we shall assume that the perturbation occurs at the image. That is, there exists a vector field X_ϱ such that

$$v_\varrho(x) := \partial_\alpha f_\alpha(x)\Big|_{\alpha=\varrho} = X_\varrho\circ f_\varrho(x), \quad \alpha,\varrho\in V,$$

and V a small neighbourhood of 0. Since f_α is defined in the neighbourhood of 0 and is invertible on the branches $i=\{1,d\}$, from the above equation we have that

$$X_{\alpha,i}(x) = v_\alpha\circ g_{\alpha,i}(x), \quad i=\{1,d\}. \tag{3.8}$$

For x in the neighbourhood of 0, $\varrho\in[0,1)$. We have from equation (3.2) that

$$v_\varrho(x) = \partial_\varrho f_\varrho(x) \approx \operatorname{sgn}(x)|x|^{\varrho+1}\ln(|x|), \tag{3.9}$$

$$X_{\varrho,i}(x) \approx \operatorname{sgn}(g_{\varrho,i}(x))|g_{\varrho,i}(x)|^{\varrho+1}\ln(|g_{\varrho,i}(x)|). \tag{3.10}$$

Next, we shall make some assumptions on the map $f_\varrho|_{I_i} : I_i \to S^1 \setminus \{0\}$.

(A1) There exists $C \geq 1$ such that

$$v_\varrho(x) = \partial_\varrho f_\varrho(x) \leq C\,\mathrm{sgn}(x)|x|^{\varrho+1}\ln(|x|), \qquad \text{for } x \in I_i, i = 1, d. \tag{3.11}$$

(A2) For a $d \geq 2$ branch map, we define the right end point of the first branch by $I_{1,+}$ and the left end point of the last branch by $I_{d,-}$ and assume that

$$\begin{aligned} v_\varrho(I_{1,+}) &= 0; \\ v_\varrho(I_{d,-}) &= 0. \end{aligned} \tag{3.12}$$

(A3) $\varrho \mapsto f_{\varrho,i} \in C^2$ and the following partial derivatives exist, and also satisfy the commutation relation

$$\partial_\varrho g'_{\varrho,i} \approx (\partial_\varrho g_{\varrho,i})' \quad \text{and} \quad \partial_\varrho f'_{\varrho,i} \approx (\partial_\varrho f_{\varrho,i})', \quad i = 1, d. \tag{3.13}$$

From equation (3.7), and $x \in S^1$, $i = 1, d$, there exists $C \geq 1$ such that,

$$\begin{aligned} &g_{\varrho,i}(x) \leq C\,x, \\ &g'_{\varrho,i}(x) \text{ is bounded}, \\ &g''_{\varrho,i}(x) \leq C|x|^{\varrho-1}, \\ &g'''_{\varrho,i}(x) \leq C|x|^{\varrho-2}. \end{aligned} \tag{3.14}$$

Suppose that f_α is the one parameter family of map with $d \geq 2$ branches, satisfying the assumptions (A1)-(A3). Let $f_{\alpha,1} : [0,\kappa] \to S^1$, $f_{\alpha,d} : [1-\kappa, 1] \to S^1, \kappa = 1/d$, be its first and last branches respectively. The middle $(d-2)$ branches are piecewise expanding. Using equation (3.14), there exists $\tilde{C} > 1$ such that we bound equation (3.10) as follows

$$|X_{\varrho,i}(x)| \leq \tilde{C}|x|^{\varrho+1}(1 + |\ln(|x|)|). \tag{3.15}$$

Subsequently, differentiating equation (3.10)

$$X'_{\varrho,i}(x) \approx g'_{\varrho,i}(x)|g_{\varrho,i}(x)|^\varrho \left[1 + (1+\varrho)\ln|g_{\varrho,i}(x)|\right], \tag{3.16}$$

we bound the above equation using the bounds in equation (3.14)

$$\begin{aligned} |X'_{\varrho,i}(x)| &\leq C|x|^\varrho \left[1 + (1+\varrho)(\ln C + |\ln(|x|)|)\right] \\ &\leq \tilde{C}|x|^\varrho(1 + |\ln(|x|)|). \end{aligned} \tag{3.17}$$

Differentiating equation (3.16),

$$X''_{\varrho,i}(x) \approx |g_{\varrho,i}(x)|^{\varrho-1}\Big\{(1+\varrho)\mathrm{sgn}(g_{\alpha,i}(x))(g'_{\varrho,i}(x))^2 + \Big[\varrho\,\mathrm{sgn}(g_{\varrho,i}(x))(g'_{\varrho,i}(x))^2$$

$$+ |g_{\varrho,i}(x)|g''_{\varrho,i}(x)\Big] \cdot [1 + (1+\varrho)\ln(|g_{\varrho,i}(x)|)]\Big\},$$
(3.18)

which we bound using equation (3.14) as

$$|X''_{\varrho,i}(x)| \leq \tilde{C}|x|^{\varrho-1}(1 + |\ln(|x|)|). \tag{3.19}$$

Differentiating equation (3.18)

$$\begin{aligned}
X'''_{\varrho,i}(x) \approx &|g_{\varrho,i}(x)|^{\varrho-1}\Big\{2(1+\varrho)\,\mathrm{sgn}(g_{\varrho,i}(x))g'_{\varrho,i}(x)\,g''_{\varrho,i}(x) + (1+\varrho)[\varrho\,\mathrm{sgn}(g_{\varrho,i}(x))\,(g'_{\varrho,i}(x))^2 \\
&+ |g_{\varrho,i}(x)|\,g''_{\varrho,i}(x)]\,\frac{g'_{\varrho,i}(x)}{g_{\varrho,i}(x)} + [1 + (1+\varrho)\,\ln(|g_{\varrho,i}(x)|)] \cdot [2\varrho\mathrm{sgn}(g_{\varrho,i}(x))g'_{\varrho,i}(x)g''_{\varrho,i}(x) \\
&+ |g_{\varrho,i}(x)|\,g'''_{\varrho,i}(x) + \mathrm{sgn}(g_{\varrho,i}(x))\,g'_{\varrho,i}(x)\,g''_{\varrho,i}(x)]\Big\} \\
&+ (\varrho-1)\mathrm{sgn}(g_{\varrho,i}(x))g'_{\varrho,i}(x)|g_{\varrho,i}(x)|^{\varrho-2}\Big\{(1+\varrho)\,\mathrm{sgn}(g_{\varrho,i}(x))\,(g'_{\varrho,i}(x))^2 \\
&+ [\varrho\,\mathrm{sgn}(g_{\varrho,i}(x))(g'_{\varrho,i}(x))^2 + |g_{\varrho,i}(x)|g''_{\varrho,i}(x)] \cdot [1 + (1+\varrho)\ln(|g_{\varrho,i}(x)|)]\Big\},
\end{aligned}$$

using equation (3.14), we have the following bounds

$$|X'''_{\varrho,i}(x)| \leq \tilde{C}|x|^{\varrho-2}(1 + |\ln(|x|)|). \tag{3.20}$$

We now state here the main result of this chapter.

Theorem A. *Suppose that f_α is the family of circle maps described above for $\alpha \in (0,1)$ and satisfy the assumptions (A1)-(A3). Then for any $\psi \in L^q(m)$ with $q > (1-\alpha)^{-1}$,*

$$\lim_{\varepsilon \to 0} \frac{\int_{S^1} \psi\, d\mu_{\alpha+\varepsilon} - \int_{S^1} \psi\, d\mu_\alpha}{\varepsilon} = \int_{S^1} \psi(\mathrm{id} - \mathcal{L}_\alpha)^{-1} \left[\sum_{i\in\{1,d\}} (X_{\alpha,i}\,\mathcal{N}_{\alpha,i}(h_\alpha))'\right] dx. \tag{3.21}$$

Taking limit $\varepsilon \to 0^+$, (3.21) holds for $\alpha = 0$.

Where $\mathcal{L}_\alpha$ is the Perron-Frobenius operator associated with f_α and $\mathcal{N}_{\alpha,i}$ the transfer operator associated to f_α in the *ith* branch, defined as

$$\mathcal{L}_\alpha\varphi(x) = \sum_{f_\alpha(y)=x} \frac{\varphi(y)}{f'_\alpha(y)}, \quad \varphi \in L^1(m), \tag{3.22}$$

this follows from equation (2.9) and (s2). Next, for $\varphi \in L^1(m)$,

$$\mathcal{N}_{\alpha,i}\varphi(x) = g'_{\alpha,i}(x) \cdot \varphi(g_{\alpha,i}(x)), \quad i \in \{1,d\}. \tag{3.23}$$

3.3.1 Invariant cones

Definition 3.3.1 (Cone)**.** *Let E be a vector space. A* cone *in E is a subset $C \subset E \setminus \{0\}$ such that for $\varphi \in C$ then $\lambda\varphi \in C$, for each $\lambda > 0$.*

Following the idea of [65] we define certain cones and show that they are invariant with respect to the operators defined in equations (3.22) and (3.23). We denote the Lebesgue measure on S^1 by m and define

$$m(\varphi) = \int_{S^1} \varphi(x)\,dm = \int_{S^1} \varphi(x)\,dx, \tag{3.24}$$

then, by Proposition 2.3.4,

$$m(\mathcal{L}_\alpha\varphi) = m(\varphi). \tag{3.25}$$

Let $a_1, b_1 > 0$, and define the cone

$$\mathcal{C}_{*,1} = \left\{\varphi \in C^1(S^1 \setminus \{0\}) \middle| 0 \le \varphi(x) \le 2h_\alpha(x)\int_{S^1}\varphi\,dx, |\varphi'(x)| \le \left(\frac{a_1}{|x|} + b_1\right)\varphi(x)\right\}. \tag{3.26}$$

It is straightforward to check that this is indeed a cone. Since $0 < \alpha < 1$ for the bounds on the density in equation (3.5), it follows that $\mathcal{C}_{*,1} \subset L^1(m)$. Also, observe that

$$\varphi(x) \le \frac{2c_2}{|x|^\alpha} m(\varphi), \quad \forall \varphi \in \mathcal{C}_{*,1}, x \in S^1 \setminus \{0\}, \tag{3.27}$$

and for $\beta \ge \alpha \ge 0$,

$$\mathcal{C}_{*,1}(\alpha, 1, a_1, b_1) \subset \mathcal{C}_{*,1}\left(\beta, \frac{c_2}{c_1}, a_1, b_1\right). \tag{3.28}$$

Lemma 3.3.1. *$\mathcal{C}_{*,1}$ is invariant with respect to the Perron-Frobenius operator, provided we choose a_1, b_1/a_1 big enough.*

Proof. For $\varphi \in \mathcal{C}_{*,1}$, we show that the first condition is invariant by $\mathcal{L}_\alpha$. Indeed, from equation (3.22), we have that by (C2) and equation (3.25),

$$\begin{aligned}
\mathcal{L}_\alpha\varphi(x) &= \sum_{f_\alpha(y)=x} \frac{\varphi(y)}{f'_\alpha(y)} \\
&\le \sum_{f_\alpha(y)=x} \frac{2h_\alpha(y)\int_{S^1}\varphi(y)\,dy}{f'_\alpha(y)} \\
&\le 2\int_{S^1}\varphi(y)\,dy \sum_{f_\alpha(y)=x} \frac{h_\alpha(y)}{f'_\alpha(y)} \\
&= 2\mathcal{L}_\alpha h_\alpha(x) m(\varphi) = 2h_\alpha(x) m(\mathcal{L}_\alpha\varphi).
\end{aligned}$$

Proposition 2.3.3 allows us to conclude the invariance of the first condition. Next, we show the invariance by $\mathcal{L}_\alpha$ of the second condition

$$|(\mathcal{L}_\alpha\varphi)'(x)| = \left|\sum_{f_\alpha(y)=x} \frac{f''_\alpha(y)}{(f'_\alpha(y))^3}\varphi(y) + \frac{1}{(f'_\alpha(y))^2}\varphi'(y)\right|$$

$$
\begin{aligned}
&= \left| \sum_{f_\alpha(y)=x} \left(\frac{f''_\alpha(y)}{(f'_\alpha(y))^2} + \frac{1}{\varphi(y)(f'_\alpha(y))} \varphi'(y) \right) \frac{\varphi(y)}{(f'_\alpha(y))} \right| \\
&\le \sum_{f_\alpha(y)=x} \frac{\varphi(y)}{(f'_\alpha(y))} \left(\frac{|f''_\alpha(y)|}{(f'_\alpha(y))^2} + \frac{1}{\varphi(y)(f'_\alpha(y))} |\varphi'(y)| \right) \\
&\le \sum_{f_\alpha(y)=x} \frac{\varphi(y)}{(f'_\alpha(y))} \left(\frac{C|y|^{\alpha-1}}{(f'_\alpha(y))^2} + \frac{a_1 + b_1|y|}{|y|(f'_\alpha(y))} \right) \\
&\le \left(\frac{a_1}{|x|} + b_1 \right) \sum_{f_\alpha(y)=x} \frac{\varphi(y)}{(f'_\alpha(y))} \sup_{y \in S^1} \left[\frac{|f_\alpha(y)|}{a_1 + b_1|f_\alpha(y)|} \cdot \left(\frac{C|y|^{\alpha-1}}{(f'_\alpha(y))^2} + \frac{a_1 + b_1|y|}{|y|(f'_\alpha(y))} \right) \right] \\
&\le \left(\frac{a_1}{|x|} + b_1 \right) \mathcal{L}_\alpha \varphi(x) \sup_{y \in S^1} \left[\left(\frac{|f_\alpha(y)|}{a_1 + b_1|f_\alpha(y)|} \cdot \frac{C|y|^{\alpha-1}}{(f'_\alpha(y))^2} + \frac{|f_\alpha(y)|}{|y| \cdot f'_\alpha(y)} \cdot \frac{a_1 + b_1|y|}{a_1 + b_1|f_\alpha(y)|} \right) \right].
\end{aligned}
$$

We set

$$
\Omega_1(y) = \frac{|f_\alpha(y)|}{a_1 + b_1|f_\alpha(y)|} \cdot \frac{C|y|^{\alpha-1}}{(f'_\alpha(y))^2} + \frac{|f_\alpha(y)|}{|y| \cdot f'_\alpha(y)} \cdot \frac{a_1 + b_1|y|}{a_1 + b_1|f_\alpha(y)|}. \tag{3.29}
$$

To complete the proof, we need to show that $\Omega_1(y) \le 1$. We do this for y in the neighbourhood of δ, δ small, and also for $\delta < y < 1 - \delta$.

For y in the neighbourhood of δ, we only need to show that

$$
\underbrace{\frac{|f_\alpha(y)|}{|y| f'_\alpha(y)} \left(\frac{C|y|^{\alpha+1}}{|y| f'_\alpha(y)(a_1 + b_1|f_\alpha(y)|) - |f_\alpha(y)|(a_1 + b_1|y|)} \right)}_{\Lambda_1(y)} \le 1
$$

$$
\begin{aligned}
\Lambda_1(y) &= \frac{|f_\alpha(y)|}{|y| f'_\alpha(y)} \left(\frac{C|y|^{\alpha+1}}{a_1(|y| f'_\alpha(y) - |f_\alpha(y)|) + b_1|y||f_\alpha(y)|(f'_\alpha(y) - 1)} \right) \\
&\le \frac{|f_\alpha(y)|}{|y| f'_\alpha(y)} \left(\frac{C|y|^{\alpha+1}}{a_1(|y| f'_\alpha(y) - |f_\alpha(y)|)} \right)
\end{aligned}
$$

From equations (3.2) and (3.1), $\dfrac{|f_\alpha(y)|}{|y|\, f'_\alpha(y)} \lesssim 1$. Hence, there exists $\delta > 0$ such that choosing a_1 big enough, $\Omega_1(y) \le 1$.

For $1 - \delta > y > \delta$, there exists γ such that $f'_\alpha(y) \ge \gamma > 1$, which implies that $\dfrac{1}{f'_\alpha(y)} \le \dfrac{1}{\gamma} < 1$. Therefore,

$$
|y| > \delta \ \Rightarrow \ |y|^{\alpha-1} < \delta^{\alpha-1}, \quad \text{for } \alpha \in [0, 1).
$$

For $b_1 > 0$, $\dfrac{|f_\alpha(y)|}{a_1 + b_1|f_\alpha(y)|} \le \dfrac{1}{a_1}$. Simplifying, we have that

$$
|f_\alpha(y)| \left(\frac{a_1 + b_1|y|}{a_1 + b_1|f_\alpha(y)|} \right) \le \frac{a_1}{b_1} + |y|.
$$

Substituting values into equation (3.29),

$$\begin{aligned}\Omega(y) &\leq \frac{1}{a_1} \cdot \frac{C|y|^{\alpha-1}}{|f'_\alpha(y)|^2} + \frac{1}{|y||f'_\alpha(y)|} \cdot \left(\frac{a_1}{b_1} + |y|\right) \\ &\leq \frac{1}{a_1} \cdot \frac{C\delta^{\alpha-1}}{\gamma^2} + \frac{a_1}{b_1} \cdot \frac{1}{\delta\gamma} + \frac{1}{\gamma}.\end{aligned}$$

$\Omega(y) \leq 1$, provided we choose a_1 and b_1/a_1, big enough. □

Proposition 3.3.2. *If $\varphi \in \mathcal{C}_{*,1}$, then*

$$\min_{x\in S^1\setminus B_\delta(0)} \varphi(x) \geq \frac{\delta^{a_1}}{2\,e^{b_1(1-\delta)}} \int_{S^1} \varphi(x)\,dx,$$

choosing δ small enough.

Proof. Since $\varphi \in \mathcal{C}_{*,1}$, we have the bounds

$$\begin{aligned}\varphi(x) &\leq 2h_\alpha(x) \int_{S^1} \varphi\,dx \leq 2c_2|x|^{-\alpha} \int_{S^1} \varphi(x)\,dx \\ |\varphi'(x)| &\leq \left(\frac{a_1}{|x|} + b_1\right)\varphi(x).\end{aligned}$$

Now, for $x, y \in S^1$, $x \geq y > \delta$, the second inequality gives

$$|x|^{-a_1\,\mathrm{sgn}(x)}\mathrm{e}^{-b_1 x} \leq \varphi(x) \leq |x|^{a_1\,\mathrm{sgn}(x)}\mathrm{e}^{b_1 x}. \tag{3.30}$$

For simplicity, we suppose that $\int_{S^1} \varphi(x)\,dx = 1$. Then, for $x \in B_\delta(0)$,

$$\int_{B_\delta(0)} \varphi(x)\,dx \leq 2\int_0^\delta 2c_2|x|^{-\alpha}\,dx \leq \frac{4c_2}{1-\alpha}\mathrm{sgn}(\delta)|\delta|^{1-\alpha}. \tag{3.31}$$

For $x \in S^1 \setminus B_\delta(0)$, from equation (3.30), we immediately see that the function is bounded from below by a decreasing function and from above by an increasing function. Without loss of generality, we make the calculations for $x \in [\delta, 1-\delta]$

$$\begin{aligned}\delta^{-a_1}\mathrm{e}^{-b_1\delta} &\leq \max\varphi(x) \leq (1-\delta)^{a_1}\mathrm{e}^{(1-\delta)b_1} \leq \mathrm{e}^{(1-\delta)b_1} \\ \mathrm{e}^{-b_1(1-\delta)} \leq (1-\delta)^{-a_1}\mathrm{e}^{((1-\delta)b_1)} &\leq \min\varphi(x) \leq \delta^{a_1}\mathrm{e}^{b_1\delta}\end{aligned}$$

$$\begin{aligned}\int_\delta^{1-\delta} \varphi(x)\,dx &= \max\varphi(x) \cdot (1-2\delta) \leq \max\varphi(x) \\ &\leq \mathrm{e}^{(1-\delta)b_1} \cdot \mathrm{e}^{-b_1(1-\delta)} \cdot \mathrm{e}^{b_1(1-\delta)}\delta^{a_1} \cdot \delta^{-a_1} \\ &\leq \delta^{-a_1}\mathrm{e}^{b_1(1-\delta)} \min\varphi(x) \cdot \delta^{a_1}\mathrm{e}^{b_1(1-\delta)} \\ &\leq \delta^{-a_1}\mathrm{e}^{b_1(1-\delta)} \min_{x\in S^1\setminus B_\delta(0)} \varphi(x),\end{aligned}$$

provided we choose δ small enough.

$$\begin{aligned}\int_{S^1}\varphi(x)\,dx &= \int_{B_\delta(0)}\varphi(x)\,dx + \int_{S^1\setminus B_\delta(0)}\varphi(x)\,dx\\ &\le \frac{4c_2}{1-\alpha}\mathrm{sgn}(\delta)|\delta|^{1-\alpha} + \delta^{-a_1}\mathrm{e}^{b_1(1-\delta)}\min_{x\in S^1\setminus B_\delta(0)}\varphi(x). \qquad (3.32)\end{aligned}$$

Taking δ small enough, we may bound $\frac{4c_2}{1-\alpha}\mathrm{sgn}(\delta)|\delta|^{1-\alpha} \le \frac{1}{2}$. Equation (3.32) now becomes

$$\min_{x\in S^1\setminus B_\delta(0)}\varphi(x) \ge \frac{\delta^{a_1}}{2\mathrm{e}^{b_1(1-\delta)}}.$$

□

Lemma 3.3.3. *There exists a $\delta > 0$ and $\gamma > 0$, such that*

$$\mathcal{C}_{*,2} = \left\{\varphi \in \mathcal{C}_{*,1}\middle|\varphi(x) \ge \gamma\int_{S^1}\varphi(x)\,dx,\ \textit{for}\ |x| \le \delta\right\} \qquad (3.33)$$

is invariant with respect to the Perron-Frobenius operator.

Proof. For $|x| \le \delta$, let $f_{\alpha,i}^{-1}(x) = y_i$, $i = 1,\cdots,d$. Denote by y_* the y_i on the first or last branch such that $|y_*| \le \delta$. Suppose also that $\mu = \|f_\alpha'(y_i)\|_\infty$. We choose δ small enough such that by equation (3.1), $f_\alpha'(y) \approx (1+|y|^\alpha)$, so that

$$\frac{1}{f_\alpha'(y_*)} \ge \frac{1}{C(1+|y_*|^\alpha)} \ge \frac{1}{C}(1-\delta^\alpha), \quad C \ge 1,$$

Proposition 3.3.2 and $\frac{1}{C}(1-\delta^\alpha)+\mu^{-1} > 1$ holds. From equation (3.22), we have that

$$\begin{aligned}\mathcal{L}_\alpha\varphi(x) &\ge \frac{\varphi(y_*)}{f_\alpha'(y_*)} + \frac{\varphi(y_i)}{\|f_\alpha'(y_i)\|_\infty} = (f_\alpha'(y_*))^{-1}\varphi(y_*) + \|f_\alpha'(y_i)\|_\infty^{-1}\varphi(y_i)\\ &\ge \left[\frac{1}{C}(1-\delta^\alpha)\cdot\gamma\int_{S^1}\varphi(x)\,dx + \mu^{-1}\varphi(y_i)\right]\\ &\ge \left[\frac{1}{C}(1-\delta^\alpha)\cdot\gamma\int_{S^1}\varphi(x)\,dx + \mu^{-1}\min\left\{\gamma\int_{S^1}\varphi(x)\,dx, \frac{\delta^a}{2\mathrm{e}^{b(1-\delta)}}\int_{S^1}\varphi\,dx\right\}\right]\\ &\ge \left[\frac{1}{C}(1-\delta^\alpha)\cdot\gamma + \mu^{-1}\min\left\{\gamma, \frac{\delta^a}{2\mathrm{e}^{b(1-\delta)}}\right\}\right]\int_{S^1}\varphi\,dx\\ &\ge \left[\frac{1}{C}(1-\delta^\alpha)\cdot\gamma + \mu^{-1}\min\left\{\gamma, \frac{\delta^a}{2\mathrm{e}^{b(1-\delta)}}\right\}\right]\int_{S^1}\mathcal{L}_\alpha\varphi\,dx\\ &\ge \gamma\int_{S^1}\mathcal{L}_\alpha\varphi\,dx.\end{aligned}$$

□

From Proposition 3.3.2 and Lemma 3.3.3 we have that $\inf_{S^1}\varphi(x) \ge \gamma\int_{S^1}\varphi(x)dx$, which implies that

$$\inf_{n\ge 0}\inf_{S^1}\mathcal{L}^n 1 \ge \gamma > 0, \qquad (3.34)$$

particularly, since the constant function $1 \in \mathcal{C}_{*,1}$.

In the spirit of [28], we define the following cone for higher order derivatives, and show that it is invariant with respect to the Perron-Frobenius operator. For $a_1, a_2, a_3, b_1, b_2, b_3 > 0$, define the cone

$$\mathcal{C} = \Big\{\varphi \in C^{(3)}(S^1 \setminus \{0\}) \Big| \varphi(x) \geq 0, |\varphi'(x)| \leq \left(\frac{a_1}{|x|} + b_1\right)\varphi(x), |\varphi''(x)| \leq \left(\frac{a_2}{x^2} + b_2\right)\varphi(x), |\varphi'''(x)| \leq \left(\frac{a_3}{|x|^3} + b_3\right)\varphi(x), \forall x \in S^1 \setminus \{0\}\Big\}. \tag{3.35}$$

Lemma 3.3.4. *Suppose that* $\frac{\min\{a_2,b_2\}}{\max\{a_1,b_1\}}, \frac{\min\{a_3,b_3\}}{\max\{a_1,b_1\}}, \frac{\min\{a_3,b_3\}}{\max\{a_2,b_2\}}$ *are large enough. Then the cone is invariant with respect to the operators* $\mathcal{L}_\alpha$ *and* $\mathcal{N}_{\alpha,i}$*, for* $i \in \{1, d\}$*.*

Proof. From the definition of the operators in equation (3.22) and equation (3.23),

$$\mathcal{L}_\alpha\varphi(x) = \sum_{y \in f_{\alpha,i}^{-1}(x),\, 2 \leq i \leq d-1} \frac{\varphi(y)}{f'_\alpha(y)} + \sum_{i \in \{1,d\}} \mathcal{N}_{\alpha,i}\varphi(x). \tag{3.36}$$

Hence, we only need to show the invariance with respect to $\mathcal{N}_{\alpha,i}$ since the invariance with respect to $\mathcal{L}_\alpha$ follows immediately. Indeed, from equation (3.36) and (s2),

$$\mathcal{L}_\alpha\varphi(x) - \sum_{i \in \{1,d\}} \left(\mathcal{N}_{\alpha,i}\varphi\right)(x) \leq \sum_{y \in f_\alpha^{-1}(x),\, 2 \leq i \leq d-1} \varphi(y).$$

The proof of the invariance of the cone with respect to the first derivative of $\mathcal{N}_{\alpha,i}\varphi(x)$ is exactly as in Lemma 3.3.1. We remark that the first summation on the right hand side of equation (3.36) is not applicable when $d = 2$.

$$\mathcal{N}_{\alpha,i}\varphi(x) < \sum_{i \in \{1,d\}} \mathcal{N}_{\alpha,i}\varphi(x) \leq \mathcal{L}_\alpha\varphi(x) \leq 2h_\alpha(x)m(\varphi).$$

This together with what we shall show next implies that the cone $\mathcal{C}$ is invariant with respect to this operator.

$$(\mathcal{N}_{\alpha,i}\varphi)'(x) = \frac{\varphi(y) \cdot f''_{\alpha,i}(y)}{(f'_{\alpha,i}(y))^3} - \frac{\varphi'(y)}{(f'_{\alpha,i}(y))^2} \tag{3.37}$$

$$\begin{aligned} |(\mathcal{N}_{\alpha,i}\varphi)'(x)| &= \left|\frac{\varphi(y) \cdot f''_{\alpha,i}(y)}{(f'_{\alpha,i}(y))^3} - \frac{\varphi'(y)}{(f'_{\alpha,i}(y))^2}\right| \\ &\leq \left(\frac{a_1}{|x|} + b_1\right)(\mathcal{N}_{\alpha,i}\varphi)(x) \sup_{y \in I_i}\left[\frac{|f_\alpha(y)|}{a_1 + b_1|f_\alpha(y)|}\left(\frac{|f''_{\alpha,i}(y)|}{(f'_{\alpha,i}(y))^2} + \frac{a_1 + b_1|y|}{|y|(f'_{\alpha,i}(y))}\right)\right]. \end{aligned}$$

We set

$$\Omega_1(y) = \frac{|f_\alpha(y)|}{a_1 + b_1|f_\alpha(y)|}\left(\frac{|f''_{\alpha,i}(y)|}{(f'_{\alpha,i}(y))^2} + \frac{a_1 + b_1|y|}{|y|(f'_{\alpha,i}(y))}\right).$$

