Chapter 10
An Introduction to Solid Tessellations with Students of Architecture

João Pedro Xavier, José Pedro Sousa, Alexandra Castro, and Vera Viana

Abstract This paper intends to describe an educational experiment accomplished in the *Geometry and Architecture* course of the first year in the Faculty of Architecture of the University of Porto in 2017. In this activity, students were introduced to digital three-dimensional modelling as an additional tool to develop their knowledge of geometry. The subject of solid tessellations was selected as leitmotif because of the structural and architectonic interest and creative potential that the situations in which polyhedra, other than the cuboid, fill space may have for aspiring architects. The time limitations of the academic year impaired the desired breadth for the task, so students had to focus their attention only in six uniform solid tessellations, out of the possible 28. Besides acquiring digital design modelling skills useful for their scholarly and professional practice, this experiment and collaborative assignment allowed students to improve knowledge of polyhedral theory and apply newfound IT skills in architectural design.

Introduction

For several years, the syllabus *Geometry and Architecture* of the first year in the Faculty of Architecture of the University of Porto (FAUP) was entirely concerned with projective geometry and such traditional systems of representation as cartography, topography, orthographic projections, axonometry, and perspective. In FAUP, digital design as a supporting tool for the study of representation and architectural design is usually introduced only in the third year. Recognizing the limitations of this methodological option and, above all, its divergence with contemporary reality and the most updated architectural practices, the professors of the syllabus decided, in 2015, to introduce the computer in the classroom as a tool for students to learn geometry and actively explore digital three-dimensional modelling. Since then, the

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program of the course has clearly been divided into two parts: the first, corresponding to ca. 2/3 of the total lecturing time, in which traditional representation systems are approached through individual assignments and hand-drawing is taken as privileged learning and investigation tool; in the second, a collaborative assignment is proposed, and the exploration of three-dimensional modelling with computer-aided design (CAD) stimulated.

The investigation of certain concepts in geometry usually takes some time to accomplish within traditional representational systems, while digital modelling allows students to address a broader range of themes and enhance their abilities for spatial visualization, mental rotation, and geometric reasoning, since it provides an accurate representation of geometric objects easily navigable within viewports and representational systems. The geometrical concepts and transformations are then taken as the fundamental concern of their inquiry, rather than the technical procedures of the chosen representational system.

An introduction to digital 3D modelling in the *Geometry and Architecture* syllabus is scheduled for a teamwork activity to be developed in 5–6 weeks so students can explore CAD processes to study and research certain geometric subjects that, from the professors' perspective, may have a sturdy impact on architectural design. Each year, a different topic is proposed so that students may be introduced into themes generally unexplored and whose complexity clearly justifies the use of digital media. So far, the topics for the students' investigations were anamorphosis, in 2015–2016 and 2020–2021, solid tessellations in 2016–2017, surfaces in 2017–2018, and vaults, in 2019–2020.

This paper presents the assignment accomplished by the students on the topic of solid tessellations. A brief insight on the theoretical framework that supported the investigation developed and the goals and methodology proposed for the task will be addressed. In the end, a selection of the students' final works will be presented from a critical standpoint, acknowledging the creative potential of didactic experiments that aim to bridge polyhedral theory and architectural design.

Polyhedra and Solid Tessellations

Polyhedra is a stimulating subject to introduce in any educational context, as Pedersen denotes [1, p. 133] when referring to how students react when instructed to make their models in paper; they usually do not question why they should make them, quite the contrary, they become interested in their characteristics and show a keen interest in modelling other examples. Studying polyhedra may also be a gateway to enhance the students' geometrical knowledge, not only because of the myriad of mathematical contents and branches into which students are introduced [1, p. 135] but also because, by modelling and manipulating geometry in such a tangible form, students seem to recognize geometric concepts and operations more efficiently, becoming capable of developing their geometric reasoning to a different level.[1] Moreover, studying and virtually modelling different classes of polyhedra within an architecture program help students to widen the repertoire of possibilities to conceive and design space beyond the rigidity of cuboidal forms [2, 3].

To a more in-depth approach of the subject from a theoretical perspective, an expert on polyhedral geometry, who co-authored this study, collaborated with the team of professors to establish a content basis that came to be fundamental for the research done by the students.

