Entropy functions for semigroup actions

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Abstract

We consider continuous actions of finitely generated semigroups and countable sofic groups, generated either by continuous self-maps

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or homeomorphisms of a compact metric space. For each known topological pressure operator associated to these actions, we provide a measure-theoretic entropy map which is concave, upper semi-continuous and satisfies a variational principle whose maximum is always attained. In the case of countable amenable group actions whose amenable entropy is concave and upper semi-continuous, we show that, for any sofic approximation sequence, the amenable metric entropy and the previous measure-theoretic entropy coincide in the space of invariant probability measures, and are equal to the upper semi-continuous envelope of the sofic entropy.

1 Introduction

Measure-theoretical and topological entropies have an intrinsic connection, which is provided by the classical variational principle for the topological pressure (cf. [43]). This principle asserts that, if f is a continuous map acting on a compact metric space X, $\varphi\colon X \to \mathbb{R}$ is a continuous potential and $P_{\mathrm{top}}(f,\varphi)$ its topological pressure, then

$$P_{\mathrm{top}}(f,\varphi) = \sup_{\mu \in \mathcal{P}_f(X)} \left\{ h_\mu(f) + \int \varphi \, d\mu \right\}$$

where $\mathcal{P}_f(X)$ denotes the space of f-invariant probability measures defined on the σ-algebra $\mathfrak{B}(X)$ of the Borel subsets of X. The thermodynamic formalism aims to establish the existence of measures which maximize the previous equality, besides reporting on their statistical properties. Such measures, called equilibrium states, include as specific examples Gibbs measures, physical measures and measures of maximal entropy.

We may find in the literature several proposals to extend the previous notions of measure-theoretic entropy and topological pressure to the setting of finitely generated group or semigroup actions, which in general depend on the type of (semi)group and on the fixed set of generators. The thermodynamic formalism of these actions has received much attention in recent years, though there is still no axiomatic frame for it. A main difficulty in setting up a unified thermodynamic formalism for (semi)group actions is the following: although the adapted notion of pressure matches the kind of (semi)group under study, a general concept of measure-theoretic entropy should make no reference to probability measures invariant by all the elements of the (semi)group, since typical finitely generated (semi)groups are expected to be free (see [24]) and to carry no common invariant measure.

In [7], given a pressure function on a suitable Banach space, the authors used methods from Convex Analysis to obtain upper semi-continuous measure-theoretic entropy maps, defined either on the space of probability measures or on the space of finitely additive set functions. This strategy does not summon any dynamics and is unspecific enough to be applied to either the classical topological pressure associated to a dynamical system (as done in [7]) or to the recently defined notions of topological pressure for the (semi)group actions we address in this work: finitely generated semigroup actions and countable sofic group actions.

For finitely generated semigroup actions, the first notion of topological pressure we will analyse was inspired by foliations on compact manifolds. More precisely, E. Ghys, R. Langevin and P. Walczak introduced in [23] the notion of entropy of a foliation, which conveyed the concept of topological entropy of the pseudogroup induced by the foliation. The main property of the former notion for codimensional one foliations $\mathcal{F}$ is its connection with the geometry of the manifold. For instance, if this geometric entropy vanishes, then so does the Godbillon-Vey class of $\mathcal{F}$. On the other hand, if the geometric entropy of $\mathcal{F}$ is positive then there exists a so called resilient leaf, which is a counterpart of a horseshoe for classical dynamical systems. Moreover, any foliation with vanishing geometric entropy admits a nontrivial transverse invariant measure (cf. [42]). However, no measure-theoretic counterpart was proposed in [23]. One may instead define the topological pressure of a finitely generated semigroup action by means of the Carathéodory structures developed in [35, Chapter 4]. These structures, which also appear in the definition of Hausdorff or box-counting dimensions, have been extensively used by several authors (see, for instance [6, 46, 47]) but, while the topological objects are already well understood, neither a measure-theoretic entropy map nor a variational principle have yet been described.

For countable sofic groups, introduced by M. Gromov in [25], L. Bowen proposed in [11] the concept of metric entropy of the action of such a group acting on a probability space. In rough terms, it measures the exponential growth of the complexity seen through a reference measure, and does not require invariance whatsoever. Later, a topological entropy for this kind of actions was defined by D. Kerr and H. Li in [27], where the authors also established a variational principle under mild assumptions. More recently, a notion of sofic pressure and a corresponding variational principle have been introduced in [18]. These are quite general concepts, which generalize the metric entropy, pressure and variational principle for countable amenable group actions proved in [33]. We are naturally led to ask whether equilibrium states exist and are unique. As far as we know, there are few classes of sofic group actions for which equilibrium states are known to exist. Existence is guaranteed under weak forms of expansiveness, which imply upper semi-continuity of the sofic metric entropy with respect to the weak* topology on the space of probability measures. This holds, for instance, whenever a countable amenable group acts on a set $X = \mathcal{A}^G$ by shifting, where $\mathcal{A}$ is a finite alphabet (cf. [18, Theorem 5.3] and [13, Example 7]). Regarding uniqueness, we may find in [13, §8.2] an example of a mixing Markov chain over a free group with more than one measure of maximal sofic entropy. We refer the reader to [13] for a comprehensive survey on sofic entropy and related subjects.

2 Main results

Let (X, d) be a locally compact metric space with a distance d, $\mathfrak{B}$ be its σ-algebra of Borel sets, $\mathcal{P}(X)$ denote the space of Borel probability measures on X with the weak*-topology, $\mathcal{P}_a(X)$ stand for the set of Borel real-valued normalized finitely additive set functions (which we will simply call *finitely additive probabilities*) with the total variation norm, and $C(X)$ the space of continuous maps $\psi\colon X \to \mathbb{R}$.

2.1 An abstract variational principle

In what follows we will consider a Banach space $\mathbf{B}$ over the field $\mathbb{R}$ equal to either

$$
\begin{aligned}
L^\infty(X) &= \big\{\varphi\colon X \to \mathbb{R} \mid \varphi \text{ is measurable and bounded}\big\} \\
&\text{or} \\
C_c(X) &= \big\{\varphi \in C(X) \mid \varphi \text{ has compact support}\big\}
\end{aligned}
$$

endowed with the supremum norm $\|\varphi\|_\infty = \sup_{x\in X} |\varphi(x)|$, whose elements will be called potentials. The Riesz representation theorem asserts that the dual of $C_c(X)$ is identified with the collection of all finite signed measures on $(X, \mathfrak{B})$, whose positive normalized continuous functionals correspond to the space $\mathcal{P}(X)$. It is known that the latter set is compact when equipped with the weak* topology (cf. [21, Theorem 2, V.4.2]). On the other hand, the dual of $L^\infty(X)$ is isometrically isomorphic to the space of regular finitely additive bounded signed measures on $(X, \mathfrak{B})$ with the total variation norm, whose subset of positive normalized elements is represented by $\mathcal{P}_a(X)$. We refer the reader to [2] for more information on these dualities.

Definition 2.1. *A function* $\Gamma\colon \mathbf{B} \to \mathbb{R}$ *is called a* pressure function *if it satisfies the following conditions:*

(C_1) Increasing*:* $\varphi \leqslant \psi \quad \Rightarrow \quad \Gamma(\varphi) \leqslant \Gamma(\psi) \quad \forall\, \varphi,\, \psi \in \mathbf{B}$.

(C_2) Translation invariant*:* $\Gamma(\varphi + c) = \Gamma(\varphi) + c \quad \forall\, \varphi \in \mathbf{B} \quad \forall\, c \in \mathbb{R}$.

(C_3) Convex*:* $\Gamma(t\,\varphi + (1-t)\,\psi) \leqslant t\,\Gamma(\varphi) + (1-t)\,\Gamma(\psi) \quad \forall\, \varphi,\, \psi \in \mathbf{B}$ $\forall\, t \in [0,1]$.

Properties (C_1) and (C_2) imply that any pressure function is Lipschitz continuous, meaning that
$|\Gamma(\varphi) - \Gamma(\psi)| \leqslant \|\varphi - \psi\|_\infty$ for every $\varphi,\, \psi \in \mathbf{B}$. The following is the abstract variational principle we will use further on. It associates to each pressure function Γ a measure-theoretic entropy map which is upper semi-continuous and upper bounded by $\Gamma(0)$.

Theorem 2.2. *[7, Theorem 1] Let (X,d) be a locally compact metric space, $\Gamma\colon \mathbf{B} \to \mathbb{R}$ be a pressure function and $\mathcal{A}_\Gamma$ be the set*

$$\mathcal{A}_\Gamma \,=\, \big\{\varphi \in \mathbf{B}\colon\; \Gamma(-\varphi) \leqslant 0\big\}.$$

Then there exists a concave, upper semi-continuous map $\mathfrak{h}$ which satisfies:

(a) If $\mathbf{B} = C_c(X)$, then

$$\Gamma(\varphi) \;=\; \max_{\mu \in \mathcal{P}(X)} \; \Big\{\mathfrak{h}(\mu) + \int \varphi\, d\mu\Big\} \qquad \forall\, \varphi \in C_c(X) \tag{2.1}$$

and

$$\mathfrak{h}(\mu) \;=\; \inf_{\varphi \in \mathcal{A}_\Gamma} \; \Big\{\int \varphi\, d\mu\Big\} \;=\; \inf_{\varphi \in C_c(X)} \; \Big\{\Gamma(\varphi) - \int \varphi\, d\mu\Big\} \tag{2.2}$$

for every $\mu \in \mathcal{P}(X)$. Moreover, if $\mathcal{E}_\varphi(\Gamma)$ denotes the set of maximizing elements in (2.1)*, then there is a residual subset $\mathfrak{R} \subset C_c(X)$ such that $\#\,\mathcal{E}_\varphi(\Gamma) = 1$ for every $\varphi \in \mathfrak{R}$.*

(b) If $\mathbf{B} = L^\infty(X)$, then

$$\Gamma(\varphi) \;=\; \max_{\mu \in \mathcal{P}_a(X)} \; \Big\{\mathfrak{h}(\mu) + \int \varphi\, d\mu\Big\} \qquad \forall\, \varphi \in L^\infty(X) \tag{2.3}$$

and

$$\mathfrak{h}(\mu) \;=\; \inf_{\varphi \in \mathcal{A}_\Gamma} \; \Big\{\int \varphi\, d\mu\Big\} \;=\; \inf_{\varphi \in L^\infty(X)} \; \Big\{\Gamma(\varphi) - \int \varphi\, d\mu\Big\} \tag{2.4}$$

for every $\mu \in \mathcal{P}_a(X)$*. Furthermore, if* $\alpha\colon \mathcal{P}(X) \to \mathbb{R} \cup \{-\infty\}$ *(respectively* $\alpha\colon \mathcal{P}_a(X) \to \mathbb{R} \cup \{-\infty\}$*) is another function taking the role of* $\mathfrak{h}$ *in* (2.1) *(respectively* (2.3)*), then* $\alpha \leqslant \mathfrak{h}$*.*

Comparing Theorem 2.2 to a similar result in [26], one notes that we neither require the pressure function to preserve co-boundary type maps, nor we need to consider a dynamical system somehow determining the pressure function. This will be crucial later on, since we will deal with pressure functions associated to group and semigroup actions. Item (a) in Theorem 2.2 was used in [7] to produce a class of generalized equilibrium states for maps that do not admit equilibrium states within the classical setting. Item (b) of Theorem 2.2 turned out to be fundamental to construct equilibrium states for certain sub-additive sequences of potentials, appearing naturally in the context of linear cocycles, through the reduction of the thermodynamic formalism for sub-additive sequences of continuous maps into another one for bounded measurable maps (cf. [7, §8]).

As the function Γ is increasing and translation invariant, one has

$$\Gamma(0) + \inf_{x \in X} \varphi(x) \leqslant \Gamma(\varphi) \leqslant \Gamma(0) + \sup_{x \in X} \varphi(x) \qquad \forall\, \varphi \in \mathbf{B}.$$

Consequently, from Theorem 2.2 we deduce that, for every $\mu \in \mathcal{P}(X)$ (respectively, $\mathcal{P}_a(X)$),

$$\Gamma(0) + \inf_{\varphi \in \mathbf{B}} \left\{ \inf_{x \in X} \varphi(x) - \int \varphi \, d\mu \right\} \leqslant \mathfrak{h}(\mu) \leqslant \Gamma(0).$$

It is immediate from (2.3) that the map $\mathfrak{h}$ is upper bounded by $\Gamma(0)$. Since the pointwise infimum of concave functions is concave, and affine maps are themselves concave, we get from (2.2) (respectively (2.4)) that $\mathfrak{h}$ is concave. The upper semi-continuity of $\mathfrak{h}$ is also an immediate consequence of (2.2) (respectively (2.4)), since $\mathfrak{h}$ is the infimum of the family $\left(\int \varphi \, d\mu\right)_{\varphi \in \mathcal{A}_\Gamma}$ of continuous maps.

