VARIATIONAL PRINCIPLES FOR FINITELY GENERATED PSEUDOGROUP ACTIONS

ANDRZEJ BIŚ, MARIA CARVALHO, AND PAULO VARANDAS

Abstract. The aim of this work is the thermodynamic formalism of finitely generated pseudogroup actions on compact metric spaces. We introduce the notions of topological and measure-theoretical pressures for those actions, which may not admit invariant measures. We prove a variational principle and discuss the impact of the existence of a homogeneous probability measure. To clarify the scope of our main results and the relevance of our approach, we address several examples and list a few applications of interest.

1. Introduction

The thermodynamical formalism was brought from Statistical Mechanics to Dynamical Systems by the pioneering works of Sinai, Ruelle and Bowen [10, 11, 42]. These authors established a fruitful correspondence between one-dimensional lattices and uniformly hyperbolic maps, which conveyed the notion of Gibbs measure and the role of equilibrium states into the realm of dynamical systems. The study of the thermodynamical formalism has since been advancing in two main complementary directions, namely the theory of non-uniformly hyperbolic dynamical systems and the study of semigroup actions. The latter can be used to model neutral behavior in partially hyperbolic dynamical systems [26] and appear naturally in the theory of foliations [23]. Our work contributes to the development of the thermodynamic formalism of finitely generated pseudogroup actions (we refer the reader to Subsection 2.1 for a precise definition), for whom the extension of the classical theory has raised several difficulties and a global description is still far from complete.

Regarding group and semigroup actions of continuous endomorphisms of a compact metric space, it is often the case that there are several definitions of topological and measure-theoretic pressures which are suitable for each specific type of action. Most of them are unrelated (see for instance [13, 23, 21, 28, 9, 27, 29, 35, 37, 41, 47, 3] and references therein). A common aim of them all is to link topological and ergodic properties by some variational principle. For instance, Ruelle [40] considered finitely generated abelian groups on compact metric spaces, introduced notions of topological and measure-theoretic pressures, established a variational principle between them and gave sufficient conditions for the existence and uniqueness of equilibrium states. Later, Ollagnier and Pinchon [37] generalized that information, having obtained a variational principle for the entropy of countable amenable group actions, which are known to have invariant Borel probability measures. More recently, a unified approach for the thermodynamical formalism of continuous finitely generated group and semigroup actions was established in [8] using methods of

Date: March 26, 2025.

2010 *Mathematics Subject Classification.* Primary: 37D35, 37C85, Secondary: 43A05.

Key words and phrases. Pseudogroup actions; Topological pressure; Variational principle; Carathéodory–Pesin structure; Weighted topological pressure.

convex analysis introduced in [6, 7]. Yet, in general, the known connections between topological and measure-theoretic properties of group and semigroup actions are not enough to provide a complete description of their complexity. This is mainly due to the fact that these actions may fail to have Borel probability measures invariant by all the generators, as happens, for instance, with the semigroup action on the unit circle generated by the north-pole/south-pole diffeomorphism and an irrational rotation.

Our motivation to develop a thermodynamic formalism for finitely generated pseudogroup actions builds over the fact that the latter appear naturally in some dynamical and non-dynamical frameworks. Let us mention two of the most relevant. Firstly, a special class of pseudogroup actions arises as holonomy maps of foliations and became essential in the study of the geometry of foliated manifolds since the work of Haefliger [25]. Moreover, whereas Ghys, Langevin and Walczak introduced in [23] a notion of geometric entropy of a foliation and related it to a new concept of topological entropy of the holonomy pseudogroup induced by the foliation, a variational principle linking the entropy of the foliation to a well suited notion of measure-theoretical entropy remains unproven. Secondly, the local iterated function systems studied in fractal geometry are strongly related to pseudogroup actions (cf. Subsection 2.1 and [1, 31]), hence we expect that the Hausdorff dimension of the corresponding attractors may be estimated using the thermodynamical formalism developed here. Such a characterization is already known for both dynamical systems and iterated function systems (cf. [12, 18] and references therein).

By developing a thermodynamic formalism for pseudogroup actions we expect to contribute for a solution to these problems. However, extending the previous research to the context of pseudogroup actions presents an additional difficulty: whereas semigroup actions deal with endomorphisms of the same space, pseudogroup actions are defined by collections of local homeomorphisms whose domains may be distinct. Therefore, these domains may diminish under composition, and some compositions are not even admissible. A major effect of this complexity is that the classical Bowen's dynamical metric (cf. [46, Section 7.2]) may no longer be a distance. Thus, we had to find a well suited dynamical metric for finitely generated pseudogroups, which ought to coincide with Bowen's definition when the pseudogroup is a group.

Inspired by [4, 29, 30], and aiming at a variational principle for finitely generated pseudogroup actions acting on a compact metric space (X, d), in this work we explore the virtues of the following concepts:

(a) Carathéodory-Pesin structures, which will be used to define a pressure function (we denote by CP-pressure) for finitely generated pseudogroup actions, as done in [4] for the topological entropy.

(b) Brin-Katok local metric entropy (cf. [14]), to find a matching measure-theoretic pressure function for finitely generated pseudogroup actions, following a similar approach in [4] for the topological entropy. This new notion will be connected with the previous CP-pressure by a variational principle.

(c) Ghys-Langevin-Walczak topological entropy of a pseudogroup (cf. [23]), to extend it to a notion of topological pressure after selecting an adequate dynamical average for the potentials. This way, we benefit from the role of homogeneous measures (see Definition 2.30), as suggested by [4, Theorem 4.12].

This paper is organized as follows. In Section 2 we gather a few definitions. The main contributions of this work are stated in Section 3. Preliminary information on the topological and measure-theoretical pressures of pseudogroup actions may be read in Section 4. After these general considerations, Sections 5 and 6 are devoted to enlighten the reader about the role of

Carathéodory-Pesin structures within the thermodynamic formalism of pseudogroup actions. The proofs of the main results will occupy Sections 7, 8 and 9. In the latter section we also analyze the impact of the existence of a homogeneous probability measure on X. Finally, in Section 10 we address some examples and applications.

2. Main definitions

In this section we introduce finitely generated pseudogroup actions.

2.1. **Pseudogroups of local homeomorphisms.** Given a compact metric space (X, d), let $\mathrm{Homeo}(X)$ stand for the family of homeomorphisms between open subsets of X such that any $g \in \mathrm{Homeo}(X)$ is uniformly continuous. For $g \in \mathrm{Homeo}(X)$, denote by D_g its domain and by $R_g = g(D_g)$ its range.

Definition 2.1. [44] *A set $G \subset Homeo(X)$ is a* pseudogroup *if it satisfies the following properties:*

(P1) If $g, f \in G$ and $R_f \cap D_g \neq \emptyset$, then $g \circ f\colon f^{-1}(R_f \cap D_g) \to g(R_f \cap D_g)$ belongs in G.
(P2) If $g \in G$, then $g^{-1} \in G$.
(P3) The identity map of X, say $\mathrm{id}_X\colon (X, d) \to (X, d)$, is in G.
(P4) If $g \in G$ and $W \subset D_g$ is an open subset of D_g, then $g_{|W} \in G$.
(P5) If $g\colon D_g \to R_g$ is a homeomorphism between open subsets of X and if, for each point $p \in D_g$, there exists a neighborhood $\mathcal{N}$ of p inside D_g such that $g_{|\mathcal{N}} \in G$, then $g \in G$.

For any set $G_1 \subset \mathrm{Homeo}(X)$ for which $\bigcup_{g\in G}\{D_g \cup R_g : g \in G\} = X$, there exists a unique smallest (in the sense of inclusion) pseudogroup G which contains G_1, thus called the *pseudogroup generated by* G_1. By definition, $g \in G$ if and only if $g \in \mathrm{Homeo}(X)$ and for any $x \in D_g$ there are a positive integer k, maps $g_1, ..., g_k \in G_1$, exponents $e_1, ..., e_k \in \{-1, 1\}$ and an open neighborhood U of x in X such that

$$U \subset D_g \qquad \text{and} \qquad g_{|U} \;=\; \big(g_1^{e_1} \circ ... \circ g_k^{e_k}\big)_{|U}.$$

If the set G_1 is finite, we say that G is *finitely generated*. Throughout this paper we will always consider finitely generated pseudogroups with symmetric generating sets, that is,

$$G_1 = \{\mathrm{id}_X,\, g_1,\, g_1^{-1},\, g_2,\, g_2^{-1}, ..., g_L,\, g_L^{-1}\}$$

for some $L \in \mathbb{N}$.

A finitely generated pseudogroup (G, G_1) on a compact metric space (X, d) is said to be *a finitely generated group* if $D_g = R_g = X$ for every $g \in G$. A finitely generated *free group* G with generator set G_1 consists of all the finite compositions that can be built with elements of G_1, where different compositions, even if they yield the same map on X, are considered distinct elements of G.

Given an integer $n > 1$, write

$$G_n = \big\{g_{i_n} \circ ... \circ g_{i_2} \circ g_{i_1} : \;\; g_{i_j} \in G_1 \quad \forall\, j \in \{1, \cdots, n\}\big\}$$

and $|G_n|$ for its cardinality. Since $\mathrm{id}_X \in G_1$, one has $\mathrm{id}_X \in G_n$ for every $n \in \mathbb{N}$, $G_m \subset G_n$ whenever $m \leqslant n$, and $\cup_{g\in G} D_g = X$.

The notion of pseudogroup action is related to iterated function systems (IFS for short). Indeed, given a compact metric space (X, d), a finitely generated *local iterated function system* (LIFS) is determined by a finite collection of continuous maps $G_1 = \{g_1, g_2, \ldots, g_k\}$, where

$g_i \colon X_i \to X$, for some nonempty subsets $X_i \subset X$, and the set G of their compositions is defined as in (P1) above. Local iterated function systems are similar to the semi-pseudogroups introduced by Waliszewski [45], and are interesting both from the theoretical point of view and the applications. We refer the reader to [1, 2, 31, 32, 33] and references therein, where one may find detailed information on connections between LIFS and fractal dimension, model stock market returns, fractal image compression and biometric identification.

2.2. **Topological entropy.** There have been several extensions of the notion of entropy for a map to the setting of finitely generated semigroup actions; for an account on this topic we refer the reader to [3, 16, 43]. In the context of pseudogroup actions, the topological entropy $h_{\text{top}}(G, G_1)$ of a finitely generated pseudogroup action (G, G_1) was introduced in [23] and defined as follows.

Definition 2.2. [23] *Given $\varepsilon > 0$ and $n \in \mathbb{N}$, two points $x, y \in X$ are $(n, \varepsilon)-$separated by (G, G_1) if there exists $g \in G_n$ such that $x, y \in D_g$ and $d(g(x), g(y)) \geqslant \varepsilon$. A subset E of X is said to be $(n, \varepsilon)-$separated if any two distinct points in E are $(n, \varepsilon)-$separated. Denote by $s(n, \varepsilon)$ the maximal number of $(n, \varepsilon)-$separated points in X.*

We observe that the condition $d(g(x), g(y)) \geqslant \varepsilon$ for some $g \in G_n$ means that, if g is given by a composition $g_{i_n} \circ \cdots \circ g_{i_1}$ where $g_{i_n}, \ldots, g_{i_1} \in G_1$, then

$$d\big(g_{i_n} \circ \cdots \circ g_{i_1}(x),\ g_{i_n} \circ \cdots \circ g_{i_1}(y)\big) \geqslant \varepsilon.$$

Definition 2.3. [23] The *topological entropy of a finitely generated pseudogroup (G, G_1) is the limit*

$$h_{\text{top}}(G, G_1) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log s(n, \varepsilon). \tag{2.4}$$

The previous limit as ε goes to 0^+ exists, since the map $\varepsilon > 0 \to \limsup_{n \to +\infty} \frac{1}{n} \log s(n, \varepsilon)$ is monotone. It is known (cf. [44, Section 3.2]) that the topological entropy of a finitely generated pseudogroup depends on the generating set. However, if $G1$ and G_1' are two generating sets of the same pseudogroup G, then $h_{\text{top}}(G, G1) = 0$ if and only if $h_{\text{top}}(G, G_1') = 0$.

Definition 2.5. *Given $\varepsilon > 0$ and $n \in \mathbb{N}$, a subset F of X is said to $(n, \varepsilon)-$span X with respect to the pseudogroup (G, G_1) if for every $x \in X$ there exists $y \in F$ with $d\big(g(x), g(y)\big) < \varepsilon$ for every $g \in G_n$ such that x, y are both in the domain of g. The minimal cardinality of the $(n, \varepsilon)-$spanning subsets of X is denoted by $r(n, \varepsilon)$.*

The condition $d(g(x), g(y)) < \varepsilon$ for every $g \in G_n$ such that x, y are both in the domain of g ensures that, if such a g is given by a composition $g = g_{i_n} \circ \cdots \circ g_{i_1}$ where $g_{i_n}, \ldots, g_{i_1} \in G_1$, then

$$\max\big\{d(x, y),\ d(g_{i_1}(x), g_{i_1}(y)),\ \ldots,\ d(g_{i_n} \circ \cdots \circ g_{i_1}(x), g_{i_n} \circ \cdots \circ g_{i_1}(y))\big\} < \varepsilon.$$

The topological entropy can be defined in terms of $(n, \varepsilon)-$spanning sets, and the two approaches are equivalent if (G, G_1) is a finitely generated group (cf. Subsection 4.4), that is,

$$\lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log s(n, \varepsilon) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log r(n, \varepsilon).$$

2.3. **Topological pressure.** We now define the topological pressure of a finitely generated pseudogroup action (G, G_1) with respect to a continuous potential, which generalizes the notion of topological entropy (2.4).

For every $x \in X$ and $n \in \mathbb{N}$, consider the set

$$G_n^x = \Big\{ g \in G_n \colon\ x \in D_g \Big\}. \tag{2.6}$$

We note that $g \in G_n^x$ if and only if $x \in D_g$ and there exist $g_{i_1}, g_{i_2}, \ldots, g_{i_n} \in G_1$ such that $g = g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1}$. Moreover, $G_n^x \neq \emptyset$ for every $x \in X$ and $n \in \mathbb{N}$, since $\mathrm{id}_X \in G_n^x$ due to property (P3) of Definition 2.1.

Denote by $C^0(X)$ the space of continuous maps $\psi \colon X \to \mathbb{R}$ endowed with the uniform norm, which we abbreviate into $\|\cdot\|_\infty$.

Definition 2.7. *Given $\psi \in C^0(X)$, the* topological pressure of the finitely generated pseudogroup (G, G_1) with respect to ψ *is the limit*

$$P_{\mathrm{top}}((G, G_1), \psi) \;=\; \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sup_{E_{n,\varepsilon}} \Big\{ \sum_{x \in E_{n,\varepsilon}} e^{\frac{1}{|G_n^x|} \sum_{g \in G_n^x} S_\psi^g(x)} \Big\} \right) \tag{2.8}$$

where $g = g_{i_n} \circ \ldots \circ g_{i_2} \circ g_{i_1}$ for some $g_{i_n}, \ldots, g_{i_1} \in G_1$, the supremum is taken over the $(n, \varepsilon)-$separated subsets $E_{n,\varepsilon}$ of X with respect to (G, G_1) and

$$S_\psi^g(x) \;=\; \psi(x) \;+\; \psi\big(g_{i_1}(x)\big) \;+\; \ldots \;+\; \psi\big(g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1}(x)\big).$$

In Definition 2.7, it suffices to take the supremum over those $(n, \varepsilon)-$separated sets which cannot be enlarged to a $(n, \varepsilon)-$separated set. We also observe that, if (G, G_1) is a finitely generated group, then the previous topological pressure can be defined by using spanning sets (cf. Subsection 4.4 for more information).

Remark 2.9. It is straightforward to show that the map $\Gamma \colon \psi \in C^0(X) \mapsto P_{\mathrm{top}}((G, G_1), \psi)$ is increasing and translation invariant. More precisely, if $\varphi,\, \psi \in C^0(X)$ and $c \in \mathbb{R}$, then:

(a) $\varphi \leqslant \psi \;\Rightarrow\; P_{\mathrm{top}}((G,\, G_1),\, \varphi) \;\leqslant\; P_{\mathrm{top}}((G,\, G_1),\, \psi)$.

(b) $P_{\mathrm{top}}((G,\, G_1),\, \psi + c) \;=\; P_{\mathrm{top}}((G,\, G_1),\, \psi) + c$.

From the previous properties we conclude that Γ is continuous since, for every $\varphi,\, \psi \in C^0(X)$, one has

$$\Gamma(\psi) - \|\varphi - \psi\|_\infty \;=\; \Gamma(\psi - \|\varphi - \psi\|_\infty) \;\leqslant\; \Gamma(\varphi) \;\leqslant\; \Gamma(\psi + \|\varphi - \psi\|_\infty) \;=\; \Gamma(\psi) + \|\varphi - \psi\|_\infty.$$

Therefore, the map $\psi \in C^0(X) \mapsto P_{\mathrm{top}}((G, G_1), \psi) - \max_{x \in X}\, \psi(x)$ is continuous as well.

2.4. **Dynamical metrics.** We expected to be able to define the topological entropy and pressure of a pseudogroup using dynamical metrics, thereby generalizing to this setting Bowen's approach with dynamical balls [46, Section 7.2]. However, the classical Bowen's metric, used to define the pressure function for a single dynamics, may no longer be a distance in the context of pseudogroup actions. In this subsection we will address this problem.

Recall that $\mathrm{id}_X \in G_1$, so

$$\forall\, x, y \in X, \;\; \forall\, n \in \mathbb{N} \quad \mathrm{id}_X \in G_n^x \cap G_n^y. \tag{2.10}$$

Therefore, the next map is well defined.

Definition 2.11. *Given $n \in \mathbb{N}$, let $\tau_n \colon X \times X \to [0, +\infty[$ be the map defined by*

$$\tau_n(x,y) \;=\; \max\big\{d(g(x), g(y))\colon\;\; g \in G_n^x \cap G_n^y\big\}.$$

For every $n \in \mathbb{N}$, the map τ_n is symmetric, non-negative and, due to (2.10), it satisfies

$$d(x,y) \;\leqslant\; \tau_n(x,y) \quad \forall\, x, y \in X. \tag{2.12}$$

It is reminiscent of Bowen's metric, and it is indeed a metric if G is a finitely generated group. We also note that, a subset E of X is $(n, \varepsilon)-$separated if and only if for any two distinct points x, y in E one has $\tau_n(x,y) \geqslant \varepsilon$.

Definition 2.13. *Given $x, y \in X$ and $k \in \mathbb{N}$, $k \geqslant 2$, let*

$$\mathcal{A}_k(x,y) \;=\; \big\{(a_1, a_2, \ldots, a_k) \in X^k\colon\; [a_1 = x \text{ and } a_k = y] \text{ or } [a_1 = y \text{ and } a_k = x]\big\}.$$

For each fixed $n \in \mathbb{N}$, define $d_n \colon X \times X \to [0, +\infty[$ by

$$d_n(x,y) \;=\; \inf\,\Big\{\sum_{j=2}^{k} \tau_n(a_{j-1}, a_j)\colon\;\; k \,\in\, \mathbb{N}\setminus\{1\} \;\text{ and }\; (a_j)_{1 \leqslant j \leqslant k} \,\in\, \mathcal{A}_k(x,y)\Big\}.$$

We remark that, for every $n \in \mathbb{N}$ and all $x, y \in X$, one has

$$d_n(x,y) \;\leqslant\; \tau_n(x,y) \tag{2.14}$$

since we may choose $k = 2$ and $a_1 = x, a_2 = y$.