The subject was first introduced from a theoretical standpoint,[2] through which students widened their knowledge on polyhedra and were given the chance to manipulate physical models of less common polyhedra and learn how to model some of them virtually. Students then explored geometric transformations and symmetry operations in a digital framework, such as translation, rotation, and reflection, to better understand important notions, such as the duality of polyhedra and the spherical symmetry groups. The words of Coxeter [5, p. 68] were explained to illustrate how, in solid tessellations or honeycombs, polyhedra fit together to fill space just once and every face of each polyhedron belongs to one other polyhedron, as well as the conditions necessary for polyhedra to do so (Fig. 10.1). To better understand these notions, the students modelled the six convex parallelohedra that

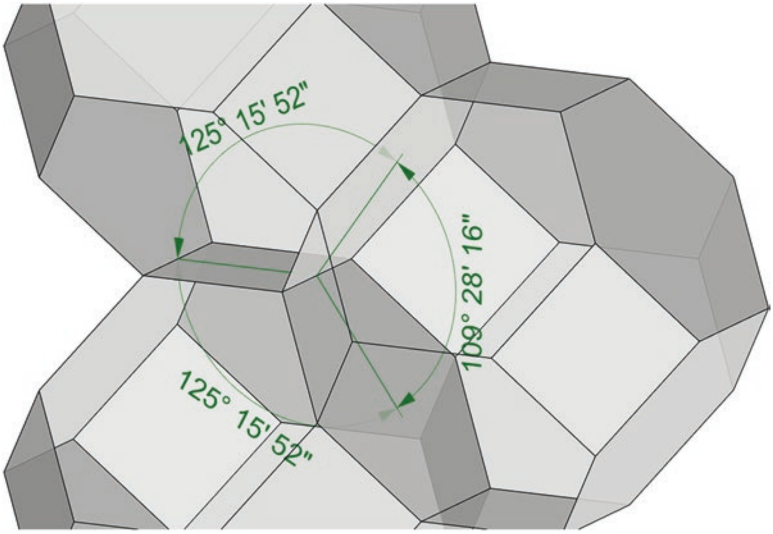

Fig. 10.1 Three truncated octahedra with a common edge fill space because the sum of the dihedral angles around that edge equals 360°

[1] The authors recognize that the extent through which the students' geometric reasoning is enhanced within a digitally driven educational approach is yet to be fully understood or certified, but this experiment aims to introduce a modest contribution to the discussion and, at the same time, to stimulate the exploration of polyhedral theory in other higher education settings.

[2] A report on the educational resources explored and the students' opinion on the theoretical segment of the classes accomplished for this didactical experiment has been included in [4].

fill space by translation of their replicas [6], besides other examples of convex and concave polyhedra that infinitely fill space.

In subsequent classes, the notion of uniformity in a tessellation, through which every vertex, equally surrounded, is superimposable under symmetries onto any other, was explained. All 28 convex uniform tessellations [7] were illustrated and some of them virtually modelled by the students. Special attention was given to the 13 honeycombs and their duals, respectively, categorized by Conway, Burgiel, and Goodman-Strauss [8, p. 292–298] as architectonic and catoptric tessellations because of the symmetry properties of their cells.

The Assignment

Out of the 28 convex uniform tessellations, six of the architectonic (Fig. 10.2) were proposed by the team of professors as the leitmotif of the collaborative assignment, not only for the sake of concision and feasibility but also because of their potential interest for architectural design. The idea was for students to overcome the characteristic abstraction of the underlying geometrical structure in each solid tessellation and regard them from an architectural—spatial – standpoint, exploring their intrinsic spatiality. With the intention of adopting computer modelling as a driving force for the students and preparing the presentation of their response to the assignment, Rhinoceros®,[3] one of the most efficient software used in architectural practice and powerful research tool, was selected due to its versatility and easiness to approach geometric modelling.

The time scheduled for the exercise was 6 weeks, and six classes, with an average of 24 students each, organized in groups of 4 or 5 elements, were involved. Each group had to explore one of the six tessellations, with the purpose of creating a spatial structure to be placed at the Garden of Quinta da Póvoa, one of the most interesting exterior spaces in the faculty's campus (Fig. 10.3).

The work was developed following a methodology subdivided into steps: (1) presentation of the theme and theoretical framework; (2) a brief introduction to the software and the commands that would be relevant for the assignment; (3) analysis of the geometric concepts involved based on preliminary modelling experiments; (4) design and development of the project itself; and (5) presentation of the final projects. Each group of students had to manipulate virtual models of the polyhedra involved in their tessellation and find out how those could fill space uniformly. One of the proposals for the task was for students to consider the geometric structure under analysis as a possibility of conceiving a small inhabitable structure but slightly deviating from its rigid configuration. After the theoretical introduction, the work in the CAD environment was developed according to the following phases:

[3] *Rhinoceros®* is a 3D computer graphics and computer-aided design application software developed by Robert McNeel & Associates (https://www.rhino3d.com/).