Following exactly the steps in Lemma 3.3.1, we have

$$\Omega_1(y) \le 1, \tag{3.38}$$

if we choose $a_1 > 0$ and $b_1/a_1 > 0$ big enough. From (s3),

$$x^2|f_\alpha'''(x)| \approx |x|^\alpha. \tag{3.39}$$

To show the invariance of the third condition in $\mathcal{C}$ with respect to the operator, we have that from equation (3.37),

$$
\begin{aligned}
(\mathcal{N}_{\alpha,i}\varphi)''(x) &= -3\frac{\varphi'(y)f_\alpha''(y)}{(f_\alpha'(y))^4} - \frac{\varphi(y)f_\alpha'''(y)}{(f_\alpha'(y))^4} + 3\frac{\varphi(y)(f_\alpha''(y))^2}{(f_\alpha'(y))^5} + \frac{\varphi''(y)}{(f_\alpha'(y))^3} \qquad (3.40)\\
|(\mathcal{N}_{\alpha,i}\varphi)''(x)| &\le \frac{\varphi(y)}{f_\alpha'(y)}\left(3\frac{|\varphi'(y)||f_\alpha''(y)|}{\varphi(y)(f_\alpha'(y))^3} + \frac{|f_\alpha'''(y)|}{(f_\alpha'(y))^3} + 3\frac{(f_\alpha''(y))^2}{(f_\alpha'(y))^4} + \frac{|\varphi''(y)|}{\varphi(y)(f_\alpha'(y))^2}\right)\\
&\le \frac{\varphi(y)}{f_\alpha'(y)}\left(3\frac{|f_\alpha''(y)|}{(f_\alpha'(y))^3}\cdot\left(\frac{a_1+b_1|y|}{|y|}\right) + \frac{|f_\alpha'''(y)|}{(f_\alpha'(y))^3} + 3\frac{(f_\alpha''(y))^2}{(f_\alpha'(y))^4} + \frac{a_2+b_2y^2}{y^2\cdot(f_\alpha'(y))^2}\right)\\
&\le \frac{a_2+b_2y^2}{y^2}\mathcal{N}_{\alpha,i}\varphi(x)\sup_{y\in I_i}\left[\frac{f_\alpha(y)^2}{(a_2+b_2f_\alpha(y)^2)}\left(\frac{3C|y|^{\alpha-1}}{(f_\alpha'(y))^3}\cdot\left(\frac{a_1+b_1|y|}{|y|}\right)\right.\right.\\
&\qquad\left.\left.+\frac{C|y|^{\alpha-2}}{(f_\alpha'(y))^3} + \frac{3|y|^{2(\alpha-1)}}{(f_\alpha'(y))^4} + \frac{a_2+b_2y^2}{y^2\cdot(f_\alpha'(y))^2}\right)\right].
\end{aligned}
$$

We define the expression in the square bracket as

$$\Omega_2(y) = \frac{f_\alpha(y)^2}{(a_2+b_2f_\alpha(y)^2)}\left(\frac{3C|y|^{\alpha-1}}{(f_\alpha'(y))^3}\cdot\left(\frac{a_1+b_1|y|}{|y|}\right) + \frac{C|y|^{\alpha-2}}{(f_\alpha'(y))^3} + \frac{3|y|^{2(\alpha-1)}}{(f_\alpha'(y))^4} + \frac{a_2+b_2y^2}{y^2\cdot(f_\alpha'(y))^2}\right),$$

then show that $\Omega_2(y) \le 1$. We show this for y in the neighbourhood of δ, δ small enough. Since $a_2, b_2 > 0$, we have the following estimates

$$\frac{1}{a_2+b_2(f_\alpha(y))^2} \le \frac{1}{a_2}.$$

From equation (3.2), there exists $C \ge 1$ and $\lambda_2 \ge 0$ such that

$$\frac{a_2+b_2y^2}{a_2+b_2(f_\alpha(y))^2} = 1 - \frac{b_2\left(|f_\alpha(y)|^2-|y|^2\right)}{a_2+b_2|f_\alpha(y)|^2} = 1-\lambda_2,$$

where $\lambda_2 = \dfrac{b_2\left(|f_\alpha(y)|^2-|y|^2\right)}{a_2+b_2|f_\alpha(y)|^2}$, $\lambda_2 \le 1$. We estimate the following as

$$\frac{a_1+b_1|y|}{a_2+b_2(f_\alpha(y))^2} \le \frac{\max\{a_1,b_1\}}{\min\{a_2,b_2\}}\left(\frac{1+|y|}{(1+(f_\alpha(y))^2)}\right) = \rho_2\frac{\max\{a_1,b_1\}}{\min\{a_2,b_2\}}. \tag{3.41}$$

$$\Omega_2(y) = \left(1 + \frac{3C|y|^\alpha}{f'_\alpha(y)}\rho_2 \frac{\max\{a_1,b_1\}}{\min\{a_2,b_2\}} + \frac{1}{a_2}\frac{C|y|^\alpha}{f'_\alpha(y)} + \frac{1}{a_2}\frac{3|y|^{2\alpha}}{(f'_\alpha(y))^2} - \lambda_2\right)\left(\frac{f_\alpha(y)}{|y|f'_\alpha(y)}\right)^2 \leq 1,$$

provided we choose $\frac{\min\{a_2,b_2\}}{\max\{a_1,b_1\}}$ and a_2 large enough, observing from equations (3.1) and (3.2) that $\frac{|f_\alpha(y)|}{|y|\,f'_\alpha(y)} \lesssim 1$.

Now, for $y \in (\delta,\kappa] \cup [1-\kappa,-\delta)$, we have that $f'_\alpha(y) \geq \gamma > 1$, which implies that $\frac{1}{f'_\alpha(y)} \leq \frac{1}{\gamma} < 1$. Suppose that $0 < b_1 < b_2$,

$$\frac{(f_\alpha(y))^2}{(a_2 + b_2 f_\alpha(y)^2)} \leq \frac{1}{a_2},$$
$$f_\alpha(y)^2\left(\frac{a_1 + b_1|y|}{a_2 + b_2 f_\alpha(y)^2}\right) \leq \frac{a_1}{b_2} + \frac{b_1}{b_2}|y|,$$
$$f_\alpha(y)^2\left(\frac{a_2 + b_2 y^2}{a_2 + b_2 f_\alpha(y)^2}\right) \leq \frac{a_2}{b_2} + |y|^2.$$

Now, for

$$|y| > \delta \ \Rightarrow\ |y|^{\alpha-1} < \delta^{\alpha-1}, \quad \text{for } \alpha \in [0,1).$$

$$\begin{aligned}\Omega_2(y) &= \frac{f_\alpha(y)^2}{(a_2 + b_2 f_\alpha(y)^2)}\left(\frac{3C|y|^{\alpha-1}}{(f'_\alpha(y))^3}\cdot\left(\frac{a_1 + b_1 y^2}{y^2}\right) + \frac{C|y|^{\alpha-2}}{(f'_\alpha(y))^3} + \frac{3|y|^{2(\alpha-1)}}{(f'_\alpha(y))^4} + \frac{a_2 + b_2 y^2}{y^2\cdot(f'_\alpha(y))^2}\right)\\ &\leq \frac{3C\delta^{\alpha-3}}{\gamma^3}\cdot\frac{a_1}{b_2} + \frac{3C\delta^{\alpha-2}}{\gamma^3}\cdot\frac{b_1}{b_2} + \frac{C\delta^{\alpha-2}}{\gamma^3}\cdot\frac{1}{a_2} + \frac{3\delta^{2(\alpha-1)}}{\gamma^4}\cdot\frac{1}{a_2} + \frac{1}{\delta^2\gamma^2}\cdot\frac{a_2}{b_2} + \frac{1}{\gamma^2}\\ &\leq 1,\end{aligned}$$

for $a_2, \frac{b_2}{b_1}, \frac{b_2}{a_1}, \frac{b_2}{a_2}$ large enough. From equation (3.39),

$$x^3 f_\alpha^{(iv)}(x) \approx |x|^\alpha. \tag{3.42}$$

Differentiating equation (3.40), we have that

$$\begin{aligned}(\mathcal{N}_{\alpha,i}\varphi)'''(x) = &-4\frac{\varphi'(y)f'''_\alpha(y)}{(f'_\alpha(y))^5} - 6\frac{\varphi''(y)f''_\alpha(y)}{(f'_\alpha(y))^5} + 15\frac{\varphi'(y)(f''_\alpha(y))^2}{(f'_\alpha(y))^6} + 10\frac{\varphi(y)f''_\alpha(y)f'''_\alpha(y)}{(f'_\alpha(y))^6}\\ &- 15\frac{\varphi(y)(f''_\alpha(y))^3}{(f'_\alpha(y))^7} - \frac{\varphi(y)f_\alpha^{(iv)}(y)}{(f'_\alpha(y))^5} + \frac{\varphi'''(y)}{(f'_\alpha(y))^4}\end{aligned}$$

$$\begin{aligned}|(\mathcal{N}_{\alpha,i}\varphi)'''(x)| \leq \frac{\varphi(y)}{f'_\alpha(y)}\Bigg(&4\frac{|\varphi'(y)||f'''_\alpha(y)|}{\varphi(y)(f'_\alpha(y))^4} + 6\frac{|\varphi''(y)||f''_\alpha(y)|}{\varphi(y)(f'_\alpha(y))^4} + 15\frac{|\varphi'(y)||f''_\alpha(y)|^2}{\varphi(y)(f'_\alpha(y))^5}\\ &+ 10\frac{|f''_\alpha(y)||f'''_\alpha(y)|}{(f'_\alpha(y))^5} + 15\frac{|f''_\alpha(y)|^3}{(f'_\alpha(y))^6} + \frac{|f_\alpha^{(iv)}(y)|}{(f'_\alpha(y))^4} + \frac{|\varphi'''(y)|}{\varphi(y)(f'_\alpha(y))^3}\Bigg)\end{aligned}$$

$$\leq \left(\frac{a_3}{|y|^3}+b_3\right)\mathcal{N}_{\alpha,i}\varphi(x)\sup_{y\in I_i}\Bigg[\frac{|f_\alpha(y)|^3}{(a_3+b_3|f_\alpha(y)|^3)}\Bigg(4\frac{|\varphi'(y)||f_\alpha'''(y)|}{\varphi(y)(f_\alpha'(y))^4}+6\frac{|\varphi''(y)||f_\alpha''(y)|}{\varphi(y)(f_\alpha'(y))^4}$$
$$+15\frac{|\varphi'(y)||f_\alpha''(y)|^2}{\varphi(y)(f_\alpha'(y))^5}+10\frac{|f_\alpha''(y)||f_\alpha'''(y)|}{(f_\alpha'(y))^5}+15\frac{|f_\alpha''(y)|^3}{(f_\alpha'(y))^6}+\frac{|f_\alpha^{(iv)}(y)|}{(f_\alpha'(y))^4}+\frac{|\varphi'''(y)|}{\varphi(y)(f_\alpha'(y))^3}\Bigg)\Bigg].$$

We define the expression in the square bracket as $\Omega_3(y)$ and show that $\Omega_3(y)\leq 1$. From (s3), equation (3.39), equation (3.42) and $\varphi\in\mathcal{C}$,

$$\Omega_3(y)=\frac{|f_\alpha(y)|^3}{(a_3+b_3|f_\alpha(y)|^3)}\Bigg(4\frac{C|y|^{\alpha-2}}{(f_\alpha'(y))^4}\cdot\frac{a_1+b_1|y|}{|y|}+6\frac{C|y|^{\alpha-1}}{(f_\alpha'(y))^4}\cdot\frac{a_2+b_2|y|^2}{|y|^2}+15\frac{C|y|^{2(\alpha-1)}}{(f_\alpha'(y))^5}$$
$$\cdot\frac{a_1+b_1|y|}{|y|}+10\frac{C|y|^{2\alpha-3}}{(f_\alpha'(y))^5}+15\frac{C|y|^{3(\alpha-1)}}{(f_\alpha'(y))^6}+\frac{C|y|^{\alpha-3}}{(f_\alpha'(y))^4}+\frac{1}{(f_\alpha'(y))^3}\cdot\frac{a_3+b_3|y|^3}{|y|^3}\Bigg).$$

We show that $\Omega_3(y)\leq 1$ first for y in the neighbourhood of δ, δ small enough. Since $a_3,b_3>0$, we have the following estimate

$$\frac{1}{a_3+b_3(f_\alpha(y))^2}\leq\frac{1}{a_3}$$

$$\frac{a_3+b_3|y|^3}{a_3+b_3|f_\alpha(y)|^3}=1-\frac{b_3\left(|f_\alpha(y)|^3-|y|^3\right)}{a_3+b_3|f_\alpha(y)|^3}=1-\lambda_3,$$

where $\lambda_3=\dfrac{b_3\left(|f_\alpha(y)|^3-|y|^3\right)}{a_3+b_3|f_\alpha(y)|^3}$, $\lambda_3\leq 1$. In a similar calculation as (3.41),

$$\frac{a_1+b_1|y|}{a_3+b_3|f_\alpha(y)|^3}\leq\frac{\max\{a_1,b_1\}}{\min\{a_3,b_3\}}\left(\frac{1+|y|}{1+|f_\alpha(y)|^3}\right)=\rho_3\frac{\max\{a_1,b_1\}}{\min\{a_3,b_3\}},$$

$$\frac{a_2+b_2|y|^2}{a_3+b_3|f_\alpha(y)|^3}\leq\frac{\max\{a_2,b_2\}}{\min\{a_3,b_3\}}\left(\frac{1+|y|^2}{1+|f_\alpha(y)|^3}\right)=\rho_4\frac{\max\{a_2,b_2\}}{\min\{a_3,b_3\}}$$

$$\Omega_3(y)=\Bigg(1+4\frac{C|y|^\alpha}{f_\alpha'(y)}\rho_3\frac{\max\{a_1,b_1\}}{\min\{a_3,b_3\}}+6\frac{C|y|^\alpha}{f_\alpha'(y)}\rho_4\frac{\max\{a_2,b_2\}}{\min\{a_3,b_3\}}+15\frac{C|y|^{2\alpha}}{(f_\alpha'(y))^2}\rho_3\frac{\max\{a_1,b_1\}}{\min\{a_3,b_3\}}$$
$$+10\frac{C|y|^{2\alpha}}{(f_\alpha'(y))^2}\cdot\frac{1}{a_3}+15\frac{C|y|^{3\alpha}}{(f_\alpha'(y))^3}\cdot\frac{1}{a_3}+\frac{C|y|^\alpha}{f_\alpha'(y)}\cdot\frac{1}{a_3}-\lambda_3\Bigg)\left(\frac{|f_\alpha(y)|}{|y|f_\alpha'(y)}\right)^3$$

$\Omega_3(y)\leq 1$ provided we choose $\dfrac{\min\{a_3,b_3\}}{\max\{a_1,b_1\}}$, $\dfrac{\min\{a_3,b_3\}}{\max\{a_2,b_2\}}$ and a_3, large enough. Now, for $y\in(\delta,\kappa]\cup[1-\kappa,-\delta)$, we have that $f_\alpha'(y)\geq\gamma>1$, which implies that $\frac{1}{f_\alpha'(y)}\leq\frac{1}{\gamma}<1$,

$$\frac{f_\alpha(y)^3}{(a_3+b_3f_\alpha(y)^3)}\leq\frac{1}{a_3},\ |f_\alpha(y)|^3\left(\frac{a_1+b_1|y|}{a_3+b_3|f_\alpha(y)|^3}\right)\leq\frac{a_1}{b_3}+\frac{b_1}{b_3}|y|,$$
$$|f_\alpha(y)|^3\left(\frac{a_2+b_2|y|^2}{a_3+b_3|f_\alpha(y)|^3}\right)\leq\frac{a_2}{b_3}+\frac{b_2}{b_3}|y|^2\text{ and }|f_\alpha(y)|^3\left(\frac{a_3+b_3|y|^3}{a_3+b_3|f_\alpha(y)|^3}\right)\leq\frac{a_3}{b_3}+|y|^3.$$

Now, for

$$|y| > \delta \ \Rightarrow \ |y|^{\alpha-1} < \delta^{\alpha-1}, \quad \text{for } \alpha \in [0,1).$$

$$\Omega_3(y) = \Bigg(4\frac{C\delta^{\alpha-3}}{\gamma^4}\frac{a_1}{b_3} + 4\frac{C\delta^{\alpha-2}}{\gamma^4}\frac{b_1}{b_3} + 6\frac{C\delta^{\alpha-3}}{\gamma^4}\frac{a_2}{b_3} + 6\frac{C\delta^{\alpha-1}}{\gamma^4}\frac{b_2}{b_3} + 15\frac{C\delta^{2\alpha-3}}{\gamma^5}\frac{a_1}{b_3} + 15\frac{C\delta^{2\alpha-2}}{\gamma^5}\frac{b_1}{b_3} + 10\frac{C\delta^{2\alpha-3}}{\gamma^5}\cdot\frac{1}{a_3} + 15\frac{C\delta^{3(\alpha-1)}}{\gamma^6}\cdot\frac{1}{a_3} + \frac{C\delta^{\alpha-3}}{\gamma^4}\cdot\frac{1}{a_3} + \frac{1}{\delta^3\gamma^3}\frac{a_3}{b_3} + \frac{1}{\gamma^3}\Bigg)$$

$\Omega_3 \leq 1$ provided we choose $a_3, \frac{b_3}{a_1}, \frac{b_3}{b_1}, \frac{b_3}{b_2}, \frac{b_3}{a_2}, \frac{b_3}{a_1}, \frac{b_3}{a_3}$ large enough. □

3.3.2 A random perturbed operator and distortion property.

Before we state the rate of decay result with respect to the Lebesgue measure. Following the approach in [65] we introduce the concept of random perturbation

$$B_\varepsilon(x) = \left\{y \in S^1 : |x-y| \leq \varepsilon\right\},$$

$$\mathbf{A}_\varepsilon\varphi(x) = \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)} \varphi(y)\,dy, \quad \varepsilon > 0, \tag{3.43}$$

$$\mathbf{P}_\varepsilon = \mathcal{L}_\alpha^{n_\varepsilon}\mathbf{A}_\varepsilon, \quad n_\varepsilon \in \mathbb{N}, \tag{3.44}$$

where $B_\varepsilon(x)$ is a ball centred around x, $\mathbf{A}_\varepsilon$ and $\mathbf{P}_\varepsilon$ are the averaging operator and the perturbed operator respectively, with $n_\varepsilon = \mathcal{O}(\varepsilon^{-\alpha})$. In the next lemma, we show that for observables $\varphi \in \mathcal{C}_{*,1}$, the Perron-Frobenius operator is approximated by the random perturbed operator.

Lemma 3.3.5. *For* $\varphi \in \mathcal{C}_{*,1}$,

$$\|\mathcal{L}_\alpha^{n_\varepsilon}\varphi - \mathbf{P}_\varepsilon\varphi\|_1 \leq k_1\|\varphi\|_1\varepsilon^{1-\alpha},$$

where $k_1 = \dfrac{18c_2\max\{a_1, b_1, 1\}}{\alpha(1-\alpha)}$.

Proof. From the definition of the perturbed operator and the property (C2) of the Perron-Frobenius operator,

$$\|\mathcal{L}_\alpha^{n_\varepsilon}\varphi - \mathbf{P}_\varepsilon\varphi\|_1 \leq \|\varphi - \mathbf{A}_\varepsilon\varphi\|_1.$$

Assuming that $m(\varphi) = 1$, the estimates in equation (3.27) gives that

$$\varphi(x) \leq 2c_2|x|^{-\alpha},$$

which would enable us get the desired bounds.

$$\begin{aligned}\|\varphi - \mathbf{A}_\varepsilon\varphi\|_1 &= \left\|\varphi(x) - \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)}\varphi(y)\,dy\right\|_1 \\ &= \int_0^1 \left|\varphi(x) - \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)}\varphi(y)\,dy\right| dx\end{aligned}$$

$$
\begin{aligned}
&= \int_{\varepsilon}^{1-\varepsilon} \left|\varphi(x) - \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)} \varphi(y)\,dy\right| dx + \int_{B_\varepsilon(0)} \left|\varphi(x) - \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)} \varphi(y)\,dy\right| dx \\
&= \int_{\varepsilon}^{1-\varepsilon} \left|\frac{1}{2\varepsilon}\int_{B_\varepsilon(x)} [\varphi(x) - \varphi(y)]\,dy\right| dx + \int_{B_\varepsilon(0)} \left|\varphi(x) - \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)} \varphi(y)\,dy\right| dx \\
&\le \frac{1}{2\varepsilon}\int_{\varepsilon}^{1-\varepsilon} \int_{B_\varepsilon(x)} |\varphi(x) - \varphi(y)|\,dy\,dx + \int_{B_\varepsilon(0)} \left|\varphi(x) - \frac{1}{2\varepsilon}\int_{B_\varepsilon(x)} \varphi(y)\,dy\right| dx \\
&\le \frac{1}{2\varepsilon}\int_{\varepsilon}^{1-\varepsilon} \int_{B_\varepsilon(x)} |\varphi(x) - \varphi(y)|\,dy\,dx + \int_{B_\varepsilon(0)} |\varphi(x)|\,dx \\
&\qquad + \frac{1}{2\varepsilon}\int_{B_\varepsilon(0)}\int_{B_\varepsilon(x)} |\varphi(y)|\,dy\,dx.
\end{aligned}
$$

By changing the order of integration in the last integral, we have that

$$
\begin{aligned}
\|\varphi - \mathbf{A}_\varepsilon\varphi\|_1 &\le \frac{1}{2\varepsilon}\int_{\varepsilon}^{1-\varepsilon} \int_{B_\varepsilon(x)} |\varphi(x) - \varphi(y)|\,dy\,dx + \int_{B_\varepsilon(0)} \varphi(x)\,dx + \int_{B_{2\varepsilon}(0)} \varphi(y)\,dy \\
&\le \frac{1}{2\varepsilon}\int_{\varepsilon}^{1-\varepsilon} \int_{B_\varepsilon(x)} |\varphi(x) - \varphi(y)|\,dy\,dx + 2\int_{-2\varepsilon}^{2\varepsilon} \varphi(y)\,dy.
\end{aligned}
$$

The integrand in the first integral is bounded as follows. For $x, y \in S^1$ such that $|x-y| \le \varepsilon$, by equations (3.26) and (3.27),

$$
|\varphi(x) - \varphi(y)| \le \sup_{z\in[x,y]} |\varphi'(z)|\varepsilon \le 2c_2\varepsilon(a_1|x|^{-1-\alpha} + b_1|x|^{-\alpha}).
$$

We have that

$$
\begin{aligned}
\|\varphi - \mathbf{A}_\varepsilon\varphi\|_1 &\le c_2\int_{\varepsilon}^{1-\varepsilon}\int_{B_\varepsilon(x)} (a_1|x|^{-1-\alpha} + b_1|x|^{-\alpha})\,dy\,dx + 8c_2\int_0^{2\varepsilon} |y|^{-\alpha}\,dy \\
&= 2c_2\varepsilon\int_{\varepsilon}^{1-\varepsilon} (a_1|x|^{-1-\alpha} + b_1|x|^{-\alpha})\,dx + 8c_2\int_0^{2\varepsilon} |y|^{-\alpha}\,dy \\
&= 2c_2\varepsilon\left[a_1 \frac{\operatorname{sgn}(x)|x|^{-\alpha}}{\alpha}\Big|_{1-\varepsilon}^{\varepsilon} + b_1\frac{\operatorname{sgn}(x)|x|^{1-\alpha}}{1-\alpha}\Big|_{\varepsilon}^{1-\varepsilon}\right] + 8c_2\frac{\operatorname{sgn}(x)|x|^{1-\alpha}}{1-\alpha}\Big|_0^{2\varepsilon} \\
&= 2c_2\varepsilon\left[a_1\left(\frac{\varepsilon^{-\alpha}}{\alpha} - \frac{(1-\varepsilon)^{-\alpha}}{\alpha}\right) + b_1\left(\frac{(1-\varepsilon)^{1-\alpha}}{1-\alpha} - \frac{\varepsilon^{1-\alpha}}{1-\alpha}\right)\right] + 8c_2\frac{(2\varepsilon)^{1-\alpha}}{1-\alpha} \\
&\le 2c_2\max\{a_1, b_1\}\varepsilon\cdot \\
&\qquad \left(\frac{(1-\alpha)\varepsilon^{-\alpha} - (1-\alpha)(1-\varepsilon)^{-\alpha} + \alpha(1-\varepsilon)^{1-\alpha} - \alpha\varepsilon^{1-\alpha}}{\alpha(1-\alpha)}\right) + 8c_2\frac{(2\varepsilon)^{1-\alpha}}{1-\alpha} \\
&\le 2c_2\max\{a_1, b_1\}\varepsilon\cdot \\
&\qquad \left(\frac{\varepsilon^{-\alpha}}{\alpha(1-\alpha)} + \frac{\alpha(1-\varepsilon)^{1-\alpha} - (1-\alpha)(1-\varepsilon)^{-\alpha} - \alpha\varepsilon^{-\alpha} - \alpha\varepsilon^{1-\alpha}}{\alpha(1-\alpha)}\right) + 8c_2\frac{(2\varepsilon)^{1-\alpha}}{1-\alpha} \\
&\le 2c_2\max\{a_1, b_1\}\frac{\varepsilon^{1-\alpha}}{\alpha(1-\alpha)} + \frac{16c_2\varepsilon^{1-\alpha}}{\alpha(1-\alpha)}
\end{aligned}
$$

$$
\|\varphi - \mathbf{A}_\varepsilon\varphi\|_1 \le \frac{18c_2\max\{a_1, b_1, 1\}}{\alpha(1-\alpha)}\varepsilon^{1-\alpha}.
$$

□

From equations (3.22), (3.43), (3.44) and the kernel $\mathcal{K}_\varepsilon(x,z) := \dfrac{1}{2\varepsilon}\mathcal{L}_\alpha^{n_\varepsilon}\chi_{B_\varepsilon(z)}(x)$,

$$\begin{aligned}
\mathbf{P}_\varepsilon\varphi(x) &= \mathcal{L}_\alpha^{n_\varepsilon}\frac{1}{2\varepsilon}\int_{B_\varepsilon(x)}\varphi(y)\,dy\\
&= \frac{1}{2\varepsilon}\sum_{f_\alpha^{n_\varepsilon}(y)=x}\frac{\int_0^1 \chi_{B_\varepsilon(y)}(z)\varphi(z)\,dz}{(f_\alpha^{n_\varepsilon})'(y)}\\
&= \frac{1}{2\varepsilon}\sum_{f_\alpha^{n_\varepsilon}(y)=x}\frac{\int_0^1 \chi_{B_\varepsilon(z)}(y)\varphi(z)\,dz}{(f_\alpha^{n_\varepsilon})'(y)}\\
&= \frac{1}{2\varepsilon}\int_0^1 \mathcal{L}_\alpha^{n_\varepsilon}\chi_{B_\varepsilon(z)}(x)\varphi(z)\,dz\\
&= \int_0^1 \mathcal{K}_\varepsilon(x,z)\varphi(z)\,dz. \qquad (3.45)
\end{aligned}$$

Next, for an appropriate choice of n_ε, we verify the positivity of the kernel $\mathcal{K}_\varepsilon(x,z)$, an estimate that plays a crucial role in establishing the desired decay properties. Since the Perron-Frobenius operator for this class of maps lacks the spectral gap property, the positivity of this kernel provides a key estimate for demonstrating the decay of correlation despite this absence.

Proposition 3.3.6. *There exists $n_\varepsilon = \mathcal{O}(\varepsilon^{-\alpha})$ and $\gamma > 0$ such that for each $\varepsilon > 0$, $x, z \in S^1$,*

$$\mathcal{K}_\varepsilon(x,z) \geq \gamma. \qquad (3.46)$$

Proof. Firstly, recall the definition

$$2\varepsilon\mathcal{K}_\varepsilon(x,z) = \mathcal{L}_\alpha^{n_\varepsilon}\chi_{B_\varepsilon(z)}(x).$$

We have from equation (3.22) that,

$$\begin{aligned}
\mathcal{L}_\alpha^{n_\varepsilon}\chi_{B_\varepsilon(z)}(x) &= \sum_{f_\alpha^{n_\varepsilon}(y)=x}\frac{\chi_{B_\varepsilon(z)}(y)}{(f_\alpha^{n_\varepsilon})'(y)}\\
&= \chi_{f_\alpha^{n_\varepsilon}(B_\varepsilon(z))}(x)\sum_{f_\alpha^{n_\varepsilon}(y)=x}\frac{1}{(f_\alpha^{n_\varepsilon})'(y)}\\
&\geq \chi_{f_\alpha^{n_\varepsilon}(B_\varepsilon(z))}(x)\inf_{y\in B_\varepsilon(z)}\frac{1}{(f_\alpha^{n_\varepsilon})'(y)}.
\end{aligned}$$

Hence, we have to control

$$\inf_{y\in B_\varepsilon(z)}\frac{1}{(f_\alpha^{m})'(y)},$$

where m is the time needed for an interval $J = B_\varepsilon(z)$ of length at least 2ε to cover the whole circle. In addition, we estimate

$$n_\varepsilon := \inf\{n \geq 1 : f_\alpha^n(J) = S^1, \text{ for all } J \text{ with } |J| \geq 2\varepsilon\}.$$

To check the distortion and thus the positivity of the kernel, we follow closely the strategy of proof in [2, 65]. Now, we fix the notation that we shall be using. Recall the definition of z_k (resp. z'_k) in Subsection 3.2.1, setting $-z_k = z'_k$ and let $I_0 = B_{z_k}(0)$ (for a fixed k) be the *intermittent region* and $I_0^c = S^1 \setminus B_{z_k}(0)$ be the *hyperbolic region*, we note that the map is uniformly expanding in the hyperbolic region, and possesses a uniformly bounded second derivative.