Lemma 2.15. *For every $n \in \mathbb{N}$, the map d_n is a metric in X.*

Proof. Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d).

Claim 1: $\quad \forall\, x, y \in X \quad d_n(x,y) = 0 \Leftrightarrow x = y.$

Clearly, $d_n(x,x) = 0$ since $0 \leqslant d_n(x,x) \leqslant \tau_n(x,x) = 0$. Conversely, assume that $d_n(x,y) = 0$. Then, given $\varepsilon > 0$, there are $k \geqslant 2$ and $(a_1, a_2, \ldots, a_k) \in \mathcal{A}_k(x,y)$ such that

$$\sum_{j=2}^{k} \tau_n(a_{j-1}, a_j) \;<\; \varepsilon.$$

Using the triangular identity for the metric d and (2.12), we obtain

$$d(x,y) \;\leqslant\; \sum_{j=2}^{k} d(a_{j-1}, a_j) \;\leqslant\; \sum_{j=2}^{k} \tau_n(a_{j-1}, a_j) \;<\; \varepsilon.$$

Since $\varepsilon > 0$ is arbitrarily small, we conclude that $d(x,y) = 0$, hence $x = y$.

Claim 2: $\quad \forall\, x, y \in X \quad d_n(x,y) \;=\; d_n(y,x).$

This property of d_n is an immediate consequence of the equalities

$$\forall\, x, y \in X \qquad \mathcal{A}_k(x,y) \;=\; \mathcal{A}_k(y,x) \quad \text{and} \quad \tau_n(x,y) \;=\; \tau_n(y,x).$$

Claim 3: $\quad \forall\, x, y, z \in X \quad d_n(x,y) \;\leqslant\; d_n(x,z) + d_n(z,y).$

Given $x, y, z \in X$ and $\varepsilon > 0$, there are integers $k, \ell \geqslant 2$, $(a_1, a_2, \dots, a_k) \in \mathcal{A}_k(x,z)$ and $(b_1, b_2, \dots, b_\ell) \in \mathcal{A}_\ell(z,y)$ such that

$$\sum_{j=2}^{k} \tau_n(a_{j-1},\, a_j) - d_n(x,z) \;<\; \varepsilon/2 \qquad \text{and} \qquad \sum_{j=2}^{\ell} \tau_n(b_{j-1},\, b_j) - d_n(z,y) \;<\; \varepsilon/2.$$

We may assume that $a_1 = x$, $a_k = z$, $b_1 = z$ and $b_\ell = y$, since the maps τ_n and d_n are symmetric. Therefore,

$$d_n(x,y) \;\leqslant\; \sum_{j=2}^{k} \tau_n(a_{j-1},\, a_j) \;+\; \sum_{j=2}^{\ell} \tau_n(b_{j-1},\, b_j)$$

since $\big(a_1 = x,\, a_2, \dots, a_k = z = b_1,\, b_2, \dots, b_\ell = y\big) \in \mathcal{A}_{k+\ell}(x,y)$. Therefore,

$$d_n(x,y) \;\leqslant\; d_n(x,z) + \varepsilon/2 + d_n(z,y) + \varepsilon/2 \;=\; d_n(x,z) + d_n(z,y) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrarily small, we deduce that

$$d_n(x,y) \;\leqslant\; d_n(x,z) + d_n(z,y).$$

□

2.5. **Dynamical balls.** In what follows, given $n \in \mathbb{N}$ and $\varepsilon > 0$, the *open dynamical $n-$ball* centered at x with radius ε, determined by the pseudogroup action (G, G_1), is the set

$$B_n(x,\varepsilon) = \Big\{ y \in X \colon\; d_n(x,y) < \varepsilon \Big\}. \tag{2.16}$$

We might have defined the notions of separated and spanning sets using the metric d_n, though they would convey a new concept of topological entropy. Those new definitions would state that:

(a) Given $\varepsilon > 0$ and $n \in \mathbb{N}$, two points $x, y \in X$ are $(n, d_n, \varepsilon)-$separated if $d_n(x,y) \geqslant \varepsilon$. A subset E of X is said to be $(n, d_n, \varepsilon)-$separated if any two distinct points $x, y \in E$ are $(n, d_n, \varepsilon)-$separated. In particular, one has

$$B_n(x, \varepsilon/2) \,\cap\, B_n(y, \varepsilon/2) = \emptyset \qquad \forall\, x, y \in E.$$

Denote by $\overline{s}(n,\varepsilon)$ the maximal number of $(n, d_n, \varepsilon)-$separated points in X.

(b) Given $\varepsilon > 0$ and $n \in \mathbb{N}$, a subset F of X is said to $(n, d_n, \varepsilon)-$span X if for every $x \in X$ there exists $y \in F$ with $d_n(x,y) < \varepsilon$. In particular, one has

$$X \;=\; \bigcup_{x \in F} B_n(x, \varepsilon). \tag{2.17}$$

The minimal cardinality of the $(n, d_n, \varepsilon)-$spanning subsets of X is denoted by $\overline{r}(n,\varepsilon)$.

From (2.14) we deduce that, given $\varepsilon > 0$ and $n \in \mathbb{N}$, any $(n, d_n, \varepsilon)-$separated subset E of X is $(n,\varepsilon)-$separated; likewise, any $(n,\varepsilon)-$spanning subset F of X is $(n, d_n, \varepsilon)-$spanning. Therefore,

$$s(n,\varepsilon) \;\geqslant\; \overline{s}(n,\varepsilon) \qquad \text{and} \qquad r(n,\varepsilon) \;\geqslant\; \overline{r}(n,\varepsilon).$$

Consequently:

(i) Due to (2.17),

$$F \subset X \;\text{ is } (n,\varepsilon)\text{−spanning} \qquad \Rightarrow \qquad X \;=\; \bigcup_{x \in F} B_n(x,\varepsilon). \tag{2.18}$$

(2i) $h_{\mathrm{top}}(G, G_1) \;\geqslant\; \lim_{\varepsilon \to 0^+} \, \limsup_{n \to +\infty} \frac{1}{n} \log \, \overline{s}(n,\varepsilon)$.

We note that, if (G, G_1) is a finitely generated group on a compact metric space, then the map τ_n is a metric for every $n \in \mathbb{N}$. We will show that, in this setting, $\tau_n = d_n$, and so

$$h_{\rm top}(G, G_1) \,=\, \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log \overline{s}(n, \varepsilon).$$

Lemma 2.19. *Let (G, G_1) is a finitely generated group on a compact metric space (X, d). Then,*

$$\forall\, n \in \mathbb{N}, \quad \forall\, x, y \in X \quad d(x, y) \,\leqslant\, d_n(x, y) \,=\, \tau_n(x, y)$$

and the metric τ_n is uniformly equivalent to d.

Proof. Fix $n \in \mathbb{N}$. We start by showing that the metrics τ_n and d are uniformly equivalent. Since $\mathrm{id}_X \in G_n$, we already know (cf. (2.12)) that

$$\forall\, x, y \in X \quad d(x, y) \,\leqslant\, \tau_n(x, y).$$

Therefore, the identity map $id_1 \colon (X, \tau_n) \to (X, d)$ is continuous. Moreover, since each $g \in G_n$ is uniformly continuous, given $\varepsilon > 0$ there is $\delta_g > 0$ such that

$$d(x, y) \,<\, \delta_g \quad \Rightarrow \quad d\big(g(x), g(y)\big) \,<\, \varepsilon.$$

As G_n is finite, we may take $\delta = \min\{\delta_g \colon g \in G_n\} > 0$ and deduce that

$$d(x, y) \,<\, \delta \quad \Rightarrow \quad \tau_n(x, y) \,=\, \max\big\{d\big(g(x), g(y)\big) \colon\, g \in G_n\big\} \,<\, \varepsilon.$$

Thus, the identity map $id_2 \colon (X, d) \to (X, \tau_n)$ is continuous. In particular, the space (X, τ_n) is compact.

Let us now prove that $d_n = \tau_n$. We already know (cf. (2.14)) that

$$\forall\, x, y \in X \quad d_n(x, y) \,\leqslant\, \tau_n(x, y).$$

In addition, since τ_n is a metric (in particular, it satisfies the triangular inequality), given $x, y \in X$, an integer $k \geqslant 2$ and $(a_1, a_2, \ldots, a_k) \in \mathcal{A}_k(x, y)$ such that $a_1 = x$ and $a_k = y$, one has

$$\tau_n(x, a_2) + \tau_n(a_2, a_3) + \cdots + \tau_n(a_{k-1}, y) \,\geqslant\, \tau_n(x, y) \,\geqslant\, d(x, y).$$

Consequently, as τ_n is symmetric, both $\tau_n(x, y)$ and $d(x, y)$ are lower bounds of the set

$$\Big\{ \sum_{j=2}^{k} \tau_n(a_{j-1}, a_j) \colon\; k \,\in\, \mathbb{N} \setminus \{1\} \;\text{ and }\; (a_j)_{1 \leqslant j \leqslant k} \,\in\, \mathcal{A}_k(x, y) \Big\}.$$

For this reason, its infimum is bigger than or equal to $\tau_n(x, y)$. That is,

$$\forall\, x, y \in X \quad d_n(x, y) \,\geqslant\, \tau_n(x, y).$$

The proof of the lemma is complete. □

2.6. **Measure-theoretic entropy.** Fix a finitely generated pseudogroup (G, G_1) on a compact metric space (X, d), generated by a finite set G_1. Denote by $\mathcal{M}_1(X)$ the space of Borel probability measures on X and by $\mathcal{M}_G(X) \subset \mathcal{M}_1(X)$ the subset of probability measures which are invariant by every element of G.

Definition 2.20. *For each probability measure $\mu \in \mathcal{M}_1(X)$ and $x \in X$, the* lower local metric entropy of μ at x with respect to the pseudogroup (G, G_1) *is defined by*

$$\underline{h}_\mu((G, G_1), x) = \lim_{\varepsilon \to 0^+} \liminf_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) \tag{2.21}$$

where we specify that, if for some $\varepsilon > 0$ *one has* $\mu(B_n(x,\varepsilon)) = 0$*, then* $\log \mu(B_n(x,\varepsilon)) = 0$*. Similarly, the* upper local metric entropy of of μ at x with respect to the pseudogroup (G, G_1) *is given by*

$$\overline{h}_\mu((G,G_1),x) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x,\varepsilon)). \tag{2.22}$$

The maps $\underline{h}_\mu((G,G_1),\cdot)\colon X \to [0,+\infty]$ and $\overline{h}_\mu((G,G_1),\cdot)\colon X \to [0,+\infty]$ are measurable (cf. Lemma 4.2 in Section 4), so we may define the *lower and upper metric entropy of* μ *with respect to the pseudogroup* (G, G_1) by the averages

$$\underline{h}_\mu(G,G_1) = \int_X \underline{h}_\mu((G,G_1),x)\, d\mu(x) \tag{2.23}$$

$$\overline{h}_\mu(G,G_1) = \int_X \overline{h}_\mu((G,G_1),x)\, d\mu(x). \tag{2.24}$$

2.7. **Measure-theoretic pressure.** The previous concept is generalized to every continuous potential as follows.

Definition 2.25. *Given* $\mu \in \mathcal{M}_1(X)$*,* $\psi \in C^0(X)$ *and* $x \in X$*, the* upper local metric pressure at x of μ, with respect to the pseudogroup (G, G_1) and the potential ψ*, is defined by*

$$\overline{P}_\mu((G,G_1),\psi,x) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} -\frac{1}{n} \Big[\log \mu(B_n(x,\varepsilon)) - \frac{1}{|G_n^x|} \sum_{g \in G_n^x} S_\psi^g(x) \Big]. \tag{2.26}$$

Similarly, the lower local metric pressure at x of μ, with respect to the pseudogroup (G, G_1) and the potential ψ*, is given by*

$$\underline{P}_\mu((G,G_1),\psi,x) = \lim_{\varepsilon \to 0^+} \liminf_{n \to +\infty} -\frac{1}{n} \Big[\log \mu(B_n(x,\varepsilon)) - \frac{1}{|G_n^x|} \sum_{g \in G_n^x} S_\psi^g(x) \Big]. \tag{2.27}$$

The previous limits, when ε goes to 0^+, exist and are measurable functions of $x \in X$ (see Lemma 4.4 in Section 4). Therefore, we may define the *lower and upper measure-theoretic pressure of* (G, G_1) *with respect to* $\mu \in \mathcal{M}_1(X)$ *and* $\psi \in C^0(X)$ by

$$\underline{P}_\mu((G,G_1),\psi) = \int_X \underline{P}_\mu((G,G_1),\psi,x)\, d\mu(x)$$

$$\overline{P}_\mu((G,G_1),\psi) = \int_X \overline{P}_\mu((G,G_1),\psi,x)\, d\mu(x).$$

Remark 2.28. We observe that, given $\psi \in C^0(X)$, one has for every $x \in X$

$$n \min_{t \in X} \psi(t) \leqslant \frac{1}{|G_n^x|} \sum_{g \in G_n^x} S_\psi^g(x) \leqslant n \max_{t \in X} \psi(t).$$

Therefore, for every $\mu \in \mathcal{M}_1(X)$,

$$\overline{h}_\mu(G,G_1) + \min_{t \in X} \psi(t) \leqslant \overline{P}_\mu((G,G_1),\psi) \leqslant \overline{h}_\mu(G,G_1) + \max_{t \in X} \psi(t). \tag{2.29}$$

Similar estimates are valid for the corresponding notions $\underline{h}_\mu(G, G_1)$ and $\underline{P}_\mu((G, G_1), \psi)$.

2.8. **Homogeneous measures.** Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d). The following is an important class of Borel measures on X.

Definition 2.30. *A probability measure μ on the $\sigma-$algebra of Borel subsets of X is said to be* $G-$homogeneous *if the following conditions are valid:*

(H1) *For every compact set $K \subset X$ one has $\mu(K) < +\infty$.*

(H2) *There exists a compact set $K_0 \subset X$ with $\mu(K_0) > 0$.*

(H3) *For every $\varepsilon > 0$ there are $0 < \delta = \delta(\varepsilon) < \varepsilon$ and $\lambda = \lambda(\varepsilon) > 0$ such that*

$$\mu(B_n(y, \delta)) \leqslant \lambda\, \mu(B_n(x, \varepsilon)) \qquad \forall\, x, y \in X \quad \forall\, n \in \mathbb{N}.$$

The canonical volume form on a closed, compact, oriented Riemannian manifold determines a $G-$homogeneous measure with respect to any finitely generated group of isometries on the manifold. Another example of a pseudogroup with a $G-$homogeneous probability measure is described in [4, Proposition 4.6].

3. Statement of the main results

Our first result is strongly inspired by [4], which concerns the topological and measure-theoretic entropies, and by [30], whose results were stated in the context of finitely generated semigroup actions.

Theorem A. *Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d). For every $\psi \in C^0(X)$, one has:*

(a) $P_{\text{top}}((G, G_1), \psi) \geqslant \sup_{\mu \in \mathcal{M}_1(X)} \underline{P}_\mu((G, G_1), \psi)$.

(b) *If, in addition, G is a free group and there is a $G-$homogeneous, $G-$invariant, ergodic probability measure $\eta \in \mathcal{M}_G(X)$, then*

$$\overline{P}_\eta((G, G_1), \psi) \;=\; h_{\text{top}}(G, G_1) \,+\, \int_X \psi \, d\eta.$$

The precise computation of the topological pressure of a pseudogroup action usually demands a substantial knowledge of the minimal cardinality of spanning sets, and so it is not always feasible. An advantage of Theorem A is the fact that it provides a bound from below for the topological pressure of a pseudogroup action, as illustrated by Examples 10.3 – 10.5.

The first part of Theorem A is a corollary of a more general statement that will be proved in Sections 6 and 7. It addresses a notion of pressure conveyed by Carathéodory-Pesin structures, whose precise definition will be recalled in Section 5.

Theorem B. *Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d). Then:*

(a) *For every $\psi \in C^0(X)$ and any Borel subset Z of X,*

$$P_Z((G, G_1), \psi) \;=\; \sup\big\{\underline{P}_\mu((G, G_1), \psi)\colon\; \mu \in \mathcal{M}_1(X) \text{ and } \mu(Z) = 1\big\}.$$

(b) *For every $\psi \in C^0(X)$,*

$$P_{\text{top}}((G, G_1), \psi) \;\geqslant\; P_X((G, G_1), \psi).$$

We note that, even if the pseudogroup (G, G_1) is a group generated by $G_1 = \{\mathrm{id}_X, f, f^{-1}\}$, where $f\colon X \to X$ is a homeomorphism of a compact metric space X, one cannot expect to prove a general variational principle like item (a) of Theorem B with $\mathcal{M}_1(X)$ replaced by the set of $f-$invariant Borel probability measures on X. Indeed, in [20, Example 1.5] we find a set Z with zero measure with respect to any $f-$invariant probability measure, though $h_Z(G, G_1) > 0$. Moreover, even if there are $G-$invariant probability measures supported on such a set Z, it may happen that their information is not enough to estimate $h_Z((G, G_1))$: see Example 10.6.

4. Auxiliary lemmas

For future use, we gather in this section a few lemmas whose proofs are known in other settings but need to be adapted to the context of pseudogroup actions.

4.1. **Measurability of the local metric entropy.** Fix a finitely generated pseudogroup (G, G_1) by local homeomorphisms of a compact metric space (X, d). Recall from Subsection 2.6 that, given a probability measure $\mu \in \mathcal{M}_1(X)$ and $x \in X$, the lower local metric entropy of μ with respect to the pseudogroup (G, G_1) at x is defined by

$$\underline{h}_\mu((G, G_1), x) = \lim_{\varepsilon \to 0^+} \liminf_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)). \tag{4.1}$$

Lemma 4.2. *The function $\underline{h}_\mu((G, G_1), \cdot)\colon X \to [0, +\infty]$ is measurable.*

Proof. Take a point $x_0 \in X$, a positive integer n, a measure $\mu \in \mathcal{M}_1(X)$ and an $\varepsilon > 0$. It is known that any semi-continuous function is measurable, and that the pointwise limit of measurable functions is a measurable function as well. Therefore, we start by showing that the function $x \in X \mapsto f_n(x) = \mu(B_n(x, \varepsilon))$ is lower semi-continuous at x_0. For this purpose, fix $a \in [0, 1[$ such that $f_n(x_0) > a$. Notice that, for any positive integers k and ℓ with $k < \ell$, one has

$$B_n(x_0, \varepsilon - 1/k) \,\subset\, B_n(x_0, \varepsilon - 1/\ell) \quad \text{and} \quad \bigcup_{k \in \mathbb{N}} B_n(x_0, \varepsilon - 1/k) \,=\, B_n(x_0, \varepsilon).$$

Therefore,

$$\lim_{k \to +\infty} \mu\big(B_n(x_0, \varepsilon - 1/k)\big) \,=\, \mu\big(B_n(x_0, \varepsilon)\big)$$

and, since $f_n(x_0) > a$, we can choose $\varepsilon_1 \in]0, \varepsilon[$ such that $\mu(B_n(x_0, \varepsilon_1)) > a$.