Although Theorem 2.2 is stated within the general setting of locally compact metric spaces, in what follows we will restrict to actions of semigroups on compact metric spaces.

2.2 Semigroup and group actions

Given a semigroup G of continuous self-maps of a compact metric space (X, d), with the supremum norm and the composition operation, the semigroup action of G on X is the continuous map $\mathbb{S} : G \times X \to X$, defined by $\mathbb{S}(\underline{g}, x) = \underline{g}(x)$ for every $x \in X$ and every $\underline{g} \in G$, which satisfies the condition

$$\forall\, \underline{g},\, \underline{h} \in G \quad \forall\, x \in X \qquad \mathbb{S}(\underline{g}\,\underline{h}, x) = \mathbb{S}(\underline{g},\, \mathbb{S}(\underline{h}, x)).$$

Since we will address several types of (semi)groups G and their corresponding known notions of topological pressure, one needs to treat them separately, showing for each case that the topological pressure operator is a pressure function. In this regard, we summarize in the following table the main sources of information we will need further on.

Action type	*Topological pressure*	*Metric entropy*	*Variational principle*
$\mathbb{Z}^d$	[32, 39]	[32, 39]	[32, 39]
CAG	[33]	[33, 48]	[33]
CSG	[18]	[11, 27]	[18]
FFGS	[15, 16, 23]	[16]	[16]
FGS	[23, 5, 6, 46]	–	–

CAG=Countable amenable group, CSG=Countable sofic group, FFGS=Free finitely generated semigroup, FGS=Finitely generated semigroup

Denote by $\mathcal{G}$ the family of the (semi)groups we have just mentioned. For each $G \in \mathcal{G}$, let $P_{\text{top}}(G, \cdot)\colon C(X) \to \mathbb{R}$ be a matching notion of topological pressure as specified in the previous table, and $h_{\text{top}}(G) = P_{\text{top}}(G, 0)$ be the topological entropy of G. The next statement guarantees that the assumptions of Theorem 2.2 are fulfilled by the known definitions of topological pressure for the type of group and semigroup actions we address in this work.

Main Theorem 2.3. *If $G \in \mathcal{G}$ and $h_{\text{top}}(G) < +\infty$, then $P_{\text{top}}(G, \cdot)$ is a pressure function.*

Therefore, under the assumptions of Theorem 2.3, the unifying statement of Theorem 2.2 provides a concave upper semi-continuous function $\mathfrak{h}_G : \mathcal{P}(X) \to \mathbb{R}$ such that

$$P_{\text{top}}(G,\, \varphi) = \max_{\mu \in \mathcal{P}(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi \, d\mu \right\} \qquad \forall\, \varphi \in C(X) \tag{2.5}$$

and

$$\mathfrak{h}_G(\mu) = \inf_{\varphi \in C(X)} \left\{ P_{\text{top}}(G,\, \varphi) - \int \varphi \, d\mu \right\} \qquad \forall\, \mu \in \mathcal{P}(X). \tag{2.6}$$

Consequently, a measure-theoretic entropy map (namely, $\mathfrak{h}_G$), a variational principle and equilibrium states are now available for all these semigroup and group actions.

2.3 Coexisting variational principles

In the cases of semigroups $G \in \mathcal{G}$ for which there is already in the literature a notion of measure-theoretic entropy h_G, defined on the compact (in the weak*-topology) and convex subset $\mathcal{P}_G(X) \subseteq \mathcal{P}(X)$ of Borel probability measures in X invariant by all the elements of the group (which we will call G-invariant), and whose topological pressure satisfies a variational principle with respect to h_G, we have now two variational principles and two metric entropy maps (possibly coincident) to deal with. This happens, for instance, when G is generated by a single map (cf. [43, §9.3]) or, more generally, when G is a countable amenable group (cf. [33]). The next result generalizes Theorem 5 in [7], relating the restriction to $\mathcal{P}_G(X)$ of the variational metric entropy $\mathfrak{h}_G$ with:

(a) The star-entropy of these semigroup actions, defined at each $\mu \in \mathcal{P}_G(X)$ by

$$h_G^*(\mu) \;=\; \sup\Big\{\limsup_{n\to+\infty} h_G(\mu_n) \mid \mu_n \in \mathcal{P}_G(X) \text{ and } \lim_{n\to+\infty} \mu_n = \mu\Big\}$$

where the convergence of $(\mu_n)_{n\in\mathbb{N}}$ happens in the weak*-topology. It is known (cf. [20, page 467]) that h_G^* is the upper semi-continuous envelope of h_G in $\mathcal{P}_G(X)$, that is, for every $\mu \in \mathcal{P}_G(X)$ one has

$$h_G^*(\mu) \;=\; \inf\,\big\{T(\mu) \mid T\colon \mathcal{P}_G(X) \to \mathbb{R} \text{ is continuous and } T \geqslant h_G\big\}.$$

(b) The double-star entropy, which is the upper semi-continuous concave envelope of the metric entropy h_G (cf. [20]), defined in $\mathcal{P}_G(X)$ by

$$h_G^{**}(\mu) \;=\; \inf\Big\{\mathcal{T}(\mu) \mid \mathcal{T}\colon \mathcal{P}_G(X) \to \mathbb{R} \text{ is continuous, affine, } \mathcal{T} \geqslant h_G\Big\}.$$

Clearly,

$$h_G^*(\mu) \;\leqslant\; h_G^{**}(\mu) \quad \forall\, \mu \in \mathcal{P}_G(X).$$

Since the star-entropy h_G^* is the upper semi-continuous envelope of h_G, the next result states, in particular, that the variational measure-theoretic entropy $\mathfrak{h}_G$ is a regularization of the map h_G in the sense of [20, 9].

Main Theorem 2.4. *Consider a semigroup (resp. a group) G of continuous self-maps (resp. homeomorphisms) of a compact metric space X. Let $\mathfrak{h}_G\colon \mathcal{P}(X) \to \mathbb{R}$ denote its variational metric entropy, provided by* (2.6) *for $\Gamma = P_{\text{top}}(G,\cdot)\colon C(X) \to \mathbb{R}$. Assume that $P_G(X) \neq \emptyset$ and that the pressure function $P_{\text{top}}(G,\cdot)$ satisfies another variational principle*

$$P_{\text{top}}(G,\varphi) = \sup_{\mu\in\mathcal{P}_G(X)} \Big\{h_G(\mu) + \int \varphi\, d\mu\Big\} \qquad \forall\, \varphi \in C(X) \tag{2.7}$$

with respect to a non-negative metric entropy map $h_G\colon \mathcal{P}_G(X) \to [0,+\infty[$. Then:

(a) If $\varphi \in C(X)$ and $\mu_\varphi \in \mathcal{P}(X)$ attains the maximum in (2.5), then μ_φ is G-invariant, that is,

$$\int (\psi \circ g)\, d\mu_\varphi = \int \psi\, d\mu_\varphi \qquad \forall\, g \in G \quad \forall\, \psi \in C(X).$$

(b) Given $\mu \in \mathcal{P}(X)$, one has $\mu \in \mathcal{P}_G(X)$ if and only if $\mathfrak{h}_G(\mu) \geqslant 0$.

(c) For every $\mu \in \mathcal{P}_G(X)$,

$$0 \leqslant h_G(\mu) \leqslant h^*_G(\mu) \leqslant \mathfrak{h}_G(\mu) \quad \textit{and} \quad h^{**}_G(\mu) \leqslant h_{\mathrm{top}}(G).$$

(d) If h_G is concave and upper semi-continuous, then $h_G = h^*_G = \mathfrak{h}_G$ in $\mathcal{P}_G(X)$.

(e) For every $\varphi \in C(X)$,

$$P_{\mathrm{top}}(G,\varphi) = \max_{\mu \in \mathcal{P}_G(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi\, d\mu \right\} = \max_{\mu \in \mathcal{P}_G(X)} \left\{ h^*_G(\mu) + \int \varphi\, d\mu \right\}.$$

Remark 2.5. If $\mu_0 \in \mathcal{P}_G(X)$ is an equilibrium state for φ with respect to the variational principle (2.7), that is

$$P_{\mathrm{top}}(G,\varphi) = h_G(\mu_0) + \int \varphi\, d\mu_0,$$

then μ_0 attains the maximum at (2.5) as well. Indeed,

$$P_{\mathrm{top}}(G,\varphi) \geqslant \mathfrak{h}_G(\mu_0) + \int \varphi\, d\mu_0 \geqslant h_G(\mu_0) + \int \varphi\, d\mu_0 = P_{\mathrm{top}}(G,\varphi).$$

The converse is not true in general (see [7, §7.4]).

In the next sections, we will prove Theorem 2.3 and Theorem 2.4. The proof of the former is quite long, inasmuch as it has to be done separately for each type of (semi)group in $\mathcal{G}$ and corresponding topological pressure operator. In the last section we will verify that, under mild assumptions which are satisfied by the pressure function of any $G \in \mathcal{G}$, the variational metric entropy map $\mathfrak{h}_G$ is an invariant by isomorphism of the (semi)group actions (see Section 8 for the definitions and more details).

3 Proof of Theorem 2.4

Let G be a semigroup of continuous self-maps of a compact metric space X and $P_{\mathrm{top}}(G,\cdot)$ be a pressure function satisfying the assumptions of Theorem 2.4. The following argument also works if G is a group of homeomorphisms of X.

(a) As $P_{\mathrm{top}}(G, \cdot)$ satisfies the variational principle (2.7) with respect to G-invariant probability measures, it is easy to conclude that, for every observables φ, $\psi \in C(X)$ and every $g \in G$, we have

$$\begin{aligned} P_{\mathrm{top}}(G,\, \varphi + \psi \circ g - \psi) &= \sup_{\mu \in \mathcal{P}_G(X)} \left\{ h_G(\mu) + \int (\varphi + \psi \circ g - \psi)\, d\mu \right\} \\ &= \sup_{\mu \in \mathcal{P}_G(X)} \left\{ h_G(\mu) + \int \varphi\, d\mu \right\} \end{aligned}$$

and, consequently,

$$P_{\mathrm{top}}(G,\, \varphi + \psi \circ g - \psi) \;=\; P_{\mathrm{top}}(G,\, \varphi) \;=\; P_{\mathrm{top}}(G, \varphi + \psi - \psi \circ g). \tag{3.1}$$

We claim that every probability measure attaining the supremum in (2.5) is G-invariant. Consider $\psi \in C(X)$, $g \in G$ and fix $\mu_\varphi, \mu_1,\, \mu_2 \in \mathcal{P}(X)$ provided by (2.5) such that

$$\begin{aligned} P_{\mathrm{top}}(G, \varphi) &= \mathfrak{h}_G(\mu_\varphi) + \int \varphi\, d\mu_\varphi \\ P_{\mathrm{top}}(G, \varphi + \psi \circ g - \psi) &= \mathfrak{h}_G(\mu_1) + \int \varphi\, d\mu_1 + \int (\psi \circ g)\, d\mu_1 - \int \psi\, d\mu_1 \\ P_{\mathrm{top}}(G, \varphi + \psi - \psi \circ g) &= \mathfrak{h}_G(\mu_2) + \int \varphi\, d\mu_2 + \int \psi\, d\mu_2 - \int (\psi \circ g)\, d\mu_2. \end{aligned}$$

The first two equalities, property (3.1) and the variational principle (2.5) now yield

$$\begin{aligned} \mathfrak{h}_G(\mu_\varphi) + \int \varphi\, d\mu_\varphi &= \mathfrak{h}_G(\mu_1) + \int \varphi\, d\mu_1 + \int (\psi \circ g)\, d\mu_1 - \int \psi\, d\mu_1 \\ &\geqslant \mathfrak{h}_G(\mu_\varphi) + \int \varphi\, d\mu_\varphi + \int (\psi \circ g)\, d\mu_\varphi - \int \psi\, d\mu_\varphi \end{aligned}$$

and so $\int (\psi \circ g)\, d\mu_\varphi - \int \psi\, d\mu_\varphi \leqslant 0$. In a similar way, we deduce that

$$\begin{aligned} \mathfrak{h}_G(\mu_\varphi) + \int \varphi\, d\mu_\varphi &= \mathfrak{h}_G(\mu_2) + \int \varphi\, d\mu_2 + \int \psi\, d\mu_2 - \int (\psi \circ g)\, d\mu_2 \\ &\geqslant \mathfrak{h}_G(\mu_\varphi) + \int \varphi\, d\mu_\varphi + \int \psi\, d\mu_\varphi - \int (\psi \circ g)\, d\mu_\varphi \end{aligned}$$

so $\int \psi\, d\mu_\varphi - \int (\psi \circ g)\, d\mu_\varphi \leqslant 0$. Therefore, μ_φ is G-invariant.