Consider the interval J and its iterates which we call $K = f_\alpha^n(J)$, for some n. Controlling the distortion, we explore different possibilities that the dynamics might take. K takes one of the following

1 $K \cap I_0 = \emptyset$;

2 $K \cap I_0 \neq \emptyset$ and K contains, at most, one z_l or z'_l for $l > k$;

3 $K \cap I_0 \neq \emptyset$ and K contains more than one z_l or z'_l for $l > k$.

Remark 3.3.7. *The proof for the above cases when $d = 2$ and $d \geq 3$ are similar, since for $d \geq 3$, the middle branches are covered by the case 1. Thus, we proceed with the proof for $d = 2$.*

Case 1: Now, suppose that we are in the scenario 1 we let $n_1 \geq 1$ be the time spent iterating the interval K in the region I_0^c before it enters the I_0 region and case 2 or 3 occurs.

Let $D := \sup_{\xi \in I_0^c} \frac{f''_\alpha(\xi)}{(f'_\alpha(\xi))^2}$. By the property (s2) of the map, for $y \in I_0^c$, $\frac{1}{f'_\alpha(y)} \leq \frac{1}{\lambda} < 1$, by the standard distortion estimate we have that, for all $x, y \in K$, using the mean value theorem twice, there exists $\eta, \xi \in K$, such that

$$\log \frac{|f'_\alpha(x)|}{|f'_\alpha(y)|} = \log\left(1 + \frac{|f'_\alpha(x) - f'_\alpha(y)|}{|f'_\alpha(y)|}\right) \leq \frac{|f'_\alpha(x) - f'_\alpha(y)|}{|f'_\alpha(y)|} = \frac{f''_\alpha(\xi)\,|x-y|}{|f'_\alpha(y)|}$$
$$= \frac{|f''_\alpha(\xi)|}{|f'_\alpha(y)|}\frac{|f_\alpha(x) - f_\alpha(y)|}{f'_\alpha(\eta)}.$$

Since we are in the hyperbolic region, $f'_\alpha > 1$, also f_α is C^2 on a compact space, $|f''_\alpha(\xi)|$ is bounded. Therefore,

$$\log \frac{|f'_\alpha(x)|}{|f'_\alpha(y)|} \leq D|f_\alpha(x) - f_\alpha(y)|.$$

Now, by the chain rule,

$$\log \frac{(f_\alpha^{n_1})'(x)}{(f_\alpha^{n_1})'(y)} \leq \sum_{j=0}^{n_1-1} \left|\log f'_\alpha(f_\alpha^j(x)) - \log f'_\alpha(f_\alpha^j(y))\right| \leq D \sum_{j=0}^{n_1-1} \left|f_\alpha^j(x) - f_\alpha^j(y)\right|$$
$$\leq D \sum_{j=0}^{n_1-1} \frac{1}{\lambda^{n_1-j}} \left|f_\alpha^{n_1}(x) - f_\alpha^{n_1}(y)\right| \leq \frac{D}{\lambda - 1} \left|f_\alpha^{n_1}(K)\right|.$$

Hence,

$$\frac{(f_\alpha^{n_1})'(x)}{(f_\alpha^{n_1})'(y)} \leq \exp\left(\frac{D}{\lambda-1}\left|f_\alpha^{n_1}(K)\right|\right).$$

Integrating with respect to y,

$$\frac{(f_\alpha^{n_1})'(x)\,|K|}{|f_\alpha^{n_1}(K)|} \le \exp\left(\frac{D}{\lambda-1}\,|f_\alpha^{n_1}(K)|\right),$$

we therefore deduce that

$$\mathcal{L}_\alpha^{n_1}\chi_{B_\varepsilon(z)}(x) \ge \chi_{f_\alpha^{n_1}(B_\varepsilon(z))}(x)\,\frac{|K|}{|f_\alpha^{n_1}(K)|}\exp\left(-N_1\,|f_\alpha^{n_1}(K)|\right),$$

taking $N_1 = \frac{D}{\lambda-1}$.

Case 2: Let us assume that K is in I_0, such that $K \subset (z_l, z_{l-2})$ or $K \subset (z'_{l-2}, z'_l)$, where $l = k + k_1$. Here, after k_1 iterations, the image will be in the hyperbolic region I_0^c and we continue the algorithm as in case 1. We control the distortion while K traverses I_0 using the Koebe principle, which we state below for completeness.

Lemma 3.3.8. *(Koebe Principle, [44, Theorem IV.1.2]) Let g be a C^3 diffeomorphism with non-positive Schwarzian derivative. Then for constants $\tau > 0$ and $C = C(\tau) > 0$. For any subinterval $J_1 \subset J_2$ such that $g(J_2)$ contains a τ-scaled neighbourhood of $g(J_1)$, then*

$$\frac{g'(x)}{g'(y)} \le \exp\left(C\frac{|g(x)-g(y)|}{|g(J_1)|}\right) \quad \textit{for all } x, y \in J_1.$$

Remark 3.3.9. *The Schwarzian derivative of a C^3 diffeomorphism f, $\mathbf{S}g(\cdot)$ is given by*

$$\mathbf{S}g(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2.$$

*Let $U \subset V$ be two intervals, V is said to contain a τ-*scaled neighbourhood *of U if both components of $V \setminus U$ has a length of at least $\tau \cdot |U|$. Where $|U|$ is the length of U.*

There exists a $\delta > 0$ such that the Schwarzian derivative of f_α is non-positive for x close to 0. Indeed, this is so since by (s3), $f_\alpha''' < 0$ close to 0 and $f_\alpha' > 0$. This particularly implies that we can fix a k such that $\mathbf{S}f_\alpha \le 0$ on $[0, z_{k-3}]$ (resp. $[z'_{k-3}, 1]$). We define $g(\cdot) = f_\alpha^{k_1}(\cdot)$ on $[0, z_{l-3}]$ i.e $g : [0, z_{l-3}] \to [0, z_{k-3}]$. We define $J_1 = [z_l, z_{l-2}]$, hence $g(J_1) = [z_k, z_{k-2}]$. Now, we choose β small enough such that $\beta < z_l$ and $g(\beta) < \frac{z_k}{2}$. Next, we choose $J_2 = [\beta, z_{l-3}]$, with $g(J_2) = [g(\beta), z_{k-3}]$. The Schwarzian derivative of g is non-positive on J_2, since the composition of maps with non-positive Schwarzian derivative is also non-positive. Next, we show that $g(J_2)$ contains a τ-scaled neighbourhood of $g(J_1)$. Indeed, if we refer to the left and right components of $g(J_2) \setminus g(J_1)$ as K_l and K_r respectively,

$$|K_l| \ge \frac{z_k}{2} \ge \tau|g(J_1)| = \tau|z_{k-2} - z_k|,$$

taking $\tau \le \frac{z_k}{2(|z_{k-2}-z_k|)}$.

$$|K_r| \ge |z_{k-3} - z_{k-2}| \ge \tau|z_{k-2} - z_k|,$$

where $\tau \leq \frac{|z_{k-3}-z_{k-2}|}{|z_{k-2}-z_k|}$. If we choose $\tau \leq \min\left\{\frac{z_k}{2(|z_{k-2}-z_k|)}, \frac{|z_{k-3}-z_{k-2}|}{|z_{k-2}-z_k|}\right\}$, then by the Koebe principle, there exists $C = C(\tau) > 0$ such that

$$\begin{aligned}\frac{(f_\alpha^{k_1})'(x)}{(f_\alpha^{k_1})'(y)} &\leq \exp\left(C\frac{|(f_\alpha^{k_1})(x) - (f_\alpha^{k_1})(y)|}{|(f_\alpha^{k_1})(J_1)|}\right)\\ &\leq \exp\left(M|(f_\alpha^{k_1})(x) - (f_\alpha^{k_1})(y)|\right) \qquad \text{for all } x, y \in J_1,\end{aligned}$$

taking $M = \frac{C}{f_\alpha^{k_1}(J_1)}$. Since $K \subset J_1$, we have that for all $x, y \in K$,

$$\frac{(f_\alpha^{k_1})'(x)}{(f_\alpha^{k_1})'(y)} \leq \exp\left(M|f_\alpha^{k_1}(K)|\right).$$

Integrating with respect to y implies that

$$\mathcal{L}_\alpha^{k_1}\chi_K(x) \geq \chi_{f_\alpha^{k_1}(K)}(x)\,\frac{|K|}{|f_\alpha^{k_1}(K)|}\exp\left(-M_1\left|f_\alpha^{k_1}(K)\right|\right).$$

A similar calculation also applies when $x, y \in K \subset (z'_{l-2}, z'_l)$.

Case 3: Suppose that K contains more than one z_l or z'_l for $l > k$ and more than one-third of K is in I_0^c, then we consider $K \cap I_0^c$, such that the fixed k is sufficiently large to contain z_{k-1}, which brings us to case 1, such that after a finite number of iterations, is sent to the whole of S^1 and ultimately ends the algorithm. Otherwise, we split this into sub-cases. To present these sub-cases, we define l' as the least integer such that $[z_{l'+1}, z_{l'}]$ belongs to K. The first of the sub-cases we consider is when $|b - z_{l'}| > \frac{|K|}{3}$, where b is the right end-point of K. This then leads us back to case 2. Now, set $K' = [z_{l'}, b]$ such that $|K'| \geq \frac{|K|}{3}$. Since, $K' \subset [z_{l'}, z_{l'-1}]$ and after $l' - k$ iterations, the image of K' will be in the hyperbolic region, we use the estimate from case 2

$$\begin{aligned}\mathcal{L}_\alpha^{l'-k}\chi_K(x) \geq \mathcal{L}_\alpha^{l'-k}\chi_{K'}(x) &\geq \chi_{f_\alpha^{l'-k}(K')}(x)\,\frac{|K'|}{|f_\alpha^{l'-k}(K')|}\exp\left(-M_1\left|f_\alpha^{l'-k}(K')\right|\right)\\ &\geq \chi_{[z_k,z_{k-1}]}(x)\,\frac{|K'|}{|f_\alpha^{l'-k}(K')|}\exp\left(-M_1\left|f_\alpha^{l'-k}(K')\right|\right)\\ &\geq \frac{1}{3}\chi_{[z_k,z_{k-1}]}(x)\,\frac{|K|}{|f_\alpha^{l'-k}(K')|}\exp\left(-M_1\left|f_\alpha^{l'-k}(K')\right|\right),\end{aligned}$$

we note that $f_\alpha^{k+1}([z_k, z_{k-1}]) = S^1$ and we have that f'_α is bounded from above by $N > 0$

$$\mathcal{L}_\alpha^{l'+1}\chi_K(x) \geq \frac{1}{3N^{k+1}}\frac{|K|}{|f_\alpha^{l'-k}(K')|}\exp\left(-M_1\left|f_\alpha^{l'-k}(K')\right|\right).$$

Next, suppose that $K = [a, z_{l'}]$, where $a > 0$, and we choose K' in such a way that $|K'| \geq \frac{|K|}{3}$, $K' \supset \bigsqcup_{l=l^*}^{l'} [z_{l+1}, z_l]$,

$$\left|\bigsqcup_{l=l^*}^{l'} [z_{l+1}, z_l]\right| \geq \frac{|K'|}{3} \geq \frac{|K|}{9},$$

taking the minimal number of z_l to make this happen. We therefore estimate l^* as follows, $|[a, z_{l^*-1}]| \geq \frac{2|K'|}{3} \geq \frac{2|K|}{9}$. From (3.3), we have that $C(l^*-1)^{1/\alpha} \geq z_{l^*-1} \geq z_{l^*-1} - a \geq \frac{2|K|}{9}$, which leads to $l^* = \mathcal{O}(|K|^{-\alpha})$.

$$\mathcal{L}_\alpha^l \chi_{[z_{l+1}, z_l]} \geq \chi_{f_\alpha^l([z_{l+1}, z_l])}(x) \frac{|z_l - z_{l+1}|}{|f_\alpha^l([z_{l+1}, z_l])|} \exp\left(-M|f_\alpha^l([z_{l+1}, z_l])|\right)$$

Hence, by the computation in case 2, we have that

$$\begin{aligned}
\mathcal{L}_\alpha^{l^*+1}\chi_K(x) &\geq \sum_{l=l^*}^{l'} \mathcal{L}_\alpha^{l^*+1}\chi_{[z_{l+1}, z_l]}(x) \\
&= \sum_{l=l^*}^{l'} \mathcal{L}_\alpha^{l^*-l}\mathcal{L}_\alpha^{l+1}\chi_{[z_{l+1}, z_l]}(x) \\
&\geq \sum_{l=l^*}^{l'} \mathcal{L}_\alpha^{l^*-l}\chi_{f_\alpha^{l+1}([z_{l+1}, z_l])}(x) \frac{(z_l - z_{l+1})}{|f_\alpha^{l+1}([z_{l+1}, z_l])|} \exp\left(-M_1 \left|f_\alpha^{l+1}([z_{l+1}, z_l])\right|\right) \\
&= \sum_{l=l^*}^{l'} \mathcal{L}_\alpha^{l^*-l}\chi_{[1/2,1]}(x)\, 2(z_l - z_{l+1}) \exp\left(-\frac{M_1}{2}\right) \\
&\geq 2\mathcal{L}_\alpha^{l^*-l}\chi_{[1/2,1]}(x) \exp\left(-\frac{M_1}{2}\right) \sum_{l=l^*}^{l'} (z_l - z_{l+1}) \\
&\geq \frac{\gamma}{9}|K| \exp\left(-\frac{M_1}{2}\right)
\end{aligned}$$

We note that $f_\alpha^{l+1}([z_{l+1}, z_l]) = [1/2, 1]$. Where γ is as defined in equation (3.34).

Let J be as defined starting out from any part of S^1, we associate to J a sequence of integers $n_1, m_1, n_2, m_2, \cdots, n_p$, such that iterating it n_1 times, we are in I_0^c (if J starts out from I_0^c, then $n_1 = 0$) and hence, satisfies case 1. Then after m_1 iterations, it is in case 2. Taking n_2 iterations, we leave I_0 and are back in the hyperbolic region and so on, until we fall into case 3 (for $d = 2$) or the iterates contains at least one I_i (for $d \geq 3$). These two situations lead to the end of the algorithm. However, we only focus on the situation where it leads to case 3, such that $[z_{l'}, b] < \frac{|K|}{3}$.

For $n \geq n_1 + m_1 + \cdots + n_p + l^* + 1$, we have that

$$\begin{aligned}
\mathcal{L}_\alpha^n \chi_J(x) &\geq \mathcal{L}_\alpha^{n-(n_1+m_1+\cdots+n_p+l^*+1)}\mathcal{L}_\alpha^{l^*+1}\mathcal{L}_\alpha^{n_p}\cdots\mathcal{L}_\alpha^{m_1}\mathcal{L}_\alpha^{n_1}\chi_J \\
&\geq \mathcal{L}_\alpha^{n-(n_1+m_1+\cdots+n_p+l^*+1)}\mathcal{L}_\alpha^{l^*+1}\mathcal{L}_\alpha^{n_p}\cdots\mathcal{L}_\alpha^{m_1}\chi_{f_\alpha^{n_1}(J)} \frac{|J|}{|f_\alpha^{n_1}(J)|} \exp\left(-N_1 \left|f_\alpha^{n_1}(J)\right|\right) \\
&\geq \mathcal{L}_\alpha^{n-(n_1+m_1+\cdots+n_p+l^*+1)}\mathcal{L}_\alpha^{l^*+1}\mathcal{L}_\alpha^{n_p}\cdots\chi_{f_\alpha^{n_1+m_1}(J)} \frac{|f_\alpha^{n_1}(J)|}{|f_\alpha^{n_1+m_1}(J)|}\frac{|J|}{|f_\alpha^{n_1}(J)|} \\
&\exp\left(-M_1|f_\alpha^{n_1+m_1}(J)| - N_1 \left|f_\alpha^{n_1}(J)\right|\right) \\
&\geq (\mathcal{L}_\alpha^{n-(n_1+m_1+\cdots+n_p+l^*+1)}\chi)\frac{\gamma}{9}|f_\alpha^{n_1+m_1+\cdots+n_p}(J)| \frac{|f_\alpha^{n_1+\cdots+m_{p-1}}(J)|}{|f_\alpha^{n_1+m_1+\cdots+m_{p-1}+n_p}(J)|}\cdots\frac{|f_\alpha^{n_1}(J)|}{|f_\alpha^{n_1+m_1}(J)|} \\
&\frac{|J|}{|f_\alpha^{n_1}(J)|} \exp\left(-M_1|f_\alpha^{n_1+m_1+\cdots+n_p}(J)| - \cdots - M_1|f_\alpha^{n_1+m_1}(J)| - N_1 \left|f_\alpha^{n_1}(J)\right| - \frac{M_1}{2}\right)
\end{aligned}$$

$$\geq \frac{\gamma^2}{9}|J|\exp\left(-M_1|f_\alpha^{n_1+m_1+\cdots+n_p}(J)| - \cdots - M_1|f_\alpha^{n_1+m_1}(J)| - N_1\,|f_\alpha^{n_1}(J)| - \frac{M_1}{2}\right)$$
$$\geq \frac{\gamma^2}{9}|J|\exp\left(-2\max\{M_1,N_1\}(\lambda^{n_1}+\lambda^{n_1+n_2}+\cdots+\lambda^{n_1+n_2+\cdots+n_p})\right)$$
$$\geq \frac{\gamma^2}{9}|J|\exp\left(\frac{-2\max\{M_1,N_1\}\lambda}{1-\lambda}\right)$$
$$=: \hat{\gamma}|J|.$$

Furthermore, we have that $n_1+m_1+\cdots+n_p+l^*+1=\mathcal{O}(\varepsilon^{-\alpha})$. Observe that $n_1+m_1+\cdots+n_p \leq n_\varepsilon = \mathcal{O}(\varepsilon^{-\alpha})$ and hence has not covered the whole of S^1. As seen in case 3, we showed that $l^*=\mathcal{O}(|K|^{-\alpha}) = \mathcal{O}(\varepsilon^{-\alpha})$. We claim that $n_\varepsilon = \mathcal{O}(\varepsilon^{-\alpha})$, to prove the claim, let us go through possible scenarios that the dynamics may take. If after iteration, a given scenario coincides with a previous one, we use the estimate of the previous scenario, even though the image of J is bigger this time. We assume that the length of J is at least ε.

a. $J \supset [0,\varepsilon]$ (or $J \subset [1-\varepsilon,1]$): there exists $l \geq 1$ such that $z_l \leq \varepsilon \leq z_{l-1}$ (resp. $z'_{l-1} \leq \varepsilon \leq z'_l$). Such that $f_\alpha^l J = S^1$. From equation (3.3), we have that $z_{l-1} \leq C(l-1)^{-1/\alpha}$ (resp. $z'_{l-1} \leq C(l-1)^{-1/\alpha}$), thus we have that $(l-1)=\mathcal{O}(\varepsilon^{-\alpha})$.

b. $J \supset [\varepsilon, 1-\varepsilon]$ and contains at least one I_i, $i=2,\cdots,d-1$, then in one step, it covers S^1. Otherwise, that is for $d=2$ the image of J will cover S^1 after $\mathcal{O}(\log\frac{1}{\varepsilon})$ steps.

c. $J \supset [\delta,\delta+\varepsilon]$ (or $J \supset [\delta-\varepsilon,\delta]$) with $\delta \leq z_0 \leq \delta+\varepsilon$ (or $\delta-\varepsilon \leq z'_0 \leq \delta$) then after one iteration, we are back to the scenario a, b or contain at least one I_i, $i=2,\cdots,d-2$ and will therefore cover S^1 in fewer steps than $\mathcal{O}(\varepsilon^{-\alpha})$.

d. $J \subset I_1 \setminus \{0,z_0\}$ (or $J \subset I_d \setminus \{z'_0,1\}$) and $J=[a,b]$ contains at least two z_l, $l \geq 2$, we choose l as the smallest integer such that $[z_l,z_{l-1}] \subset J$, that is we have the following inequality $a \leq z_l < z_{l-1} \leq b < z_{l-2}$. Here, there is a possibility that $|b-z_{l-1}| > \frac{|J|}{3}$ or $|b-z_{l-1}| < \frac{|J|}{3}$. For $|b-z_{l-1}| > \frac{|J|}{3}$, we define $J_1 = J \cap [z_{l-1},z_{l-2}]$. Iterating $l-1$ times, leads us to the previous scenario with size bigger than or equal to $\varepsilon/3$ which terminates in $\mathcal{O}(\varepsilon^{-\alpha})$ steps. And thus, $\varepsilon/3 \leq \frac{|J|}{3} \leq |J_1| \leq z_{l-2}-z_{l-1} \leq z_{l-2} \leq C(l-2)^{-1/\alpha}$ and thus $(l-2)=\mathcal{O}(\varepsilon^{-\alpha})$. Next, in the sub-case $|b-z_{l-1}| < \frac{|J|}{3}$, we define $J_1 = J\cap[a,z_{l-1}] = [a,z_{l-1}]$. Iterating l times, we have that $f_\alpha^l J_1 \supset f_\alpha^l I_l$, hence, it takes $l+1$ iterations to cover S^1. We estimate the time it takes to cover S^1 by taking $J_2=[0,|J_1|]$, we choose m such that $z_m \leq |J_2| \leq z_{m-1}$. Just as in scenario a, we have that $(m-1)=\mathcal{O}(|J_2|^{-\alpha}) = \mathcal{O}(\varepsilon^{-\alpha})$, assuming that $l<m$.

e. $J \subset I_1 \setminus \{0,z_0\}$ (or $J \subset I_d \setminus \{z'_0,1\}$) and J contains exactly one z_l, $l \geq 2$. We observe that after l iterations we are in scenario c, which ends in $\mathcal{O}(\varepsilon^{-\alpha})$ steps. But $J \subset [z_{l+1},z_{l-1}]$ and thus we have that $\varepsilon \leq |J| \leq z_{l-1}-z_{l+1} \leq C(l-1)^{-1/\alpha}$, which leads to $l=\mathcal{O}(\varepsilon^{-\alpha})$.

f. $J \subset I_1 \setminus \{0,z_0\}$ (or $J \subset I_d \setminus \{z'_0,1\}$), and J contains no z_l, $l \geq k$, that is $J \subset (z_{l+1},z_l)$ (or resp. $J \subset (z'_l,z'_{l+1})$). After iterating $l+1$ times, we have that $f_\alpha^{(l+1)} J \subset (z'_0,1)$ (or $f_\alpha^{(l+1)} J \subset (0,z_0)$) or contains at least one $I_i, i=2,\cdots,d-1$, $d \geq 3$ and after a finite iteration ends the algorithm and thus cover S^1 or in one of the cases showed above. We estimate l as follows: $\varepsilon \leq |J| \leq z_l - z_{l+1} \leq z_l \leq Cl^{-1/\alpha}$ and thus $l = \mathcal{O}(\varepsilon^{-\alpha})$. Observe that there is also a possibility of looping between this and $J \subset I_d \setminus \{z_0', 1\}$, and after a finite number of iterations, we come out of this loop by condition (s2) into one of the previous scenarios given, since the length of J grows with time, and thus cover S^1 in $\mathcal{O}(\varepsilon^{-\alpha})$ steps.

□

Using the previous results, we prove that the random perturbed transfer operator decays at an exponential rate.

Proposition 3.3.10. *For $\varphi \in L^1$, with $\int_\Omega \varphi(x)\,dx = 0$, we have that*

$$\|\mathbf{P}_\varepsilon^k \varphi\|_1 \leq (1-\gamma)^k \|\varphi\|_1, \quad \text{for all } k \in \mathbb{N}.$$

Proof. From equations (3.43) and (3.44), $\mathbf{P}_\varepsilon 1 = \mathcal{L}_\alpha^{n_\varepsilon} 1 = \mathcal{L}_\alpha^{*\,n_\varepsilon} 1 = \mathbf{P}_\varepsilon 1 = 1$. Now, set $\Omega = S^1$, and define $\Omega_0 = \{x \in \Omega : \varphi(x) \geq 0\}, \Omega_1 = \{x \in \Omega : \mathbf{P}_\varepsilon \varphi(x) \geq 0\}$. We observe that

$$\int_\Omega |\mathbf{P}_\varepsilon \varphi(x)|\,dx = 2\int_{\Omega_1} \mathbf{P}_\varepsilon \varphi(x)\,dx.$$

We use the bound in equation (3.46) and (3.45) to get the following estimate

$$\begin{aligned}
\|\mathbf{P}_\varepsilon \varphi\|_1 &= \int |\mathbf{P}_\varepsilon \varphi|\,dx = 2\int_{\Omega_1}\left(\int_\Omega \mathcal{K}_\varepsilon(x,y)\varphi(y)\,dy\right)\,dx \\
&= 2\int_{\Omega_1}\left(\int_\Omega \mathcal{K}_\varepsilon(x,y)\varphi(y)\,dy\right)\,dx - 2\Omega_1\gamma\int_\Omega \varphi(x)\,dx, \quad \left(\Leftarrow \int_\Omega \varphi(x)\,dx = 0\right) \\
&= 2\int_\Omega\left(\int_{\Omega_1}(\mathcal{K}_\varepsilon(x,y)-\gamma)\,dx\right)\varphi(y)\,dy \\
&\leq 2\int_\Omega\left(\int_\Omega(\mathcal{K}_\varepsilon(x,y)-\gamma)\,dx\right)\varphi(y)\,dy \\
&\leq 2\int_{\Omega_0}\left(\int_\Omega(\mathcal{K}_\varepsilon(x,y)-\gamma)\,dx\right)\varphi(y)\,dy \\
&= 2\int_{\Omega_0}(\mathbf{P}_\varepsilon 1-\gamma)\varphi(y)\,dy \\
&= 2\int_{\Omega_0}(1-\gamma)\varphi(y)\,dy \\
&= (1-\gamma)\|\varphi\|_1.
\end{aligned}$$

Iterating the above estimate, we get

$$\|\mathbf{P}_\varepsilon^k \varphi\|_1 \leq (1-\gamma)^k \|\varphi\|_1, \quad \forall k \in \mathbb{N}. \tag{3.47}$$

□

3.3.3 Decay estimate

The success of the mechanism we shall employ depends on the decay estimate (with respect the Lebesgue measure) of the system under consideration. The estimates in Lemma 3.3.6 and Lemma 3.3.10 will be particularly useful in proving the following result.