By assumption, every $g \in G$ is uniformly continuous and its domain D_g is open. Thus, for each $g \in G$ such that $x_0 \in D_g$ there exists $\delta_g > 0$ such that

$$x \in B(x_0, \delta_g) \quad \Rightarrow \quad x \in D_g \;\text{ and }\; g(x) \in B\big(g(x_0), \varepsilon - \varepsilon_1\big).$$

As the pseudogroup is finitely generated, we may take

$$\delta_n \,=\, \min\{\delta_g \colon\, g \in G_n^{x_0}\}.$$

This way, for every $x \in B(x_0, \delta_n)$ and any $g \in G_n^{x_0}$, we are sure that $x \in D_g$ and that $d(g(x), g(x_0)) < \varepsilon - \varepsilon_1$. In particular, $\tau_n(x, x_0) < \varepsilon - \varepsilon_1$. Hence, by the inequality (2.14), one has $d_n(x, x_0) < \varepsilon - \varepsilon_1$. That is, $x \in B_n(x_0, \varepsilon - \varepsilon_1)$.

Now take a point $y \in B_n(x_0, \varepsilon_1)$. Then

$$d_n(y, x) \,\leqslant\, d_n(y, x_0) + d_n(x_0, x) \,\leqslant\, \varepsilon_1 + \varepsilon - \varepsilon_1 = \varepsilon.$$

Consequently, $y \in B_n(x,\varepsilon)$. We have showed that $B_n(x_0,\varepsilon_1) \subset B_n(x,\varepsilon)$, and this inclusion yields

$$f_n(x) \;=\; \mu\big(B_n(x,\varepsilon)\big) \;\geqslant\; \mu\big(B_n(x_0,\varepsilon_1)\big) \;>\; a.$$

So, f_n is lower semi-continuous at $x_0 \in X$.

□

A similar reasoning shows that the map $\overline{h}_\mu((G,G_1),\cdot)\colon X \to [0,+\infty]$ is measurable.

4.2. **Measurability of the local metric pressure.** Recall from Subsection 2.7 that, given $\mu \in \mathcal{M}_1(X)$, $\psi \in C^0(X)$ and $x \in X$, the upper local metric pressure of μ at x with respect to the pseudogroup (G,G_1) and the potential ψ is defined by

$$\overline{P}_\mu((G,G_1),\psi,x) = \lim_{\varepsilon\to 0^+}\, \limsup_{n\to+\infty}\, -\frac{1}{n}\Big[\log\,\mu(B_n(x,\varepsilon)) - \frac{1}{|G_n^x|}\sum_{\underline{g}\in G_n^x} S_\psi^{\underline{g}}(x)\Big]. \tag{4.3}$$

Lemma 4.4. *For every $\mu \in \mathcal{M}_1(X)$, $\psi \in C^0(X)$ and $x \in X$, the previous limit when ε goes to 0^+ exists. Moreover, the map*

$$x \in X \quad\mapsto\quad \overline{P}_\mu((G,G_1),\psi,x)$$

is measurable.

Proof. Given $x \in X$, if $0 < \varepsilon_1 < \varepsilon_2$ then

$$\log\,\mu(B_n(x,\varepsilon_1)) - \frac{1}{|G_n^x|}\sum_{\underline{g}\in G_n^x} S_\psi^{\underline{g}}(x) \;\leqslant\; \log\,\mu(B_n(x,\varepsilon_2)) - \frac{1}{|G_n^x|}\sum_{\underline{g}\in G_n^x} S_\psi^{\underline{g}}(x)$$

so, taking the lim sup as n goes to $+\infty$, we conclude that the function

$$\varepsilon \,\in\, \mathbb{R}_+ \quad\mapsto\quad \limsup_{n\to+\infty} -\frac{1}{n}\Big[\log\,\mu(B_n(x,\varepsilon)) - \frac{1}{|G_n^x|}\sum_{\underline{g}\in G_n^x} S_\psi^{\underline{g}}(x)\Big]$$

is non-increasing. Therefore, the limit as $\varepsilon \to 0^+$ exists.

We proceed to show that the previous limit varies measurably with x. Fix $\mu \in \mathcal{M}_1(X)$ and define the following sequences of functions

$$\begin{aligned} x \in X \quad&\mapsto\quad f_n^1(x) = \mu(B_n(x,\varepsilon)) \\ x \in X \quad&\mapsto\quad f_n^2(x) = \frac{1}{|G_n^x|}\sum_{\underline{g}\in G_n^x} S_\psi^{\underline{g}}(x). \end{aligned}$$

Then, for every $n \in \mathbb{N}$:

(i) The function $x \in X \mapsto f_n^1(x)$ is measurable (cf. the proof of Lemma 4.2).

(ii) The domains of the generators $g_i \in G_1$ are open subsets of X, thus the map $x \in X \mapsto |G_n^x|$ is locally constant and the map $x \mapsto f_n^2(x)$ is continuous.

Altogether, we conclude that $x \in X \mapsto -[\frac{1}{n}\log f_n^1(x) - \frac{1}{n} f_n^2(x)]$ is measurable. Since pointwise limits of measurable functions are measurable functions, the map $x \in X \mapsto \overline{P}_\mu((G,G_1),\psi,x)$ is measurable. □

Analogously, one shows that the map $x \in X \to \underline{P}_\mu((G,G_1),\psi,x)$, introduced in (2.27), is well defined and measurable.

4.3. **Vitaly covering lemmas.** A metric space (X, d) is called *boundedly compact* if all bounded closed subsets of X are compact (cf. [24, p.9]). In particular, compact metric spaces, Euclidean spaces $\mathbb{R}^n$ and Riemannian manifolds are boundedly compact. We denote the diameter of a set $A \subset X$ by $\operatorname{diam}(A)$. The following classical covering lemma for boundedly compact metric spaces (see e.g. [34, Theorem 2.1]) is essential on further sections.

Lemma 4.5. (Vitaly covering lemma) *Let X be a boundedly compact metric space and consider a set $\mathbb{B} \subset X \times \mathbb{R}_+$ such that*

$$\sup\big\{ diam(B(x,r))\colon (x,r) \in \mathbb{B}\big\} < +\infty.$$

Then there is a finite or countable subset $\widetilde{\mathbb{B}} \subset \mathbb{B}$ such that $\big\{\overline{B(x,r)}\colon (x,r) \in \widetilde{\mathbb{B}}\big\}$ is a pairwise disjoint collection of closed balls and

$$\bigcup_{(x,\,r)\,\in\,\mathbb{B}} \overline{B(x,\,r)} \;\subset\; \bigcup_{(x,\,r)\,\in\,\widetilde{\mathbb{B}}} \overline{B(x,\,5r)}.$$

A dynamically defined Vitaly covering lemma was established by Ma and Wen [30] in the case of topological dynamical systems, yielding an analogous of Lemma 4.5 where balls were replaced by Bowen dynamical balls. The argument to prove it extends easily to the dynamical balls defined in Subsection 2.5 for a finitely generated pseudogroup (G, G_1) acting on a compact metric space (X, d). Therefore:

Lemma 4.6. *Given $r > 0$, let $\mathcal{B}(r) = \big\{B_n(x, r)\colon x \in X, n \in \mathbb{N}\big\}$ be a collection of dynamical balls of radius r, determined by a finitely generated pseudogroup (G, G_1) acting on a compact metric space (X, d). For any family $\mathcal{F} \subset \mathcal{B}(r)$ there exists a subfamily $\mathcal{G} \subset \mathcal{F}$ by pairwise disjoint dynamical balls such that*

$$\bigcup_{B_n(x,r)\,\in\,\mathcal{F}} B_n(x,\,r) \;\subset\; \bigcup_{B_n(x,r)\,\in\,\mathcal{G}} B_n(x,\,3r).$$

4.4. **Spanning *vs.* separated.** Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d). Fix $\psi \in C^0(X)$. Recall that, in Subsection 2.3, we introduced the notion of topological pressure of (G, G_1) at ψ, defined by

$$P_{\text{top}}((G,G_1),\psi) \;=\; \lim_{\varepsilon\,\to\,0^+}\; \limsup_{n\,\to\,+\infty} \frac{1}{n}\log\left(\sup_{E_{n,\varepsilon}}\Big\{\sum_{x\,\in\,E_{n,\varepsilon}} e^{\frac{1}{|G_n^x|}\sum_{\underline{g}\,\in\,G_n^x} S_\psi^{\underline{g}}(x)}\Big\}\right)$$

where the supremum is taken over $(n, \varepsilon)-$separated sets $E_{n,\varepsilon}$ of X with respect to (G, G_1).

As done with the topological entropy, we could have used separated sets instead of spanning ones, defining another concept of topological pressure by

$$Q_{\text{top}}((G,G_1),\psi) \;=\; \lim_{\varepsilon\,\to\,0^+}\; \limsup_{n\,\to\,+\infty} \frac{1}{n}\log\left(\inf_{F_{n,\varepsilon}}\Big\{\sum_{x\,\in\,F_{n,\varepsilon}} e^{\frac{1}{|G_n^x|}\sum_{\underline{g}\,\in\,G_n^x} S_\psi^{\underline{g}}(x)}\Big\}\right)$$

where the infimum is taken over $(n, \varepsilon)-$spanning sets $F_{n,\varepsilon}$ of X with respect to (G, G_1). One can easily check that the functions

$$\varepsilon > 0 \quad\mapsto\quad \sup_{E_{n,\varepsilon}}\Big\{\sum_{x\,\in\,E_{n,\varepsilon}} \exp\big[P_n^\psi(x)\big]\colon\; E_{n,\varepsilon} \text{ is } (n,\varepsilon)-\textit{separated}\Big\}$$

$$\varepsilon > 0 \quad \mapsto \quad \inf_{F_{n,\varepsilon}} \Big\{ \sum_{x \,\in\, F_{n,\varepsilon}} \exp\left[P_n^{\psi}(x)\right] \colon \;\; F_{n,\varepsilon} \; \text{ is } (n,\varepsilon) - \textit{spanning} \Big\}$$

are monotone, hence their limits as ε goes to 0^+ exist. In this subsection we will compare these two definitions of pressure.

Lemma 4.7. *Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d). Then*

$$Q_{\mathrm{top}}((G, G_1), \psi) \;\leqslant\; P_{\mathrm{top}}((G, G_1), \psi) \qquad \forall\, \psi \in C^0(X).$$

Proof. Fix $\psi \in C^0(X)$. We start by observing that, in the previous definitions of $P_{\mathrm{top}}((G, G_1), \psi)$ and $Q_{\mathrm{top}}((G, G_1), \psi)$, it suffices to take, for the former, the supremum over those $(n, \varepsilon)-$separated sets which cannot be enlarged to a $(n, \varepsilon)-$separated set, and, for the latter, the infimum over those $(n, \varepsilon)-$spanning sets which do not have proper subsets that $(n, \varepsilon)-$span X.

To simplify the notation, for each $x \in X$ and $n \in \mathbb{N}$ we will write

$$P_n^{\psi}(x) \;=\; \frac{1}{|G_n^x|} \sum_{\underline{g} \,\in\, G_n^x} S_{\psi}^{\underline{g}}(x). \tag{4.8}$$

Since an $(n, \varepsilon)-$ separated set E of maximal cardinality is $(n, \varepsilon)-$spanning and the summands $\exp\left[P_n^{\psi}(x)\right]$ are positive, for every $\varepsilon > 0$ and $n \in \mathbb{N}$ one has

$$\inf_{F_{n,\varepsilon}} \Big\{ \sum_{x \,\in\, F_{n,\varepsilon}} \exp\left[P_n^{\psi}(x)\right] \Big\} \;\leqslant\; \sup_{E_{n,\varepsilon}} \Big\{ \sum_{x \,\in\, E_{n,\varepsilon}} \exp\left[P_n^{\psi}(x)\right] \Big\}.$$

Consequently, for every $\varepsilon > 0$,

$$\limsup_{n \to +\infty} \frac{1}{n} \log \left(\inf_{F_{n,\varepsilon}} \Big\{ \sum_{x \,\in\, F_{n,\varepsilon}} \exp\left[P_n^{\psi}(x)\right] \Big\} \right) \;\leqslant\; \limsup_{n \to +\infty} \frac{1}{n} \log \left(\sup_{E_{n,\varepsilon}} \Big\{ \sum_{x \,\in\, E_{n,\varepsilon}} \exp\left[P_n^{\psi}(x)\right] \Big\} \right)$$

and so, taking the limit as ε goes to 0^+, we get

$$Q_{\mathrm{top}}((G, G_1), \psi) \;\leqslant\; P_{\mathrm{top}}((G, G_1), \psi). \tag{4.9}$$

$\square$

We now address the converse inequality, which is harder to show. Next lemma proves it in the particular case of finitely generated groups (though its reasoning is also valid if (G, G_1) is a finitely generated semigroup).

Lemma 4.10. *Let (G, G_1) be a finitely generated group on a compact metric space (X, d). Then*

$$Q_{\mathrm{top}}((G, G_1), \psi) \;\geqslant\; P_{\mathrm{top}}((G, G_1), \psi) \qquad \forall\, \psi \in C^0(X).$$

Proof. The following proof is an adaptation for finitely generated groups of the argument on page 209 of [46], which concerns the topological pressure of a single map. The additional assumption that G is a group simplifies the definition of pressure, hence the next computations, since $G_n^x = G_n$ for every $x \in X$.

Given $\varepsilon > 0$ and $n \in \mathbb{N}$, let E be an $(n, \varepsilon)-$separated set with maximal cardinality and F be an $(n, \varepsilon/2)-$spanning set. Define the map $\xi \colon E \to F$ by choosing, for each $x \in E$, some point $\xi(x) \in F$ satisfying

$$\max\,\Big\{ d\big(g(x),\, g(\xi(x))\big) \colon \;\; g \in G_n \Big\} \;<\; \varepsilon/2$$

whose existence is guaranteed since F is $(n, \varepsilon/2)-$spanning. We claim that ξ is injective. Otherwise, if there are two distinct points x_1, $x_2 \in E$ for which $\xi(x_1) = \xi(x_2)$, then for any $g \in G_n$ one has

$$d\big(g(x_1),\, g(x_2)\big) \,\leqslant\, d\big(g(x_1),\, g(\xi(x_1))\big) \,+\, d\big(g(x_2),\, g(\xi(x_1))\big) \,<\, \varepsilon/2 + \varepsilon/2 \,=\, \varepsilon$$

and so

$$\max\,\big\{d\big(g(x_1),\, g(x_2)\big):\; g \in G_n\big\} \,<\, \varepsilon$$

contradicting the assumption that E is $(n, \varepsilon)-$separated.

Fix a potential $\psi \in C^0(X)$. As ψ is uniformly continuous, given $\delta > 0$ there is $0 < \varepsilon(\delta) < \delta$ such that, for every $0 < \varepsilon \leqslant \varepsilon(\delta)$,

$$d(x, y) \,<\, \varepsilon/2 \qquad \Rightarrow \qquad \big|\psi(x) - \psi(y)\big| \,<\, \delta.$$

Then, for every $x \in X$, $n \in \mathbb{N}$ and $\underline{g} = g_{i_n} \circ g_{i_{n-1}} \circ .... \circ g_1 \in G_n$,

$$\begin{aligned}
&\big|S_\psi^{\underline{g}}\,(x) \,-\, S_\psi^{\underline{g}}\,(\xi(x))\big| \,= \\
=\;\; &\big|\big[\psi(x) + \cdots + \psi(g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1}(x))\big] \,-\, \big[\psi(\xi(x)) + \cdots + \psi(g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1}(\xi(x)))\big]\big| \\
\leqslant\;\; &\big|\psi(x) - \psi(\xi(x))\big| + \cdots + \big|\psi(g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1}(x)) - \psi(g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1}(\xi(x)))\big| \\
<\;\; &(n+1)\delta.
\end{aligned}$$

Therefore,

$$\Big|\frac{1}{|G_n|}\,\sum_{\underline{g}\,\in\, G_n} S_\psi^{\underline{g}}\,(x) \,-\, \frac{1}{|G_n|}\,\sum_{\underline{g}\,\in\, G_n} S_\psi^{\underline{g}}\,(\xi(x))\Big| \,<\, (n+1)\delta$$

that is,

$$\big|P_n^\psi(x) \,-\, P_n^\psi(\xi(x))\big| \,<\, (n+1)\delta. \tag{4.11}$$

Now, by the injectivity of ξ, we have $\mathrm{card}(\xi(E)) \leqslant \mathrm{card}(F)$; hence

$$\sum_{y\,\in\, F} \exp\,[P_n^\psi(y)] \,\geqslant\, \sum_{x\,\in\, E} \exp\,[P_n^\psi(\xi(x))].$$

Moreover,

$$\begin{aligned}
\sum_{x\,\in\, E} \exp\,[P_n^\psi(\xi(x))] \;&=\; \sum_{x\,\in\, E} \exp[P_n^\psi(\xi(x)) \,-\, P_n^\psi(x)]\;\exp\,[P_n^\psi(x)] \\
&\geqslant\; \min\,\big\{\exp\,[P_n^\psi(\xi(x)) \,-\, P_n^\psi(x)]:\; x \in E\big\}\,\sum_{x\,\in\, E} \exp\,[P_n^\psi(x)] \\
&\geqslant\; \exp\,[-(n+1)\delta]\;\sum_{x\,\in\, E} \exp\,[P_n^\psi(x)]
\end{aligned}$$

where the last inequality is due to (4.11). Consequently,

$$\begin{aligned}
\sum_{y\,\in\, F} \exp\,[P_n^\psi(y)] \;&\geqslant\; \exp[-(n+1)\delta]\;\sum_{x\,\in\, E} \exp\,[P_n^\psi(x)] \\
\inf_F\,\sum_{y\,\in\, F} \exp\,[P_n^\psi(y)] \;&\geqslant\; \exp[-(n+1)\delta]\;\sum_{x\,\in\, E} \exp\,[P_n^\psi(x)] \\
\inf_F\,\sum_{y\,\in\, F} \exp\,[P_n^\psi(y)] \;&\geqslant\; \exp[-(n+1)\delta]\;\sup_E\;\sum_{x\,\in\, E} \exp\,[P_n^\psi(x)].
\end{aligned}$$

Taking logarithms of both sides and lim sup as n goes to $+\infty$, we get

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{n}\log\Big(\inf_F\sum_{y\in F}\exp\left[P_n^\psi(y)\right]\Big) &\geqslant \limsup_{n\to+\infty}\frac{1}{n}\log\Big(\exp[-(n+1)\delta]\,\sup_E\sum_{x\in E}\exp\left[P_n^\psi(x)\right]\Big)\\ &= -\delta+\limsup_{n\to+\infty}\frac{1}{n}\log\Big(\sup_E\sum_{x\in E}\exp\left[P_n^\psi(x)\right]\Big).\end{aligned}$$

Finally, letting $\varepsilon\to 0^+$ we obtain

$$Q_{\mathrm{top}}((G,G_1),\psi)\ \geqslant\ -\delta+P_{\mathrm{top}}((G,G_1),\psi).$$

Since $\delta>0$ is arbitrarily small, the proof of the lemma is complete. □

Bringing together Lemmas 4.7 and 4.10, we conclude that:

Proposition 4.12. *Let (G,G_1) be a finitely generated group on a compact metric space (X,d). Then*

$$Q_{\mathrm{top}}((G,G_1),\psi)\ =\ P_{\mathrm{top}}((G,G_1),\psi)\qquad\forall\,\psi\in C^0(X).$$

5. Carathéodory-Pesin structures

In this section we introduce a notion of pressure of a finitely generated pseudogroups using Carathéodory-Pesin structures (cf. [38, 39]). Carathéodory-Pesin structures were somehow inspired by Bowen's definition of the topological entropy of a continuous map in a way entirely similar to the definition of Hausdorff measure and dimension. More precisely, in [38] Pesin elaborated over the theory of the so called Carathéodory structures and proved that a continuous endomorphism of a compact metric space dynamically defines a Carathéodory structure whose upper capacity coincides with the topological entropy of the map. This approach has then been successfully applied to many other classes of dynamical systems and group actions, using a strategy very similar to Pesin's but often skipping the details, and thus raising many doubts on the mathematical validity of those generalizations. Therefore, for the sake of completeness and elucidation, we proceed with a self-contained and comprehensive verification of all axioms that define a Carathéodory-Pesin structure.