Remark 3.1. The only property we used so far is (3.1), which in general may be established either before or without proving that the pressure function $P_{\mathrm{top}}(G, \cdot)$ satisfies a variational principle like (2.7) with respect to a non-negative metric entropy map. This is the case of both the topological pressure for a map [43, §9.2] and the pressure function defined by Ollagnier and Pinchon in [33] for countable amenable group actions.

Remark 3.2. Whenever an action of a (semi)group G on a compact metric space X does not admit a G-invariant Borel probability measure, although its topological entropy is finite, we conclude from item (a) of Theorem 2.4 that the pressure function we are dealing with does not satisfy (3.1). See Examples 5.5 and 5.6.

(b) The proof of assertion (b) in Theorem 2.4 is analogous to the arguments in [43, page 222], for a map, and in [16, page 464] for free finitely generated semigroup actions. Indeed, from the assumptions that $h_{\mathrm{top}}(G) < +\infty$ and that the topological pressure satisfies a variational principle with the non-negative metric entropy map h_G, we conclude that, given $\mu \in \mathcal{P}(X)$,

$$\mu \in \mathcal{P}_G(X) \quad \Leftrightarrow \quad \int \varphi \, d\mu \leqslant P_{\mathrm{top}}(G, \varphi) \qquad \forall \varphi \in C(X).$$

Therefore, by formula (2.6), we get

$$\mu \in \mathcal{P}_G(X) \quad \Leftrightarrow \quad \mathfrak{h}_G(\mu) \geqslant 0.$$

This ends the proof that the map $\mathfrak{h}_G$ determines the set $\mathcal{P}_G(X)$, as claimed.

(c) By definition, one has $0 \leqslant h_G(\mu) \leqslant h^*_G(\mu)$ for every $\mu \in \mathcal{P}_G(X)$. Moreover, by the variational principle (2.7), for each $\mu \in \mathcal{P}_G(X)$

$$h_G(\mu) \leqslant P_{\mathrm{top}}(G, \varphi) - \int \varphi \, d\mu \qquad \forall \varphi \in C(X)$$

and so

$$\mathfrak{h}_G(\mu) = \inf_{\varphi \in C(X)} \left\{ P_{\mathrm{top}}(G, \varphi) - \int \varphi \, d\mu \right\} \geqslant h_G(\mu). \tag{3.2}$$

Consequently, if $h^*_G(\mu) > \mathfrak{h}_G(\mu)$ for some $\mu \in \mathcal{P}_G(X)$, then there would exist $\nu \in \mathcal{P}_G(X)$ satisfying $h_G(\nu) > \mathfrak{h}_G(\nu)$, contradicting (3.2). This proves that

$$0 \leqslant h_G(\mu) \leqslant h^*_G(\mu) \leqslant \mathfrak{h}_G(\mu) \qquad \forall \mu \in \mathcal{P}_G(X). \tag{3.3}$$

By the variational principle (2.7), we know that the continuous affine (constant) map $h_{\mathrm{top}}(G)$ satisfies $h_{\mathrm{top}}(G) \geqslant h_G$. Thus, by definition, one has $h^{**}_G(\mu) \leqslant h_{\mathrm{top}}(G)$ for every $\mu \in \mathcal{P}_G(X)$.

(d) Assume that h_G is concave and upper semi-continuous. Then, using the variational principle (2.7) and [43, Theorem 9.12], we conclude that

$$h_G(\mu) = \inf_{\varphi \in C(X)} \left\{ P_{\mathrm{top}}(G, \varphi) - \int \varphi \, d\mu \right\} \qquad \forall \mu \in \mathcal{P}_G(X).$$

Since $\mathfrak{h}_G$ also satisfies this formula (cf. (2.2)), one has $h_G = \mathfrak{h}_G$ in $\mathcal{P}_G(X)$. Hence, the inequalities (3.3) yield $h_G = h^*_G = \mathfrak{h}_G$ in $\mathcal{P}_G(X)$.

(e) We start by observing that the inequalities (3.3) imply that

$$\sup_{\mu \in \mathcal{P}_G(X)} \left\{ h_G(\mu) + \int \varphi \, d\mu \right\} \leqslant \max_{\mu \in \mathcal{P}_G(X)} \left\{ h^*_G(\mu) + \int \varphi \, d\mu \right\} \leqslant \max_{\mu \in \mathcal{P}_G(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi \, d\mu \right\}.$$

On the other hand, as $\mathcal{P}_G(X) \subset \mathcal{P}(X)$ and $\mathfrak{h}_G$ is upper semi-continuous (hence it has a maximum on the compact $\mathcal{P}_G(X)$), for every $\varphi \in C(X)$ one has

$$\max_{\mu \in \mathcal{P}_G(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi \, d\mu \right\} \leqslant \max_{\mu \in \mathcal{P}(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi \, d\mu \right\}.$$

Therefore, from the variational principles (2.5) and (2.7) we obtain

$$\begin{aligned} \max_{\mu \in \mathcal{P}_G(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi \, d\mu \right\} &\leqslant \max_{\mu \in \mathcal{P}(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi \, d\mu \right\} \\ &= P_{\text{top}}(G, \varphi) \\ &= \sup_{\mu \in \mathcal{P}_G(X)} \left\{ h_G(\mu) + \int \varphi \, d\mu \right\} \\ &\leqslant \max_{\mu \in \mathcal{P}_G(X)} \left\{ h^*_G(\mu) + \int \varphi \, d\mu \right\} \\ &\leqslant \max_{\mu \in \mathcal{P}_G(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi \, d\mu \right\}. \end{aligned}$$

Consequently,

$$P_{\text{top}}(G, \varphi) = \max_{\mu \in \mathcal{P}_G(X)} \left\{ \mathfrak{h}_G(\mu) + \int \varphi \, d\mu \right\} = \max_{\mu \in \mathcal{P}_G(X)} \left\{ h^*_G(\mu) + \int \varphi \, d\mu \right\}.$$

The proof of Theorem 2.4 is complete. □

In the next section we will prove Theorem 2.3, addressing each type of (semi)group action separately: finitely generated semigroup actions in Sections 4 and 5; Carathéodory structures for finitely generated group actions in Section 6; and countable sofic group actions in Section 7.

4 Finitely generated free semigroup actions

Consider a compact metric space (X, d) and a semigroup $(G, \circ)$ of continuous endomorphisms of X, where the semigroup operation $\circ$ is the composition of maps. Assume that G is finitely generated, that is, there exists a finite set $G_1 = \{id_X, g_1, \cdots, g_p\} \subset G$ such that $G = \bigcup_{n=1}^{+\infty} G_n$, where, for each $n \in \mathbb{N}$,

$$G_n = \left\{ g_{j_n} \circ \cdots \circ g_{j_1} : \quad g_{j_i} \in G_1 \quad \forall\, 1 \leqslant i \leqslant n \right\}.$$

Each element g of G_n may be seen as a word which originates from the concatenation of n elements in G_1. Yet, different concatenations may generate the same element in G, unless $G = \mathbb{F}_p^+$ is the free semigroup. So, when considering free semigroup actions, we will regard the different concatenations instead of the elements in G they create. One way to interpret this statement is to define the itinerary map

$$\begin{array}{rccc} \iota\colon & \mathbb{F}_p & \to & G_n \subset G \\ & j_1 \dots j_n & \mapsto & g_{j_n} \circ \dots \circ g_{j_1} \end{array}$$

where $\mathbb{F}_p$ is the free semigroup with p generators, thus addressing concatenations on G as images by ι of finite words on $\mathbb{F}_p$. Thereby, each $x \in X$ is endowed by the pair $(G,\, G_1)$ with infinitely many path-orbits, whose union describes accurately the action of $(G,\, G_1)$ on X.

There have been several proposals to generalize the notions of entropy and pressure for a continuous map to the setting of finitely generated semigroup actions; for an account on some of them we refer the reader to [4], [16] and references therein. We shall start by considering the notion introduced by Ghys, Langevin and Walczak in [23], which we denote by $h_{\text{top}}(G,\, G_1)$ to emphasize its dependence on the set G_1 of generators. In what follows we will define a corresponding pressure $P_{\text{top}}(G,\, G_1,\, \cdot)$ on $C(X)$ and prove that it is a pressure function provided that $P_{\text{top}}(G,\, G_1,\, \varphi)$ is finite for every $\varphi \in C(X)$.

4.1 Topological entropy

Given $n \in \mathbb{N}$ and $\varepsilon > 0$, the (n, ε)-Bowen ball generated by the semigroup action and centered at x is defined by

$$B_n^G(x, \varepsilon) = \Big\{ y \in X\colon\; d(g(x), g(y)) < \varepsilon \quad \forall\, g \in G_n \Big\}.$$

One says that two points $x,\, y \in X$ are (n, ε)-separated by G if there exists $g \in G_n$ such that $d(g(x),\, g(y)) \geqslant \varepsilon$, that is, y does not belong to $B_n^G(x, \varepsilon)$. A subset E of X is (n, ε)-separated if any two distinct points of E are (n, ε)-separated by G. Having fixed $n \in \mathbb{N}$ and $\varepsilon > 0$, consider

$$s_n(G, G_1, \varepsilon) = \max \{|E|\colon\;\; E \subset X \text{ is } (n, \varepsilon)\text{-separated}\}.$$

Since X is compact, $s_n(G, G_1, \varepsilon)$ is finite for every $n \in \mathbb{N}$ and $\varepsilon > 0$. Moreover, the map

$$\varepsilon > 0 \quad \mapsto \quad \limsup_{n \to +\infty} \frac{1}{n} \log\, s_n(G, G_1, \varepsilon)$$

is monotone.

Definition 4.1. *The* topological entropy *of the semigroup G generated by G_1 is given by*

$$h_{\text{top}}(G,\, G_1) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log\, s_n(G, G_1, \varepsilon).$$

4.2 Topological pressure for finitely generated free semigroup actions

Our aim is to generalize the previous notion to any potential $\varphi \in C(X)$. Given $n \in \mathbb{N}$, $\varepsilon > 0$ and $g \in G_n$ presented by the concatenation $g = g_{j_n} \circ \cdots \circ g_{j_1}$ (which may be one of many such presentations), where $g_{j_i} \in G_1$ for every $i \in \{1, \cdots, n\}$, define

$$x \in X \quad \mapsto \quad S_n^g\, \varphi(x) = \varphi(x) + \varphi(g_{j_1}(x)) + \varphi(g_{j_2}\, g_{j_1}(x)) + \cdots + \varphi(g_{j_n} \cdots g_{j_1}(x)).$$

Definition 4.2. *The* topological pressure *of a free finitely generated semigroup* (G, G_1) *and the potential* φ *is given by*

$$P_{\text{top}}(G,\, G_1,\, \varphi) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log P_n(G,\, G_1,\, \varphi,\, \varepsilon)$$

where, for every $n \in \mathbb{N}$ *and* $\varepsilon > 0$*,*

$$P_n(G,\, G_1,\, \varphi,\, \varepsilon) = \frac{1}{|G_n|} \sum_{g \in G_n} \sup_E \Big\{ \sum_{x \in E} e^{S_n^g\, \varphi(x)} : \;\; E \subset X \text{ is } (n, \varepsilon)\text{-separated} \Big\}.$$

4.3 Properties of the pressure

We now study the behavior of the operator $P_{\text{top}}(G,\, G_1, .)$ when we change either the potential or the semigroup. The following properties are similar to the ones listed on [43, Theorem 9.7] and [43, Theorem 9.8] for the pressure associated to one dynamics. Their proofs are also identical to [43, §9].