Lemma 3.3.11. *For $C_1 > 0$, $\varphi \in \mathcal{C}_{*,1} + \mathbb{R}$, $\int \varphi\, dx = 0$ and $\psi \in L^\infty(m)$,*

$$\left| \int \psi \mathcal{L}_\varrho^n \varphi \, dx \right| \le C_1 \|\varphi\|_1 \|\psi\|_\infty n^{1-1/\varrho} (\log n)^{1/\varrho}.$$

Proof. For each $n = kn_\varepsilon + j$ with $k \in \mathbb{N}$, $j < n_\varepsilon$, in order to get the required estimate, we decompose the Perron-Frobenius operator as follows

$$\left| \int \psi \mathcal{L}_\varrho^n \varphi \, dx \right| \le \|\psi\|_\infty \left(\|\mathcal{L}_\varrho^n \varphi - \mathbf{P}_\varepsilon^k \mathcal{L}_\varrho^j \varphi\|_1 + \|\mathbf{P}_\varepsilon^k \mathcal{L}_\varrho^j \varphi\|_1 \right). \tag{3.48}$$

By Proposition 3.3.10,

$$\|\mathbf{P}_\varepsilon^k \mathcal{L}_\varrho^j \varphi\|_1 \le (1-\gamma)^k \|\mathcal{L}_\varrho^j \varphi\|_1 \le (1-\gamma)^k \|\varphi\|_1 \le \exp(-\gamma k) \|\varphi\|_1. \tag{3.49}$$

From Lemma 3.3.5, we have that

$$\begin{aligned} \|\mathcal{L}_\varrho^n \varphi - \mathbf{P}_\varepsilon^k \mathcal{L}_\varrho^j \varphi\|_1 &\le \sum_{i=0}^{k-1} \left\| \mathcal{L}_\varrho^{(i+1)n_\varepsilon} \mathcal{L}_\varrho^j \varphi - \mathbf{P}_\varepsilon^k \mathcal{L}_\varrho^{in_\varepsilon} \mathcal{L}_\varrho^j \varphi \right\|_1 \\ &\le Ck \|\varphi\|_1 \varepsilon^{1-\varrho} \\ &\le C \|\varphi\|_1 \frac{n}{n_\varepsilon} \varepsilon^{1-\varrho}. \end{aligned} \tag{3.50}$$

From equations (3.49) and (3.50), we obtain the following estimate for equation (3.48)

$$\begin{aligned} \left| \int \psi \mathcal{L}_\varrho^n \varphi \, dx \right| &\le C \|\psi\|_\infty \left(C \|\varphi\|_1 \frac{n}{n_\varepsilon} \varepsilon^{1-\varrho} + \exp(-\gamma k) \|\varphi\|_1 \right) \\ &\le C \|\psi\|_\infty \|\varphi\|_1 \left(C \frac{n}{n_\varepsilon} \varepsilon^{1-\varrho} + \exp\left[-\gamma \left(\frac{n}{n_\varepsilon} - \frac{j}{n_\varepsilon} \right) \right] \right) \\ &\le C \|\psi\|_\infty \|\varphi\|_1 \left(C \frac{n}{n_\varepsilon} \varepsilon^{1-\varrho} + \exp(\gamma) \exp\left(-\gamma \frac{n}{n_\varepsilon} \right) \right) \\ &\le C \|\psi\|_\infty \|\varphi\|_1 n^{1-1/\varrho} (\log n)^{1/\varrho}, \end{aligned} \tag{3.51}$$

provided we take $\varepsilon = C_{\gamma,\varrho} n^{-1/\varrho} (\log n)^{1/\varrho}$. □

Define the cone

$$\mathcal{C}_0 = \left\{ \varphi \in C^0(S^1 \setminus \{0\}) | \varphi \ge 0 \text{ and } \varphi \text{ is decreasing} \right\},$$

it is easy to check that $\mathcal{C}_0$ is invariant with respect to $\mathcal{L}_\alpha$. Let $\kappa = \frac{1}{d}$ as defined, then

$$(1-\kappa) \int_0^\kappa \varphi \, dx + \kappa \int_{1-\kappa}^1 \varphi \, dx \ge \kappa m(\varphi), \quad \forall \varphi \in \mathcal{C}_0. \tag{3.52}$$

Indeed,

$$\kappa m(\varphi) = \kappa \int_{S^1} \varphi(x)\,dx = \kappa \int_0^\kappa \varphi(x)\,dx + \kappa \sum_{i=1}^{d-2} \int_{l_i}^{l_{i+1}} \varphi(x)\,dx + \kappa \int_{1-\kappa}^1 \varphi(x)\,dx, \quad l_i = \frac{i}{d},$$

since $\varphi(x) \geq 0$ and decreasing, we have that $\kappa \sum_{i=1}^{d-2} \int_{l_i}^{l_{i+1}} \varphi(x)\,dx \leq \kappa(d-2) \int_0^\kappa \varphi(x)\,dx$

$$\kappa m(\varphi) \leq (1-\kappa) \int_0^\kappa \varphi(x) + \kappa \int_{1-\kappa}^1 \varphi(x) dx.$$

Remark 3.3.12. *We have equality in equation* (3.52) *when* $d = 2$. $h_\alpha \in \mathcal{C}_0 \cap \mathcal{C} \cap \mathcal{C}_{*,1}$, *for* a_1, b_1 *large enough. In addition,* h_α *is Lipschitz.*

Proposition 3.3.13. *For* $\alpha \in (0,1)$, a_1 *and* b_1 *big enough. Then*

$$\mathcal{N}_{\alpha,i}\left(\mathcal{C}_{*,1}(\alpha, 1, a_1, b_1) \cap \left\{(1-\kappa)\int_0^\kappa \varphi\,dx + \kappa \int_{1-\kappa}^1 \varphi\,dx \geq \kappa m(\varphi)\right\}\right) \subset \mathcal{C}_{*,1}(\alpha, (d-1), a_1, b_1),$$

d the number of branches and $\kappa = \frac{1}{d}$. *Furthermore, for any* $\psi \in L^\infty(m)$ *and* $\varphi \in \mathcal{C}_{*,1}(\alpha) + \mathbb{R}$, *with zero average, there exists* $C > 0$ *independent of* α, a_1, b_1, *such that*

$$\left|\int_0^1 \psi \mathcal{L}_0^k(\varphi)\,dx\right| \leq \frac{Cab_1}{(1-\beta)(\log k)k^{-2+1/\beta}} \|\psi\|_\infty \|\varphi\|_1, \quad \forall k \geq 1, \beta \in (0,1).$$

Proof.

$$\begin{aligned}(1-\kappa)m(\mathcal{N}_{\alpha,i}\varphi(x)) &= (1-\kappa)\int_0^\kappa \varphi\,dx + (1-\kappa)\int_{1-\kappa}^1 \varphi\,dx \\ &\geq (1-\kappa)\int_0^\kappa \varphi\,dx + \kappa \int_{1-\kappa}^1 \varphi\,dx \\ &\geq \kappa m(\varphi) \quad \text{(using equation (3.52))}.\end{aligned}$$

Hence, $m(\varphi) \leq (d-1)m(\mathcal{N}_{\alpha,i}\varphi(x))$, we remark that when $d = 2$ this is an equality and we are back to equation (3.25). By equation (3.38) and the fact that $\mathcal{N}_{\alpha,i}h_\alpha \leq h_\alpha$, the invariance of $\mathcal{N}_{\alpha,i}$ follows. Next, we show the decay of correlations at $\alpha = 0$.

We fix β for any $\beta \in (0,1)$. Recall the inclusion in equation (3.28) for a parameter say $\varrho = 0$, we then have from Lemma 3.3.5, for $\varphi \in \mathcal{C}_{*,1}(\beta)$ that

$$\|\mathcal{L}_0^{n_\varepsilon}(\mathrm{id} - \mathbf{A}_\varepsilon)\varphi\|_1 \leq \frac{18c_2 \max\{a_1, b_1, 1\}}{\beta(1-\beta)} \|\varphi\|_1 \varepsilon^{1-\beta}.$$

we may take $n_\varepsilon = \frac{|\log \varepsilon|}{\log 2}$ in the proof of Proposition 3.3.6. From Lemma 3.3.11, taking $\varepsilon = n^{-1/\alpha}$, we get the result.

□

Theorem 3.3.14. *[79, Theorem 5] Let* $\mathcal{L}_\alpha$ *be the Perron-Frobenius operator associated with* f_α *the circle map with parameter* $\alpha \in (0,1)$, *and* h_α *its density, then for all Hölder continuous* *function* $\varphi : S^1 \to \mathbb{R}$, $\psi \in L^\infty(m)$ *with* $\int \varphi\, dm = 1$,

$$\int |\mathcal{L}_\alpha^n(\varphi) - h_\alpha|\, dm \approx n^{1-1/\alpha}. \tag{3.53}$$

For $\alpha = 0$

$$\left| \int (\psi \circ f_0^n)\varphi\, d\mu - \int \varphi\, d\mu \int \psi\, d\mu \right| \leq C\theta^n, \tag{3.54}$$

$\theta < 1$ *depending only on the Hölder exponents of the observables.*

Theorem 3.3.15. *[6, Theorem 3.62] Let* f_α *be the circle map with* $\alpha \in (0,1)$*. Then* f_α *has a unique SRB measure* μ_α*. Moreover,* μ_α *is exact, equivalent to* m*, its basin covers* m *almost all of* S^1 *and for every Hölder continuous function* $\varphi : S^1 \to \mathbb{R}$ *and* $\psi \in L^\infty(m)$

$$\left| \int (\psi \circ f_\alpha^n)\varphi d\mu - \int \varphi d\mu \int \psi d\mu \right| \lesssim \left(\frac{1}{n^{1/\alpha - 1}} \right) \tag{3.55}$$

Proposition 3.3.16. *[50, Corollary 2.4.6] For all Hölder function* φ *with mean zero,* $\alpha \in (0,1)$ *and* ψ *a bounded function which cancel each other in the vicinity of* 0*, we have*

$$\int \varphi \cdot \psi \circ f_\alpha^n\, dx = \mathcal{O}\left(\frac{1}{n^{1/\alpha}} \right).$$

For the particular mechanism we shall deploy, the rate of decay given in Proposition 3.3.16 shall play a pivotal role when $1/2 \leq \alpha < 1$.

Theorem 3.3.17. *[50, Theorem 2.4.14] Let* f_α *be the intermittent circle map with the parameter* $\alpha \in (0,1)$ *and* φ *be a zero average Hölder function with* $\varphi(0) = 0$*, satisfying* $|\varphi(x)| \leq Cx^\gamma$*, for a certain* $\gamma > 0$*. Then*

$$\|\mathcal{L}^n \varphi\|_1 = \mathcal{O}\left(\frac{1}{n^{\min\{\lambda, \lambda(1+\gamma)-1\}}} \right),$$

where $\lambda = \frac{1}{\alpha}$.

Remark 3.3.18. *Although the above results in [50] were stated for the LSV map, the theorems are written in the general setting of the Young tower and applies to the circle map with indifferent fixed points we are considering.*

3.3.4 Some properties of the transfer operator

For $\alpha \mapsto \mathcal{L}_\alpha \varphi(x)$, and $\varphi : S^1 \to \mathbb{R}$ sufficiently regular, we give some properties of $\partial_\alpha \mathcal{L}_\alpha$ that will be particularly useful going forward.

Lemma 3.3.19. *For* $\alpha \in (0,1)$*,* $\alpha \mapsto g_{\alpha,i}(y)$*, and for all* $x \in S^1 \setminus \{0\}$*,*

$$\partial_\alpha g_{\alpha,i}(x) = -\frac{X_{\alpha,i}(x)}{f'_\alpha(g_{\alpha,i}(x))}, \quad i \in \{1, d\}; \tag{3.56}$$

$$\partial_\alpha g'_{\alpha,i}(x) = -\frac{X'_{\alpha,i}(x)}{f'_{\alpha,i}(g_{\alpha,i}(x))} + X_{\alpha,i}(x) \frac{f''_{\alpha,i}(g_{\alpha,i}(x))}{(f'_{\alpha,i}(g_{\alpha,i}(x))^3}, \quad i \in \{1, d\}. \tag{3.57}$$

Proof. Since for $x \in S^1 \setminus \{0\}$, $f_{\alpha,i}(g_{\alpha,i}(x)) = x$. Then by the chain rule, we have that

$$(\partial_\alpha g_{\alpha,i}(x))\, f'_\alpha(g_{\alpha,i}(x)) + \partial_\alpha f_\alpha(g_{\alpha,i}(x)) = 0,$$

using equation (3.8), gives equation (3.56). Using the chain rule, we have that

$$g'_{\alpha,i}(x) = \frac{1}{f'_{\alpha,i}(g_{\alpha,i}(x))}. \tag{3.58}$$

Now,

$$\begin{aligned} g'_{\alpha+\varepsilon,i}(x) - g'_{\alpha,i}(x) &= \frac{1}{f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x))} - \frac{1}{f'_{\alpha,i}(g_{\alpha,i}(x))} \\ &= \frac{f'_{\alpha,i}(g_{\alpha,i}(x)) - f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x))}{f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x)) \cdot f'_{\alpha,i}(g_{\alpha,i}(x))} \end{aligned} \tag{3.59}$$

To simplify equation (3.59), we first multiply by $\frac{f'_{\alpha+\varepsilon,i}(g_{\alpha,i}(x))}{f'_{\alpha+\varepsilon,i}(g_{\alpha,i}(x))}$, then add and subtract $f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x)) \cdot f'_{\alpha,i}(g_{\alpha,i}(x))$ to the numerator, we get

$$g'_{\alpha+\varepsilon,i}(x) - g'_{\alpha,i}(x) = \underbrace{\frac{f'_{\alpha,i}(g_{\alpha,i}(x)) - f'_{\alpha+\varepsilon,i}(g_{\alpha,i}(x))}{f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x)) \cdot f'_{\alpha+\varepsilon,i}(g_{\alpha,i}(x))}}_{\text{(I)}} + \underbrace{\frac{f'_{\alpha+\varepsilon,i}(g_{\alpha,i}(x)) - f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x))}{f'_{\alpha+\varepsilon,i}(g_{\alpha,i}(x)) \cdot f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x))}}_{\text{(II)}}. \tag{3.60}$$

But $\partial_\alpha g'_{\alpha,i}(x) = \lim_{\varepsilon\to 0} \frac{g'_{\alpha+\varepsilon,i}(x) - g'_{\alpha,i}(x)}{\varepsilon}$. To simplify (I) and (II) above, recall that by Taylor's formula, we have that

$$f'_{\alpha+\varepsilon,i}(y) = f'_{\alpha,i}(y) + \varepsilon \cdot \partial_\alpha f'_{\alpha,i}(y) + \mathcal{O}(\varepsilon^2), \tag{3.61}$$

hence, (I) simplifies to

$$\text{(I)} = \frac{-\varepsilon \cdot \partial_\alpha f'_{\alpha,i}(g_{\alpha,i}(x)) - \mathcal{O}(\varepsilon^2)}{f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x)) \cdot f'_{\alpha,i}(g_{\alpha,i}(x))}$$

$$\lim_{\varepsilon\to 0} \frac{\text{(I)}}{\varepsilon} = \frac{-\partial_\alpha f'_{\alpha,i}(g_{\alpha,i}(x))}{(f'_{\alpha,i}(g_{\alpha,i}(x)))^2}.$$

Since $\alpha \mapsto f_{\alpha,i} \in C^2$, therefore by the definition in equation (3.8)

$$\lim_{\varepsilon\to 0} \frac{\text{(I)}}{\varepsilon} = \frac{-v'_{\alpha,i}(g_{\alpha,i}(x))}{(f'_{\alpha,i}(g_{\alpha,i}(x)))^2} = \frac{-X'_{\alpha,i}(x)}{f'_{\alpha,i}(g_{\alpha,i}(x))}, \tag{3.62}$$

the above follows since from (3.8), we have that

$$X'_{\alpha,i}(x) = \frac{v'_{\alpha,i}(g_{\alpha,i}(x))}{f'_{\alpha,i}(g_{\alpha,i}(x))}. \tag{3.63}$$

Next, to simplify (II), we note that from Taylor's formula,

$$f'_{\alpha+\varepsilon,i}(g_{\alpha+\varepsilon,i}(x)) = f'_{\alpha,i}(g_{\alpha,i}(x))+\varepsilon\cdot\partial_\alpha f'_{\alpha,i}(g_{\alpha,i}(x))+\varepsilon\cdot\partial_\alpha g_{\alpha,i}(x)\cdot f''_{\alpha,i}(g_{\alpha,i}(x))+\mathcal{O}(\varepsilon^2). \tag{3.64}$$

Using equation (3.61) and equation (3.64),

$$\lim_{\varepsilon\to 0}\frac{(\mathrm{II})}{\varepsilon} = \frac{-\partial_\alpha g_{\alpha,i}(x)\cdot f''_{\alpha,i}(g_{\alpha,i}(x))}{(f'_{\alpha,i}(g_{\alpha,i}(x)))^2}$$

and by equation (3.56),

$$\lim_{\varepsilon\to 0}\frac{(\mathrm{II})}{\varepsilon} = \frac{X_{\alpha,i}(x)}{f'_{\alpha,i}(g_{\alpha,i}(x))}\cdot\frac{f''_{\alpha,i}(g_{\alpha,i}(x))}{(f'_{\alpha,i}(g_{\alpha,i}(x)))^2}. \tag{3.65}$$

From equation (3.62) and equation (3.65),

$$\partial_\alpha g'_{\alpha,i}(x) = -\frac{X'_{\alpha,i}(x)}{f'_{\alpha,i}(g_{\alpha,i}(x))} + X_{\alpha,i}(x)\frac{f''_{\alpha,i}(g_{\alpha,i}(x))}{(f'_{\alpha,i}(g_{\alpha,i}(x))^3}, \quad i\in\{1,d\}.$$

□

Lemma 3.3.20. *For $\varphi\in C^1(S^1\setminus\{0\})$, then for all $x\in S^1\setminus\{0\}$, $\alpha\in(0,1)$, and $i\in\{1,d\}$,*

$$\partial_\alpha\mathcal{L}_\alpha\varphi(x) = -\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)'(x). \tag{3.66}$$

In particular, $m(\partial_\alpha\mathcal{L}_\alpha\varphi(x)) = 0$.

Proof. From the assumption on f_α and equation (3.23),

$$\begin{aligned}\partial_\alpha\mathcal{L}_\alpha\varphi(x) &= \sum_{i\in\{1,d\}}\partial_\alpha\mathcal{N}_{\alpha,i}\varphi(x)\\ &= \sum_{i\in\{1,d\}}\Big[\partial_\alpha g'_{\alpha,i}(x)\cdot\varphi(g_{\alpha,i}(x)) + \varphi'(g_{\alpha,i}(x))\cdot\partial_\alpha g_{\alpha,i}(x)\cdot g'_{\alpha,i}(x)\Big].\end{aligned}$$

Substituting equation (3.56) and equation (3.58) into the above,

$$\partial_\alpha\mathcal{L}_\alpha\varphi(x) = \sum_{i\in\{1,d\}}\left[\partial_\alpha g'_{\alpha,i}(x)\cdot\varphi(g_{\alpha,i}(x)) - X_{\alpha,i}(x)\cdot\frac{\varphi'(g_{\alpha,i}(x))}{(f'_{\alpha,i}(g_{\alpha,i}(x)))^2}\right].$$

To simplify the right hand side of the above equation, we use equation (3.57) to get that

$$\partial_\alpha g'_{\alpha,i}(x)\cdot\varphi(g_{\alpha,i}(x)) = -X'_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi(x) + X_{\alpha,i}\mathcal{N}_{\alpha,i}(\varphi f''_{\alpha,i}/(f'_{\alpha,i})^2)(x)$$

and observe that

$$X_{\alpha,i}(x)\cdot\frac{\varphi'(g_{\alpha,i}(x))}{(f'_{\alpha,i}(g_{\alpha,i}(x)))^2} = X_{\alpha,i}(x)\mathcal{N}_{\alpha,i}(\varphi'/f'_{\alpha,i})(x).$$

Therefore,

$$\begin{aligned}\partial_\alpha \mathcal{L}_\alpha \varphi(x) &= \sum_{i\in\{1,d\}} \left[-X'_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi(x) - X_{\alpha,i}(x)\mathcal{N}_{\alpha,i}(\varphi'/f'_{\alpha,i})(x) + X_{\alpha,i}\mathcal{N}_{\alpha,i}(\varphi f''_{\alpha,i}/(f'_{\alpha,i})^2)(x)\right]\\ &= \sum_{i\in\{1,d\}} \left[-X'_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi(x) - X_{\alpha,i}(x)\left(\mathcal{N}_{\alpha,i}(\varphi'/f'_{\alpha,i})(x) - \mathcal{N}_{\alpha,i}(\varphi f''_{\alpha,i}/(f'_{\alpha,i})^2)\right)(x)\right]\\ &= -\sum_{i\in\{1,d\}} (X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)'(x).\end{aligned}$$

Using integration by parts, for $i \in \{1, d\}$,

$$\int_0^1 (X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)'dx = X_{\alpha,i}(1)\mathcal{N}_{\alpha,i}\varphi(1) - X_{\alpha,i}(0)\mathcal{N}_{\alpha,i}\varphi(0) = 0. \tag{3.67}$$

Indeed, by equation (3.12) and for $i = 1$, $g_{\alpha,1}(0) = 0$, from equation (3.10),

$$X_{\alpha,1}(0) = \ 0 = \ X_{\alpha,1}(1),$$

for $i = d$, $g_{\alpha,d}(1) = 1$, similarly, from equation (3.10),

$$X_{\alpha,d}(0) = 0 = X_{\alpha,d}(1).$$

Therefore, from equation (3.66), we have that $m(\partial_\alpha \mathcal{L}_\alpha \varphi(x)) = 0$. □

Lemma 3.3.21. *For $\alpha \mapsto X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi(x) \in C^3$, for $i \in \{1, d\}$,*

$$\partial^2_\alpha \mathcal{L}_\alpha \varphi(x) = \sum_{i\in\{1,d\}} \left[-((\partial_\alpha X_{\alpha,i})(\mathcal{N}_{\alpha,i}\varphi))'(x) + X'_{\alpha,i}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)'(x) + X_{\alpha,i}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)''(x)\right]. \tag{3.68}$$

Proof. From Lemma 3.3.20, we have that

$$\begin{aligned}\partial^2_\alpha \mathcal{L}_\alpha \varphi(x) &= -\sum_{i\in\{1,d\}} \partial_\alpha(X_{\alpha,i}\cdot \mathcal{N}_{\alpha,i}\varphi)'(x)\\ &= -\sum_{i\in\{1,d\}} \Big(\partial_\alpha(X'_{\alpha,i}\cdot \mathcal{N}_{\alpha,i}\varphi + X_{\alpha,i}\cdot \mathcal{N}'_{\alpha,i}\varphi)(x)\Big)\\ &= -\sum_{i\in\{1,d\}} \Big((\partial_\alpha(X'_{\alpha,i})\mathcal{N}_{\alpha,i}\varphi + X'_{\alpha,i}\partial_\alpha\mathcal{N}_{\alpha,i}\varphi + \partial_\alpha(X_{\alpha,i})\cdot \mathcal{N}'_{\alpha,i}\varphi + X_{\alpha,i}\cdot \partial_\alpha\mathcal{N}'_{\alpha,i}\varphi)(x)\Big)\\ &= -\sum_{i\in\{1,d\}} \Big((\partial_\alpha(X'_{\alpha,i})\mathcal{N}_{\alpha,i}\varphi + \partial_\alpha(X_{\alpha,i})\cdot \mathcal{N}'_{\alpha,i}\varphi + X'_{\alpha,i}\partial_\alpha\mathcal{N}_{\alpha,i}\varphi + X_{\alpha,i}\cdot \partial_\alpha\mathcal{N}'_{\alpha,i}\varphi)(x)\Big)\\ &= -\sum_{i\in\{1,d\}} \Big(((\partial_\alpha X_{\alpha,i}\,\mathcal{N}_{\alpha,i}\varphi)' + X'_{\alpha,i}\partial_\alpha\mathcal{N}_{\alpha,i}\varphi + X_{\alpha,i}\cdot (\partial_\alpha\mathcal{N}_{\alpha,i}\varphi)')(x)\Big)\\ &\overset{\text{Lemma3.3.20}}{=} -\sum_{i\in\{1,d\}} \left[\Big((\partial_\alpha X_{\alpha,i}\,\mathcal{N}_{\alpha,i}\varphi)' - X'_{\alpha,i}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)' - X_{\alpha,i}\cdot (X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)''\Big)(x)\right]\\ &= \sum_{i\in\{1,d\}} \left[-(\partial_\alpha X_{\alpha,i}\,\mathcal{N}_{\alpha,i}\varphi)'(x) + (X'_{\alpha,i}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)')(x) + (X_{\alpha,i}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)'')(x)\right].\end{aligned}$$

By the Leibniz integral rule, $m(\partial_\alpha^2\mathcal{L}_\alpha\varphi(x)) = \partial_\alpha m(\partial_\alpha\mathcal{L}_\alpha\varphi(x)) = 0$, from Lemma 3.3.20. □

Remark 3.3.22. *The use of the Leibniz integral rule above is justified by the bound shown in equation* (3.88) *that* $|\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))| < \infty$.

3.4 The linear response formula

The remainder of this chapter will be dedicated to proving the linear response formula in Theorem A, firstly for observables $\psi \in L^\infty(m)$, thereafter show how the result can be extended to $\psi \in L^q(m)$. We achieve the proof in three main steps outlined as follows. Firstly, we shall show that our claimed linear response formula is well-defined. Subsequently, we will show that the map $\beta \mapsto \int \psi \circ f_\beta^n\, dx$ is locally Lipschitz continuous at $\beta = \alpha \in [0,1)$. Finally, we show the linear response formula indeed holds.

3.4.1 Well-defined formula

Let $\psi \in L^\infty(m)$, we show that the right hand side of equation (3.21) is well-defined. We first recall equation (3.67),

$$\int_0^1 \sum_{i=1,d}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\varphi)'dx = 0.$$

Next, we show that it is Lebesgue integrable, for each $i \in \{1,d\}$ and $\alpha \in [0,1)$,

$$\|(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\|_1 = \|X_{\alpha,i}(\mathcal{N}_{\alpha,i}(h_\alpha))' + X'_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha)\|_1 < \infty. \tag{3.69}$$

Indeed, the above is true for $\alpha = 0$. From equation (3.2), we have that $g_{0,i}(x) \le \hat{C}\, x$, $\hat{C} > 0$ and $g'_{\alpha,i}(x) \le k$, $k > 0$. Since $h_0|_{S^1} = 1$ we have from equation (3.23) that

$$\mathcal{N}_{0,i}h_0(x) \le k \quad \text{for } i = 1,d,$$

from the above equation and equation (3.17),

$$\begin{aligned}\int |(X_{0,i}\mathcal{N}_{0,i}(h_0))'|\, dx &\le \int \left|X'_{0,i}(x)\mathcal{N}_{0,i}h_0(x)\right| dx\\ &\le \hat{C}\int (1 + |\ln(|x|)|)\, dx, \quad \forall x \in S^1,\end{aligned}$$

whose integral is finite.

For $0 < \alpha < 1$, equation (3.5) gives the following bounds,

$$\mathcal{N}_{\alpha,i}(h_\alpha)(x) \le h_\alpha(x) \le c_2|x|^{-\alpha}, \qquad |h'_\alpha(x)| \le c_2|x|^{-(1+\alpha)}, \quad c_2 > 0. \tag{3.70}$$

From Lemma 3.3.4, it implies that

$$|(\mathcal{N}_{\alpha,i}(h_\alpha))'(x)| \le \left(\frac{a_1}{|x|} + b_1\right)\mathcal{N}_{\alpha,i}(h_\alpha)(x) \le c_2(a_1|x|^{-(1+\alpha)} + b_1|x|^{-\alpha}), \quad a_1, b_1, c_2 > 0. \tag{3.71}$$

Recalling equations (3.15) and (3.17), together with equations (3.70) and (3.71)

$$
\begin{aligned}
&\|(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\|_1 \\
&= \|X_{\alpha,i}(\mathcal{N}_{\alpha,i}(h_\alpha))' + X'_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha)\|_1 \\
&\le \int_0^1 \left[\tilde{C}|x|^{\alpha+1}(1+|\ln(|x|)|)\cdot c_2|x|^{-(1+\alpha)}(a_1+b_1|x|) + \tilde{C}|x|^{\alpha}(1+|\ln(|x|))\cdot c_2|x|^{-\alpha}\right] dx \\
&\le \tilde{C}\int_0^1 \left[1+(a_1+b_1|x|)\right](1+|\ln(|x|)|)\, dx < \infty.
\end{aligned}
$$

By the Neumann series,

$$
(\mathrm{id} - \mathcal{L}_\alpha)^{-1} = \sum_{j=0}^{\infty} \mathcal{L}_\alpha^j,
$$

hence, the right hand side of equation (3.21) may be written as

$$
\left|\sum_{j=0}^{\infty}\int_{S^1}\psi\mathcal{L}_\alpha^j\left[\sum_{i\in\{1,d\}}(X_{\alpha,i}(\mathcal{N}_{\alpha,i}(h_\alpha))')\right]dx\right| \le \sum_{j=0}^{\infty}\left|\int_{S^1}\psi\mathcal{L}_\alpha^j\left[\sum_{i\in\{1,d\}}(X_{\alpha,i}(\mathcal{N}_{\alpha,i}(h_\alpha))')\right]dx\right|. \tag{3.72}
$$

Our next task is to show that this series is absolutely convergent for $\alpha \in [0,1)$. We achieve this in two parts. Firstly for $\alpha \in (0,1)$, then for $\alpha = 0$. For $\alpha \in (0,1)$, we check the hypothesis of Theorem 3.3.17. We define the function

$$
F_\alpha(x) = \sum_{i\in\{1,d\}} \frac{(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'(x)}{h_\alpha(x)},
$$

the bounds for h_α in equation (3.5) gives that $F_\alpha(0) = 0$. By Lemma 3.3.20 we have that $\int F_\alpha h_\alpha dx = 0$. We also claim that F_α is in fact Lipschitz. Indeed,

$$
F_\alpha(x) = \sum_{i\in\{1,d\}} X_{\alpha,i}(x)\frac{(\mathcal{N}_{\alpha,i}(h_\alpha))'(x)}{h_\alpha(x)} + \sum_{i\in\{1,d\}} X'_{\alpha,i}(x)\frac{\mathcal{N}_{\alpha,i}(h_\alpha)(x)}{h_\alpha(x)}, \tag{3.73}
$$

$X'_{\alpha,i}$ (resp. $X_{\alpha,i}$) being Lipschitz, follows from equation (3.19) (resp. (3.17)) that $X''_{\alpha,i}$ (resp. $X'_{\alpha,i}$) is bounded. We simplify the second sum above using equation (3.36),

$$
\begin{aligned}
\sum_{i\in\{1,d\}} X'_{\alpha,i}(x)\frac{\mathcal{N}_{\alpha,i}(h_\alpha)(x)}{h_\alpha(x)} &\le \max_i X'_{\alpha,i}(x)\sum_{i\in\{1,d\}}\frac{(\mathcal{N}_{\alpha,i}(h_\alpha))(x)}{h_\alpha(x)} \\
&= \max_i X'_{\alpha,i}(x) - \max_i X'_{\alpha,i}(x)\sum_{y\in f_{\alpha,i}^{-1}(x), 2\le i\le d-1}\frac{h_\alpha(y)}{h_\alpha(x)f'_\alpha(y)}.
\end{aligned}
$$

We have used the fact that $\mathcal{L}_\alpha h_\alpha = h_\alpha$ in the last equation. $h_\alpha(y)$ is Lipschitz and $\dfrac{1}{h_\alpha(x)}$ is Lipschitz, with f'_α bounded. From equation (3.71), we have that the first sum in (3.73) is also Lipschitz continuous. The product and sum of bounded Lipschitz functions is Lipschitz.