Consider a finitely generated pseudogroup (G,G_1) generated by a finite set G_1 of local homeomorphisms of a compact metric space (X,d). Fix an arbitrary subset $Z\subset X$. Given $N\in\mathbb{N}$ and $\varepsilon>0$ denote by $I_N(\varepsilon)\subset Z\times\{n\in\mathbb{N}\colon n\geqslant N\}$ a finite or countable set such that Z is covered by dynamical balls $B_{n_j}(x_j,\varepsilon)$, where $(x_j,n_j)\in I_N(\varepsilon)$ and $n_j\geqslant N$. In other words,

$$Z\subset\bigcup_{(x_j,n_j)\,\in\, I_N(\varepsilon)}B_{n_j}(x_j,\varepsilon).$$

Denote by $C_Z(N,\varepsilon)$ the family of all such subsets $I_N(\varepsilon)$. For notational simplicity, we will write $j\in I_N(\varepsilon)$ to identify the pair $(x_j,n_j)\in I_N(\varepsilon)$.

Definition 5.1. For a subset $Z\subset X$, a continuous map $\psi\colon X\to\mathbb{R}$, a positive integer $N\in\mathbb{N}$, $s\geqslant 0$ and $\varepsilon>0$, define

$$M_Z(s,\varepsilon,\psi,N)=\inf\Bigg\{\sum_{j\,\in\, I_N(\varepsilon)}\exp\Big[-s\,n_j+\frac{1}{|G_{n_j}^{x_j}|}\sum_{\underline{g}\,\in\, G_{n_j}^{x_j}}S_\psi^{\underline{g}}\,(x_j)\Big]\colon\ I_N(\varepsilon)\in C_Z(N,\varepsilon)\Bigg\}$$

where $\underline{g}=g_{i_{n_j}}\circ\cdots\circ g_{i_2}\circ g_{i_1}$ and $|G_{n_j}^{x_j}|$ stands for the cardinality of the set $G_{n_j}^{x_j}$.

Lemma 5.2. *Given $s \geqslant 0$ and $\varepsilon > 0$, the limit*

$$M_Z(s,\varepsilon,\psi) \;=\; \lim_{N \to +\infty} M_Z(s,\varepsilon,\psi,N)$$

is well defined and satisfies

$$M_Z(s,\varepsilon,\psi) \;=\; \sup\big\{M_Z(s,\varepsilon,\psi,N)\colon\; N \in \mathbb{N}\big\}.$$

Proof. If $I_{N+1}(\varepsilon) \in C_Z(N+1,\varepsilon)$, then it clearly satisfies $I_{N+1}(\varepsilon) \in C_Z(N,\varepsilon)$. Thus

$$\begin{aligned}
M_Z(s,\varepsilon,\psi,N+1) \;&=\; \inf_{I_{N+1}(\varepsilon)\,\in\, C_Z(N+1,\varepsilon)} \left\{ \sum_{j\,\in\, I_{N+1}(\varepsilon)} \exp\Big[-s\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j) \right\} \\
&\geqslant\; \inf_{I_{N+1}(\varepsilon)\,\in\, C_Z(N,\varepsilon)} \left\{ \sum_{j\,\in\, I_{N+1}(\varepsilon)} \exp\Big[-s\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big] \right\} \\
&\geqslant\; \inf_{I_{N}(\varepsilon)\,\in\, C_Z(N,\varepsilon)} \left\{ \sum_{j\,\in\, I_{N}(\varepsilon)} \exp\Big[-s\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big] \right\} \\
&=\; M_Z(s,\varepsilon,\psi,N).
\end{aligned}$$

This shows that the sequence $(M_Z(s,\varepsilon,\psi,N))_{N\,\in\,\mathbb{N}}$ is non-decreasing and proves the lemma. $\square$

We claim that the function $s \to M_Z(s,\varepsilon,\psi)$ behaves like an $s-$Hausdorff measure: there exists a unique critical parameter where it drops from infinity to zero. This is a consequence of the following result.

Lemma 5.3. *Given $s < t$, then*

$$\begin{aligned}
M_Z(s,\varepsilon,\psi) < +\infty \qquad &\Rightarrow \qquad M_Z(t,\varepsilon,\psi) = 0 \\
M_Z(t,\varepsilon,\psi) > 0 \qquad &\Rightarrow \qquad M_Z(s,\varepsilon,\psi) = +\infty.
\end{aligned}$$

Proof. Fix $t > s$ and assume that $M_Z(s,\varepsilon,\psi) < +\infty$. Notice that, for each family $I_N(\varepsilon)$,

$$\begin{aligned}
&\sum_{j\,\in\, I_N(\varepsilon)} \exp\Big[-t\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big] \\
&= \sum_{j\,\in\, I_N(\varepsilon)} \exp\Big[-(t-s)\,n_j - s\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big] \\
&\leqslant \exp[-N\,(t-s)] \sum_{j\,\in\, I_N(\varepsilon)} \exp\Big[-s\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big].
\end{aligned}$$

Therefore, taking the infimum, one gets the inequalities

$$M_Z(t,\varepsilon,\psi,N) \;\leqslant\; \exp[-N\,(t-s)]\, M_Z(s,\varepsilon,\psi,N) \;\leqslant\; \exp[-N\,(t-s)]\, M_Z(s,\varepsilon,\psi)$$

which, letting N go to $+\infty$, yield $M_Z(t,\varepsilon,\psi) = 0$. This proves item (a) of Lemma 5.3.

We proceed to prove item (b). Assume that $M_Z(t,\varepsilon,\psi)>0$. We start by noticing that

$$\begin{aligned}
&\sum_{j\,\in\, I_N(\varepsilon)} \exp\Big[-s\,n_j+\frac{1}{|G^{x_j}_{n_j}|}\sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big]\\
&\geqslant \exp[N\,(t-s)] \sum_{j\,\in\, I_N(\varepsilon)} \exp\Big[-t\,n_j+\frac{1}{|G^{x_j}_{n_j}|}\sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big]\\
&\geqslant \exp[N\,(t-s)] \inf_{I_N(\varepsilon)\,\in\, C_Z(N,\varepsilon)} \sum_{j\,\in\, I_N(\varepsilon)} \exp\Big[-t\,n_j+\frac{1}{|G^{x_j}_{n_j}|}\sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big].
\end{aligned}$$

Taking the infimum with respect to $I_N(\varepsilon)\in C_Z(N,\varepsilon)$ and then letting $N\to+\infty$, we obtain

$$M_Z(s,\varepsilon,\psi,N)\;\geqslant\;\exp[N\,(t-s)]\,M_Z(t,\varepsilon,\psi,N)$$

which guarantees that

$$M_Z(s,\varepsilon,\psi)\;\geqslant\;\lim_{N\,\to\,+\infty}\exp[N\,(t-s)]\,M_Z(t,\varepsilon,\psi)\;=\;+\infty.$$

This completes the proof of the lemma. □

Lemma 5.3 indicates that the parameter $M_Z(\varepsilon,\psi)$ given by

$$M_Z(\varepsilon,\psi)\;=\;\sup\big\{s\geqslant 0\colon\;M_Z(s,\varepsilon,\psi)=+\infty\big\}\;=\;\inf\big\{s\geqslant 0\colon\;M_Z(s,\varepsilon,\psi)=0\big\}\tag{5.4}$$

is well defined. Moreover:

Lemma 5.5. *The limit* $\lim_{\varepsilon\,\to\,0^+}\,M_Z(\varepsilon,\psi)$ *exists.*

Proof. Fix a positive integer N, $Z\subset X$ and $0<\varepsilon_1\leqslant\varepsilon_2$. Afterwards, choose a covering $\{B_{n_j}(x_j,\varepsilon_1)\}_{j\,\in\, I}$ of the set Z with $n_j\geqslant N$ for all $j\in I$. As $0<\varepsilon_1\leqslant\varepsilon_2$, we have

$$Z\;\subset\;\bigcup_{j\,\in\, I} B_{n_j}(x_j,\varepsilon_1)\;\subset\;\bigcup_{j\,\in\, I} B_{n_j}(x_j,\varepsilon_2)$$

hence there exists a subset $I_2\subset I$ such that

$$Z\;\subset\;\bigcup_{j\,\in\, I_2} B_{n_j}(x_j,\varepsilon_2).$$

Consequently,

$$\sum_{j\,\in\, I}\exp\Big[-s\,n_j+\frac{1}{|G^{x_j}_{n_j}|}\sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big]\;\geqslant\;\sum_{j\,\in\, I_2}\exp\Big[-s\,n_j+\frac{1}{|G^{x_j}_{n_j}|}\sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big]$$

from which we deduce that

$$M_Z(s,\varepsilon_1,\psi)\;\geqslant\;M_Z(s,\varepsilon_2,\psi).$$

The last inequality implies that

$$M_Z(\varepsilon_1,\psi)\;=\;\inf\{s\geqslant 0\colon\;M_Z(s,\varepsilon_1,\psi)=0\}\;\geqslant\;\inf\{s\geqslant 0\colon\;M_Z(s,\varepsilon_2,\psi)=0\}\;=\;M_Z(\varepsilon_2,\psi).$$

Thus, the function $\varepsilon\mapsto M_Z(\varepsilon,\psi)$ is non-increasing, and so the limit $\lim_{\varepsilon\,\to\,0^+}M_Z(\varepsilon,\psi)$ does exist. □

The previous lemma motivates the following definition.

Definition 5.6. Given $\psi \in C^0(X)$ and $Z \subset X$, the *CP-pressure of a finitely generated pseudogroup (G, G_1), when restricted to Z, with respect to the potential ψ*, is the limit

$$P_Z((G,G_1),\psi) \,=\, \lim_{\varepsilon \,\to\, 0^+} \, M_Z(\varepsilon,\psi).$$

It is not hard to check that this pressure function has the following properties (details in [38]).

Lemma 5.7. *Consider a continuous potential $\psi\colon X \to \mathbb{R}$ and sets $Z_1, Z_2 \subset X$.*

(a) If $Z_1 \subset Z_2$, then $P_{Z_1}((G,G_1),\psi) \leqslant P_{Z_2}((G,G_1),\psi)$.

(b) If $Z = \bigcup_{k \,\in\, \mathbb{N}} Z_k$, then $P_Z((G,G_1),\psi) \,=\, \sup\big\{P_{Z_k}((G,G_1),\psi)\colon\, k \in \mathbb{N}\big\}$.

6. A partial variational principle

In this section we adapt the argument in [30, Theorem 1] to finitely generated pseudogroups. Analogous properties were obtained for the topological entropy of finitely generated semigroups and for non-autonomous dynamical systems in [4, 5], respectively, and for the measure-theoretic pressure of finitely generated free semigroup actions in [43].

Theorem 6.1. *Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d) and Z be a Borel subset of X. For every $\mu \in \mathcal{M}_1(X)$, $s \geqslant 0$ and $\psi \in C^0(X)$, one has:*

(a) If $\overline{P}_\mu((G,G_1),\psi,x) \leqslant s$ for every $x \in Z$, then $P_Z((G,G_1),\psi) \leqslant s$.

(b) If $\mu(Z) > 0$ and $\underline{P}_\mu((G,G_1),\psi,x) \geqslant s$ for every $x \in Z$, then $P_Z((G,G_1),\psi) \geqslant s$.

Proof. Let (G, G_1) be a finitely generated pseudogroup, Z be a Borel subset of X and μ be a Borel probability measure on X.

(a) Assume that there is $s \geqslant 0$ such that $\overline{P}_\mu((G,G_1),\psi,x) \leqslant s$ for every $x \in Z$. Given $\varepsilon > 0$ and $k \in \mathbb{N}$, consider the set

$$Z_k^\varepsilon \,=\, \Big\{x \in Z\colon\, \limsup_{n \,\to\, +\infty} -\frac{1}{n}\Big[\log\, \mu(B_n(x,r)) - \frac{1}{|G_n^x|} \sum_{\underline{g} \,\in\, G_n^x} S_\psi^{\underline{g}}\,(x)\Big] \leqslant s+\varepsilon,\ \forall r \in \,]0, 1/k[\Big\}.$$

Thus,

$$Z \,=\, \bigcup_{k \,\in\, \mathbb{N}} Z_k^\varepsilon.$$

Now fix k and $r \in \,]0, 1/3k[$. Notice that, by the definition of Z_k^ε, for any $x \in Z_k^\varepsilon$ there exists a strictly increasing sequence $(n_j(x))_{j \,\in\, \mathbb{N}}$ satisfying

$$\log\, \mu(B_{n_j(x)}(x,r)) - \frac{1}{|G_{n_j(x)}^x|} \sum_{\underline{g} \,\in\, G_{n_j(x)}^x} S_\psi^{\underline{g}}\,(x) \geqslant -(s+\varepsilon)\, n_j(x). \tag{6.2}$$

Moreover, for any $N \in \mathbb{N}$, the set Z_k^ε is contained in the union of the elements of the family

$$\mathcal{F} \,=\, \big\{B_{n_j(x)}(x,r)\colon x \in Z_k^\varepsilon \quad \text{and} \quad n_j(x) \geqslant N\big\}.$$

Combining equation (6.2) with Lemma 4.6, we find a (at most countable) subfamily by pairwise disjoint dynamical balls, say $\mathcal{G} \,=\, \big\{B_{n_j}(x_j, r)\big\}_{j \,\in\, J} \subset \mathcal{F}$, such that

$$Z_k \,\subset\, \bigcup_{j \,\in\, J} B_{n_j}(x_j, 3\,r)$$

and

$$\mu\big(B_{n_j}(x_j,r)\big) \,\geqslant\, \exp\Big[-(s+\varepsilon)\,n_j + \frac{1}{|G^{x_j}_{n_j}|}\sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big].$$

Therefore,

$$\begin{aligned} M_{Z^\varepsilon_k}(s+\varepsilon,3\,r,\psi,N) \,&\leqslant\, \sum_{j\,\in\, J}\exp\Big[-(s+\varepsilon)\,n_j + \frac{1}{|G^{x_j}_{n_j}|}\sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big] \\ &\leqslant\, \sum_{j\,\in\, J}\mu\big(B_{n_j}(x_j,r)\big) \,\leqslant\, 1 \end{aligned}$$

where the last inequality is due to the disjointness of the elements in $\mathcal{G}$. Taking the limit as $N\to+\infty$, we get

$$M_{Z^\varepsilon_k}(s+\varepsilon,3r,\psi) \,\leqslant\, 1$$

so, by (5.4), we conclude that

$$M_{Z^\varepsilon_k}(3r,\psi) \,\leqslant\, s+\varepsilon \qquad \forall r\in\,]0,1/3k[.$$

Letting $r\,\to\,0^+$, we obtain

$$P_{Z^\varepsilon_k}((G,G_1),\psi) \,\leqslant\, s+\varepsilon.$$

Thus, by Lemma 5.7,

$$P_Z((G,G_1),\psi) \,=\, \sup\big\{P_{Z^\varepsilon_k}((G,G_1),\psi)\colon\; k\in\mathbb{N}\big\} \,\leqslant\, s+\varepsilon.$$

As $\varepsilon>0$ may be chosen arbitrary, the proof of item (a) of Theorem 6.1 is complete.