4.3.1 Variation of $P_{\text{top}}(G,\, G_1, .)$ with the potential

We start by verifying that the map $P_{\text{top}}(G,\, G_1, .) : \; C(X) \; \to \; \mathbb{R}$ satisfies the three axioms requested on the definition of a pressure function.

Lemma 4.3. *If* $h_{\text{top}}(G,\, G_1) < +\infty$*, then* $P_{\text{top}}(G,\, G_1, .)$ *is a pressure function.*

Proof. Consider $\varphi,\, \psi \in C(X)$, $c \in \mathbb{R}$ and $\varepsilon > 0$. From Definition 4.2 it is clear that

1. $P_{\text{top}}(G,\, G_1,\, 0) = h_{\text{top}}(G,\, G_1)$.

2. $P_{\text{top}}(G,\, G_1,\, \varphi + c) = P_{\text{top}}(G,\, G_1,\, \varphi) + c$.

3. $\varphi \leqslant \psi \;\Rightarrow\; P_{\text{top}}(G,\, G_1,\, \varphi) \leqslant P_{\text{top}}(G,\, G_1,\, \psi)$.

In particular,

$$h_{\text{top}}(G,\, G_1) + \min\, \varphi \;\leqslant\; P_{\text{top}}(G,\, G_1,\, \varphi) \;\leqslant\; h_{\text{top}}(G,\, G_1) + \max\, \varphi$$

from whose inequalities we also conclude that $P_{\text{top}}(G, G_1, .)$ is either finite valued or identically $+\infty$.

The pressure map $P_{\text{top}}(G, G_1, .)$ is also convex. Clearly, given $0 < t < 1$, $n \in \mathbb{N}$, $g \in G_n$ and $\varphi, \psi \in C(X)$, one has

$$S_n^g \big(t\varphi + (1-t)\psi\big) \;=\; t\, S_n^g\, \varphi + (1-t)\, S_n^g\, \psi.$$

Moreover, for every finite subset E of X, by the Holder's inequality (using $p = 1/t$ and $q = 1/(1-t)$) we know that

$$\sum_{x \in E} e^{S_n^g \big(t\varphi + (1-t)\psi\big)(x)} \;\leqslant\; \Big(\sum_{x \in E} e^{S_n^g\, \varphi(x)}\Big)^t \, \Big(\sum_{x \in E} e^{S_n^g\, \psi(x)}\Big)^{1-t}.$$

Besides, since the maps $u > 0 \to u^t$ and $u > 0 \to u^{1-t}$ are increasing, having fixed $\varepsilon > 0$ the previous inequality yields

$$\sup_E \sum_{x \in E} e^{S_n^g \big(t\varphi + (1-t)\psi\big)(x)} \;\leqslant\; \Big(\sup_E \sum_{x \in E} e^{S_n^g\, \varphi(x)}\Big)^t \, \Big(\sup_E \sum_{x \in E} e^{S_n^g\, \psi(x)}\Big)^{1-t}$$

where the supremum is taken over all (n, ε)-separated subsets E of X. So,

$$\frac{1}{|G_n|} \sum_{g \in G_n} \sup_E \Big\{\sum_{x \in E} e^{S_n^g \big(t\varphi + (1-t)\psi\big)(x)} : \;\; E \subset X \text{ is } (n,\varepsilon)\text{-separated}\Big\}$$
$$\leqslant\; \frac{1}{|G_n|} \sum_{g \in G_n} \Big(\sup_E \sum_{x \in E} e^{S_n^g\, \varphi(x)}\Big)^t \, \Big(\sup_E \sum_{x \in E} e^{S_n^g\, \psi(x)}\Big)^{1-t}.$$

Applying again the Holder's inequality (with the same values of p and q), we obtain

$$\frac{1}{|G_n|} \sum_{g \in G_n} \sup_E \Big\{\sum_{x \in E} e^{S_n^g \big(t\varphi + (1-t)\psi\big)(x)} : \;\; E \subset X \text{ is } (n,\varepsilon)\text{-separated}\Big\}$$
$$\leqslant\; \Big(\frac{1}{|G_n|} \sum_{g \in G_n} \sup_E \sum_{x \in E} e^{S_n^g\, \varphi(x)}\Big)^t \, \Big(\frac{1}{|G_n|} \sum_{g \in G_n} \sup_E \sum_{x \in E} e^{S_n^g\, \psi(x)}\Big)^{1-t}$$

where the supremum is taken over the (n, ε)-separated subsets of X. Thus,

$$P_n(G, G_1, t\varphi + (1-t)\psi, \varepsilon) \;\leqslant\; \big(P_n(G, G_1, \varphi, \varepsilon)\big)^t \, \big(P_n(G, G_1, \psi, \varepsilon)\big)^{1-t}$$

which implies that

$$P_{\text{top}}(G, G_1, t\varphi + (1-t)\psi) \;\leqslant\; tP_{\text{top}}(G, G_1, \varphi) + (1-t)P_{\text{top}}(G, G_1, \psi).$$

$\square$

4.3.2 A variational principle for a finitely generated free semigroup action

Firstly note that

$$P_{\text{top}}(G,\,G_1,\,\varphi)\,\leqslant\,h_{\text{top}}(G,\,G_1)+\max_{x\,\in\,X}\,\varphi\,<\,+\infty\qquad\forall\,\varphi\in C(X).$$

Consequently, Theorem 2.2 and the information from Subsection 4.3 yield:

Proposition 4.4. *Let $(G,\,G_1)$ be a finitely generated free semigroup with $h_{\text{top}}(G,\,G_1)\,<\,+\infty$. Then there exists a concave, upper semi-continuous function $\mathfrak{h}_G\colon\,\mathcal{P}(X)\,\to\,\mathbb{R}$ such that, for every $\varphi\in C(X)$,*

$$P_{\text{top}}(G,\,G_1,\,\varphi)=\max_{\mu\,\in\,\mathcal{P}(X)}\,\left\{\mathfrak{h}_G(\mu)+\int\varphi\,d\mu\right\}. \tag{4.1}$$

As far as we know, the previous result establishes the first variational principle for the notion of topological entropy introduced by Ghys, Langevin and Walczak, whose maximal entropy measures have been discussed in [5]. Due to Proposition 4.4, new concepts are now available for finitely generated free semigroup actions with finite topological entropy. For instance, we may define invariant measures for the semigroup action as follows: $\mu\,\in\,\mathcal{P}(X)$ is a G-invariant probability measure if $\mathfrak{h}_G(\mu)\,\geqslant\,0$, or, equivalently, if $P_{\text{top}}(G,\,G_1,\,\varphi)\geqslant\int\varphi\,d\mu$ for every $\varphi\in C(X)$). Afterwards, we may take $\mathfrak{h}_G(\mu)$ as a natural notion of measure-theoretic entropy of a free semigroup action, finitely generated and with finite topological entropy, with respect to a G-invariant probability measure $\mu\in\mathcal{P}(X)$. We refer the reader to Section 8 for an extra property of $\mathfrak{h}_G$ which confirms its entropy-like nature.

5 Finitely generated semigroup actions

The pressure operator $P_{\text{top}}(G,\,G_1,\,.)$ that we analyzed in Section 4 suits free finitely generated semigroup actions. It may be reshaped, though, to comply with more general finitely generated semigroups. Consider a semigroup G generated by a finite set G_1 of continuous self-maps of X which contains the identity map. For each $n\in\mathbb{N}$, recall that $G_n=\big\{g_{i_n}\circ g_{i_{n-1}}\circ\ldots\circ g_{i_1}:\,g_{i_j}\in G_1\big\}$ and that each element $g\,\in\,G_n$ is represented by a concatenation $g_{i_n},g_{i_{n-1}},\ldots,g_{i_1}$, not necessarily in a unique way. Given a continuous potential $\varphi\colon\,X\,\to\,\mathbb{R}$, $n\,\in\,\mathbb{N}$, $x\,\in\,X$ and $g\,=\,g_{i_n}\,\circ\,g_{i_{n-1}}\,\circ\ldots\circ\,g_{i_1}\,\in\,G_n$, define

$$S_\varphi^{(g_{i_n},g_{i_{n-1}},\ldots,g_{i_1})}(x)\;=\;\varphi(x)+\varphi(g_{i_1}(x))+\ldots+\varphi(g_{i_n}\circ g_{i_{n-1}}\circ\ldots\circ g_{i_1}(x))$$

and, for each $g\in G_n$,

$$\mathrm{Max}_n^g(\varphi(x))\;=\;\max\Big\{S_\varphi^{(g_{i_n},g_{i_{n-1}},\ldots,g_{i_1})}(x)\colon\;\;g_{i_n}\circ g_{i_{n-1}}\circ\ldots\circ g_{i_1}=g\Big\}.$$

Definition 5.1. *The* topological pressure *of a finitely generated semigroup* (G, G_1) *and a potential* $\varphi \in C(X)$ *is given by*

$$P_{\text{top}}((G,G_1),\varphi) = \lim_{\varepsilon \to 0^+} \limsup_{n \to +\infty} \frac{1}{n} \log P_n((G,G_1),\varphi,\varepsilon)$$

where

$$P_n((G,G_1),\varphi,\varepsilon) = \frac{1}{|G_n|} \sum_{g \in G_n} \sup_E \Big\{ \sum_{x \in E} e^{\text{Max}_n^g(\varphi(x))} : E \subset X \textit{ is } (n,\varepsilon)\textit{-separated} \Big\}.$$

Observe that, when G is a free, finitely generated semigroup and $\varphi = 0$, the notion of topological entropy we considered in Section 4 coincides with $P_{\text{top}}((G,G_1), 0)$. In the following lemmas we will verify that $P_{\text{top}}((G,G_1),\cdot)$ defines a pressure function.

Lemma 5.2. *For any potentials* ϕ, ψ *with* $\phi \leqslant \psi$ *one has*

$$P_{\text{top}}((G,G_1),\varphi) \leqslant P_{\text{top}}((G,G_1),\psi).$$

Proof. If $\phi \leqslant \psi$ then

$$\begin{aligned} S_\phi^{(g_{i_n}, g_{i_{n-1}}, \dots, g_{i_1})}(x) &= \phi(x) + \phi(g_{i_1}(x)) + \dots + \phi(g_{i_n} \circ g_{i_{n-1}} \circ \dots \circ g_{i_1}(x)) \\ &\leqslant \psi(x) + \psi(g_{i_1}(x)) + \dots + \psi(g_{i_n} \circ g_{i_{n-1}} \circ \dots \circ g_{i_1}(x)) \\ &= S_\psi^{(g_{i_n}, g_{i_{n-1}}, \dots, g_{i_1})}(x). \end{aligned}$$

Therefore, $\text{Max}_n^g(\phi(x)) \leqslant \text{Max}_n^g(\psi(x))$ for any $g \in G_n$, and consequently

$$\begin{aligned} P_n((G,G_1),\phi,\epsilon) &= \frac{1}{|G_n|} \sum_{g \in G_n} \sup_E \Big\{ \sum_{x \in E} e^{\text{Max}_n^g(\phi(x))} : E \subset X \text{ is } (n,\varepsilon)\text{-separated} \Big\} \\ &\leqslant \frac{1}{|G_n|} \sum_{g \in G_n} \sup_E \Big\{ \sum_{x \in E} e^{\text{Max}_n^g(\psi(x))} : E \subset X \text{ is } (n,\varepsilon)\text{-separated} \Big\} \\ &= P_n((G,G_1),\psi,\varepsilon). \end{aligned}$$

Applying logarithms, dividing by n, and letting $n \to +\infty$ and $\varepsilon \to 0^+$, we obtain $P_{\text{top}}((G,G_1),\phi) \leqslant P_{\text{top}}((G,G_1),\psi)$. □

Lemma 5.3. *For any potentials* ϕ *and constant* $c \in \mathbb{R}$ *one has*

$$P_{\text{top}}((G,G_1),\phi + c) = P_{\text{top}}((G,G_1),\phi) + c.$$

Proof. Firstly, notice that

$$S_{\phi+c}^{(g_{i_n}, g_{i_{n-1}}, \dots, g_{i_1})}(x) = S_\phi^{(g_{i_n}, g_{i_{n-1}}, \dots, g_{i_1})}(x) + (n+1)c.$$