We observe from equations (3.15), (3.17), (3.70) and (3.71) that for any $\varepsilon > 0$, there exists some $M_\varepsilon > 0$ such that for $x \in S^1 \setminus \{0\}$,

$$\begin{aligned}|F_\alpha(x)| &\le |x|^\alpha \left[1 + (a_1 + b_1|x|)\right](1 + |\ln(|x|)|)\\ &\le M_\varepsilon |x|^{\alpha(1-\varepsilon/2)}.\end{aligned}$$

From Theorem 3.3.17, taking $\gamma = \alpha(1-\varepsilon/2) > 0$, then

$$\min\left\{\frac{1}{\alpha}, \frac{1}{\alpha}(1+\gamma) - 1\right\} = \frac{1}{\alpha}(1+\gamma) - 1 > 1/\alpha - \varepsilon.$$

By the duality of the Perron-Frobenius operator, we have that

$$\begin{aligned}\sum_{j=0}^{\infty}\left|\int_{S^1}\psi\mathcal{L}_\alpha^j\left[\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\right]dx\right| &= \sum_{j=0}^{\infty}\left|\int_{S^1}\psi\mathcal{L}_\alpha^j\left(F_\alpha\cdot h_\alpha\right)dx\right|\\ &= \sum_{j=0}^{\infty}\left|\int_{S^1}\psi\mathcal{L}_\alpha^j F_\alpha\,d\mu_\alpha\right|\\ &\le \|\psi\|_\infty\sum_{j=0}^{\infty}\int_{S^1}\left|\mathcal{L}_\alpha^j F_\alpha\right|d\mu_\alpha\\ &\le CK_\alpha\|\psi\|_\infty\frac{1}{j^{(1/\alpha)-\varepsilon}}.\end{aligned} \tag{3.74}$$

This series is only summable only when $\varepsilon < \frac{1}{\alpha} - 1$.

Now, for $\alpha = 0$, applying Proposition 3.3.13 with $\beta \in (0,1)$ fixed, there is a constant $C_q > 0$ such that to $(X_{0,i}\mathcal{N}_{0,i}(h_0))' + C_q \in \mathcal{C}_{*,1}(\beta, 1, a, b_1)$. For

$$\varphi = (X_{0,i}\mathcal{N}_{0,i}(h_0))' \le \tilde{C}(1 + |\ln(|x|)|) \in \mathcal{C}_{*,1} + \mathbb{R},$$

$$\left|\int_0^1\psi\mathcal{L}_0^j\left[\sum_{i\in\{1,d\}}(X_{0,i}\mathcal{N}_{0,i}(h_0))'\right]dx\right| \le \frac{Cab_1}{(1-\beta)(\log j)j^{-2+1/\beta}}\|\psi\|_\infty, \quad \forall j \ge 1,$$

hence, the claimed linear response formula is well-defined.

3.4.2 Local Lipschitz continuity

Next, we assume without loss of generality that $\int \psi d\mu_\alpha = 0$, for $\psi \in L^\infty(m)$. Our aim is to show that $\beta \mapsto \int \psi \circ f_\beta^n\,dx$ is Lipschitz continuous at $\beta = \alpha$. We may write the zero average function F_ϱ as

$$F_\varrho = h_\varrho - \mathbf{1} \in \mathcal{C}_{*,1} + \mathbb{R}, \quad \varrho > 0,$$

then apply the rate of decay result in Lemma 3.3.11 to get an estimate for the decay of correlation

$$\begin{aligned}\mathrm{Cor}(F_\varrho, \psi\circ f_\alpha^n) &= \left|\int F_\varrho(\psi\circ f_\alpha^n)\,dx - \int F_\varrho\,dx\int\psi\circ f_\alpha^n\,dx\right|\\ &= \left|\int\psi\mathcal{L}_\varrho^n F_\varrho\,dx\right| \qquad\qquad \left(\Leftarrow \int F_\varrho\,dx = 0\right)\end{aligned}$$

$$= \left| \int \psi \mathcal{L}_\varrho^n h_\varrho \, dx - \int \psi \mathcal{L}_\varrho^n \mathbf{1} \, dx \right|$$
$$= \left| \int \psi \, d\mu_\varrho - \int \mathbf{1}(\psi \circ f_\varrho^n) \, dx \right|. \tag{3.75}$$

In the last equality, we use the fact that $\mathcal{L}_\varrho^n h_\varrho = h_\varrho$ and the duality property of the Perron-Frobenius operator.

Let $\varrho = \alpha > 0$, then from Lemma 3.3.11, we have that

$$\left| \int \mathbf{1}(\psi \circ f_\alpha^n) \, dx \right| \leq C_\alpha \|\psi\|_\infty \frac{(\log n)^{1/\alpha}}{n^{1/\alpha - 1}}. \tag{3.76}$$

By Theorem 3.3.14, the unperturbed system, $\varrho = \alpha = 0$ has the following estimate

$$\left| \int \mathbf{1}(\psi \circ f_0^n) \, dx \right| \leq C\theta^n, \quad \theta < 1. \tag{3.77}$$

Suppose that ϱ is any $\beta > 0$, equation (3.75) becomes

$$\left| \int \psi \mathcal{L}_\beta^n F_\beta \, dx \right| = \left| \int \psi \, d\mu_\beta - \int \mathbf{1}(\psi \circ f_\beta^n) \, dx \right| \leq C_\beta \|\psi\|_\infty \frac{(\log n)^{1/\beta}}{n^{1/\beta - 1}}. \tag{3.78}$$

Choosing n large enough, depending on α and β. Equations (3.76), (3.77), (3.78) are $\mathcal{O}(\beta - \alpha)$. That is, fixing $\xi > 0$, there is $C > 0$ such that for all

$$n(\alpha, \beta, \xi) =: n > C \left(C_{\max\{\alpha,\beta\}} (\beta - \alpha)^{-(1+\xi)} \right)^{1/(-1+1/\max\{\alpha,\beta\})}, \tag{3.79}$$

we have that

$$\left| \int (\psi \circ f_\alpha^n) \, dx \right| + \left| \int \psi \, d\mu_\beta - \int (\psi \circ f_\beta^n) \, dx \right| \leq C(\beta - \alpha)^{1+\xi}. \tag{3.80}$$

What we want to ultimately show is that

$$\lim_{\beta \to \alpha} \frac{1}{\beta - \alpha} \left[\left(\int \psi \, d\mu_\beta - \int \psi \, d\mu_\alpha \right) - \left(\int (\psi \circ f_\beta^n) \, dx - \int (\psi \circ f_\alpha^n) \, dx \right) \right] = 0.$$

We have that the term in the square bracket is bounded by (3.80). Suppose that $n := n(\alpha, \beta, \xi)$, and let $\mathbf{1}$ be the constant function $\equiv 1$, we have by using the duality property of the Perron-Frobenius operator and telescoping sum that for every α, β, n,

$$\mathcal{L}_\beta^n \mathbf{1} - \mathcal{L}_\alpha^n \mathbf{1} = \sum_{j=0}^{n-1} \mathcal{L}_\beta^j (\mathcal{L}_\beta - \mathcal{L}_\alpha) \mathcal{L}_\alpha^{n-1-j}(\mathbf{1}).$$

Hence, we have that

$$\frac{1}{\beta - \alpha} \left(\int (\psi \circ f_\beta^n) \, dx - \int (\psi \circ f_\alpha^n) \, dx \right) = \frac{1}{\beta - \alpha} \int_0^1 \psi (\mathcal{L}_\beta^n \mathbf{1} - \mathcal{L}_\alpha^n \mathbf{1}) \, dx$$

$$= \frac{1}{\beta-\alpha}\int_0^1 \psi \sum_{j=0}^{n-1} \mathcal{L}_\beta^j(\mathcal{L}_\beta - \mathcal{L}_\alpha)\mathcal{L}_\alpha^{n-1-j}(\mathbf{1})\,dx$$
$$= \sum_{j=0}^{n-1}\int_0^1 \psi\mathcal{L}_\beta^j\Big(\frac{(\mathcal{L}_\beta-\mathcal{L}_\alpha)}{\beta-\alpha}(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\Big)\,dx. \quad (3.81)$$

Taylor's formula gives that for any $\varphi \in C^2\left(S^1\setminus\{0\}\right), \beta\neq\alpha, x\neq 0$

$$\frac{(\mathcal{L}_\beta-\mathcal{L}_\alpha)\varphi(x)}{\beta-\alpha} = \partial_\alpha\mathcal{L}_\alpha\varphi(x) + \frac{1}{\beta-\alpha}\int_\alpha^\beta (\beta-t)\partial_t^2\mathcal{L}_t\varphi(x)\,dt. \quad (3.82)$$

Assuming that $\beta>\alpha>0$, $n\geq 1$ in equation (3.81), Lemma 3.3.20 and Lemma 3.3.21 gives that

$$\sum_{j=0}^{n-1}\int_0^1 \psi\mathcal{L}_\beta^j\Big(\frac{(\mathcal{L}_\beta-\mathcal{L}_\alpha)}{\beta-\alpha}(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\Big)dx = \sum_{j=0}^{n-1}\int_0^1 \psi\mathcal{L}_\beta^j\Big(\partial_\alpha\mathcal{L}_\alpha(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1})) + \frac{1}{\beta-\alpha}\int_\alpha^\beta (\beta-t)\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\,dt\Big)\,dx$$

$$\sum_{j=0}^{n-1}\int_0^1 \psi\mathcal{L}_\beta^j\left(\frac{(\mathcal{L}_\beta-\mathcal{L}_\alpha)}{\beta-\alpha}(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\right)dx = \underbrace{-\sum_{j=0}^{n-1}\int_0^1 \psi\mathcal{L}_\beta^j\left(\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))'\right)dx}_{A_n} + \underbrace{\int_\alpha^\beta \frac{\beta-t}{\beta-\alpha}\sum_{j=0}^{n-1}\int_0^1 \psi\mathcal{L}_\beta^j\left(\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\right)dx\,dt}_{B_n}. \quad (3.83)$$

Case 1: We use the Theorem 3.3.17 to show that A_n and B_n are summable, for $0<\alpha<1$.

Summability of A_n: Checking the assumptions of Theorem 3.3.17, allows us to give conclusions about the summability of this series. By equation (3.67) $(X_{\alpha,i}\mathcal{N}_{\alpha,i}\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))'$ is a zero average function. Next, $(X_{\alpha,i}\mathcal{N}_{\alpha,i}\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))'$ is bounded. Finally, to check that $H_\alpha(x) = \sum_{i\in\{1,d\}}\frac{(X_{\alpha,i}\mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1})))'}{h_\alpha}$ is Hölder, it is sufficient to check that there exists an $M\geq 0$ such that

$$|H_\alpha'(x)| \leq \sum_{i\in\{1,d\}}\left|\frac{h_\alpha(x)(X_{\alpha,i}\mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1})))''(x) - (X_{\alpha,i}\mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1})))'(x)h_\alpha'(x)}{h_\alpha^2(x)}\right| \leq M.$$

Indeed, by equation (3.5) and the calculations below, this is bounded. Also, $H_\alpha(0)=0$, hence, same as in the calculations of equation (3.74), and the fact that we assumed that $0<\alpha<\beta<1$, we have that

$$\sum_{j=0}^{n-1}\int_{S^1}\left|\psi\mathcal{L}_\beta^j\Big(\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1})))'\Big)\right|dx\le C_\beta\|\psi\|_\infty\sum_{j=0}^{n-1}\frac{1}{j^{1/\beta-\varepsilon}},$$

which is summable as $n\to\infty$.

Summability of B_n : Now, we have that from equation (3.68), let $\varphi=\mathcal{L}_\alpha^{n-1-j}(\mathbf{1})\in\mathcal{C}\cap\mathcal{C}_{*,1}$, by the invariance of the cone, for any $\alpha\le t\le\beta$, $\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\in\mathcal{C}$.

Claim: $|\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))|<\infty$.

Proof of claim: We find the bounds on $\partial_\alpha X_{\alpha,i}(x)$ and $\partial_\alpha X'_{\alpha,i}(x)$. Using the chain rule, we note from equation (3.8) that,

$$\partial_\alpha X_{\alpha,i}(x)=(\partial_\alpha g_{\alpha,i}(x))\,(v'_\alpha\circ g_{\alpha,i}(x))+\partial_\alpha^2 f_\alpha\circ g_{\alpha,i}(x).$$

By equations (3.1) and (3.9),

$$\begin{aligned}\partial_\alpha f'_\alpha(x)&\approx|x|^\alpha\ln(|x|),\\ \partial_\alpha^2 f_\alpha(x)&\approx\ln(|x|)\partial_\alpha f_\alpha(x).\end{aligned}\tag{3.84}$$

Hence, from equations (3.56), (3.84), (3.8)

$$\partial_\alpha X_{\alpha,i}(x)\approx\left(-\frac{1}{f'_\alpha(g_{\alpha,i}(x))}|g_{\alpha,i}(x)|^\alpha+1\right)\ln(|g_{\alpha,i}(x)|)X_{\alpha,i}(x).\tag{3.85}$$

Using the estimate $g_{\alpha,i}(x)\le Cx$, equation (3.15) and (s2),

$$|\partial_\alpha X_{\alpha,i}(x)|\le\tilde{C}|x|^{\alpha+1}(1+|\ln(|x|)|)^2,\tag{3.86}$$

next, differentiate equation (3.85) with respect to x, to get the following bounds

$$|\partial_\alpha X'_{\alpha,i}(x)|\le C|x|^\alpha(1+|\ln(|x|)|)^2.\tag{3.87}$$

Now, we see that since $\varphi\in\mathcal{C}_{*,1}(\alpha)\cap\mathcal{C}$, Lemma 3.3.4 implies that

$$|(\mathcal{N}_{\alpha,i}\varphi)''(x)|\le\left(\frac{a_2}{x^2}+b_2\right)\mathcal{N}_{\alpha,i}\varphi(x)\le\left(\frac{a_2}{x^2}+b_2\right)\cdot\frac{2c_2}{|x|^\alpha}m(\varphi)\le 2c_2|x|^{-(\alpha+2)}(a_2+b_2x^2).$$

From Lemma 3.3.21, we have that

$$\partial_t^2\mathcal{L}_t\varphi(x)=\sum_{i\in\{1,d\}}[\underbrace{-((\partial_tX_{t,i})(\mathcal{N}_{t,i}\varphi))'(x)}_{\text{(I)}}+\underbrace{X'_{t,i}(X_{t,i}\mathcal{N}_{t,i}\varphi)'(x)}_{\text{(II)}}+\underbrace{X_{t,i}(X_{t,i}\mathcal{N}_{t,i}\varphi)''(x)}_{\text{(III)}}]$$

Now, differentiating and simplifying the terms above, we have that

$$\text{(I)}=-\left[\partial_tX_{t,i}(x)\,(\mathcal{N}_{t,i}\varphi(x))'+\partial_tX'_{t,i}(x)\,(\mathcal{N}_{t,i}\varphi(x))\right],$$

$$\text{(II)} = (X'_{t,i}(x))^2\,\mathcal{N}_{t,i}\varphi(x) + X'_{t,i}(x)\,X_{t,i}(x)\,(\mathcal{N}_{t,i}\varphi(x))',$$
$$\text{(III)} = X_{t,i}(x)\left[X''_{t,i}(x)\,\mathcal{N}_{t,i}\varphi(x) + 2X'_{t,i}(x)\,(\mathcal{N}_{t,i}\varphi(x))' + X_{t,i}(x)\,(\mathcal{N}_{t,i}\varphi(x))''\right].$$

From equations (3.15), (3.17), (3.19), (3.26), (3.3.4), (3.86) and (3.87), we bound the above as follows

$$\begin{aligned}|\text{(I)}| &\le \left(\tilde{C}|x|^{t+1}(1+|\ln(|x|)|)^2\cdot 2c_2|x|^{-(\alpha+1)}(a_1+b_1|x|)m(\varphi)\right)\\ &\qquad + \left(\tilde{C}|x|^{t}(1+|\ln(|x|)|)^2\cdot 2c_2|x|^{-\alpha}m(\varphi)\right)\\ &\le C|x|^{t-\alpha}(1+|\ln(|x|)|)^2[\max\{a_1,b_1\}(1+|x|)+1].\end{aligned}$$

$$\begin{aligned}|\text{(II)}| &\le \left(C|x|^{2t}(1+|\ln(|x|)|)^2\cdot 2c_2|x|^{-\alpha}m(\varphi)\right)\\ &\qquad + \left(C|x|^{t}(1+|\ln(|x|)|)\cdot|x|^{t+1}(1+|\ln(|x|)|)\cdot 2c_2|x|^{-(\alpha+1)}(a_1+b_1|x|)m(\varphi)\right)\\ &\le C|x|^{2t-\alpha}(1+|\ln(|x|)|)^2[\max\{a_1,b_1\}(1+|x|)+1].\end{aligned}$$

$$\begin{aligned}|\text{(III)}| &\le |x|^{t+1}(1+|\ln(|x|)|)\Big[|x|^{t-1}(1+|\ln(|x|)|)\cdot 2c_2|x|^{-\alpha} + 4Cc_2|x|^{t}(1+|\ln(|x|)|)\cdot\\ &\qquad |x|^{-(\alpha+1)}(a_1+b_1|x|) + |x|^{t+1}(1+|\ln(|x|)|)\cdot 2c_2|x|^{-(\alpha+2)}(a_2+b_2x^2)\Big]m(\varphi)\\ &\le C|x|^{2t-\alpha}(1+|\ln(|x|)|)^2[1+\max\{a_1,a_2\}+\max\{b_1,b_2\}|x|].\end{aligned}$$

Since $x\in S^1\setminus\{0\}$ and $0<\alpha\le t\le\beta<1$,

$$|\partial_t^2\mathcal{L}_t\varphi(x)| \le C|x|^{t-\alpha}(1+|\ln(|x|)|)^2[1+\max\{a_1,a_2\}+\max\{b_1,b_2\}|x|], \tag{3.88}$$

which proves our claim. Next, we check the hypothesis of Theorem 3.3.17. Firstly, observe that just as in section 3.4.1, we can find a $\gamma>0$ such that

$$|\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))| \le C_\gamma|x|^\gamma,$$

where C_γ is independent of n and j. We see also that from Lemma 3.3.21, that $\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))$ has zero mean. Define a Hölder function

$$G_\alpha(x) = \frac{\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))(x)}{h_\alpha(x)}.$$

Indeed, it is sufficient to show that there exists $M\ge 0$ such that

$$|G'(x)| = \left|\frac{h_\alpha(x)(\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))(x))' - \partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))(x)h'_\alpha(x)}{(h_\alpha(x))^2}\right| \le M.$$

From the bounds we got above and the bounds in equation (3.5), to get the bounds on $|G'_\alpha(x)| \le M$, we only need get the bounds on $(\partial_t^2 \mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))(x))'$. By the commutation relation equation (3.13), we have to differentiate each of the three terms (I) – (III) above. The bounds on $\partial_\alpha X''_{\alpha,i}(x)$, is

$$\partial_\alpha X''_{\alpha,i}(x) \le C|x|^{\alpha-1}(1+|\ln(|x|)|)^2. \tag{3.89}$$

We have that

$$\begin{aligned}
(\mathrm{I}') &= -\left[\partial_t X_{t,i}\,(\mathcal{N}_{t,i}\varphi)'' + 2\partial_t X'_{t,i}\,(\mathcal{N}_{t,i}\varphi)' + \partial_t X''_{t,i}\,(\mathcal{N}_{t,i}\varphi)\right],\\
(\mathrm{II}') &= 2(X'_{t,i})^2(\mathcal{N}_{t,i}\varphi)' + 2X'_{t,i}X''_{t,i}\mathcal{N}_{t,i}\varphi + X'_{t,i}X_{t,i}(\mathcal{N}_{t,i}\varphi)'' + X''_{t,i}X_{t,i}(\mathcal{N}_{t,i}\varphi)',\\
(\mathrm{III}') &= X_{t,i}\left[X'''_{t,i}\mathcal{N}_{t,i}\varphi + X''_{t,i}(\mathcal{N}_{t,i}\varphi)' + 4X'_{t,i}\cdot(\mathcal{N}_{t,i}\varphi)'' + 2X''_{t,i}\cdot(\mathcal{N}_{t,i}\varphi)' + X_{t,i}\cdot(\mathcal{N}_{t,i}\varphi)'''\right]\\
&\quad + X'_{t,i}\left[X''_{t,i}\mathcal{N}_{t,i}\varphi + 2X'_{t,i}\cdot(\mathcal{N}_{t,i}\varphi)'\right],
\end{aligned}$$

which we bound as follows, taking $\varphi = \mathcal{L}_\alpha^{n-j-1}(\mathbf{1}) \in \mathcal{C}_{*,1}(\alpha)\cap\mathcal{C}$. Using the equations (3.15), (3.17), (3.19), (3.20), (3.86), (3.87), (3.89), there is a $C > 0$ such that

$$\begin{aligned}
|(\mathrm{I}')| &\le C|x|^{t-\alpha-1}(1+|\ln(|x|)|)^2\,[1+\max\{a_1,a_2\}+\max\{b_1,b_2\}|x|],\\
|(\mathrm{II}')| &\le C|x|^{2t-\alpha-1}\,(1+|\ln(|x|)|)^2\,[1+\max\{a_1,a_2\}+\max\{b_1,b_2\}|x|],\\
|(\mathrm{III}')| &\le C|x|^{2t-\alpha-1}\,(1+|\ln(|x|)|)^2\,[1+\max\{a_1,a_2,a_3\}+\max\{b_1,b_2,b_3\}|x|],
\end{aligned}$$

for $x \in S^1\setminus\{0\}$ and $0 < \alpha \le t \le \beta < 1$,

$$|(\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))(x))'| \le C|x|^{t-\alpha-1}(1+|\ln(|x|)|)^2\,[1+\max\{a_1,a_2,a_3\}+\max\{b_1,b_2,b_3\}|x|],$$

which is finite and thus allows us to conclude that $G(x)$ is in fact Lipschitz continuous. Also, $G(0) = 0$. Since it satisfies the assumptions of Theorem 3.3.17, we easily bound B_n as follows

$$\begin{aligned}
\int_\alpha^\beta \frac{\beta-t}{\beta-\alpha}\sum_{j=0}^{n-1}\int_0^1 \psi\mathcal{L}_\beta^j\left(\partial_t^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\right)\,dx\,dt &\le \frac{\|\psi\|_\infty}{\beta-\alpha}\int_\alpha^\beta(\beta-t)\left(C_\beta\sum_{j=0}^{n-1}\frac{1}{j^{1/\beta-\varepsilon}}\right)dt\\
&= \|\psi\|_\infty\cdot(\beta-\alpha)\left(C_\beta\sum_{j=0}^{n-1}\frac{1}{j^{1/\beta-\varepsilon}}\right),
\end{aligned} \tag{3.90}$$

as in equation (3.74), the term in the bracket is summable, and equation (3.90) $\to 0$ as $\beta \to \alpha$. In the same vein, for $\alpha \in (0,1)$, with $\beta < \alpha$, we get the same estimate as before, on substituting

$$\sum_j^{n-1}\mathcal{L}_\beta^j(\mathcal{L}_\beta-\mathcal{L}_\alpha)\mathcal{L}_\alpha^{n-j-1} = -\sum_j^{n-1}\mathcal{L}_\alpha^j(\mathcal{L}_\alpha-\mathcal{L}_\beta)\mathcal{L}_\beta^{n-j-1}$$

into equation (3.81). Hence, the Taylor's formula now is for any $\varphi \in C^2(S^1\setminus\{0\}), \beta \ne \alpha, x \ne 0$

$$\frac{(\mathcal{L}_\alpha-\mathcal{L}_\beta)\varphi(x)}{\alpha-\beta} = \partial_\beta\mathcal{L}_\beta\varphi(x) + \frac{1}{\alpha-\beta}\int_\beta^\alpha(t-\beta)\partial_t^2\mathcal{L}_t\varphi(x)dt \tag{3.91}$$

Case 2: For $\alpha = 0$, the equation (3.81) can also be bounded in a similar manner, using instead, Proposition 3.3.13. The decomposition in the Taylor's formula can be done as thus,

$$\sum_{j}^{n-1} \mathcal{L}_\beta^j(\mathcal{L}_\beta - \mathcal{L}_0)\mathcal{L}_0^{n-j-1} = -\sum_{j}^{n-1} \mathcal{L}_0^j(\mathcal{L}_0 - \mathcal{L}_\beta)\mathcal{L}_\beta^{n-j-1}. \tag{3.92}$$

Substituting $\alpha = 0$ into equation (3.81),

$$\frac{1}{\beta}\left(\int (\psi \circ f_\beta^n)\,dx - \int (\psi \circ f_0^n)\,dx\right) = \sum_{j=0}^{n-1} \int_0^1 \psi \mathcal{L}_\beta^j \left(\frac{(\mathcal{L}_\beta - \mathcal{L}_0)}{\beta}(\mathcal{L}_0^{n-1-j}(\mathbf{1}))\right) dx,$$

and make the substitution from equation (3.92), we have

$$\frac{1}{\beta}\left(\int (\psi \circ f_\beta^n)\,dx - \int (\psi \circ f_0^n)\,dx\right) = -\sum_{j=0}^{n-1} \int_0^1 \psi \mathcal{L}_0^j \left(\frac{(\mathcal{L}_0 - \mathcal{L}_\beta)}{\beta}(\mathcal{L}_\beta^{n-1-j}(\mathbf{1}))\right) dx.$$

From equation (3.91), we have that

$$\begin{aligned}\frac{1}{\beta}\left(\int (\psi \circ f_\beta^n)dx - \int (\psi \circ f_0^n)\,dx\right) = \sum_{j=0}^{n-1} \int_0^1 \psi \mathcal{L}_0^j \Big(\partial_\beta \mathcal{L}_\beta(\mathcal{L}_\beta^{n-1-j}(\mathbf{1}))& \\ + \frac{1}{\beta}\int_0^\beta (t-\beta)\partial_t^2 \mathcal{L}_t(\mathcal{L}_\beta^{n-1-j}(\mathbf{1}))\,dt\Big)\,dx,&\end{aligned}$$

next, using Lemma 3.3.20 and Lemma 3.3.21 as before, we have

$$\begin{aligned}\frac{1}{\beta}\left(\int (\psi \circ f_\beta^n)\,dx - \int (\psi \circ f_0^n)\,dx\right) = \underbrace{-\sum_{j=0}^{n-1} \int_0^1 \psi \mathcal{L}_0^j \Big(\sum_{i\in\{1,d\}} (X_{\beta,i}\mathcal{N}_{\beta,i}\mathcal{L}_\beta^{n-1-j}(\mathbf{1}))'\Big)\,dx}_{A_{n_0}}& \\ \underbrace{+ \int_0^\beta \frac{t-\beta}{\beta} \sum_{j=0}^{n-1} \int_0^1 \psi \mathcal{L}_0^j \Big(\partial_t^2 \mathcal{L}_t(\mathcal{L}_\beta^{n-1-j}(\mathbf{1}))\Big)\,dx\,dt}_{B_{n_0}}.&\end{aligned}$$

The summability of A_{n_0} and B_{n_0} follows from Proposition 3.3.13. This shows that for $\psi \in L^\infty(m)$ and $\beta \in [0,1)$, $\beta \mapsto \int \psi \circ f_\beta^n\,dx$ is locally Lipschitz.