(b) Assume now that $\mu(Z)>0$ and there is $s\geqslant 0$ such that $\underline{P}_\mu((G,G_1),\psi,x)\geqslant s$ for any $x\in Z$. Fix $\varepsilon>0$ and define the sequence of sets $(E^\varepsilon_k)_{k\,\in\,\mathbb{N}}$ by

$$E^\varepsilon_k \,=\, \Big\{x\in Z\colon\; \liminf_{n\,\to\,+\infty} -\frac{1}{n}\Big[\log\,\mu(B_n(x,r)) - \frac{1}{|G^x_n|}\sum_{\underline{g}\,\in\, G^x_n} S^{\underline{g}}_{\psi}\,(x)\Big] > s-\varepsilon,\;\forall r\in\,]0,1/k[\Big\}.$$

By the assumption, the sequence $(E^\varepsilon_k)_{k\,\in\,\mathbb{N}}$ increases to Z, so, by the continuity of μ,

$$\lim_{k\,\to\,+\infty}\,\mu(E^\varepsilon_k) \,=\, \mu(Z)>0.$$

Thus, there exists $k_0\,\in\,\mathbb{N}$ such that $\mu(E^\varepsilon_{k_0})\,>\,\frac{1}{2}\mu(Z)\,>\,0$. Take the sequence of subsets $(E^\varepsilon_{k_0,N})_{N\,\in\,\mathbb{N}}$ defined by

$$E^\varepsilon_{k_0,N} \,=\, \Big\{x\in Z\colon\; -\frac{1}{n}\Big[\log\,\mu(B_n(x,r)) - \frac{1}{|G^x_n|}\sum_{\underline{g}\,\in\, G^x_n} S^{\underline{g}}_{\psi}\,(x)\Big] > s-\varepsilon,\;\forall\, n\geqslant N,\,\forall\, r\in\,]0,1/k_0[\Big\}.$$

The sequence $(E^\varepsilon_{k_0,N})_{N\,\in\,\mathbb{N}}$ increases to $E^\varepsilon_{k_0}$ and so, using once more the continuity of the measure μ, we can choose a positive integer N^* such that

$$\mu(E^\varepsilon_{k_0,N^*}) \,>\, \frac{1}{2}\mu(E^\varepsilon_{k_0}) \,>\, 0.$$

Therefore, for any $x_j\in E^\varepsilon_{k_0,N^*}$, every $n_j\geqslant N^*$ and all $r\in\,]1,1/k_0[$, one has

$$\log\,\mu\big(B_{n_j}(x_j,r)\big) - \frac{1}{|G^{x_j}_{n_j}|}\sum_{\underline{g}\,\in\, G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j) < -(s-\varepsilon)\,n_j$$

or, equivalently,

$$\mu\big(B_{n_j}(x_j,r)\big) \,<\, \exp\Big[-(s-\varepsilon)\,n_j + \frac{1}{|G_{n_j}^{x_j}|}\sum_{\underline{g}\,\in\,G_{n_j}^{x_j}} S_{\psi}^{\underline{g}}\,(x_j)\Big]. \tag{6.3}$$

Now consider the countable covering $\mathcal{H}$ of $E^{\varepsilon}_{k_0,N^*}$ defined by

$$\mathcal{H} \,=\, \Big\{B_{n_j}(y_j,r/2)\colon\; y_j\in E^{\varepsilon}_{k_0,N^*},\; n_j\geqslant N^*,\; r\in\,]0,1/k_0[\; \text{ and } \; B_{n_j}(y_j,r/2)\cap E^{\varepsilon}_{k_0,N^*}\neq\emptyset\Big\}_{j\,\in\,J}.$$

For any $j\in J$, one can choose $x_j\in B_{n_j}(y_j,r/2)\cap E_{k_0,N^*}$ such that $B_{n_j}(y_j,r/2)\subset B_{n_j}(x_j,r)$ and, by (6.3),

$$\mu\big(B_{n_j}(y_j,r/2)\big) \,\leqslant\, \mu\big(B_{n_j}(x_j,r)\big) \,<\, \exp\Big[-(s-\varepsilon)\,n_j + \frac{1}{|G_{n_j}^{x_j}|}\sum_{\underline{g}\,\in\,G_{n_j}^{x_j}} S_{\psi}^{\underline{g}}\,(x_j)\Big].$$

Consequently,

$$\begin{aligned} M_Z(s-\varepsilon,r/2,\psi,N) \,&\geqslant\, M_{E^{\varepsilon}_{k_0,N^*}}(s-\varepsilon,r/2,\psi,N)\\ &\geqslant\, \sum_{j\,\in\,J}\exp\Big[-(s-\varepsilon)\,n_j + \frac{1}{|G_{n_j}^{x_j}|}\sum_{\underline{g}\,\in\,G_{n_j}^{x_j}} S_{\psi}^{\underline{g}}\,(x_j)\Big]\\ &\geqslant\, \sum_{j\,\in\,J}\mu\big(B_{n_j}(y_j,r/2)\big)\\ &\geqslant\, \mu(E^{\varepsilon}_{k_0,N^*}) \,>\, 0. \end{aligned}$$

Taking the limit as $N\to+\infty$, we conclude that $M_Z(s-\varepsilon,r/2,\psi)>0$, which yields

$$M_Z(r/2,\psi) \,>\, s-\varepsilon.$$

As $\varepsilon>0$ is arbitrary, taking the limit as $r\to 0^+$ we obtain $P_Z((G,G_1),\psi)>s$. This proves item (b) of Theorem 6.1. ☐

7. A variational principle

The first part of Theorem A is a consequence of the variational principle that Theorem B establishes, besides the connection between the topological pressure and the CP-pressure of a pseudogroup action which we will show in the next section.

Theorem 7.1. *Let (G,G_1) be a finitely generated pseudogroup acting on a compact metric space (X,d). Then:*

(a) *For every $\psi\in C^0(X)$ and any Borel subset Z of X,*

$$P_Z((G,G_1),\psi) \,=\, \sup\big\{\underline{P}_\mu((G,G_1),\psi)\colon\; \mu\in\mathcal{M}_1(X) \text{ and } \mu(Z)=1\big\}.$$

(b) *For every $\psi\in C^0(X)$,*

$$P_{\text{top}}((G,G_1),\psi) \,\geqslant\, P_X((G,G_1),\psi).$$

To prove item (a) of Theorem 7.1, we start by introducing the weighted topological pressure for finitely generated pseudogroups, which yields an alternative formulation of the CP-pressure. This strategy has similarities with the one of [43, Theorem 1.2], in the sense that it builds over a modification of a very elegant argument by Feng and Huang (cf. [20, Lemma 4.3]). The proof of item (b) of Theorem 7.1 will be postponed to Section 8, after recalling the dynamically defined Carathéodory-Pesin structures and the notion of capacity pressure.

7.1. **Weighted topological pressure.** Fix a subset $Z \subset X$ and a pseudogroup (G, G_1) of local homeomorphisms of a compact metric space (X, d), generated by a finite set G_1. Let N be a natural number and $\varepsilon > 0$. Denote by $I_N(\varepsilon) \subset Z \times \mathbb{N}$ a finite or countable set indexing a covering of Z by dynamical balls $B_{n_j}(x_j, \varepsilon)$ such that $x_j \in Z$ and $n_j \geqslant N$ for every j. Denote by $C_Z(N, \varepsilon)$ the family of all coverings $I_N(\varepsilon)$ of the set Z. Unless it leads to misunderstandings, we will write I instead of $I_N(\varepsilon)$.

For a set $A \subset X$ denote by $\chi_A \colon X \to [0,1]$ the map defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. Consider the sets

$$\mathcal{F}_{N,\varepsilon} \, = \, \Big\{ \big(c_j, B_{n_j}(x_j, \varepsilon)\big) \colon \; c_j \in \mathbb{R} \;\text{ and }\; B_{n_j}(x_j, \varepsilon) \in I_N(\varepsilon) \Big\}$$

and let $\mathcal{G}_{N,\varepsilon}$ be the family of all sets $\mathcal{F}_{N,\varepsilon}$.

Definition 7.2. For a subset $Z \subset X$, a continuous map $\psi \colon X \to \mathbb{R}$, $N \in \mathbb{N}$, $s \geqslant 0$ and $\varepsilon > 0$,

$$W_Z(s, \varepsilon, \psi, N) \, = \, \inf \Big\{ \sum_{j \,\in\, I} c_j \, \exp\Big[- s\, n_j + \frac{1}{|G_{n_j}^{x_j}|} \sum_{\underline{g} \,\in\, G_{n_j}^{x_j}} S_{\psi}^{\underline{g}}\,(x_j) \Big] \colon \; \chi_Z \leqslant \sum_{j \,\in\, I} c_j \, \chi_{B_{n_j}(x_j,\varepsilon)} \Big\}$$

where the infimum is taken over all finite or countable families $\mathcal{F}_{N,\varepsilon}$ of $\mathcal{G}_{N,\varepsilon}$.

Lemma 7.3. *The limit*

$$W_Z(s, \varepsilon, \psi) \, = \, \lim_{N \,\to\, +\infty} W_Z(s, \varepsilon, \psi, N)$$

exists and

$$\lim_{N \,\to\, +\infty} W_Z(s, \varepsilon, \psi, N) \, = \, \sup_{N \,\in\, \mathbb{N}} W_Z(s, \varepsilon, \psi, N).$$

Proof. The proof is similar to the one of Lemma 5.2, so we shall omit it. □

Lemma 7.4. *Given $s < t$, then*

$$\begin{aligned} W_Z(s, \varepsilon, \psi) < +\infty \qquad &\Rightarrow \qquad W_Z(t, \varepsilon, \psi) = 0 \\ W_Z(t, \varepsilon, \psi) > 0 \qquad &\Rightarrow \qquad W_Z(s, \varepsilon, \psi) = +\infty. \end{aligned}$$

Proof. The proof is similar to the one of Lemma 5.3, so we shall omit it. □

One can now define the weighted $\varepsilon-$pressure of the pseudogroup (G, G_1) with respect to the potential ψ on the subset Z as the unique critical point of the function

$$s \geqslant 0 \quad \mapsto \quad W_Z(s, \varepsilon, \psi)$$

thus generalizing the concept of weighted topological entropy introduced in [20].

Definition 7.5. The *weighted topological pressure of* (G, G_1) *restricted to* $Z \subset X$*, with respect to the potential* $\psi \in C^0(X)$ is given by

$$W_Z((G, G_1), \psi) \, = \, \lim_{\varepsilon \,\to\, 0^+} W_Z(\varepsilon, \psi)$$

where

$$W_Z(\varepsilon, \psi) \, = \, \sup \big\{ s \geqslant 0 \colon \; W_Z(s, \varepsilon, \psi) = +\infty \big\} \, = \, \inf \big\{ s \geqslant 0 \colon \; W_Z(s, \varepsilon, \psi) = 0 \big\}.$$

We proceed by showing that, for finitely generated pseudogroup actions, the weighted topological pressure coincides with the CP-pressure.

Proposition 7.6. *For any finitely generated pseudogroup (G, G_1), acting on a compact metric space (X, d), any subset $Z \subset X$ and any potential $\psi \in C^0(X)$, one has*

$$W_Z((G, G_1), \psi) \,=\, P_Z((G, G_1), \psi).$$

This proposition is a direct consequence of the following uniform estimates between weighted pressure and the CP-pressure at small scales.

Lemma 7.7. *Consider a finitely generated pseudogroup (G, G_1) acting on a compact metric space (X, d), a subset $Z \subset X$ and $\psi \in C^0(X)$. For every $s \geqslant 0$, $\varepsilon > 0$ and $\delta > 0$, there exists $N_0 \in \mathbb{N}$ such that*

$$M_Z(s+\delta, 4\varepsilon, \psi, N) \,\leqslant\, W_Z(s, \varepsilon, \psi, N) \,\leqslant\, M_Z(s, \varepsilon, \psi, N) \quad \forall\, N \geqslant N_0.$$

Proof. Our reasoning is similar to the proofs of Proposition 3.2 in [20] and Proposition 4.1 in [43] (which also follows closely the former).

We start by showing the first inequality. This part of the argument is an adaptation of the proof of Lemma 8.16 in [34], where the relation between Hausdorff dimension and Hausdorff weighted dimension is established.

Fix a subset $Z \subset X$, $\psi \in C^0(X)$, $s \geqslant 0$, $\varepsilon > 0$ and $\delta > 0$. Select $N_0 \in \mathbb{N}$ such that $N_0 \geqslant 2$ and $n^2 \exp[-n\,\delta] \leqslant 1$ for every $n \geqslant N_0$. Choose a family

$$\big\{(c_j, B_{n_j}(x_j, \varepsilon))\big\}_{j \in I}$$

where I is countable, $x_j \in X$, $c_j \in \,]0, +\infty[$ and $n_j \geqslant N_0$. Take $t > 0$ and, for each $k, n \in \mathbb{N}$, consider the sets

$$\begin{aligned}
I_n &= \big\{j \in \mathbb{N} \colon \; n_j = n\big\} \\
I_{n,k} &= \big\{j \in I_n \colon \; j \leqslant k\big\} \\
Z_{n,t} &= \Big\{x \in Z \colon \; \sum_{j \in I_n} c_j \, \chi_{B_n(x_j,\varepsilon)} > t\Big\} \\
Z_{n,k,t} &= \Big\{x \in Z \colon \; \sum_{j \in I_{n,k}} c_j \, \chi_{B_n(x_j,\varepsilon)} > t\Big\}.
\end{aligned}$$

Since $I_{n,k}$ is a finite set, by approximating the c_j's from above we may assume that each $c_j \in \mathbb{N}$. Let m be the least positive integer with $m > t$. Given $k \in \mathbb{N}$, choose a family of balls

$$\mathcal{B} \,=\, \big\{B_n(x_j, \varepsilon) \colon \; j \in I_{n,k}\big\}.$$

By induction, we define $m+1$ functions

$$v_0, \, v_1, \, ..., \, v_m \colon \, \mathcal{B} \quad \to \quad \mathbb{Z}$$

and m subfamilies $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_m$ of $\mathcal{B}$ in the following way:

(i) For every $B_n(x_j, \varepsilon) \in \mathcal{B}$, let

$$v_0(B_n(x_j, \varepsilon)) \,=\, c_j$$

and define the subfamily $\mathcal{B}_1$ as the union of pairwise disjoint dynamical balls satisfying the condition

$$\bigcup_{B_n(x_j,\varepsilon)\,\in\,\mathcal{B}} B_n(x_j,\varepsilon) \,\subset\, \bigcup_{B_n(x_j,\varepsilon)\,\in\,\mathcal{B}_1} B_n(x_j,3\varepsilon)$$

a valid step due to Lemma 4.6.

(2i) Proceed recursively, defining, for $j = 1, 2, ..., m$, pairwise disjoint subfamilies $\mathcal{B}_j$ of $\mathcal{B}$ such that

$$\begin{aligned} \mathcal{B}_j \;&\subset\; \big\{B_n(x_j,\varepsilon)\in\mathcal{B}\colon\; v_{j-1}(B_n(x_j,\varepsilon)) \geqslant 1\big\} \\ Z_{n,k,t} \;&\subset\; \bigcup_{B_n(x_j,\varepsilon)\,\in\,\mathcal{B}_j} B_n(x_j,3\varepsilon), \end{aligned}$$

as well as a finite sequence of functions v_j such that

$$v_j(B_n(x_j,\varepsilon)) \;=\; \begin{cases} v_{j-1}(B_n(x_j,\varepsilon)) - 1, & \text{if } B_n(x_j,\varepsilon)\in\mathcal{B}_j \\ v_{j-1}(B_n(x_j,\varepsilon)), & \text{if } B_n(x_j,\varepsilon)\in\mathcal{B}\setminus\mathcal{B}_j. \end{cases}$$

Notice that, for $j < m$, we have

$$Z_{n,k,t} \,\subset\, \Big\{x_j\colon \sum_{B_n(x_j,\varepsilon)\,\in\,\mathcal{B}} v_j(B_n(x_j,\varepsilon)) \geqslant m - j\Big\}$$

and so every point $x \in Z_{n,k,t}$ belongs to some ball $B_n(x,\varepsilon)\in\mathcal{B}$ with $v_j(B_n(x,\varepsilon)) \geqslant 1$. In particular,

$$\begin{aligned} &\sum_{j=1}^{m} |\mathcal{B}| \,\exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|}\sum_{\underline{g}\,\in\, G_n^{x_j}} S_\psi^{\underline{g}}\,(x_j)\Big] \\ &= \sum_{j=1}^{m}\;\sum_{B_n(x_j,\varepsilon)\,\in\,\mathcal{B}_j} [v_{j-1}(B_n(x_j,\varepsilon)) - v_j(B_n(x_j,\varepsilon))]\,\exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|}\sum_{\underline{g}\,\in\, G_n^{x_j}} S_\psi^{\underline{g}}\,(x_j)\Big] \\ &\leqslant \sum_{B_n(x_j,\varepsilon)\,\in\,\mathcal{B}_j}\;\sum_{j=1}^{m} [v_{j-1}(B_n(x_j,\varepsilon)) - v_j(B_n(x_j,\varepsilon))]\,\exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|}\sum_{\underline{g}\,\in\, G_n^{x_j}} S_\psi^{\underline{g}}\,(x_j)\Big] \\ &\leqslant \sum_{B_n(x_j,\varepsilon)\,\in\,\mathcal{B}} v_0(B_n(x_j,\varepsilon))\,\exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|}\sum_{\underline{g}\,\in\, G_n^{x_j}} S_\psi^{\underline{g}}\,(x_j)\Big] \\ &= \sum_{j\,\in\, I_{n,k}} c_j\,\exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|}\sum_{\underline{g}\,\in\, G_n^{x_j}} S_\psi^{\underline{g}}\,(x_j)\Big]. \end{aligned}$$

Select now $j_0 \in \{1,2,...,m\}$ such that $|\mathcal{B}_{j_0}| = \min\{|\mathcal{B}_j| \colon j \in \{1,2,...,m\}\}$. Then

$$\begin{aligned} & |\mathcal{B}_{j_0}| \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g} \in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \\ \leqslant\ & \frac{1}{m} \sum_{j \in I_{n,k}} c_j \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g} \in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \\ \leqslant\ & \frac{1}{t} \sum_{j \in I_{n,k}} c_j \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g} \in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big]. \end{aligned}$$

Let

$$J_{n,k,t} \;=\; \big\{j \in I \colon\; B_n(x_j,\varepsilon) \in \mathcal{B}_{j_0}\big\}.$$

Since $Z_{n,k,t}$ is a nested sequence of sets convergent to $Z_{n,t}$ as $k \to +\infty$, there exists $k_0 \geqslant 1$ such that $Z_{n,k,t} \neq \emptyset$ for every $k > k_0$. Define

$$E_{n,k,t} \;=\; \big\{x_j \colon\; j \in J_{n,k,t}\big\}.$$

From the sequence of compact sets $\{E_{n,k,t}\}_{k \in \mathbb{N}}$ in the compact metric space X one can choose a subsequence $\{E_{n,k_\ell,t}\}_{\ell \in \mathbb{N}}$ converging in the Hausdorff distance to some compact set $E_{n,t} \subset X$ as ℓ goes to $+\infty$. Since, for every $\ell \in \mathbb{N}$, any two points in $E_{n,k_\ell,t}$ are $(n,\varepsilon)-$separated, so do the points in $E_{n,t}$. Thus, $E_{n,t}$ is a finite set. Additionally, $E_{n,k_\ell,t} = E_{n,t}$ for large ℓ. Therefore, for large enough ℓ one has

$$Z_{n,k_\ell,t} \;\subset \bigcup_{j \in J_{n,k_\ell,t}} B_n(x_j, 3\varepsilon) \;=\; \bigcup_{x_j \in E_{n,k_\ell,t}} B_n(x_j, 3\varepsilon) \;\subset\; \bigcup_{x_j \in E_{n,t}} B_n(x_j, 7\varepsilon/2).$$

Thus, for large enough ℓ,

$$Z_{n,k_\ell,t} \;\subset \bigcup_{x_j \in E_{n,t}} B_n(x_j, 4\varepsilon).$$

Note that the equality $E_{n,k_\ell,t} = E_{n,t}$, which holds for large ℓ, yields

$$|E_{n,t}| \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g} \in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \;\leqslant\; \frac{1}{t} \sum_{j \in I_n} c_j \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g} \in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big]$$

and

$$\begin{aligned} M_{Z_{n,t}}(s+\delta, 4\varepsilon, \psi, N) \;&\leqslant\; |E_{n,t}| \, \exp\Big[-n(s+\delta) + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g} \in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \\ &\leqslant\; \frac{1}{te^{n\delta}} \sum_{j \in I_n} c_j \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g} \in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \\ &\leqslant\; \frac{1}{tn^2} \sum_{j \in I_n} c_j \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g} \in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \end{aligned}$$

where the last inequality is a consequence of the assumption that $n^2 \, \exp[-n\,\delta] \,\leqslant\, 1$ for all $n \geqslant N_0$.