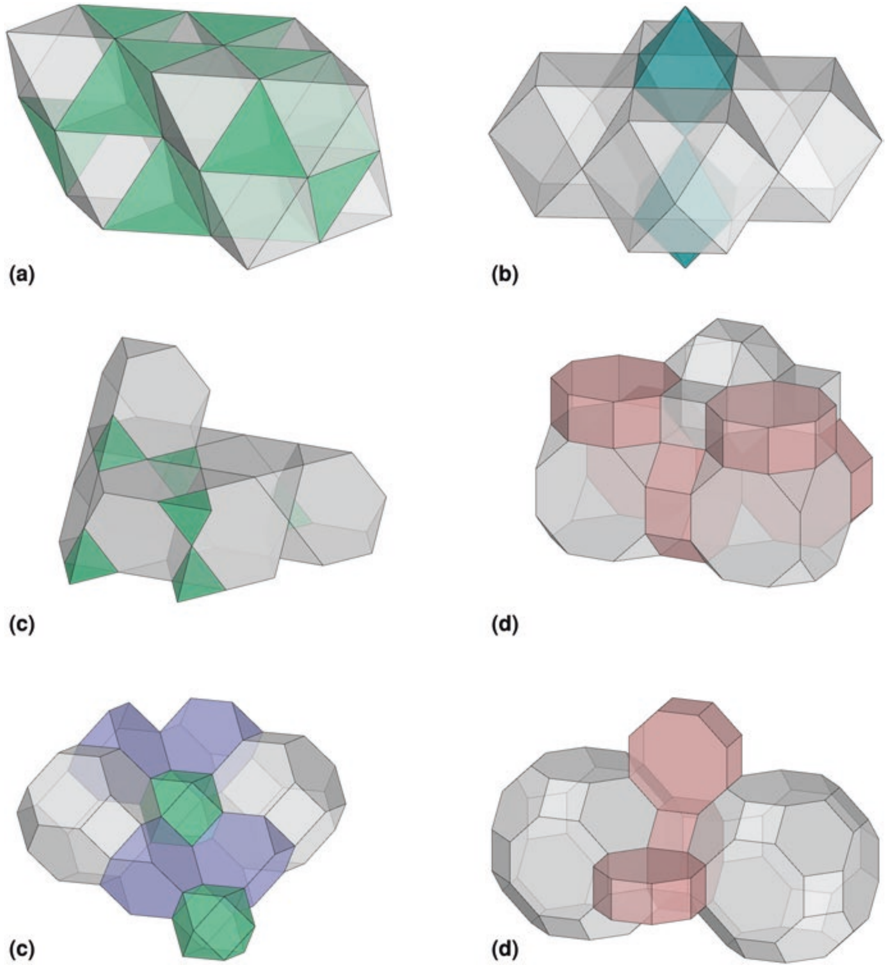

Fig. 10.2 The six architectonic tessellations selected for the teamwork. The designations of polyhedra per vertex according to Wenninger [8]. (**a**) Tetroctahedrille (8 Tetrahedra and 6 Octahedra). (**b**) Cuboctahedrille (2 Octahedra and 4 Cuboctahedra). (**c**) Trunctetrahedrille (2 Tetrahedra and 6 Truncated Tetrahedra). (**d**) 1-RCO-Hedrille (1 Rhombicuboctahedron, 1 Truncated Cube, 1 Cube, and 2 Octagonal Prisms). (**e**) Truncated Tetroctahedrille (1 Cuboctahedron, 2 Truncated Tetrahedra, and 2 Truncated Octahedra). (**f**) b-tCO-Hedrillle (2 Octagonal Prisms and 2 Rhombitruncated Cuboctahedra)

Phase 1: Modelling the Tessellation

The initial approach began with students modelling each polyhedron in the tessellation, after which the possibilities of multiplication through translation and rotations were analyzed, exploring different operations and geometric transformations. This

Fig. 10.3 Site Plan of "Casa Cor-de-Rosa, Cavalariças and Pavilhão Carlos Ramos" [10] (left) and Faculty of Architecture – University of Porto, Casa Cor-de-Rosa/Álvaro Siza [11] (right)

allowed students to get to know in further detail the tessellation assigned to them and begin to recognize its intrinsic spatial characteristics.

Phase 2: Incorporating the Polyhedral Composition within the garden's Context

Students chose a specific area of the garden and began thinking about adapting the structure to the location by refining its overall scale, dimension, and orientation. In this stage, architectonic topics, such as the configuration of the structure, its spatial features, and how it would physically interact with the site itself, began to be considered.