3.4.3 Convergence to the limit

Finally, we show the differentiability of $\beta \mapsto \int \psi d\mu_\beta$, at $\beta = \alpha \in [0,1)$. The idea is to show that as $\beta \to \alpha$, $B_n \to 0$ (resp. $B_{n_0} \to 0$) , and A_n (resp. A_{n_0}) converges to the claimed entity. Recall equation (3.80), with n as in equation (3.79), setting $n(\beta) = n(\alpha,\beta,\xi)$ for small $\xi > 0$. It suffices to check that when $\beta \to \alpha^+ = 0$,

$$\sum_{j=0}^{n(\beta)}\int_0^1 \psi\mathcal{L}_\beta^j\Big(\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}\mathcal{L}_\alpha^{n-j}(\mathbf{1}))'\Big)\,dx$$
$$+\int_\alpha^\beta \frac{\beta-t}{\beta-\alpha}\sum_{j=0}^{n(\beta)-1}\int_0^1 \psi\mathcal{L}_\beta^j\Big(\partial_\alpha^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\Big)\,dx\,dt, \quad (3.93)$$

converges to

$$\sum_{j=0}^{\infty}\int_{S^1}\psi\mathcal{L}_\alpha^j\Big(\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\Big)dx=\int_{S^1}\psi\sum_{j=0}^{\infty}\mathcal{L}_\alpha^j\Big(\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\Big)\,dx, \quad (3.94)$$

the linear response formula. By the summability of B_n (see equation (3.90)), the second term of equation (3.93)

$$\int_\alpha^\beta \frac{\beta-t}{\beta-\alpha}\sum_{j=0}^{n(\beta)-1}\int_0^1 \psi\mathcal{L}_\beta^j\Big(\partial_\alpha^2\mathcal{L}_t(\mathcal{L}_\alpha^{n-1-j}(\mathbf{1}))\Big)\,dx\,dt\to 0, \qquad \text{as } \beta\to\alpha.$$

Next, for $\alpha\in[0,1)$, fix $\eta>0$, we can take $R=R_\eta$ large enough so that the tail of the series in equation (3.94) $<\frac{\eta}{4}$, while the tail of the first term of equation (3.93) $<\frac{\eta}{4}$ uniformly in β. Let $\eta>0$ (small),

$$\sum_{j=R_\eta}^{\infty}\int_0^1 \psi\mathcal{L}_\alpha^j\Big(\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\Big)\,dx<\frac{\eta}{4},$$
$$\sum_{j=R_\eta}^{n(\beta)}\int_0^1 \psi\mathcal{L}_\beta^j\Big(\sum_{i\in\{1,d\}}\Big(X_{\alpha,i}\mathcal{N}_{\alpha,i}\mathcal{L}_\alpha^{n-1-j}(\mathbf{1})\Big)'\Big)\,dx<\frac{\eta}{4},$$

since both series converges (from section 3.4.1 and section 3.4.2). Now for every fixed $0\le j\le R_\eta$, we show that the difference tends to 0, as $\beta\to\alpha^+=0$

$$\sum_{R_\eta}^{\infty}\Big\{\int_0^1\Big[(\psi\circ f_\beta^j)\Big(\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^{n(\beta)-j}(\mathbf{1}))')\Big)-(\psi\circ f_\alpha^j)\Big(\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\Big)\Big]\,dx\Big\}$$
$$=\sum_{R_\eta}^{\infty}\varsigma. \quad (3.95)$$

It is enough now to show that $|\varsigma|<\frac{\eta}{2R_\eta}$ i.e $\forall j\le R_\eta$ and for $\beta=\beta(\eta)\to\alpha$ as $\eta\to 0$. Naturally, as $\eta\to 0$, $\beta(\eta)\to\alpha$. So it is sufficient to show that $\exists\ N_\eta\ge 1$ so that

$$\Big\|\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^n(\mathbf{1})))'-\sum_{i\in\{1,d\}}(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\Big\|_1$$
$$=\Big\|\sum_{i\in\{1,d\}}((X_{\alpha,i}\mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^n(\mathbf{1})))'-(X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))')\Big\|_1\le\frac{\eta}{2R_\eta}, \quad \forall n\ge N_\eta. \quad (3.96)$$

Case 1: At $\alpha=0$, this holds trivially by equation (3.96) and since $h_0=\mathbf{1}$.

Case 2: $\alpha \in (0,1)$, fix α. Now, set

$$\phi_n := \mathcal{L}_\alpha^n(\mathbf{1})$$

By the Leibniz rule, we get

$$\begin{aligned}
&\left\| \sum_{i\in\{1,d\}} \left[(X_{\alpha,i}\mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^n(\mathbf{1})))' - (X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))' \right] \right\|_1 \\
&= \left\| \sum_{i\in\{1,d\}} \left[X'_{\alpha,i}\mathcal{N}_{\alpha,i}(\phi_n) + X_{\alpha,i}(\mathcal{N}_{\alpha,i}(\phi_n))' - X'_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha) - X_{\alpha,i}(\mathcal{N}_{\alpha,i}(h_\alpha))' \right] \right\|_1 \\
&\leq \left\| \sum_{i\in\{1,d\}} \left[X'_{\alpha,i}\mathcal{N}_{\alpha,i}(\phi_n) - X'_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha) \right] \right\|_1 + \left\| \sum_{i\in\{1,d\}} \left[X_{\alpha,i}(\mathcal{N}_{\alpha,i}(\phi_n))' - X_{\alpha,i}\left(\mathcal{N}_{\alpha,i}(h_\alpha)\right)' \right] \right\|_1 \\
&\leq \underbrace{\left\| \max_i X'_{\alpha,i} \sum_{i\in\{1,d\}} \mathcal{N}_{\alpha,i}(\phi_n - h_\alpha) \right\|_1}_{\text{(I)}} + \underbrace{\left\| \sum_{i\in\{1,d\}} \left[X_{\alpha,i}\left(\mathcal{N}_{\alpha,i}(\phi_n - h_\alpha)\right)' \right] \right\|_1}_{\text{(II)}} \qquad (3.97)
\end{aligned}$$

$X'_{\alpha,i} \in L^\infty(m)$, by the Hölder inequality, equation (3.36), and (C2)

$$\begin{aligned}
\text{(I)} &\leq \left\| \max_i X'_{\alpha,i} \right\|_\infty \left\| \sum_{i\in\{1,d\}} \mathcal{N}_{\alpha,i}(\phi_n - h_\alpha) \right\|_1 \\
&\leq \left\| \max_i X'_{\alpha,i} \right\|_\infty \|\mathcal{L}_\alpha(\phi_n - h_\alpha)\|_1 \\
&\leq C_\alpha \|\phi_n - h_\alpha\|_1.
\end{aligned}$$

From Theorem 3.3.14, we have that

$$\text{(I)} \leq C_\alpha n^{1-1/\alpha}, \qquad (3.98)$$

which is only summable for $\alpha < 1/2$. However, it tends to zero for all $\alpha \in (0,1)$.

Remark 3.4.1. *Setting $\psi = \mathbf{1}$ and $\varphi = \mathbf{1} - h_\alpha$, we may as well use Lemma 3.3.11 to bound (I) above.*

Since $X_{\alpha,i} \in L^\infty(m)$ we need only show that (II) $< \dfrac{\eta}{2R_\eta}$. Now, same as before, we let $\varphi \in \{\mathbf{1}, h_\alpha\}$, for any $\bar{x} \in S^1 \setminus \{0\}$, $n \geq 0$, $X_{\alpha,i} \in L^\infty(m)$, by the Hölder inequality

$$\begin{aligned}
\text{(II)} &\leq \left\| \sum_{i\in\{1,d\}} \left[X_{\alpha,i}\left(\mathcal{N}_{\alpha,i}(\phi_n - h_\alpha)\right)' \right] \right\|_1 \\
&= \int_{S^1} \left| \sum_{i\in\{1,d\}} \left[X_{\alpha,i}\left(\mathcal{N}_{\alpha,i}(\phi_n - h_\alpha)\right)' \right] \right| dz
\end{aligned}$$

$$= \underbrace{\int_0^{\bar{x}} \left| \sum_{i\in\{1,d\}} \left[X_{\alpha,i} \left(\mathcal{N}_{\alpha,i}(\phi_n - h_\alpha)\right)' \right] \right| dz}_{\text{(A)}} + \underbrace{\int_{\bar{x}}^{1-\bar{x}'} \left| \sum_{i\in\{1,d\}} \left[X_{\alpha,i} \left(\mathcal{N}_{\alpha,i}(\phi_n - h_\alpha)\right)' \right] \right| dz}_{\text{(B)}}$$
$$+ \underbrace{\int_{\bar{x}'}^{1} \left| \sum_{i\in\{1,d\}} \left[X_{\alpha,i} \left(\mathcal{N}_{\alpha,i}(\phi_n - h_\alpha)\right)' \right] \right| dz}_{\text{(C)}} .$$

We recall the bounds equations (3.15) and the definition of the cone equation (3.26) and in Lemma 3.3.4 with $\varphi = \{h_\alpha, \mathbf{1}\}$, there exists $a_1, b_1, c_2 > 0$ such that

$$|(\mathcal{N}_{\alpha,i}\mathcal{L}^n(\varphi))'(x)| \leq \left(\frac{a_1}{|x|} + b_1\right) \mathcal{N}_{\alpha,i}(\varphi)(x) \leq c_2(a_1|x|^{-(1+\alpha)} + b_1|x|^{-\alpha}), \quad \forall n \geq 1, x \in S^1\backslash\{0\}.$$

$$\begin{aligned}
\text{(A)} &\leq 2 \sum_{i\in\{1,d\}} \int_0^{\bar{x}} \left| X_{\alpha,i}(z) \left(\mathcal{N}_{\alpha,i}\mathcal{L}^n(\varphi)\right)'(z) \right| dz \\
&\leq \tilde{C} \int_0^{\bar{x}} |z|^{\alpha+1}(1 + |\ln(|z|)|)(a_1|z|^{-(1+\alpha)} + b_1|z|^{-\alpha})\, dz, \quad \tilde{C} \geq 1 \\
&\leq \tilde{C} \int_0^{\bar{x}} (1 + |\ln(|z|)|)(a_1 + b_1|z|)\, dz \\
&\leq \tilde{C}\bar{x} \left[a_1 \left(|\ln(|\bar{x}|)| + 2 \right) + \frac{b_1|\bar{x}|}{2} \left(2 + \frac{|\bar{x}|}{2} \right) \right].
\end{aligned} \tag{3.99}$$

The estimate for (C) is similar. We observe that equation (3.99) is small for small $\bar{x}$. To estimate (B), we choose n large enough, so that this is so small, and we bound it. Now, by Hölder inequality and equation (3.36)

$$\begin{aligned}
\text{(B)} \leq & \int_{\bar{x}}^{1-\bar{x}'} \left| \max_i X_{\alpha,i} \sum_{i\in\{1,d\}} \left(\mathcal{N}_{\alpha,i}\mathcal{L}^n(\mathbf{1} - h_\alpha)\right)' \right| dz \\
\leq & \, C_\alpha \int_{\bar{x}}^{1-\bar{x}'} \left| (\mathcal{L}_\alpha^{n+1}(\mathbf{1} - h_\alpha))'(z) \right| dz + C_\alpha \sum_{y\in f_{\alpha,i}^{-1}(x), 2\leq i\leq d-1} \int_{\bar{x}}^{1-\bar{x}'} \left| \frac{(\mathcal{L}_\alpha^{n}(\mathbf{1} - h_\alpha))'(g_{\alpha,i}(x))}{f_\alpha'(g_{\alpha,i}(x))} \right| dz.
\end{aligned} \tag{3.100}$$

The second integral is ignored, of course, if it is a $d = 2$ branch circle map. To estimate the first integral in the above equation, we proceed as follows. For x_0 (resp. x_0') in the neighbourhood of the intermittent fixed point, define $x_0 \in (0, \varepsilon_0]$ and

$$\bar{x} = x_l = g_{\alpha,1}^l(x_0) \quad \text{and} \quad \bar{x}' = x_l' = g_{\alpha,d}^l(x_0'), \quad l \geq 1, \tag{3.101}$$

we define similarly $g_{\alpha,d}^l$ in the neighbourhood around $[-\varepsilon_0, 0)$. This gives rise to the sequence, $(x_l)_l$ (resp. $(x_l')_l$). From [6, Remark 3.63],

$$x_l \approx l^{-1/\alpha} \quad \text{and} \quad x_l' \approx l^{-1/\alpha} \qquad \text{for } l \geq 1. \tag{3.102}$$

Next, for every $1 \leq m \leq n$, and $l \geq 1$, set

$$\begin{aligned} y_m(x_l) &= g^m_{\alpha,1}(x_l) = g^m_{\alpha,1}(g^l_{\alpha,1})(x_0) = g^{l+m}_{\alpha,1}(x_0) = x_{l+m} && \text{for } l \geq 1, \\ y_m(x'_l) &= g^m_{\alpha,d}(x'_l) = g^m_{\alpha,d}(g^l_{\alpha,d})(x'_0) = g^{l+m}_{\alpha,d}(x'_0) = x'_{l+m} && \text{for } l \geq 1. \end{aligned} \tag{3.103}$$

Set

$$\phi = \phi_{n+1-m} - h_\alpha, \tag{3.104}$$

with $\mathcal{L}^n_\alpha(\mathbf{1}) = \phi_n$. Hence,

$$\mathcal{L}^m_\alpha(\phi) = \mathcal{L}^m_\alpha(\mathcal{L}^{n+1-m}_\alpha(\mathbf{1}) - h_\alpha) = \mathcal{L}^{n+1}_\alpha(\mathbf{1} - h_\alpha),$$

and

$$(\mathcal{L}^m_\alpha(\phi))'(x) = \sum_{f^m_\alpha(y)=x} \frac{1}{(f^m_\alpha)'(y)} \cdot \frac{\phi(y)}{(f^m_\alpha)'(y)} \left(\frac{\phi'(y)}{(f^m_\alpha)'(y)} - \phi(y) \frac{(f^m_\alpha)''(y)}{((f^m_\alpha)'(y))^2} \right),$$

taking $-\bar{x} = \bar{x}'$, we see however that

$$\begin{aligned} \int_{\bar{x}}^{1-\bar{x}'} |(\mathcal{L}^{n+1}_\alpha(\mathbf{1} - h_\alpha))'(z)| \, dz &= \int_{S^1} \left| \chi_{|x|>\bar{x}} \left[(\mathcal{L}^{n+1}_\alpha(\mathbf{1} - h_\alpha))'(z) \right] \right| dz \\ &\leq \left\| \chi_{|x|>\bar{x}} \left[(\mathcal{L}^{n+1}_\alpha(\mathbf{1} - h_\alpha))'(z) \right] \right\|_1 \\ &= \left\| \chi_{|x|>\bar{x}} \left[(\mathcal{L}^m_\alpha(\phi))'(z) \right] \right\|_1 \\ &\leq \left\| \mathcal{L}^m_\alpha \left(\chi_{|y|>y_m} \frac{|\phi'|}{(f^m_\alpha)'} \right) \right\|_1 + \left\| \mathcal{L}^m_\alpha \left(\chi_{|y|>y_m} \frac{|\phi| |(f^m_\alpha)''|}{((f^m_\alpha)')^2} \right) \right\|_1 \\ &\leq \underbrace{\left\| \chi_{|y|>y_m} |\phi'| \cdot ((f^m_\alpha)')^{-1} \right\|_1}_{M} + \underbrace{\left\| \chi_{|y|>y_m} \cdot |\phi| \left| (f^m_\alpha)''((f^m_\alpha)')^{-2} \right| \right\|_1}_{N} \end{aligned}$$

$$M \leq ((f^m_\alpha)')^{-1} \| \chi_{|y|>y_m} |\phi'| \|_1$$

To estimate $((f^m_\alpha)')^{-1}$ for some $|y| \geq y_m(x_l)$, we use the bounded distortion property, of f_α [79, Lemma 5] on $(y_m, f_\alpha(y_m)) = (y_m, y_{m-1})$, by equations (3.2), (3.102) and (3.103)

$$\begin{aligned} ((f^m_\alpha)')^{-1} &\leq C \frac{f_\alpha(y_m) - y_m}{f_\alpha(x_l) - x_l} \leq C \frac{\operatorname{sgn}(y_m)|y_m|^{\alpha+1}}{\operatorname{sgn}(x_l)|x_l|^{\alpha+1}} = C \frac{\operatorname{sgn}(x_{l+m})|x_{l+m}|^{\alpha+1}}{\operatorname{sgn}(x_l)|x_l|^{\alpha+1}} \\ &\leq C_\alpha \left(1 + \frac{m}{l} \right)^{-(1+1/\alpha)}. \end{aligned} \tag{3.105}$$

From equations (3.71), (3.103) and (3.104)

$$\left\| \chi_{|y|>y_m} |\phi'| \right\|_1 \leq \left\| \chi_{|y|>y_m} \left(|\phi'_{n+1-m}| + |h'_\alpha| \right) \right\|_1 \leq C_\alpha \int_{y_m(x_l)}^{y_m(x'_l)} c_2 \left(a_1 |x|^{-(1+\alpha)} + b_1 |x|^{-\alpha} \right) dy$$

$$\leq C_\alpha \left(\frac{-a_1 c_2 \text{sgn}(y)}{\alpha} |y|^{-\alpha} + \frac{b_1 c_2 \text{sgn}(y)}{(1-\alpha)} |y|^{(1-\alpha)} \right) \Big|_{y_m(x_l)}^{y_m(x_l')} \leq C_\alpha y_m^{-\alpha} \leq C_\alpha (l+m). \tag{3.106}$$

Hence, we have from equations (3.105) and (3.106) that

$$M \leq C_\alpha \, l \Big(1 + \frac{m}{l} \Big)^{-1/\alpha}. \tag{3.107}$$

From equations (3.98), we have that

$$N \leq C_{\alpha,m} \|\phi_{n+1-m} - h_\alpha\|_1 \leq C_{\alpha,m} (n+1-m)^{1-1/\alpha}, \tag{3.108}$$

where $C_{\alpha,m} = \sup_m \sup_x |(f_\alpha^m)''((f_\alpha^m)')^{-2}|$.

To estimate the second integral in equation (3.100), we use the change of variables, and this gives

$$\int_{g_{\alpha,i}(\bar{x})}^1 X_{\alpha,i}(f_\alpha(v)) \mathcal{L}_\alpha^n(\varphi))'(v) \, dv$$

and then proceed as in the first integral.

We choose $l \geq 1$, such that $\bar{x}$ is small enough to make (A) and (C) (see equation (3.99)) small, $m \geq l$ so that equation (3.107) is small and we choose β close enough to α, so that $n(\beta)$ is large enough to make equation (3.108) small. Which shows what we want for $\psi \in L^\infty(m)$.

3.4.4 Observables in L^q

In this section, we generalize the result to any $\psi \in L^q(m)$, $(1-\alpha)^{-1} < q < \infty$, for $\alpha \in [0, 1)$. Let us define a bounded function

$$\psi_M(x) = \min\{\psi(x), M\},$$

by the Chebyshev's inequality, for $\psi \in L^q(m)$,

$$\text{Leb}(\{\psi(x) > M\}) \leq \frac{\|\psi\|_q^q}{M^q}.$$

Letting $\|\psi\|_q = 1$ in the above equation, we have

$$\text{Leb}(\{\psi(x) > M\}) \leq M^{-q}, \tag{3.109}$$

and $\|\psi - \psi_M\|_r \leq M^{1-q/r}$, for $r \geq 1$. Indeed,

$$\psi_M(x) = \begin{cases} \psi(x), & \text{if } \psi(x) \leq M; \\ M, & \text{if } \psi(x) > M, \end{cases}$$

From equation (3.109), we have that

$$\begin{aligned}\|\psi-\psi_M\|_r &\leq \left(\int_{\psi(x)>M} dx\right)^{1/p}\left(\int_{\psi(x)>M}|\psi-\psi_M|^q\,dx\right)^{1/q}, \qquad \frac{1}{p}+\frac{1}{q}=\frac{1}{r},\\ &\leq (\mathrm{Leb}(\psi(x)>M))^{1/p}\|\psi\|_q\\ &\leq M^{1-q/r}. \end{aligned} \tag{3.110}$$

Decomposing $\psi(x)$ into

$$\psi(x)=(\psi(x)-\psi_M(x))+\psi_M(x), \tag{3.111}$$

we show that the limit is well defined (just as in subsection 3.4.1) for $\psi\in L^q(m)$. We only need show that $\left|\int \psi\mathcal{L}_\alpha^j\left(F_\alpha h_\alpha\right)dx\right|$ is summable.

$$\left|\int \psi\mathcal{L}_\alpha^j\left(F_\alpha h_\alpha\right)dx\right| = \left|\int [(\psi(x)-\psi_M(x))+\psi_M(x)]\mathcal{L}_\alpha^j\left(F_\alpha h_\alpha\right)dx\right|.$$

If $\psi_M(x)=\psi(x)$, the above is trivially true. So, for $\psi_M(x)=M(j):=j^\eta$

$$\left|\int \psi\mathcal{L}_\alpha^j\left(F_\alpha h_\alpha\right)dx\right| \leq \underbrace{\left|\int (\psi(x)-\psi_M(x))\mathcal{L}_\alpha^j(F_\alpha h_\alpha)dx\right|}_{\text{(I)}} + \underbrace{\left|\int \psi_M(x)\mathcal{L}_\alpha^j(F_\alpha h_\alpha)\,dx\right|}_{\text{(II)}},$$

by the Hölder inequality, and from equation (3.110)

$$\text{(I)} \leq \|\psi-\psi_M\|_r\cdot\|F_\alpha\|_{r'} \leq Cj^{\eta(1-q/r)},$$

where $1/r+1/r'=1$ and F_α is bounded.

We simplify (II) as in equation (3.74), using Theorem 3.3.17

$$\begin{aligned}\text{(II)} &\leq j^\eta\int|\mathcal{L}_\alpha^j(F_\alpha h_\alpha)|dx\\ &\leq j^\eta\cdot K_\alpha\frac{1}{j^{(1/\alpha)-\varepsilon}}\\ &\leq K_\alpha\frac{1}{j^{(1/\alpha)-\varepsilon-\eta}}.\end{aligned}$$

(I) and (II) are summable for $\dfrac{1}{(q/r-1)}<\eta<1/\alpha-\varepsilon-1$.

To extend Subsection 3.4.2 to $\psi\in L^q(m)$, we use the same decomposition of ψ and follow a similar calculation as above to show the summability of A_n and B_n. In subsection 3.4.3, we take $M(n)=\ n^\eta$, $\eta\in(0,1/\alpha-1)$. Using the decomposition of ψ from equation (3.111) in equation (3.95), we have that

$$\varsigma = \left\{ \int_{S^1} \left[((\psi - \psi_M) \circ f_\beta^j) \Big(\sum_{i \in \{1,d\}} (X_{\alpha,i} \mathcal{N}_{\alpha,i} (\mathcal{L}_\alpha^{n(\beta)-j}(\mathbf{1}))') \Big) - \Big((\psi - \psi_M) \circ f_\alpha^j \Big) \cdot \Big(\sum_{i \in \{1,d\}} (X_{\alpha,i} \mathcal{N}_{\alpha,i}(h_\alpha))' \Big) \right] dx \right\} + \left\{ \int_{S^1} \left[(\psi_M \circ f_\beta^j) \Big(\sum_{i \in \{1,d\}} (X_{\alpha,i} \mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^{n(\beta)-j}(\mathbf{1}))') \Big) - (\psi_M \circ f_\alpha^j) \Big(\sum_{i \in \{1,d\}} (X_{\alpha,i} \mathcal{N}_{\alpha,i}(h_\alpha))' \Big) \right] dx \right\}$$

$$=: (\mathrm{I}) + (\mathrm{II}).$$

$$(\mathrm{I}) \le M^{(1-q/r)} \left(\|X_{\alpha,i} \mathcal{N}_{\alpha,i}(\mathcal{L}_\alpha^{n(\beta)-j}(\mathbf{1}))'\|_{r'} + \|X_{\alpha,i}\mathcal{N}_{\alpha,i}(h_\alpha))'\|_\infty \right).$$

Hence, choosing $M(n)$ big enough, we can bound the above by $\eta/2R_\eta$. (II) $\to 0$ as $\beta \to \alpha$. Now, we can choose $\beta = \beta(\eta, M(n))$ such that $|(\mathrm{II})| < \eta/2R_\eta$.

3.5 Example

Here, we present an example of a degree 2 circle map that satisfies the conditions (s1)-(s3) in section 3.2. For $0 < \alpha < 1$, Let

$$f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha), & 0 \le x < \frac{1}{2} \\ x - 2^\alpha (1-x)^{(\alpha+1)}, & \frac{1}{2} \le x \le 1. \end{cases}$$

We now proceed to verify the assumptions (A1)-(A3). Indeed, for $\alpha \in (0,1)$ and $x \in I_1$, $\partial_\alpha f_{\alpha,1}(x) = 2^\alpha x^{\alpha+1} \ln(2x)$, and for $x \in I_2$, $\partial_\alpha f_{\alpha,2}(x) = -2^\alpha (1-x)^{\alpha+1} \ln(2(1-x))$. This verifies the assumption (A1). To verify assumption (A2), observe that $I_{1,+} = I_{2,-} = \frac{1}{2}$, $\partial_\alpha f_{\alpha,1}(\frac{1}{2}) = \partial_\alpha f_{\alpha,2}(\frac{1}{2}) = 0$. The assumption (A3) is easily verified.

3.6 Statistical stability for the solenoid map

Consider a solid torus $M = S^1 \times \mathbb{D}$, where $\mathbb{D}$ is the unit disk, $F_\alpha : M \to M$ defined by

$$(x, \mathbf{y}) \mapsto (f_\alpha(x), g_x(\mathbf{y})) := F_\alpha(x, \mathbf{y}) = \left(f_\alpha(x), \frac{1}{2}\cos(2\pi x) + \frac{1}{5}y, \frac{1}{2}\sin(2\pi x) + \frac{1}{5}z \right) \quad (3.112)$$

$f_\alpha : S^1 \to S^1$, the intermittent circle map, with an ergodic SRB measure μ_α.

The solenoid map contained in the solid torus $M \in \mathbb{R}^3$, defined as a skew product between a solid disc and a circle. Geometrically, the transformation stretches the torus to twice its length, shrinks its diameter by a factor of 5, then twists it and doubles it over, placing the object back into the solid torus without self-intersecting. F_α an embedding into itself by means of the a degree $d = 2$ circle map and intersects each disk in two smaller disks of 1/5 the diameter. At each iterate, F_α contracts volume by a factor of 2, yet there is expansion in the x direction albeit non-uniform, since contrary to the classical solenoid attractor, the solenoid with intermittency introduced in [13] has the intermittent map in the base dynamics.

It is easy to see from (3.112) that there exists a semi-conjugacy between f_α and F_α, given by

$$f_\alpha \circ \pi = \pi \circ F_\alpha, \tag{3.113}$$

with $\pi : M \to S^1$, the natural projection. The next result gives the existence of a probability measure on M.

Lemma 3.6.1. *[6, Lemma 4.31] For $0 < \alpha < 1$, there exists an F-invariant Borel probability measure η_α on M such that $\pi_* \eta_\alpha = \mu_\alpha$. Moreover, the support of η_α coincides with Ω.*

Where Ω is a compact attractor. To keep the notations simple, we drop the subscript for μ_α, η_α and simply write μ, η. In what follows, we will show that the measure η in Lemma 3.6.1 is in fact the unique SRB measure of F. Firstly, we shall state results that enables us to establish this claim.

Theorem 3.6.2. *[6, Theorem 3.18] Let $f^R : \Delta_0 \to \Delta_0$ be an induced map for $f : S^1 \to S^1$ and μ_0 an f^R-invariant probability measure. If*

$$\nu = \sum_{j=0}^{\infty} f_*^j(\mu_0 | \{R > j\}),$$

and μ_0 is ergodic, then ν is ergodic.

We remark that taking $\Delta_0 = S^1$, the circle map induced over the S^1 was shown to be a weak Gibbs-Markov induced map, with return times $R = n$ for the first and last branches and $R = 1$ for the middle branches.

In the Gibbs Markov case, we have the following result.

Corollary 3.6.3. *[6, Corollary 3.21] Let $f^R : \Delta_0 \to \Delta_0$ be a Gibbs-Markov map and $\mu_0 \ll m$ its unique f^R-invariant probability measure. If $f_* m \ll m$ and $\nu = \sum_{j=0}^{\infty} f_*^j(\mu_0|\{R > j\})$ and ν is finite, then*

$$\mu = \frac{1}{\sum_{j=0}^{\infty} \mu_0\{R > j\}} \sum_{j=0}^{\infty} f_*^j(\mu_0|\{R > j\})$$

is the unique ergodic f-invariant probability measure such that $\mu \ll m$ and $\mu(\Delta_0) > 0$.