In particular, $\mathrm{Max}_n^g(\phi(x)+c) = \mathrm{Max}_n^g(\phi(x)) + (n+1)\,c$ for any $g \in G_n$, so

$$\begin{aligned}
&P_n((G,G_1),\phi+c,\varepsilon)\\
&\quad= \frac{1}{|G_n|}\sum_{g\in G_n}\sup_E\Big\{\sum_{x\in E} e^{\mathrm{Max}_n^g(\phi(x)+c)}\colon E\subset X \text{ is } (n,\varepsilon)\text{-separated}\Big\}\\
&\quad= \frac{e^{(n+1)\,c}}{|G_n|}\sum_{g\in G_n}\sup_E\Big\{\sum_{x\in E} e^{\mathrm{Max}_n^g(\phi(x))}\colon E\subset X \text{ is } (n,\varepsilon)\text{-separated}\Big\}\\
&\quad= e^{(n+1)\,c}\,P_n((G,G_1),\phi,\varepsilon).
\end{aligned}$$

Applying logarithms, dividing by n, and
taking the limits with $n\to+\infty$ and $\varepsilon\to 0^+$, we get

$$P_{\mathrm{top}}((G,G_1),\phi+c) = P_{\mathrm{top}}((G,G_1),\phi)+c.$$

□

Lemma 5.4. *For any potentials ϕ,ψ and $t\in[0,1]$ one has*

$$P_{\mathrm{top}}((G,G_1),\,t\,\phi+(1-t)\,\psi)\ \leqslant\ t\,P_{\mathrm{top}}((G,G_1),\phi)+(1-t)\,P_{\mathrm{top}}((G,G_1),\psi).$$

Proof. Clearly,

$$S_{t\phi+(1-t)\psi}^{(g_{i_n},g_{i_{n-1}},\dots,g_{i_1})}(x) = tS_{\phi}^{(g_{i_n},g_{i_{n-1}},\dots,g_{i_1})}(x)+(1-t)S_{\psi}^{(g_{i_n},g_{i_{n-1}},\dots,g_{i_1})}(x).$$

Therefore, for any $g\in G_n$,

$$\begin{aligned}
&\mathrm{Max}_n^g((t\phi+(1-t)\,\psi)(x))\\
&\qquad= \max\Big\{S_{t\,\phi+(1-t)\,\psi}^{(g_{i_n},g_{i_{n-1}},\dots,g_{i_1})}(x)\colon\ g_{i_n}\circ g_{i_{n-1}}\circ\ldots\circ g_{i_1}=g\Big\}\\
&\qquad\leqslant t\,\mathrm{Max}_n^g(\phi(x))+(1-t)\,\mathrm{Max}_n^g(\psi(x)).
\end{aligned}$$

We now proceed as in the proof of the convexity of the pressure in Lemma 4.3, replacing S_n^g by Max_n^g.

□

From the previous lemmas we conclude that a statement similar to Proposition 4.4 holds in this setting.

5.1 Examples

The next examples, together with item (a) of Theorem 2.4, show that the pressure functions we have considered in this and the previous section do not satisfy property (3.1).

Example 5.5. Consider the unit circle $X = \mathbf{S}^1 = \{e^{2\pi i\,t}\colon\ t \in [0,1[\}$ and the action by the free semigroup $\mathbb{F}_2^+$ generated by

$$g_1\colon\ e^{2\pi i\,t} \in \mathbf{S}^1\ \mapsto\ e^{2\pi i\,t^2} \qquad \text{and} \qquad g_2\colon\ e^{2\pi i\,t} \in \mathbf{S}^1\ \mapsto\ e^{2\pi i\,(t+\alpha)}$$

where $\alpha \in\ ]0,1[$. The only probability measure invariant by g_1 is the Dirac measure supported in the fixed point 1. However, this measure is not invariant by the rotation g_2. So the joint action by g_1 and g_2 has no common invariant probability measure, which implies that the maximizing elements of the generalized variational principle (4.1) are not G-invariant. We note that these two generators are Lipschitz maps and therefore the topological entropy of the corresponding action is finite (cf. [17, Proposition 3.1]).

Example 5.6. The special linear group $SL(2,\mathbb{Z})$, generated by the two linear maps with matrices

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

acts continuously on the projective space $\mathbb{R}P^1$ through the projectivizations $g_1, g_2\colon \mathbb{R}P^1 \to \mathbb{R}P^1$ of A_1 and A_2, respectively. It is known that this action does not admit an $SL(2,\mathbb{Z})$-invariant Borel probability measure (cf. [50, page 62]). Yet, the two generators are Lipschitz maps and so their action has finite topological entropy.

5.2 Homogeneous and Gibbs probability measures

In [5, 10], to overcome the absence of a unifying notion of metric entropy for finitely generated group actions, the authors propose a local entropy formula which is similar to the Brin-Katok formula in [14], and prove that, when the group G is amenable, the corresponding group action is finitely generated and G admits a homogeneous probability measure μ, then μ is a measure with maximal entropy. Let us be more precise.

Following [29], a countable group G is said to be *amenable* if for any compact set $\mathcal{K} \subset G$ and $\delta > 0$ there exists a compact set $F \subset G$ such that $m_L(F \,\Delta\, \mathcal{K}F) < \delta\, m_L(F)$, where m_L stands for the left Haar measure on G or the counting measure in the case of a discrete group G, and Δ is the symmetric difference between the two sets. Such a set F is said to be $(\mathcal{K},\delta)$-invariant. A strictly increasing sequence $(F_n)_n$ of non-empty compact subsets of G which exhausts G is said to be *Følner* if, for every compact $\mathcal{K} \subset G$, any $\delta > 0$ and all large enough $n \in \mathbb{N}$, the set F_n is $(\mathcal{K},\delta)$-invariant. We observe that, as $(F_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of finite non-empty subsets of G, then $(|F_n|)_{n\in\mathbb{N}}$ is an injective sequence of natural numbers, and so $\lim_{n\to+\infty} |F_n| = +\infty$. A Følner sequence $(F_n)_n$ is called *tempered* if there exists $C > 0$ such that

$$m_L\Big(\bigcup_{1\leqslant k<n} F_k^{-1}\, F_n\Big) \leqslant C\, m_L(F_n) \qquad \forall n \in \mathbb{N}.$$

It is known that every Følner sequence has a tempered subsequence and that every amenable group has a tempered Følner sequence (cf. [29, Proposition 1.4]). In [41, 34] one finds other equivalent definitions of amenability.

For any countable group G we may find a compact metric space X where the action of G has a probability measure invariant by all the elements of the group (see [13, §9]): for instance, $X = \mathcal{A}^G$ on a finite alphabet $\mathcal{A}$ and the action given by the left shift. Yet, as seen in Examples 5.5 and 5.6, there are groups whose actions do not admit such common invariant probability measures. Regarding general sufficient conditions for the nonexistence of G-invariant probability measures, see for instance [30, 38]. In contrast, countable amenable groups acting on any compact metric space always have probability measures which are invariant by all the elements of the group (cf. [49]). Actually, a countable group G is amenable if and only if its action on any compact metric space X has a G-invariant Borel probability measure (see [44]).

Regarding a variational principle for amenable group actions, we refer the reader to [33, 48] and references therein. In the special case of expansive $\mathbb{Z}^d$-actions with the specification property, Ruelle constructed equilibrium states and proved that they are Gibbs measures [39]. The Pointwise Ergodic Theorem for amenable group actions (cf. [29, Theorem 1.2]) ensures that, if μ is G-invariant and $(F_n)_n$ is a tempered Følner sequence, then for every $\varphi \in L^1(\mu)$ the limit

$$\bar{\varphi}(x) \,=\, \lim_{n\,\to\,+\infty} \frac{1}{|F_n|} \sum_{g\,\in\,F_n} \varphi(g(x))$$

exists for μ-almost every $x \in X$, and it is G-invariant. If, in addition, μ is ergodic then $\bar{\varphi}(x) = \int \varphi\, d\mu$ at μ almost everywhere.

We say that a probability μ on the Borel subsets of a compact metric space (X, d) is *G-homogeneous* if given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ and $C_\varepsilon > 0$ such that

$$\mu(B^G_n(y,\, \delta_\varepsilon)) \;\leqslant\; C_\varepsilon\; \mu(B^G_n(x,\, \varepsilon)) \qquad \forall\, n \in \mathbb{N} \quad \forall\, x, y \in X.$$

The space of finitely generated groups which admit G-homogeneous measures includes both finitely generated groups of isometries on a Riemannian manifold and finitely generated groups of homeomorphisms on a compact topological group (cf. [5, §4.2]).

Proposition 5.7. *[5, Lemma 4.10 and Corollary 4.13] Let G be a group generated by a finite collection G_1 of homeomorphisms of a compact metric space X. If the corresponding group action admits a G-homogeneous probability measure μ, then the limit*

$$h_\mu(G,\, G_1,\, x) \,=\, \lim_{\varepsilon\,\to\,0^+}\, \limsup_{n\,\to\,+\infty}\, -\frac{1}{n}\, \log \mu\big(B^G_n(x, \varepsilon)\big)$$

does not depend on $x \in X$ and is equal to $h_{\mathrm{top}}(G,\, G_1)$.

This result may be generalized to the context of G-Gibbs probability measures as follows. We say that a probability measure μ is a *G-Gibbs measure*, with respect to a continuous potential φ and a strictly increasing sequence $(F_n)_n$ of compact subsets of G which exhausts G, if given $\varepsilon > 0$ there exists a constant $K_\varepsilon > 0$ such that

$$K_\varepsilon^{-1} \leqslant \frac{\mu\big(B_n^G(x,\,\varepsilon)\big)}{e^{-P_{\mathrm{top}}((G,G_1),\,\varphi)\,n+\frac{1}{|F_n|}\sum_{g\,\in\,F_n}\,\varphi(g(x))}} \leqslant K_\varepsilon \qquad \forall\, n \in \mathbb{N} \quad \forall\, x \in X.$$

Notice that a G-Gibbs measure for $\varphi \equiv 0$ is a G-homogeneous measure.

Corollary 5.8. *Let G be an amenable group generated by a finite collection G_1 of homeomorphisms of a compact metric space X, and take $\varphi \in C(X)$. If the group action admits an invariant ergodic G-Gibbs measure μ with respect to the potential φ and a tempered Følner sequence $(F_n)_n$, then for μ-almost every $x \in X$ one has*

$$P_{\mathrm{top}}((G,G_1),\,\varphi) = \lim_{\varepsilon\,\to\,0^+}\,\limsup_{n\,\to\,+\infty}\,-\frac{1}{n}\,\log\mu\big(B_n^G(x,\varepsilon)\big) + \int \varphi\,d\mu.$$

6 Carathéodory structures for finitely generated group actions

Pesin and Pitskel introduced in [36] a notion of pressure for invariant but not necessarily compact sets by a continuous map. This is a particular case of the so-called Carathéodory structures (also known as Carathéodory capacities) described in great generality later in [35, Chapter 4], which turned to have a wide range of applications in many different dynamical contexts. We refer the reader e.g. to [6, 40, 47] and references therein for some of these applications arising in the context of non-uniform hyperbolicity, free and amenable group actions.