Phase 3: Definition of the Structure's Materiality

While combining polyhedra to achieve the desired shape, students were proposed to include a cutting plane, meant as a disruptive yet creative element in the overall structure. The plane would trim the form either at its base, to facilitate its integration in the garden's ground or elsewhere. Once the spatial structure was outlined, students had to conceive its materialization, attending to the following requirements: (1) having structural bars in every edge of each polyhedron while some of its faces are left open; (2) enclosing the space within some cells in the structure with the faces as panels.

Phase 4: Presentation of the Project

In the last phase of their project, the students presented a poster based on a template provided for the assignment, where they inserted: CAD-produced images aimed to display; the architectonic tessellation assigned for the group; the geometric composition conceived in a wireframe and surface appearance; and the overall result shown from complementary points of view. The poster had to include a conceptual hand-sketch, a plan and an elevation, and a photomontage of the inhabitable structure inserted in the garden's context. In addition to the poster, students produced a small video showing the architectonic composition both from the outside and the inside or, alternatively, a cardboard model.

A Selection of the Projects Developed by the Students

The results were diverse, given that, for this assignment, the six classes involved conceived a total of 33 geometric structures from six different uniform tessellations. The variety in the solutions found by the students for the inhabitable structures and the interest they have shown throughout the whole project were a clear testimony of the creative potential that polyhedra and solid tessellations meant for the aspiring undergraduate architects involved. We will analyze in more detail three of their works[4] conceived from the same tessellation, the cuboctahedrille (Fig. 10.2b). This tessellation turned out to be the one that led to the more interesting results, and the reasons for this might be the fact that only two different polyhedra were involved or its clear connection to the tetraoctahedral spaceframe and the face-centered cubic lattice. Nonetheless, each group conducted their studies independently and conceived very different results, both in the spatial concept and the architectonic image of the geometric structure devised.

Project a: Stopping Point

In this project (Fig. 10.4), the group conceived a small pavilion, very compact in its overall form. Placed under a big tree, the pavilion is strongly marked by its closed panels and the entrances opposed to each other. One interesting detail of the project is the configuration of the cells sectioned by the cutting plane proposed in the assignment. Segmenting the overall form was a decisive step in the design process that generated an interesting irregularity for the structure, which included three small openings on the upper part of the volume so that natural light might penetrate its interior.

[4] Other examples of the students' work besides Projects A, B and C, are shown in [9].

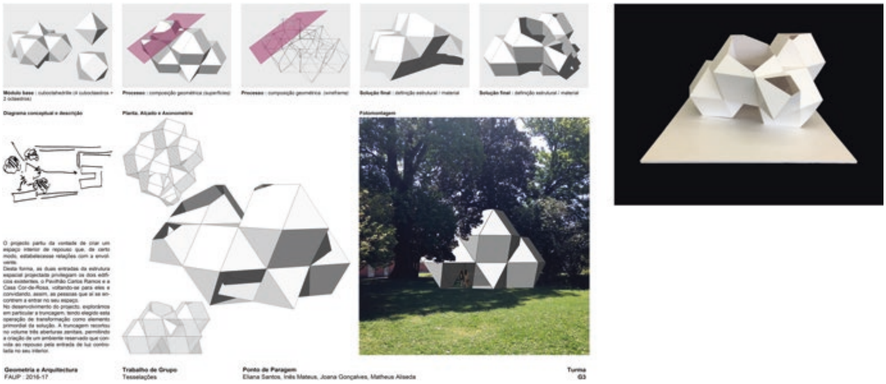

Fig. 10.4 Project A: "Stopping point" by Eliana Santos, Inês Mateus, Joana Gonçalves and Matheus Aliseda, 2016/2017 (Presentation poster and cardboard model)

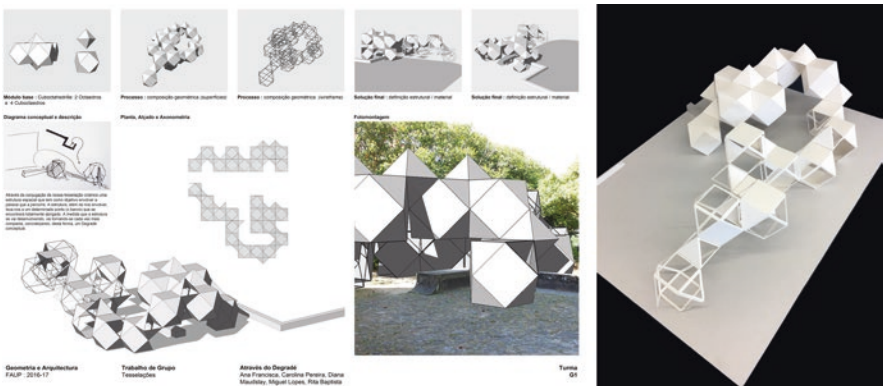

Fig. 10.5 Project B: "Through the gradient" by Ana Francisca, Carolina Pereira, Diana Maudslay, Miguel Lopes and Rita Baptista, 2016/2017 (Presentation poster and cardboard model)