In the next result, take $M = S^1 \times \mathbb{D}$ and F to be the solenoid map with intermittency.

Theorem 3.6.4. *[6, Lemma 4.9] If $F : M \to M$ has a set Λ with a Young structure with integrable recurrence time R, then F has a unique ergodic SRB measure η with $\eta(\Lambda) > 0$. Moreover, the measure η is given by*

$$\eta = \frac{1}{\sum_{j=0}^{\infty} \eta_0\{R > j\}} \sum_{j=0}^{\infty} F_*^j(\eta_0|\{R > j\}),$$

where η_0 is the unique SRB measure for f^R, and $R \in L^1(\eta_0)$.

The next result shows that our claim is true for the induced system.

Lemma 3.6.5. *[6, Lemma 4.5] Let Λ be a set with Young structure and $F : \gamma_0 \cap \Lambda \to \gamma_0 \cap \Lambda$ a quotient of the return map $f^R : \Lambda \to \Lambda$. If η_0 is an SRB measure for f^R, then*

1. *$\mu_0 = (\Theta_{\gamma_0})_* \eta_0$ is the unique F-invariant probability measure such that $\mu_0 \ll m_{\gamma_0}$;*
2. *η_0 is ergodic.*

Proposition 3.6.6. *Suppose that μ is an SRB measure and $\pi_* \eta = \mu$, then η is the unique SRB measure.*

Proof. We know that R is constant on stable disks, hence we have that

$$\sum_{j=0}^{\infty} \eta_0\{R > j\} = \sum_{j=0}^{\infty} \eta_0\{R \circ \pi > j\} = \sum_{j=0}^{\infty} \pi_* \eta_0\{R > j\} = \sum_{j=0}^{\infty} \mu_0\{R > j\}. \tag{3.114}$$

Let $A \subset S^1$ be any Borel measurable set. using (3.114) From Corollary 3.6.3, Theorem 3.6.4, Lemma 3.6.5, and the semi-conjugacy property of the projection map, $\pi : M \to S^1$, we get

$$\begin{aligned}
\pi_* \eta(A) &= \frac{1}{\sum_{j=0}^{\infty} \mu_0\{R > j\}} \sum_{j=0}^{\infty} F_*^j(\eta_0(\pi^{-1}(A))|\{R > j\}) \\
&= \frac{1}{\sum_{j=0}^{\infty} \mu_0\{R > j\}} \sum_{j=0}^{\infty} \eta_0(F^{-j} \circ \pi^{-1}(A)|\{R > j\}) \\
&= \frac{1}{\sum_{j=0}^{\infty} \mu_0\{R > j\}} \sum_{j=0}^{\infty} f_*^j \pi_*(\eta_0(A)|\{R > j\}) \\
&= \frac{1}{\sum_{j=0}^{\infty} \mu_0\{R > j\}} \sum_{j=0}^{\infty} f_*^j(\mu_0(A)|\{R > j\}) = \mu(A).
\end{aligned}$$

□

Since the η is the unique SRB measure of F, we now show that it is statistically stable. For any continuous $\phi : M \to \mathbb{R}$, let $\phi^{+/-} : S^1 \to \mathbb{R}$ defined for each $x \in S^1$ by

$$\phi^-(x) = \inf_{p \in \pi^{-1}(x)} \phi(p) \qquad \text{and} \qquad \phi^+(x) = \sup_{p \in \pi^{-1}(x)} \phi(p).$$

From [6, Lemma 4.31],

$$\int \phi \, d\eta_\alpha = \lim_{k \to \infty} \int (\phi \circ F_\alpha^k)^+ \, d\mu_\alpha = \lim_{k \to \infty} \int (\phi \circ F_\alpha^k)^- \, d\mu_\alpha. \tag{3.115}$$

For ease of notation, we shall write the subscript α_n simply as n and the map and measure at α_0 as F and μ respectively. The density of f_α, $h_\alpha \in L^p(m)$.

Lemma 3.6.7. *For any $k \geq 1$, we have*

$$\lim_{n \to \infty} \int (\phi \circ F_n^k)^+ \, d\mu_n = \int (\phi \circ F^k)^+ \, d\mu$$

Proof.

$$\left|\int(\phi\circ F_n^k)^+\,d\mu_n - \int(\phi\circ F^k)^+\,d\mu\right| \le \underbrace{\left|\int\left((\phi\circ F_n^k)^+ - (\phi\circ F^k)^+\right)\,d\mu_n\right|}_{(A)} + \underbrace{\left|\int(\phi\circ F^k)^+\,d\mu_n - \int(\phi\circ F^k)^+\,d\mu\right|}_{(B)}.$$

By the Hölder inequality, we have that

$$\begin{aligned}(A) &\le \int\left|h_n\left[(\phi\circ F_n^k)^+ - (\phi\circ F^k)^+\right]\right|\,dm\\ &\le \|h_n\|_p\left(\int\left|(\phi\circ F_n^k)^+ - (\phi\circ F^k)^+\right|^q\,dm\right)^{1/q}, \qquad \text{with } \frac{1}{p}+\frac{1}{q}=1.\end{aligned}$$

For a fixed k, since the solenoid maps are continuous, we have that for n big enough, $(A)\to 0$. The linear response result in Theorem A implies statistical stability on S^1. Hence, $(B)\to 0$ as $n\to\infty$. □

Theorem B. $\lim_{n\to\infty}\int\phi\,d\eta_n = \int\phi\,d\eta.$

Proof. Given an arbitrary $\varepsilon>0$, take $\delta>0$ such that $d(p,q)<\delta \Rightarrow |\phi(p)-\phi(q)|<\varepsilon$, for $p,q\in M$. We have that each disk $\pi^{-1}(x)$, is contracted by F_n by a factor of $1/5$. There is a $k_0\ge 1$ such that for all $k\ge k_0$ and $x\in S^1$

$$\operatorname{diam}(F_n^k(\pi^{-1}(x)))<\delta.$$

For all $k\ge k_0$ and $m\ge 0$

$$(\phi\circ F_n^{k+m})^+(x) - (\phi\circ F_n^k)^+(f^m(x)) = \sup(\phi\circ F_n^{k+m}|_{\pi^{-1}(x)}) - \sup(\phi\circ F_n^k|_{\pi^{-1}(f^m(x))}).$$

Since $F_n^{k+m}(\pi^{-1}(x))\subset F_n^k(\pi^{-1}(f^m(x)))$, this then implies that the above equation is bounded by

$$\begin{aligned}&\sup(\phi\circ F_n^{k+m}|_{\pi^{-1}(x)}) - \sup(\phi\circ F_n^k|_{\pi^{-1}(f^m(x))})\\ &\qquad\le \sup(\phi\circ F_n^{k+m}|_{\pi^{-1}(x)}) - \inf(\phi\circ F_n^{k+m}|_{\pi^{-1}(x)})<\varepsilon.\end{aligned}$$

By the invariance of μ_n, we have that $\left(\int(\phi\circ F_n^{k+m})^+(x)\,d\mu_n\right)_{k,n}$ is uniformly Cauchy. Therefore,

$$\lim_{n\to\infty}\lim_{k\to\infty}\int(\phi\circ F_n^k)^+(x)\,d\mu_n = \lim_{k\to\infty}\lim_{n\to\infty}\int(\phi\circ F_n^k)^+(x)\,d\mu_n,$$

and from equation (3.115) and Lemma 3.6.7

$$\lim_{n\to\infty}\int\phi d\eta_n = \lim_{k\to\infty}\int(\phi\circ F^k)^+\,d\mu,$$

using equation (3.115) again, completes the proof. □

Chapter 4

Hölder continuity for piecewise expanding maps

4.1 Introduction

We study the Hölder continuity of the densities and the entropies of a multi-dimensional family of piecewise expanding maps with countable domains of smoothness, under small perturbations.

We focus on *absolutely continuous invariant probability measures* (*ACIP*s), a class of measures that are absolutely continuous with respect to the Lebesgue measure, and often coincide with SRB measures in the context of non-uniformly hyperbolic systems. Specifically, we present general results concerning the Hölder continuity of the densities and metric entropies of ergodic ACIPs for certain classes of piecewise expanding maps in any finite dimension. We achieve this by studying the spectral properties of the perturbed Perron-Frobenius operator associated with this family. More precisely, we employ the abstract result of Keller and Liverani in [58], which provides a powerful framework for establishing continuity properties of dynamical quantities in systems with expanding behavior.

Our primary application of these theoretical results is focused on a particular family of two-dimensional tent maps introduced in [67]. This family is especially interesting because it is related to limit return maps that arise when a homoclinic tangency is unfolded by a family of three-dimensional diffeomorphisms, as discussed in [67, 75]. The existence of ergodic ACIPs for these tent maps was established in [68], and the continuity of the densities of these measures, along with their entropies, was demonstrated in [14, 15]. Building on these foundational results, we now strengthen the previous conclusions by showing that the densities and metric entropies associated with these ACIPs vary Hölder continuously with the dynamics.

4.2 Hölder continuity of the densities

Here we present the general setting under which our main results will be obtained. Let Ω be a compact subset of $\mathbb{R}^d$, for some $d \geq 1$. Consider m the *Lebesgue* measure on Ω and, for each $1 \leq p \leq \infty$, the respective space $L^p(\Omega)$ endowed with its usual norm $\| \quad \|_p$. Absolute continuity will be always meant with respect to m. Let $(\phi_t)_{t\in I}$ be a family of transformations $\phi_t : \Omega \to \Omega$, where I is a metric space. We assume that there exists $N \in \mathbb{N} \cup \{\infty\}$, and for each $t \in I$, there exists an m mod 0 partition $\{R_{t,i}\}_{i=1}^{N-1}$ of Ω such that each $R_{t,i}$ is a closed domain with piecewise C^2 boundary of finite $(d-1)$-dimensional measure. Assume also that each

$$\phi_{t,i} = \phi_t|_{R_{t,i}} \tag{4.1}$$

is a C^2 bijection from $\mathrm{int}(R_{t,i})$, the interior of $R_{t,i}$, onto its image, with a C^2 extension to the boundary of $R_{t,i}$. Consider the *Jacobian* function

$$J_t = |\det(D\phi_t)|,$$

defined on the (full Lebesgue measure) subset of points in Ω where ϕ_t is differentiable. Next we state some properties for our family of maps.

(P1) there exists $\sigma_t > 0$ such that for all $1 \le i < N$ and all $x \in \mathrm{int}(\phi_t(R_{t,i}))$

$$\|D\phi_{t,i}^{-1}(x)\| \le \sigma_t.$$

(P2) there exists $\Delta_t \ge 0$ such that for all $1 \le i < N$ and all $x, y \in \mathrm{int}(R_{t,i})$

$$\log \frac{J_t(x)}{J_t(y)} \le \Delta_t \, \|\phi_t(x) - \phi_t(y)\|.$$

(P3) there exist $\alpha_t, \beta_t > 0$ and, for each $1 \le i < N$, there exists a C^1 unitary vector field $X_{t,i}$ on $\partial\phi_t(R_{t,i})$[1] such that:

- (a) the line segments joining each $x \in \partial\phi_t(R_{t,i})$ to $x + \alpha_t X_{t,i}(x)$ are pairwise disjoint, contained in $\phi_t(R_{t,i})$ and their union is a neighborhood of $\partial\phi_t(R_{t,i})$ in $\phi_t(R_{t,i})$;
- (b) for each $x \in \partial\phi_t(R_{t,i})$ and $v \in T_x\partial\phi_t(R_{t,i}) \setminus \{0\}$, we have $|\sin\angle(v, X_{t,i}(x))| \ge \beta_t$, where $\angle(v, X_{t,i}(x))$ denotes the angle between v and $X_{t,i}(x)$.

(P3) is a geometric condition which allows the extension made on [38] in [4]. This condition is basically that the image of the domains of smoothness $\phi_{t,i}$ should not be too small (sizes uniformly bounded away from zero), and the angles at the border corners should also be bounded from below.

Remark 4.2.1. *For a finite number of domains of smoothness, the constant $\alpha_t > 0$ always exists. The condition (P3)(a) is satisfied in the one dimensional case, when the elements in $\{R_{t,i}\}_{i=1}^{N}$ are intervals whose images have sizes of that are uniformly bounded away from zero. To make sense of the condition (P3)(b) in one dimension, an optimal value of β_t is $\beta_t = 1$ [15, Remark 3.2].*

[1] At the points $x \in \partial\phi_t(R_{t,i})$ where $\partial\phi_t(R_{t,i})$ is not smooth the vector $X_{t,i}(x)$ is a common C^1 extension of $X_{t,i}$ restricted to each $(d-1)$-dimensional smooth component of $\partial\phi_t(R_{t,i})$ having x in its boundary. The tangent space at any such point is the union of the tangent spaces to the $(d-1)$-dimensional smooth components that point belongs to.

Under the conditions (P1)-(P3), it was established in [4, Theorem 5.2] and under similar conditions for a finite domain of smoothness [38] that each ϕ_t possess some ergodic absolutely continuous invariant probability measure. Assuming the uniqueness of this measure for each t, the continuity of the measure in relation to the parameter t was proven in [15] under the following uniformity condition:

(U) there exists $\ell \geq 1$ such that ϕ_t^j satisfies (P1)-(P3) for each $1 \leq j \leq \ell$; moreover, there exist $0 < \theta < 1$ and $M > 0$ such that, for all $t \in I$ and $1 \leq j \leq \ell$,

$$\sigma_{t,\ell}\left(1+\frac{1}{\beta_{t,\ell}}\right) \leq \theta, \quad \sigma_{t,j}\left(1+\frac{1}{\beta_{t,j}}\right) \leq M \quad \text{and} \quad \Delta_{t,j}+\frac{1}{\alpha_{t,j}\beta_{t,j}}+\frac{\Delta_{t,j}}{\beta_{t,j}} \leq M,$$

where $\sigma_{t,j}, \Delta_{t,j}, \alpha_{t,j}, \beta_{t,j}$ are the constants in (P1)-(P3) for the map ϕ_t^j.

We give extra conditions that allows us to establish the Hölder continuous variation of these measures and their entropies. Set for each $s, t \in I$ and $1 \leq i < N$

$$K_{t,s,i} = \phi_{s,i}^{-1}(\phi_t(R_{t,i}) \cap \phi_s(R_{s,i})) \quad \text{and} \quad \psi_{t,s,i} = \phi_{t,i}^{-1} \circ \phi_{s,i}|_{K_{t,s,i}}.$$

Naturally, we consider $\psi_{t,s,i}$ only when $K_{t,s,i} \neq \emptyset$. In fact, assumptions (1)-(3) of Theorem C below essentially mean that the sets $\phi_t(R_{t,i})$ and $\phi_s(R_{s,i})$ are close to each other and the maps $\phi_{s,i}$ and $\phi_{t,i}$ are close to each other. Let id denote the identity map on $\mathbb{R}^d$, possibly restricted to some subset of $\mathbb{R}^d$. Define the difference set

$$A = \{s - t : s, t \in I\}.$$

Given a compact set $K \subset \mathbb{R}^d$ and a function $\psi : K \to \mathbb{R}^d$, let

$$\|\psi\|_0 = \sup_{x \in K} \|\psi(x)\|,$$

where $\| \quad \|$ is for the Euclidean norm in $\mathbb{R}^d$.

Theorem C. *Let $(\phi_t)_{t\in I}$ be a family of maps for which (U) holds and each ϕ_t has a unique ergodic absolutely continuous invariant probability measure μ_t. Assume that there exists a function $\mathcal{E} : A \to \mathbb{R}^+$ such that, for all $s, t \in I$*

1. $\displaystyle\sum_{i=1}^{N} m\left(\phi_{t,i}^{-1}\left(\phi_t(R_{t,i}) \setminus \phi_s(R_{s,i})\right)\right)^{1/d} \leq \mathcal{E}(t-s);$

2. $\displaystyle\sum_{i=1}^{N} \|\psi_{t,s,i} - \mathrm{id}\,\|_0 \leq \mathcal{E}(t-s);$

3. $\displaystyle\sum_{i=1}^{N} \left|\frac{J_s}{J_t \circ \psi_{t,s,i}} - 1\right| \leq \mathcal{E}(t-s).$

Then, there exist $C > 0$ and $0 < \eta < 1$ such that for all $s, t \in I$,

$$\left\|\frac{d\mu_t}{dm} - \frac{d\mu_s}{dm}\right\|_1 \leq C[\mathcal{E}(t-s)]^\eta.$$

The factor $1/d$ in assumption (1) is related to an application of Sobolev and Hölder inequalities in the proof of Proposition 4.2.4. Notice that, in the case that N is finite, we just need the bounds for the summands.

The main ingredient for the proof of the above theorem is the notion of variation for functions in multidimensional spaces (see Section 2.4). Firstly, we obtain a general fact about bounded variation functions that plays a key role in the proof of Proposition 4.2.4.

Lemma 4.2.2. *If K is a compact subset of $\mathbb{R}^d$ and $\psi : K \to \mathbb{R}^d$ is a diffeomorphism onto its image, then there exists $C > 0$ such that, for all $f \in BV(\mathbb{R}^d)$,*

$$\int_K |f \circ \psi - f| \, dm \leq C \|\psi - \mathrm{id}\,\|_0 V(f).$$

Proof. We start by proving the result for a continuous piecewise affine function f. More precisely, suppose that the support Δ of f can be decomposed into a finite number of domains $\Delta_1, \ldots, \Delta_N$ such that the gradient ∇f of f is a constant vector $\nabla_i f$ on each Δ_i. Using (B1), we obtain

$$\int_K |f \circ \psi - f| \, dm \leq \int_K \|\psi - \mathrm{id}\,\|_0 \cdot \|\nabla f\| \, dm \leq \|\psi - \mathrm{id}\,\|_0 V(f).$$

The next step is to deduce the result for any C^1 function f. For this, we take a sequence $(f_n)_n$ of continuous piecewise affine functions such that

$$\|f - f_n\|_0 \to 0 \quad \text{and} \quad \|Df - Df_n\|_0 \to 0, \quad \text{when} \quad n \to \infty$$

(the derivatives Df_n are defined only in the interior of the smoothness domains). Then, using (2.10) and the dominated convergence theorem, we have

$$V(f) = \int \|Df\| \, dm = \lim_{n\to\infty} \int \|Df_n\| \, dm = \lim_{n\to\infty} V(f_n)$$

and

$$\int_K |f \circ \psi - f| \, dm = \lim_{n\to\infty} \int_K |f_n \circ \psi - f_n| \, dm.$$

Using the case already seen, we get the conclusion also for f.

For the general case, we know by (B2) that given $f \in BV(\mathbb{R}^d)$ there is a sequence $(f_n)_n$ of C^1 maps for which

$$\lim_{n\to\infty} \int |f - f_n| \, dm = 0 \quad \text{and} \quad \lim_{n\to\infty} V(f_n) = V(f). \tag{4.2}$$

We have

$$\int_K |f \circ \psi - f| \, dm \leq \int_K |f \circ \psi - f_n \circ \psi| \, dm + \int_K |f_n \circ \psi - f_n| \, dm + \int_K |f_n - f| \, dm.$$

Taking $\rho = 1/|\det D\psi| \circ \psi^{-1}$, we may write

$$\int_K |f_n \circ \psi - f \circ \psi| \, dm = \int_{\psi(K)} |f_n - f| \cdot \rho \, dm \leq \|\rho\|_0 \int |f_n - f| \, dm.$$

The conclusion in this case follows from equation (4.2) and the previous case. □

Next, we study the action of the Perron-Frobenius operators on the space of *bounded variation* functions in Ω. Let $\Omega \subset \mathbb{R}^d$ be the common domain of the maps in the family $(\phi_t)_{t\in I}$. For each $t \in I$, consider the *Perron-Frobenius operator*

$$\mathcal{L}_t : L^1(\Omega) \longrightarrow L^1(\Omega),$$

defined for each $f \in L^1(\Omega)$ by

$$\mathcal{L}_t f = \sum_{i=1}^{N} \frac{f \circ \phi_{t,i}^{-1}}{J_t \circ \phi_{t,i}^{-1}} \chi_{\phi_t(R_{t,i})},$$

where $\{R_{t,i}\}_{i=1}^N$ are the domains of smoothness of $\phi_t : \Omega \to \Omega$ and $\phi_{t,i}$ are the maps introduced in equation (4.1).

Proposition 4.2.3. *Under assumption (U), there exist* $0 < \lambda < 1$ *and* $C > 0$ *such that, for all* $t \in I$, $f \in BV(\Omega)$ *and* $n \geq 1$, *we have*

$$\|\mathcal{L}_t^n f\|_{BV} \leq C\lambda^n \|f\|_{BV} + C\|f\|_1.$$

Proof. Take $\ell \geq 1$ as in (U). It is a standard fact that $\mathcal{L}_t^\ell$ is the Perron-Frobenius operator for ϕ_t^ℓ. By [4, Lemma 5.4], we have for any $f \in BV(\Omega)$

$$V(\mathcal{L}_t^\ell f) \leq \sigma_{t,\ell}\left(1 + \frac{1}{\beta_{t,\ell}}\right) V(f) + M\|f\|_1 \leq \theta V(f) + M\|f\|_1, \tag{4.3}$$

and so

$$\|\mathcal{L}_t^\ell f\|_{BV} \leq \theta V(f) + (M+1)\|f\|_1. \tag{4.4}$$

Given $n \geq 1$, consider $q \geq 0$ and $0 \leq r < \ell$ such that $n = \ell q + r$. It follows from (C2) and equation (4.3) that

$$\begin{aligned} V(\mathcal{L}_t^{\ell q} f) &\leq \theta V(\mathcal{L}_t^{\ell(q-1)} f) + M\|f\|_1 \\ &\leq \theta^2 V(\mathcal{L}_t^{\ell(q-2)} f) + (\theta+1)M\|f\|_1 \\ &\;\vdots \\ &\leq \theta^q V(f) + (\theta^{q-1} + \theta^{q-2} + \cdots + 1)M\|f\|_1. \end{aligned}$$

It follows that

$$\|\mathcal{L}_t^{\ell q} f\|_{BV} = V(\mathcal{L}_t^{\ell q} f) + \|\mathcal{L}_t^{\ell q} f\|_1 \leq \theta^q V(f) + \left(1 + M\sum_{j\geq 0}\theta^j\right)\|f\|_1. \tag{4.5}$$

On the other hand,

$$V(\mathcal{L}_t^r f) \leq \sigma_{t,r}\left(1 + \frac{1}{\beta_{t,r}}\right) V(f) + M\|f\|_1 \leq MV(f) + M\|f\|_1. \tag{4.6}$$

Finally, using equations (4.5) and (4.6), we get

$$\begin{aligned}\|\mathcal{L}_t^n f\|_{BV} &= \|\mathcal{L}_t^{\ell q}\mathcal{L}_t^r f\|_{BV} \\ &= \theta^q V(\mathcal{L}_t^r f) + \Big(1 + M\sum_{j\geq 0}\theta^j\Big)\|f\|_1 \\ &\leq \theta^q M V(f) + \Big(M + 1 + M\sum_{j\geq 0}\theta^j\Big)\|f\|_1.\end{aligned}$$

Now, observe that

$$\theta^q = \theta^{(n-r)/\ell} = \left(\theta^{1/\ell}\right)^n \theta^{-r/\ell} \leq \left(\theta^{1/\ell}\right)^n \theta^{-1}.$$

Take

$$\lambda = \theta^{1/\ell} \quad \text{and} \quad C = \max\left\{\frac{M}{\theta}, M + 1 + M\sum_{j\geq 0}\theta^j\right\}$$

and recall that $V(f) \leq \|f\|_{BV}$. □

It follows from the previous result that $\mathcal{L}_t(BV(\Omega)) \subset BV(\Omega)$. From here on, we assume $\mathcal{L}_t$ as an operator from the space $BV(\Omega)$ into itself. Given a bounded linear operator $T : BV(\Omega) \to BV(\Omega)$, consider

$$|||T||| = \sup_{\{f\in BV(\Omega):\|f\|_{BV}\leq 1\}} \|Tf\|_1.$$

Next, we show that for $s, t \in I$, $\mathcal{L}_t$ is close to $\mathcal{L}_s$ in an appropriate topology.

Proposition 4.2.4. *Under the assumptions of Theorem C, there exists $C > 0$ such that, for all $s, t \in I$,*

$$|||\mathcal{L}_t - \mathcal{L}_s||| \leq C\mathcal{E}(t - s).$$

Proof. We need to show that there exists some constant $C > 0$ such that, for all $f \in BV(\Omega)$, we have

$$\|\mathcal{L}_t f - \mathcal{L}_s f\|_1 \leq C\mathcal{E}(t - s)\|f\|_{BV}.$$

Indeed,

$$\begin{aligned}\|\mathcal{L}_t f - \mathcal{L}_s f\|_1 &\leq \sum_{i=1}^N \int_\Omega \left|\frac{f\circ\phi_{t,i}^{-1}}{J_t\circ\phi_{t,i}^{-1}}\chi_{\phi_t(R_{t,i})} - \frac{f\circ\phi_{s,i}^{-1}}{J_s\circ\phi_{s,i}^{-1}}\chi_{\phi_s(R_{s,i})}\right| dm \\ &= \underbrace{\sum_{i=1}^N \int_{\phi_t(R_{t,i})\cap\phi_s(R_{s,i})} \left|\frac{f\circ\phi_{t,i}^{-1}}{J_t\circ\phi_{t,i}^{-1}} - \frac{f\circ\phi_{s,i}^{-1}}{J_s\circ\phi_{s,i}^{-1}}\right| dm}_{\text{(I)}} + \\ &\quad + \underbrace{\sum_{i=1}^N \int_{\phi_t(R_{t,i})\setminus\phi_s(R_{s,i})} \left|\frac{f\circ\phi_{t,i}^{-1}}{J_t\circ\phi_{t,i}^{-1}}\right| dm}_{\text{(II)}} + \underbrace{\sum_{i=1}^N \int_{\phi_s(R_{s,i})\setminus\phi_t(R_{t,i})} \left|\frac{f\circ\phi_{s,i}^{-1}}{J_s\circ\phi_{s,i}^{-1}}\right| dm}_{\text{(III)}}\end{aligned}$$

We just need to obtain the appropriate bounds for (I), (II) and (III). To estimate (I), note that by the change of variables $y = \phi_{s,i}(x)$, we have

$$\int_{\phi_t(R_{t,i})\cap\phi_s(R_{s,i})} \left| \frac{f\circ\phi_{t,i}^{-1}}{J_t\circ\phi_{t,i}^{-1}} - \frac{f\circ\phi_{s,i}^{-1}}{J_s\circ\phi_{s,i}^{-1}} \right| dm = \int_{\phi_{s,i}^{-1}(\phi_t(R_{t,i})\cap\phi_s(R_{s,i}))} \left| \frac{f\circ\phi_{t,i}^{-1}\circ\phi_{s,i}}{J_t\circ\phi_{t,i}^{-1}\circ\phi_{s,i}} - \frac{f}{J_s} \right| J_s\, dm.$$

Set $\psi_{t,s,i} = \phi_{t,i}^{-1}\circ\phi_{s,i}|_{\phi_{s,i}^{-1}(\phi_t(R_{t,i})\cap\phi_s(R_{s,i}))}$. Therefore,

$$\begin{aligned}
(\mathrm{I}) &= \sum_{i=1}^{N}\int_{K_{t,s,i}} \left| \frac{f\circ\phi_{t,i}^{-1}\circ\phi_{s,i}}{J_t\circ\phi_{t,i}^{-1}\circ\phi_{s,i}} - \frac{f}{J_s} \right| J_s\, dm \\
&\le \sum_{i=1}^{N}\int_{K_{t,s,i}} |f\circ\psi_{t,s,i} - f| \left| \frac{J_s}{J_t\circ\psi_{t,s,i}} \right| dm + \sum_{i=1}^{N}\int_{K_{t,s,i}} \left| \frac{J_s}{J_t\circ\psi_{t,s,i}} - 1 \right| |f|\, dm.
\end{aligned}$$

By assumption (3) of Theorem C, there exists some $C_0 > 0$ such that, for each $1 \le i < N$,

$$\left| \frac{J_s}{J_t\circ\psi_{t,s,i}} \right| \le 1 + \sup_{1\le i<N} \left| \frac{J_s}{J_t\circ\psi_{t,s,i}} - 1 \right| \le 1 + \sum_{i=1}^{N} \left| \frac{J_s}{J_t\circ\psi_{t,s,i}} - 1 \right| \le C_0.$$

By Lemma 4.2.2 and the assumptions of Theorem C, we may write

$$\begin{aligned}
(\mathrm{I}) &\le C_0 \sum_{i=1}^{N} \|\psi_{t,s,i} - \mathrm{id}\,\|_0\, V(f) + \sum_{i=1}^{N} \left| \frac{J_s}{J_t\circ\psi_{t,s,i}} - 1 \right| \|f\|_1 \\
&\le C\mathcal{E}(t-s)\|f\|_{BV},
\end{aligned}$$

for some uniform constant $C > 0$. To estimate (II), note that, by change of variables,

$$\int_{\phi_t(R_{t,i})\setminus\phi_s(R_{s,i})} \left| \frac{f\circ\phi_{t,i}^{-1}}{J_t\circ\phi_{t,i}^{-1}} \right| dm = \int_{\phi_{t,i}^{-1}(\phi_t(R_{t,i})\setminus\phi_s(R_{s,i}))} |f|\, dm = \int_\Omega \chi_{\phi_{t,i}^{-1}(\phi_t(R_{t,i})\setminus\phi_s(R_{s,i}))} |f|\, dm.$$

Observe that, by the Sobolev inequality (B3), we have $f \in L^p(\Omega)$, with $p = d/(d-1)$ and d being conjugate. It follows from Hölder and Sobolev inequalities and assumption (1) of Theorem C that there exists some $C > 0$ such that

$$\begin{aligned}
(\mathrm{II}) &= \sum_{i=1}^{N} \int_\Omega \chi_{\phi_{t,i}^{-1}(\phi_t(R_{t,i})\setminus\phi_s(R_{s,i}))} |f|\, dm \\
&\le \sum_{i=1}^{N} \|\chi_{\phi_{t,i}^{-1}(\phi_t(R_{t,i})\setminus\phi_s(R_{s,i}))}\|_d \|f\|_p \\
&\le C \sum_{i=1}^{N} m\left(\phi_{t,i}^{-1}(\phi_t(R_{t,i})\setminus\phi_s(R_{s,i}))\right)^{1/d} \|f\|_{BV} \\
&\le C\mathcal{E}(t-s)\|f\|_{BV}.
\end{aligned}$$

We are done for (II). The calculations follow similarly for (III). □

The density $\rho_t \in BV(\Omega)$ [15, Corollary 3.4].