Fix $t \in]0,1[$ and $N \geqslant N_0$. Since $Z \mapsto M_Z(s,\varepsilon,\psi,N)$ is an outer measure and

$$Z \subset \bigcup_{n=N}^{+\infty} Z_{n,n^{-2}t}$$

then

$$\begin{aligned} M_Z(s+\delta,4\varepsilon,\psi,N) &\leqslant \sum_{n=N}^{+\infty} M_{Z_{n,n^{-2}t}}(s+\delta,4\varepsilon,\psi,N) \\ &\leqslant \sum_{n=N}^{+\infty} \frac{1}{t} \sum_{j\in I_n} c_j \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g}\in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \\ &\leqslant \frac{1}{t} \sum_{j\in I} c_j \, \exp\Big[-s\,n + \frac{1}{|G_n^{x_j}|} \sum_{\underline{g}\in G_n^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \\ &\leqslant W_Z(s,\varepsilon,\psi,N). \end{aligned}$$

The previous estimate is precisely the first inequality in the statement of Lemma 7.7.

The second inequality is simpler to prove. Fix $N \in \mathbb{N}$ and notice that, taking $n_j \geqslant N$ and $c_j = 1$ for $j \in I$, we get

$$\sum_{j\in I} \chi_{B_{n_j}(x_j,\varepsilon)} \geqslant \chi_Z \quad \Rightarrow \quad Z \subset \bigcup_{j\in I} B_{n_j}(x_j,\varepsilon).$$

Consequently,

$$\begin{aligned} &\inf\Big\{\sum_{j\in I} c_j \, \exp\Big[-s\,n_j + \frac{1}{|G_{n_j}^{x_j}|} \sum_{\underline{g}\in G_{n_j}^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \colon \sum_{j\in I} c_j\,\chi_{B_{n_j}(x_j,\varepsilon)} \geqslant \chi_Z\Big\} \\ &\quad\leqslant \inf\Big\{\sum_{j\in I} \exp\Big[-s\,n_j + \frac{1}{|G_{n_j}^{x_j}|} \sum_{\underline{g}\in G_{n_j}^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \colon \sum_{j\in I} \chi_{B_{n_j}(x_j,\varepsilon)} \geqslant \chi_Z\Big\} \\ &\quad\leqslant \inf\Big\{\sum_{j\in I} \exp\Big[-s\,n_j + \frac{1}{|G_{n_j}^{x_j}|} \sum_{\underline{g}\in G_{n_j}^{x_j}} S_\psi^{\underline{g}}(x_j)\Big] \colon Z \subset \bigcup_{j\in I} B_{n_j}(x_j,\varepsilon)\Big\}. \end{aligned}$$

This ensures that $W_Z(s,\varepsilon,\psi,N) \leqslant M_Z(s,\varepsilon,\psi,N)$, as claimed. The proof of Lemma 7.7 is complete. □

7.2. **Proof of Theorem 7.1(a).** Fix a non-empty compact set $Z \subset X$ and $\psi \in C^0(X)$. Let $\mu \in \mathcal{M}_1(X)$ be a Borel probability measure such that $\mu(Z) = 1$. First we will prove that

$$P_Z((G,G_1),\psi) \geqslant \underline{P}_\mu((G,G_1),\psi). \tag{7.8}$$

Taking into account that

$$\underline{P}_\mu((G,G_1),\psi) = \int_Z \underline{P}_\mu((G,G_1),\psi,x)\,d\mu(x)$$

we deduce that, given $\delta > 0$, the set

$$Z_\delta^\mu = \big\{x \in Z \colon \underline{P}_\mu((G,G_1),\psi,x) \geqslant \underline{P}_\mu((G,G_1),\psi) - \delta\big\}$$

is a positive $\mu-$measure subset of X and, by definition,

$$\underline{P}_\mu((G,G_1),\psi,x) \geqslant \underline{P}_\mu((G,G_1),\psi) - \delta \qquad \forall\, x \in Z^\mu_\delta.$$

Therefore, Theorem 6.1 implies that

$$P_{Z^\mu_\delta}((G,G_1),\psi) \geqslant \underline{P}_\mu((G,G_1),\psi) - \delta.$$

Taking the supremum over all Borel probability measures μ satisfying $\mu(Z) = 1$, we conclude that

$$P_Z((G,G_1),\psi) \geqslant P_{Z^\mu_\delta}((G,G_1),\psi) \geqslant \sup\big\{\underline{P}_\mu((G,G_1),\psi) - \delta\colon\ \mu \in \mathcal{M}_1(X) \text{ and } \mu(Z) = 1\big\}.$$

Since $\delta > 0$ is arbitrary, this proves (7.8).

To finish the proof of Theorem 7.1 (a), we are left to show the converse inequality, namely

$$P_Z((G,G_1),\psi) \leqslant \sup\big\{\underline{P}_\mu((G,G_1),\psi)\colon\ \mu \in \mathcal{M}_1(X) \text{ and } \mu(Z) = 1\big\}. \tag{7.9}$$

Taking into account that, for every $t \in \mathbb{R}$,

$$\begin{aligned} P_Z((G,G_1),\psi + t) &= P_Z((G,G_1),\psi) + t \\ \underline{P}_\mu((G,G_1),\psi + t) &= \underline{P}_\mu((G,G_1),\psi) + t \end{aligned}$$

we may assume without loss of generality that $P_Z((G,G_1),\psi) > 0$. Therefore, by Proposition 7.6,

$$W_Z((G,G_1),\psi) = P_Z((G,G_1),\psi) > 0.$$

Take $s \in \,]0, W_Z((G,G_1),\psi)[$, $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $W_Z(s,\varepsilon,\psi,N) > 0$. We observe that the fact that $W_Z(s,\varepsilon,\psi,N) > 0$ is essential in the proof of the next proposition; we will use it when $K = Z$.

Proposition 7.10. *Consider a finitely generated pseudogroup (G,G_1) acting on a compact metric space (X,d), a non-empty compact subset $Z \subset X$ and a continuous potential $\psi : X \to \mathbb{R}$ such that $P_Z((G,G_1),\psi) > 0$. Assume that $s \geqslant 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$ are such that $c = W_K(s,\varepsilon,\psi,N) > 0$. Then there exists a Borel probability measure $\mu_0 \in \mathcal{M}_1(X)$ satisfying $\mu(Z) = 1$ and*

$$\mu_0(B_n(x,\varepsilon)) \leqslant \frac{1}{c} \exp\Big[-s\,n + \frac{1}{|G^x_n|} \sum_{\underline{g} \in G^x_n} S^{\underline{g}}_\psi(x)\Big] \qquad \forall\, x \in Z,\ \forall\, n \geqslant N.$$

Proof. Fix a compact subset $Z \subset X$ and $s \geqslant 0$, $N \in \mathbb{N}$ and $\varepsilon > 0$ as in the statement of the proposition. Define a functional $\mathcal{P}\colon C^0(X) \to \mathbb{R}$ by

$$\mathcal{P}(h) = \frac{1}{c} \inf\Big\{\sum_{j\in I} c_j \exp\Big[-s\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g} \in G^{x_j}_{n_j}} S^{\underline{g}}_\psi(x_j)\Big]\colon\ h\,\chi_Z \leqslant \sum_{j\in I} c_j\,\chi_{B_{n_j}(x_j,\varepsilon)}\Big\}$$

where the infimum is taken over all finite or countable families $I = \mathcal{F}_{N,\varepsilon}$ of $\mathcal{G}_{N,\varepsilon}$ (recall Definition 7.2). Let $\mathbf{1}$ denote the constant function equal to 1 with domain X.

Claim 1: For every $f_1, f_2 \in C^0(X)$, one has:

(C1) $\mathcal{P}(f_1 + f_2) \leqslant \mathcal{P}(f_1) + \mathcal{P}(f_2)$.
(C2) $\mathcal{P}(t\,f_1) = t\,\mathcal{P}(f_1)$ for every $t \geqslant 0$.
(C3) $\mathcal{P}(\mathbf{1}) = 1$.
(C4) $0 \leqslant \mathcal{P}(f_1) \leqslant \|f_1\|$.
(C5) $\mathcal{P}(f_1) = 0$ for every $f_1 \in C^0(X)$ with $f_1 \leqslant 0$.

The proof of the previous properties is not hard and is left to the reader. Denote by $\mathrm{Cons}(X)$ the space of all constant functions defined on X. By the Hahn-Banach Theorem we can extend the linear functional $t \in \mathbb{R} \mapsto t\,\mathcal{P}(\mathbf{1})$ from the space $\mathrm{Cons}(X)$ to a linear functional $L\colon C^0(X) \to \mathbb{R}$ satisfying

$$L(\mathbf{1}) = \mathcal{P}(\mathbf{1}) = 1 \qquad \text{and} \qquad -\mathcal{P}(-f) \leqslant L(f) \leqslant \mathcal{P}(f) \quad \forall\, f \in C^0(X).$$

Notice that, if $f \in C^0(X)$ is such that $f \geqslant 0$, then $\mathcal{P}(-f) = 0$, and so $L(f) \geqslant 0$. Thus, by the Riesz representation theorem there exists a Borel probability measure $\mu_0 \in \mathcal{M}_1(X)$ such that

$$L(h) \,=\, \int_X h\, d\mu_0 \qquad \forall\, h \in C^0(X).$$

Claim 2: $\mu_0(Z) = 1$.

Indeed, take a compact set $E \subset X \setminus Z$. By the Urysohn's lemma there exists $h_0 \in C^0(X)$ such that $0 \leqslant h_0 \leqslant 1$, $h_0(x) = 1$ for $x \in E$ and $h_0(x) = 0$ for $x \in Z$. Thus, $h_0\,\chi_Z \equiv 0$ and, consequently, $\mathcal{P}(h_0) = 0$. Hence, $\mu_0(E) \leqslant L(h_0) \leqslant \mathcal{P}(h_0) = 0$. Since $E \subset X \setminus Z$ is an arbitrary compact set, by the regularity of the measure μ_0 we conclude that $\mu_0(X \setminus Z) = 0$, thus proving the claim.

We are left to estimate the measure μ_0 of dynamical balls. Fix $x \in Z$ and $n \in \mathbb{N}$. Using Urysohn's lemma once more, for each compact subset $E \subset B_n(x,\varepsilon)$ there exists $f_E \in C^0(X)$ such that $0 \leqslant f_E \leqslant 1$, $f_E(y) = 1$ for any $y \in E$ and $f_E(y) = 0$ for any $y \in X \setminus B_n(x,\varepsilon)$. Then

$$\mu_0(E) \,\leqslant\, \int \chi_E\, d\mu_0 \,\leqslant\, L(f_E) \,\leqslant\, \mathcal{P}(f_E).$$

Notice that, for every (at most) countable set I and $n_j \geqslant N$, one has

$$\begin{aligned}
&\inf\Big\{ \sum_{j\in I} c_j\, \exp\Big[-s\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g}\in G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big] \colon\; f_E\,\chi_Z \leqslant \sum_{j\in I} c_j\,\chi_{B_{n_j}(x_j,\varepsilon)} \Big\} \\
&\qquad \leqslant \inf\Big\{ \sum_{j\in I} \exp\Big[-s\,n_j + \frac{1}{|G^{x_j}_{n_j}|} \sum_{\underline{g}\in G^{x_j}_{n_j}} S^{\underline{g}}_{\psi}\,(x_j)\Big] \colon\; f_E\,\chi_Z \leqslant \sum_{j\in I} \chi_{B_{n_j}(x_j,\varepsilon)} \Big\} \\
&\qquad \leqslant \exp\Big[-s\,n + \frac{1}{|G^{x}_{n}|} \sum_{\underline{g}\in G^{x}_{n}} S^{\underline{g}}_{\psi}\,(x)\Big]
\end{aligned}$$

where the last inequality is due to the fact that $f_E\,\chi_Z \leqslant \chi_{B_n(x,\varepsilon)}$. So

$$\mu_0(E) \,\leqslant\, \mathcal{P}(f_E) \,\leqslant\, \frac{1}{c}\,\exp\Big[-s\,n + \frac{1}{|G^{x}_{n}|} \sum_{\underline{g}\in G^{x}_{n}} S^{\underline{g}}_{\psi}\,(x)\Big].$$

Taking into account the regularity of μ_0, we may add that, for every $x \in Z$ and every $n \geqslant N$,

$$\begin{aligned}
\mu_0\big(B_n(x,\varepsilon)\big) \,&=\, \sup\big\{\mu_0(E)\colon\; E \subset B_n(x,\varepsilon) \text{ is compact }\big\} \\
&\leqslant\, \frac{1}{c}\,\exp\Big[-s\,n + \frac{1}{|G^{x}_{n}|} \sum_{\underline{g}\in G^{x}_{n}} S^{\underline{g}}_{\psi}\,(x)\Big]. \qquad (7.11)
\end{aligned}$$

The proof of the proposition is complete. $\square$

Take the Borel probability measure $\mu_0 \in \mathcal{M}_1(X)$ such that $\mu_0(Z) = 1$ provided by Proposition 7.10. From (7.11), we easily deduce that, for every $n \in \mathbb{N}$,

$$-\frac{1}{n}\left[\log\,\mu_0(B_n(x,\varepsilon)) - \frac{1}{|G_n^x|}\sum_{\underline{g}\,\in\,G_n^x} S_\psi^{\underline{g}}\,(x)\right] \geqslant s + \frac{1}{n}\log c$$

and so $\underline{P}_{\mu_0}((G,G_1),\psi,x) \geqslant s$ for every $x \in Z$. Therefore,

$$\underline{P}_{\mu_0}((G,G_1),\psi) \;=\; \int_X \underline{P}_{\mu_0}((G,G_1),\psi,x)d\mu_0(x) \;\geqslant\; s$$

and

$$\sup\Big\{\underline{P}_\mu((G,G_1),\psi)\colon\;\; \mu \in \mathcal{M}_1(X) \text{ and } \mu(Z) = 1\Big\} \;\geqslant\; s.$$

Finally, since s can be chosen arbitrarily close to $P_Z((G,G_1),\psi) = W_Z((G,G_1),\psi)$, we obtain the inequality (7.9). The proof of Theorem 7.1 (a) is complete. $\square$

We are left to show Theorem 7.1 (b). We postpone this proof to the next section, after the presentation of a special dynamically defined Carathéodory-Pesin structure which conveys a reformulation of the notion of topological pressure of a pseudogroup.

8. Capacity pressure *vs.* topological pressure

Consider a finitely generated pseudogroup (G,G_1) acting on a compact metric space (X,d) and a potential $\psi \in C^0(X)$. Recall from Subsection 4.4 that

$$Q_{\text{top}}((G,G_1),\psi) \;=\; \lim_{\varepsilon\,\to\,0^+}\;\limsup_{n\,\to\,+\infty}\;\frac{1}{n}\log\left(\inf_{F_{n,\varepsilon}}\Big\{\sum_{x\,\in\,F_{n,\varepsilon}} e^{\frac{1}{|G_n^x|}\sum_{\underline{g}\,\in\,G_n^x}\;S_\psi^{\underline{g}}\,(x)}\Big\}\right)$$

where $\underline{g} = g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1}$, the supremum is taken over all $(n,\varepsilon)-$spanning sets $F_{n,\varepsilon} \subset X$ and

$$S_\psi^{\underline{g}}\,(x) \;=\; \psi(x) + \psi(g_{i_1}(x)) + ... + \psi(g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1}(x)).$$

Taking into account that, for every $t \in \mathbb{R}$,

$$Q_{\text{top}}((G,G_1),\psi\,+\,t) \;=\; Q_{\text{top}}((G,G_1),\psi)\,+\,t$$

we may assume without loss of generality that $Q_{\text{top}}((G,G_1),\psi) > 0$.

In what follows we will keep the abbreviation (4.8). Given $\varepsilon > 0$ and a positive integer N, denote by $\big\{B_N(x_i,\varepsilon)\big\}_{x_i\,\in\,I_N^*(\varepsilon)}$ a finite or countable family of dynamical balls satisfying

$$\bigcup_{x_i\,\in\,I_N^*(\varepsilon)} B_N(x_i,\varepsilon) \;=\; X$$

and by $C_X^*(N,\varepsilon)$ the collection of all sets $I_N^*(\varepsilon)$. For any $s \geqslant 0$ and $\psi \in C^0(X)$ let

$$\begin{aligned}
M^*(s,\varepsilon,\psi,N) \;\;&=\;\; \inf\Big\{\sum_{x_j\,\in\,I_N^*(\varepsilon)} \exp[-s\,N + P_n^\psi(x_j)]\colon\; I_N^*(\varepsilon) \in C_X^*(N,\varepsilon)\Big\} \\
M^*(s,\varepsilon,\psi) \;\;&=\;\; \limsup_{N\,\to\,+\infty} M^*(s,\varepsilon,\psi,N).
\end{aligned}$$

By an argument entirely similar to the one used to show Lemma 5.3 we deduce that the map $s \mapsto M^*(s,\varepsilon,\psi)$ behaves like as $s-$Hausdorff measure, that is, it has a unique critical point $c_\psi(\varepsilon)$, where it drops from $+\infty$ to 0; hence

$$c_\psi(\varepsilon) \;=\; \sup\{s \geqslant 0\colon\; M^*(s,\varepsilon,\psi) = +\infty\}.$$

Moreover, by a standard reasoning, as in the proof of Lemma 5.5, we guarantee the existence of the limit

$$c_\psi \;=\; \lim_{\varepsilon \to 0^+} c_\psi(\varepsilon).$$

Definition 8.1. The number $c_\psi = c\,(G,G_1)_\psi$ is called *the (upper) capacity pressure* of (G,G_1) with respect to the potential $\psi \in C^0(X)$.