Project B: Through the Gradient

In this project (Fig. 10.5), the students decided to expand the composition and place it along an existing path close to a sitting area of the garden so that people would walk along and throughout the architectonic structure before reaching the meeting point enclosed by a granite bench. A distinctive aspect of the proposal is the composition configuration that departed from an idea of gradation. As the poster illustrates, the inhabitable structure starts with a single *open* polyhedral cell to which more and more elements were added, and the base module multiplied until the structure finishes in the preexistent sitting area. Much the same way, the initial cells do not include panels to be read as open and progressively developed into closed cells. This detail ensures transparency as an important structure element and enhances the honeycomb's legibility.

Project C: (In)Tangible

In this example (Fig. 10.6), the students developed a spatial concept very different from the last two. Envisaging the structure as a kind of tent-like shape, they placed the faceted volume in an open area of the garden, along a connecting path, that visually links a small door existing on the facade of the Pavilion designed by the architect Álvaro Siza Vieira and an old gazebo from which one can look out over the river and the sea.

Conclusions

From the results here presented, we may conclude that the impact of this didactic experiment was extremely positive on many levels. First, because of the students' engagement with the challenge itself and the easiness and spontaneity with which they manipulated the geometric structures in search of an architectural shape. Secondly, for the important contribution of the digital tools to support the investigation and how they provided students the control of such complex forms. In the context of the *Geometry and Architecture* course, this possibility posed a stimulating challenge since it allowed to open the syllabus to different topics which are less commonly explored yet very pertinent in the context of contemporary architectural

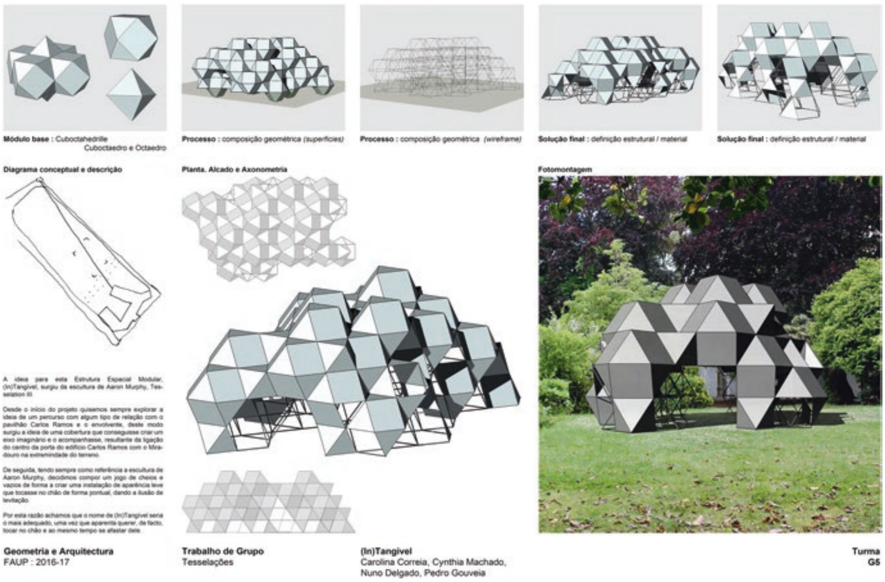

Fig. 10.6 Project C: "(In)Tangible" by Carolina Correia, Cynthia Machado, Nuno Delgado and Pedro Gouveia, 2016/2017 (Presentation poster)

practice. Thirdly, for the contribution of a brief introduction to polyhedral theory to the development of the spatial intelligence of the students' and their overall knowledge on geometry.

In this regard, we must highlight how studying and modelling polyhedra in higher education are an interesting subject through which geometry and architecture may be creatively explored, not only for the tangibility of the operations involved but also for the numerous mathematical concepts and transformations that the students are able to interact and better understand. For higher education courses in which modelling, materializing and structuring space is a topmost concern, such as Architecture, research on polyhedra is certainly a valuable subject matter, especially if students are given the possibility to explore 3Dmodelling software to improve their knowledge of geometry.

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10. Site Plan of "Casa Cor-de-Rosa, Cavalariças and Pavilhão Carlos Ramos" – FAUP © CEFA.
11. Faculty of Architecture – University of Porto, Casa Cor-de-Rosa/Álvaro Siza © Luis Ferreira Alves/FAUP.

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