4.2.1 Spectral decomposition of $\mathcal{L}_t$

In the situation where 1 is the only eigenvalue of $\mathcal{L}$, with modulus 1. We decompose the Perron-Frobenius operator as follows

$$\mathcal{L}_t = \Pi_t + R$$

where $R : BV(\Omega) \to BV(\Omega)$ is a bounded operator on $BV(\Omega)$ with a spectral radius strictly less than 1, satisfying

$$\Pi_t R = R\,\Pi_t = 0,$$

and Π_t is the projection. In this setting, we have the following result.

Theorem 4.2.5. *[72, Theorem 3] There exists non-negative functions $\varphi_1, \dots, \varphi_\kappa \in BV(\Omega)$ and $\psi_1, \dots \psi_\kappa \in L^\infty(\Omega)$ such that:*

(a) For every $f \in L^1(\Omega)$,

$$\Pi f = \sum_{i=1}^{\kappa} \varphi_i \int_\Omega \psi_i f \, dm. \tag{4.7}$$

(b) $\mathcal{L}_t \varphi_i = \varphi_i$, $\psi_i \circ T = \psi_i$ for $i = 1, \dots, \kappa$.

(c) $\int_\Omega \varphi_i \psi_i \, dm = \delta_{ij}$ $\psi_i \wedge \psi_j = 0 = \psi_j \wedge \psi_i$ as $i \neq j$; $\int_\Omega \varphi_i \, dm = 1$ for $i = 1, \dots, \kappa$.

(d) There exists measurable sets $C_1, \dots C_\kappa \subset X$ such that $\psi_i = \chi_{C_i}$ a.e. for $i = 1, \dots, \kappa$ and $X = \cup_i^\kappa C_i$ a.e.

(e) $\cap_{n=1}^\infty T^n(L^1) = \cap_{n=1}^\infty T^n(L^\infty) = span\{\psi_1, \dots, \psi_\kappa\}$;

(f) for every $f \in L^1(\Omega), f \circ T^n \to \Pi^* f$ in the $\sigma(L^1, BV)$-topology; for every $f \in L^\infty(\Omega)$, $f \circ T^n \to \Pi^* f$ in $\sigma(L^\infty, L^1)$- topology and

$$\Pi^* f = \sum_{i=1}^{\kappa} \psi_i \int_\Omega \varphi_i f \, dm.$$

We remark that m is a d dimensional Lebesgue measure.

4.2.2 Proof of Theorem C

In this section we prove Theorem C. We will use the Keller-Liverani stability result in [58]. Recall that there exists $C > 0$ such that, for all $t, s \in I$,

(KL1) $|||\mathcal{L}_t - \mathcal{L}_s||| \leq \mathcal{E}(t - s)$;

(KL2) $\|\mathcal{L}_t^n\|_1 \leq C$, *for all* $n \geq 1$;

(KL3) $\|\mathcal{L}_t^n f\|_{BV} \leq C\lambda^n \|f\|_{BV} + C\|f\|_1$, *for all* $n \geq 1$ *and* $f \in BV(\Omega)$;

(KL4) 1 *is an isolated eigenvalue of* $\mathcal{L}_t$ *and has multiplicity one.*

Now we state a result by Keller-Liverani

Proposition 4.2.6. *[58, Corollary 1] Suppose that* $\{P_t\}_{t\in I}$ *is a family of bounded operators satisfying (KL1)-(KL4). Fix* $\delta > 0$ *and* $r \in (\alpha, M)$ *and set* $\eta := 1 - \frac{\log r/M}{\log \alpha/M}$. *If in addition,* λ *is an isolated eigenvalue of* P_s *with* $|\lambda| > r$ *and if* $\delta > 0$ *is such that* $B_\delta(\lambda) \cap \sigma_\alpha(P_s) = \{\lambda\}$. *Then there are constants* $C = C(\delta, r) > 0$ *and* $\varepsilon_0 = \varepsilon_0(\delta, r) > 0$ *such that* $0 \leq s, t \leq \varepsilon_0$

$$|||\Pi_s - \Pi_t||| \leq C\,[\mathcal{E}(t-s)]^\eta \quad \textit{for all } s, t \in I.$$

Where $B_\delta(\lambda)$ is the unit open ball of radius δ around λ, $\sigma(P_s)$ the spectrum of P_s and $\sigma_\alpha(P_s) := \{z \in \mathbb{C} : |z| \leq \alpha\} \cup \sigma(P_s)$.

In fact, (KL1) is given by Proposition 4.2.4, (KL2) follows from (C2), and (KL3) is given by Proposition 4.2.3. Regarding (KL4), notice that 1 is an isolated eigenvalue by the theorem of Ionescu Tulcea and Marinescu [53](Theorem 2.4.2), since $\mathcal{L}_t$ is quasicompact; the multiplicity one holds because we assume that each ϕ_t has a unique ergodic absolutely continuous invariant probability measure; recall (C3).

Since (KL1)-(KL4) hold, we conclude using Proposition 4.2.6 that there exist $C > 0$ and $0 < \eta < 1$ such that

$$|||\Pi_s - \Pi_t||| \leq C[\mathcal{E}(t-s)]^\eta, \tag{4.8}$$

where Π_t, Π_s are the projections onto the eigenspaces of the eigenvalue 1 with respect to the operators $\mathcal{L}_t, \mathcal{L}_s$, respectively. By virtue of property (KL4), it follows from Theorem 4.2.5 that, for all $s \in I$, there exists $\rho_s \in BV(\Omega)$ with $\rho_s \geq 0$ and $\int \rho_s\, dm = 1$ such that, for every $f \in L^1(\Omega)$,

$$\Pi_s f = \rho_s \int f\, dm.$$

This allows us to conclude that

$$\|\rho_s - \rho_t\|_1 = \left\|\rho_s - \rho_t \int \rho_s\, dm\right\|_1 = \|\Pi_s\rho_s - \Pi_t\rho_s\|_1 \leq |||\Pi_s - \Pi_t|||.$$

The conclusion of Theorem C then follows from equation (4.8).

4.3 Hölder continuity of the entropies

For some basic notions on entropy, we refer the reader to Section 2.2.

4.3.1 Existence of entropy formula

For the sake of completeness, in this section we present conditions under which the existence of an entropy formula for an absolutely continuous invariant probability of piecewise expanding maps is guaranteed, see [11, 12] for the full branch Markov maps setting. The Markovian property was weakened in [14] to quasi-Markovian property (see [45] for a similar condition), which is defined as follows.

Definition 4.3.1. *The partition* $\{R_{t,i}\}_{i=1}^N$ *into domains of smoothness of* $\phi_t : \Omega \to \Omega$ *is said to be quasi-Markovian with respect to a measure* μ_t, *if there exits* $\eta > 0$ *such that for* μ_t *almost every* $x \in \Omega$ *there are infinitely many values of* $n \in \mathbb{N}$ *for which*

$$m(\phi_t^n(R_t^n(x))) \geq \eta.$$

Where $R_t^n(x)$ is the element in $\{R_{t,i}^n\}_{i=1}^N$ containing $x \in \Omega$. The following important criterion was set in order to establish the quasi-Markovian property for the partition of a piecewise expanding map with absolutely continuous invariant probability measure.

Definition 4.3.2. *The singular set of a piecewise expanding map $\phi_t : \Omega \to \Omega$ is defined as*

$$\mathcal{S}_{\phi_t} = \overline{\bigcup_{R \in \mathcal{R}_t} \partial R}.$$

Where ∂ is the boundary and the bar stands for the closure ($\overline{A}$, is the closure of set A).

Remark 4.3.1. *If $N < \infty$, then the singular set is the finite union of $(d-1)$-dimensional submanifolds of $\mathbb{R}^d$.*

The piecewise expanding maps ϕ_t *behaves as a power of the distance* close to $\mathcal{S}_{\phi_t}$ if there exists constants $K, \rho > 0$ such that

(S1) $\|D\phi_t(x)\| \leq \dfrac{K}{\text{dist}(x, \mathcal{S}_{\phi_t})^\rho}$;

(S2) $\log \dfrac{\|D\phi_t(x)^{-1}\|}{\|D\phi_t(y)^{-1}\|} \leq \dfrac{K}{\text{dist}(x, \mathcal{S}_{\phi_t})^\rho} \text{dist}(x, y)$;

for every $x, y \in M \setminus \mathcal{S}_{\phi_t}$, M a compact manifold, with $\text{dist}(x, y) < \dfrac{\text{dist}(x, \mathcal{S}_{\phi_t})}{2}$.

In [14, Proposition 3.4], an explicit criterion to check the quasi-Markovian property for a C^1-piecewise expanding map was given, and subsequently gave a sufficient condition for establishing the entropy formula for such classes of maps. As a by product, the following result gives a sufficient condition for the existence of an entropy formula for our class of maps of interest (piecewise expanding maps with long branches).

Proposition 4.3.2 ([14])**.** *Let $\phi_t : \Omega \to \Omega$ with $\Omega \subset \mathbb{R}^d$ be a C^1 piecewise expanding map with bounded distortion and large branches for which (S1)-(S2) and (U) hold. If $\log dist(\cdot, \mathcal{S}_{\phi_t}) \in L^d(m)$ and μ_t is an ergodic absolutely continuous invariant probability measure for ϕ_t such that $H_{\mu_t}(\mathcal{R}_{\phi_t}) < \infty$, then*

$$h_{\mu_t}(\phi) = \int \log J_t d\mu_t. \tag{4.9}$$

4.3.2 Hölder continuity of the entropies

Let $\phi_t : \Omega \to \Omega$ be a family of piecewise expanding maps and $\mathcal{P}$ the partition of Ω, it has been established in [11, 12, 14] under different settings that for a finite entropy of the partition $H_{\mu_t}(\mathcal{P})$, the entropy formula for an absolutely continuous invariant probability measure μ_t is,

$$h_{\mu_t}(\phi_t) = \int \log J_t d\mu_t.$$

For each $t \in I$, let $h_{\mu_t}(\phi_t)$ denote the *entropy* of the transformation ϕ_t with respect to the ϕ_t-invariant measure μ_t.

Theorem D. *Let $(\phi_t)_{t\in I}$ be a family of maps for which (U) holds and each ϕ_t has a unique absolutely continuous invariant probability measure μ_t. Assume that*

1. *there exists a function $\mathcal{E} : A \to \mathbb{R}^+$ such that, for all $s, t \in I$,*

$$\|\log J_s - \log J_t\|_d \le \mathcal{E}(t-s) \quad \textit{and} \quad \left\| \frac{d\mu_t}{dm} - \frac{d\mu_s}{dm} \right\|_1 \le \mathcal{E}(t-s);$$

2. *$h_{\mu_t}(\phi_t) = \int_\Omega \log J_t \, dm$, and there is $M > 0$ such that $\| \log J_t\|_\infty \le M$, for all $t \in I$.*

Then, there exists some constant $C > 0$ such that, for all $s, t \in I$,

$$|h_{\mu_t}(\phi_t) - h_{\mu_s}(\phi_s)| \le C\mathcal{E}(t-s).$$

Proof Theorem D. For each $t \in I$, let ρ_t denote the density of μ_t with respect to m. Since the entropy formula in assumption (2) holds, we have for all $s, t \in I$

$$\begin{aligned} |h_{\mu_s}(\phi_s) - h_{\mu_t}(\phi_t)| &= \left| \int \log J_s d\mu_s - \int \log J_t \, d\mu_t \right| \\ &\le \left| \int (\log J_s - \log J_t) \, d\mu_s \right| + \left| \int \log J_t \, d\mu_s - \int \log J_t \, d\mu_t \right| \\ &\le \left| \int (\log J_s - \log J_t) \, \rho_s \, dm \right| + \left| \int \log J_t (\rho_s - \rho_t) \, dm \right|. \end{aligned}$$

Using Hölder inequality and the bound in assumption (2), we get

$$\left| \int \log J_t (\rho_s - \rho_t) \, dm \right| \le M \|\rho_s - \rho_t\|_1. \tag{4.10}$$

On the other hand, it follows from Proposition 4.2.3 that, for all $n \ge 1$,

$$\|\mathcal{L}_s^n f\|_{BV} \le C\lambda^n \|f\|_{BV} + C\|f\|_1,$$

for all $f \in BV(\Omega)$ and $s \in I$. Since ρ_s is a fixed point for $\mathcal{L}_s$ with $\|\rho_s\|_1 = 1$ and the last inequality holds for all $n \ge 1$, we get

$$\|\rho_s\|_{BV} \le C.$$

Taking $p = d/(d-1)$, it follows from Sobolev inequality that there exists a constant $C' > 0$ such that

$$\|\rho_s\|_p \le C'. \tag{4.11}$$

Hence, using Hölder Inequality and equation (4.11) we get

$$\left| \int (\log J_s - \log J_t) \, \rho_s \, dm \right| \le \|\rho_s\|_p \, \|\log J_s - \log J_t\|_d \le C' \, \|\log J_s - \log J_t\|_d \, . \tag{4.12}$$

The conclusion follows from equations (4.10), (4.12) and assumption (1). □

4.4 Application to two dimensional tent maps

The result in this section will be obtained as an application of previous theorems to a family of two-dimensional piecewise expanding maps introduced in [67]. Consider the triangle $\Omega \subset \mathbb{R}^2$, which is the union of the two triangles

$$R_1 = \{(x_1, x_2) : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq x_1\}$$

and

$$R_2 = \{(x_1, x_2) : 1 \leq x_1 \leq 2,\ 0 \leq x_2 \leq 2 - x_1\}.$$

Consider the map $\phi_1 : \Omega \to \Omega$, given by

$$\phi_1(x_1, x_2) = \begin{cases} (x_1 + x_2, x_1 - x_2), & \text{if } (x_1, x_2) \in R_1; \\ (2 - x_1 + x_2, 2 - x_1 - x_2), & \text{if } (x_1, x_2) \in R_2. \end{cases}$$

$\phi_1(x, y)$ is conjugate to the one dimensional tent map $f_a : [-1, 1] \to [-1, 1]$ defined by $f_a(x) = 1 - a|x|$, for $a = 2$ with a the rate of expansion, and hence displays some nice properties such as the consecutive pre-images of the critical set $\{\phi_1^{-n}(\mathcal{C})\}_{n\in\mathbb{N}}$ define a sequence of partitions such that $\operatorname{diam}(\phi_1^{-n}(\mathcal{C})) \to 0$, as $n \to \infty$ of Ω, therefore, ϕ_1 is conjugate to the one sided shift map with two symbols, Hence, it follows that ϕ_1 is transitive in Ω. ϕ_1 posses an absolutely continuous ergodic invariant probability measure [69].

The *family of two dimensional tent maps* $\phi_t : \Omega \to \Omega$ are defined for $0 < t \leq 1$ by

$$\phi_t = t\phi_1. \tag{4.13}$$

Note that R_1 and R_2 are the smoothness domains of ϕ_t, separated by the common straight line segment $\mathcal{C} = \{(x_1, x_2) \in \Omega : x_1 = 1\}$. These tent maps can be described geometrically as follows: first the triangle Ω is folded through $\mathcal{C}$, making R_2 overlap R_1; then a flip of this domain is made and expanded to Ω, thus obtaining $\phi_1(\Omega)$; for the other maps ϕ_t, we apply a final contraction by the factor t.

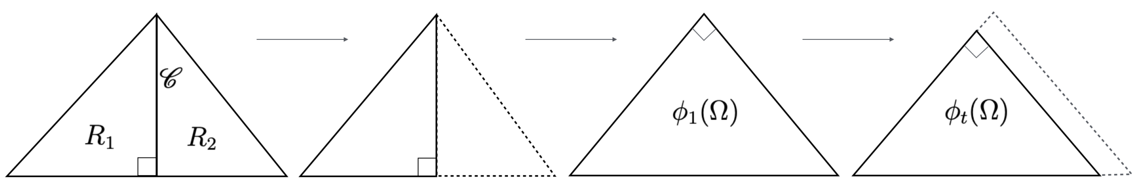

Fig. 4.1 The tent maps

It was proved in [68] that, for each $t \in [\tau, 1]$, with

$$\tau = \frac{1}{\sqrt{2}}(\sqrt{2} + 1)^{1/4}, \tag{4.14}$$

the map ϕ_t exhibits a *strange attractor* in Ω, thereby extending the results obtained in [69] only for $t = 1$. The existence and uniqueness of an ergodic absolutely continuous ϕ_t-invariant probability measure μ_t was obtained in [68], for each $t \in [\tau, 1]$. Moreover, any exponent of ϕ_t also possess an ACIEP which coincides with μ_t. In the next result we improve the conclusions of [14, 15], on the continuity of the measures and respective entropies for this family of maps, by showing that they in fact vary Hölder continuously.

Theorem E. *There exists $C > 0$ and $0 < \eta < 1$ such that, for all $s, t \in [\tau, 1]$,*

$$\left\| \frac{d\mu_t}{dm} - \frac{d\mu_s}{dm} \right\|_1 \leq C|t-s|^\eta \quad \text{and} \quad |h_{\mu_t}(\phi_t) - h_{\mu_s}(\phi_s)| \leq C|t-s|^\eta.$$

According to [15, Section 4], the uniformity condition (U) is satisfied with $\ell = 6$. For the sake of completeness, We shall show it below

$$(\phi_t^\ell)_{t\in[\tau,1]}.$$

For ϕ_t^6, we have 64 domain of smoothness, hence $\{R_{t,i}\}_{i=1}^{64}$, the $(m \bmod 0)$ partition of Ω.

$$\|(D\phi_t^6)^{-1}(x)\| = \frac{1}{8t^6} := \sigma_{t,6}.$$

For each $t \in [\tau, 1]$, $\sigma_{t,6} < 1$, hence (P1) is satisfied. (P2) is satisfied with $\Delta_{t,6} = 0$ on each $\{R_{t,i}\}_{i=1}^{64}$, since we have that ϕ_t^6 is linear. the linearity of ϕ_t^6 on each t in $R_{t,i}$ and preserves angles, this will allow to show the (P3) condition for the domains of smoothness instead on their images. Since, the boundary of each domain of smoothness is formed by at most five straight line segments having slope $-1, 0, 1$ and ∞, meeting at an angle at least $\frac{\pi}{4}$. Then there is a piecewise C^1 unitary vector field $X_{t,i}$ in $\partial R_{t,i}$ such that

$$\beta_{t,6} =: \sin\frac{\pi}{8} \leq |\sin\angle(v, X_{t,i}(x))|$$

for every $x \in \partial R_{t,i}$ and $v \in T_x\partial R_{t,i} \setminus \{0\}$. Next, since the $\{R_{t,i}\}_{i=1}^{64}$ depend continuously on t, so it is possible to choose a uniform α such that (P3) holds for each $t \in [\tau, 1]$. Hence, we have that for each $t \in I$

$$\sigma_{t,\ell}\left(1 + \frac{1}{\beta_{t,\ell}}\right) = \frac{1}{8t^6}\left(1 + \frac{1}{\sin\pi/8}\right) \leq \theta$$

$$\Delta_{t,\ell} + \frac{1}{\alpha_{t,\ell}\beta_{t,\ell}} + \frac{\Delta_{t,\ell}}{\beta_{t,\ell}} = \frac{1}{\alpha_{t,6}\sin(\pi/8)} \leq M.$$

Now that we have established the conditions (P1)-(P3) and (U), we are in the setting to prove Theorem E. We achieve this by applying Theorem C and Theorem D to the family of tent maps $(\phi_t)_{t\in[\tau,1]}$ presented above. We know that R_1 and R_2 are the only domains of smoothness of every ϕ_t. Therefore, for each $t \in [\tau, 1]$ and $i = 1, 2$, we have

$$\phi_{t,i} = \phi_t|_{R_i}, \quad K_{t,s,i} = \phi_{s,i}^{-1}(\phi_t(R_i) \cap \phi_s(R_i)) \quad \text{and} \quad \psi_{t,s,i} = \phi_{t,i}^{-1} \circ \phi_{s,i}|_{K_{t,s,i}}.$$

Existence and uniqueness of an ergodic absolutely continuous ϕ_t-invariant probability measure μ_t was obtained in [68], for all $t \in [\tau, 1]$. Moreover, the entropy formula holds for this family of maps, by [14, Theorem G]. We are left to verify the assumptions of Theorem C and Theorem D with adequate estimates to deduce Theorem E. It is enough to show that there exists some constant $M > 0$ such that, for all $s, t \in [\tau, 1]$ and $i = 1, 2$, we have

(a) $m\left(\phi_{t,i}^{-1}\left(\phi_t(R_i) \setminus \phi_s(R_i)\right)\right) \leq M|t-s|$;

(b) $\|\psi_{t,s,i} - \mathrm{id}\,\|_0 \leq M|t-s|$;

(c) $\left|\dfrac{J_s}{J_t \circ \psi_{t,s,i}} - 1\right| \leq M|t-s|$;

(d) $\|\log J_s - \log J_t\|_d \leq M|t-s|$;

(e) $\|\log J_t\|_\infty \leq M$.

Indeed, from equation (4.13), we easily deduce that, for all $(y_1, y_2) \in \phi_{t,1}(R_1)$, we have

$$\phi_{t,1}^{-1}(y_1, y_2) = \left(\frac{1}{2t}(y_1 + y_2), \frac{1}{2t}(y_1 - y_2)\right)$$

and, for all $(y_1, y_2) \in \phi_{t,2}(R_2)$, we have

$$\phi_{t,2}^{-1}(y_1, y_2) = \left(\frac{1}{2t}(4t - y_1 - y_2), \frac{1}{2t}(y_1 - y_2)\right).$$

Moreover, each map ϕ_t is piecewise linear with

$$D\phi_t(x_1, x_2) = \begin{pmatrix} t & t \\ t & -t \end{pmatrix}$$

for all $(x_1, x_2) \in R_1 \setminus \mathcal{C}$, and

$$D\phi_t(x_1, x_2) = \begin{pmatrix} -t & t \\ -t & -t \end{pmatrix}$$

for all $(x_1, x_2) \in R_2 \setminus \mathcal{C}$. Therefore, we have

$$J_t = 2t^2,$$

for all $(x_1, x_2) \in \Omega \setminus \mathcal{C}$ and $\tau \leq t \leq 1$.

Proof of (a) Observe from the dynamics of ϕ_t, that $\phi_{t,1}(R_1) = \phi_{t,2}(R_2)$ and, moreover the Jacobian of $\phi_{t,1}$ is constant and equal to the Jacobian of $\phi_{t,2}$. Therefore, it is enough to show the conclusion for $i = 1$. In fact, for $t > s$ (and for $t < s$ there is nothing to be proved, since in that case $\phi_t(R_1) \subset \phi_s(R_1)$), we have

$$m(\phi_t(R_1) \setminus \phi_s(R_1)) \leq \mathrm{lenght}(\phi_t(\mathcal{C}))\, \|\phi_t(1,0) - \phi_s(1,0)\|$$

$$= \sqrt{2}t\|(t,t)-(s,s)\| = 2t(t-s). \tag{4.15}$$

Since the Jacobian of $\phi_{t,1}$ is constant and equal to $2t^2$, we deduce that the Jacobian of $\phi_{t,1}^{-1}$ is $1/(2t^2)$, which together with equation (4.15) yields

$$m\left(\phi_{t,i}^{-1}\left(\phi_t(R_i)\setminus\phi_s(R_i)\right)\right) \leq \frac{(t-s)}{t} \leq \frac{(t-s)}{\tau}.$$

Proof of (b) For each $(x_1,x_2)\in R_1$ and $\tau\leq t\leq 1$, we have

$$\begin{aligned}\|\psi_{t,s,i}-\mathrm{id}\,\|_0 &= \sup_{(x_1,x_2)\in R_1}\|\phi_{t,1}^{-1}\circ\phi_{s,1}(x_1,x_2)-(x_1,x_2)\| \\ &= \sup_{(x_1,x_2)\in R_1}\left\|\left(\frac{s}{t}-1\right)(x_1,x_2)\right\| \\ &\leq \frac{\sqrt{2}}{\tau}|t-s|,\end{aligned}$$

and for each $(x_1,x_2)\in R_2$, we have

$$\begin{aligned}\|\psi_{t,s,i}-\mathrm{id}\,\|_0 &= \sup_{(x_1,x_2)\in R_2}\|\phi_{t,2}^{-1}\circ\phi_{s,2}(x_1,x_2)-(x_1,x_2)\| \\ &= \sup_{(x_1,x_2)\in R_2}\left\|\left(\frac{s}{t}-1\right)(x_1-2,x_2)\right\| \\ &\leq \frac{\sqrt{2}}{\tau}|t-s|.\end{aligned}$$

Proof of (c) For all $\tau\leq s,t\leq 1$, we have

$$\left|\frac{J_s}{J_t\circ\psi_{t,s,i}}-1\right| = \left|\frac{s^2}{t^2}-1\right| = \left|\frac{(s-t)(s+t)}{t^2}\right| \leq \frac{2}{\tau^2}|t-s|.$$

Proof of (d) By the mean value theorem, we have that for all $\tau\leq s,t\leq 1$

$$\|\log J_s-\log J_t\|_2 = \left(\int_\Omega\left(\frac{2}{\tau}(s-t)\right)^2 dm\right)^{1/2} \leq \frac{2}{\tau}m(\Omega)|t-s|.$$

Proof of (e) For all $\tau\leq t\leq 1$, we have

$$\|\log J_t\|_\infty = |\log(2t^2)| \leq \log 2.$$

Recall that the expression for τ in (4.14) gives $1<2\tau^2\leq 2t^2\leq 2$.

Chapter 5

Conclusion

The regularity of physical measures and their metric entropies play a fundamental role in understanding the stability of dynamical systems, particularly those exhibiting complex or chaotic behavior. By addressing their regularity, we aim to provide deeper insights into the intricate relationship between the dynamics of a system and the statistical properties of its invariant measures. Understanding this relationship is key to predicting the robustness of complex dynamical systems under perturbations, a question that has far-reaching implications in various fields, from mathematical theory to applied sciences. This research sheds light on the subtle and quantitative ways in which physical measures and their metric entropy responds to changes in the underlying system, offering a more nuanced understanding of the stability and variability of chaotic systems. Through this investigation, we contribute to the broader goal of characterizing the resilience of dynamical systems to fluctuations, and thus advancing the overall theory of dynamical stability.

We now present some questions that would be interesting to study in the future. In [34] the problem of linear response was considered for solenoidal attractors where the base dynamics is a uniformly expanding map. An interesting problem would be to study linear response in the solenoid map where in the base dynamics a nonuniformly expanding map is considered, particularly the circle map with intermittency. An ultimate goal would be to extend the result to partially hyperbolic attractors.

In the case of the one dimensional tent maps studied by Baladi and Smania (see [25, 27]), the linear response fails under certain transversality condition of the topological class. A recent result by Bahsoun and Galatolo shows that linear response holds if the critical point in the one dimensional tent map is replaced by a singularity (see [20]). As a next step, an interesting problem to tackle would be to check whether changing the dimension of the system affects the existence of a linear response formula. In particular, it would be interesting to investigate whether linear response holds within the higher dimensional family of tent maps that was considered in Chapter 4.

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