6.1 Upper Carathéodory capacities

Let G be a finitely generated group acting on a compact metric space (X,d). Consider a finite generating set $\mathbb{G}_1$, made up from homeomorphisms of X, which is symmetric (that is, $\mathbb{G}_1 = \mathbb{G}_1^{-1}$) and does not contain the identity map id_X. For each $n \in \mathbb{N}$, let $G_n \subset G$ denote the ball of radius n centered at id_X in the Cayley graph of the group G with respect to the distance

$$D(f,g) = \begin{cases} \min\,\Big\{k\,\in\,\mathbb{N}\colon\; fg^{-1} = g_{i_k}\ldots g_{i_1}\ \text{ and }\ g_{i_j}\in\mathbb{G}_1\Big\} & \text{if exists} \\ 0 & \text{otherwise.}\end{cases}$$

For instance, $G_1 = \mathbb{G}_1 \cup \{\mathrm{id}_X\}$. For each finite set $F \subset G$, consider the F-dynamical ball centered at $x \in X$ defined by

$$B_F(x,\varepsilon)\,=\,\Big\{y \in X\colon\; d(g(x),g(y)) < \varepsilon \quad \forall\, g \in F\Big\}.$$

and, given $\varphi \in C(X)$, set $S_F\varphi(x) = \sum_{h \in F} \varphi(h(x))$. Take $\varphi \in C(X)$ and fix a subset $Z \subset X$, a real number $s \geqslant 0$, a natural $N \in \mathbb{N}$, and a strictly increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite non-empty subsets of G which exhausts G. Define

(6.1)
$$M_\varphi(Z, N, \varepsilon, s, (F_n)_n) \;=\; \inf_{\mathcal{C}} \Big\{ \sum_{B_{F_n}(x,\varepsilon) \in \mathcal{C}} \exp\big(-s\,|F_n| + \sup_{y \in B_{F_n}(x,\varepsilon)} S_{F_n}\varphi(y)\big) \Big\}$$

where the infimum is taken over the collection $\mathcal{C}_N(Z, \varepsilon, (F_n)_n)$ of all finite or countable covers $\mathcal{C} = \big\{B_{F_{n_i}}(x_i, \varepsilon)\colon\; n_i \geqslant N \text{ and } x_i \in X\big\}$ of Z. The quantity $M_\varphi(Z, N, \varepsilon, s, (F_n)_n)$ is non-decreasing as N increases, so the limit

$$M_\varphi(Z, \varepsilon, s, (F_n)_n) \;=\; \lim_{N \to \infty} M_\varphi(Z, N, \varepsilon, s, (F_n)_n)$$

exists. It is known that the function $s \mapsto M_\varphi(Z, \varepsilon, s, (F_n)_n)$ has a unique critical point where it jumps from infinity to zero. Thus one defines

$$P(Z, \varepsilon, \varphi, (F_n)_n) \;=\; \inf\,\Big\{s \geqslant 0\colon\; M_\varphi(Z, \varepsilon, s, (F_n)_n) = 0\Big\}.$$

One can prove that the function $\varepsilon \mapsto P(Z, \varepsilon, \varphi, (F_n)_n)$ is monotone, therefore the following limit exists

$$P(Z, \varphi, (F_n)_n) \;=\; \lim_{\varepsilon \to 0^+} P(Z, \varepsilon, \varphi, (F_n)_n).$$

The *topological pressure* we will deal with in this section for finitely generated group actions is precisely $P\colon\; \varphi \mapsto P(X, \varphi, (F_n)_n)$.

6.2 P is a pressure function

In this section we show that, having fixed Z and $(F_n)_n$ as before, the function $\varphi \mapsto P(Z, \varphi, (F_n)_n)$ is a pressure function if it is always finite. We remark that the first term appearing in the summands in (6.1) might be more generally written as $e^{-s\,a_n}$ for some sequence $(a_n)_n$ of real numbers. Yet, as may be attested during the proof of Lemma 6.2, the map P is translation invariant only if the sequences $(a_n)_n$ and $(|F_n|)_n$ have the same growth rate.

Lemma 6.1. *$P(Z, \varphi, (F_n)_n) \leqslant P(Z, \psi, (F_n)_n)$ for every φ, $\psi \in C(X)$ satisfying $\varphi \leqslant \psi$.*

Proof. Consider φ, $\psi \in C(X)$ with $\varphi \leqslant \psi$. Therefore $S_{F_n}\varphi(y) \leqslant S_{F_n}\psi(y)$ for every $y \in Y$ and $n \in \mathbb{N}$. Consequently, if $\mathcal{C}$ is an arbitrary finite or countable cover of Z, then for $s \geqslant 0$

$$\sum_{B_{F_n}(x,\varepsilon) \in \mathcal{C}} \exp\big(-s|F_n| + \sup_{y \in B_{F_n}(x,\varepsilon)} S_{F_n}\varphi(y)\big) \leqslant \sum_{B_{F_n}(x,\varepsilon) \in \mathcal{C}} \exp\big(-s|F_n| + \sup_{y \in B_{F_n}(x,\varepsilon)} S_{F_n}\psi(y)\big).$$

Taking the infimum over covers $\mathcal{C} \in \mathcal{C}_N(Z,\varepsilon,(F_n)_n)$ and letting $N \to +\infty$, we get

$$\begin{aligned} M_\varphi(Z,N,\varepsilon,s,(F_n)_n) &\leqslant M_\psi(Z,N,\varepsilon,s,(F_n)_n) \\ M_\varphi(Z,\varepsilon,s,(F_n)_n) &\leqslant M_\psi(Z,\varepsilon,s,(F_n)_n) \end{aligned}$$

which imply that

$$\begin{aligned} P(Z,\varepsilon,\varphi,(F_n)_n) &= \inf\Big\{s\colon M_\varphi(Z,\varepsilon,s,(F_n)_n)=0\Big\} \\ &\leqslant \inf\Big\{s\colon M_\psi(Z,\varepsilon,s,(F_n)_n)=0\Big\} = P(Z,\varepsilon,\psi,(F_n)_n). \end{aligned}$$

Letting $\varepsilon \to 0^+$, we get $P(Z,\varphi,(F_n)_n) \leqslant P(Z,\psi,(F_n)_n)$, as claimed. □

Lemma 6.2. *$P(Z,\varphi+c,(F_n)_n) = P(Z,\varphi,(F_n)_n) + c$ for every $\varphi \in C(X)$ and $c \in \mathbb{R}$.*

Proof. It is immediate that, for every $\varphi \in C(X)$, $c \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$S_{F_n}(\varphi+c)(x) = \sum_{h \in F_n} (\varphi+c)(h(x)) = c\,|F_n| + \sum_{h \in F_n} \varphi(h(x)).$$

Thus, evaluating on dynamical balls and summing over each arbitrary finite or countable cover $\mathcal{C}$, we conclude that, for any $s \geqslant 0$,

$$\begin{aligned} &\sum_{B_{F_n}(x,\varepsilon)\in\mathcal{C}} \exp(-s\,|F_n| + \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}(\varphi+c)(x)) \\ &\qquad = \sum_{B_{F_n}(x,\varepsilon)\in\mathcal{C}} \exp((c-s)\,|F_n| + \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}\varphi(x)). \end{aligned}$$

Taking the infimum over covers $\mathcal{C} \in \mathcal{C}_N(Z,\varepsilon,(F_n)_n)$ and letting $N \to +\infty$, we deduce that

$$M_{(\varphi+c)}(Z,\varepsilon,s,(F_n)_n) = M_\varphi(Z,\varepsilon,s-c,(F_n)_n).$$

Thus,

$$\begin{aligned} &\inf\Big\{s\colon\ M_{\varphi+c}(Z,\varepsilon,s,(F_n)_n)=0\Big\} \\ =\ &\inf\Big\{s\colon\ M_\varphi(Z,\varepsilon,s-c,(F_n)_n)=0\Big\} \\ =\ &\inf\Big\{(s-c)+c\colon\ M_\varphi(Z,\varepsilon,s-c,(F_n)_n)=0\Big\} \\ =\ &\inf\Big\{(s-c)\colon\ M_\varphi(Z,\varepsilon,s-c,(F_n)_n)=0\Big\} + c. \end{aligned}$$

Finally, taking the limit with $\varepsilon \to 0^+$ we obtain the claimed equality. □

The most laborious step to verify that the Carathéodory structure defines a pressure function is to prove the convexity condition.

Lemma 6.3. *For every finite set I, $a \in [0,1]$ and arbitrary choices $(x_i)_{i\in I}$, $(y_i)_{i\in I}$ and $(z_i)_{i\in I}$, the following inequality holds*

$$\sum_{i\in I} e^{z_i}\, e^{a\,x_i+(1-a)\,y_i} \;\leqslant\; \Big(\sum_{i\in I} e^{z_i+x_i}\Big)^a \Big(\sum_{i\in I} e^{z_i+y_i}\Big)^{1-a}.$$

Proof. Write $\sum_{i\in I} e^{z_i}\, e^{a\,x_i+(1-a)\,y_i} = \sum_{i\in I} e^{a\,(z_i+x_i)+(1-a)\,(z_i+y_i)}$ and apply Holder's inequality. □

Lemma 6.4. *For any φ, $\psi \in C(X)$ and arbitrary $a \in [0,1]$, one has*

$$P(Z, a\,\varphi + (1-a)\,\psi, (F_n)_n) \;\leqslant\; a\,P(Z,\varphi,(F_n)_n) + (1-a)\,P(Z,\psi,(F_n)_n).$$

Proof. As the generalized Birkhoff sums used on the definition of P are affine, we have

$$S_{F_n}(a\,\varphi + (1-a)\,\psi)(y) = a\,S_{F_n}\varphi(y) + (1-a)\,S_{F_n}\psi(y) \qquad \forall\, y \in X.$$

Thus, for any dynamical ball $B_{F_n}(x,\varepsilon)$ and arbitrary finite or countable cover $\mathcal{C}$, we can write

$$\sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}(a\varphi+(1-a)\psi)(y) \leqslant a \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}\varphi(y)+(1-a) \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}\psi(y).$$

Therefore,

$$\sum_{B_{F_n}(x,\varepsilon)\in\mathcal{C}} \exp\Big(-s|F_n| + \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}(a\varphi+(1-a)\psi)(y)\Big)$$
$$\leqslant \sum_{B_{F_n}(x,\varepsilon)\in\mathcal{C}} \exp\Big(-s|F_n| + a \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}\varphi(y) + (1-a) \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}\psi(y)\Big).$$

Lemma 6.3 now implies that the right-hand side is bounded above by the product of

$$\Big(\sum_{B_{F_n}(x,\varepsilon)\in\mathcal{C}} \exp\Big(-s\,|F_n| + \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}\varphi(y)\Big)\Big)^a$$

and

$$\Big(\sum_{B_{F_n}(x,\varepsilon)\in\mathcal{C}} \exp\Big(-s\,|F_n| + \sup_{y\in B_{F_n}(x,\varepsilon)} S_{F_n}\psi(y)\Big)\Big)^{1-a}.$$

To complete the proof it is enough to show that if

$$s \;>\; a\,P(Z,\varphi,(F_n)_n) + (1-a)\,P(Z,\psi,(F_n)_n)$$

then

$$P(Z, a\,\varphi + (1-a)\,\psi, (F_n)_n) \;\leqslant\; s.$$

Given such an s, either $s > P(Z, \varphi, (F_n)_n)$ or $s > P(Z, \psi, (F_n)_n)$. Assume that the first inequality holds and consider $\varepsilon > 0$ and a family of covers $\tilde{\mathcal{C}} \in \mathcal{C}_N(Z, \varepsilon, (F_n)_n)$ such that

$$M_\varphi(Z, \varepsilon, s, (F_n)_n) = \inf_{\tilde{\mathcal{C}}} \sum_{B_{F_n}(x,\varepsilon) \in \tilde{\mathcal{C}}} \exp\Big(- s\,|F_n| + \sup_{y \in B_{F_n}(x,\varepsilon)} S_{F_n}\varphi(y)\Big).$$

Then

$$\begin{aligned}
&M_{a\varphi+(1-a)\psi}(Z, \varepsilon, s, (F_n)_n) = \\
&= \inf_{\mathcal{C} \in \mathcal{C}_N(Z,\varepsilon,(F_n)_n)} \sum_{B_{F_n}(x,\varepsilon) \in \mathcal{C}} \exp\Big(- s|F_n| + \sup_{y \in B_{F_n}(x,\varepsilon)} S_{F_n}(a\varphi + (1-a)\psi)(y)\Big) \\
&\leqslant \inf_{\tilde{\mathcal{C}}} \sum_{B_{F_n}(x,\varepsilon) \in \tilde{\mathcal{C}}} \exp\Big(- s|F_n| + \sup_{y \in B_{F_n}(x,\varepsilon)} S_{F_n}(a\varphi + (1-a)\psi)(y)\Big) \\
&\leqslant M_\varphi(Z, \varepsilon, s, (F_n)_n)^a \cdot \inf_{\tilde{\mathcal{C}}} \Big(\sum_{B_{F_n}(x,\varepsilon) \in \tilde{\mathcal{C}}} \exp\Big(- s|F_n| + \sup_{y \in B_{F_n}(x,\varepsilon)} S_{F_n}\psi(y)\Big)\Big)^{1-a}
\end{aligned}$$

so $M_{a\varphi+(1-a)\psi}(Z, \varepsilon, s, (F_n)_n) = 0$. Thus, $P(Z, a\varphi+(1-a)\,\psi, (F_n)_n) \leqslant s$. $\square$

6.3 An alternative variational principle for finitely generated group actions

We can now apply Theorem 2.2 and deduce the following consequence.