Proposition 8.2. *Consider a finitely generated pseudogroup (G,G_1) acting on a compact metric space (X,d) and an arbitrary $\psi \in C^0(X)$. Then*

$$c\,(G,G_1)_\psi \;\leqslant\; Q_{\mathrm{top}}((G,G_1),\psi).$$

Proof. Fix $\psi \in C^0(X)$ and, to simplify the notation, let $\alpha(\psi) \;=\; c\,(G,G_1)_\psi$. Given $\varepsilon > 0$ and $\gamma > 0$, choose a sequence $(m_k)_{k\,\in\,\mathbb{N}}$ of positive integers such that

$$M^*(\alpha(\psi)-\gamma,\varepsilon) \;=\; \lim_{k\,\to\,+\infty} M^*(\alpha(\psi)-\gamma,\varepsilon,m_k) \;=\; +\infty.$$

Thus, for every sufficiently large k, the quantity $M^*(\alpha(\psi)-\gamma,\varepsilon,m_k)$ is arbitrary large. In particular,

$$M^*(\alpha(\psi)-\gamma,\varepsilon,m_k) \;>\; 1.$$

Therefore,

$$\inf\Big\{\sum_{x_j\,\in\, I^*_{m_k}(\varepsilon)} \exp\big[-(\alpha(\psi)-\gamma)\,m_k + P^\psi_{m_k}(x_j)\big]\colon\; I^*_{m_k}(\varepsilon) \in C^*_X(m_k,\varepsilon)\Big\} \;>\; 1$$

and

$$\inf_{F_{m_k,\varepsilon}}\Big\{\sum_{x_j\,\in\, F_{m_k,\varepsilon}} \exp\big[-(\alpha(\psi)-\gamma)\,m_k + P^\psi_{m_k}(x_j)\big]\colon\; X = \bigcup_{x_j\,\in\, F_{m_k,\varepsilon}} B_{m_k}(x_j,\varepsilon)\Big\} \;>\; 1$$

where the infimum is taken over all $(m_k,\varepsilon)-$spanning subsets of X and we are summoning the infomation in (2.18) regarding the connection between spanning sets and coverings of X by open dynamical balls. Consequently,

$$\inf_{F_{m_k,\varepsilon}}\Big\{\sum_{x_j\,\in\, F_{m_k,\varepsilon}} \exp\big[P^\psi_{m_k}(x_j)\big]\colon\; X = \bigcup_{x_j\,\in\, F_{m_k,\varepsilon}} B_{m_k}(x_j,\varepsilon)\Big\} \;>\; \exp\big[-(\alpha(\psi)-\gamma)\,m_k\big]$$

Taking logarithms and dividing both sides by m_k, we obtain

$$\frac{1}{m_k}\,\log\Big(\inf_{F_{m_k,\varepsilon}}\Big\{\sum_{x_j\,\in\, F_{m_k,\varepsilon}} \exp\big[P^\psi_{m_k}(x_j)\big]\colon\; X = \bigcup_{x_j\,\in\, F_{m_k,\varepsilon}} B_{m_k}(x_j,\varepsilon)\Big\}\Big) \;\geqslant\; \alpha(\psi)-\gamma.$$

Since γ is arbitrarily small, taking $\limsup$ as k goes to $+\infty$ and $\lim$ when $\varepsilon \to 0^+$, we deduce that

$$\begin{aligned}
&Q_{\text{top}}((G,G_1),\psi) \geqslant \\
&\geqslant \lim_{\varepsilon \to 0^+} \limsup_{k \to +\infty} \frac{1}{m_k} \log\Big(\inf_{F_{m_k,\varepsilon}} \Big\{ \sum_{x_j \in F_{m_k,\varepsilon}} \exp\,[P^{\psi}_{m_k}(x_j)] \colon\; X = \bigcup_{x_j \in F_{m_k,\varepsilon}} B_{m_k}(x_j,\varepsilon) \Big\} \Big) \\
&\geqslant \alpha(\psi) \,=\, c\,(G,G_1)_{\psi}.
\end{aligned}$$

□

8.1. **Proof of Theorem 7.1(b).** We can now connect the three notions $P_X((G,G_1),\,\psi)$, $c\,(G,G_1)_\psi$ and $P_{\text{top}}((G,G_1),\psi)$, for every $\psi \in C^0(X)$, and thereby conclude the proof of Theorem 7.1.

Proposition 8.3. *For a finitely generated pseudogroup (G,G_1) acting on a compact metric space (X,d) and an arbitrary $\psi \in C^0(X)$, one has*

$$P_X((G,G_1),\psi) \,\leqslant\, c\,(G,G_1)_\psi \,\leqslant\, P_{\text{top}}((G,G_1),\psi).$$

Proof. Fix $\psi \in C^0(X)$ and $\varepsilon > 0$. Notice that, for every positive integer N, any finite or countable covering $\{B_N(x_i,\varepsilon) : x_i \in X\}$ belongs both to $I^*_N(\varepsilon)$ and $I_N(\varepsilon)$. Thus, for every $s \geqslant 0$,

$$\begin{aligned}
M_X(s,\varepsilon,\psi,N) \;&=\; \inf\Big\{ \sum_{j \in I_N(\varepsilon)} \exp\,[-s\,N + P^{\psi}_N(x_j)] \colon\; I_N(\varepsilon) \in C_X(N,\varepsilon) \Big\} \\
&\leqslant\; \inf\Big\{ \sum_{j \in I^*_N(\varepsilon)} \exp\,[-s\,N + P^{\psi}_N(x_j)] \colon\; I^*_N(\varepsilon) \in C^*_X(N,\varepsilon) \Big\} \\
&=\; M^*(s,\varepsilon,\psi,N).
\end{aligned}$$

Therefore,

$$M_X(s,\varepsilon,\psi) \,\leqslant\, M^*(s,\varepsilon,\psi) \qquad \forall\, s \geqslant 0$$

and so

$$M_X(\varepsilon,\psi) \,\leqslant\, c_\psi(\varepsilon) \qquad \forall\, \varepsilon > 0.$$

Taking the limit when $\varepsilon \to 0^+$, we get

$$P_X((G,G_1),\psi) \,\leqslant\, c\,(G,G_1)_\psi.$$

Consequently, Proposition 8.2 and Lemma 4.7 imply that

$$P_X((G,G_1),\psi) \,\leqslant\, c\,(G,G_1)_\psi \,\leqslant\, Q_{\text{top}}((G,G_1),\psi) \,\leqslant\, P_{\text{top}}((G,G_1),\psi).$$

The proof of Theorem 7.1(b) is complete. □

An immediate consequence of Theorem 7.1 is Theorem A(a), namely:

Corollary 8.4. *Let (G,G_1) be a finitely generated pseudogroup acting on a compact metric space (X,d). Then, for every $\psi \in C^0(X)$,*

$$P_{\text{top}}((G,G_1),\psi) \,\geqslant\, \sup_{\mu \in \mathcal{M}_1(X)} \underline{P}_\mu((G,G_1),\psi).$$

9. $G-$Homogeneous probability measures

In this section we complete the proof of Theorem A. We start by extending [4, Lemma 4.10] to the upper local measure-theoretic pressure defined in Subsection 2.7.

Lemma 9.1. *Let (G, G_1) be a finitely generated pseudogroup acting on a compact metric space (X, d) and endowed with a $G-$homogeneous probability measure η. Then, for every $x, y \in X$,*

$$\overline{P}_\eta((G, G_1), \psi, x) \,=\, \overline{P}_\eta((G, G_1), \psi, y).$$

Proof. Fix $\psi \in C^0(X)$ and $\gamma > 0$. Since η is $G-$homogeneous, there exist $0 < \varepsilon < \gamma$ and $\lambda_1 > 0$ such that

$$\eta\big(B_n(y, \varepsilon)\big) \,\leqslant\, \lambda_1\, \eta\big(B_n(x, \gamma)\big)$$

and there are $0 < \delta < \varepsilon$ and $\lambda_2 > 0$ such that

$$\eta\big(B_n(x, \delta)\big) \,\leqslant\, \lambda_2\, \eta\big(B_n(y, \varepsilon)\big) \qquad \forall\, x, y \in X \quad \forall\, n \in \mathbb{N}$$

for every $x, y \in X$ and $n \in \mathbb{N}$. Hence,

$$\eta\big(B_n(x, \delta)\big) \,\leqslant\, \lambda_2\, \eta\big(B_n(y, \varepsilon)\big) \,\leqslant\, \lambda_1\, \lambda_2\, \eta\big(B_n(x, \gamma)\big)$$

and (recall the abbreviation (4.8))

$$\eta\big(B_n(x, \delta)\big)\, \exp -[P_n^\psi(x)] \,\leqslant\, \lambda_2\, \eta\big(B_n(y, \varepsilon)\big)\, \exp -[P_n^\psi(x)] \,\leqslant\, \lambda_1\, \lambda_2\, \eta\big(B_n(x, \gamma)\big)\, \exp -[P_n^\psi(x)]$$

for every $x, y \in X$ and $n \in \mathbb{N}$. Applying logarithm to both sides of the above inequalities, dividing by $-n$ and taking the $\limsup$ when $n \to +\infty$, we get, respectively,

$$\begin{aligned} \log\, \eta\big(B_n(x, \delta)\big) - P_n^\psi(x) \quad &\leqslant \quad \log\, \lambda_2 + \log\, \eta\big(B_n(y, \varepsilon)\big) - P_n^\psi(x) \\ &\leqslant \quad \log\,(\lambda_1\, \lambda_2) + \log\, \eta\big(B_n(x, \gamma)\big) - P_n^\psi(x) \end{aligned}$$

and

$$\begin{aligned} \limsup_{n \to +\infty}\, -\frac{1}{n}\Big[\log\, \eta\big(B_n(x, \delta)\big) - P_n^\psi(x)\Big] \quad &\geqslant \quad \limsup_{n \to +\infty}\, -\frac{1}{n}\Big[\log\, \eta\big(B_n(y, \varepsilon)\big) - P_n^\psi(x)\Big] \\ &\geqslant \quad \limsup_{n \to +\infty}\, -\frac{1}{n}\Big[\log\, \eta\big(B_n(x, \gamma)\big) - P_n^\psi(x)\Big]. \end{aligned}$$

Let

$$A_\varepsilon(x, y) \,=\, \limsup_{n \to +\infty}\, -\frac{1}{n}\Big[\log\, \eta\big(B_n(y, \varepsilon)\big) - P_n^\psi(x)\Big].$$

Since $\delta < \varepsilon < \gamma$, taking the limit as $\gamma \to 0^+$ we obtain

$$\overline{P}_\eta((G, G_1), \psi, x) \,\leqslant\, \lim_{\varepsilon \to 0^+} A_\varepsilon(x, y) \,\leqslant\, \overline{P}_\eta((G, G_1), \psi, x).$$

This means that the transformation

$$(x, y) \quad \mapsto \quad \lim_{\varepsilon \to 0^+} A(x, y)$$

does not depend on y. In particular, when $y = x$ we get

$$\lim_{\varepsilon \to 0^+} A(y, y) \,=\, \overline{P}_\eta((G, G_1), \psi, y) \qquad \text{and} \qquad \overline{P}_\eta((G, G_1), \psi, x) \,=\, \overline{P}_\eta((G, G_1), \psi, y).$$

Thus,

$$\overline{P}_\eta((G, G_1), \psi, x) \,=\, \overline{P}_\eta((G, G_1), \psi) \qquad \forall\, x \in X.$$

$\square$

9.1. **Proof of Theorem A(b).** We are ready to prove the second part of Theorem A. It generalizes Corollary 4.13 of [4], whose proof depends on the validity of Proposition 4.12, and which asserts that, if (G, G_1) is a finitely generated group on a compact metric space endowed with a $G-$homogeneous probability measure η, then

$$h_{\text{top}}(G, G_1) \,=\, \overline{h}_\eta\,(G, G_1).$$

Fix $\psi \in C^0(X)$ and consider, for every $x \in X$, the sequence of averages (so called $n-$th Cesàro averages of the $n-$th spherical averages of ψ at x) defined by

$$n \in \mathbb{N} \quad \mapsto \quad \Re^\psi_n(x) \,=\, \frac{1}{n}\,\frac{1}{|G^x_n|}\,\sum_{\underline{g}\,\in\, G^x_n} S^{\underline{g}}_{\psi}\,(x).$$

Given an ergodic probability measure $\mu \in \mathcal{M}_G(X)$, it is known that, under suitable assumptions on the pseudogroup action, the sequence of averages $\big(\Re^\psi_n(x)\big)_{n\,\in\,\mathbb{N}}$ converges at $\mu-$almost every $x \in X$ to $\int \psi\, d\mu$. We refer the reader e.g. to [22, 36] for more information regarding those pointwise ergodic theorems.

Assume that (G, G_1) is a free group and that $\eta \in \mathcal{M}_G(X)$ is a $G-$homogeneous ergodic probability measure. By [15], the sequence $\big(\Re^\psi_n(x)\big)_{n\,\in\,\mathbb{N}}$ converges to $\int_X \psi\, d\eta$ for $\eta-$almost every x. Denote by Y the subset with $\eta(Y) = 1$ of those points x. Select $x \in Y$. Then, by Lemma 9.1 and Corollary 4.32 of [4], one has

$$\begin{aligned}
\overline{P}_\eta((G, G_1), \psi) &= \overline{P}_\eta((G, G_1), \psi, x)\\
&= \lim_{\varepsilon\,\to\,0^+}\ \limsup_{n\,\to\,+\infty}\ -\frac{1}{n}\Big[\log\,\mu(B_n(x,\varepsilon)) - \frac{1}{|G^x_n|}\sum_{\underline{g}\,\in\,G^x_n} S^{\underline{g}}_{\psi}\,(x)\Big]\\
&= \lim_{\varepsilon\,\to\,0^+}\Big[\limsup_{n\,\to\,+\infty}\ -\frac{1}{n}\,\log\,\mu(B_n(x,\varepsilon)) - \int_X \psi\, d\eta\Big].\\
&= \overline{h}_\eta((G, G_1), x) \,+\, \int_X \psi\, d\eta\\
&= \overline{h}_\eta(G, G_1) \,+\, \int_X \psi\, d\eta\\
&= h_{\text{top}}(G, G_1) + \int_X \psi\, d\eta.
\end{aligned}$$

This ends the proof of Theorem A.

Question: Assume that (G, G_1) be a finitely generated pseudogroup, acting on a compact metric space (X, d) and endowed with a $G-$homogeneous probability measure η. Is it true that, for every $\psi \in C^0(X)$, one has

$$P_{\text{top}}((G, G_1), \psi) \,=\, \overline{P}_\eta((G, G_1), \psi)\ ?$$

10. Examples

In this final section we provide several examples by which we illustrate the complexity of computing the topological pressure for pseudogroup actions and some applications of our main results.

Example 10.1. Our first example indicates that the topological entropy of a finitely generated pseudogroup action may be positive even if all its generators have zero entropy.

Consider the interval $[0,1]$ with the Euclidean distance and the open subintervals $J_1 = \,]0,1/3[$ and $J_2 = \,]2/3,1[$. Let $g_1\colon\ ]0,1[\to J_1$ and $g_2\colon\ ]0,1[\to J_2$ be the homeomorphisms given by

$$\begin{array}{rccc} g_1\colon &]0,1[& \to &]0,\frac{1}{3}[\\ & x & \mapsto & \frac{x}{3} \end{array} \qquad \text{and} \qquad \begin{array}{rccc} g_2\colon &]0,1[& \to &]\frac{2}{3},1[\\ & x & \mapsto & \frac{x}{3}+\frac{2}{3} \end{array}.$$

Take $G_1 = \{\mathrm{id}_{[0,1]},\, g_1,\, g_1^{-1},\, g_2,\, g_2^{-1}\}$ and let G be the pseudogroup generated by G_1. We will show, by adapting a reasoning in [23, page 131], that $h_{\mathrm{top}}(G,G_1) \geqslant \log 2$.

Given $n\in\mathbb{N}$ and positive integers $i_1,i_2,\cdots,i_n$ equal to 1 or 2, define

$$J_{i_1,\,i_2,\,\cdots,\,i_n} \,=\, g_{i_1}\circ g_{i_2}\circ\cdots\circ g_{i_n}\big(]0,1[\big).$$

Fix $x_0\in\,]0,1[$ and denote

$$x\,(i_1,i_2,\cdots,i_n) \,=\, g_{i_1}\circ g_{i_2}\circ\cdots\circ g_{i_n}\,(x_0).$$

Then the set

$$E_{n,\,x_0} \,=\, \big\{x\,(i_1,i_2,\cdots,i_n)\colon \quad i_\ell\in\{1,2\}\ \ \forall 1\leqslant \ell\leqslant n\big\}$$

has cardinal 2^n and is $(n,\varepsilon)-$separated for every $0<\varepsilon<\frac{1}{3}$. Indeed, let $z\neq w$ be two points in $E_{n,\,x_0}$, say

$$z \,=\, x\,(i_1,i_2,\cdots,i_k,\,i_{k+1},\cdots,i_n) \qquad \text{and} \qquad w \,=\, x\,(i_1,i_2,\cdots,i_k,\,j_{k+1},\cdots,j_n)$$

where $0\leqslant k\leqslant n$ and $j_{k+1}\neq i_{k+1}$. Then the map

$$g \,=\, g_{i_k}^{-1}\,\circ\, g_{i_{k+1}}^{-1}\,\circ\cdots\circ\, g_{i_1}^{-1}$$

is in G and is defined in the interval $J_{i_1,\,i_2,\,\cdots,\,i_n}$, to which both z and w belong. Moreover, $g(z)\in J_{i_{k+1}}$ and $g(w)\in J_{j_{k+1}}$. Since $j_{k+1}\neq i_{k+1}$, if $0<\varepsilon<1/3$ then

$$\big|\,g(z)-g(w)\,\big| \,>\, 1/3 \,>\, \varepsilon.$$

So, z and w are $(n,\varepsilon)-$separated. Thus, $s(n,\varepsilon)\geqslant 2^n$, which implies that $h_{\mathrm{top}}(G,G_1) \geqslant \log 2$.

Example 10.2. In view of Example 10.1 we may inquire whether the topological entropy of a pseudogroup reflects the existence of generators with zero topological entropy, even in the context where some of the generators have positive topological entropy. This is precisely what happens in the following example.

Consider the space $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ with the distance

$$d\big((a_n)_{n\,\in\,\mathbb{Z}},\ (b_n)_{n\,\in\,\mathbb{Z}}\big) \,=\, \sum_{n\,\in\,\mathbb{Z}} \frac{|a_n-b_n|}{2^{|n|}}.$$

Let $\sigma\colon \Sigma_2\to\Sigma_2$ and $T\colon\Sigma_2\to\Sigma_2$ be the homeomorphisms given by

$$\sigma\big((a_n)_{n\,\in\,\mathbb{N}}\big) \,=\, \big(a_{n+1}\big)_{n\,\in\,\mathbb{N}} \qquad \text{and} \qquad T\big((a_n)_{n\,\in\,\mathbb{N}}\big) \,=\, \big(\widehat{a}_n\big)_{n\,\in\,\mathbb{N}}$$

where $\widehat{a}_n = 1$ if $a_n = 0$ and $\widehat{a}_n = 0$ if $a_n = 1$. Let G be the group generated by $G_1 = \big\{\mathrm{id}_{\Sigma_2},\,\sigma,\,\sigma^{-1},\,T,\,T^{-1}\big\}$ and $\mu = \big(\frac{1}{2},\frac{1}{2}\big)^{\mathbb{Z}}$ be the symmetric Bernoulli measure in Σ_2. Since T is an isometry, for every $x\in\Sigma_2$ and $\varepsilon>0$, one has

$$B_n(x,\varepsilon) \,=\, \big\{y\in\Sigma_2\colon\ d\big(\sigma^j(x),\,\sigma^j(y)\big) \,<\, \varepsilon,\ \forall -n\leqslant j\leqslant n\big\} \,=\, \sigma^n\Big(\widetilde{B}_{2n+1}\big(\sigma^{-n}(x),\varepsilon\big)\Big)$$

where

$$\widetilde{B}_k\big(z,\varepsilon\big) = \big\{y \in \Sigma_2\colon\ d\big(\sigma^j(y),\, \sigma^j(z)\big)\ <\ \varepsilon, \quad \forall\, 0 \leqslant j \leqslant k\big\}$$

denotes the Bowen's one sided dynamical ball for the shift σ, with length $k \geqslant 1$, radius $\varepsilon > 0$ and center $z \in \Sigma_2$. Now observe that

$$\underline{h}_\mu((G, G_1), x)\ =\ \lim_{\varepsilon\to 0^+}\ \liminf_{n\to+\infty} -\frac{1}{n} \log \mu(B_n(x,\varepsilon)) = 2\log 2$$

for every $x \in \Sigma_2$ while, using the Brin-Katok formula [14],

$$h_\mu(\sigma) = \underline{h}_\mu(\sigma, x) = \lim_{\varepsilon\to 0^+}\ \liminf_{n\to+\infty} -\frac{1}{n} \log \mu(\tilde{B}_n(x,\varepsilon)) = \log 2$$

at μ-almost every $x \in \Sigma_2$. Altogether, this shows that

$$\underline{h}_\mu(G, G_1)\ =\ 2\log 2\ =\ 2\,\underline{h}_\mu(\sigma)\ =\ 2\,h_\mu(\sigma)$$

which turns out to be the expected value if G_1 was reduced to $\{\mathrm{id}_{\Sigma_2},\, \sigma,\, \sigma^{-1}\}$.