Proposition 6.5. *Let (G, G_1) be a group finitely generated by a finite set G_1, which is symmetric and contains the identity, of homeomorphisms of a compact metric space X. Consider a strictly increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite non-empty subsets of G which exhausts G. Assume that $P(X, \varphi, (F_n)_n)$ is finite for every $\varphi \in C(X)$. Then there exists a concave, upper semi-continuous function $\mathfrak{h}_G\colon \mathcal{P}(X) \to \mathbb{R}$ such that*

$$P(X, \varphi, (F_n)_n) = \max_{\mu \in \mathcal{P}(X)} \Big\{ \mathfrak{h}_G(\mu) + \int \varphi\, d\mu \Big\} \qquad \forall\, \varphi \in C(X).$$

7 Countable sofic group actions

For any integer $\ell \geqslant 1$, let $\mathrm{Sym}(\ell)$ denote the group of permutations of the set $\{1, 2, \ldots, \ell\}$. Following Gromov [25] and Weiss [45], a countable group G is called *sofic* if there exist a sequence of positive integers $(\ell_i)_{i \in \mathbb{N}}$ with limit $+\infty$ and a sequence of permutations, called *sofic approximation sequence of* G and denoted by $\Sigma = \{\sigma_i\colon G \to \mathrm{Sym}(\ell_i) \mid i \in \mathbb{N}\}$, which satisfies

(a) $\lim_{i \to +\infty} \frac{1}{\ell_i} \#\big\{1 \leqslant k \leqslant \ell_i\colon\ \sigma_i(h) \circ \sigma_i(g)(k) = \sigma_i(hg)(k)\big\} = 1$ for all $g, h \in G$;

(b) $\lim_{i\to+\infty} \frac{1}{\ell_i} \#\{1 \leqslant k \leqslant \ell_i\colon\ \sigma_i(h)(k) \neq \sigma_i(g)(k)\} = 1$ for all distinct $g, h \in G$.

Thus, a sofic approximation sequence to a countable group is a sequence of partial actions on finite sets that asymptotically approximates the action of the group on itself by left-translations.

When no confusion arises, to simplify the notation we will write $\sigma_g(\cdot)$ instead of $\sigma(g)(\cdot)$ for every map $\sigma\colon G \to \mathrm{Sym}(\ell)$. Consider a metric ρ on the space $\mathcal{F}(\{1, 2, \dots, \ell\}, X)$ of functions from $\{1, 2, \dots, \ell\}$ to X defined by

$$\rho(\psi_1,\, \psi_2) = \frac{1}{\ell}\Big(\sum_{1\leqslant j\leqslant \ell} d(\psi_1(j),\, \psi_2(j))^2\Big)^{\frac{1}{2}}.$$

Having fixed $\ell \in \mathbb{N}$, a finite subset $F \subset G$ and $\sigma\colon G \to \mathrm{Sym}(\ell)$, define

$$\mathrm{Map}(F, \sigma, \delta) = \Big\{\psi\colon \{1, 2, \dots, \ell\} \to X \ \mid\ \max_{g\in F}\ \rho\big(\sigma_g \circ \psi,\, \psi \circ \sigma_g\big) < \delta\Big\}.$$

Given a probability measure μ on the Borel subsets of X, non-empty finite subsets $F \subset G$ and $L \subset C(X)$, a map $\sigma\colon G \to \mathrm{Sym}(\ell)$ and $\delta > 0$, consider the set

$$\mathrm{Map}_\mu(F, \sigma, L, \delta) = \Big\{\psi \in \mathrm{Map}(F, \sigma, \delta)\colon \Big|\frac{1}{\ell}\sum_{j=0}^{\ell-1}\varphi(\psi(j)) - \int \varphi\, d\mu\Big| < \delta,\ \forall\, \varphi \in L\Big\}.$$

Definition 7.1. *Given a probability measure μ on the Borel subsets of X and a countable sofic group G with a sofic approximation sequence Σ, the* sofic metric entropy *of the continuous action of G on X with respect to μ is defined by*

$$h_{\Sigma,\, G}(\mu) = \sup_{\varepsilon>0}\ \inf_F\ \inf_L\ \inf_{\delta>0}\ h^{\varepsilon}_{\Sigma,\, \mu}(G, \varphi, F, L, \delta),$$

where

$$h^{\varepsilon}_{\Sigma,\, \mu}(G, \varphi, F, L, \delta) = \limsup_{i\to+\infty} \frac{1}{\ell_i}\, \log\, s^{\varepsilon}_{\Sigma,\, \mu}(F, \sigma_i, L, \delta)$$

and $s^{\varepsilon}_{\Sigma,\, \mu}(F, \sigma_i, L, \delta)$ denotes the maximal cardinality of the (ρ, ε)-separated subsets of maps which belong to the family $\mathrm{Map}_\mu(F, \sigma_i, L, \delta)$.

We have omitted the dependence of $h_{\Sigma,\, G}(\mu)$ on ρ since this notion turns out to be independent of the pseudo-metric as far as we keep it compatible with the topology induced by ρ (cf. [27]).

7.1 Pressure function for countable sofic group actions

The concept of sofic pressure of a continuous action, which we will now recall, was introduced in [18] as an extension of the sofic entropy of [11]. To simplify the notation we shall omit the dependence of this notion on the space X and the pseudo-metric ρ. Let $\mathbb{S}$ be a continuous action of a countable sofic group G of homeomorphisms of a compact metric space (X, d), and let Σ be a sofic approximation sequence of G. Given a non-empty finite subset $F \subset G$, $\varphi \in C(X)$, $\sigma \colon G \to \mathrm{Sym}(\ell)$, $\delta > 0$ and $\varepsilon > 0$, denote

$$M^{\varepsilon}_{\Sigma}(\varphi, F, \delta, \sigma) = \sup_{E} \Big\{ \sum_{\psi \in E} e^{\sum_{j=1}^{\ell} \varphi(\psi(j))} \Big\}$$

where the supremum is taken over all (ρ, ε)-separated subsets E of $\mathrm{Map}(F, \sigma, \delta)$. Moreover, set

$$P^{\varepsilon}_{\Sigma}(G, \varphi, F, \delta) \; = \; \limsup_{i \to +\infty} \frac{1}{\ell_i} \log M^{\varepsilon}_{\Sigma}(\varphi, F, \delta, \sigma_i).$$

Definition 7.2. *The* sofic topological pressure *of φ under the action of G is defined by*

$$P_{\Sigma}(G, \varphi) = \sup_{\varepsilon > 0} \, \inf_{F} \, \inf_{\delta > 0} \, P^{\varepsilon}_{\Sigma}(G, \varphi, F, \delta)$$

where the sets $F \subset G$ are chosen non-empty and finite.

It is known that the sofic entropy $h_{\Sigma}(G) = P_{\Sigma}(G, 0)$ of an action may depend on the choice of the sofic approximation, and may have different positive values even for mixing subshifts of finite type (see [1]).

7.2 A variational principle

Denote by $\mathcal{P}_G(X)$ the set of probability measures on the Borel subsets of X which are preserved by all elements of a group G and by $h_{\Sigma,G}(\mu)$ the extension in [18] of the concept of sofic metric entropy of $\mu \in \mathcal{P}_G(X)$ defined by L. Bowen in [11]. The next result establishes a variational principle and shows that the finiteness of the sofic pressure is a sufficient condition for $\mathcal{P}_G(X) \neq \emptyset$.

Theorem 7.3. *[18, Theorem 1.2] Given a countable sofic group G with a sofic approximation sequence Σ, let $\mathbb{S}$ be a continuous action of G on a metric space (X, d) and $\varphi \colon X \to \mathbb{R}$ be a continuous potential. Then*

$$P_{\Sigma}(G, \varphi) = \sup_{\mu \in \mathcal{P}_G(X)} \Big\{ h_{\Sigma,G}(\mu) + \int \varphi \, d\mu \Big\}. \tag{7.1}$$

Moreover, if $P_{\Sigma}(G, \varphi) \neq -\infty$ then $\mathcal{P}_G(X) \neq \emptyset$.

The existence of equilibrium states for countable sofic group actions is not known under great generality. If G is a countable sofic group and $X = \{1, 2, \dots, d\}^G$, then every local potential has an equilibrium state which is a Gibbs measure (cf. [18, Theorem 5.3] and [3]). More generally, if the group action is expansive then the metric entropy function $h_{\Sigma,G}(\cdot)$ varies upper semi-continuously and equilibrium states do exist for continuous potentials (cf. [19]). In the next subsection we discuss the existence of finitely additive equilibrium states for countable sofic group actions.

Remark 7.4. On [27, page 541], one finds an example of an action of the sofic free group $G = \mathbb{F}_2$ on a closed $\mathbb{F}_2$-invariant subset X of the shift $\{0,1,2\}^{F_2}$ whose sofic topological entropy is equal to $-\infty$ for every sofic approximation sequence. In particular, $\mathcal{P}_{\mathbb{F}_2}(X) = \emptyset$. In this example, since $h_{\text{top}}(\mathbb{F}_2) < +\infty$, after choosing a finite generator set G_1 it is advantageous to use instead the notion of entropy we have considered in Section 4. This way, Proposition 4.4 may be applied and provides a concave, upper semi-continuous function $\mathfrak{h}_{\mathbb{F}_2} \colon \mathcal{P}(X) \to \mathbb{R}$ such that

$$P_{\text{top}}(\mathbb{F}_2, G_1, \varphi) = \max_{\mu \in \mathcal{P}(X)} \left\{ \mathfrak{h}_{\mathbb{F}_2}(\mu) + \int \varphi \, d\mu \right\} \qquad \forall \, \varphi \in C(X).$$

7.3 Sofic equilibrium states

Firstly, we observe that the sofic pressure satisfies the axioms of a pressure function listed in Definition 2.1.

Lemma 7.5. *[18, Proposition 6.1] The function $\varphi \in C(X) \mapsto P_\Sigma(G, \varphi)$ is monotone, translation invariant and convex, provided that $P_\Sigma(G, \cdot) \neq \pm\infty$.*

Therefore, we may apply Theorem 2.2 and Theorem 2.4, thus concluding that:

Proposition 7.6. *Given a countable sofic group G of homeomorphisms on a compact metric space X, assume that $P_\Sigma(G, \cdot) \neq \pm\infty$ for a sofic approximation sequence Σ. Then there exists a concave, upper semi-continuous map $\mathfrak{h}_{\Sigma,G} \colon \mathcal{P}(X) \to \mathbb{R}$ satisfying*

$$\mathfrak{h}_{\Sigma,G}(\mu) = \inf_{\varphi \in C(X)} \left\{ P_\Sigma(G, \varphi) - \int \varphi \, d\mu \right\} \qquad \forall \, \mu \in \mathcal{P}(X)$$

and such that, for every $\varphi \in C(X)$,

$$P_\Sigma(G, \varphi) = \max_{\mu \in \mathcal{P}(X)} \left\{ \mathfrak{h}_{\Sigma, G}(\mu) + \int \varphi \, d\mu \right\} = \max_{\mu \in \mathcal{P}_G(X)} \left\{ \mathfrak{h}_{\Sigma, G}(\mu) + \int \varphi \, d\mu \right\}. \tag{7.2}$$

Moreover:

(a) Every measure $\mu \in \mathcal{P}(X)$ which attains the maximum is G-invariant and $\mathfrak{h}_{\Sigma, G}(\mu) \geqslant 0$.

(b) There is a Baire residual subset $\mathfrak{R} \subset C(X)$ such that every $\varphi \in \mathfrak{R}$ has a unique G-invariant maximizing probability measure.

(c) If the sofic metric entropy map $\mu \in \mathcal{P}_G(X) \mapsto h_{\Sigma, G}(\mu)$ is concave and upper semi-continuous, then $h_{\Sigma, G}(\mu) = \mathfrak{h}_{\Sigma, G}(\mu)$ for every $\mu \in \mathcal{P}_G(X)$.