Example 10.3. The precise computation of the topological pressure of a pseudogroup action usually demands a thorough understanding of the collection of spanning sets with respect to the pseudogroup action. The next example explore Theorem A as a tool to provide a lower bound for the topological pressure of a pseudogroup action.

Consider the interval $[0, 1]$ with the Euclidean distance and the subintervals $I_1 = [0, 1/3[$, $I_2 =]1/3, 2/3[$ and $I_3 =]2/3, 1]$. Let G be the pseudogroup generated by the collection of homeomorphisms $\{\mathrm{id}_{[0,1]},\, g_1,\, g_1^{-1},\, g_2,\, g_2^{-1},\, g_3,\, g_3^{-1}\}$, where

$$g_1\colon \begin{array}{ccc} [0,\frac{1}{3}[& \to & [0,\frac{2}{3}[\\ x & \mapsto & 2x \end{array} \qquad g_2\colon \begin{array}{ccc}]\frac{1}{3},\frac{2}{3}[& \to &]\frac{2}{3},1[\\ x & \mapsto & x+\frac{1}{3} \end{array} \qquad g_3\colon \begin{array}{ccc}]\frac{2}{3},1] & \to & [0,1[\\ x & \mapsto & 3-3x \end{array}.$$

These generators have a Markov property, in the sense that

$$g_1(I_1)\ \supset\ I_1 \cup I_2, \quad g_2(I_2)\ =\ I_3 \quad \text{and} \quad g_3(I_3)\ \supset\ I_1 \cup I_2 \cup I_3.$$

In particular, the equality $g_2(I_2) = I_3$ guarantees that, for every $x \in]0, 1[$, $n \in \mathbb{N} \setminus \{1\}$ and $\underline{g} = g_{i_n} \circ \cdots \circ g_{i_2} \circ g_{i_1} \in G_n^x$, one has

$$\#\big\{1 \leqslant j \leqslant n\colon\ i_j = 2\big\}\ <\ n/2.$$

Indeed, taking into account the domains of g_1, g_2 and g_3, we are sure that

$$g_{i_j}\ =\ g_2 \quad \text{and} \quad j < n \quad \Rightarrow \quad g_{i_{j+1}}\ =\ g_3.$$

Since g_2 is an isometry and the absolute value of the derivatives of both g_1 and g_3 is bounded from below by 2, then

$$B_n(x,\, \varepsilon)\ \subset\ B(x,\, 2^{-\frac{n}{2}}\,\varepsilon) \qquad \forall\, x \in\,]0, 1[, \quad \forall\, \varepsilon > 0, \quad \forall\, n \in \mathbb{N} \setminus \{1\}.$$

Thus, taking μ as the Lebesgue measure on the interval $[0, 1]$, Theorem A yields, when $\psi \equiv 0$,

$$h_{\mathrm{top}}(G, G_1)\ \geqslant\ \underline{h}_\mu(G, G_1)\ \geqslant\ \inf_{x\,\in\,]0,1[}\ \underline{h}_\mu((G, G_1), x)\ \geqslant\ (\log 2)/2.$$

Therefore, from Remark 2.9 we conclude that, for every $\psi \in C^0([0, 1])$ which is C^0-close enough to the constant map equal to zero,

$$P_{\mathrm{top}}((G, G_1), \psi)\ >\ \max_{x\,\in\,[0,1]}\ \psi.$$

We stress that, within the setting of dynamical systems and under mild assumptions, the latter inequality is sufficient for the existence of conformal measures (see [19] for more details).

Example 10.4. Consider now a compact metric space (X, d) and let G be a pseudogroup generated by a finite set of local isometries on X. Then, for every $x \in X$, $n \in \mathbb{N}$ and $\varepsilon > 0$, one has $B_n(x, \varepsilon) = B(x, \varepsilon)$ and the maximal number of $(n, \varepsilon)-$separated points in X does not depend on n. Therefore, $h_{\text{top}}(G, G_1) = 0$. Moreover, for every $\mu \in \mathcal{M}_1(X)$ and $x \in X$,

$$-\frac{1}{n} \log \mu(B_n(x, \varepsilon)) = -\frac{1}{n} \log \mu(B(x, \varepsilon)).$$

So, for every $x \in X$,

$$\overline{h}_\mu((G, G_1), x) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} -\frac{1}{n} \log \mu(B(x, \varepsilon)) = 0.$$

Thus, $\overline{h}_\mu(G, G_1) = 0$. In a similar way we show that $\underline{h}_\mu(G, G_1) = 0$.

As stated in Remark 2.28, for every $\mu \in \mathcal{M}_1(X)$ and every $\psi \in C^0(X)$, one has

$$\overline{h}_\mu(G, G_1) + \min_{x \in X} \psi(x) \leqslant \overline{P}_\mu((G, G_1), \psi) \leqslant \overline{h}_\mu(G, G_1) + \max_{x \in X} \psi(x).$$

and similar inequalities for $\underline{h}_\mu(G, G_1)$ and $\underline{P}_\mu((G, G_1), \psi)$. In this example, since $\overline{h}_\mu(G, G_1) = 0 = \underline{h}_\mu(G, G_1)$ for every $\mu \in \mathcal{M}_1(X)$, we obtain

$$\min_{x \in X} \psi(x) \leqslant \underline{P}_\mu((G, G_1), \psi) \leqslant \overline{P}_\mu((G, G_1), \psi) \leqslant \max_{x \in X} \psi(x)$$

for every $\mu \in \mathcal{M}_1(X)$ and $\psi \in C^0(X)$. Therefore, by Theorem A, for every $\psi \in C^0(X)$ one has

$$P_{\text{top}}((G, G_1), \psi) \geqslant \min_{x \in X} \psi(x).$$

Example 10.5. Consider the $2-$dimensional sphere $\mathbb{S}^2$ of radius one, with the Haar measure, which we denote by η. Take two irrational rotations $R_1 \colon \mathbb{S}^2 \to \mathbb{S}^2$ and $R_2 \colon \mathbb{S}^2 \to \mathbb{S}^2$. Let G be the free group generated by $G_1 = \{\text{id}_{\mathbb{S}^2}, R_1, R_1^{-1}, R_2, R_2^{-1}\}$. Since G is generated by isometries, one has $B_n(x, \varepsilon) = B(x, \varepsilon)$ for every $x \in \mathbb{S}^2$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Thus, the measure η is $G-$homogeneous. Moreover, like in Example 10.4 we have $\overline{h}_\eta(G, G_1) = 0$. More generally, given $\psi \in C^0(S^2)$,

$$\min_{x \in S^2} \psi(x) \leqslant \overline{P}_\eta((G, G_1), \psi) \leqslant \max_{x \in S^2} \psi(x).$$

In addition, as η is a $G-$homogeneous, $G-$invariant and ergodic probability measure, we also conclude from [4, Theorem 4.12] and Theorem A that, for every $\psi \in C^0(X)$,

$$\overline{P}_\eta((G, G_1), \psi) = \int_{\mathbb{S}^2} \psi \, d\eta.$$

Example 10.6. The next example shows that even if G-invariant probability measures exist they may carry low entropy, hence fail to be the candidates to describe the thermodynamic formalism.

Consider the $2-$dimensional sphere $\mathbb{S}^2$ and an Axiom A diffeomorphism $g_1 \colon \mathbb{S}^2 \to \mathbb{S}^2$ with $h_{\text{top}}(g_1) > 0$. Take two distinct periodic points of g_1, say p, q. Choose now a Morse-Smale diffeomorphism $g_2 \colon \mathbb{S}^2 \to \mathbb{S}^2$ so that p and q are fixed points of g_2 and the only periodic orbits of g_2. Take the group G generated by $G_1 = \left\{\text{id}_{\mathbb{S}^2}, \, g_1, \, g_2, \, g_1^{-1}, \, g_2^{-1}\right\}$ and note that

$$h_{\text{top}}(G, G_1) \geqslant h_{\text{top}}(g_1) > 0.$$

Moveover, the space of $G-$invariant probability measures is precisely

$$\mathcal{M}_G(\mathbb{S}^2) \,=\, \big\{t\delta_p \,+\, (1-t)\delta_q \colon\; t \in [0,1]\big\}.$$

In addition, one has

$$\underline{h}_{\,t\delta_p\,+\,(1-t)\delta_q}\,(G, G_1) = 0$$

for every $t \in [0,1]$. Consequently,

$$\begin{array}{rcll} h_{\text{top}}(G,G_1) & \geqslant & h_{\mathbb{S}^2}(G,G_1) & \text{by Theorem B(b)} \\ & = & \sup\limits_{\mu\,\in\,\mathcal{M}_1(\mathbb{S}^2)} \big\{\, \underline{h}_\mu(G,G_1)\,\big\} & \text{by Theorem B(a)} \\ & = & h_{\text{top}}(g_1) & \text{by [38, Theorem 11.5]} \\ & > & 0 & \\ & = & \sup\limits_{\mu\,\in\,\mathcal{M}_G(\mathbb{S}^2)} \big\{\, \underline{h}_\mu(G,G_1)\,\big\}. & \end{array}$$

Acknowledgments

We thank the referees for the valuable questions and comments which helped us to significantly improve this work. AB was partially supported by the grant with reference 11/IDUB/DOS/2021 from the University of Łodz. The research of MC and PV was partially supported by CMUP, member of LASI, which is financed by national funds through FCT - Fundação para a Ciência e a Tecnologia, I.P., under the projects with references UIDB/00144/2020 and UIDP/00144/2020. MC and PV also acknowledge financial support from the project PTDC/MAT-PUR/4048/2021. PV has benefited from the grant CEECIND/03721/2017 of the Stimulus of Scientific Employment, Individual Support 2017 Call, awarded by FCT.

References

[1] M. Barnsley and L. Hurd. *Fractal Image Compression.* AK Peters Ltd., Wellesly, MA, 1993. 1, 2.1

[2] M. Barnsley, M. Hegland and P. Massopust. *Numerics and fractals.* Bull. Inst. Math. Acad. Sin. (N.S.) 9:3 (2014), 389–430. 2.1

[3] A. Biś. *Entropies of a semigroup of maps.* Discrete Contin. Dyn. Syst. 11:2&3 (2004), 639–648. 1, 2.2

[4] A. Biś. *An analogue of the variational principle for group and pseudogroup actions.* Ann. Inst. Fourier 63:3 (2013), 839–863. 1, 2.8, 3, 6, 9, 9.1, 10.5

[5] A. Biś. *Topological and measure-theoretical entropies of nonautonomous dynamical systems.* J. Dyn. Diff. Equat. 30 (2018), 273–285. 6

[6] A. Biś, M. Carvalho, M. Mendes and P. Varandas. *A convex analysis approach to entropy functions, variational principles and equilibrium states.* Comm. Math. Phys. 394 (2022), 215–256. 1

[7] A. Biś, M. Carvalho, M. Mendes, P. Varandas and X. Zhong. *Correction:"Convex analysis approach to entropy functions, variational principles and equilibrium states".* Comm. Math. Phys. 401 (2023), 3335–3342. 1

[8] A. Biś, M. Carvalho, M. Mendes and P. Varandas. *Entropy functions for semigroup actions.* Preprint. 1

[9] A. Biś and M. Urbański. *Some remarks on topological entropy of a semigroup of continuous maps.* Cubo 8:2 (2006), 63–71. 1

[10] R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms.* Lectures Notes in Math. 470, Springer-Verlag, Berlin, 1975. 1

[11] R. Bowen and D. Ruelle. *The ergodic theory of Axiom A flows.* Invent. Math. 29 (1975), 181–202. 1

[12] R. Bowen, *Hausdorff dimension of quasicircles.* Inst. Hautes Études Sci. Publ. Math. 50 (1979), 11–25. 1

[13] L. Bowen. *Measure conjugacy invariants for actions of countable sofic groups.* J. Amer. Math. Soc. 23 (2010), 217–245. 1

[14] M. Brin and A. Katok. *On local entropy.* Geometric Dynamics (Rio de Janeiro, 1981), Lecture Notes in Math. 1007, Springer-Verlag, Berlin, 1983, pp. 30–38. 1, 10.2

[15] A. Bufetov. *Convergence of spherical averages for actions of free groups.* Ann. Math. 155 (2002), 929–944. 9.1

[16] M. Carvalho, F.B. Rodrigues and P. Varandas. *A variational principle for free semigroup actions.* Adv. Math. 334 (2018), 450–487. 2.2

[17] T. Ceccherini-Silberstein, R. Grigorchuk and P. de la Harpe. *Amenability and paradoxical decompositions for pseudogroups and for discrete metric spaces.* Proc. Steklov Inst. Math. 224:1 (1999), 57–97.

[18] T. Das, L. Fishman, D. Simmons and M. Urbański, *Hausdorff dimensions of perturbations of a conformal iterated function system via thermodynamic formalism.* Selecta Math. New Ser. 29 (2023), 19. 1

[19] M. Denker and M. Urbański. *On the existence of conformal measures.* Trans. Amer. Math. Soc. 328:2 (1991), 563–587. 10.3

[20] D.-J. Feng and W. Huang. *Variational principles for topological entropies of subsets.* J. Funct. Anal. 263:8 (2012), 2228–2254. 3, 7, 7.1, 7.1

[21] S. Friedland. *Entropy of graphs, semigroups and groups.* Ergodic Theory of $\mathbb{Z}^d$ Actions, Warwick, 1991, London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press. Cambridge, 1996, pp. 319–343. 1

[22] K. Fujiwara and A. Nevo. *Maximal and pointwise ergodic theorems for word-hyperbolic groups.* Ergodic Theory Dynam. Sys. 18 (1998), 843–858. 9.1

[23] E. Ghys, R. Langevin and P. Walczak. *Entropie geometrique des feuilletages.* Acta Math. 16 (1988), 105–142. 1, 2.2, 2.2, 2.3, 10.1

[24] M. Gromov. *Endomorphisms of symbolic algebraic varieties.* J. Eur. Math. Soc. 1 (1999), 109–197. 4.3

[25] A. Haeflinger. *Grupoides d'holonomie et classifiants.* Astérisque 116 (1984), 70–97. 1

[26] B. Hasselblatt and Y. Pesin Partially Hyperbolic Dynamical Systems. Handbook of Dynamical Systems 1B, 1–55, Elsevier North Holland, 2005 1

[27] D. Kerr and H. Li. *Entropy and the variational principle for actions of sofic groups.* Invent. Math. 186 (2011), 501–558. 1

[28] D. Lind and K. Schmidt. *Symbolic and algebraic dynamical systems.* Handbook of Dynamical Systems, vol. 1, eds. H.W. Broer, B. Hasselblatt and F. Takens, 2005. 1

[29] D. Ma and M. Wu. *Topological pressure and topological entropy of a semigroup of maps.* Discrete Contin. Dynam. Sys. 31 (2011), 545–557. 1

[30] J.-H. Ma and Z.-Y. Wen. *A Billingsley type theorem for Bowen entropy.* C. R. Math. Acad. Sci. Paris 346:9&10 (2008), 503–507. 1, 3, 4.3, 6

[31] P. Massopust. *Fractal functions, fractal surfaces and wavelets.* Academic Press, Inc., San Diego, CA, 1994, xii+383 pp. 1, 2.1

[32] P. Massopust. *Local fractal interpolation on unbounded domains.* Proc. Edinb. Math. Soc. 61:1 (2018), 151–167. 2.1

[33] P. Massopust. *On local fractal functions in Besov and Triebel-Lizorkin spaces.* J. Math. Anal. Appl. 436:1 (2016), 393–407. 2.1

[34] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge Studies in Advanced Mathematics, vol. 44, Cambridge University Press, Cambridge, 1995. 4.3, 7.1

[35] R. Miles and T. Ward. *Directional uniformities, periodic points and entropy.* Discrete Contin. Dyn. Syst. 20:10 (2015), 3525–3545. 1

[36] A. Nevo. *Pointwise ergodic theorems for actions of groups.* In Handbook of Dynamical Systems, vol. 1B, 871–982, Elsevier B. V., Amsterdam, 2006. 9.1

[37] J. Ollagnier and D. Pinchon. *The variational principle.* Studia Math. 72 (1982), 151–159. 1

[38] Ya. Pesin. *Dimension Theory in Dynamical Systems.* Contemporary Views and Applications, University of Chicago Press, 1997. 5, 5, 10.7

[39] Ya. Pesin and B. Pitskel. *Topological pressure and the variational principle for noncompact sets.* Funct. Anal. Appl. 18:4 (1984), 307–318. 5

[40] D. Ruelle. *Statistical mechanics on a compact set with $\mathbb{Z}^d$ action satisfying expansiveness and specification.* Trans. Amer. Math. Soc. 187 (1973), 237–251. 1

[41] F. M. Schneider. *Topological entropy of continuous actions of compactly generated groups.* Preprint ArXiv:1502.03980 1

[42] Ya. Sinai. *Gibbs measures in ergodic theory.* Russian Math. Surveys 27 (1972), 21–69. 1

[43] Q. Xiao and D. Ma. *Variational principle of topological pressure of free semigroup actions for subsets*. Qual. Th. Dynam. Sys. 21 (2022), 21–99 2.2, 6, 7, 7.1

[44] P. Walczak. *Dynamics of Foliations, Groups and Pseudogroups.* Springer Basel AG, 2004. 2.1, 2.2

[45] W. Waliszewski. *Categories, groupoids, pseudogroups and analytical structures.* Rozprawy Mat. 45 (1965), 40 pp. 2.1

[46] P. Walters. *An Introduction to Ergodic Theory.* Springer-Verlag, New York, 1975. 1, 2.4, 4.4

[47] Y. Zhao. *Measure theoretic pressure for amenable group actions.* Colloq. Math. 148:1 (2017), 87–106. 1

Faculty of Mathematics, Łódź University, Poland
Email address: andrzej.bis@wmii.uni.lodz.pl

CMUP & Departamento de Matemática, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre s/n, 4169–007 Porto, Portugal.
Email address: mpcarval@fc.up.pt

CMUP & Departamento de Matemática, Universidade Federal da Bahia, Salvador, Brazil.
Email address: paulo.varandas@ufba.br