Proof. Taking into account the proof of Theorem 2.4, we are left to show the claim in item (b). Firstly, we observe that, using [7, Theorem 2], a G-invariant probability measure μ attains the maximum in (7.2) if and only if it is a tangent functional to the convex function $P_\Sigma(G,\cdot)$ at φ; that is, if and only if

$$P_\Sigma(G,\varphi+\psi) - P_\Sigma(G,\varphi) \geqslant \int \psi\, d\mu \qquad \forall\, \psi \in C(X).$$

Thus, item (b) is a consequence of [31] (see also [37, page 12]), which ensures that the convex function $P_\Sigma(G,\cdot)$, acting on the separable Banach space $\mathbf{B} = C(X)$, admits a unique tangent functional for every φ in a residual subset of $\mathbf{B}$. □

We note that, if one considers a pressure function $P_\Sigma^{\text{top}}(G,\cdot)$ whose domain is the space $L^\infty(X)$ of bounded measurable maps on X, a direct use of Theorem 2.2 also provides an upper semi-continuous entropy map $\mathfrak{h}^a_{\Sigma,G}\colon \mathcal{P}_a(X) \to \mathbb{R}$ such that, for each $\varphi \in L^\infty(X)$,

$$P_\Sigma^{\text{top}}(G,\varphi) = \max_{\mu\in\mathcal{P}_a(X)} \Big\{ \mathfrak{h}^a_{\Sigma,G}(\mu) + \int \varphi\, d\mu \Big\}.$$

Even though the maximum in the right-hand side above is taken over the whole space of finitely additive measures, a further step can be performed to show that such maximum is attained in the space of G-invariant finitely additive measures. We leave the details to the interested reader.

It is also worth mentioning that, while the sofic metric entropy map $\mu \in \mathcal{P}_G(X) \mapsto h_{\Sigma,G}(\mu)$ may be not affine for general approximation sequences Σ, it is affine for uniformly diffuse sofic approximations. We refer the reader to [13, §6.1] for the precise definitions and more information.

The sofic entropy map $h_{\Sigma,G}$ of the action of a sofic group G on a shift $\mathcal{A}^G$, with a finite alphabet $\mathcal{A}$, is upper semi-continuous ([19]). So, the existence of equilibrium states for the variational principle (7.1) is guaranteed. In [13, §8.2] we find an example of one such action by shifting, associated to a free group G, with more than one measure of maximal sofic entropy. By Remark 2.5, this example also provides an instance where uniqueness of equilibrium states for the generalized variational principle (7.2) fails.

7.4 Uniqueness of sofic equilibrium states

This characterization of equilibrium states, as tangent functionals to a convex pressure function, turns out to be extremely useful in order to obtain a criterion for uniqueness of equilibrium states in terms of the differentiability of the sofic pressure function (we refer the reader e.g. to [7] for the notions of Fréchet and Gateaux differentiability). Indeed, Lemma 7.5 and [7, Theorem 3, Corollary 4] have the following immediate consequence.

Corollary 7.7. *Let G be a countable sofic group of homeomorphisms on a compact metric space X, and assume that $P_\Sigma(G,\cdot) \neq \pm\infty$ for a sofic approximation sequence Σ. Then:*

(a) $P_\Sigma(G,\cdot)$ is locally affine at φ if and only if $P_\Sigma(G,\cdot)$ is Fréchet differentiable at φ.

(b) $P_\Sigma(G,\cdot)$ has a unique tangent functional at φ if and only if $P_\Sigma(G,\cdot)$ is Gateaux differentiable at φ.

7.5 A particular case: countable amenable group actions

Let G be a countable amenable group (hence sofic) of homeomorphisms of a compact metric space X. The amenable metric entropy, defined by Ollagnier and Pinchon in [33], and the sofic metric entropy (respectively, the amenable and sofic topological pressures) coincide for this class of group actions, as proved in [12, 28]. Thus, in the case of amenable groups, the sofic metric entropy does not depend on the chosen sofic approximation sequence Σ, and the conclusions of Proposition 7.6, valid in this particular case, bridges between the classical and the sofic thermodynamic objects.

Let us briefly recall the definition of the amenable metric entropy (more details in [33]). Consider a tempered Følner sequence $(F_n)_{n\in\mathbb{N}}$ in G. Take a probability measure $\mu \in \mathcal{P}_G(X)$ and a finite measurable partition $\mathcal{A}$ of X with finite Shannon conditional entropy with respect to μ (definition in [43, §4.3]). Given $n \in \mathbb{N}$, let $\bigvee_{g\in F_n} g^{-1}(\mathcal{A})$ denote the refinement of $\mathcal{A}$ by $g \in F_n$. Then the limit, called the *entropy of G with respect to μ and the partition $\mathcal{A}$*,

$$h_\mu(G,\mathcal{A}) \,=\, \lim_{n\to+\infty} \frac{1}{|F_n|}\, H\Big(\bigvee_{g\in F_n} g^{-1}(\mathcal{A})\Big)$$

exists and does not depend on the chosen Følner sequence (cf. [13]).

Definition 7.8. *Given $\mu \in \mathcal{P}_G(X)$, the amenable metric entropy of μ with respect to the action of G is defined by*

$$h_G^{(\mathrm{am})}(\mu) \,=\, \sup_{\mathcal{A}}\, h_\mu(G,\mathcal{A})$$

where the supremum is taken over all the finite measurable partitions of X with finite Shannon entropy.

As mentioned previously, the sofic measure-theoretic entropy may not be affine, though modified versions of it are. However, the amenable metric entropy map is affine. Actually, the proof of [43, Theorem 8.1], showing that the measure-theoretic entropy map is affine when we consider a continuous self-map of a compact metric space, may be repeated in this setting (replacing $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ by $\left(\frac{1}{|F_n|}\right)_{n\in\mathbb{N}}$ in the definition of entropy with respect to a partition) to show that $h_G^{(\mathrm{am})}$ is an affine map.

According to item (c) of Proposition 7.6, if the sofic entropy map is concave and upper-semicontinuous, then one has $\mathfrak{h}_{\Sigma,G}(\mu) = h_{\Sigma,G}(\mu)$, for every $\mu \in \mathcal{P}_G(X)$ and every sofic approximation sequence Σ. As $h_G^{(\mathrm{am})}$ is affine, item (d) of Theorem 2.4 informs that, whenever $h_G^{(\mathrm{am})}$ is upper semi-continuous, one also has $h_G^{(\mathrm{am})} = \mathfrak{h}_{\Sigma,G}$ in $\mathcal{P}_G(X)$.

8 Variational measure-theoretic entropy

Let X and Y be compact metric spaces. Assume that $\Gamma_X\colon C(X)\to\mathbb{R}$ and $\Gamma_Y\colon C(Y)\to\mathbb{R}$ are pressure functions, and $\mathfrak{h}_X$ and that $\mathfrak{h}_Y$ are the concave, upper semi-continuous maps provided by Theorem 2.2, which satisfy

$$\begin{aligned}\mathfrak{h}_X(\mu) &= \inf_{\varphi\in C(X)}\left\{\Gamma_X(\varphi) - \int \varphi\,d\mu\right\} \qquad \forall\,\mu\in\mathcal{P}(X)\\ \mathfrak{h}_Y(\nu) &= \inf_{\psi\in C(Y)}\left\{\Gamma_Y(\psi) - \int \psi\,d\nu\right\} \qquad \forall\,\nu\in\mathcal{P}(Y).\end{aligned}$$

Lemma 8.1. *Suppose that there exists a homeomorphism $\mathfrak{F}\colon X\to Y$ and that*

$$\Gamma_Y(\psi) = \Gamma_X(\psi\circ\mathfrak{F}) \qquad \forall\,\psi\in C(Y). \tag{8.1}$$

Then $\mathfrak{h}_X(\mu) = \mathfrak{h}_Y(\mathfrak{F}_*\mu)$ for every $\mu\in\mathcal{P}(X)$, where $\mathfrak{F}_*\mu$ stands for the probability measure in $\mathcal{P}(Y)$ defined by $\mathfrak{F}_*\mu(B) = \mu(\mathfrak{F}^{-1}(B))$ for every Borel subset B of Y.

Proof. We start by observing that, for every $\varphi\in C(X)$, there is a unique $\psi\in C(Y)$, namely $\psi = \varphi\circ\mathfrak{F}^{-1}$, such that $\varphi = \psi\circ\mathfrak{F}$. Therefore, given $\mu\in\mathcal{P}(X)$,

$$\begin{aligned}\mathfrak{h}_Y(\mathfrak{F}_*\mu) &= \inf_{\psi\in C(Y)}\left\{\Gamma_Y(\psi) - \int_Y \psi\,d\mathfrak{F}_*\mu\right\}\\ &= \inf_{\psi\in C(Y)}\left\{\Gamma_X(\psi\circ\mathfrak{F}) - \int_X (\psi\circ\mathfrak{F})\,d\mu\right\}\\ &= \inf_{\varphi\in C(X)}\left\{\Gamma_X(\varphi) - \int \varphi\,d\mu\right\} = \mathfrak{h}_X(\mu).\end{aligned}$$

□

Let $G\times X\to X$ and $H\times Y\to Y$ be two semigroup actions by continuous self-maps on compact metric spaces X and Y, respectively. These actions may be rewritten using the continuous maps $S_1\colon\, G\to C(X,X)$ and

$S_2\colon H \to C(Y,Y)$. Following [22], one says that the actions S_1 and S_2 are *isomorphic* if there exist a semigroup isomorphism $\tau\colon G \to H$ and a homeomorphism $\mathfrak{F}\colon X \to Y$ such that $S_2 \circ \tau = T \circ S_1$, where $T(f) = \mathfrak{F} \circ f \circ \mathfrak{F}^{-1}$ for every $f \in C(X,X)$. In this case, we will use the same notion of topological pressure for both G and H.

For instance, assume that $G = H = \mathbb{Z}$, $\tau = \mathrm{id}_{\mathbb{Z}}$, that $f_1\colon X \to X$ and $f_2\colon Y \to Y$ are homeomorphisms and let $S_1\colon \mathbb{Z} \to \mathrm{Homeo}(X,X)$ and $S_2\colon \mathbb{Z} \to \mathrm{Homeo}(Y,Y)$ be given by $S_1(n) = f_1^n$ and $S_2(n) = f_2^n$, respectively. Then S_1 and S_2 are isomorphic by the pair $(\tau, \mathfrak{F})$ if and only if f_1 is topologically conjugate to f_2 by $\mathfrak{F}$.

Proposition 8.2. *Let X and Y be compact metric spaces and consider two semigroup actions by semigroups G, $H \in \mathcal{G}$, denoted by $S_1\colon G \to C(X,X)$ and $S_2\colon H \to C(Y,Y)$. Let $\mathfrak{h}_{X,G}$ and $\mathfrak{h}_{Y,H}$ be the concave, upper semi-continuous measure-theoretic entropy maps assigned by Theorem 2.2 to the pressure functions $P_{\mathrm{top}}(G,.)\colon C(X) \to \mathbb{R}$ and $P_{\mathrm{top}}(H,.)\colon C(Y) \to \mathbb{R}$. If S_1 and S_2 are isomorphic by the pair $(\tau,\mathfrak{F})$, then $P_{\mathrm{top}}(G,.)$ and $P_{\mathrm{top}}(H,.)$ satisfy the condition* (8.1)*, and so*

$$\mathfrak{h}_{X,G}(\mu) \;=\; \mathfrak{h}_{Y,H}(\mathfrak{F}_*(\mu)) \qquad \forall\, \mu \in \mathcal{P}(X).$$

Proof. To show that $P_{\mathrm{top}}(G,.)$ and $P_{\mathrm{top}}(H,.)$ satisfy the condition (8.1) we just have to adapt the argument to prove [43, Theorem 9.8 (iv)], which was done for a semigroup generated by a single continuous map. Afterwards, we apply Lemma 8.1 to $\Gamma_X = P_{\mathrm{top}}(G,.)$ and $\Gamma_Y = P_{\mathrm{top}}(H,.)$. □

For example, if $X = Y$, G is an abelian group in $\mathcal{G}$ and $\tau\colon G \to G$ is defined by $\tau(g) = g^{-1}$, then $P_{\mathrm{top}}(G,.) = P_{\mathrm{top}}(\tau(G),.)$ and the condition (8.1) holds. Given an action $S_1\colon G \to \mathrm{Homeo}(X,X)$, then S_1 and $S_2 = S_1 \circ \tau$ are isomorphic by the pair (τ, id_X); thus $\mathfrak{h}_{X,G} = \mathfrak{h}_{X,\tau(G)}$.

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